REFLECTION POSITIVITY AND INVERTIBLE TOPOLOGICAL PHASES

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Abstract. We implement an extended version of reflection positivity (Wick-rotated unitarity) for invertible topological quantum field theories and compute the abelian group of deformation classes using stable homotopy theory. We apply these field theory considerations to lattice systems, assuming the existence and validity of low energy effective field theory approximations, and thereby produce a general formula for the group of Symmetry Protected Topological (SPT) phases in terms of Thom's bordism spectra; the only input is the dimension and symmetry type. We provide computations for fermionic systems in physically relevant dimensions. Other topics include symmetry in quantum field theories, a relativistic 10-fold way, the homotopy theory of relativistic free fermions, and a topological spin-statistics theorem.

Contents

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1. Introduction

The moduli space, or stack, of a geometric object with fixed discrete invariants is a central object of interest in geometry. A typical example is the moduli stack of Riemann surfaces of fixed genus. Here the underlying topological space is connected, but moving up to complex dimension two the moduli stack of complex surfaces of general type with fixed Euler number and signature is not necessarily connected. It has finitely many components [Ca], so there are finitely many *deformation types*. If singular objects are permitted, then sometimes connectivity can be restored. For example, Reid [Re] speculates that the moduli stack of three-dimensional Calabi-Yau varieties is connected if one allows certain singularities. To illustrate further, consider the moduli stack of one-dimensional Riemannian manifolds. If we allow simple singularities, such as the figure eight, then we can connect a single circle to two circles by a path (standard Morse function on a two-dimensional torus). We can also connect one circle to two circles if we allow noncompact smooth manifolds: elongate a circle to an ellipse to two lines and then each line to a circle. On the other hand, the set of path components of the moduli stack of smooth closed Riemannian 1-manifolds is isomorphic to $\mathbb{Z}^{\geqslant 0}$; the isomorphism maps a 1-manifold to the cardinality of π_0.

In theoretical physics one contemplates moduli stacks of quantum systems with fixed discrete invariants, such as dimension and symmetry type. If we remove the singular locus of *phase transitions*, then path components of the moduli stack are identified with *phases* of the quantum system.[1] In condensed matter physics the quantum systems are modeled discretely, using lattices, and the classification of phases is an active topic of current interest. As far as we know there is not a robust mathematical theory of lattice systems and their moduli which leads to rigorous computations of sets of phases. Quantum field theories also exhibit phases and phase transitions, and those too are topical. Physicists often pass back and forth between lattice models and field theories using various mechanisms. In this paper we envision passing from a lattice system to an effective low-energy field theory using two heuristic principles to argue that the set of phases is conserved:

(i) the deformation class of a quantum system is determined by its low energy behavior;
(ii) the low energy physics of a *gapped*[2] system is well-approximated by a *topological*[3] field theory.

A stronger version of (i) asserts that the entire homotopy type of the moduli stack is determined by the low energy behavior. These two principles are applied by physicists to quantum systems of all kinds: condensed matter systems, quantum field theories, string theories. For discrete lattice systems we also assume an emergent low energy relativistic symmetry. We remark that fracton models [NH] are thought not to satisfy (ii), nor to have any sort of emergent relativistic symmetry, but those are not relevant here. The lattice models that motivate this paper belong to a special class, often called *short-range entangled*, for which the long-range effective topological field theory is *invertible*. In particular, there is a unique ground state for the lattice model on any compact manifold. Early discussions of this property may be found in [CGW, K1]. (Now 'invertible' is used in place of 'short-range entangled' to describe the lattice model.)

One reason to pass to continuum models is that there *is* a mathematical Axiom System for Wick-rotated quantum field theory; it encodes the structural properties of correlation functions and linear spaces of quantum states. It was first introduced in the mid 1980's for scale-independent theories: by Segal [Se1] for 2-dimensional conformal field theories and later by Atiyah [A1] for topological field theories. With modifications these axioms are now believed to be relevant to scale-dependent theories as well. In this framework a quantum field theory is a linear representation of a *bordism category*. The latter categorifies Thom's bordism groups [T], and a field theory categorifies integer-valued bordism invariants, such as the signature of a compact oriented manifold.

The twin pillars of quantum field theory are *locality* and *unitarity*. These fundamental properties persist after Wick rotation: locality manifests as factorization laws for correlation functions and unitarity manifests as reflection positivity. Locality is encoded in the Axiom System using composition of morphisms: gluing bordisms along codimension one submanifolds. In the early 1990's, especially motivated by 3-dimensional Chern-Simons theory, an *extended* notion of locality was

[1]There is a tight analogy with the example of Riemannian 1-manifolds above: a figure eight corresponds to a first-order phase transition, while a noncompact manifold corresponds to a higher-order phase transition.

[2]A quantum mechanical system is gapped if its minimum energy is an eigenvalue of finite multiplicity of the Hamiltonian, assumed bounded below, and is an isolated point of the spectrum. For quantum field theory 'spectrum' means the spectrum of representations of the translation group of Minkowski spacetime. For lattice systems the spectral gap must be bounded below independent of the lattice size.

[3]We allow a topological field theory tensored with a non-topological invertible field theory; see §5.4. A field theory is *topological* if it does not depend on any continuously varying (background) fields, such as a metric or conformal structure. We give a precise definition of a topological field theory in §2.2.

introduced by gluing bordisms with corners along higher codimension submanifolds, and this led naturally to formulations involving higher categories; see [F1, La, BD, L], for example. Extended locality is a characteristic feature of both physical and mathematical applications of field theory, whereas unitarity is often not present in purely mathematical contexts. Unitarity in field theory, or rather its Wick-rotated manifestation—reflection positivity—is the first main subject of this paper. It is straightforward to implement reflection positivity in the non-extended Axiom System. A natural question arises: What is the *extended* notion of reflection positivity that goes with extended locality? We offer a solution in a very special case: *invertible topological* field theories. These theories can be studied using stable homotopy theory [FHT1], and indeed we define[4] a theory of this type as a map of spectra. Spectra are the main characters in stable homotopy theory, a mathematical field that partly grew out of Thom's work. The domain of an invertible topological field theory is a Madsen-Tillmann bordism spectrum, and our main result tells that extended reflection positivity brings us full circle to the bordism spectra introduced by Thom in his thesis [T].

Theorem 1.1. *There is a 1:1 correspondence*

$$\left\{\begin{matrix}\text{deformation classes of reflection positive}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with symmetry group } H_n\end{matrix}\right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tor}}. \tag{1.2}$$

The right hand side is the torsion subgroup of homotopy classes of maps from a Thom spectrum to a shift of the Anderson dual to the sphere spectrum. There are standard computational techniques which we employ in the latter part of this paper to illustrate the efficacy of the theorem. Often field theories are classified by enumerating lagrangians with specified background and fluctuating fields that are consistent with a given symmetry group. By contrast, Theorem 1.1 is a direct *quantum* classification of correlation functions and state spaces, as encoded by the Axiom System. The only inputs are the discrete invariants: the spacetime dimension n and the Wick-rotated vector symmetry group[5] H_n. We prove Theorem 1.1 in §8 as a corollary of a more general result (Theorem 8.20). There is a related assertion which remains conjectural in this paper: the abelian group of deformation classes of *all* reflection positive invertible field theories, including those that are not topological, is obtained by simply omitting 'tor' on the right hand side of (1.2). We make some comments about this generalization in §5.4 and Remark 8.41; we use it in the computations of §9. More to the point, we introduce "continuous invertible topological field theories" as a substitute for invertible non-topological theories, and prove theorems for those.[6] We remark that for general reasons nontorsion only arises if the spacetime dimension n is odd.

We apply Theorem 1.1 to compute the abelian group of phases of invertible lattice systems with fixed dimension and symmetry type. This implicitly assumes that every possible deformation class of invertible topological theory can be realized by a lattice model, something not implied by the

[4]A better starting point is the topological version of the Axiom System, and then Theorem 5.12 brings us to stable homotopy theory. But as the literature is still in flux we opt for Ansatz 5.14 instead; see the remarks following Theorem 5.12.

[5]The basic case is $H_n = SO_n$. In general there is a homomorphism $\rho_n\colon H_n \to O_n$ whose image includes SO_n; the kernel consists of internal global symmetries. There is a unique associated stable symmetry group H independent of dimension, as we prove in Theorem 2.19.

[6]We thank Peter Teichner for his encouragement to adopt this point of view.

heuristic principles (i) and (ii) above. We emphasize the algorithmic nature of our classification: given a spacetime dimension n and a symmetry group H_n the right hand side of (1.2) is the group of topological phases and is computable. We provide concrete evidence for this application of Theorem 1.1: in §9.3 we undertake detailed computations for some fermionic systems and compare to results in the physics literature, the latter derived by means of physical arguments. Some readers may wish to examine our tables of computations before tackling the more theoretical parts of the paper. In unpublished work Kitaev [K1, K2, K3] develops a classification of invertible phases based on microscopic considerations, and he too is led to stable homotopy theory and results consonant with our effective field theory classification. Kapustin [Ka1] initiated computations of topological phases via character groups of bordism groups, and he used them and subsequent computations, for example [KTTW], as phenomenological evidence for a general classification along these lines. Gaiotto-Kapustin [GK], following on Gu-Wen [GW], show that some invertible fermionic phases defined by lattice models are characterized by spin bordism groups; see also Brumfiel-Morgan [BrMo]. Campbell [C] and Guo-Putrov-Wang [GPW] carry out computations for other bosonic and fermionic cases of interest, providing further affirmative checks against the condensed matter literature.

A second subject of this paper, after extended reflection positivity, is the study of symmetry groups in relativistic quantum field theory, and that is where we begin in §2. Our starting point is a theory on n-dimensional Minkowski spacetime with global symmetry group $H_{1,n-1}$, after dividing out by translations. The analytic continuations of correlation functions, which exist as a consequence of positivity of energy, are invariant under the complex Lie group $H_n(\mathbb{C})$, and the entire Wick-rotated theory is symmetric under the compact real form $H_n \subset H_n(\mathbb{C})$ that appears on the left hand side of (1.2). In an appendix §A.3 we discuss Wick rotation and the CRT theorem[7] for general symmetry types. We use the rigidity of compact Lie groups to constrain possible symmetry groups (Theorem 2.7) *à la* Coleman-Mandula [CM]. One key result in this section (Theorem 2.19) is the existence and uniqueness of a stabilization H, which is the group in the Thom spectrum on the right hand side of (1.2). When we move to curved Riemannian manifolds—i.e., couple the theory to background gravity—the symmetry becomes infinitesimal in the sense of Cartan: an H_n-structure on the tangent bundle. In §3 we formulate reflection symmetry in terms of a group extension

$$1 \longrightarrow H_n \longrightarrow \widehat{H}_n \longrightarrow \{\pm 1\} \longrightarrow 1; \tag{1.3}$$

elements in $\widehat{H}_n \backslash H_n$ are a Wick-rotated analog of anti-unitary symmetries in quantum mechanics. We use this extension in §4.1 to define an involution on the bordism category of H_n-manifolds. In the basic case $H_n = SO_n$ the involution is orientation-reversal; our uniform treatment gives analogs for any symmetry group. For example, fermionic theories with time-reversal symmetry (and no other symmetry) have $H_n = \mathrm{Pin}_n^{\pm}$: the involution takes a pin structure to its "w_1-flipped" pin structure. Topological field theories are independent of the Riemannian metric, so we can replace H_n by a noncompact analog, which we construct in Appendix C.

Three basic lessons we learned about reflection positivity: (i) 'reflection' and 'positivity' are distinct; (ii) 'reflection' is a structure whereas 'positivity' is a condition; and (iii) 'extended positivity' is a structure, not a condition. In the Axiom System a field theory is defined to be a

[7] There is a subtlety concerning double covers of the Lorentz signature isometry group, uncovered in [GT], which we explicate in the context of Wightman quantum field theory for general symmetry types; see §A.2.

homomorphism—a symmetric monoidal functor—

$$F\colon \operatorname{Bord}_{\langle n-1,n\rangle}(H_n) \longrightarrow \operatorname{Vect}_{\mathbb{C}} \tag{1.4}$$

from the bordism category to the category of complex vector spaces and linear maps. A *reflection structure* (§4.3) is equivariance data for F with respect to the generalized orientation-reversal involution on $\operatorname{Bord}_{\langle n-1,n\rangle}(H_n)$ and the involution of complex conjugation on $\operatorname{Vect}_{\mathbb{C}}$. (We briefly review involutions on categories and equivariant functors in Appendix B.) A reflection structure induces a hermitian metric on the vector space of states attached to an $(n-1)$-manifold, and *positivity* is the condition that these hermitian structures be positive definite. Analogous to reflection positivity in Euclidean space (§3.2) we see that the partition function of the *double* of a manifold with boundary must be positive in order that a reflection structure be positive. Our treatment of this material using general symmetry groups means it applies to all theories, including those with time-reversal symmetry and fermions which, after Wick rotation, involve nonorientable manifolds with pin structure.

To proceed to *extended* field theories we specialize in §5 to the invertible case. (Invertible field theories were first singled out in [FM2] in an application to string theory.) In §5.2 we review how invertibility catalyzes a transition to stable homotopy theory: the analog of (1.4) for an invertible topological field theory is a map of spectra

$$F\colon \Sigma^n MTH_n \longrightarrow \mathcal{I}. \tag{1.5}$$

The domain is the invertible quotient of a higher bordism category, a Madsen-Tillmann spectrum. There is freedom to choose the codomain spectrum, and in §5.3 we introduce two universal choices. The first is (a shift of) $I\mathbb{C}^\times$, a "character dual" to the sphere spectrum, which is used to track topological theories on the nose: theories with unequal partition functions are distinct. The second universal target spectrum is (a shift of) the *Anderson dual* $I\mathbb{Z}(1)$ to the sphere spectrum. It tracks deformation classes of invertible theories rather than individual theories. Significantly, in the spirit of "derived geometry", maps into $I\mathbb{Z}(1)$ classify deformation classes of invertible theories that are not necessarily topological; the topological theories have finite order in the abelian group of homotopy classes of maps. For the application to topological phases one should include the non-topological theories, as they incorporate nonzero thermal Hall response. An example is Kitaev's E_8 phase [K5]. See §5.4 for a general discussion, including an interpretation of maps into $I\mathbb{Z}(1)$ as a continuous invertible topological field theory. In this paper we only use non-topological field theories heuristically and posit that their deformation classes are encoded in continuous topological field theories, which we treat rigorously.

The main arguments about *extended positivity* occur in §6–§8. Madsen-Tillmann spectra filter Thom spectra, which leads to a notion of a *stable* invertible topological field theory: a map out of a Thom spectrum. For invertible theories a reflection structure is a lift of (1.5) to an equivariant map of $\mathbb{Z}/2$-equivariant spectra. Section 6 begins with a brief exposition of spectra and Borel equivariant stable homotopy theory, sufficient for the considerations in this paper. The involution on the domain that models generalized orientation-reversal is straightforward to construct from the group extension (1.5). On the other hand, it is not clear *a priori* how to model complex conjugation on the codomain, so in §6.3 we give an extended discussion motivating our choice, Definition 6.30. We conclude §6 by introducing spectra and spaces of "higher super lines", including Hermitian structures and a higher notion of positivity (Definition 6.41, Definition 6.45). There is a basic link between *non-extended* positivity and stability, which we establish in Theorem 7.22 and Theorem 7.30 using obstruction theory arguments. This results in an intermediate classification (Corollary 7.33) of invertible topological theories with reflection structure satisfying non-extended positivity. We undertake a more systematic study in §9. There we define extended positivity for invertible field theories in terms of higher super lines and their embellishments. We give an intuitive construction of the space of invertible reflection positive theories, and then we identify its homotopy type in Theorem 8.20, whose proof occupies the second half of §6. Theorem 1.1 is a corollary.

The third main subject of this paper is what might be called the homotopy theory of relativistic free fermions.[8] There are two distinct scenarios in which a free fermion field theory gives rise to a deformation class of n-dimensional reflection positive invertible theories. First scenario: an $(n-1)$-dimensional free fermion theory has an associated n-dimensional invertible anomaly theory, which is not necessarily topological; our concern here is its deformation class.[9] Second scenario: an n-dimensional *massive* free fermion theory has a long-range effective invertible topological field theory approximation, according to the general principle (ii) invoked above, applied to a quantum field theory rather than a lattice system. We sketch the first scenario in some detail in §9.2, culminating in a formula (Conjecture 9.70) for the deformation class of the anomaly theory. Since massive free fermions have trivial anomaly, the starting point is the group of free fermionic data under direct sum modulo massive free fermionic data. The existence of a mass term has a meaning in terms of Clifford modules (Lemma 9.55), and this produces an identification of the quotient as a homotopy group of the KO-theory spectrum (Theorem 9.63). The formula for the deformation class of the associated anomaly theory is, conjecturally, a product of the Atiyah-Bott-Shapiro map [ABS] with the KO-theory class of the spinor data, followed by a Pfaffian map (Conjecture 9.70). In this paper we provide a detailed sketch of these ideas; we hope to give a thorough mathematical treatment in the future. There is a huge literature on relativistic free fermion field theories and associated anomalies; the recent paper [W1], which describes several particular cases in detail, provided motivation and guidance for the general story here. By contrast, we only comment briefly (§9.2.6) on the second scenario, beginning from a massive n-dimensional free fermion theory, enough to show that the starting and ending data match those in the first scenario. In fact, it is this second scenario that is relevant to this paper, and in particular the conjecture (9.75) about its low energy effective field theory is used in the computations which follow.

To enable detailed comparisons with the physics literature we carry out the discussion of relativistic free fermions for 10 cases simultaneously. To enumerate them we resume group theoretical arguments in §9.1 to classify relativistic symmetry groups whose internal subgroup is the unit reals $\{\pm 1\}$, unit complexes $\mathbb{T}$, or unit quaternions SU_2. Restricting to fermionic theories in which $(-1)^F$ embeds in this internal subgroup—which implements the "spin/charge relation" [SeWi]—we obtain the 10 groups in question. They include Spin, $\mathrm{Pin}^{\pm}$, and semidirect products with the various unit scalars. This "relativistic 10-fold way" is a variation on the nonrelativistic case, which is described in many works: a sample includes [D, AZ, HHZ, K6, SRFL, FM1, KZ, WS]. Remark 9.32

[8] A free fermion field theory is neither topological nor invertible, but it has an associated invertible field theory.

[9] The anomaly theory lies in *differential* KO-theory, whereas its deformation class lies in topological KO-theory.

provides a link to this condensed matter literature: we compute a group I of symmetries that preserve points of *space* in a nonrelativistic setting. It is this group I which acts at each lattice site in a discrete model, and it can be used to compare to the ubiquitous symmetry tables for fermion lattice systems. Our uniform treatment is based on Lemma 9.27, which embeds each symmetry group in a Clifford algebra. Usual constructions with Clifford modules—the Atiyah-Bott-Shapiro-Thom class, Dirac operators and their indices—then generalize easily. There is a purely geometric application that we do not pursue here: index theory on pin and pin^c manifolds is straightforward using this embedding.

The results of the homotopy theory computations are reported in §9.3. We provide a table for each of the 10 fermionic symmetry groups. In each spacetime dimension $n \leqslant 5$ we compute the group of free fermion theories (Theorem 9.63), the group of deformation classes of interacting theories (Theorem 1.1), and the map between them (Conjecture 9.70). We make comparisons with the condensed matter literature where available and find almost total agreement; in the few cases with a discrepancy we motivate a reexamination of the physics assertions. In §10 we outline how the calculations are done and supply Ext charts that encode the E_2-term of the relevant Adams spectral sequences. The Ext charts also encode the map to KO-theory; in fact, one of the main tasks in this section is to rewrite the “twisted” Atiyah-Bott-Shapiro maps in a more accessible form. We provide more explanation of the charts in Appendix D. In that appendix we also illustrate the use of Margolis homology to derive information from the Adams spectral sequence. Papers by Campbell [C] and Beaudry-Campbell [BeC] give pedagogical introductions to the Adams spectral sequence and flesh out the details of our computations. Notice that whereas Theorem 1.1 computes the group of interacting phases for any symmetry type, the 10 fermionic symmetry types are special in that there is a notion of a free fermionic phase which does not exist in general. This leads to a richer application of homotopy theory and a more stringent test against the condensed matter literature.

The sections of the paper not yet mentioned contain complements or background material. An analog of the spin-statistics theorem in relativistic quantum field theory holds for reflection positive invertible topological theories, as we explain in §11. Section A.1 contains a review of pin groups and Clifford algebras, background for the discussion of the CRT theorem later in Appendix A and for some of the material in §9.

Beyond the immediate relevance to the study of topological phases, the successful application of bordism computations to quantum systems is evidence—perhaps the first substantial test against physics—that the sparse Axiom System initiated by Segal and Atiyah captures essential features of quantum field theory.

The lecture series [F4] provides additional background and discussion on many of the topics treated here.

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2. Symmetry groups in relativistic quantum field theory

The analytic extension of correlation functions, a consequence of positivity of energy, provides a powerful constraint on symmetry groups. We explore the general structure in §2.1 from the Wick-rotated point of view. The rigidity of compact Lie groups is the key idea that underlies our proofs of structure theorems, such as Theorem 2.7. One important result is Theorem 2.19, which constructs a stable group H from an n-dimensional symmetry group H_n, assuming the spacetime dimension satisfies $n \geqslant 3$. In the expository §2.2 we recall the axiomatization of a field theory as a categorified bordism invariant. We accommodate general symmetry groups on curved manifolds using reductions of frame bundles, an analog of the passage from Klein's Erlangen Programm [BB] to Cartan's H-structures [S].

2.1. Stabilization of Wick-rotated symmetry groups

The Poincaré group is the connected double cover of the identity component of the isometry group $\mathcal{I}_{1,n-1}$ of n-dimensional Minkowski spacetime M^n. Minkowski spacetime M^n is assumed equipped with a time orientation, a choice of component of timelike vectors in the inner product space $\mathbb{R}^{1,n-1}$ of translations. Let $\mathcal{I}^{\uparrow}_{1,n-1} \subset \mathcal{I}_{1,n-1}$ denote the subgroup of isometries that preserve the time orientation. Assume $n \geqslant 2$. Many treatments of quantum field theory, for example those based on S-matrix theory, begin with the assumption that the Poincaré group is a *subgroup* of the (unbroken) global symmetry group $\mathcal{H}_{1,n-1}$ of the theory. Then the Coleman-Mandula theorem [CM] asserts that on the level of Lie algebras there is a splitting as a direct sum of the Lie algebra of Poincaré with the Lie algebra of a *compact* Lie group K. We find it more natural to posit from the beginning a homomorphism $\rho_n\colon \mathcal{H}_{1,n-1} \to \mathcal{I}^{\uparrow}_{1,n-1}$. After all, $g \in \mathcal{H}_{1,n-1}$ acts on the operators in the theory, and so on the supports of those operators. For a single point operator, or local operator, that action is $\rho_n(g)$. The relativistic invariance of the theory is the hypothesis that the image of ρ_n contains the identity component of $\mathcal{I}^{\uparrow}_{1,n-1}$. Therefore, the image is either the identity component or the entire two-component group $\mathcal{I}^{\uparrow}_{1,n-1}$. The kernel of ρ_n is the group K of *internal* symmetries—symmetries that fix the points of spacetime. Note that K contains the central element of the Lorentz group $\mathrm{Spin}_{1,n-1}$ if that element acts effectively, which by the spin-statistics theorem happens if and only if the theory contains fermionic states. (That element is often denoted '$(-1)^F$'. Below we deduce in general a central element $k_0 \in K$ with $(k_0)^2 = 1$, and it is identified with either the central element of Spin or the identity element.) The internal symmetry group K is assumed to be a *compact* Lie group.[10]

Assume the translation subgroup $\mathbb{R}^{1,n-1} \subset \mathcal{I}^{\uparrow}_{1,n-1}$ lifts to a normal subgroup of $\mathcal{H}_{1,n-1}$; see [FM1, Remark 2.13] for a justification of this hypothesis. Let $H_{1,n-1}$ denote the quotient of $\mathcal{H}_{1,n-1}$ by this normal subgroup of translations. There is a short exact sequence[11]

$$1 \longrightarrow K \longrightarrow H_{1,n-1} \xrightarrow{\rho_n} O^{\uparrow}_{1,n-1} \tag{2.1}$$

[10]The global symmetry group of a "noncompact field theory", such as for a free massless $\mathbb{R}$-valued scalar field theory, may be noncompact. Our discussion does not include supersymmetries or higher symmetries.

[11]We overload the symbol 'ρ_n'. Here it denotes the homomorphism induced from the previous ρ_n after modding out translations. Below we use it for the complexification, restriction to the Euclidean real form, and various lifts.

where the image of ρ_n contains the identity component of $O^{\uparrow}_{1,n-1} \subset O_{1,n-1}$, by the relativistic invariance of the theory. The CRT theorem, reviewed in §A.3, gives a larger symmetry group. A fundamental consequence of the positivity of energy[12] in quantum field theory, also reviewed in §A.3, is a holomorphic extension[13] of correlation functions on which the complexification $H_n(\mathbb{C})$ of $H_{1,n-1}$ acts as symmetries. There is an exact sequence

$$1 \longrightarrow K(\mathbb{C}) \longrightarrow H_n(\mathbb{C}) \xrightarrow{\rho_n} O_n(\mathbb{C}) \tag{2.2}$$

of complex Lie groups. The *Wick-rotated theory* has a *compact* real form H_n of $H_n(\mathbb{C})$ as symmetry group such that H_n fits into the exact sequence

$$1 \longrightarrow K \longrightarrow H_n \xrightarrow{\rho_n} O_n \tag{2.3}$$

of compact Lie groups with the *same* compact kernel K as in (2.1). The image of this ρ_n is either O_n or SO_n, depending on whether the relativistic theory has spatial reflections or not; equivalently, by the CRT theorem, whether it has time-reversal symmetry or not.

Definition 2.4. The *symmetry type* of a quantum field theory is a pair (H_n, ρ_n) of a compact Lie group H_n and a homomorphism $\rho_n \colon H_n \to O_n$ whose image contains $SO_n \subset O_n$. The kernel K of ρ_n is called the *group of internal symmetries*. We require that the anti-Wick rotation to Minkowski spacetime has a Lorentzian real form (2.1) with compact internal symmetry group $K = \ker \rho_n$.

The caveats in footnote 10 apply. See Remark 2.13 below for an example of a pair (H_n, ρ_n) that does not satisfy the anti-Wick rotation condition. The symmetry type is a basic structure in a quantum field theory, useful to articulate explicitly in any example.

Define $SH_n = \rho_n^{-1}(SO_n)$ and let $\widetilde{SH}_n$ be the double cover of SH_n constructed from the spin double cover of SO_n. These compact Lie groups are usefully encoded in the pullback diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{SH}_n \arrow[r, "\rho_n"] \arrow[d, "2:1"] & \mathrm{Spin}_n \arrow[r] \arrow[d, "2:1"] & 1 \\ 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & SH_n \arrow[r, "\rho_n"] \arrow[d, hook, "1:2"] & SO_n \arrow[r] \arrow[d, hook, "1:2"] & 1 \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] & O_n & \end{tikzcd} \tag{2.5}$$

If $\rho_n \colon H_n \to O_n$ is surjective, define $\widetilde{H}_n$ as the pullback[14]

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}_n \arrow[r, "\rho_n"] \arrow[d] & \mathrm{Pin}^+_n \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] & O_n \arrow[r] & 1 \end{tikzcd} \tag{2.6}$$

[12] The dual to the cone of forward timelike vectors determines the notion of positive energy.
[13] See [KS] for a geometric version on curved manifolds.
[14] See §A.1 for a review of pin groups.

The restriction of $\widetilde{H}_n$ over $\mathrm{Spin}_n \subset \mathrm{Pin}_n^+$ is $\widetilde{SH}_n$. Let $\mathfrak{k}, \mathfrak{h}_n, \mathfrak{o}_n$ denote the Lie algebras of K, H_n, O_n, respectively. The following theorem makes precise the sense in which the entire symmetry group is nearly the product of (Wick-rotated) spacetime symmetries and internal symmetries. In our approach to symmetry it plays the role of the Coleman-Mandula theorem.

Theorem 2.7.

(1) *There is a splitting $\mathfrak{h}_n \cong \mathfrak{o}'_n \oplus \mathfrak{k}$, and ρ_n induces an isomorphism of Lie algebras $\mathfrak{o}'_n \xrightarrow{\cong} \mathfrak{o}_n$.*

(2) *If $n \geqslant 3$ there is an isomorphism $\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$. Hence there exists a central element $k_0 \in K$ with $(k_0)^2 = 1$ and an isomorphism*

$$SH_n \cong \mathrm{Spin}_n \times K \;/\; \langle(-1, k_0)\rangle, \tag{2.8}$$

where $\langle(-1, k_0)\rangle$ is the cyclic group generated by $(-1, k_0)$.

(3) *If $n \geqslant 3$ and $\rho_n\colon H_n \to O_n$ is surjective, then there exists a group extension*

$$1 \longrightarrow K \longrightarrow J \longrightarrow \{\pm 1\} \longrightarrow 1 \tag{2.9}$$

and a pullback diagram of group extensions

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}_n \arrow[r, "\rho_n"] \arrow[d] & \mathrm{Pin}_n^+ \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & J \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{2.10}$$

There is an isomorphism

$$H_n \cong \widetilde{H}_n \;/\; \langle(-1, k_0)\rangle. \tag{2.11}$$

The pullback (2.10) shows that the failure of $\widetilde{H}_n$ to be a product is encoded in the group extension (2.9), which is independent of n.

Corollary 2.12. *There is a canonical homomorphism $\mathrm{Spin}_n \to H_n$ under which the image of the central element $-1 \in \mathrm{Spin}_n$ is $k_0 \in K$.*

This homomorphism anti-Wick rotates back to a homomorphism of the Poincaré group into the total symmetry group $\mathcal{H}_{1,n-1}$ of the relativistic theory, the traditional starting point for discussions of symmetry in quantum field theory.

Remark 2.13. For $n = 2$ we can only conclude that $\widetilde{SH}_2$ is isomorphic to a semidirect product of Spin_2 and K. An example is $SH_2 = SO_2 \ltimes O_2$, where a rotation $R \in SO_2$ acts on O_2 by the automorphism that is the identity on $SO_2 \subset O_2$ and composes a reflection with R. Alternatively, $SH_2 \cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathbb{T} \times \mathbb{T})$ where the involution on $\mathbb{T} \times \mathbb{T}$ is $(\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_1^{-1}\lambda_2^{-1})$.

Proof of Theorem 2.7. Split the Lie algebra $\mathfrak{h}_n = [\mathfrak{h}_n, \mathfrak{h}_n] \oplus \mathfrak{z}$, where $\mathfrak{z} \subset \mathfrak{h}_n$ is the center, and let $\mathfrak{o}'_n$ be the orthogonal complement of the ideal $\mathfrak{k} \cap [\mathfrak{h}_n, \mathfrak{h}_n] \subset [\mathfrak{h}_n, \mathfrak{h}_n]$ with respect to the nondegenerate Killing form on the semisimple Lie algebra $[\mathfrak{h}_n, \mathfrak{h}_n]$. Then ρ_n induces an isomorphism $\mathfrak{o}'_n \to \mathfrak{o}_n$, which proves (1). The exponential of $\mathfrak{o}'_n$ is a closed Lie subgroup $S \subset \widetilde{SH}_n$ which locally projects diffeomorphically onto Spin_n under ρ_n, so is isomorphic to Spin_n. It follows that $\widetilde{SH}_n \cong S \ltimes K$.

We claim this semidirect product is a direct product if $n \geqslant 3$. To see this observe that conjugation by $s \in S$ induces an automorphism $\alpha(s)$ of K which is the identity on the identity component $K^0 \subset K$, since the Lie algebra of S commutes with the Lie algebra of K. Since S is connected, the induced automorphism of $\pi_0 K$ is also trivial. Hence on each component of K the automorphism $\alpha(s)$ is left multiplication by an element $z(s) \in Z^0$ in the center of K^0. (Proof: Write $\alpha = \alpha(s)$ and suppose $\alpha(k) = zk$ for some k in that component and $z \in K^0$. Any other element of that component has the form kk_0 for $k_0 \in K^0$, and $\alpha(kk_0) = z(kk_0)$. But we can also write any element in the component as $k'_0 k$ for some $k'_0 \in K^0$, and $\alpha(k'_0 k) = k'_0 zk = (k'_0 z {k'_0}^{-1})(k'_0 k)$ from which $k'_0 z {k'_0}^{-1} = z$. This holds for every $k'_0 \in K^0$, from which we deduce $z \in Z^0$.) Next, Spin_n acts trivially on Z^0; this follows since the outer automorphism group of a compact Lie group is discrete, every inner automorphism of the abelian group Z^0 is trivial, and Spin_n is connected. Hence the map $s \mapsto z(s)$ is a homomorphism $S \to Z^0$. But if $n \geqslant 3$ the Lie group $S \cong \mathrm{Spin}_n$ has no nontrivial homomorphisms to an abelian Lie group.

Assume $\rho_n \colon H_n \to O_n$ is surjective. We claim $\mathrm{Spin}_n \subset \widetilde{SH}_n \subset \widetilde{H}_n$ is a normal subgroup. Fix $\tilde{h} \in \widetilde{H}_n$ such that $\rho_n(\tilde{h}) = e_2 \in \mathrm{Pin}^+_n$. Conjugation by e_2 induces an involution $\alpha : \mathrm{Spin}_n \to \mathrm{Spin}_n$. It lifts to an automorphism of $\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$ defined as conjugation by $\tilde{h}$, so there is an induced automorphism $\beta \colon K \to K$ and a homomorphism $\gamma \colon \mathrm{Spin}_n \to K$.

Lemma 2.14. *If $n \geqslant 3$, then the homomorphism γ is trivial.*

Proof. Define $\widetilde{H}_n(\mathbb{C})$ by pulling back as in (2.6) using the complexified groups (2.2); pullback over the Lorentzian real forms to obtain the first of the pair of real forms $\widetilde{H}_{1,n-1} \subset \widetilde{H}_n(\mathbb{C}) \supset \widetilde{H}_n$. Note that $\tilde{h}$ lies in each of these groups, and conjugation by $\tilde{h}$ preserves both real forms. Thus we obtain a homomorphism $\mathrm{Spin}_n(\mathbb{C}) \to K(\mathbb{C})$ that restricts to $\gamma \colon \mathrm{Spin}_n \to K$ and to a homomorphism $\mathrm{Spin}_{1,n-1} \to K$. Now if γ is nontrivial, then so is the induced map on Lie algebras, and since $\mathfrak{o}_n$ is simple, $\dot{\gamma} \colon \mathfrak{o}_n \to \mathfrak{k}$ is injective. It follows that the Lie algebra map $\mathfrak{o}_{1,n-1} \to \mathfrak{k}$ is also injective. Hence $\mathfrak{k}$ contains a subalgebra isomorphic to $\mathfrak{o}_{1,2} \cong \mathfrak{sl}_2\mathbb{R}$. The Killing form on $\mathfrak{k}$ induces a nonzero semidefinite invariant symmetric bilinear form on the simple Lie algebra $\mathfrak{sl}_2\mathbb{R}$, which is impossible since every invariant symmetric form on $\mathfrak{sl}_2\mathbb{R}$ is a multiple of the Killing form, which is indefinite and nondegenerate. $\square$

It follows that $\mathrm{Spin}_n \subset \widetilde{H}_n$ is a normal subgroup. Set $J = \widetilde{H}_n / \mathrm{Spin}_n$. Then (2.10) follows from (2.6) and (2.11) follows from the fact that the kernel of $\widetilde{H}_n \to H_n$ equals the kernel of $\widetilde{SH}_n \to SH_n$. This completes the proof of Theorem 2.7. $\square$

Remark 2.15. Lemma 2.14 is not true without using the anti-Wick rotation back to Lorentzian signature. Namely, let $n = 3$ and $H_3 = \mathbb{Z}/2\mathbb{Z} \ltimes (SO_3 \times SO_3)$, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by shearing $(g_1, g_2) \mapsto (g_1, g_1 g_2)$; the homomorphism ρ_3 that kills the last factor $K = SO_3$ maps $H_3 \to O_3$ and sends the generator of $\mathbb{Z}/2\mathbb{Z}$ to the central element $-1 \in O_3$. The reader can check that $\gamma\colon \mathrm{Spin}_3 \to SO_3$ is surjective. But H_3 is not a possible symmetry group because of the anti-Wick rotation, as in the proof of Lemma 2.14.

If we restrict the internal symmetry group to only include the image of the central element $-1 \in \mathrm{Spin}_n$ under $\mathrm{Spin}_n \to H_n$, then there are five possibilities. In these cases K is trivial or $K \cong \{\pm 1\}$. Let $\mu_4 = \{\pm 1, \pm\sqrt{-1}\}$ be the multiplicative group of fourth roots of unity, and define $E_n \subset O_n \times \mu_4$ as the subgroup of (A, λ) such that $\det A = \lambda^2$.

Proposition 2.16. *Assume $n \geqslant 3$. If the internal symmetry group K is trivial, then $H_n \cong SO_n$ or $H_n \cong O_n$. If $K \cong \{\pm 1\}$ is cyclic of order two, then there are six possibilities for H_n up to isomorphism: $SO_n \times \{\pm 1\}$, Spin_n, $O_n \times \{\pm 1\}$, E_n, Pin_n^+, and Pin_n^-.*

Proof. The first statement is clear from the fact that the image of ρ_n in (2.3) is either SO_n or O_n. The group extensions by $\{\pm 1\}$ are central and are classified up to isomorphism by the cohomology group $H^2(BSO_n; \{\pm 1\}) \cong \mathbb{Z}/2\mathbb{Z}$ or $H^2(BO_n; \{\pm 1\}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, depending on the image of ρ_n, and it is not difficult to work out what the groups H_n are. ☐

The non-identity element of K in $SO_n \times \{\pm 1\}$, $O_n \times \{\pm 1\}$, and E_n is not the image of the central element $-1 \in \mathrm{Spin}_n$. This leaves the five basic symmetry types listed in the following table:

(2.17)

states/symmetry	H_n	K	k_0
bosons only	SO_n	$\{1\}$	1
fermions allowed	Spin_n	$\{\pm 1\}$	-1
bosons, time-reversal (T)	O_n	$\{1\}$	1
fermions, $T^2 = (-1)^F$	Pin_n^+	$\{\pm 1\}$	-1
fermions, $T^2 = \mathrm{id}$	Pin_n^-	$\{\pm 1\}$	-1

Appendix A reviews the pin groups and justifies the Wick rotation of time-reversal that leads to the last three lines in the first column of the table.

The main result in this section is a stabilization of H_n for increasing dimensions, as needed in Theorem 1.1. Throughout this paper for $k < \ell$ we use the embedding

$$\begin{aligned} O_k &\longrightarrow O_\ell \\ A &\longmapsto \begin{pmatrix} I_{\ell-k} & \\ & A \end{pmatrix} \end{aligned} \tag{2.18}$$

of orthogonal groups, where I denotes the identity matrix.

Theorem 2.19. *Assume $n \geqslant 3$. There exist compact Lie groups H_m, $m > n$, and homomorphisms i_n, ρ_n which fit into the commutative diagram*

$$\begin{tikzcd} H_n \arrow[r, hook, "i_n"] \arrow[d, "\rho_n"] & H_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\rho_{n+1}"] & H_{n+2} \arrow[r, hook] \arrow[d, "\rho_{n+2}"] & \dots \\ O_n \arrow[r, hook] & O_{n+1} \arrow[r, hook] & O_{n+2} \arrow[r, hook] & \dots \end{tikzcd} \tag{2.20}$$

in which squares are pullbacks.

The stabilization is usually apparent, even when $n = 2$ and Theorem 2.19 does not apply. For example, if $H_n = \mathrm{Pin}_n^+ \ltimes \mathbb{T} \,/\, \langle(-1,-1)\rangle$, where Pin_n^+ acts on $\mathbb{T} = U_1$ through its components by conjugation, then $H_m = \mathrm{Pin}_m^+ \ltimes \mathbb{T} \,/\, \langle(-1,-1)\rangle$. (We encounter this and related groups in §9.)

Remark 2.21. For $m < n$, define H_m and the homomorphism $\rho_m \colon H_m \to O_m$ by a pullback square:

$$\begin{tikzcd} H_m \arrow[r, dashed] \arrow[d, dashed, "\rho_m"'] & H_n \arrow[d, "\rho_n"] \\ O_m \arrow[r, hook] & O_n \end{tikzcd} \tag{2.22}$$

Remark 2.23. The pullback diagram (2.20) and the fact that $\rho_{m+1}(H_{m+1})$ acts transitively on the m-sphere imply diffeomorphisms

$$H_{m+1}/H_m \cong O_{m+1}/O_m \cong S^m \tag{2.24}$$

Proof of Theorem 2.19. In view of (2.8), for $m > n$ define $SH_m := \mathrm{Spin}_m \times K \,/\, \langle(-1,k_0)\rangle$, and so obtain a stabilization over SO_m. If $\rho_n(H_n) = SO_n$ this completes the proof. If not, define $\widetilde{H}_m$ as the pullback

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}_m \arrow[r] \arrow[d, two heads] & \mathrm{Pin}_m^+ \arrow[r] \arrow[d, two heads] & 1 \\ 1 \arrow[r] & K \arrow[r] & J \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{2.25}$$

and

$$H_m \cong \widetilde{H}_m \,/\, \langle(-1,k_0)\rangle. \tag{2.26}$$ □

Theorem 2.19 allows us to speak about symmetry types in quantum field theory independent of dimension. Set

$$H = \operatorname*{colim}_{n\to\infty} H_n. \tag{2.27}$$

For $H_n = SO_n$ we obtain $H = SO_\infty = SO$. Thus we can speak of 'oriented theories'='SO theories', 'Spin theories', 'Pin^+ theories', etc. The colimit of (2.20) is a homomorphism

$$\rho\colon H \longrightarrow O. \tag{2.28}$$

The *symmetry type* of a theory (Definition 2.4) can be taken to be the pair (H,ρ) in place of (H_n,ρ_n).

2.2. Curved manifolds and bordism categories with H_n-structure

Fix an n-dimensional relativistic quantum field theory with symmetry type (H_n, ρ_n). A "coupling to background gravity" means that we define the theory on each n-dimensional smooth Riemannian manifold X. The H_n-symmetry is no longer global; it is tangential and encoded in a reduction of the orthonormal frame bundle to H_n. Let $\mathcal{B}_O(X) \to X$ denote the principal O_n-bundle of frames: a point of $\mathcal{B}_O(X)$ is an orthonormal basis of the tangent space at a point of X. If $P \to X$ is a principal H_n-bundle, define the principal O_n-bundle $\rho_n(P) = P \times_{H_n} O_n \to X$ via mixing: $[ph, g] = [p, \rho_n(h)g]$ for all $p \in P$, $g \in O_n$, and $h \in H_n$.

Definition 2.29. An H_n*-structure* is a pair (P, θ) consisting of a principal H_n-bundle $P \to X$ equipped with an isomorphism of principal O_n-bundles $\mathcal{B}_O(X) \xrightarrow{\theta} \rho_n(P)$. An H_n*-manifold* is a Riemannian n-manifold endowed with an H_n-structure. A *differential* H_n*-structure* is a connection Θ on $P \to X$ with the property that θ maps the Levi-Civita connection to $\rho_n(\Theta)$.

It also makes sense to have an H_n-structure on a Riemannian manifold of dimension $\ell > n$, via the composition $H_n \xrightarrow{\rho_n} O_n \hookrightarrow O_\ell$, and on a manifold of dimension $k < n$ by stabilizing the O_k-frame bundle to a principal O_n-bundle via the inclusion $O_k \hookrightarrow O_n$. The Stability Theorem 2.19 implies that an H_n-manifold has an induced H_m-structure for all $m \geqslant n$. The same applies to the differential refinements.

Example 2.30. In bosonic theories of electromagnetism, $K = \mathbb{T}$ is the group U_1 of unit norm complex numbers, at least in the absence of further global symmetries. If there is no time-reversal symmetry, then $H_n = SO_n \times \mathbb{T}$. Thus $P \to X$ is the fiber product of the frame bundle with a principal $\mathbb{T}$-bundle, which is usually equipped with a connection, or gauge field. In theories of electromagnetism with fermions we still have $K = \mathbb{T}$, but now the center $-1 \in \mathrm{Spin}_n$ of the spin group is identified[15] with $-1 \in \mathbb{T}$ and so

$$H_n = \mathrm{Spin}_n^c = \mathrm{Spin}_n \times \mathbb{T} \,/\, \{\pm 1\} \tag{2.31}$$

is the group introduced in [ABS]. In other words, the Riemannian manifold X has a Spin^c-structure. If, in addition, there is time-reversal symmetry, then there are several different extensions, including the Atiyah-Bott-Singer group Pin_n^c; see Proposition 9.4 for the complete classification.

Example 2.32. For $H_n = O_n \times K$ an H_n-structure on a Riemannian manifold is an auxiliary principal K-bundle, and a *differential* H_n-structure is a connection on that bundle. For $H_n = \mathrm{Spin}_n^c$ the differential structure is usually called a spin^c *connection*.

The basic properties of Wick-rotated correlation functions on all *compact* manifolds simultaneously are encoded in the powerful framework of *bordism categories*, following the fundamental work of Segal [Se1] and Atiyah [A1]. Topological field theories do not depend on the metric, nor do they require differential structures, and for the most part we focus on topological theories and so on topological bordism categories. The geometric case is used as motivation; we make some comments in Remark 2.39.

[15] This assumes the spin/charge relation that particles of even electromagnetic charge are bosons while those of odd electromagnetic charge are fermions; see [SeWi] for more discussion.

For the topological bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ defined in the next paragraph, we drop the connection. We can also drop the Riemannian metric, as just mentioned, and to do so we would replace the compact Lie group H_n and homomorphism $\rho_n\colon H_n \to O_n$ with a canonically associated noncompact real Lie group $\underline{H}_n$ and homomorphism $\underline{H}_n \to GL_n\mathbb{R}$. We give the construction in Appendix C. Our field theories are *discrete* in the sense that the partition function in $\mathbb{C}$-valued and $\mathbb{C}$ has the discrete topology. Hence the theories factor through the topological bordism category built with $\underline{H}_n$-manifolds in place of H_n-manifolds. So we follow standard usage ("spin theories", etc.) and use the compact Lie group H_n, but no connections.

Define a *topological bordism category* $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ as follows. An object is a compact $(n-1)$-manifold Y without boundary, equipped with an H_n-structure $Q \to Y$ and an "arrow of time". To make sense of an H_n-structure on an $(n-1)$-manifold we stabilize the tangent bundle of Y to a rank n bundle $\underline{\mathbb{R}} \oplus TY \to Y$ by summing with a trivial line bundle, thought of as a normal direction into n dimensions. In this topological setting the Riemannian metric is not present; in the geometric setting of Remark 2.39, an object in a geometric bordism category is an $(n-1)$-manifold with a germ of an embedding in an n-manifold. The arrow of time is a normal orientation. In the topological setting only the tangential information is relevant—we can drop the germ—and the arrow of time is an orientation of the trivial subbundle $\underline{\mathbb{R}} \to Y$ of $\underline{\mathbb{R}} \oplus TY \to Y$. Nonetheless, even in this topological case it is illuminating to use the product germ $(-\epsilon, \epsilon) \times Y$ for some $\epsilon > 0$ and replace $\underline{\mathbb{R}} \oplus TY \to Y$ by the tangent bundle to the germ. A morphism $X\colon Y_0 \to Y_1$ is an equivalence class of compact n-manifolds X with H_n-structure $P \to X$ and an isomorphism of the boundary $\partial X \xrightarrow{\cong} Y_0 \amalg Y_1$ with the disjoint union of the incoming Y_0 and the outgoing Y_1; the equivalence relation is diffeomorphism commuting with all of the data. The isomorphisms include the H_n-structures and under those isomorphisms the orientation of the trivial bundle $\underline{\mathbb{R}} \to Y_i$ must line up with the incoming normal to the boundary for $i = 0$ and with the outgoing normal to the boundary for $i = 1$. In other words, the arrow of time is used to distinguish incoming and outgoing boundary components of morphisms. Composition of morphisms is gluing of bordisms. There is a additional commutative composition law on the category—disjoint union—and with this structure $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ is a *symmetric monoidal category.* See [L, CS] for detailed accounts.

A Wick-rotated field theory is a linear representation of a bordism category.

Definition 2.33. A *topological field theory with Wick-rotated vector symmetry group* H_n is a symmetric monoidal functor

$$F\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{2.34}$$

to the symmetric monoidal category of complex vector spaces under tensor product.

Much has been written about this definition, and we defer to previous accounts—such as the original [A1] and the recent survey [F2, §§2–4] and the references therein—for more exposition. Here we simply make the connection to point operators[16] and their correlation functions.

Remark 2.35 (Vector spaces of point operators). The sphere S^{n-1} is the link of a point in n dimensions, i.e., it is the boundary of a small ball about the point. Therefore, the vector space $V :=$

[16]These are usually called 'local operators' in the physical literature, but we use 'point' rather than 'local' to distinguish point operators from line operators and higher dimensional analogs, since those too are local.

$F(S^{n-1})$ is the space of point operators in a topological field theory; in a geometric theory we take a limit as the radius of the sphere shrinks to zero. If the theory has total symmetry group H_n, then the sphere has an H_n-structure and the vector space of point operators depends on it. If $H_n = SO_n \times K$ or $H_n = O_n \times K$, the extra data is a principal K-bundle $Q \to S^{n-1}$ (with connection). So there is a vector space V_Q of point operators for each Q. The group $\operatorname{Aut} Q$ of global gauge transformations acts on V_Q. For the trivial K-bundle this is the familiar representation of the global symmetry group K on local operators. If K is finite, then the "twist operators" for $Q \to S^1$ nontrivial are familiar in $n = 2$. They are also familiar when $H_2 = \mathrm{Spin}_2$, in which case the operators associated to the nonbounding spin circle create a defect at the excised point which changes the spin structure on the punctured surface. In $n = 3$ dimensions, if H_3 is a Cartesian product of SO_3 and $K = \mathbb{T}$, then the twist operators in some sense create a magnetically charged instanton for the global symmetry group K; the $\mathbb{Z}$-grading from the action of K on the point operators measures the electric charge.

Remark 2.36 (Correlation functions of point operators). Let M be a closed n-manifold. Fix points $x_1, \dots, x_k$ of M at which we place local operators. Let X be the compact manifold with boundary obtained from M by removing small open balls about each x_i; regard X as a bordism

$$X\colon \coprod_i S^{n-1}(x_i) \longrightarrow \varnothing^{n-1} \tag{2.37}$$

from the disjoint union of the k boundary spheres to the empty manifold. Equip the manifold X with an H_n-structure P, and let Q_i denote its restriction to the i^{th} sphere. Applying the theory (2.34) we obtain a homomorphism

$$F(X;P)\colon \underbrace{V_{Q_1} \otimes \cdots \otimes V_{Q_k}}_{k \text{ times}} \longrightarrow \mathbb{C} \tag{2.38}$$

which, evaluated on operators $\mathcal{O}_1, \dots, \mathcal{O}_k$, is usually written $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_k(x_k) \rangle_M$.

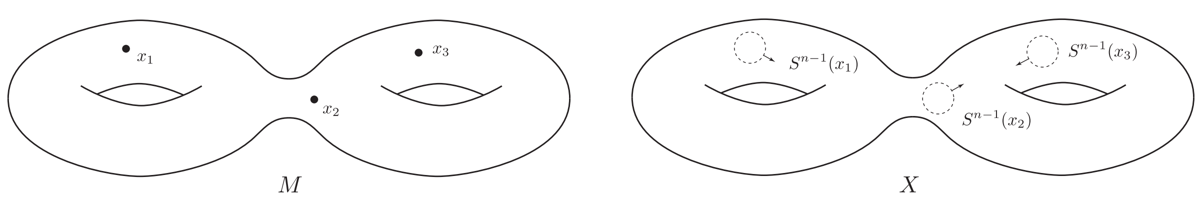


FIGURE 1. Correlation functions

Remark 2.39 (Remark about non-topological theories). Wick-rotated field theories which are not topological can also be formulated as functors on bordism categories, but now the objects and morphisms have a geometric structure. The references [Se2, KS, ST] develop this idea in various directions. We confine ourselves here to a few heuristic formal remarks. Analogous to the topological bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ we envision a *geometric bordism category* $\mathrm{Bord}^{\nabla}_{\langle n-1,n\rangle}(H_n)$ whose objects and morphisms are smooth manifolds with *differential* H_n-structures (Definition 2.29). An object is a closed $(n-1)$-manifold equipped with an infinite jet of an embedding into an n-dimensional manifold with differential H_n-structure and an arrow of time. A morphism is a compact n-manifold with differential H_n-structure together with a partition of the boundary and boundary isomorphisms as in the topological case. As in the topological case (2.34), a field theory is a functor with domain $\mathrm{Bord}^{\nabla}_{\langle n-1,n\rangle}(H_n)$ and codomain a suitable symmetric monoidal category of topological vector spaces. We want the correlation functions and vector spaces to vary smoothly in smooth families, so the whole structure must be "sheafified" over the category of smooth manifolds and smooth maps [ST, §2].

3. Unitarity and Wick rotation

We recall in §3.1 how positivity of energy leads to Wick rotation in quantum mechanics, and describe reflection positivity in that context. The usual quantum mechanical context for reflection positivity is recollected in §3.2, with attention paid to nontrivial internal symmetry groups. These preliminaries are motivation for §3.3, where we encode the reflection structure in a novel way via a coextension of the Wick-rotated vector symmetry group to a $\mathbb{Z}/2\mathbb{Z}$-graded group, constructed from a hyperplane reflection. The new components act antilinearly on the Hilbert space of states. It is this formulation that we use in the rest of the paper.

3.1. Wick rotation in quantum mechanics

A quantum mechanical system, according to basic axioms, consists of a complex separable Hilbert space $\mathcal{H}$ equipped with a self-adjoint operator H, the Hamiltonian. The group $\mathbb{R}$ of time translations is represented unitarily on $\mathcal{H}$:

$$\begin{aligned} \mathbb{R} &\longrightarrow U(\mathcal{H}) \\ t &\longmapsto e^{-itH/\hbar}, \end{aligned} \tag{3.1}$$

where i is a choice of complex number such that $i^2 = -1$. If we assume positivity of energy—that H is a nonnegative self-adjoint operator—then real time evolution (3.1) is the boundary value of a holomorphic semigroup of bounded operators defined on the lower half plane $\mathcal{T} = \mathbb{R} - \sqrt{-1}\,\mathbb{R}^{>0} \subset \mathbb{C}$. The semigroup of *imaginary time evolution* is the restriction to $-\sqrt{-1}\,\mathbb{R}^{>0}$, which is the semigroup

$$\tau \longmapsto e^{-\tau H/\hbar}, \qquad \tau > 0. \tag{3.2}$$

The transition from (3.1) to (3.2) is called *Wick rotation*.

The unitarity of time evolution manifests in the reality of the semigroup (3.2).

Example 3.3 (Particle on the circle)**.** Let $\mathbb{A}^1$ denote the affine[17] time line. The trajectory of a particle on the circle is a function $\lambda(s) = e^{ix(s)}$, $s \in \mathbb{A}^1$; the lagrangian density is $L = \frac{1}{2}\dot{x}^2\,|ds|$. The ensuing quantum mechanical system has Hilbert space $\mathcal{H} = L^2(S^1;\mathbb{C})$, Hamiltonian the Laplace operator $H = \Delta$ (up to a constant), and imaginary time evolution the heat operator $\tau \mapsto e^{-\tau\Delta}$.

It is illuminating to add a "θ-angle" to this system; see [GKKS, Appendix D], for example. Orient S^1 and fix $\omega \in \Omega^1(S^1)$ with $\int_{S^1}\omega = 1$. Then for a fixed constant $\theta \in \mathbb{R}$ define the lagrangian

$$L = \frac{1}{2}\dot{x}^2\,|ds| - \theta\lambda^*(\omega). \tag{3.4}$$

In this classical theory we must orient time in order to integrate L; time-reversal exchanges the theories labeled by θ and $-\theta$. Upon quantization we obtain the Hilbert space $\mathcal{H} = L^2(S^1;\mathcal{L}_{e^{i\theta}})$ of sections of the complex line bundle $\mathcal{L}_{e^{i\theta}}$ with holonomy $e^{i\theta}$. The Hamiltonian is the Laplace operator on this space, and imaginary time evolution is by the associated heat operator. Now time-reversal ($\theta \mapsto -\theta$) acts as complex conjugation:

$$\mathcal{H} \longmapsto \overline{\mathcal{H}} \tag{3.5}$$

$$e^{-\tau\Delta} \longmapsto \overline{e^{-\tau\Delta}} \tag{3.6}$$

We encode the formal structure in terms of oriented compact Riemannian 1-manifolds, as described in §2.2, though we emphasize that this is not a topological theory. The interval of length $\tau > 0$ maps to the imaginary time evolution $e^{-\tau H/\hbar}\colon \mathcal{H} \to \mathcal{H}$. The semigroup law is manifest by gluing intervals. The circle of length τ maps to $\operatorname{Trace}(e^{-\tau H/\hbar}) \in \mathbb{C}$. We interpret these oriented Riemannian 1-manifolds as morphisms in a geometric bordism category whose objects are, roughly, compact oriented 0-manifolds. More precisely, they are 0-manifolds embedded in the germ of an oriented Riemannian 1-manifold, and there is an arrow of time, or orientation of the normal bundle. The simplest object is a single point, which we can view as $0 \in \mathbb{R}$ embedded in a small interval $(-\epsilon, \epsilon)$ with its standard orientation; in the quantum mechanics it maps to the Hilbert space $\mathcal{H}$. According to (3.5) we have

$$\text{orientation-reversal} \longmapsto \text{complex conjugation} \tag{3.7}$$

More precisely, the orientation-reversal on objects in the geometric bordism category reverses the orientation and reverses the arrow of time. This is the 'reflection' part of 'reflection positivity'; the positivity is the positive definiteness of the Hilbert space $\mathcal{H}$.

3.2. Reflection positivity in Euclidean quantum field theory

Positivity of energy in a relativistic quantum field theory also results in an analytic continuation and restriction to Euclidean space, as we review in §A.3. Here we focus on the Wick rotation of correlation functions and the Wick rotation of unitarity as manifested in reflection positivity.

[17] We (pedantically) distinguish the affine time line $\mathbb{A}^1$ from the group $\mathbb{R}$ of translations of time, which appears in (3.1): after all, a 1-hour seminar and a seminar ending at 1:00 can be quite different.

(See [GJ, §6], [Kaz, §2.2] for an account.) Let n be the spacetime dimension and $\mathbb{E}^n$ Euclidean n-space. In this subsection we restrict to the basic symmetry type $H_n = SO_n$; we take up general symmetry types in the next subsection (see Remark 3.22). Fix an affine hyperplane $\Pi \subset \mathbb{E}^n$ and let σ denote (affine) reflection about Π. Let $\mathcal{O}$ denote an operator, or product of operators, in the quantum theory which is supported in the open half-space $\mathbb{E}^n_+$ on one side of Π; the reflected operator $\sigma(\mathcal{O})$ has support in the complementary half-space $\mathbb{E}^n_-$. Let $\langle \mathcal{O} \rangle_{\mathbb{E}^n_+} \in \mathcal{H}$ denote the half-space correlation function, which is a vector in the Hilbert space of the theory. In a lagrangian field theory it is the functional integral over the half-space $\mathbb{E}^n_+$. Then the *reflection* part of 'reflection positivity' is

$$\langle \sigma(\mathcal{O}) \rangle_{\mathbb{E}^n_-} = \overline{\langle \mathcal{O} \rangle_{\mathbb{E}^n_+}} \tag{3.8}$$

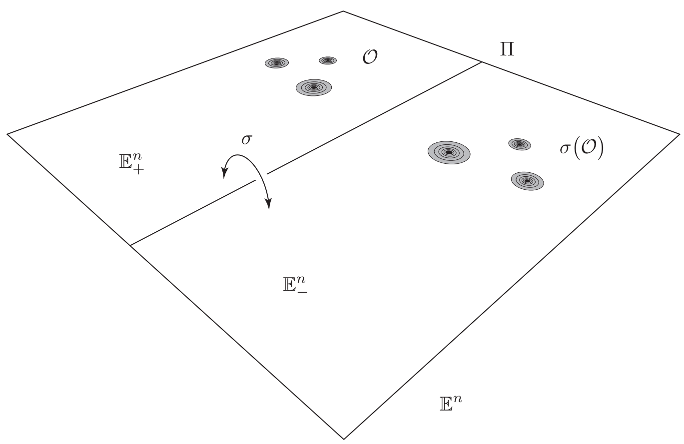


FIGURE 2. Reflection positivity in Euclidean space

in accordance with (3.7); see (3.6) for the analog in quantum mechanics. The Hilbert space $\mathcal{H}$ is associated to $(\Pi, \mathfrak{o})$, where $\mathfrak{o}$ is an orientation of the normal line to Π, the arrow of time in §2.2. The reflection σ reverses $\mathfrak{o}$, and the Hilbert space associated to $(\Pi, -\mathfrak{o})$ is the complex conjugate

$$\mathcal{H}_{(\Pi,-\mathfrak{o})} \xrightarrow{\cong} \overline{\mathcal{H}_{(\Pi,\mathfrak{o})}}, \tag{3.9}$$

according to the dictum (3.7); cf. (3.5). Therefore, $\langle \sigma(\mathcal{O}) \rangle_{\mathbb{E}^n_-} \in \overline{\mathcal{H}}$ and (3.8) is an equation in the complex conjugate Hilbert space $\overline{\mathcal{H}}$. The *positivity* part of 'reflection positivity' is the positive definiteness of $\mathcal{H}$, which implies that the norm square of the vector $\langle \mathcal{O} \rangle_{\mathbb{E}^n_+}$ is nonnegative:

$$\langle \sigma(\mathcal{O})\, \mathcal{O} \rangle_{\mathbb{E}^n} \geqslant 0 \tag{3.10}$$

A theorem of Osterwalder-Schrader [OS] reconstructs the relativistic theory in Minkowski spacetime from the Euclidean theory; reflection positivity is an important ingredient.

Remark 3.11. In theories with fermionic states the Hilbert space $\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$-graded. The norm square of an odd vector is then purely imaginary [DM, §4.4] and positive definiteness requires a sign choice; see Example 6.49 for details in the invertible case.

Remark 3.12 (Internal symmetry and reflection positivity). Suppose the full Wick-rotated vector symmetry group H_n has a nontrivial internal symmetry group K, and for simplicity take $H_n = SO_n \times K$. Let X be Euclidean space with an open neighborhood of the support of the operators $\mathcal{O}$, $\sigma(\mathcal{O})$ removed. Let $Y = \partial X \cap \mathbb{H}_+$ and assume $\sigma(Y) = \partial X \cap \mathbb{H}_-$. In general there are twist operators that are defined by a principal K-bundle $P \to X$, as in Remark 2.35. The reflection σ must account for the K-bundle, and it might seem at first that σ should "reverse" it by an involution on K. But that does not happen; rather σ lifts to $P \to X$. We give three arguments.

(1) If $\mathcal{O}$ is a point operator, then Y is a sphere. Identifying $\sigma(Y)$ with Y via a translation, σ acts on Y as reflection in the equatorial plane parallel to Π. If we one-point compactify X to S^n minus the two balls and assume P extends over the compactification, then the restrictions of P to Y and $\sigma(Y)$ are isomorphic, since the compactification is diffeomorphic to $[0,1] \times S^{n-1}$.

(2) Continuing, suppose $P \to X$ is the trivial bundle and V is the vector space of local operators attached to Y. (In a geometric theory we take a limit as the radius of the removed ball shrinks to zero.) The automorphism group K of the trivial bundle over Y acts on V, producing K-multiplets of point operators. The hyperplane reflection σ induces an isomorphism $V \to \overline{V}$ that commutes with the K-action, since geometrically the lift of reflection to the trivial bundle commutes with the global gauge transformations. So a K-multiplet in V is mapped to a K-multiplet in $\overline{V}$ that transforms in the complex conjugate representation.

(3) Let $n = 1$ and $H_1 = SO_1 \times \mathbb{Z}/3\mathbb{Z}$. Let $\alpha\colon \mathrm{Bord}_{\langle 0,1\rangle}(H_1) \to \mathrm{Vect}_{\mathbb{C}}$ be the invertible theory which attaches a nontrivial character $\chi\colon \mathbb{Z}/3\mathbb{Z} \to \mathbb{T}$ to the positively oriented point with its trivial $\mathbb{Z}/3\mathbb{Z}$ bundle. (That object Y of the bordism category has automorphism group $\mathbb{Z}/3\mathbb{Z}$, which then acts on the vector space $\alpha(Y)$.) This theory is unitary. Now $\alpha(P \to S^1)$ is χ applied to the holonomy of the principal $\mathbb{Z}/3\mathbb{Z}$-bundle $P \to S^1$. Reflection reverses the orientation of S^1, and if the bundle stays the same under reflection, then the holonomy complex conjugates, which is precisely what it should do in a reflection positive theory.

3.3. The extended symmetry group $\widehat{H}_n$

Let (H_n, ρ_n) be a symmetry type (Definition 2.4). We use reflection symmetry (3.8) to construct a larger symmetry group $\widehat{H}_n$ from H_n by adjoining an involution. In the special case $H_n = \mathrm{Spin}_n$, we define $\widehat{H}_n = \mathrm{Pin}_n^+$; the general case is a bootstrap from this, following the proof of Theorem 2.19. The arguments in Remark 3.12 motivate the triviality of the hyperplane reflection automorphism of K in our construction. We view $\widehat{H}_n$ as a symmetry group of the Euclidean quantum field theory; the action of an element in $\widehat{H}_n \backslash H_n$ on the Hilbert space $\mathcal{H}$ is by an anti-unitary transformation.

Proposition 3.13. *There exists a canonical group extension*

$$1 \longrightarrow H_n \xrightarrow{j_n} \widehat{H}_n \longrightarrow \{\pm 1\} \longrightarrow 1, \tag{3.14}$$

split (noncanonically) by a choice of hyperplane reflection $\sigma \in O_n$, such that the splitting induces the automorphism of $\widetilde{SH}_n \cong \mathrm{Spin}_n \times K$ that is the product of conjugation by σ on Spin_n and the *identity automorphism of K. There is a homomorphism $\hat\rho_n$ that fits into the pullback diagram*

$$\begin{tikzcd} H_n \arrow[r, "j_n"] \arrow[d, "\rho_n"'] & \widehat{H}_n \arrow[d, "\hat\rho_n"] \\ O_n \arrow[r] & \{\pm1\}\times O_n \end{tikzcd} \tag{3.15}$$

Finally, there are inclusions $\hat\imath_n\colon \widehat{H}_n \to \widehat{H}_{n+1}$ which, together with the inclusions $i_n\colon H_n \to H_{n+1}$, induce a commutative diagram linking (3.15) for varying n.

A hyperplane reflection $\sigma \in O_n$ induces an automorphism of SO_n by conjugation in O_n, and it lifts uniquely to an automorphism of Spin_n, which is realized as conjugation by $\tilde\sigma \in \mathrm{Pin}_n^+$, where $\tilde\sigma$ is a lift of σ. However, it is the *twisted* conjugation [ABS, §3] by $\tilde\sigma$ in Pin_n^+ that lifts conjugation by σ in O_n, where the twist is multiplication by the nontrivial character

$$\mathrm{Pin}_n^+ \longrightarrow \pi_0\,\mathrm{Pin}_n^+ \xrightarrow{\;\cong\;} \{\pm1\}. \tag{3.16}$$

Note $\tilde\sigma$ is only determined up to sign; the splitting of (3.14) associated to σ is determined up to multiplication by k_0.

Proof. Define

$$\widehat{SH}_n = \mathrm{Pin}_n^+ \times K \;/\; \langle(-1,k_0)\rangle \tag{3.17}$$

and project onto $\pi_0\,\mathrm{Pin}_n^+$ to define the quotient map in the extension

$$1 \longrightarrow SH_n \longrightarrow \widehat{SH}_n \longrightarrow \{\pm1\} \longrightarrow 1 \tag{3.18}$$

If $\rho_n(H_n) = SO_n$, then set $\widehat{H}_n = \widehat{SH}_n$. If ρ_n is surjective, then define the double cover of $\widehat{H}_n$ as the mixing construction

$$(\mathrm{Pin}_n^+ \times K) \times_{(\mathrm{Spin}_n \times K)} \widetilde{H}_n, \tag{3.19}$$

where $\widetilde{H}_n$ is defined in (2.6). Let $\widehat{H}_n$ be the quotient by the cyclic subgroup $\langle[-1,k_0;1]\rangle$ of order two.

Reflection through the hyperplane perpendicular to $\xi \in S^{n-1} \subset \mathrm{Pin}_n^+$ lifts to

$$[\pm\xi,1;1] \in (\mathrm{Pin}_n^+ \times K) \times_{(\mathrm{Spin}_n \times K)} \widetilde{H}_n, \tag{3.20}$$

so passes to an element of order two in $\widehat{H}_n$, which gives the splittings of (3.14).

For any $s \in \mathrm{Pin}_n^+$, $k \in K$, $\tilde h \in \widetilde{H}_n$ set

$$\hat\rho_n[s,k;\tilde h] = \big(\det(\bar s)\,,\,\bar s\rho_n(h)\big) \in \{\pm1\}\times O_n, \tag{3.21}$$

where $\bar s \in O_n$ is the image of $s \in \mathrm{Pin}_n^+$ and h the image of $\tilde h$ in H_n. This passes to a homomorphism with domain the mixing construction (3.19), and then to its quotient $\widehat{H}_n$. $\square$

Remark 3.22. Now we formulate reflection positivity on Euclidean space for a theory with symmetry type (H_n,ρ_n). Adjoining translations via the pullback

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \mathcal{H}_n \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Euc}_n \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & H_n \arrow[r, "\rho_n"] & O_n \arrow[r] & 1 \end{tikzcd} \tag{3.23}$$

we obtain a larger group $\mathcal{H}_n$ and a homomorphism $\mathcal{H}_n \to \mathrm{Euc}_n$ to the Euclidean group. The complex point observables form a vector bundle $\mathcal{O} \to \mathbb{E}^n$, and the action of Euc_n on $\mathbb{E}^n$ lifts to an action of $\mathcal{H}_n$ on $\mathcal{O}$. Proposition 3.13 gives a co-extension $\widehat{\mathcal{H}}_n$ of $\mathcal{H}_n$ and a homomorphism $\widehat{\mathcal{H}}_n \to \{\pm1\} \times \mathrm{Euc}_n$. As before fix a hyperplane reflection σ and now fix a lift $\hat\sigma \in \widehat{\mathcal{H}}_n$ of $(-1,\sigma) \in \{\pm1\} \times \mathrm{Euc}_n$. Then part of the data of a reflection structure is a lift of $\hat\sigma$ to an antilinear map of the complex vector bundle $\mathcal{O} \to \mathbb{E}^n$. Therefore (3.8)–(3.10) apply, with $\hat\sigma$ replacing σ.

Proposition 3.24. *For each $n \geqslant 1$ there is an inclusion of group extensions*

$$\begin{tikzcd} 1 \arrow[r] & H_n \arrow[r] \arrow[d, hook, "i_n"] & \{\pm1\} \times H_n \arrow[r] \arrow[d, hook, "s_{n+1} * j_{n+1} i_n"] & \{\pm1\} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H_{n+1} \arrow[r, "j_{n+1}"] & \widehat{H}_{n+1} \arrow[r] & \{\pm1\} \arrow[r] & 1 \end{tikzcd} \tag{3.25}$$

in which i_n is the inclusion in (2.20) and j_n the inclusion in (3.14). Furthermore, the inclusions i_n and $\hat\imath_n$ induce a commutative diagram linking (3.25) for varying n.

Proof. Define $s_n\colon \{\pm1\} \to \widehat{H}_n$ as the splitting of (3.14) induced by the hyperplane reflection that reverses the first coordinate of $\mathbb{R}^n$ and fixes the others; use $[e_1, 1; 1]$ in (3.20). Then the s_n fit (3.14) into a commutative diagram of split short exact sequences as n varies, using the inclusions $i_n, \hat\imath_n$. With all maps defined the rest is a systematic verification. ☐

For the basic symmetry groups in (2.17) the extended symmetry groups are listed here:

(3.26)

states/symmetry	H_n	$\widehat{H}_n$
bosons only	SO_n	O_n
fermions allowed	Spin_n	Pin^+_n
bosons, time-reversal (T)	O_n	$\{\pm1\} \times O_n$
fermions, $T^2 = (-1)^F$	Pin^+_n	$\widehat{\mathrm{Pin}}^+_n$
fermions, $T^2 = \mathrm{id}$	Pin^-_n	$\widehat{\mathrm{Pin}}^-_n$

The splitting of $\widehat{O}_n$ is a consequence of the fact that hyperplane reflections are inner in O_n. A similar argument proves that the 4-component group $\widehat{\mathrm{Pin}}^\pm_n$ can be constructed from $\mathrm{Pin}^\pm_n$ by adjoining the automorphism that is the identity on $\mathrm{Spin}_n \subset \mathrm{Pin}^\pm_n$ and multiplication by the central element $-1 \in \mathrm{Spin}_n$ on the off-component of $\mathrm{Pin}^\pm_n$. (This argument is echoed in Remark A.9.)

4. Reflection symmetry on manifolds

The enhanced symmetry group $\widehat{H}_n$ produces an involution (§4.1) on H_n-manifolds that generalizes orientation-reversal for $H = SO$. In the field theory context it induces an involution on bordism categories that we call 'bar'. (See Appendix B for a general discussion of involutions on categories and other relevant background.) In §4.2 we prove that the dual of an object in a bordism category is isomorphic to its bar. The definitions of *reflection structure* and *positive reflection structure* for non-extended field theories are in §4.3. In a reflection positive theory the partition function of any double is nonnegative, as we prove in §4.4. We work as always with arbitrary symmetry groups.[18]

Kevin Walker has introduced theories with more general reflection structures in which, possibly, the group extension (3.14) that controls anti-unitarity is not split. In particular, he allows $\widehat{H}_n = \mathrm{Pin}_n^-$ when $H_n = \mathrm{Spin}_n$. This leads to exotic hermitian structures. Our more restrictive framework is based on Wick rotation of relativistic theories.

4.1. An involution on H_n-manifolds

Recall from §2.2 that an H_n-manifold is a Riemannian n-manifold equipped with a reduction (P, θ) of its orthonormal frame bundle $\mathcal{B}_O(X) \to X$ to H_n. Extend the principal H_n-bundle $P \to X$ to a principal $\widehat{H}_n$-bundle $j_n(P) \to X$, where j_n is the inclusion of groups in (3.14). Using (3.15) extend the isomorphism $\theta\colon \mathcal{B}_O(X) \to \rho_n(P)$ to an isomorphism $\hat{\theta}\colon \{\pm 1\} \times \mathcal{B}_O(X) \to \hat{\rho}_n\big(j_n(P)\big)$.

Definition 4.1. The *opposite H_n-structure* (P', θ') is the principal H_n-bundle $P' := j_n(P)\backslash P \to X$ and the restriction θ' of $\hat{\theta}$ to $\{-1\} \times \mathcal{B}_O(X)$.

Taking opposites is involutive: there is a canonical isomorphism $(P, \theta) \xrightarrow{\cong} (P'', \theta'')$.

Remark 4.2. Let $\sigma \in O_n$ be a hyperplane reflection and ϕ_σ the automorphism of H_n resulting from the splitting of (3.14). Then we can identify the principal H_n-bundle $P' \to X$ as the projection $P \to X$ of manifolds with the original H_n-action on P precomposed with the automorphism ϕ_σ. For if $\tilde{\sigma} \in \widehat{H}_n$ is the splitting element, then we map $P \to j_n(P)\backslash P$ by $p \mapsto p \cdot \tilde{\sigma}$.

Example 4.3. An SO_n-structure is an orientation, and the opposite SO_n-structure is the reverse orientation. In this case $P \to X$ is the bundle of oriented orthonormal frames, $j_n(P) \to X$ the bundle $\mathcal{B}_O(X) \to X$ of all orthonormal frames, and $j_n(P)\backslash P \to X$ the bundle of oppositely oriented orthonormal frames.

Example 4.4. For simplicity, we sometimes abbreviate '$\mathrm{Pin}_n^{\pm}$-structure' to 'pin structure', just as 'Spin_n-structure' is abbreviated to 'spin structure'. The opposite of a pin structure is obtained by tensoring with the orientation double cover; see Definition A.8, Remark A.9, and the text following (3.26). One motivation for our general study of symmetry groups (§2.1) and involutions (§3.2) is to explain the appearance of this opposite pin structure in the formulation of reflection positivity for Wick-rotated quantum field theories with fermions and time-reversal symmetry.

We use the involution in Definition 4.1 to construct an involution of categories

$$\beta_{\mathcal{B}} = \beta\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathrm{Bord}_{\langle n-1,n\rangle}(H_n). \tag{4.5}$$

[18] The definition of the double of a (s)pin manifold is somewhat tricky, for example; the general setting is clarifying.

In Appendix B we explain that an involution on a category $\mathcal{B}$ is a functor $\beta\colon \mathcal{B} \to \mathcal{B}$ and a natural transformation of functors $\eta\colon \mathrm{id}_{\mathcal{B}} \to \beta^2$. The objects and morphisms in $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ are Riemannian manifolds with H_n-structure: the functor β fixes the underlying Riemannian manifold and flips the H_n-structure to its opposite. The equivalence η implements the canonical isomorphism indicated after Definition 4.1. We emphasize that the "bar involution" β is covariant: a morphism $X\colon Y_0 \to Y_1$ maps to a morphism $\beta X\colon \beta Y_0 \to \beta Y_1$. Put differently, the arrows of time on objects are unchanged under β.

Remark 4.6. One can envisage other involutions on the bordism category, and so other notions of reflection structure (Definition 4.14 below), especially for mathematical applications. The heuristics in Remark 3.12 are meant to illustrate why we feel the involution defined here correctly models Wick-rotated unitarity in relativistic field theories.

4.2. Duals and opposites

An object Y in a symmetric monoidal category, such as $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$, may have a dual $Y^\vee$, which is equipped with duality data; see Definition B.8 for a quick review. In a topological bordism category every object has a dual. The underlying smooth manifold of the dual $Y^\vee$ equals that of Y, but the arrow of time is reversed. This reversal is evident in the coevaluation and evaluation duality data. For example, evaluation is the bordism

$$e_Y = [0,1] \times Y\colon\; Y^\vee \amalg Y \longrightarrow \varnothing^{n-1} \tag{4.7}$$

with the entire boundary incoming. The H_n-structure is the same at the two ends, but the arrows of time are opposite. If the boundary at $0 \in [0,1]$ is the object Y, with its arrow of time, then the boundary at $1 \in [0,1]$ is the object $Y^\vee$. See Figure 3, where the coevaluation c_Y and the "S-diagram" (B.9) are also depicted.

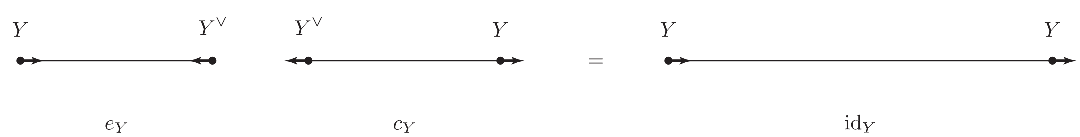


FIGURE 3. Evaluation, coevaluation, and the gluing to the identity

An object Y in a topological bordism category has a canonical product germ (see §2.2), namely the germ of $\{0\} \times Y$ in $X = (-\epsilon, \epsilon) \times Y$, where we fix $\epsilon > 0$. Let σ be the diffeomorphism of X that reflects $t \mapsto -t$ and fixes Y. The splitting in Proposition 3.13 leads to an alternative construction of the opposite H_n-structure and the following important identification.

Proposition 4.8. *For any object Y in $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ there is an isomorphism*

$$h\colon \beta Y \xrightarrow{\;\cong\;} Y^\vee \tag{4.9}$$

Also, $\beta h^\vee = h$.

Reversing the H_n-structure (βY) is equivalent to reversing the arrow of time ($Y^\vee$). Or, in the language of Definition B.14, every object in $\mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$ carries a *hermitian structure.*

Proof. Set $X = (-\epsilon, \epsilon) \times Y$. The reflection

$$\begin{aligned} \sigma\colon (-\epsilon,\epsilon)\times Y &\longrightarrow (-\epsilon,\epsilon)\times Y \\ (t,y) &\longmapsto (-t,y) \end{aligned} \tag{4.10}$$

lifts to the frame bundle $\mathcal{B}_O(X)$. We now construct a diagram of principal K-bundles:

$$\begin{tikzcd} Q' \arrow[r, hook] \arrow[d] & P' \arrow[r, hook] \arrow[d, "\pi'"'] & j_n(P) \arrow[d] & P \arrow[l, hook'] \arrow[d, "\pi"] & Q^\vee \arrow[l, hook'] \arrow[d] \\ \mathcal{B}_Y \arrow[r, hook] & \mathcal{B}_O(X) \arrow[r, hook, "-1\times\mathrm{id}"] & \{\pm1\}\times\mathcal{B}_O(X) & \mathcal{B}_O(X) \arrow[l, hook', "1\times\mathrm{id}"'] & \mathcal{B}_Y^\vee \arrow[l, hook'] \end{tikzcd} \tag{4.11}$$

Let $\mathcal{B}_Y \subset \mathcal{B}_O(X)$ be the O_{n-1}-subbundle of frames with first vector $\pm\partial/\partial t$, the sign chosen to align with the arrow of time of the object Y. Let $\mathcal{B}_Y^\vee$ be the compatible frames with the opposite arrow of time. Then σ induces an isomorphism $\mathcal{B}_Y \to \mathcal{B}_Y^\vee$ which is realized inside $\mathcal{B}_O(X)$ as multiplication by the hyperplane reflection $\sigma_1 \in O_n$ in the orthogonal complement to the vector $e_1 \in \mathbb{R}^n$. (Observe that σ_1 centralizes $O_{n-1} \subset O_n$.) Let $P \xrightarrow{\pi} \mathcal{B}_O(X) \to X$ be the H_n-structure: the composition is a principal H_n-bundle and the first map is a principal K-bundle over its image. Set $Q^\vee = \pi^{-1}(\mathcal{B}_Y^\vee)$; then $Q^\vee \to X$ is a principal H_{n-1}-bundle. Let $j_n(P), P'$ be as in Definition 4.1, so that $P' \xrightarrow{\pi'} \mathcal{B}_O(X) \to X$ is the opposite H_n-structure. Set $Q' = {\pi'}^{-1}(\mathcal{B}_Y)$, so that $Q' \to X$ is an H_{n-1}-bundle that encodes the opposite H_n-structure. Let $\hat\sigma_1 = [e_1, 1; 1] \in \widehat{H}_n$ be the lift of $\sigma_1 \in O_n$, as defined in (3.19) and the text that follows; then $\hat\sigma_1$ centralizes H_{n-1} and has order two. The action of multiplication by $\hat\sigma_1$ on $j_n(P)$ restricts to an isomorphism of H_{n-1}-bundles $Q' \to Q^\vee$. (It covers multiplication by $(-1, \sigma_1) \in \{\pm1\} \times O_n$ on $\{\pm1\} \times \mathcal{B}_O(X)$, which restricts to an isomorphism $\mathcal{B}_Y \to \mathcal{B}_Y^\vee$.)

$\beta h^\vee$ is the inverse of the involution $\hat\sigma_1$ on $j_n(P)$, restricted to the bar dual bundles. Since $\hat\sigma_1$ is its own inverse, we conclude $\beta h^\vee = h$. ☐

Remark 4.12. In a geometric bordism category not every germ admits a reflection which is an isometry. It is only for germs which do admit such a reflection that we expect the associated topological vector space of a field theory to have a Hilbert space structure; see [KS]. This is the case for the (noncompact) affine hyperplane in Figure 2, consistent with (3.9).

4.3. Reflection structures and positivity

Let

$$\beta_{\mathcal{C}} = \beta\colon \mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathrm{Vect}_{\mathbb{C}} \tag{4.13}$$

be the involution of complex conjugation (Example B.2). Recall (2.34) that a topological field theory is a symmetric monoidal functor $F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathrm{Vect}_{\mathbb{C}}$.

Definition 4.14. A *reflection structure* on F is equivariance data for the involutions $\beta_{\mathcal{B}}, \beta_{\mathcal{C}}$.

Equivariance data is spelled out in Definition B.6. For every closed $(n-1)$-manifold Y with H_n-structure we have an isomorphism of vector spaces

$$F(\beta Y) \xrightarrow{\cong} \overline{F(Y)}, \tag{4.15}$$

the curved space analog of (3.9). Combining with the isomorphism (4.9), we see that $F(e_Y)$ is a hermitian form

$$h_Y : F(Y^\vee) \otimes F(Y) \cong F(\beta Y) \otimes F(Y) \cong \overline{F(Y)} \otimes F(Y) \longrightarrow \mathbb{C}, \tag{4.16}$$

which by the usual "S-diagram" argument (Figure 3) is nondegenerate. Sesquilinearity is a consequence of the isomorphism

$$\begin{aligned} e_Y &\longrightarrow \beta(e_Y) \\ (t, y) &\longmapsto (1-t, y) \end{aligned} \tag{4.17}$$

where recall as a manifold $e_Y = [0,1] \times Y$.

Definition 4.18. A reflection structure is *positive* if the induced hermitian form h_Y is positive definite for all $Y \in \mathrm{Bord}_{\langle n-1,n\rangle}(H_n)$.

Remark 4.19. In a non-extended field theory reflection is *data* and positivity is a *condition*. In the extended case considered later, both reflection and positivity are data.

Remark 4.20. There is also a notion of positivity if the domain is the category of *super* vector spaces; see Example 6.49.

Example 4.21. To avoid trivialities, suppose the spacetime dimension n is even. Fix a nonzero complex number $\lambda \in \mathbb{C}$. There is a simple invertible field theory of unoriented manifolds ($H_n = O_n$) whose partition function on a closed n-manifold X is $\lambda^{\mathrm{Euler}(X)}$, where $\mathrm{Euler}(X)$ is the Euler number of X. The vector space $F_\lambda(Y)$ attached to any closed $(n-1)$-manifold Y is the trivial line $\mathbb{C}$: the Euler characteristic of a compact manifold with boundary is a well-defined number. In the bordism category we can write the closed manifold S^n as the composition $\varnothing^{n-1} \xrightarrow{D^n} S^{n-1} \xrightarrow{D^n} \varnothing^{n-1}$ of two closed balls. Denote the first arrow as X and apply the theory F_λ:

$$\lambda^2 = F_\lambda(S^n) = h_{S^{n-1}}(F_\lambda(X), F_\lambda(X)). \tag{4.22}$$

Therefore, a necessary condition for positivity is that λ be real.

A reflection structure imposes a curved space analog of (3.8), which for an n-dimensional H_n-bordism X, asserts that

$$F(\beta X) = \overline{F(X)}. \tag{4.23}$$

For example, if $H = SO_n$ then the partition function complex conjugates when the orientation of spacetime is reversed. For a theory of unoriented manifolds ($H_n = O_n$), condition (4.23) implies that every partition function is real. For theories of pin manifolds ($H_n = \mathrm{Pin}_n^\pm$) the partition function of the w_1-twisted pin structure (Definition A.8) is the complex conjugate of the original partition function.

4.4. Doubles

The reflection-conjugation equation (4.23) also applies to manifolds with boundary. We use it to derive a necessary condition for reflection positivity.

Definition 4.24. Let X be a compact H_n-manifold with boundary, viewed as a bordism $\varnothing^{n-1} \to \partial X$. The *double* of X is the closed H_n-manifold

$$\Delta X = e_{\partial X}(\beta X, X). \tag{4.25}$$

The double is illustrated in Figure 4. In that picture $Y = \partial X$.

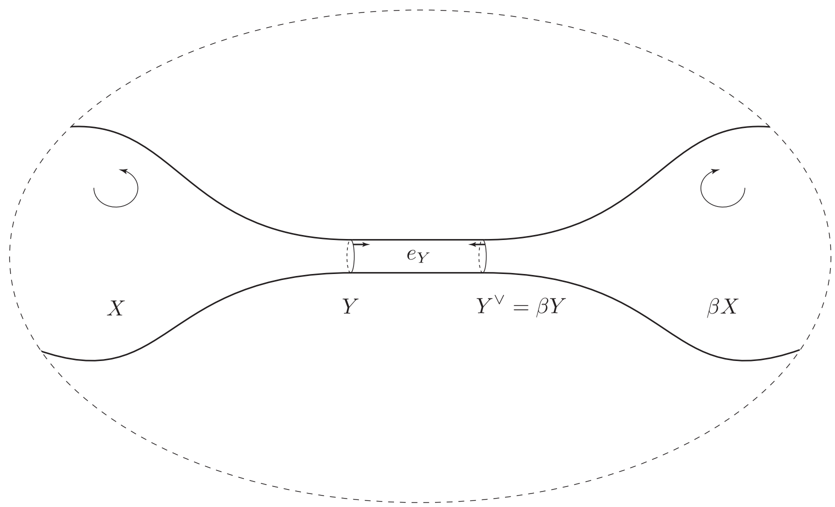


FIGURE 4. The double of X

Proposition 4.26. *If a theory $F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to \mathrm{Vect}_{\mathbb{C}}$ admits a positive reflection structure, then $F(\Delta X) \geqslant 0$ for all compact H_n-manifolds X with boundary.*

Note that the value of a theory on a closed n-manifold does not depend on the reflection structure. The necessary condition for positivity in Proposition 4.26 is the compact manifold analog of the usual reflection positivity statement (3.10) in Euclidean space.

Proof. From (4.25) and (4.23) we deduce

$$F(\Delta X) = F(e_{\partial X})\bigl(F(\beta X), F(X)\bigr) = h_{\partial X}\bigl(\overline{F(X)}, F(X)\bigr) = \|F(X)\|^2_{F(\partial X)} \geqslant 0. \tag{4.27}$$ □

The double construction is standard for unoriented and oriented manifolds. It is a bit trickier for spin and pin manifolds, so we give a recognition principle and illustrate with some examples. Observe that the double has an obvious (anti-)involution $\Delta X \xrightarrow{\sigma} \beta\Delta X$ with fixed point set $Y = \{1/2\} \times \partial X$, and σ induces multiplication by -1 on the normal bundle. Set $X' = X \cup_{\partial X} [0,1/2] \times \partial X$ and cut along $Y = \partial X'$ to write

$$\Delta X = \beta X' \cup_{\partial X'} X', \tag{4.28}$$

which is the typical description of a double. But we must account for the H_n-structure as well.

Proposition 4.29. *Let X be a closed H_n-manifold, $\sigma\colon X \to \beta X$ an anti-involution with fixed point set Y such that*

(*i*) *There exists a submanifold $N \subset X$ with boundary Y such that X is the union of N and σN along Y and σ induces a diffeomorphism $\beta N \cong \sigma N$ of H_n-manifolds; and*

(*ii*) *$\sigma\big|_Y$ induces the hyperplane reflection isomorphism of the H_n-structure on Y to its opposite.*

Then $X \cong \Delta N$ as H_n-manifolds

The isomorphism in (ii) is left multiplication by $[e_1, 1; 1] \in \widehat{H}_n$; see (3.20).

Proof. Use the tubular neighborhood theorem to replace Y with $[0, 1] \times Y$ and so construct the desired H_n-isomorphism. ☐

Corollary 4.30. *The sphere S^n with H_n-structure $H_{n+1} \to H_{n+1}/H_n$ is a double*

We note from Remark 2.23 that the homogeneous space H_{n+1}/H_n is diffeomorphic to S^n.

Proof. Reflection σ in the hyperplane perpendicular to e_1 is an involution of S^n with fixed point set the equatorial S^{n-1} perpendicular to e_1. The reflection lifts to an isomorphism of the principal H_n-bundle $H_{n+1} \to H_{n+1}/H_n$ with the pullback of its opposite. (The isomorphism is globally left multiplication by $[e_1, 1; 1] \subset \widehat{H}_{n+1}$.) ☐

Example 4.31. For $H_m = \mathrm{Spin}_m$ the circle $\mathrm{Spin}_2 / \mathrm{Spin}_1$ has the *bounding* spin structure: the Spin_1-bundle $\mathrm{Spin}_2 \to \mathrm{Spin}_2 / \mathrm{Spin}_1$ is the nontrivial double cover of the circle. The nonbounding spin circle is *not* a double. Indeed, there is a reflection positive invertible 1-dimensional spin topological field theory α into super vector spaces that attaches the odd line to a positively oriented spin point; it follows that $\alpha(S^1_{\text{nonbounding}}) = -1$. This does not violate Proposition 4.26 since $S^1_{\text{nonbounding}}$ is not a double. Turning this argument around, since the oriented circle *is* a double, the 1-dimensional *oriented* topological field theory into super vector spaces that attaches the odd line to a positively oriented point does *not* admit a positive reflection structure.

Remark 4.32. The group H_{n+1} acts as symmetries of the H_n-sphere in Corollary 4.30. Topologically, then, there is a universal family of H_n-spheres parametrized by the classifying space BH_{n+1}. Field theories may be evaluated on families of manifolds and bordisms; this family of spheres enters our analysis in §7.2.

5. Invertible topological field theories and stable homotopy theory

We first recall that to fully implement locality in field theory we need to use a bordism multicategory that encodes gluing laws in arbitrary codimension. Next we recount how invertible topological field theories lie in the framework of homotopy theory: invertibility moves the discussion from abstract multicategories to topological spaces. Finally, we specify the universal target that tracks deformation classes of invertible topological theories. The main result is Theorem 5.23, which is our point of departure for implementing reflection positivity in invertible topological theories. We conclude in §5.4 with a discussion of invertible non-topological theories and their role in low energy approximations of gapped quantum systems.

The material in this section is covered in much more expository detail in many references, so we only recount essentials.

5.1. Extended field theories

There are several physics motivations for extending an n-dimensional Wick-rotated field theory to lower dimensional manifolds, and these are hardly restricted to the topological case of interest here. First, the vector space of physical states attached to an $(n-1)$-manifold Y depends locally on Y. This is familiar in $n = 2$ dimensions, where a theory not only has a vector space attached to a circle, but also to an interval with boundary conditions; the gluing laws for intervals lie in codimension two, since intervals are glued along 0-manifolds in this 2-dimensional theory. The result is sometimes called an *open-closed theory* [MS].[19] The labels on the boundary are objects in a category, so it is natural to associate that category to the 0-manifold consisting of a single point. As we are doing quantum mechanics, the category is linear and indeed the vector space associated to the interval with boundary labels β_0, β_1 is $\mathrm{Hom}(\beta_0, \beta_1)$ in the category. The objects are boundary conditions, or *D-branes*. Another common example is 3-dimensional Chern-Simons theory, in which a unitary modular tensor category is associated to the 1-manifold S^1, which is a manifold of codimension two in this theory.

Let X^n be a Riemannian n-manifold on which a theory F is defined, and fix $x \in X$. We explained in Remark 2.35 that the vector space $F(S^{n-1}_x)$ attached to a small sphere around x, in the limit of small radius, is the space of point operators at x. A field theory also has *extended operators*, whose support may be a submanifold $W \subset X$ of dimension $k > 0$. An extended operator with $k = 1$ is called a *line operator*, with $k = 2$ a *surface operator*, etc. The link of W at any $x \in W$ is a sphere S^{n-k-1}_x. In an extended field theory F there is an invariant $F(S^{n-k-1}_x)$ which is a k-category whose objects are the operators on W. Thus the line operators in a theory form a 1-category, the surface operators a 2-category, etc.; see [Ka2] for a thorough account.

We believe that every field theory of physical relevance should be fully extended. The mathematical implementation is most developed in the topological case: a sampling of references is [F1, La, BD, L, F2, AF]. Invariants of manifolds of increasing codimension are encoded in a higher categorical structure of increasing complexity. The modern framework also includes invariants for families of manifolds; see [ST] for a non-topological version. The domain of an n-dimensional topological field theory with symmetry group H_n is the bordism multicategory $\mathrm{Bord}_n(H_n)$ whose objects are 0-manifolds; 1-morphisms are bordisms of 0-manifolds, which are 1-manifolds with boundary; 2-morphisms are bordisms of bordisms, which are 2-manifolds with corners; and so on until we reach n-manifolds with arbitrary corners. At that point we continue to $(n+\ell)$-morphisms which are roughly ℓ-dimensional families of n-manifolds, where ℓ is an arbitrary positive integer. The entire structure is an *(∞, n)-category* [BM, L, BS, Ng, CS, S-P].

Definition 5.1. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. A *fully extended n-dimensional topological field theory with Wick-rotated vector symmetry group H_n and target* $\mathcal{C}$ is a symmetric

[19] There is a difference between an open-closed theory and a fully extended 2-dimensional theory [L, §4.2].

monoidal functor

$$F\colon \operatorname{Bord}_n(H_n) \longrightarrow \mathcal{C}. \tag{5.2}$$

We typically shorten this to 'topological field theory'. In general there is no preferred choice of target $\mathcal{C}$, and it is an open issue to construct suitable general targets. In the very special invertible case we study here there are two preferred targets; see §5.3.

5.2. Invertible topological field theories

There is a natural superposition of quantum systems which does not introduce interactions between them. In the framework of Wick-rotated field theories on compact manifolds this is implemented by tensoring theories together, and that tensor product makes sense for fully extended theories too. There is a unit for the tensor product: the trivial theory $\mathbf{1}$ in which the vector space attached to any $(n-1)$-manifold is $\mathbb{C}$, all correlation functions equal 1, and a similar triviality in higher codimension. A theory F is *invertible* if there exists F' such that $F\otimes F' \cong \mathbf{1}$.

Example 5.3. An $n=1$ theory F with $H_1 = SO_1$ is determined by the vector space $F(\mathrm{pt}_+)$ attached to a point with positive orientation; it is invertible if and only if this vector space is one-dimensional. (A one-dimensional vector space is called a *line*. A vector space V is invertible if and only if there exists V' such that $V\otimes V' \cong \mathbb{C}$, and this happens if and only if V is a line.) In an n-dimensional invertible field theory, the vector space attached to any $(n-1)$-dimensional manifold is a line and all correlation functions between nonzero operators are nonzero.

We first explain the transition to stable homotopy theory in the non-extended case, as in Example 5.3. The codomain, or target, of a non-extended topological field theory (Definition 2.33) is the ordinary category $\mathrm{Vect}_{\mathbb{C}}$ whose objects are complex vector spaces and whose morphisms are linear maps. To accommodate theories with fermionic states, we use instead the codomain category $s\mathrm{Vect}_{\mathbb{C}}$ of super vector spaces. An invertible theory F factors through the subcategory $s\mathrm{Line}_{\mathbb{C}}$ whose objects are complex super lines[20] and whose morphisms are invertible linear maps:

$$\begin{tikzcd} \operatorname{Bord}_{\langle n-1,n\rangle}(H_n) \arrow[rr, "F"] \arrow[dr, dashed] & & s\mathrm{Vect}_{\mathbb{C}} \\ & s\mathrm{Line}_{\mathbb{C}} \arrow[ur, hook] & \end{tikzcd} \tag{5.4}$$

The category $s\mathrm{Line}_{\mathbb{C}}$ is a *groupoid*: every morphism is invertible. Even more, it is a *Picard groupoid*: every object is invertible under tensor product. The main point is that groupoids and Picard groupoids come from topology, as we quickly review.

One of the first constructions in algebraic topology goes in the opposite direction:

$$\text{Spaces} \xrightarrow{\pi_{\leqslant 1}} \text{Groupoids} \tag{5.5}$$

[20] A $\mathbb{Z}/2\mathbb{Z}$-graded line is either even or odd, which means the single quantum state is either bosonic or fermionic.

To any topological space S is attached a groupoid $\pi_{\leqslant 1}S$ whose objects are the points of S; the set $(\pi_{\leqslant 1}S)(s_0, s_1)$ of morphisms from $s_0 \in S$ to $s_1 \in S$ is the set of homotopy classes of paths from s_0 to s_1. If the space has no higher homotopy information—S is a *homotopy 1-type*—then $\pi_{\leqslant 1}S$ captures the homotopy type of S completely. There is an inverse construction that takes a groupoid $\mathcal{G}$ (or a category) and attaches a homotopy 1-type $\|\mathcal{G}\|$, the *classifying space* [Se3].

Example 5.6. Let $S = \|s\mathrm{Line}_{\mathbb{C}}\|$. Then $\pi_0 S \cong \mathbb{Z}/2\mathbb{Z}$, since there are two isomorphism classes of super line; and $\pi_1 S \cong \mathbb{C}^\times$, since the automorphism group of any super line is the group $\mathbb{C}^\times$ of nonzero complex numbers under multiplication.

Remark 5.7. In Example 5.6 the groupoid $s\mathrm{Line}_{\mathbb{C}}$ is discrete: there is no topology on objects or morphisms. If we use the standard topology on the morphism spaces of linear maps, then the geometric realization $\|s\mathrm{Line}_{\mathbb{C}}\|$ is a homotopy 2-type with $\pi_0 \cong \mathbb{Z}/2\mathbb{Z}$, $\pi_1 = 0$, and $\pi_2 \cong \mathbb{Z}$. In other words, whereas in Example 5.6 the discrete group $\mathbb{C}^\times$ of morphisms gives rise to $\pi_1 \cong \mathbb{C}^\times$, with the usual topology the group $\mathbb{C}^\times$ deformation retracts to the circle ($\pi_0 = 0$, $\pi_1 \cong \mathbb{Z}$), and so its homotopy groups show up one degree higher in the geometric realization.

A symmetric monoidal structure on a groupoid goes over to an *infinite loop structure* on the classifying space S. That is, there exists a sequence $\mathcal{X} = \{S_0, S_1, S_2, \dots\}$ of pointed spaces equipped with homotopy equivalences $S_q \simeq \Omega S_{q+1}$, where $S_0 = S$ and ΩS_{q+1} is the based loop space. These satisfy the condition that S_q is $(q-1)$-connected. We call $\mathcal{X}$ a *spectrum* and we call S its *0-space*. See §6 for a review of spectra.

Example 5.8. The classifying space $\|\mathrm{Line}_{\mathbb{C}}\|$ has only one nontrivial homotopy group $\pi_1 \cong \mathbb{C}^\times$, so it is an Eilenberg-MacLane space $K(\mathbb{C}^\times, 1)$. The corresponding Eilenberg-MacLane spectrum is denoted $\Sigma H\mathbb{C}^\times$: the 0-space of the spectrum $H\mathbb{C}^\times$ is a $K(\mathbb{C}^\times, 0)$, for which a simple model is the discrete group $\mathbb{C}^\times$, and the 'Σ' indicates a shift.

The functor (5.5) is the first in a sequence of functors $\{\pi_0, \pi_{\leqslant 1}, \pi_{\leqslant 2}, \dots\}$ in which the zeroth maps a space to its set of path components and the higher ones map to higher groupoids. The classifying space construction also works in this context, and it produces a space with potentially nonzero homotopy groups in any degree.

A symmetric monoidal (∞, n)-category $\mathcal{B}$ has a higher Picard groupoid quotient $\overline{\mathcal{B}}$, obtained by formally adjoining inverses for every object and morphism. Also, a symmetric monoidal (∞, n)-category $\mathcal{C}$ has a maximal Picard subgroupoid $\mathcal{C}^\times \hookrightarrow \mathcal{C}$ constructed by removing the non-invertible objects and morphisms from $\mathcal{C}$.

Definition 5.9. A fully extended field theory $F\colon \mathrm{Bord}_n(H_n) \to \mathcal{C}$ is invertible if it admits a factorization

$$\begin{tikzcd} \mathrm{Bord}_n(H_n) \arrow[r, "F"] \arrow[d] & \mathcal{C} \\ \overline{\mathrm{Bord}_n(H_n)} \arrow[r, dashed, "\widetilde{F}"] & \mathcal{C}^\times \arrow[u, hook] \end{tikzcd} \tag{5.10}$$

Passing to classifying spaces, $\widetilde{F}$ is equivalent to an infinite loop map

$$\|F\|\colon \|\mathrm{Bord}_n(H_n)\| \longrightarrow \|\mathcal{C}^\times\|, \tag{5.11}$$

or equivalently a map of spectra. The homotopy type of the domain is given by the following variation of the celebrated Galatius-Madsen-Tillmann-Weiss [GMTW] Theorem.

Theorem 5.12. $\|\operatorname{Bord}_n(H_n)\|$ *is the 0-space of the Madsen-Tillmann spectrum* $\Sigma^n MTH_n$.

One version of this theorem is proved in [BM], though it is only for unoriented manifolds and is carried out for "n-uple categories" rather than (∞, n)-categories. Proofs of Theorem 5.12 in the context of (∞, n)-categories have appeared in preprint form. The theorem is stated in [L, §2.5] as a corollary of the cobordism hypothesis. A preprint of Ayala-Francis [AF] proves the cobordism hypothesis and Theorem 5.12 for framed manifolds. A preprint by Schommer-Pries [S-P] contains a complete proof of Theorem 5.12 independent of the cobordism hypothesis. Nonetheless, because there is currently no published proof, in this paper we only use Theorem 5.12 as motivation and formally define an invertible field theory as a map of spectra (Ansatz 5.14 below).

See §7.1 for a review of Madsen-Tillmann spectra.

5.3. Universal targets

There are two universal targets for invertible topological field theories, corresponding to the discrete and continuous topologies on $\mathbb{C}^\times$. These targets are spectra; there is no need to define an (∞, n)-category $\mathcal{C}$ with non-invertible morphisms and objects as we only consider invertible theories.

The first target is constructed so that invertible n-dimensional field theories with that target are determined by the partition function. The spectrum $I\mathbb{C}^\times$ is characterized in the homotopy category of spectra by a functorial isomorphism

$$\pi_0\colon [\mathcal{B}, I\mathbb{C}^\times] \longrightarrow \operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{C}^\times) \tag{5.13}$$

from the abelian group of homotopy classes of spectrum maps $\mathcal{B} \to I\mathbb{C}^\times$ to the character group of $\pi_0\mathcal{B}$, for any spectrum $\mathcal{B}$. The shift $\Sigma^n I\mathbb{C}^\times$ satisfies a similar universal property with π_0 replaced by π_n. The spectrum $I\mathbb{C}^\times$ is closely related to the Brown-Comenetz dual to the sphere spectrum [BC]. Combining with the discussion in §5.2 we arrive at the following.

Ansatz 5.14. A *discrete invertible n-dimensional extended topological field theory with symmetry group H_n* is a spectrum map

$$F\colon \Sigma^n MTH_n \longrightarrow \Sigma^n I\mathbb{C}^\times. \tag{5.15}$$

The space of theories of this type is $\operatorname{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$.

Here 'Map' indicates the *space* of maps between the indicated spectra; see (6.8) below. The word 'discrete' is meant to evoke the choice $\Sigma^n I\mathbb{C}^\times$ for the codomain: $\mathbb{C}^\times$ has the discrete topology.

Remark 5.16. The choice of codomain spectrum $\Sigma^n I\mathbb{C}^\times$, which implements the dictum 'the partition function determines the theory', holds magic derived from the first few stable homotopy groups of spheres. For example, the truncation to $\pi_{\langle n-1,n\rangle}$ is a non-extended theory, and it takes values in a groupoid equivalent to the groupoid $s\mathrm{Line}_{\mathbb{C}}$ of super lines: the homotopy groups of spheres "knows about" the bosonic/fermionic grading of quantum states. The next $\mathbb{Z}/2\mathbb{Z}$ in the stable stem also has an interpretation in terms of statistics of particles; see [GK] where objects with nontrivial $\mathbb{Z}/2\mathbb{Z}$-grading are termed 'Majorana'.

The spectrum $\Sigma^n I\mathbb{C}^\times$ is appropriate for classifying *isomorphism classes* of topological theories, but we are interested instead in *deformation classes*: we want to identify two theories if there is a continuous path of theories connecting them. For example, as maps into $\Sigma^n I\mathbb{C}^\times$ the Euler theories $F_{\lambda_0}, F_{\lambda_1}$ in Example 4.21 are nonisomorphic if $\lambda_0 \neq \lambda_1$, whereas they are always deformation equivalent. The *Anderson dual* $\Sigma^{n+1} I\mathbb{Z}(1)$ is the appropriate codomain to compute deformation classes.[21] Roughly speaking, it results from $\Sigma^n I\mathbb{C}^\times$ by taking the continuous topology on $\mathbb{C}^\times$. Its universal property is expressed in the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\pi_n\mathcal{B}, \mathbb{Z}(1)) \longrightarrow [\mathcal{B}, \Sigma^{n+1} I\mathbb{Z}(1)] \longrightarrow \mathrm{Hom}(\pi_{n+1}\mathcal{B}, \mathbb{Z}(1)) \longrightarrow 0 \tag{5.17}$$

which is non-canonically split. The kernel is the torsion subgroup:

$$[\mathcal{B}, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tor}} \cong \mathrm{Ext}^1(\pi_n\mathcal{B}, \mathbb{Z}(1)). \tag{5.18}$$

There is a map

$$\phi\colon [\mathcal{B}, \Sigma^n I\mathbb{C}^\times] \cong \mathrm{Hom}(\pi_n\mathcal{B}, \mathbb{C}^\times) \longrightarrow \mathrm{Ext}^1(\pi_n\mathcal{B}, \mathbb{Z}(1)) \tag{5.19}$$

onto the kernel of (5.17). It sends a homomorphism $\pi_n\mathcal{B} \to \mathbb{C}^\times$ to the pullback of the exponential group extension

$$1 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathbb{C} \xrightarrow{\ \exp\ } \mathbb{C}^\times \longrightarrow 1. \tag{5.20}$$

If we give $\mathbb{C}^\times$ its usual topology, then ϕ may be regarded as mapping the topological space $\mathrm{Hom}(\pi_n\mathcal{B}, \mathbb{C}^\times)$ to its group of path components.

Intuitively, to define the notion of deformation equivalence of theories (5.15) we want to consider a second topology on $\mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ induced from the continuous topology on $\mathbb{C}^\times$, and then compute π_0. Instead we make use of the fibration

$$H\mathbb{C} \xrightarrow{\ \exp\ } I\mathbb{C}^\times \longrightarrow \Sigma I\mathbb{Z}(1) \tag{5.21}$$

induced from (5.20) as follows.

Definition 5.22. Theories $\alpha_0, \alpha_1 \in \mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ are *deformation equivalent* if there exists $\xi \in H^n(\Sigma^n MTH_n; \mathbb{C})$ whose image under exp is the difference $[\alpha_1] - [\alpha_0]$ of the isomorphism classes $[\alpha_0], [\alpha_1] \in [\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times]$.

We immediately conclude the following.

Theorem 5.23. *There is a 1:1 correspondence*

$$\left\{\begin{matrix}\text{deformation classes of discrete invertible} \\ n\text{-dimensional extended topological field} \\ \text{theories with symmetry group } H_n\end{matrix}\right\} \cong [\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tor}}. \tag{5.24}$$

[21] $\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z} \subset \mathbb{C}$ avoids the choice of a particular $\sqrt{-1} \in \mathbb{C}$.

This appears, at least implicitly, in a joint paper [FHT1] of the authors and Constantin Teleman; Theorem 5.23 has been the basis of many investigations since.

It is natural to ask for a field theoretic interpretation of a map of spectra $\Sigma^n MTH_n \to \Sigma^{n+1} I\mathbb{Z}(1)$ whose homotopy class is not torsion, so does not factor through $\Sigma^n I\mathbb{C}^\times$. We give one in the next subsection (Ansatz 5.26).

5.4. Remarks on non-topological invertible theories and low energy approximations

The main immediate application of Theorem 1.1 in this paper is to low-energy approximations of gapped unitary quantum systems in case that approximation is invertible. For the heuristic discussion in this section we momentarily drop the invertibility hypothesis.

A typical example of the phenomenon we wish to highlight is 3-dimensional Yang-Mills theory with a Chern-Simons term. The coupling constant of the Chern-Simons term obeys an integrality constraint. Then the low energy effective theory is quantum "topological" Chern-Simons theory [W3]. In fact, this low energy theory is *not* topological; there is a mild metric dependence [W2]. One precise expression of the mildness is that the energy-momentum tensor[22] is a multiple of the identity operator, which is the only point operator in the theory anyhow. (See the discussion in [GK, §1.1].) Witten observes that if one is willing to introduce some sort of framing, then the long distance topological Chern-Simons theory is the tensor product of a purely topological theory and an invertible theory. The invertible theory is analogous to a gravitational Chern-Simons theory, but more precisely its partition function is the exponential of the Atiyah-Patodi-Singer η-invariant. The coupling constant does not obey the usual integrality constraint, which is why the framing is required for this global decomposition. The full quantum Yang-Mills theory with Chern-Simons term is a theory of oriented Riemannian manifolds (the Wick rotated symmetry group is $H_3 = SO_3$), and so one expects the same for the low-energy approximation. That indeed holds; it is only to make a global decomposition into topological $\times$ invertible that a framing is introduced.

This example violates the physical principle (ii) stated towards the beginning of §1. A more precise expectation is that the low energy physics of a gapped system is well-approximated by a theory whose energy-momentum tensor may depend on the the background fields, but as an operator it is a multiple of the identity at each point. Or, at least locally we suppose the low energy theory is topological $\times$ invertible. If the low energy theory happens to be invertible, then we conclude that any non-topological invertible theory can occur and that there is no shift of symmetry group, e.g., no extra tangential structure is required. We expect that choices must be made in constructing the low energy effective theory, so a potential 'low energy approximation' map from gapped theories to theories that are locally topological times invertible may only be defined up to homotopy. (See [F4, §11.4] for another perspective on the appearance of a possibly nontopological invertible theory.)

To illustrate the nature of the low energy approximation, we contemplate the following three geometric objects associated to a smooth manifold M: (a) a principal $\mathbb{C}^\times$-bundle $P \to M$ with connection, (b) a principal $\mathbb{C}^\times$-bundle $P \to M$ with flat connection, and (c) a principal $\mathbb{C}^\times$-bundle $P \to M$ (with no connection). In particular, we track what information is induced on the free loop space $LM = \mathrm{Map}(S^1, M)$ by integrating over the loop. In (a) we obtain a smooth function

[22] The energy-momentum tensor is a multiple of the Cotton tensor of the Riemannian 3-manifold.

$LM \to \mathbb{C}^\times$, the holonomy, and if there is nonzero curvature then it has nonzero derivative. In (b) the holonomy is a *locally constant* function $LM \to \mathbb{C}^\times$, and therefore we can use the *discrete* topology on $\mathbb{C}^\times$: the holonomy represents a class in $H^0(LM;\mathbb{C}^\times)$. In (c) there is no connection, so no holonomy, but nonetheless we can extract a principal $\mathbb{Z}(1)$-bundle $E_P \to LM$, a fiber bundle of $\mathbb{Z}(1)$-torsors. Namely, an element $\lambda \in \mathbb{C}^\times$ determines a $\mathbb{Z}(1)$-torsor $E_\lambda \subset \mathbb{C}$ of all $x \in \mathbb{C}$ such that $\exp(x) = \lambda$, and so the holonomy function $LM \to \mathbb{C}^\times$ of a connection $\Theta \in \mathcal{A}_P$ on $P \to M$ determines $E_{P,\Theta} \to LM$, so a $\mathbb{Z}(1)$-torsor over $\mathcal{A}_P \times LM$. Since the affine space $\mathcal{A}_P$ of connections is contractible, the principal $\mathbb{Z}(1)$-bundle over $\mathcal{A}_P \times LM$ descends to a principal $\mathbb{Z}(1)$-bundle $E_P \to LM$. It may be regarded as the homotopical information in a connection. It determines a class in the sheaf cohomology group $H^0(LM;\underline{\mathbb{C}}^\times)$ in which $\underline{\mathbb{C}}^\times$ has the *continuous* topology. Since $\underline{\mathbb{C}}^\times$ is an Eilenberg-MacLane space with $\pi_1 \cong \mathbb{Z}(1)$, there is an isomorphism

$$H^0(LM;\underline{\mathbb{C}}^\times) \xrightarrow{\;\cong\;} H^1(LM;\mathbb{Z}(1)). \tag{5.25}$$

Returning to invertible field theories[23] we have the following situations: (a) a non-topological theory, as contemplated in Remark 2.39; (b) a discrete invertible topological theory, as in Ansatz 5.14; and (c) a topological field theory whose partition "function" is a $\mathbb{Z}(1)$-torsor rather than a complex number. While (a) and (b) have clear analogs for non-invertible field theories, it is unclear what a non-invertible analog of (c) would be. In the invertible case we posit the following definition of a type (c) theory.

Ansatz 5.26. *A continuous invertible n-dimensional extended topological field theory with symmetry group H_n* is a spectrum map

$$\varphi\colon \Sigma^n MTH_n \longrightarrow \Sigma^{n+1} I\mathbb{Z}(1). \tag{5.27}$$

The space of theories of this type is $\mathrm{Map}(\Sigma^n MTH_n, \Sigma^{n+1} I\mathbb{Z}(1))$.

Remark 5.28. In differential geometry a principal $\mathbb{C}^\times$-bundle $P \to M$ has a *primary topological* invariant in $H^2\bigl(M;\mathbb{Z}(1)\bigr)$, its Chern class. A connection gives a *secondary geometric* invariant, its holonomy. If the connection is flat, the secondary invariant is also topological (discrete), and in that case the Chern class lies in the torsion subgroup of $H^2\bigl(M;\mathbb{Z}(1)\bigr)$. The *stable* continuous invertible field theories we encounter in §7.2 attach a primary $\mathbb{Z}(1)$-valued invariant to closed $(n+1)$-manifolds.

A discrete invertible topological field theory F (Ansatz 5.14) gives rise to a continuous invertible topological field theory φ, which retains the homotopical information in F, in particular its deformation class. In this paper we do not develop the theory of non-topological field theories, but in the invertible case we use instead continuous topological theories, which represent the homotopical information carried by a geometric theory.

Remark 5.29. In the application to low energy approximations of gapped theories, we expect that only this homotopical shadow of a geometric theory is well-defined, due to the choices in constructing a low energy theory.

[23] Note that each of (a), (b), and (c) above determines the corresponding type of invertible 1-dimensional field theory of oriented manifolds equipped with a map to M.

6. Equivariant stable homotopy theory

Reflection symmetry in invertible topological theories is expressed by a $\mathbb{Z}/2$-action on the constituent spectra. This requires working in $\mathbb{Z}/2$-equivariant stable homotopy theory. What we will use here is *Borel equivariant* homotopy theory. This is somewhat easier than the more general theory, and at the moment is all that seems needed for our main results. There are many places to read about equivariant stable homotopy theory. The reader may wish to consult [Ad], [GM], [HHR, Chapter 2], [Sch] and [tD, Chapter 8].

6.1. Spectra

Let $\mathcal{T}$ be the category of pointed topological spaces, and for $A, B \in \mathcal{T}$ write $\mathcal{T}(A,B)$ for the set of basepoint preserving continuous functions from A to B and $\underline{\mathcal{T}}(A,B)$ for the same set, regarded as a topological space with the compact open topology.

A *spectrum* X is a sequence $\{X_0, X_1, \dots\}$ of pointed spaces, equipped with structure maps $s_n : S^1 \wedge X_n \to X_{n+1}$. A map $X \to Y$ of spectra is a sequence of maps $X_n \to Y_n$ making the diagrams

$$\begin{tikzcd} S^1 \wedge X_n \arrow[r, "s_n^X"] \arrow[d] & X_{n+1} \arrow[d] \\ S^1 \wedge Y_n \arrow[r, "s_n^Y"'] & Y_{n+1} \end{tikzcd}$$

commute. The set of spectrum maps from X to Y is a subset of

$$\prod_n \underline{\mathcal{T}}(X_n, Y_n)$$

and so may be regarded as a topological space with the subspace topology. The space of maps between spectra X and Y will be denoted $\underline{\mathcal{S}}(X,Y)$.

The *homotopy groups* $\pi_n X$ of a spectrum X are defined for $n \in \mathbb{Z}$ by

$$\pi_n(X) = \varinjlim_k \pi_{n+k} X_{n+k} \tag{6.1}$$

in which the bonding maps are given by the suspension mapping

$$\pi_{n+k} X_{n+k} \xrightarrow{\Sigma} \pi_{n+k+1} \Sigma X_{n+k} \xrightarrow{s_{n+k}} \pi_{n+k+1} X_{n+k+1}.$$

The group $\pi_{n+k} X_{n+k}$ is defined for any $n \in \mathbb{Z}$ as soon as $k \geqslant -n$. A map $X \to Y$ is a *weak equivalence* if it induces an isomorphism of homotopy groups.

Equipped with the weak equivalences, the category $\mathcal{S}$ of spectra becomes a bona fide place for doing homotopy theory. A functor $\mathcal{S} \to \mathcal{C}$ to a category $\mathcal{C}$ is a *homotopy functor* if it takes weak equivalences to isomorphisms. There is a universal homotopy functor $\mathcal{S} \to \operatorname{ho}\mathcal{S}$ characterized by the property that the restriction mapping gives an equivalence between the category of functors

$\mathrm{ho}\,\mathcal{S} \to \mathcal{C}$ with the category of homotopy functors $\mathcal{S} \to \mathcal{C}$. The category $\mathrm{ho}\,\mathcal{S}$ is the *homotopy category of spectra*, and the set (in fact abelian group) $\mathrm{ho}\,\mathcal{S}(X,Y)$ is called the abelian group of *homotopy classes of maps* from X to Y. We will use the common abbreviation

$$[X,Y] = \mathrm{ho}\,\mathcal{S}(X,Y).$$

Example 6.2. The suspension spectrum $\Sigma^\infty Z$ of a space Z is the spectrum

$$\left(\Sigma^\infty Z\right)_n = S^n \wedge Z$$

with the structure maps derived from the equivalence $S^1 \wedge S^n = S^{n+1}$. When the context is clear it is customary to drop the Σ^∞ and not distinguish in notation between a space nd its suspension spectrum.

Example 6.3. For a non-negative integer $k \geqslant 0$ let S^k be the suspension spectrum of the k-sphere and S^{-k} be the spectrum defined by

$$\left(S^{-k}\right)_n = \begin{cases} * & n < k \\ S^{n-k} & n \geqslant k \end{cases}.$$

From the formula (6.1) one easily checks that for all $k \in \mathbb{Z}$ one has an isomorphism

$$[S^k, X] \approx \pi_k X$$

natural in X.

6.1.1. *Smash product.* Suppose that $X = \{X_n\}$ is a spectrum and Z is a space. Define $X \wedge Z$ to be the spectrum with

$$\left(X \wedge Z\right)_n = X_n \wedge Z$$

and the structure maps derived from those of X. This is the *smash product* of the spectrum X with the space Z.

Example 6.4. The spectrum $S^0 \wedge Z$ is the suspension spectrum of Z.

Example 6.5. The spectrum $S^{-k} \wedge S^k$ consists of the spaces

$$\left(S^{-k} \wedge S^k\right)_m = \begin{cases} * & m < k \\ S^m & m \geqslant k. \end{cases}$$

There is an inclusion

$$S^{-k} \wedge S^k \to S^0$$

which is easily checked to be a weak equivalence.

For a spectrum $X = \{X_n\}$ there is a functorial weak equivalence

$$\operatorname{ho}\varinjlim S^{-n} \wedge X_n \xrightarrow{\approx} X. \tag{6.6}$$

(See, for example [HHR, §2.2.1] where it is called the *canonical homotopy presentation*.)

There is an enrichment $\operatorname{ho}\underline{\mathcal{S}}$ of $\operatorname{ho}\mathcal{S}$ over the homotopy category of spaces. It is characterized by the existence of an isomorphism

$$\operatorname{ho}\mathcal{T}(Z, \operatorname{ho}\underline{\mathcal{S}}(X,Y)) \approx \operatorname{ho}\mathcal{S}(X \wedge Z, Y) \tag{6.7}$$

functorial in CW complexes Z, and spectra X and Y. We will employ the abbreviation

$$\operatorname{Map}(X,Y) = \operatorname{ho}\underline{\mathcal{S}}(X,Y). \tag{6.8}$$

Taking Z to be the *space* S^0 in (6.7) gives the isomorphism

$$[X,Y] = \pi_0 \operatorname{Map}(X,Y).$$

When the spectrum $X = \{X_n\}$ has the property that each X_n is a CW complex and Y has the property that each map

$$Y_n \to \Omega Y_{n+1}$$

is a weak equivalence, the homotopy type of $\operatorname{Map}(X,Y)$ is given by

$$\operatorname{ho}\underline{\mathcal{S}}(X,Y) = \operatorname{ho}\varprojlim \mathcal{M}(X_n, Y_n), \tag{6.9}$$

with $\mathcal{M}(X_n,Y_n)$ is the homotopy limit of the diagram

$$\begin{tikzcd}[column sep=small]
\underline{\mathcal{T}}(X_n,Y_n) \arrow[dr] & & \underline{\mathcal{T}}(X_{n-1},Y_{n-1}) \arrow[dl, "\sim"] \arrow[dr] & & {} \arrow[dl, "\sim"] & \cdots & {} \arrow[dr] & & \underline{\mathcal{T}}(X_0,Y_0) \arrow[dl, "\sim"] \\
& \underline{\mathcal{T}}(X_{n-1},\Omega Y_n) & & \underline{\mathcal{T}}(X_{n-2},\Omega Y_{n-1}) & & & & \underline{\mathcal{T}}(X_0,\Omega Y_1) &
\end{tikzcd}$$

in which the southeast arrows are given by the compositions

$$\underline{\mathcal{T}}(X_m,Y_m) \to \underline{\mathcal{T}}(S^1 \wedge X_{m-1}, Y_m) \approx \underline{\mathcal{T}}(X_{m-1}\Omega Y_m).$$

Note that the projection map $\mathcal{M}(X_n,Y_n) \to \underline{\mathcal{T}}(X_n,Y_n)$ is a weak equivalence, so that (6.9) can heuristically be interpreted as giving a presentation of $\operatorname{ho}\underline{\mathcal{S}}(X,Y)$ as a homotopy inverse limit of the spaces $\underline{\mathcal{T}}(X_n,Y_n)$.

A spectrum Y with the property that for all n the map $Y_n \to \Omega Y_{n+1}$ is a weak equivalence is called an Ω*-spectrum* (or a *loop spectrum*). Every spectrum Y is naturally weakly equivalent to an Ω-spectrum. Indeed, given Y define LY by

$$LY_n = \operatorname{ho}\varinjlim \Omega^k Y_{n+k}.$$

Using the homeomorphism $\Omega(\Omega^k Y_{n+k}) \approx \Omega^k \Omega Y_{n+k}$ one sees that LY has the structure of an Ω-spectrum and that the canonical map $Y \to LY$ is a weak equivalence.

6.1.2. *Duality.* The operation $X \wedge Z$ extends to a symmetric monoidal smash product on spectra. In fact there is a unique extension having the property that it commutes with colimits in both variables, and for spaces Z_1 and Z_2 and integers $k, \ell \geqslant 0$ one has

$$\big(S^{-k} \wedge Z_1\big) \wedge \big(S^{-\ell} \wedge Z_2\big) \simeq S^{-(k+\ell)} \wedge Z_1 \wedge Z_2.$$

The existence and uniqueness can be deduced from the canonical homotopy presentation (6.6).

Equipped with the smash product the categories $\operatorname{ho}\mathcal{S}$ and $\operatorname{ho}\underline{\mathcal{S}}$ become symmetric monoidal categories. By Example 6.5 the suspension spectra of spheres are dualizable (in fact invertible). It follows that the suspension spectrum of any finite CW complex is also dualizable.

6.1.3. *Stability.* An easy check (or an appeal to the invertibility of spheres) shows that for all k and all X the map

$$\pi_k X \to \pi_{k+1} X \wedge S^1$$

is an isomorphism. This implies a map $A \to X$ gives rise to a long exact sequence

$$\cdots \to \pi_k A \to \pi_k X \to \pi_k X \cup CA \to \pi_{k-1} A \to \dots$$

in which $X \cup CA$ is the spectrum

$$\big(X \cup CA\big)_n = X_n \cup CA_n$$

with $CA = A \times [0,1]/A \times \{1\} \cup * \times [0,1]$. This, in turn, implies that the map from A to the homotopy fiber of $X \to X \cup CA$ is a weak equivalence.

6.1.4. *Thom Spectra.* Let X be a space. Given a map $V : X \to BO$, define a sequence of maps $V_n : X_n \to BO_n$ by the homotopy pullback squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "V_n"'] & X \arrow[d, "V"] \\ BO_n \arrow[r] & BO \end{tikzcd} \tag{6.10}$$

The map $V_n : X_n \to BO_n$ classifies a vector bundle of rank n over X_n (which will also be denoted V_n). By construction, the pullback of $V_{n+1} \to X_{n+1}$ to X_n comes equipped with an isomorphism to $V_n \oplus \underline{\mathbb{R}} \to X_n$. This give a map of Thom spaces

$$\Sigma \operatorname{Thom}(X_n; V_n) = \operatorname{Thom}(X_n; V_n \oplus 1) \to \operatorname{Thom}(X_{n+1}, V_{n+1})$$

making the sequence of spaces $\{\operatorname{Thom}(X_n; V_n)\}$ into a spectrum. This is the *Thom spectrum* of V, denoted $\operatorname{Thom}(X; V)$. The canonical homotopy presentation of $\operatorname{Thom}(X; V)$ takes the form

$$\operatorname{Thom}(X; V) = \operatorname{ho}\varinjlim S^{-n} \wedge \operatorname{Thom}(X_n; V_n).$$

We will also encounter the Thom spectrum $\mathrm{Thom}(X;-V)$ associated to a map $V\colon X\to BO$ by composing with the "additive inverse" map $(-1): BO\to BO$ (see §7.1). With X_n and V_n defined as in (6.10), the isomorphism

$$V_{n+1}|_{X_n}\approx V_n\oplus\underline{\mathbb{R}}$$

becomes

$$-V_{n+1}|_{X_n}\approx -V_n-\underline{\mathbb{R}}.$$

This leads to maps

$$\mathrm{Thom}(X;-V_n)\to S^1\wedge\mathrm{Thom}(X_{n+1};-V_{n+1}),$$

and an alternative presentation

$$\mathrm{Thom}(X;-V)=\mathrm{ho}\varinjlim S^n\wedge\mathrm{Thom}(X_n;-V_n). \tag{6.11}$$

If V has virtual dimension d then $V-\underline{\mathbb{R}^d}$ has virtual dimension 0 and one defines

$$\mathrm{Thom}(X;V)=S^d\wedge\mathrm{Thom}(X;V-\underline{\mathbb{R}^d}).$$

The Thom spectrum construction is a functor on the category of spaces over the classifying space $\mathbb{Z}\times BO$ of KO-theory. It is symmetric monoidal in the sense that for $V:X\to\mathbb{Z}\times BO$ and $W:Y\to\mathbb{Z}\times BO$ there is a natural weak equivalence

$$\mathrm{Thom}(X\times Y;\pi_X^*V\oplus\pi_Y^*W)\approx\mathrm{Thom}(X;V)\wedge\mathrm{Thom}(Y;W),$$

in which π_X and π_Y are the projections.

6.2. Borel equivariant stable homotopy theory

Now suppose that G is a compact Lie group (which in our case will be $\mathbb{Z}/2$) and let $\mathcal{S}^{hG}$ be the category of spectra equipped with a G action, and equivariant maps. An object of $\mathcal{S}^{hG}$ consists of a sequence $\{X_n,s_n\}$ of left G-spaces X_n and equivariant maps $S^1\wedge X_n\to X_{n+1}$ in which S^1 has the trivial G-action. Sometimes what we are calling a G-spectrum is called a *naive G-spectrum.*

Definition 6.12. A map $X\to Y$ in $\mathcal{S}^{hG}$ is a *Borel weak equivalence* if it is a weak equivalence when regarded as a map in $\mathcal{S}$.

Equipped with the Borel weak equivalences, the category $\mathcal{S}^{hG}$ becomes a category in which one can do homotopy theory. The homotopy category $\mathrm{ho}\,\mathcal{S}^{hG}$ is defined as the target of the universal homotopy functor out of $\mathcal{S}^{hG}$. We will use the abbreviation

$$[X,Y]^{hG}=\mathrm{ho}\,\mathcal{S}^{hG}(X,Y).$$

The construction of the smash product goes through in a straightforward way for the Borel equivariant spectra, and there is a *derived equivariant mapping space* between two equivariant spectra. In fact, it follows from the expression (6.9) that when X and Y are G-spectra, the space $\operatorname{ho}\underline{\mathcal{S}}(X,Y)$ acquires the homotopy type of a G-space. The derived equivariant mapping space works out to be homotopy fixed point space

$$\operatorname{Map}^G(X,Y) = \operatorname{Map}(X,Y)^{hG},$$

and the maps in the homotopy category of G-spectra are given by

$$[X,Y]^{hG} = \pi_0 \operatorname{Map}(X,Y)^{hG}.$$

In Borel equivariant homotopy theory the suspension spectra of finite G-sets (with a disjoint base point added) are self dual. This implies that the suspension spectra of finite G-CW-complexes are dualizable and the suspension spectrum of the one point compactification S^V of a finite dimensional representation V of G is invertible. These facts are not quite immediate. If X is a finite G-set, then the evaluation map

$$X_+ \wedge X_+ \to S^0$$

is the map of suspension spectra induced by the map

$$X \times X \to S^0$$

sending the diagonal to the non base point and the complement of the diagonal to the base point. It is not so straightforward to write down the coevaluation map. Nevertheless, for G-spectra W and Z, the composite

$$\operatorname{Map}(Z, W \wedge X_+) \to \operatorname{Map}(Z \wedge X_+, W \wedge X_+ \wedge X_+) \to \operatorname{Map}(Z \wedge X_+, W)$$

is a G-equivariant map that is a weak equivalence of underlying spaces, and so gives an equivalence

$$\operatorname{Map}(Z, W \wedge X_+)^{hG} \approx \operatorname{Map}(Z \wedge X_+, W)^{hG}$$

and an isomorphism

$$[Z, W \wedge X_+]^{hG} \approx [Z \wedge X_+, W]^{hG}.$$

Once one knows that the finite G-sets are dualizable it follows that the suspension spectrum of any finite G-CW-complex is dualizable. We denote the dual of X as $D(X)$. This implies the invertibility of S^V since the map

$$D(S^V) \wedge S^V \to S^0$$

is a weak equivalence of underlying spectra. It is customary to use the notation

$$S^{-V} = DS^V.$$

For more on virtual representation spheres see Example 6.17 of §6.2.2.

6.2.1. *Homotopy fixed points and homotopy orbits.* Regarding a non-equivariant spectrum as a G-spectrum with the trivial action gives a functor

$$\mathcal{S} \to \mathcal{S}^{hG}.$$

This functor preserves weak equivalences and so induces a functor on homotopy categories. The homotopy orbit and fixed point functors provide both a left and right adjoint to this induced functor.

Recall that the *homotopy orbit space* of a pointed G-space Z is the space

$$Z_{hG} = EG_+ \underset{G}{\wedge} Z,$$

and that the *homotopy fixed point space* is the space

$$Z^{hG} = \mathcal{T}(EG_+, Z)^G$$

of equivariant basepoint preserving maps from EG_+ to Z. These notions extend component-wise to equivariant spectra. The *homotopy orbit spectrum* of a G-spectrum $X = \{X_n\}$ is the spectrum $X_{hG} = \{(X_n)_{hG}\}$ and the *pre homotopy fixed point spectrum* is the spectrum $X^{h'G} = \{(X_n)^{hG}\}$.

The functor X_{hG} preserves weak equivalences and so directly induces a functor on homotopy categories. The functor $X^{h'G}$ preserves weak equivalences between Ω-spectra and so induces a *homotopy fixed point* functor

$$(\,-\,)^{hG} : \operatorname{ho}\mathcal{S}^{hG} \to \operatorname{ho}\mathcal{S}$$

sending X to $(LX)^{h'G}$.

These functors on the homotopy category are adjoints to the inclusion

$$\operatorname{ho}\mathcal{S} \to \operatorname{ho}\mathcal{S}^{hG}$$

in the sense that there are natural isomorphisms

$$[X, A]^{hG} \approx [X_{hG}, A] \tag{6.13}$$

$$[A, Y]^{hG} \approx [A, Y^{hG}] \tag{6.14}$$

in which X and Y are G-spectra and A is a spectrum with trivial G-action. Also, the fixed point spectrum $A^{h\mathbb{Z}/2}$ is computed as

$$\operatorname{Map}^{\mathbb{Z}/2}(S^0, A) \;\simeq\; \operatorname{Map}(B\mathbb{Z}/2_+, A) \xleftarrow{\simeq} A \vee \operatorname{Map}(B\mathbb{Z}/2, A) \xrightarrow{\simeq} A \times \operatorname{Map}(B\mathbb{Z}/2, A), \tag{6.15}$$

in which the left pointing map involves a choice of a basepoint $x \in B\mathbb{Z}/2$ and is the sum of the map

$$B\mathbb{Z}/2_+ \to S^0$$

sending $B\mathbb{Z}/2$ to the non basepoint and the map

$$B\mathbb{Z}/2_+ \to B\mathbb{Z}/2$$

which is the identity map on $B\mathbb{Z}/2$ and sends the disjoint base point on the left to the new basepoint on the right.

6.2.2. *Equivariant Thom spectra.* Suppose that B is a space and $p : X \to B$ is a principal G-bundle. A map $W : B \to BO$ leads, as above, to a sequence of maps

$$\begin{tikzcd} B_n \arrow[r] \arrow[d, "W_n"] & B_{n+1} \arrow[r, dashed] \arrow[d, "W_{n+1}"] & B \arrow[d, "W"] \\ BO_n \arrow[r] & BO_{n+1} \arrow[r, dashed] & BO \end{tikzcd}$$

and a Thom spectrum $\mathrm{Thom}(B;W) = \{\mathrm{Thom}(B_n;W_n)\}$. Define principal G-bundles $X_n \to B_n$ by the pullback square

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "p_n"'] & X \arrow[d, "p"] \\ B_n \arrow[r] & B. \end{tikzcd}$$

The bundle $p_n^* W_n$ is a G-equivariant vector bundle on X_n. In fact, by descent, the data of a G-equivariant vector bundle on X_n is equivalent to the data of a vector bundle over B_n. The G-action on $(X_n, p^* W_n)$ induces a G-action on the Thom spectrum $\mathrm{Thom}(X, p^*W) = \{\mathrm{Thom}(X_n; p_n^* W_n)\}$ making it into an equivariant spectrum. By construction the homotopy orbit spectrum is given by

$$\mathrm{Thom}(X; p^*W)_{hG} = \mathrm{Thom}(B;W). \tag{6.16}$$

As in §6.1.4, equivariant Thom spectra for maps $B \to \mathbb{Z} \times BO$ are defined by subtracting a suitable trivial bundle and suspending the result.

Example 6.17 (Representation spheres)**.** An element $V \in KO^0(BG)$ is classified by a map

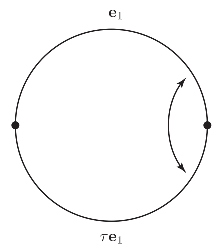
$$V : BG \to \mathbb{Z} \times BO$$

and so gives rise to an equivariant Thom spectrum. When V corresponds to a representation of G the equivariant Thom spectrum is the spectrum S^V. This construction sends sums of elements of $KO^0(BG)$ to smash products of G-spectra. Composing with the map

$$RO(G) \to KO^0(BG)$$

gives a construction of a sphere S^V associated to every virtual representation V of G. This gives another approach to the construction and invertibility of representation spheres in Borel equivariant stable homotopy theory.

6.2.3. *The σ-sphere.* We now specialize to the case $G = \mathbb{Z}/2$, and write σ for the real sign representation. The sphere S^σ has an equivariant cell decomposition with one non-basepoint fixed 0-cell, and one free 1-cell as shown here.



This gives a pushout square

$$\begin{tikzcd} \mathbb{Z}/2 \times \partial D^1 \arrow[r] \arrow[d] & \mathbb{Z}/2 \times D^1 \arrow[d] \\ S^0 \arrow[r] & S^\sigma \end{tikzcd}$$

leading to a cofibration sequence

$$\mathbb{Z}/2_+ \to S^0 \to S^\sigma \tag{6.18}$$

of equivariant spectra. Passing to duals and using the self-duality of finite G-sets gives a cofibration sequence

$$S^{-\sigma} \to S^0 \to \mathbb{Z}/2_+. \tag{6.19}$$

The map $S^0 \to \mathbb{Z}/2_+$ is the *transfer map* and, non-equivariantly, has degree 1 on each summand of $\mathbb{Z}/2_+ = S^0 \vee S^0$.

Write

$$\begin{aligned} \gamma &= 1 - \sigma \\ \delta &= \sigma - 1. \end{aligned}$$

For a $\mathbb{Z}/2$-spectrum X we define

$$\begin{aligned} X^\delta &= S^\delta \wedge X \\ X^\gamma &= S^\gamma \wedge X. \end{aligned} \tag{6.20}$$

Smashing with (6.18) and (6.19) gives for any X, (co-)fibration sequences

$$X^\delta \to \mathbb{Z}/2_+ \wedge X \to X \quad \text{and} \tag{6.21}$$

$$X \to \mathbb{Z}/2_+ \wedge X \to X^\gamma. \tag{6.22}$$

6.3. Real structures

Our next aim is to equip $I\mathbb{C}^\times$ and $I\mathbb{Z}(1)$ with $\mathbb{Z}/2$-actions corresponding to complex conjugation, in such a way that the cofibration sequence (see (5.21))

$$I\mathbb{Z}(1) \to H\mathbb{C} \xrightarrow{\exp} I\mathbb{C}^\times \tag{6.23}$$

is a cofibration sequence of $\mathbb{Z}/2$-equivariant spectra. Though there no mystery about the action on the abelian group-valued functor $[-, I\mathbb{C}^\times]$, there are infinitely many refinements of this to an action on the spectrum $I\mathbb{C}^\times$. Here we will motivate a specific choice, and check it against three situations in which there is a naturally occurring action.

6.3.1. $\mathbb{Z}/2$-*actions.* The space of $\mathbb{Z}/2$-actions on a spectrum X is the space of maps

$$B\mathbb{Z}/2 \to B\,\mathrm{hAut}(X)$$

from the classifying space of $\mathbb{Z}/2$ to the classifying space of the monoid of self homotopy equivalences of X. Smashing a map $S^0 \to S^0$ with the identity map of X gives a map

$$B\,\mathrm{hAut}(S^0) \to B\,\mathrm{hAut}(X).$$

The maps $B\mathbb{Z}/2 \to B\,\mathrm{hAut}(S^0)$ then correspond both to (i) $\mathbb{Z}/2$-actions on S^0 and (ii) $\mathbb{Z}/2$-actions on all spectra which are natural in the sense that they commute with all maps and are homotopy colimit preserving. Put more succinctly, the "natural" $\mathbb{Z}/2$-actions are homotopy colimit preserving sections of the forgetful functor

$$\mathcal{S}^{h\mathbb{Z}/2} \to \mathcal{S}. \tag{6.24}$$

Associating to a vector space its one point compactification defines a map

$$BO \to B\,\mathrm{hAut}(S^0),$$

so that a virtual representation V of $\mathbb{Z}/2$, of virtual dimension 0, determines a natural $\mathbb{Z}/2$-action via the composition

$$B\mathbb{Z}/2 \xrightarrow{V} BO \to B\,\mathrm{hAut}(S^0).$$

The corresponding section of (6.24) is the one sending a spectrum X to $S^V \wedge X$.

Remark 6.25. Because S^0 is the tensor unit in $\mathcal{S}$, the space $B\,\mathrm{hAut}(S^0)$ is actually an infinite loop space. The map $BO \to B\,\mathrm{hAut}(S^0)$ also turns out to be an infinite loop map. This means that "natural" $\mathbb{Z}/2$-actions may be composed, and that the composition of actions corresponding to virtual representations V and W is the natural action corresponding to $V \oplus W$.

Remark 6.26. From the defining property of $I\mathbb{Z}(1)$ one can check that the map

$$\begin{aligned} \operatorname{Map}(S^0, S^0) &\to \operatorname{Map}(I\mathbb{Z}(1), I\mathbb{Z}(1)) \\ f &\mapsto f \wedge \mathrm{id} \end{aligned}$$

is a weak equivalence. Now the loop space of any component of the space of maps $B\mathbb{Z}/2 \to B\,\mathrm{hAut}(S^0)$ is the space of maps $B\mathbb{Z}/2 \to \mathrm{hAut}(S^0)$. The homotopy type of this latter space falls within the purview of the Segal conjecture, and consists of the path components of $QB\mathbb{Z}/2_+ \times QS^0$ whose first component is a generator of

$$\pi_0 QB\mathbb{Z}/2_+ \approx \mathbb{Z}.$$

For this reason, one knows a lot about the space of actions of $\mathbb{Z}/2$ on $I\mathbb{Z}(1)$, and in particular that there are infinitely many inequivalent actions inducing the sign representation on $\pi_0 I\mathbb{Z}(1)$.

For the spectrum $H\mathbb{C}$ one has $B\,\mathrm{hAut}(H\mathbb{C}) \approx K(\mathrm{Aut}(\mathbb{C}), 1)$, in which $\mathrm{Aut}(\mathbb{C})$ is the group of abelian group automorphisms of $\mathbb{C}$. In this case there is no difference between $\mathbb{Z}/2$-actions on $H\mathbb{C}$ and $\mathbb{Z}/2$-actions on $\mathbb{C}$, and complex conjugation is uniquely specified.

6.3.2. *Duality.* Spectra with no negative homotopy groups are modeled by (higher) Picard groupoids. Picard groupoids come equipped with a $\mathbb{Z}/2$-action sending each object to its inverse. This corresponds to a natural $\mathbb{Z}/2$-action on spectra which we now determine.

Let $\mathcal{C}$ be a Picard category and consider the category of pairs (x, y) equipped with an isomorphism $x \otimes y \to 1$. The functor $(x, y) \mapsto x$ is an equivalence of categories, so the $\mathbb{Z}/2$-action sending x to its inverse corresponds to the action on the category of pairs sending

$$x \otimes y \to 1$$

to

$$y \otimes x \to x \otimes y \to 1.$$

If $\mathcal{C}$ corresponds to a spectrum X then the category of pairs corresponds to $X \vee X \approx X \times X$, and the category of pairs (x, y) equipped with an isomorphism $x \otimes y \to 1$ is the homotopy fiber of the map

$$X \vee X \to X.$$

Writing this in terms of equivariant spectra we are looking at the homotopy fiber of

$$\mathbb{Z}/2_+ \wedge X \to X,$$

which by (6.21) is X^δ.

Summarizing, we have the following.

Proposition 6.27. *The natural $\mathbb{Z}/2$-action corresponding to "duality" is given by the map*

$$B\mathbb{Z}/2 \xrightarrow{\delta} BO \to B\,\mathrm{hAut}(S^0)$$

and associates to a spectrum X, the $\mathbb{Z}/2$-equivariant spectrum

$$X^\delta = S^\delta \wedge X = S^{\sigma - 1} \wedge X.$$

6.3.3. *Complex conjugation.* A complex conjugation on $I\mathbb{Z}(1)$ corresponds to a map

$$\nu : B\mathbb{Z}/2 \to B\,\mathrm{hAut}(I\mathbb{Z}(1))$$

having at least the property that its effect on π_1 is the sign representation of $\mathbb{Z}/2$ on $\mathbb{Z}(1)$. Write

$$\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{Z}(1))\bigr)_c$$

for the space of maps inducing this homomorphism on π_1. The space $\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{Z}(1))\bigr)_c$ is a union of infinitely many path components of $\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{Z}(1))\bigr)$ (see Remark 6.26).

Similarly, complex conjugation on $I\mathbb{C}^\times$ corresponds to a map

$$\nu' : B\mathbb{Z}/2 \to B\,\mathrm{hAut}(I\mathbb{C}^\times),$$

whose effect on π_1 corresponds to the action of $\mathbb{Z}/2$ by complex conjugation on $\mathbb{C}^\times$. Write $\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{C}^\times)\bigr)_c$ for this space of maps.

Since the maps

$$\begin{aligned} \mathrm{Map}(I\mathbb{Z}(1), H\mathbb{C}) &\to \mathrm{Hom}(\mathbb{Z}(1), \mathbb{C}) \\ \mathrm{Map}(H\mathbb{C}, I\mathbb{C}^\times) &\to \mathrm{Hom}(\mathbb{C}, \mathbb{C}^\times) \end{aligned}$$

are weak equivalence, so are the maps

$$\begin{aligned} \mathrm{Map}(I\mathbb{Z}(1), H\mathbb{C})^{h\mathbb{Z}/2} &\to \mathrm{Hom}(\mathbb{Z}(1), \mathbb{C})^{\mathbb{Z}/2} \\ \mathrm{Map}(H\mathbb{C}, I\mathbb{C}^\times)^{h\mathbb{Z}/2} &\to \mathrm{Hom}(\mathbb{C}, \mathbb{C}^\times)^{\mathbb{Z}/2} \end{aligned}$$

for any $\mathbb{Z}/2$-actions on $I\mathbb{Z}(1)$ and $I\mathbb{C}^\times$. It follows that any action ν as above extends uniquely to a $\mathbb{Z}/2$-equivariant map

$$I\mathbb{Z}(1)^\nu \to H\mathbb{C}$$

and so induces a $\mathbb{Z}/2$-action ν' on the cofiber $I\mathbb{C}^\times$. Similarly an action ν' as above induces a $\mathbb{Z}/2$-action ν on $I\mathbb{Z}(1)$. In this way we have an equivalence

$$\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{Z}(1))\bigr)_c \approx \mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{C}^\times)\bigr)_c. \tag{6.28}$$

The space of *real structures* on $I\mathbb{Z}(1)$ and $I\mathbb{C}^\times$ will be defined to be a single path component of the above spaces. Before specifying which one, we turn to a motivating example.

Example 6.29 (Hermitian structures and positivity)**.** Let $f\mathrm{Vect}_\mathbb{C}$ be the topological groupoid of finite dimensional complex vector spaces and (complex) linear isomorphisms, endowed with the symmetric monoidal structure of $\otimes$. For $V \in f\mathrm{Vect}_{\mathbb{C}}$, let V^* be the dual vector space. We define a *covariant* "duality" functor $V \mapsto V^\vee$ by

$$\begin{aligned} V^\vee &= V^* \\ f^\vee &= \bigl(f^*\bigr)^{-1}. \end{aligned}$$

The canonical isomorphism $V^{\vee\vee} \approx V$ extends the functor $V^\vee$ to a $\mathbb{Z}/2$-action on $f\mathrm{Vect}_{\mathbb{C}}$. (See Appendix B.) There is another $\mathbb{Z}/2$-action

$$V \mapsto \overline{V}$$

gotten by redefining scalar multiplication by $x \in \mathbb{C}$ to be scalar multiplication by $\bar{x}$.

Let $f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{pos}}$ be the topological groupoid of finite dimensional complex vector spaces equipped with a positive definite Hermitian inner product, and unitary transformations. Since the inclusion $U(n) \subset GL_n(C)$ is a homotopy equivalence, the functor

$$f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{pos}} \to f\mathrm{Vect}_{\mathbb{C}}$$

is a weak equivalence of topological categories. On $f\mathrm{Vect}_{\mathbb{C}}^{\mathrm{pos}}$ the Hermitian inner product gives a natural isomorphism $\overline{V}^* \approx V$, trivializing the composition "bar star" of the two $\mathbb{Z}/2$-actions defined above. This suggests that whatever complex conjugation is, on the categories in which $\mathbb{C}$ is regarded as having a topology, the combined action (in the sense of Remark 6.25) of complex conjugation and duality should be trivializable. The trivialization is non-canonical, however. One might have chosen negative definite vector spaces, or, for each prime p made a choice of positive or negative definite Hermitian inner products on vector spaces of dimension p and then extend to all finite dimensional vector spaces by tensoring.

With Example 6.29 as motivation, and in view of Proposition 6.27, we propose the following.

Definition 6.30. The space of *real structures* on $I\mathbb{Z}(1)$ is the path component of the space

$$\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{Z}(1))\bigr)_c \tag{6.31}$$

containing the map $1 - \sigma$. The space of *real structures* on $I\mathbb{C}^\times$ is the path component of the space $\mathcal{T}\bigl(B\mathbb{Z}/2, B\,\mathrm{hAut}(I\mathbb{C}^\times)\bigr)_c$ corresponding to the space of real structures on $I\mathbb{Z}(1)$ under the equivalence (6.28).

As above, we write $I\mathbb{Z}(1)^\nu$ for the $\mathbb{Z}/2$-spectrum corresponding to a real structure $\nu : B\mathbb{Z}/2 \to B\,\mathrm{hAut}(I\mathbb{Z}(1))$. Any real structure fits canonically into a cofibration sequence

$$I\mathbb{Z}(1)^\nu \longrightarrow H\mathbb{C}^{\nu'} \xrightarrow{\exp} (I\mathbb{C}^\times)^{\nu'} \tag{6.32}$$

in which ν and ν' correspond under the equivalence (6.28); the superscript on $H\mathbb{C}$ is the unique complex conjugation, explained at the end of §6.3.1.

Remark 6.33. Since the space of real structures ν on $I\mathbb{Z}(1)$ is connected, but not contractible, any $I\mathbb{Z}(1)^\nu$ is non-canonically equivariantly equivalent to $I\mathbb{Z}(1)^\gamma = S^{1-\sigma} \wedge I\mathbb{Z}(1)$.

Ansatz 6.34. We use the basepoint in (6.31) to fix once and for all $\nu = \gamma = 1 - \sigma$. Under the equivalence (6.28) this determines a real structure ν_0' on $I\mathbb{C}^\times$. Our choices render the cofibration sequence (6.32) as

$$I\mathbb{Z}(1)^\gamma \longrightarrow H\mathbb{C}^{\nu_0'} \xrightarrow{\exp} (I\mathbb{C}^\times)^{\nu_0'} \tag{6.35}$$

Remark 6.36. The real structure γ on $I\mathbb{Z}(1)$ is the restriction of a natural action of $\mathbb{Z}/2$; the corresponding real structure ν_0' is not. However, in terms of the polar decomposition $\mathbb{C}^\times = \mathbb{T} \times \mathbb{R}^{>0}$ we have

$$(I\mathbb{C}^\times)^{\nu_0'} \approx I\mathbb{T} \wedge S^{1-\sigma} \;\vee\; H\mathbb{R}^{>0}. \tag{6.37}$$

The spectrum $I\mathbb{T}$ is characterized in the homotopy category of spectra by a functorial isomorphism

$$[\mathcal{B}, I\mathbb{T}] \xrightarrow{\cong} \operatorname{Hom}(\pi_0\mathcal{B}, \mathbb{T}) \tag{6.38}$$

for all spectra $\mathcal{B}$, analogous to (5.13). The equivariant spectrum $I\mathbb{T}^\gamma = I\mathbb{T} \wedge S^{1-\sigma}$ fits into a cofibration sequence analogous to (6.35):

$$I\mathbb{Z}(1)^\gamma \longrightarrow H\mathbb{R}(1)^{\nu_0'} \xrightarrow{\exp} I\mathbb{T}^{\nu_0'} \tag{6.39}$$

Remark 6.40. This definition of real structure fits with the three cases in which one has an algebraic interpretation of $I\mathbb{Z}(1)$ (see Remark 5.16). The zeroth space of $\Sigma I\mathbb{Z}(1)$ is modeled by the unit complex numbers with the usual topology; that of $\Sigma^2 I\mathbb{Z}(1)$ corresponds to the symmetric monoidal groupoid of $\mathbb{Z}/2$-graded complex lines; and $\Sigma^3 I\mathbb{Z}(1)$ to the Brauer-Wall symmetric monoidal 2-groupoid of $\mathbb{Z}/2$-graded simple algebras over $\mathbb{C}$, $\mathbb{Z}/2$-graded bimodules and intertwiners. These three models come equipped with natural real structures, coming from change of scalars. By direct computation one can show that the homotopy fixed points of $\Sigma^i I\mathbb{Z}(1)^\gamma$ is modeled by the corresponding real versions of the three categories described above. To check this it suffices to do so when $i = 3$ as the other cases are gotten from it by passing to loop spaces. The real Brauer-Wall category corresponds to a spectrum B with the following homotopy groups

$\pi_i B = 0$	$i \notin [0,3]$
$\pi_0 B = \mathbb{Z}/8$	(the eight real Clifford algebras)
$\pi_1 B = \mathbb{Z}/2$	(the even and odd real line)
$\pi_2 B = \{\pm 1\}$	

and has the property that the multiplication by η maps

$$\pi_0 B \to \pi_1 B \to \pi_2 B$$

are non-zero. A straightforward computation shows that any spectrum X with these properties is homotopy equivalent to B. To verify the claim it therefore suffices to show that the (-1)-connected cover of $\bigl(\Sigma^3 I\mathbb{Z}(1)^\gamma\bigr)^{h\mathbb{Z}/2}$ has these properties. We therefore need to know the groups

$$\pi_i\bigl(\Sigma^3 I\mathbb{Z}(1)^\gamma\bigr)^{h\mathbb{Z}/2} \qquad i \geqslant 0$$

and the effect of multiplication by η. Now for the real structure $\gamma = 1-\sigma$ one has

$$\begin{aligned}
\operatorname{Map}(S^0, \Sigma^3 I\mathbb{Z}(1)^\gamma)^{h\mathbb{Z}/2} &\approx \operatorname{Map}(S^0, S^{(1-\sigma)} \wedge \Sigma^3 I\mathbb{Z}(1))^{h\mathbb{Z}/2} \\
&\approx \operatorname{Map}(S^{(\sigma-1)}, \Sigma^3 I\mathbb{Z}(1))^{h\mathbb{Z}/2} \\
&\approx \operatorname{Map}(S^{(\sigma-1)}_{h\mathbb{Z}/2}, \Sigma^3 I\mathbb{Z}(1)) \\
&\approx \operatorname{Map}(\operatorname{Thom}(B\mathbb{Z}/2; \sigma-1), S^3 \wedge I\mathbb{Z}(1)),
\end{aligned}$$

by (6.13) and (6.16). We therefore need information about

$$[\operatorname{Thom}(B\mathbb{Z}/2; \sigma-1), S^i \wedge I\mathbb{Z}(1)] \quad 1 \leqslant i \leqslant 3$$

or, from the defining property of $I\mathbb{Z}(1)$, the character groups of

$$\pi_i \operatorname{Thom}(B\mathbb{Z}/2; \sigma-1) \quad 0 \leqslant i \leqslant 2.$$

As described in §10, these groups coincide with the same homotopy groups of $MT\mathrm{Pin}^-$ and are shown in Figure 5 (the case $s=1$) to be the groups $\mathbb{Z}/2$, $\mathbb{Z}/2$, and $\mathbb{Z}/8$ with both η-multiplications non-zero.

6.3.4. *Terminology.* It will be convenient in the sequel to have names for the objects assigned to closed manifolds of arbitrary codimension in an invertible field theory. In codimension 0 we have a complex number and in codimension 1 an object in the category of complex $\mathbb{Z}/2\mathbb{Z}$-graded lines with the monoidal structure of graded tensor product and the Koszul sign in the symmetry. We refer to such an object as a 'complex super line' or a '$\mathbb{Z}/2\mathbb{Z}$-graded line'. Hence in codimension k we introduce the term 'complex super k-line'.[24]

Definition 6.41.

(i) $I\mathbb{Z}(1)$ is the spectrum of *higher complex super lines*;
(ii) $\bigl(I\mathbb{Z}(1)^\gamma\bigr)^{h\mathbb{Z}/2}$ is the spectrum of *higher real super lines*;
(iii) $I\mathbb{Z}(1)_H := (I\mathbb{Z}(1)^\gamma \wedge S^{\sigma-1})^{h\mathbb{Z}/2}$ is the spectrum of *higher Hermitian super lines*;
(iv) $I\mathbb{C}^\times$ is the spectrum of *higher flat complex super lines*;
(v) The k^{th} space in the spectrum $I\mathbb{Z}(1)$ is the space of *complex super k-lines*.

[24]Kapranov [Kap, §3.4] suggests a higher use of super based on the sphere spectrum.

Example 6.29 is the motivation for (iii). There are analogs of (iv) and (v) for real and Hermitian super lines. For example, the fixed point spectrum

$$I\mathbb{C}^\times_H := \big((I\mathbb{C}^\times)^{\nu'_0} \wedge S^{\sigma-1}\big)^{h\mathbb{Z}/2} \tag{6.42}$$

is the spectrum of *higher flat Hermitian super lines*, and the k^{th} space of that spectrum is the space of *flat Hermitian super k-lines.* As for the fixed point spectrum in (iii), since $S^{1-\sigma} \wedge S^{\sigma-1}$ is the sphere spectrum with the trivial $\mathbb{Z}/2$-action—the "bar star" involution—we deduce from (6.15) a canonical identification

$$I\mathbb{Z}(1)_H = \operatorname{Map}\big(B\mathbb{Z}/2_+, I\mathbb{Z}(1)\big). \tag{6.43}$$

Pulling back along $B\mathbb{Z}/2 \to \mathrm{pt}$ we obtain a map

$$I\mathbb{Z}(1) \longrightarrow I\mathbb{Z}(1)_H; \tag{6.44}$$

the image is a summand, split by a choice of point in $B\mathbb{Z}/2$.

Definition 6.45. The image $I\mathbb{Z}(1)_{\text{pos}}$ of (6.44) is the *spectrum of higher positive definite Hermitian super lines.*

The k^{th} space in $I\mathbb{Z}(1)_{\text{pos}}$ is the space of *positive definite Hermitian super k-lines.* Define the spectrum of *higher* flat *positive definite Hermitian super lines* as the homotopy pullback

$$\begin{tikzcd} I\mathbb{C}^\times_{\text{pos}} \arrow[r] \arrow[d] & \Sigma I\mathbb{Z}(1)_{\text{pos}} \arrow[d] \\ I\mathbb{C}^\times_H \arrow[r] & \Sigma I\mathbb{Z}(1)_H. \end{tikzcd} \tag{6.46}$$

We examine this homotopy-theoretic definition of positivity by focusing on the top piece, first in the ungraded case and then in the $\mathbb{Z}/2\mathbb{Z}$-graded case.

Example 6.47 (Hermitian lines)**.** Consider the spectrum $\Sigma^2 H\mathbb{Z}$. Its zero-space represents the ordinary groupoid of complex lines; morphisms have the continuous topology. There is a contractible space of trivializable involutions, and we imagine a point in it to represent bar star. The analog of (6.43) implies that the set of components of the fixed point spectrum of any such involution is

$$\pi_0 \operatorname{Map}(B\mathbb{Z}/2_+, \Sigma^2 H\mathbb{Z}) = \pi_0\Sigma^2 H\mathbb{Z} \oplus \pi_0 \operatorname{Map}(B\mathbb{Z}/2, \Sigma^2 H\mathbb{Z}) = \{0\} \oplus \mathbb{Z}/2. \tag{6.48}$$

The zero space of $\operatorname{Map}(B\mathbb{Z}/2_+, \Sigma^2 H\mathbb{Z})$ represents the groupoid of Hermitian lines, the $\mathbb{Z}/2\mathbb{Z}$ tracks the sign of the Hermitian form. The positive subspace, obtained by pulling back along $B\mathbb{Z}/2 \to \mathrm{pt}$, picks out the positive definite forms.

Example 6.49 (super Hermitian lines)**.** The zero-space of the spectrum $\Sigma^2 I\mathbb{Z}(1)$ represents the groupoid of super lines L with continuous topology on morphisms. We compute the set of components of the fixed point spectrum of a trivializable involution:

$$\pi_0 \operatorname{Map}(B\mathbb{Z}/2_+, \Sigma^2 I\mathbb{Z}(1)) = \pi_0\Sigma^2 I\mathbb{Z}(1) \oplus \pi_0 \operatorname{Map}(B\mathbb{Z}/2, \Sigma^2 I\mathbb{Z}(1)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \tag{6.50}$$

This is the group of isomorphism classes of super Hermitian lines. The first $\mathbb{Z}/2\mathbb{Z}$ is the grading of the line, the second the "sign" of the form. But the sesquilinearity condition

$$\langle \bar\ell_1, \ell_2\rangle = (-1)^{|\ell_1||\ell_2|}\overline{\langle \bar\ell_2, \ell_1\rangle}, \qquad \ell_1, \ell_2 \in L, \tag{6.51}$$

implies that if L is odd then $\langle \bar\ell, \ell\rangle \in \sqrt{-1}\mathbb{R}$ for all $\ell \in L$. (The form is a bilinear map $\overline{L} \times L \to \mathbb{C}$.) The notion of positivity in this case chooses a ray in $\sqrt{-1}\mathbb{R}$; there is no canonical choice. In the literature, e.g. [DM, (4.4.2)], an arbitrary choice is made. In our homotopy theoretic presentation, this choice lies in the identification of the space of super Hermitian lines with the 0-space of $\Sigma^2 I\mathbb{Z}(1)$. As we descend deeper into extended field theories, there are further choices to be made; see Remark 6.26.

7. Reflection structures and stability

We begin in §7.1 by reviewing Madsen-Tillmann spectra; see [GMTW, §3]. They give a filtration (7.6) of Thom spectra, which leads to an analysis of the obstructions to extending invertible field theories to stable theories. In §7.2 we develop the relation between naive positivity and stability in two situations: non-equivariant discrete theories and equivariant continuous theories. In each case the only obstruction in n spacetime dimensions arises from the partition function of the n-sphere. But its positivity does not guarantee positive definite metrics on the state spaces attached to arbitrary $(n-1)$-manifolds (Proposition 7.37), consideration of which is deferred until §8. We conclude in §7.3 by analyzing the obstruction to extending "H-type" theories to "L-type" theories.

7.1. Madsen-Tillmann and Thom spectra

The homomorphism $\rho_n\colon H_n \to O_n$ in (2.3), which defines the symmetry type of a theory, produces a rank n vector bundle $V_n \to BH_n$ over the classifying space. We refer to §6.1.4 for the general theory of Thom spectra.

Definition 7.1. The *Madsen-Tillmann spectrum* MTH_n is the Thom spectrum of $-V_n \to BH_n$.

More natural for us is a suspension, the connective spectrum

$$\Sigma^n MTH_n = \operatorname{Thom}(BH_n; \underline{\mathbb{R}^n} - V_n). \tag{7.2}$$

The general construction of Thom spectra is described in §6.1.4. Here is a geometric description. Let $Gr_n(\mathbb{R}^{n+q})$ denote the Grassmannian of n-dimensional subspaces of $\mathbb{R}^{n+q}$. It approximates BO_n, and the pullback

$$\begin{tikzcd} X_{n,n+q} \arrow[r, dashed] \arrow[d, dashed] & BH_n \arrow[d] \\ Gr_n(\mathbb{R}^{n+q}) \arrow[r] & BO_n \end{tikzcd} \tag{7.3}$$

is a finite dimensional approximation to BH_n. The q^{th} space of the spectrum (7.2) can be taken to be the Thom *space* $\mathrm{Thom}(X_{n,n+q}; Q_q)$ of the vector bundle $Q_q \to X_{n,n+q}$, which is the pullback of the rank q "quotient bundle" over the Grassmannian: the fiber at a subspace $W \subset \mathbb{R}^{n+q}$ is $W^\perp$.

Remark 7.4. The Pontrjagin-Thom construction provides the basic relationship to H_n-manifolds. If a map $S^{k+q} \to \mathrm{Thom}(X_{n,n+q}; Q_q)$ is transverse to the 0-section of $Q_q \to X_{n,n+q}$, then the inverse image of the 0-section is a k-manifold $M \subset S^{k+q}$ whose stable tangent bundle is equipped with an isomorphism to the pullback of the "tautological bundle"[25] $V_n \to X_{n,n+q}$, which is equipped with an H_n-structure. Theorem 5.12 implies that the abelian group $\pi_k\Sigma^n MTH_n$ is generated by closed k-dimensional H_n-manifolds under disjoint union. The class of a closed manifold M^k is zero if and only if $M = \partial W$ where W is a compact $(k+1)$-manifold whose stable tangent bundle is isomorphic to a rank n bundle with an H_n-structure extending that of M. This bordism group was introduced by Reinhart [R]; see also [E, Appendix].

Remark 7.5. Not every element of the homotopy group is represented by a manifold; group completion of the semigroup of manifold classes is needed to obtain the homotopy group. For example, $\pi_0 MTO_0 \cong \mathbb{Z}$ but since a 0-dimensional manifold has a unique O_0-structure such manifolds only realize the submonoid of nonnegative integers. We also remark that the sphere S^{2m} represents a nonzero element in $\pi_{2m}\Sigma^{2m}MTSO_{2m}$, but is zero in the next group $\pi_{2m+1}\Sigma^{2m+1}MTSO_{2m+1}$: the closed ball D^{2m+1} has nonzero Euler characteristic so no SO_{2m}-structure. As another illustration, the 2-sphere and the genus 2 surface represent opposite elements of $\pi_2\Sigma^2 MTSO_2$: a genus 2 handlebody with a 3-ball excised admits an SO_2-structure.

The Stabilization Theorem 2.19 provides a sequence of spectra[26]

$$\Sigma^n MTH_n \longrightarrow \Sigma^{n+1}MTH_{n+1} \longrightarrow \Sigma^{n+2}MTH_{n+2} \longrightarrow \cdots \tag{7.6}$$

whose colimit, denoted MTH, is the Thom spectrum of the stable vector bundle

$$-V \longrightarrow BH \tag{7.7}$$

which is the negative of the classifying map of (2.28); see the construction in §6.1.4, especially the presentation (6.11) which is equivalent to (7.6). From the geometric description in Remark 7.4

[25]The fiber of the tautological bundle at a point $W \subset \mathbb{R}^{n+q}$ in $Gr_n(\mathbb{R}^{n+q})$ is W.

[26]That theorem supplies a *stable tangential structure* BH from which BH_n is constructed by pullback; recall (2.27).

the homotopy groups $\pi_k\Sigma^n MTH_n$ stabilize once $n > k$; then $\pi_k MTH$ is the bordism group of k-dimensional manifolds with a *stable tangential* H-structure. We identify MTH with the Thom spectrum $MH^\perp$ of the perpendicular[27] stable normal structure. In many cases $H^\perp = H$; however, for example, $(\mathrm{Pin}^\pm)^\perp = \mathrm{Pin}^\mp$.

Following Ansatz 5.14 an invertible topological field theory is a map with domain $\Sigma^n MTH_n$. To investigate extensions along the sequence (7.6) we will use the following in §7.2.

Proposition 7.12. *The fiber of the map $\Sigma^n MTH_n \longrightarrow \Sigma^{n+1} MTH_{n+1}$ is $\Sigma^n(BH_{n+1})_+$. The map $\Sigma^n(BH_{n+1})_+ \to \Sigma^n MTH_n$ is represented by the universal family $BH_n \to BH_{n+1}$ of H_n-spheres.*

See [GMTW, §3.1], [FHT1, Lemma 3.1] for a proof. The universal family of spheres was mentioned in Remark 4.32. We remind that spectra are built out of *based* spaces; for a based space X the spectrum $\Sigma^n X_+$ is the one-point union of S^n and the suspension spectrum $\Sigma^n X$, and the latter is $(n-1)$-connected if X is connected.

Our final task in this section is to refine Ansatz 5.14 and Ansatz 5.26, which formulate invertible field theories as maps of spectra, to include reflection structures. Recall from §4 that the reflection structure on the bordism category maps a manifold with H_n-structure to the same manifold with the opposite H_n-structure, which is defined using the group extension (3.14). Turning to bordism spectra we observe that this group extension induces a $\mathbb{Z}/2$-action on BH_n and makes the vector bundle $V_n \to BH_n$ into an equivariant vector bundle $V_n^\beta \to BH_n^\beta$. Applying the discussion in §6.2.2 we refine the Thom spectrum (7.2) to a $\mathbb{Z}/2$-equivariant spectrum we denote $\Sigma^n MTH_n^\beta$. There is an equivariant lift of (7.6). Recall the involutions on $I\mathbb{Z}(1), I\mathbb{C}^\times$ chosen after Remark 6.33.

Ansatz 7.13.

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[27] The classifying space $BH^\perp$ is the pullback

(7.8)
$$\begin{tikzcd} BH^\perp \arrow[r] \arrow[d] & BH \arrow[d] \\ BO \arrow[r] & BO \end{tikzcd}$$

in which the bottom map classifies the negative of the universal bundle (of rank zero). There is a sequence of inclusions $\cdots H_n^\perp \hookrightarrow H_{n+1}^\perp \hookrightarrow H_{n+2}^\perp \cdots$ of compact Lie groups such that $BH^\perp$ is the colimit of $BH_n^\perp$. Namely, define $\widetilde{H}_n^\perp$ as the pullback (see (2.10))

(7.9)
$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}_n^\perp \arrow[r] \arrow[d] & \mathrm{Pin}_n^- \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & J \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd}$$

and then set

(7.10)
$$H_n^\perp \cong \widetilde{H}_n^\perp \,/\, \langle(-1, k_0)\rangle.$$

One checks that $BH_n^\perp$ is the pullback

(7.11)
$$\begin{tikzcd} BH_n^\perp \arrow[r] \arrow[d] & BH \arrow[d] \\ BO_n \arrow[r] & BO \end{tikzcd}$$

(i) A discrete *invertible n-dimensional extended topological field theory with symmetry group H_n and reflection structure* is an equivariant map

$$F\colon \Sigma^n MTH_n^\beta \longrightarrow \Sigma^n (I\mathbb{C}^\times)^{\nu_0'}, \tag{7.14}$$

(ii) A continuous *invertible n-dimensional extended topological field theory with symmetry group H_n and reflection structure* is an equivariant map

$$\varphi\colon \Sigma^n MTH_n^\beta \longrightarrow \Sigma^{n+1} I\mathbb{Z}(1)^\gamma. \tag{7.15}$$

The space of theories of this type is

$$\mathcal{I}_n(H_n)_{\text{reflection}} = \operatorname{Map}^{\mathbb{Z}/2}(\Sigma^n MTH_n^\beta, \Sigma^{n+1} I\mathbb{Z}(1)^\gamma). \tag{7.16}$$

7.2. Naive positivity and stability

We first prove that the double of an H_n-manifold is null bordant through an H_{n+1}-manifold. Recall the evaluation bordism (4.7), the identification of duals and bars in Proposition 4.8, and Definition 4.24 of a double.

Proposition 7.17. *Let Y_0, Y_1 be closed $(n-1)$-dimensional H_n-manifolds and $X\colon Y_0 \to Y_1$ an H_n-bordism. Then*

$$\beta X \amalg e_{Y_1} \amalg X\colon \beta Y_0 \amalg Y_0 \longrightarrow \varnothing^{n-1} \tag{7.18}$$

is H_{n+1}-bordant to e_{Y_0}.

Proof. The bordism[28] is $[0,1]\times X$. □

Corollary 7.19. *The double ΔX of a compact H_n-manifold with boundary is null bordant through an H_{n+1}-manifold.*

By Corollary 4.30 this applies to S^n with its canonical H_n-structure, and so every double is H_{n+1}-bordant to S^n.

Proof. Apply Proposition 7.17 to $X\colon \varnothing^{n-1} \to \partial X$ (and smooth the corners of $[0,1]\times X$). □

Remark 7.20. If X is the 2-dimensional disk, viewed as a bordism from the empty 1-manifold to the circle, then ΔX is the 2-dimensional sphere S^2 and the null bordism $[0,1]\times X$ is the 3-dimensional ball D^3. The Euler characteristic obstructs the existence of an H_2-structure on D^3 which restricts to the given H_2-structure on S^2 (for any stable tangential structure H).

The sequence of bordism spectra (7.6) results in a special type of invertible field theory. The following applies to both discrete (Ansatz 5.14) and continuous (Ansatz 5.26) invertible field theories, possibly with reflection structure (Ansatz 7.13).

[28] It is a bordism of manifolds with boundary, or better a higher morphism in a multi-bordism category. We only use $Y_0 = \varnothing^{n-1}$, as in Corollary 7.19, in which case $[0,1]\times X$ is a null bordism of a closed manifold.

Definition 7.21. An n-dimensional invertible topological field theory with domain $\Sigma^n MTH_n$ is *stable* if it is the restriction of a theory defined on MTH.

Stability can be investigated one step at a time in the sequence (7.6) using obstruction theory. We first carry this out for *discrete* invertible topological field theories *without* reflection structure. Recall that the sphere has a canonical H_n-structure given by the principal bundle $H_{n+1} \to H_{n+1}/H_n$.

Theorem 7.22. *A discrete invertible theory $F\colon \Sigma^n MTH_n \to \Sigma^n I\mathbb{C}^\times$ is stable if and only if $F(S^n) = 1$. The subspace of $\mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times)$ consisting of theories F with $F(S^n) = 1$ is homotopy equivalent to the mapping space $\mathrm{Map}(MTH, \Sigma^n I\mathbb{C}^\times)$.*

By Corollary 7.19 the condition is equivalent to $F(\Delta X) = 1$ for all compact X^n with boundary.

Proof. If F is the restriction of $\widetilde{F}\colon MTH \to \Sigma^n I\mathbb{C}^\times$, then $F(S^n) = \widetilde{F}(S^n) = 1$ since S^n is null bordant as an H_{n+1}-manifold. Conversely, by Proposition 7.12 the map F extends over $\Sigma^{n+1} MTH_{n+1}$ if and only if it evaluates trivially on the universal family of H_n-spheres. But that evaluation is the constant function $BH_{n+1} \to \mathbb{C}^\times$ with value $F(S^n)$. There is no further obstruction in the sequence (7.6), because the subsequent fibers have vanishing homotopy groups in degrees $\leqslant n$ and $\pi_q \Sigma^n I\mathbb{C}^\times = 0$ for $q > n$.

To analyze the space of discrete stable theories we note that the cofibration sequence

$$\Sigma^n MTH_n \longrightarrow \Sigma^{n+1} MTH_{n+1} \longrightarrow \Sigma^{n+1}(BH_{n+1})_+ \tag{7.23}$$

of spectra induces a fibration sequence

$$\begin{aligned} \mathrm{Map}\bigl(\Sigma^{n+1}(BH_{n+1})_+, \Sigma^n I\mathbb{C}^\times\bigr) &\longrightarrow \mathrm{Map}(\Sigma^{n+1} MTH_{n+1}, \Sigma^n I\mathbb{C}^\times) \\ &\longrightarrow \mathrm{Map}(\Sigma^n MTH_n, \Sigma^n I\mathbb{C}^\times) \longrightarrow \mathrm{Map}\bigl(\Sigma^n (BH_{n+1})_+, \Sigma^n I\mathbb{C}^\times\bigr) \end{aligned} \tag{7.24}$$

of mapping spaces. The first space is contractible, since $\Sigma^{n+1}(BH_{n+1})_+$ is n-connected. The fiber of the last map is the subspace indicated in the theorem, by the obstruction argument in the previous paragraph. To pass to stable maps make a similar argument with the cofibration sequence

$$\Sigma^{n+1} MTH_{n+1} \longrightarrow MTH \longrightarrow C \tag{7.25}$$

and the induced fibration on mapping spaces. □

Remark 7.26. If X^n is a closed H_n-manifold, then $[0,1] \times X$ is a null bordism of $\beta X \amalg X$. Thus if F is stable and has a reflection structure, then $\|F(X)\|^2 = 1$.

Next, we turn to *continuous* invertible field theories *with* reflection structure, which according to Ansatz 7.13(ii) are $\mathbb{Z}/2\mathbb{Z}$-equivariant maps

$$\varphi\colon \Sigma^n MTH_n^\beta \longrightarrow \Sigma^{n+1} I\mathbb{Z}(1)^\gamma. \tag{7.27}$$

We investigate stability for these equivariant theories.

Remark 7.28. As explained after (5.25) a continuous invertible field theory assigns a $\mathbb{Z}(1)$-torsor to a closed H_n-manifold, hence an equivariant theory (7.27) assigns to a β-equivariant family $\mathcal{X}\to S$ of closed H_n-manifolds an equivariant $\mathbb{Z}(1)$-torsor over S, where the action on $\mathbb{Z}(1)$-torsors is that in Example B.5; see also Remark 6.40. The universal model is the map $\exp\colon \mathbb{C}\to\mathbb{C}^\times$, equivariant for complex conjugation, with fibers $\mathbb{Z}(1)$-torsors. Over the fixed point set $\mathbb{R}^\times=\mathbb{R}^{>0}\amalg\mathbb{R}^{<0}$ the fibers are $\mathbb{Z}(1)$-torsors of Type P and Type N; see Example B.5. As discussed in §5.4 a non-topological invertible field theory (type (a) in that discussion) has a homotopy class that is a continuous theory. If we have a reflection structure, then the partition function of a β-fixed H_n-manifold is real, and if it is positive then the corresponding $\mathbb{Z}(1)$-torsor has Type P.

Remark 7.29. A *stable* continuous theory $\tilde\varphi$ assigns an integer (better: element of $\mathbb{Z}(1)$) to a closed $(n+1)$-manifold. The universal property (5.17) of maps into the Anderson dual implies that the topological field theory associated to $\tilde\varphi$ is determined by its truncation to n- and $(n+1)$-manifolds.

Theorem 7.30. *An equivariant continuous invertible field theory $\varphi\colon \Sigma^nMTH_n^\beta\to\Sigma^{n+1}I\mathbb{Z}(1)^\gamma$ is stable if and only if $\varphi(S^n)$ has Type P. The subspace of $\operatorname{Map}^{\mathbb{Z}/2}(\Sigma^nMTH_n^\beta,\Sigma^{n+1}I\mathbb{Z}(1)^\gamma)$ consisting of equivariant continuous invertible field theories with Type P partition function on S^n is homotopy equivalent to the mapping space $\operatorname{Map}^{\mathbb{Z}/2}(MTH^\beta,\Sigma^{n+1}I\mathbb{Z}(1)^\gamma)$.*

Proof. Since S^n is diffeomorphic to βS^n, the partition function $\varphi(S^n)$ is a $\mathbb{Z}(1)$-torsor with involution. The partition function of the universal family of n-spheres is then a $\mathbb{Z}(1)$-torsor over BH_{n+1} with involution covering the trivial involution on the base. It is classified by a map $BH_{n+1}\to\mathbb{R}^\times$ whose homotopy class in $H^0(BH_{n+1};\{\pm1\})\cong\{\pm1\}$ encodes the Type (P or N) of $\varphi(S^n)$.

Now use the stabilization sequence (7.6) as before. If φ is stable, then it is trivial on the fiber $\Sigma^n(BH_{n+1})_+$ of the first map, which is represented by the universal family of n-spheres. The argument in the preceding paragraph shows that $\varphi(S^n)$ has Type P. To prove the converse, if $\varphi(S^n)$ has Type P then the first obstruction vanishes, and so φ is the restriction of a map $\Sigma^{n+1}MTH_{n+1}^\beta\to\Sigma^{n+1}I\mathbb{Z}(1)^\gamma$. The obstruction at the next stage is a map $\Sigma^{n+1}(BH_{n+2}^\beta)_+\to\Sigma^{n+1}I\mathbb{Z}(1)^\gamma$. But $\Sigma^{n+1}(BH_{n+2}^\beta)_+\simeq S^{n+1}\vee\Sigma^{n+1}BH_{n+2}^\beta$ with $\mathbb{Z}/2$ acting trivially on the suspension S^{n+1} of the basepoint. Since $\Sigma^{n+1}BH_{n+2}^\beta$ is $(n+1)$-connected, the obstruction lies in

$$\begin{aligned}[S^{n+1},\Sigma^{n+1}I\mathbb{Z}(1)^\gamma]^{\mathbb{Z}/2}&\cong[S^{\sigma-1},I\mathbb{Z}(1)]^{\mathbb{Z}/2}\\ &\cong[E\mathbb{Z}/2_+\underset{\mathbb{Z}/2}{\wedge}S^{\sigma-1},I\mathbb{Z}(1)]\\ &\cong\operatorname{Hom}(\pi_0\,E\mathbb{Z}/2_+\underset{\mathbb{Z}/2}{\wedge}S^{\sigma-1},\mathbb{Z}(1))=0,\end{aligned}\tag{7.31}$$

since

$$\pi_0\,E\mathbb{Z}/2_+\underset{\mathbb{Z}/2}{\wedge}S^{\sigma-1}=\pi_1\mathbb{RP}^\infty=\mathbb{Z}/2.$$

There are no further obstructions to extending to MTH, because the fibers have nonvanishing homotopy groups only in degrees greater than $n+1$ and $\pi_q\Sigma^{n+1}I\mathbb{Z}(1)=0$ for $q>n+1$.

The equivariant version of (7.23) with the β-involution leads to the fibration sequence

$$\begin{aligned}\operatorname{Map}^{\mathbb{Z}/2}\big(\Sigma^{n+1}(BH_{n+1}^\beta)_+,\Sigma^{n+1}I\mathbb{Z}(1)^\gamma\big)&\longrightarrow\operatorname{Map}^{\mathbb{Z}/2}(\Sigma^{n+1}MTH_{n+1}^\beta,\Sigma^{n+1}I\mathbb{Z}(1)^\gamma)\\ \longrightarrow\operatorname{Map}^{\mathbb{Z}/2}(\Sigma^nMTH_n^\beta,\Sigma^{n+1}I\mathbb{Z}(1)^\gamma)&\longrightarrow\operatorname{Map}^{\mathbb{Z}/2}\big(\Sigma^n(BH_{n+1}^\beta)_+,\Sigma^{n+1}I\mathbb{Z}(1)^\gamma\big)\end{aligned}\tag{7.32}$$

As in (7.24) the first space is contractible. The obstruction argument above identifies the fiber of the last map as equivariant continuous theories with positive sphere partition function. To pass to stable maps use an equivariant version of (7.25). ☐

Corollary 7.33. *There is a 1:1 correspondence*

$$\left\{\begin{array}{l}\text{isomorphism classes of continuous invertible}\\ n\text{-dimensional extended topological field}\\ \text{theories with (i) symmetry group } H_n,\\ \text{(ii) reflection structure, and (iii) partition}\\ \text{function on } S^n \text{ of Type P}\end{array}\right\} \cong [MTH^\beta, \Sigma^{n+1} I\mathbb{Z}(1)^\gamma]^{\mathbb{Z}/2}. \tag{7.34}$$

Example 7.35. The restriction map[29]

$$[MTSO^\beta, \Sigma^4 I\mathbb{Z}(1)^\gamma]^{\mathbb{Z}/2} \longrightarrow [\Sigma^3 MTSO_3^\beta, \Sigma^4 I\mathbb{Z}(1)^\gamma]^{\mathbb{Z}/2} \tag{7.36}$$

is an index two inclusion of infinite cyclic groups. It follows that there exist continuous invertible 3-dimensional oriented theories φ with reflection structure such that $\varphi(S^3)$ has Type N. In turn, this suggests the existence of invertible non-topological theories with reflection structure whose real-valued partition function on S^3 is negative; see 5.4. Here is an explicit example. The domain is the geometric bordism category of oriented *Riemannian* manifolds. The partition function is $F(X^3) = \exp(2\pi i \xi_X)$, where ξ_X is the Atiyah-Patodi-Singer invariant [APS].[30] To apply the arguments in Theorem 7.30 we need to use a Riemannian sphere that is a double—the round sphere does nicely—in which case the spectrum of the APS operator is symmetric about zero and so the η-invariant vanishes. The dimension of the kernel is one, $\xi_X = 1/2$, and so $F(S^3) = -1$. We remark that the corresponding integer invariant of a closed oriented 4-manifold W is $(\mathrm{Sign}(W) \pm \mathrm{Euler}(W))/2$; either sign works. Also, the square of this theory, whose deformation class generates $[MTSO^\beta, \Sigma^4 I\mathbb{Z}(1)^\gamma]^{\mathbb{Z}/2}$, represents "Kitaev's E_8-phase" [K5].

Let F be a invertible topological n-dimensional theory, and suppose that $F(S^n) > 0$. Then the hermitian form on $F(S^{n-1})$ is positive definite; see (4.27). The positivity holds for any null bordant $(n-1)$-manifold, but on other manifolds there is no guarantee of positivity (Definition 4.18), even for stable theories.

Proposition 7.37. *Let F be an invertible n-dimensional topological field theory of H_n-manifolds with $F(S^n) > 0$. Suppose F has a reflection structure. Then the sign of the hermitian form (4.16) on a closed $(n-1)$-manifold is a bordism invariant and determines a homomorphism*

$$\pi_{n-1}\Sigma^{n-1} MTH_{n-1} \longrightarrow \{\pm 1\}. \tag{7.38}$$

Proof. If $X\colon Y_0 \to Y_1$ is an H_n-bordism, then by reversing the arrow of time on the incoming boundary we obtain $X'\colon \varnothing^{n-1} \to \beta Y_0 \amalg Y_1$. Hence by Corollary 7.19 and the remark which follows, we deduce that the hermitian line $\overline{F(Y_0)} \otimes F(Y_1)$ is positive definite. Therefore, $F(Y_0)$ and $F(Y_1)$ are simultaneously positive or simultaneously negative. ☐

[29]The involution on $\pi_4 MTSO$ and $\pi_4\Sigma^3 MTSO_3$ acts as -1: both groups are detected by the signature, which negates under orientation-reversal.

[30]of the operator called 'B^{ev}' in their paper.

We conclude this section with a lemma we will use in §8.

Lemma 7.39. *The map $\Sigma^n MTH_n \to MTH$ induces a surjection on $H_{n+1}(-;\mathbb{R})$.*

We remark that $\pi_{n+1}(\mathcal{B})\otimes\mathbb{R} \to H_{n+1}(\mathcal{B};\mathbb{R})$ is an isomorphism for any spectrum $\mathcal{B}$.

Proof. Arrange the stabilization (7.6) and cofibration sequences (7.23) as follows:

$$\begin{tikzcd} \Sigma^n(BH_{n+1})_+ \arrow[d] & \Sigma^{n+1}(BH_{n+2})_+ \arrow[d, "s"] & \\ \Sigma^n MTH_n \arrow[r, "i"] \arrow[d] & \Sigma^{n+1}MTH_{n+1} \arrow[r, "j"] \arrow[d, "\chi"] & \Sigma^{n+2}MTH_{n+2} \arrow[d] \\ \Sigma^n(BH_n)_+ & \Sigma^{n+1}(BH_{n+1})_+ & \Sigma^{n+2}(BH_{n+2})_+ \end{tikzcd} \tag{7.40}$$

The two compositions with shape $\begin{tikzcd}[sep=small] \cdot \arrow[d] & \\ \cdot \arrow[r] & \cdot \arrow[d] \\ & \cdot \end{tikzcd}$ are cofibration sequences. The map s_* on π_{n+1} sends the generator of the infinite cyclic group $\pi_{n+1}\Sigma^{n+1}(BH_{n+2})_+$ to the class of S^{n+1}, and the map χ_* on π_{n+1} sends the class of a closed $(n+1)$-manifold to its Euler number. Also, $\pi_{n+1}\Sigma^{n+2}(BH_{n+2})_+ = 0$. It follows that j_* on π_{n+1} is surjective. If n is even, then $\chi_* = 0$ on π_{n+1} and by exactness i_* is surjective. If n is odd, then $\chi_*\circ s_*$ is multiplication by 2. Working now on $\pi_{n+1}\otimes\mathbb{R}$ we can lift any class in $\pi_{n+1}\Sigma^{n+2}MTH_{n+2}\otimes\mathbb{R}$ through j_* to have zero image under χ_*, hence by exactness to be in the image of $i_*\otimes\mathbb{R}$. In other words, $(j\circ i)_*\otimes\mathbb{R}$ is surjective. Finally, the stabilization map $\Sigma^{n+2}MTH_{n+2}\to MTH$ induces an isomorphism on π_{n+1}. ☐

7.3. H-type theories

Wen [Wen] and Morrison-Walker [MW] introduced the notion of n-dimensional topological field theories defined only on n-manifolds with an infinitesimal time direction. These are of Hamiltonian type, or H-type, and are the minimal expectation for the low energy effective theory describing a Hamiltonian system. In this paper we assume emergent relativistic invariance, so do not engage with H-type theories in a serious way. Nonetheless, in this subsection we indicate briefly how to analyze invertible theories of H-type.

The first issue is definitional: Do the n-manifolds in the bordism category have (i) an *oriented* time direction or merely (ii) a time direction? In unoriented theories this means a reduction of O_n to either (i) O_{n-1} or (ii) $O_1\times O_{n-1}$. We opt for (i). After all, a Hamiltonian system does have a definite orientation of time, and even in relativistic quantum field theory we assume a time orientation of Minkowski spacetime (§2.1). Then a more general symmetry group H_n is reduced to H_{n-1}, and an invertible theory of H-type is a map out of the spectrum $\Sigma^{n-1}MTH_{n-1}$.

Now the extension question: Does an equivariant map $\varphi\colon \Sigma^{n-1}MTH^{\beta}_{n-1}\to\Sigma^{n+1}I\mathbb{Z}(1)^{\gamma}$ extend to an equivariant map $\Sigma^n MTH^{\beta}_n\to\Sigma^{n+1}I\mathbb{Z}(1)^{\gamma}$? (In Wen's language this is an extension from H-type to L-type.) The obstruction is the value of φ on the universal family of H_{n-1}-spheres

S^{n-1} parametrized by BH_n. Without the equivariance the value[31] is a $\mathbb{Z}/2\mathbb{Z}$-graded *complex* line bundle over BH_n; the equivariance implies the value is a $\mathbb{Z}/2\mathbb{Z}$-graded *real* line bundle. (See Remark 6.40 for the connective cover of $\Sigma^2 I\mathbb{Z}(1)$ and its bar involution γ.) The first obstruction is the grading: the single quantum state on S^{n-1} should be bosonic. If so, the remaining obstruction is a class in $H^1(BH_n;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(H_n,\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(\pi_0 H_n,\mathbb{Z}/2\mathbb{Z})$. For example, if $H_n = O_n$ or $H_n = \mathrm{Pin}^{\pm}_n$, then a hyperplane reflection should act trivially on the line $\varphi(S^{n-1})$.

Example 7.41. Continuing Example 7.35, the restriction map

$$[\Sigma^3 MTSO_3^{\beta}, \Sigma^4 I\mathbb{Z}(1)^{\gamma}]^{\mathbb{Z}/2} \longrightarrow [\Sigma^2 MTSO_2^{\beta}, \Sigma^4 I\mathbb{Z}(1)^{\gamma}]^{\mathbb{Z}/2} \tag{7.42}$$

is an index two inclusion of infinite cyclic groups. So there exists a continuous invertible theory φ of H-type with reflection structure that does not extend to all oriented 3-manifolds. Here is an example defined on the category of oriented Riemannian 2-manifolds: assign the $\mathbb{Z}/2\mathbb{Z}$-graded determinant line $\varphi(Y)$ of the $\bar\partial$-operator to a closed 2-manifold Y. Then index $\bar\partial_{S^2} = 1$ implies that $\varphi(Y)$ is odd.

8. Positivity in extended invertible topological theories

In this section we develop the theory of extended positivity in invertible field theories. We already introduced a homotopy-theoretic manifestation of extended positivity for higher super lines in Definition 6.41. Here, in §8.1, we begin by introducing spaces of invertible field theories leading up to the space of invertible reflection positive theories. Our main result, Theorem 8.20, identifies the homotopy type of the space of invertible *continuous* reflection positive theories as the 0-space of the Anderson dual to a Thom spectrum. The homotopy type of the corresponding space in the *discrete* case, worked out in Theorem 8.29, is a corollary, as is Theorem 1.1 in the introduction. The proof of Theorem 8.20 appears in §8.2 and §8.3.

8.1. Spaces of invertible field theories, extended positivity, and stability

8.1.1. *Preliminary: splitting off a reflection.* Fix $n > 0$. Recall that if (H_n, ρ_n) is a symmetry type (Definition 2.4), then we have a canonical co-extension (3.14) of H_n by $\{\pm 1\}$ to a group $\widehat{H}_n$. It is this extension that determines the β-involution on the Madsen-Tillmann spectrum MTH_n, as in the discussion preceding Ansatz 7.13; the homotopy quotient of MTH_n^{β} is $MT\widehat{H}_n$.

The splitting of interest is contained in (3.25) (and is also implicit in Proposition 4.8). It exists whenever there is an "auxiliary" direction. The middle vertical homomorphism in (3.25) induces

$$BH_{n-1} \times B\mathbb{Z}/2 \to B\hat{H}_n,$$

which factors the projection

$$BH_{n-1} \times B\mathbb{Z}/2 \to B\hat{H}_n \to B\mathbb{Z}/2.$$

[31]Parallel to the $\mathbb{Z}(1)$-torsors attached to n-manifolds are graded gerbes attached to $(n-1)$-manifolds. The construction of a line may depend on a choice of metric, for example, so may be part of a non-topological theory.

This, in turn, gives a sequence of equivariant maps

$$\Sigma^{n-1}MTH_{n-1} \wedge S^{1-\sigma} \to \Sigma^n MTH_n^\beta \to MTH \wedge S^{1-\sigma} \tag{8.1}$$

factoring the smash product of the identity map of $S^{1-\sigma}$ with the defining inclusion of $\Sigma^{n-1}MTH_{n-1}$ into MTH.

The stable form of the splitting implies the following.

Proposition 8.2. *The $\mathbb{Z}/2$-equivariant spectra MTH^β and MTH^γ are canonically equivariantly weakly equivalent.*

We remind that, despite the similarity of notation, the β-involution is defined by the group coextension whereas the γ-involution is natural, obtained by smashing with $S^{1-\sigma}$.

Proof. Take $n \to \infty$ in (8.1). The colimit of the first term is $MTH \wedge S^{1-\sigma}$ and the composition is homotopic to the identity map. □

8.1.2. *Spaces of theories.* Let $n > 0$ be the spacetime dimension and fix a positive integer $k \leqslant n$. Let G be a Lie group equipped with a homomorphism $\rho : G \to O_k$. The map ρ is used to form the Thom spectrum $MTG = \mathrm{Thom}(BG; -\rho)$. Define the *space of continuous invertible k-truncated n-dimensional topological field theories of symmetry type* (G, ρ) as[32]

$$\mathcal{I}_n(G) = \mathcal{I}_n(G, \rho) = \mathrm{Map}(\Sigma^k MTG, \Sigma^{n+1} I\mathbb{Z}(1)).$$

Usually ρ is understood in the notation. A point of $\mathcal{I}_n(G)$ may be thought of as a k-dimensional field theory that associates to a closed ℓ-manifold M, $\ell \leqslant k$, a super $(n-\ell)$-line.

Different flavors of field theories are obtained by changing the target, as in Definition 6.41 and Definition 6.45. We give the definitions for continuous invertible theories; there are analogous definitions for discrete invertible theories.

Definition 8.3. Fix integers $n > 0$ and $k \leqslant n$.

(i) The space of *continuous invertible k-truncated n-dimensional* Hermitian *extended topological field theories with symmetry type* (G, ρ) is

$$\mathcal{I}_n(G, \rho)_{\text{Hermitian}} = \mathrm{Map}(\Sigma^k MTG, \Sigma^{n+1} I\mathbb{Z}(1)_H)$$

(ii) The space of *continuous invertible k-truncated n-dimensional* positive definite *extended topological field theories with symmetry type* (G, ρ) is

$$\mathcal{I}_n(G, \rho)_{\text{positive}} = \mathrm{Map}(\Sigma^k MTG, \Sigma^{n+1} I\mathbb{Z}(1)_{\text{pos}}).$$

[32]The 'k' usually appears in the notation for G, as in (8.9) below, so we do not adorn '$\mathcal{I}$' with it.

Note that composition with the map $I\mathbb{Z}(1)_{\mathrm{pos}} \to I\mathbb{Z}(1)_H$ induces a map

$$\mathcal{I}_n(G,\rho)_{\mathrm{positive}} \longrightarrow \mathcal{I}_n(G,\rho)_{\mathrm{Hermitian}}. \tag{8.4}$$

Assume the symmetry type is a pair (H_n,ρ_n) as in Definition 2.4. We recall the notation (7.16) for the space of theories with reflection structure:

$$\mathcal{I}_n(H_n)_{\mathrm{reflection}} = \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^n MTH_n^{\beta}, \Sigma^{n+1} I\mathbb{Z}(1)^{\gamma}). \tag{8.5}$$

Composition with the first map in (8.1) produces a map

$$\mathcal{I}_n(H_n)_{\mathrm{reflection}} \longrightarrow \mathcal{I}_n(H_{n-1})_{\mathrm{Hermitian}}. \tag{8.6}$$

Therefore, the value of a theory with reflection structure on a closed manifold of dimension $\ell \leqslant n-1$ is a *Hermitian* super $(n-\ell)$-line. (The Hermitian line for $\ell = n-1$ is described in §4.3 for not necessarily invertible theories.) Recall the stabilization $\rho\colon H \to O$ in (2.28), and define

$$\mathcal{I}_n(H)_{\mathrm{stable}} = \mathrm{Map}(MTH, \Sigma^{n+1} I\mathbb{Z}(1)), \tag{8.7}$$

the space of *stable* n-dimensional invertible topological field theories of symmetry type H.

We use the notations $\mathcal{I}_n^{\delta}(G)_{\mathrm{Hermitian}}$, $\mathcal{I}_n^{\delta}(G)_{\mathrm{positive}}$, $\mathcal{I}_n^{\delta}(H_n)_{\mathrm{reflection}}$ for the corresponding spaces of discrete field theories, which are mapping spaces with codomain $\Sigma^n I\mathbb{C}_H^{\times}$, $\Sigma^n I\mathbb{C}^{\times}_{\mathrm{pos}}$, and $\Sigma^n (I\mathbb{C}^{\times})^{\nu_0'}$, respectively. (See (6.42) and (6.46).)

The main objects of interest are invertible reflection positive theories. As stated after (8.6), an invertible theory with reflection structure has values on closed manifolds of dimension $\leqslant (n-1)$ that are higher Hermitian super lines. The following definition uses (8.4) to impose positivity, which in dimension $n-1$ is a *condition* (Definition 4.18) and in dimensions $< (n-1)$ is a *structure*.

Definition 8.8. Fix $n > 0$ and a symmetry type (H_n,ρ_n) in the sense of Definition 2.4. Define the spaces $\mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$ and $\mathcal{I}_n^{\delta}(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$ of *n-dimensional continuous (resp. discrete) invertible reflection positive topological field theories with symmetry type* (H_n,ρ_n) and maps out of these spaces so that each square in the diagram

$$\begin{tikzcd} \mathcal{I}_n^{\delta}(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \arrow[r] \arrow[d] & \mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \arrow[r] \arrow[d] & \mathcal{I}_n(H_{n-1})_{\mathrm{positive}} \arrow[d] \\ \mathcal{I}_n^{\delta}(H_n)_{\mathrm{reflection}} \arrow[r] & \mathcal{I}_n(H_n)_{\mathrm{reflection}} \arrow[r, "(8.6)"] & \mathcal{I}_n(H_{n-1})_{\mathrm{Hermitian}} \end{tikzcd} \tag{8.9}$$

is a homotopy pullback.

For the spaces of theories in the right hand column we use Definition 8.3 with $k = n-1$, $G = H_{n-1}$, and $\rho = \rho_{n-1}$. Our task is to determine the homotopy types of $\mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$ and $\mathcal{I}_n^{\delta}(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$.

8.1.3. *Extended positivity structure.* Definition 8.8 is natural given our homotopy-theoretic implementation of higher positive definite Hermitian super lines in Definition 6.45. We now make a short digression to identify extended positivity in an invertible n-dimensional field theory as a structure that trivializes an associated invertible $(n-1)$-dimensional field theory. For this we need yet an additional space of invertible field theories, based on the target spectrum of higher *real* super lines (Definition 6.41(ii)).

Definition 8.10. The space of *continuous invertible* $(n-1)$*-dimensional* real *extended topological field theories with symmetry type* (H_{n-1}, ρ_{n-1}) is

$$\mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1}) = \mathrm{Map}\bigl(\Sigma^{n-1}MTH_{n-1}, (\Sigma^n I\mathbb{Z}(1)^{\gamma})^{h\mathbb{Z}/2}\bigr). \tag{8.11}$$

The partition function on a closed $(n-1)$-manifold lies in $\{\pm 1\}$, the value on a closed $(n-2)$-manifold is a real super line, etc. (See Remark 6.40 for the top few homotopy groups of $\bigl(I\mathbb{Z}(1)^{\gamma}\bigr)^{h\mathbb{Z}/2}$.)

To begin, for any pointed space X there is an equivalence of spectra $X_+ \approx X \vee S^0$, which leads to a cofibration sequence

$$X \longrightarrow X_+ \longrightarrow S^0. \tag{8.12}$$

Set $X = B\mathbb{Z}/2$, smash with $\Sigma^{n-1}MTH_{n-1}$, and apply $\mathrm{Map}\bigl(-, \Sigma^{n+1}I\mathbb{Z}(1)\bigr)$ to obtain the fibration sequence

$$\mathcal{I}_n(H_{n-1})_{\text{positive}} \longrightarrow \mathcal{I}_n(H_{n-1})_{\text{Hermitian}} \longrightarrow \mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1}). \tag{8.13}$$

For the middle term use (6.43) and for the last the identification

$$\begin{aligned}\mathrm{Map}(\Sigma^{n-1}MTH_{n-1} \wedge B\mathbb{Z}/2, \Sigma^{n+1}I\mathbb{Z}(1)) &\approx \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^{n}MTH_{n-1} \wedge S^{\sigma-1}, \Sigma^{n+1}I\mathbb{Z}(1))\\ &\approx \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^{n}MTH_{n-1}, \Sigma^{n+1}I\mathbb{Z}(1)^{\gamma})\\ &\approx \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^{n-1}MTH_{n-1}, \Sigma^{n}I\mathbb{Z}(1)^{\gamma})\\ &\approx \mathrm{Map}(\Sigma^{n-1}MTH_{n-1}, \Sigma^{n}(I\mathbb{Z}(1)^{\gamma})^{h\mathbb{Z}/2}).\end{aligned}$$

Therefore, the space $\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}}$ may also be defined as the homotopy fiber of the composition

$$\kappa\colon \mathcal{I}_n(H_n)_{\text{reflection}} \longrightarrow \mathcal{I}_n(H_{n-1})_{\text{Hermitian}} \longrightarrow \mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1}). \tag{8.14}$$

This leads to the following definition.

Definition 8.15. An (*extended*) *positivity structure* on a continuous n-dimensional field theory $\varphi \in \mathcal{I}_n(H_n)_{\text{reflection}}$ is a trivialization of $\kappa(\varphi)$.

That is, a positivity structure is a path from $\kappa(\varphi)$ to the basepoint in $\mathcal{I}^{\mathbb{R}}_{n-1}(H_{n-1})$. This discussion identifies the space of continuous reflection positive invertible field theories as the space of continuous invertible field theories with both a reflection structure and a positivity structure.

Remark 8.16. The partition function of the field theory $\kappa(\varphi)\colon \Sigma^{n-1}MTH_{n-1} \to \Sigma^n(I\mathbb{Z}(1)^\gamma)^{h\mathbb{Z}/2}$ is the homomorphism

$$\pi_{n-1}\Sigma^{n-1}MTH_{n-1} \longrightarrow \{\pm 1\} \tag{8.17}$$

induced on π_{n-1}, and it agrees with the homomorphism (7.38) which tracks the sign of the hermitian lines in the theory φ. The highest piece of the positivity structure is therefore the standard positivity *constraint* in Definition 4.18. The theory $\kappa(\varphi)$ assigns a real super line to a closed $(n-2)$-manifold and more complicated objects in lower dimensions; their trivializations are *data*.

8.1.4. *Main theorems.* We apply the splitting of §8.1.1 to construct a map

$$\mathcal{I}_n(H)_{\mathrm{stable}} \longrightarrow \mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \tag{8.18}$$

as follows. (These spaces of invertible field theories are defined in (8.7) and (8.9).) Map

$$\Sigma^{n-1}MTH_{n-1} \wedge B\mathbb{Z}/2_+ \longrightarrow \Sigma^{n-1}MTH_{n-1} \longrightarrow MTH \tag{8.19}$$

into $\Sigma^{n+1}I\mathbb{Z}(1)$ to obtain a map of $\mathcal{I}_n(H)_{\mathrm{stable}}$ into the upper right corner of (8.9). Use equivariant maps of the sequence (8.1) into $\Sigma^{n+1}I\mathbb{Z}(1)^\gamma$ to map $\mathcal{I}_n(H)_{\mathrm{stable}}$ into the middle of the bottom row of (8.9) . The two compositions into the lower right corner are canonically homotopic, so the fact that the right square in (8.9) is a homotopy pullback yields (8.18).

Theorem 8.20. *The map $\mathcal{I}_n(H)_{\mathrm{stable}} \longrightarrow \mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}}$ in (8.18) is a homotopy equivalence.*

We give the proof of Theorem 8.20 in §8.2 and §8.3.

Corollary 8.21. *There is an isomorphism*

$$\pi_0\,\mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \cong [MTH, \Sigma^{n+1}I\mathbb{Z}(1)]. \tag{8.22}$$

Next, we turn to discrete invertible theories. First, observe that the $\mathbb{Z}/2$-action on $\mathbb{C}$ by complex conjugation is equivalent to the $\mathbb{Z}/2$-action on $\mathrm{Map}(\mathbb{Z}/2, \mathbb{R})$, so for any $\mathbb{Z}/2$-spectrum X one has

$$\mathrm{Map}^{\mathbb{Z}/2}(X, H\mathbb{C}^{\nu_0'}) \approx \mathrm{Map}(X, H\mathbb{R}). \tag{8.23}$$

The spectrum $\mathrm{Map}(X, H\mathbb{R})$ carries a residual $\mathbb{Z}/2$-action, induced from the $\mathbb{Z}/2$-action on X; it splits as a wedge of the $(+1)$- and (-1)-eigenspaces. The exponential sequence (6.35) of $\mathbb{Z}/2$-equivariant spectra implies that the left map in the bottom row of (8.9) extends to a fibration sequence

$$\begin{aligned}\mathrm{Map}^{\mathbb{Z}/2}(\Sigma^n MTH_n^\beta, \Sigma^n(I\mathbb{C}^\times)^{\nu_0'}) &\longrightarrow \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^n MTH_n^\beta, \Sigma^{n+1}I\mathbb{Z}(1)^\gamma)\\ &\longrightarrow \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^n MTH_n^\beta, \Sigma^{n+1}H\mathbb{C}^{\nu_0'}).\end{aligned} \tag{8.24}$$

Apply (8.23) to the last term and use the fact that the left hand square in (8.9) is a homotopy pullback to obtain a fibration sequence

$$\mathcal{I}_n^\delta(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \longrightarrow \mathcal{I}_n(H_n)_{\substack{\mathrm{reflection}\\ \mathrm{positive}}} \longrightarrow \mathrm{Map}(\Sigma^n MTH_n, \Sigma^{n+1}H\mathbb{R}). \tag{8.25}$$

Proposition 8.26. *The image of the homomorphism*

$$\pi_0\,\mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \longrightarrow \pi_0\,\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \tag{8.27}$$

is the torsion subgroup of $\pi_0\,\mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}}$.

Theorem 1.1 in the introduction follows from Proposition 8.26 and (8.22). In Theorem 8.29 below we determine the homotopy type of the space of discrete invertible reflection positive field theories.

Proof. Since (8.25) is a fibration sequence of spectra, applying π_0 we obtain an exact sequence of abelian groups in which, after applying (8.22), the second map is[33]

$$[MTH, \Sigma^{n+1}I\mathbb{Z}(1)] \longrightarrow [\Sigma^n MTH_n, \Sigma^{n+1}H\mathbb{R}(1)]. \tag{8.28}$$

The construction following (8.19) implies that this map is pullback along the defining inclusion of $\Sigma^n MTH_n$ into MTH. The proposition follows if we prove (8.28) is injective after tensoring the domain with $\mathbb{R}$. This follows immediately from Lemma 7.39. ☐

We parlay (8.25) into a more useful expression for the homotopy type of the space of discrete invertible reflection positive field theories. Recall the spectrum $I\mathbb{T}$ introduced in Remark 6.36.

Theorem 8.29. *For n odd there is a homotopy equivalence*

$$\operatorname{Map}(MTH, \Sigma^n I\mathbb{T}) \xrightarrow{\ \approx\ } \mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \tag{8.30}$$

For n even there is a fibration sequence

$$\operatorname{Map}(MTH, \Sigma^n I\mathbb{T}) \longrightarrow \mathcal{I}_n^\delta(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \xrightarrow{\ s\ } \mathbb{R}^{>0} \tag{8.31}$$

in which $\mathbb{R}^{>0}$ has the discrete topology and s maps a discrete theory F to $F(S^n)$.

Compare with the more rigid Theorem 7.22 in the absence of reflection structures. Also, note that for any n-manifold X the disjoint union $\beta X \amalg X$ is null bordant, and so in a stable theory the partition functions have unit norm, consistent with the appearance of $I\mathbb{T}$ in (8.30) and (8.31). There is a canonical section of s given by Euler theories (Example 4.21): given $x \in \mathbb{R}^{>0}$ define the Euler theory as the composition

$$\Sigma^n MTH_n^\beta \longrightarrow \Sigma^n(BH_n^\beta)_+ \longrightarrow \Sigma^n S^0 \xrightarrow{\ \sqrt{x}\ } \Sigma^n H\mathbb{R}^{>0} \longrightarrow \Sigma^n(I\mathbb{C}^\times)^{\nu_0'} \tag{8.32}$$

The restriction to $\Sigma^{n-1}MTH_{n-1}^\beta$ is trivialized; using (8.9) we obtain a reflection positive theory.

[33] The map (8.28) is $\mathbb{Z}/2$-equivariant for the β-involution on MTH and $\Sigma^n MTH_n$. By Proposition 8.2 the β- and γ-involutions on MTH agree, from which $\mathbb{Z}/2$-acts as -1 on the domain. It follows that the image is contained in the (-1)-eigenspace of the codomain, which is why we write '$H\mathbb{R}(1)$' in place of '$H\mathbb{R}$'.

Proof. For any pointed space C_n use the nonequivariant version of the exponential sequence (6.39) and the fibration sequence (8.25) to construct the diagram

$$\begin{tikzcd} \mathrm{Map}(MTH,\Sigma^n I\mathbb{T}) \arrow[r] \arrow[d] & \mathrm{Map}(MTH,\Sigma^{n+1}I\mathbb{Z}(1)) \arrow[r] \arrow[d] & \mathrm{Map}(MTH,\Sigma^{n+1}H\mathbb{R}(1)) \arrow[d] \\ \mathcal{I}^\delta_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \arrow[r] \arrow[d] & \mathcal{I}_n(H_n)_{\substack{\text{reflection}\\ \text{positive}}} \arrow[r] \arrow[d] & \mathrm{Map}(\Sigma^n MTH_n,\Sigma^{n+1}H\mathbb{R}) \arrow[d] \\ \Omega C_n \arrow[r] & * \arrow[r] & C_n \end{tikzcd} \tag{8.33}$$

in which the rows are fibration sequences, as is the middle column, by Theorem 8.20. We claim

$$C_n = \begin{cases} *, & n \text{ odd}, \\ K(\mathbb{R},1), & n \text{ even}, \end{cases} \tag{8.34}$$

renders the last column a fibration sequence; it follows that the first column is as well. (Here $K(\mathbb{R},1)$ is an Eilenberg-MacLane space.) There is an exponential to pass from the third column to the first column in (8.33), and so naturally $\Omega C_n \approx \mathbb{R}^{>0}$ with the discrete topology.

To prove the claim observe first that we can replace the upper right entry of (8.33) with the homotopy equivalent space $\mathrm{Map}(\Sigma^{n+2}MTH_{n+2},\Sigma^{n+1}H\mathbb{R}(1))$, using arguments similar to those in §7.2. To analyze the resulting right vertical map consider the composition

$$\pi_q\Sigma^n MTH_n\otimes\mathbb{R} \xrightarrow{i_*} \pi_q\Sigma^{n+1}MTH_{n+1}\otimes\mathbb{R} \xrightarrow{j_*} \pi_q\Sigma^{n+2}MTH_{n+2}\otimes\mathbb{R}. \tag{8.35}$$

The composition $j_*\circ i_*$ is an isomorphism for $q<n$, and since we map to $\Sigma^{n+1}H\mathbb{R}$ only $q\leqslant n+1$ is relevant. Use (7.40) and the exact sequence

$$\pi_{m+1}\Sigma^{m+1}MTH_{m+1} \xrightarrow{\text{Euler}} \mathbb{Z} \xrightarrow{[S^m]} \pi_m\Sigma^m MTH_m \longrightarrow \pi_m\Sigma^{m+1}MTH_{m+1} \longrightarrow 0 \tag{8.36}$$

to verify the following four assertions. If n is odd, then $j_*\circ i_*$ is an isomorphism for $q=n$ and $q=n+1$. If n is even, then $j_*\circ i_*$ is an isomorphism for $q=n+1$ and is surjective for $q=n$ with kernel generated by $[S^n]$. Observe that $[S^n]=[\widehat{H}_{n+1}/\widehat{H}_n]$ is fixed by the β-involution. It follows that the upper right arrow in (8.33) is injective with image the (-1)-eigenspace of the β-involution; the cokernel the $(+1)$-eigenspace generated by $[S^n]$. (Compare with the discussion in footnote [33].) The claim, and so the theorem, follows. □

We conclude this subsection with a comment about our application of these theorems to computations. Namely, the considerations in §5.4 lead to the following conjecture, which uses non-topological invertible theories (for which we do not develop mathematical foundations in this paper).

Conjecture 8.37. *There is a 1:1 correspondence*

$$\begin{Bmatrix} \text{deformation classes of reflection positive} \\ \text{invertible } n\text{-dimensional extended field} \\ \text{theories with symmetry type } (H_n,\rho_n) \end{Bmatrix} \cong [MTH,\Sigma^{n+1}I\mathbb{Z}(1)]. \tag{8.38}$$

We remark that since the rational cohomology of BH vanishes in odd degrees, elements of infinite order in (8.38) occur only for n odd.

Remark 8.39. A restatement of Corollary 8.21 is the 1:1 correspondence

$$\left\{\begin{array}{l}\text{isomorphism classes of reflection positive continuous}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with symmetry type } (H_n,\rho_n)\end{array}\right\} \cong [MTH, \Sigma^{n+1}I\mathbb{Z}(1)]. \tag{8.40}$$

If we accept that the effective low energy theory of an invertible gapped system is a continuous invertible topological field theory, as in Remark 5.29, then we can apply (8.40) to the computations in §9 rather than (8.38). This has an advantage: (8.40) is a theorem in the context of this paper.

Remark 8.41. A homotopy class of maps $MTH \to \Sigma^{n+1}I\mathbb{Z}(1)$ leads to a canonical isomorphism class of invertible field theories via the following sketch; the theories are topological if and only if the homotopy class has finite order. By the twisted Thom isomorphism the homotopy classes are elements of $I\mathbb{Z}(1)^{\tau+n+1}(BH)$, where τ is the canonical "density twisting": the pullback to manifolds with tangential H-structure can be integrated. According to the main theorem in [FH1] there is a unique lift to the differential cohomology group $\widehat{I\mathbb{Z}(1)}^{\tau+n+1}(B_\nabla H)$. Choose a "cocycle" representative. Then on any manifold with a differential H-structure we can integrate to construct an invariant, and these invariants fit to an invertible field theory on $\mathrm{Bord}_n^\nabla(H)$.

8.2. Proof of Theorem 8.20

We restate the theorem in the language of stable homotopy theory.

Proposition 8.42. *The square*

$$\begin{tikzcd} \mathrm{Map}(MTH, \Sigma^{n+1}I\mathbb{Z}(1)) \arrow[r] \arrow[d] & \mathrm{Map}(\Sigma^{n-1}MTH_{n-1}, \Sigma^{n+1}I\mathbb{Z}(1)) \arrow[d] \\ \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^n MTH_n^\beta, \Sigma^{n+1}I\mathbb{Z}(1)^\gamma) \arrow[r] & \mathrm{Map}^{\mathbb{Z}/2}(\Sigma^{n-1}MTH_{n-1}^\gamma, \Sigma^{n+1}I\mathbb{Z}(1)^\gamma) \end{tikzcd} \tag{8.43}$$

is a homotopy pullback square of spaces.

The analysis of this square becomes cleaner if every term of the form $\mathrm{Map}^{\mathbb{Z}/2}(X, \Sigma^{n+1}I\mathbb{Z}(1)^\gamma)$ is replaced with $\mathrm{Map}((X \wedge S^{\sigma-1})_{h\mathbb{Z}/2}, \Sigma^{n+1}I\mathbb{Z}(1))$. Doing so, Proposition 8.42 becomes the assertion that the square

$$\begin{tikzcd} \Sigma^{n-1}MTH_{n-1} \wedge B\mathbb{Z}/2_+ \arrow[r] \arrow[d] & \Sigma^n MT\hat{H}_n^{(\sigma-1)} \arrow[d] \\ \Sigma^{n-1}MTH_{n-1} \arrow[r] & MTH \end{tikzcd} \tag{8.44}$$

becomes a homotopy pullback square after applying $\mathrm{Map}(\,-\,, \Sigma^{n+1}I\mathbb{Z}(1))$. We use the notation

$$MT\widehat{H}_n^{(\sigma-1)} = \mathrm{Thom}(B\widehat{H}_n; -\hat{\rho}_n + \sigma - 1). \tag{8.45}$$

To clarify the argument we state this as as

Proposition 8.46. *For any $m \geqslant n$, the square*

$$\begin{tikzcd} \Sigma^{m-1} MTH_{m-1} \wedge B\mathbb{Z}/2_+ \arrow[r] \arrow[d] & \Sigma^m MT\hat{H}_m^{(\sigma-1)} \arrow[d] \\ \Sigma^{m-1} MTH_{m-1} \arrow[r] & MTH \end{tikzcd} \tag{8.47}$$

becomes a homotopy pullback square after applying $\mathrm{Map}(\,-\,, \Sigma^{n+1} I\mathbb{Z}(1))$.

The proof of Proposition 8.46 will make repeated use of the following result, which follows from the universal property (5.17) of $I\mathbb{Z}(1)$.

Lemma 8.48. *Suppose A is a spectrum having the property that $\pi_i A = 0$ for $i \leqslant n$ and $\pi_{n+1} A$ is a torsion group. If $A \to X \to Y$ is a cofibration sequence then*

$$\mathrm{Map}(Y, \Sigma^{n+1} I\mathbb{Z}(1)) \to \mathrm{Map}(X, \Sigma^{n+1} I\mathbb{Z}(1))$$

is a weak equivalence of spaces.

The proof of Proposition 8.46 is by decreasing induction on m. As $m \to \infty$ the square (8.47) becomes

$$\begin{tikzcd} MTH \wedge B\mathbb{Z}/2_+ \arrow[r] \arrow[d] & MTH \wedge B\mathbb{Z}/2_+ \arrow[d] \\ MTH \arrow[r] & MTH \end{tikzcd}$$

which is obviously a pushout. On the other hand for $m > (n+2)$ the maps

$$\Sigma^{m-1} MTH_{m-1} \to MTH$$
$$\Sigma^m MT\hat{H}_m^{(\sigma-1)} \to MTH \wedge B\mathbb{Z}/2_+$$

become equivalences after applying $\mathrm{Map}(\,-\,, \Sigma^{n+1} I\mathbb{Z}(1))$, so the result is true for all $m > n+2$. (Compare with the proof of Theorem 7.30.)

Since the homotopy fiber of the left vertical map in (8.47) is $\Sigma^{m-1} MTH_{m-1} \wedge B\mathbb{Z}/2$, Proposition 8.46 is equivalent to the assertion that for all $m \geqslant n$, the sequence

$$\Sigma^{m-1} MTH_{m-1} \wedge B\mathbb{Z}/2 \to \Sigma^m MT\hat{H}_m^{(\sigma-1)} \to MTH$$

becomes a fibration sequence after applying $\mathrm{Map}(\,-\,, \Sigma^{n+1} I\mathbb{Z}(1))$. The induction step therefore follows from

Proposition 8.49. *For $m \geqslant n$, the square*

$$\begin{tikzcd} \Sigma^{m-1} MTH_{m-1} \wedge B\mathbb{Z}/2 \arrow[r] \arrow[d] & \Sigma^m MT\hat{H}_m^{(\sigma-1)} \arrow[d] \\ \Sigma^m MTH_m \wedge B\mathbb{Z}/2 \arrow[r] & \Sigma^{m+1} MT\hat{H}_{m+1}^{(\sigma-1)} \end{tikzcd} \tag{8.50}$$

becomes a homotopy pullback square after applying $\mathrm{Map}(\,-\,, \Sigma^{n+1} I\mathbb{Z}(1))$.

What is at stake in Proposition 8.49 is to prove that the induced map

$$\Sigma^{m-1}(BH_m)_+ \wedge B\mathbb{Z}/2 \to \Sigma^m \operatorname{Thom}(B\hat{H}_{m+1}; \sigma - 1) \tag{8.51}$$

of homotopy fibers of the vertical maps in (8.50) becomes a homotopy equivalence after applying $\operatorname{Map}(-, \Sigma^{n+1} I\mathbb{Z}(1))$. The following result will be proved in §8.3.

Lemma 8.52. *The map* (8.51) *is the* $(m-1)^{\text{st}}$ *suspension of the map of Thom spectra* (*of the bundle* $(\sigma - 1)$) *associated to the map*

$$BH_m \times B\mathbb{Z}/2 \to B\hat{H}_{m+1} \tag{8.53}$$

given by the choice of reflection in the last coordinate.

Assuming Lemma 8.52 we can prove Proposition 8.49.

Proof of Proposition 8.49. It suffices to show that the induced map (8.51) becomes a weak equivalence after applying $\operatorname{Map}(\,-\,, \Sigma^{n+1} I\mathbb{Z}(1))$. The map (8.53) fits into a Cartesian square

$$\begin{tikzcd}
S^m \arrow[r, equal] \arrow[d] & S^m \arrow[d] & \\
BH_m \arrow[r] \arrow[d] & BH_m \times B\mathbb{Z}/2 \arrow[r] \arrow[d] & B\mathbb{Z}/2 \arrow[d, equal] \\
BH_{m+1} \arrow[r] & B\hat{H}_{m+1} \arrow[r] & B\mathbb{Z}/2\ ,
\end{tikzcd}$$

so Lemma 8.52 implies that the cofiber of (8.51) is $2m$-connected. Since $m \geqslant n \geqslant 1$, one has $2m \geqslant n$ and so the cofiber is n-connected. Both terms in (8.51) are rationally acyclic. The result then follows from Lemma 8.48. ☐

8.3. Transfers

Suppose that $M \to X$ is a fiber bundle with fibers closed smooth manifolds M_x of dimension n. Let $T_{M/X}$ be the vector bundle over M whose fiber at $a \in M_x$ is the tangent space $T_a M_x$. There is functorial stable map

$$\Sigma^\infty X_+ \to \operatorname{Thom}(M, -T_{M/X})$$

called the *transfer map*. When there is an embedding $M \subset X \times \mathbb{R}^n$ for some n it can be constructed from the Pontrjagin Thom collapse

$$\operatorname{Thom}(X, \underline{\mathbb{R}^n}) \to \operatorname{Thom}(M, \underline{\mathbb{R}^n} - T_{M/X})$$

by passing to suspension spectra and desuspending n times. The transfer map is constructed in the general case by passing to the colimit over the category of pairs

$$X_\alpha \to X$$
$$i_\alpha : M_\alpha \hookrightarrow X_\alpha \times R^{N_\alpha}$$

in which $M_\alpha \to X_\alpha$ is the pullback of $M \to X$ along the map $X_\alpha \to X$.

When there is an embedding $M \subset W$ over B, the Pontrjagin Thom construction leads to a *twisted* transfer map

$$\mathrm{Thom}(B; W) \to \mathrm{Thom}(X; W - T_{M/X}).$$

The twisted transfer extends in the evident manner to the case of virtual bundles W.

Proposition 8.54. *Suppose that W is a vector bundle over X, that $f\colon M \to W$ is a map over X transverse to the zero section and let N be the inverse image of 0. There is a commutative diagram*

$$\begin{tikzcd} \mathrm{Thom}(X;0) \arrow[r] \arrow[d] & \mathrm{Thom}(N; -T_{N/X}) \arrow[d] \\ \mathrm{Thom}(X;W) \arrow[r] & \mathrm{Thom}(M; W - T_{M/X}) \end{tikzcd}$$

in which the left vertical map is derived from the zero section, and the right is the natural map of Thom complexes coming from the inclusion $N \subset M$ and the isomorphism

$$T_{M/X} \approx T_{N/X} \oplus W.$$

Proof. It suffices to establish the case in which there is an embedding

$$\iota : M \hookrightarrow \mathbb{R}^n.$$

Applying the Pontrjagin-Thom constructions to the rows in the transverse pullback square

$$\begin{tikzcd} N \arrow[r] \arrow[d] & X \times \mathbb{R}^n \arrow[d] \\ M \arrow[r, "(f,\iota)"'] & W \times \mathbb{R}^n \end{tikzcd} .$$

gives a diagram

$$\begin{tikzcd} \mathrm{Thom}(X; \underline{\mathbb{R}^n}) \arrow[r] \arrow[d] & \mathrm{Thom}(N; \underline{\mathbb{R}^n} - T_{N/X}) \arrow[d] \\ \mathrm{Thom}(X; W \oplus \underline{\mathbb{R}^n}) \arrow[r] & \mathrm{Thom}(M; W + \underline{\mathbb{R}^n} - T_{M/X}) \end{tikzcd}$$

in which the left vertical map is the inclusion of the zero section. Desuspending, the claim follows easily from this. $\square$

Proof of Lemma 8.52. The idea is to apply Proposition 8.54 to the left triangle in the diagram

$$\begin{tikzcd} S(\rho_m)\times B\mathbb{Z}/2 \arrow[r] \arrow[dr] & S(\rho_m\oplus\sigma) \arrow[r] \arrow[d] & S(\hat\rho_{m+1}) \arrow[d] \\ & BH_m\times B\mathbb{Z}/2 \arrow[r] & B\hat H_{m+1} \end{tikzcd} \tag{8.55}$$

with

$$\begin{aligned} X &= BH_m\times B\mathbb{Z}/2 \\ W &= \sigma \\ M &= S(\rho_m\oplus\sigma) \\ N &= S(\rho_m)\times B\mathbb{Z}/2 \end{aligned}$$

The diagram is written in order to clarify the relationship with manifolds. Note that there are equivalences

$$\begin{aligned} S(\hat\rho_{m+1}) &\approx B\hat H_m \\ S(\rho_m) &\approx BH_{m-1}. \end{aligned}$$

Also, for a vector bundle $V\to X$ the relative tangent bundle of $p: S(V)\to X$ is given by $T_{S(V)/X}\oplus\mathbb{R}=p^*V$. Proposition 8.54 then gives the left square in the diagram

$$\begin{tikzcd} \Sigma^{m-1}(BH_m)_+\wedge B\mathbb{Z}/2_+ \arrow[r] \arrow[d] & \Sigma^{m-1}(BH_m)_+\wedge B\mathbb{Z}/2 \arrow[r] \arrow[d] & \Sigma^m\,\mathrm{Thom}(B\hat H_{m+1};\sigma-1) \arrow[d] \\ \Sigma^{m-1}MTH_{m-1}\wedge B\mathbb{Z}/2_+ \arrow[r] & Y \arrow[r] & \Sigma^m MT\hat H_m^{(\sigma-1)} \end{tikzcd} \tag{8.56}$$

with

$$Y=\Sigma^m\,\mathrm{Thom}(S(\rho_m\oplus\sigma);1-\rho_m-\sigma-1+\sigma);$$

the right square in (8.56) is the pullback of transfer maps induced from the pullback square in (8.55). The map (8.51) is the composition of

$$\Sigma^{m-1}(BH_m)_+\wedge B\mathbb{Z}/2\to\Sigma^{m-1}(BH_m)_+\wedge B\mathbb{Z}/2_+ \tag{8.57}$$

with the top row of (8.56). Lemma 8.52 now follows from the fact that the composition of (8.57) with the left map in the top row of (8.56) is the identity. $\square$

9. Fermionic theories with scalar internal symmetry group

In this section we apply Theorem 1.1 to some basic symmetry groups, namely those whose subgroup K of internal symmetries is the group O_1, U_1, Sp_1 of unit norm elements in the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. (We use the names $\{\pm 1\}, \mathbb{T}, SU_2$ for these three groups.) The internal symmetry group $K = \mathbb{T}$ is the basic charge symmetry of electromagnetism; in quantum mechanical models the presence of a so-called particle-hole symmetry "breaks"[34] it to either $K = \{\pm 1\}$ or $K = SU_2$. In §9.1 we classify the possible symmetry groups H_n with these internal symmetries, and restricting to fermionic symmetry groups we recover the 10-fold way; see Tables (9.24) and (9.25). (Wang-Senthil [WS] list many of these groups—in a nonrelativistic form (9.34), (9.35)—and the corresponding "Cartan label". Metlitski [M] introduces the group $\mathrm{Pin}^{\tilde{c}+}$, which provided guidance for our treatment here. This twisted form of Pin^c also appears implicitly in [SeWi, §A.4].) Lemma 9.27 relates the relativistic 10-fold way to the 10 real and complex Clifford algebras, thus providing a link to other 10-fold ways.

In §9.2 we sketch two ways in which a theory of free fermions in Minkowski spacetime gives rise to a deformation class of reflection positive invertible field theories, or to a reflection positive continuous invertible topological field theory. If one begins with an $(n-1)$-dimensional free fermion theory, then there is an associated n-dimensional invertible anomaly theory; if the original free fermion theory admits a mass term, then the anomaly is trivializable. In this paper we do not attempt a complete treatment, so state the main result as a conjecture, Conjecture 9.70. It expresses the deformation class of the anomaly theory as a composition of a twisted Atiyah-Bott-Shapiro map and a Pfaffian map on real K-theory. This K-theory interpretation depends on Lemma 9.55, which expresses the existence of a mass in terms of Clifford algebras.

The second scenario is to begin with a massive free fermion theory in n dimensions, as we sketch in §9.2.6. The low energy effective field theory is invertible, and (9.71) is a formula for its deformation class. It is this scenario about gapped theories that is relevant to this paper.

We carry out computations in low dimensions in §9.3. For each of the 10 electron symmetry groups we list the groups of deformation classes of reflection positive invertible topological theories and compute the map from free fermions to it. There is no further physical reasoning; we compute directly from the results in Theorem 1.1 and (9.71). The techniques lie in stable homotopy theory, and in the next section we give some details to illustrate how the computations are made. As discussed in §1 these classification results apply to invertible topological phases of condensed matter systems, often called SPT phases. The fermionic symmetry groups with $K = \mathbb{T}$ pertain to *topological insulators*; those with $K = \{\pm 1\}$ and $K = SU_2$ pertain to *topological superconductors.*

Remark 9.1. Most of the interacting groups we compute are torsion so are covered by Theorem 1.1. In the general case we interpret the computations as theorems by using (8.40), in which the interacting group is a group of isomorphism classes of reflection positive *continuous* invertible topological field theories. See §5.4 for a discussion of expectations for low energy effective field theories.

In the theoretical discussions we assume $n \geqslant 3$; in the computations we apply the results to all n.

[34] We do not have any fundamental understanding of this mechanism, especially the appearance of SU_2. In §9.1 we simply offer it as a storyline in relativistic theory that matches the condensed matter literature.

9.1. Symmetry groups of fermionic systems

We already classified symmetry groups H_n with $K = \{\pm 1\}$ in Proposition 2.16. The *fermionic* groups are the ones for which $-1 \in K$ is the distinguished element k_0 of Theorem 2.7 and Corollary 2.12.[35] (The other possibility is $k_0 = 1$, in which case the symmetry group is *bosonic.*) Those fermionic groups are Spin_n, Pin_n^+, and Pin_n^-.

Next, we classify symmetry groups with $K = \mathbb{T}$. These are group extensions

$$1 \longrightarrow \mathbb{T} \longrightarrow SH_n \longrightarrow SO_n \longrightarrow 1 \tag{9.2}$$

if there is no time-reversal symmetry and

$$1 \longrightarrow \mathbb{T} \longrightarrow H_n \longrightarrow O_n \longrightarrow 1 \tag{9.3}$$

if there is time-reversal symmetry. Recall the group E_n introduced before Proposition 2.16.

Proposition 9.4 ($K = \mathbb{T}$). *Up to isomorphism there are two distinct group extensions* (9.2) *with $n \geqslant 3$, and the groups SH_n that appear are $SO_n \times \mathbb{T}$ and Spin_n^c. Up to isomorphism there are six distinct group extensions* (9.3) *with $n \geqslant 3$, and the groups H_n that appear are mutually nonisomorphic. Three of the groups have identity component $SO_n \times \mathbb{T}$:*

$$O_n \times \mathbb{T} \tag{9.5}$$

$$O_n \ltimes \mathbb{T} \tag{9.6}$$

$$E_n \ltimes \mathbb{T} \,/\, \{\pm 1\} \tag{9.7}$$

The identity component of the remaining three groups is Spin_n^c:

$$\mathrm{Pin}_n^c \;= \mathrm{Pin}_n^+ \times \mathbb{T} \,/\, \{\pm 1\} \tag{9.8}$$

$$\mathrm{Pin}_n^{\tilde c+} = \mathrm{Pin}_n^+ \ltimes \mathbb{T} \,/\, \{\pm 1\} \tag{9.9}$$

$$\mathrm{Pin}_n^{\tilde c-} = \mathrm{Pin}_n^- \ltimes \mathbb{T} \,/\, \{\pm 1\} \tag{9.10}$$

The group Pin_n^c is also isomorphic to $\mathrm{Pin}_n^- \times \mathbb{T} \,/\, \{\pm 1\}$. It sits in the complex Clifford algebra generated by $\mathbb{R}^n$ with a nondegenerate symmetric bilinear form [ABS]. In $\mathrm{Pin}_n^{\tilde c\pm}$ the action of $\mathrm{Pin}_n^\pm$ on $\mathbb{T}$ factors through $\pi_0 \mathrm{Pin}_n^\pm$ and is via inversion $\lambda \mapsto \lambda^{-1}$. In each case we divide out by the diagonal subgroup $\{\pm 1\}$. The groups with identity component Spin_n^c are fermionic.

Proof. The extension (9.2) is central, so up to isomorphism classified by the cohomology group

$$H^2(BSO_n; \underline{\mathbb{T}}) \cong H^3(BSO_n; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}. \tag{9.11}$$

The underline indicates the sheaf cohomology of continuous functions into $\mathbb{T}$ with the standard topology. It is well-known that Spin_n^c corresponds to the nonzero element.

[35]This implies the "spin/charge relation" of condensed matter physics, which is emphasized in [SeWi]: bosons have even charge and fermions have odd charge.

The only nontrivial automorphism of $\mathbb{T}$ is inversion, so in the extension (9.3) either O_n acts trivially or it acts through its components with elements of determinant -1 acting by inversion. In each case the group extensions are classified by a cohomology group of the classifying space BO_n:

$$H^2(BO_n;\mathbb{T}) \cong H^3(BO_n;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \tag{9.12}$$

$$H^2(BO_n;\widetilde{\mathbb{T}}) \cong H^3(BO_n;\widetilde{\mathbb{Z}}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \tag{9.13}$$

The tilde indicates coefficients twisted by inversion. The product (9.5) and semi-direct product (9.6) account for (9.12) and the remaining four groups for (9.10), as can be seen from cohomological computations we omit. $\square$

According to the arguments in Appendix A, the anti-Wick rotation of $\mathrm{Pin}^{\tilde{c}+}$ contains a time-reversal symmetry T with $T^2 = (-1)^F$ and the anti-Wick rotation of $\mathrm{Pin}^{\tilde{c}-}$ contains a time-reversal symmetry T with $T^2 = 1$. More precisely, the groups (9.8) and (9.5) are Wick rotations of relativistic symmetry groups that include CT symmetry; the remaining groups are Wick rotations of relativistic symmetry groups that include T symmetry.[36]

Finally, we classify symmetry groups with $K = SU_2$. Now we have possible extensions

$$1 \longrightarrow SU_2 \longrightarrow SH_n \longrightarrow SO_n \longrightarrow 1 \tag{9.14}$$

and

$$1 \longrightarrow SU_2 \longrightarrow H_n \longrightarrow O_n \longrightarrow 1 \tag{9.15}$$

Proposition 9.16 ($K = SU_2$)**.** *Up to isomorphism there are two distinct group extensions (9.14) with $n \geqslant 3$, and the groups SH_n that appear are $SO_n \times SU_2$ and*

$$G_0 = \mathrm{Spin}_n \times_{\{\pm 1\}} SU_2. \tag{9.17}$$

Up to isomorphism there are four distinct group extensions (9.15) with $n \geqslant 3$, and the groups H_n that appear are mutually nonisomorphic. Two of the groups have identity component $SO_n \times SU_2$:

$$O_n \times SU_2 \tag{9.18}$$

$$E_n \times_{\{\pm 1\}} SU_2 \tag{9.19}$$

The identity component of the remaining two groups is G_0:

$$G_n^+ = \mathrm{Pin}_n^+ \times_{\{\pm 1\}} SU_2 \tag{9.20}$$

$$G_n^- = \mathrm{Pin}_n^- \times_{\{\pm 1\}} SU_2 \tag{9.21}$$

[36]This is our interpretation of [W1, §3.7]. There are more general possibilities with larger internal symmetry group K. This occurs in [SeWi, §3], for example, in a theory with both T and CT symmetry.

The symmetry groups with identity component G^0 are fermionic.

Proof. The classification of the identity component SH_n follows from Theorem 2.7(2): there are two central elements $k_0 \in SU_2$ with $k_0^2 = 1$. To classify the two-component group H_n we apply a useful general result [FHT2, Corollary 7.3]. Namely, for any compact Lie group H, let H^0 denote the component of the identity element, $Z^0 \subset H^0$ its center, and $\pi = \pi_0 H$ the abelian group of components. Then there exists a group L that fits into the diagram

$$\begin{tikzcd} 1 \arrow[r] & Z^0 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H^0 \arrow[r] & H \arrow[r] & \pi \arrow[r] & 1 \end{tikzcd} \tag{9.22}$$

of group extensions. Furthermore, the group L acts on H^0 by conjugation—the action descends to an action of π since Z^0 is central, but it depends on the choice of L—and the group H is reconstructed from H^0 and L as a semidirect product

$$H \cong L \ltimes_{Z^0} H^0 = L \ltimes H^0 \big/ Z^0. \tag{9.23}$$

By the Stabilization Theorem 2.19 we may assume that n is odd, since for n even H_n is obtained by pullback, so the center of SO_n is trivial and the center of Spin_n is $\{\pm 1\}$. First, assume $H^0 = SH_n = SO_n \times SU_2$, so that $Z^0 = \{\pm 1\}$. There are two possibilities: $L \cong \{\pm 1\}^{\times 2}$ or $L \cong \mu_4$. We can take the image of L in O_n to be the central subgroup $\{\pm 1\}$. The conjugation action on SO_n is trivial, and as all automorphisms of SU_2 are inner we can take the entire action on H^0 to be trivial. Then (9.23) (with a direct product in place of a semidirect product) yields the two groups (9.18) and (9.19). The argument for $H^0 = \mathrm{Spin}_n \times_{\{\pm 1\}} SU_2$ is similar; again $Z^0 \cong \{\pm 1\}$. $\square$

9.2. Free fermions and twisted Dirac operators

In this section we take up the homotopy theory of relativistic free fermions. We treat the 10 fermionic symmetry groups simultaneously via embeddings into Clifford algebras (§9.2.1). For each we define a twisted Atiyah-Bott-Shapiro map (§9.2.2) that encodes the index of twisted Dirac operators (§9.2.3) on compact Riemannian manifolds. The relativistic story begins on Minkowski spacetime in Lorentz signature, where a free fermion theory is specified by a real Clifford module for a Lorentz signature Clifford algebra (§9.2.4). We develop that algebraic theory for the fermionic symmetry groups, and in particular determine those theories that admit a nondegenerate mass term (Lemma 9.55). A *massless* theory has an anomaly, which is an invertible field theory, and we conjecture its deformation class in §9.2.5. A formally similar setup (§9.2.6) attaches an invertible field theory to a *massive* free fermion theory, and we conjecture that its deformation class is given by the same formula. It is this formula that we use in the computations in §9.3.

9.2.1. *A relativistic 10-fold way.* Proposition 2.16, Proposition 9.4, and Proposition 9.16 combine to yield $3+4+3=10$ fermionic symmetry groups, which we arrange into two tables:

(9.24)

s	H^c	K	Cartan	D
0	Spin^c	$\mathbb{T}$	A	$\mathbb{C}$
1	Pin^c	$\mathbb{T}$	AIII	$\mathrm{Cliff}^{\mathbb{C}}_{-1}$

(9.25)

s	H	K	Cartan	D
0	Spin	$\{\pm 1\}$	D	$\mathbb{R}$
-1	Pin^+	$\{\pm 1\}$	DIII	Cliff_{-1}
-2	$\mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AII	Cliff_{-2}
-3	$\mathrm{Pin}^- \times_{\{\pm 1\}} SU_2$	SU_2	CII	Cliff_{-3}
4	$\mathrm{Spin} \times_{\{\pm 1\}} SU_2$	SU_2	C	$\mathbb{H}$
3	$\mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$	SU_2	CI	Cliff_{+3}
2	$\mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AI	Cliff_{+2}
1	Pin^-	$\{\pm 1\}$	BDI	Cliff_{+1}

In addition to the fermionic symmetry group H or H^c and its internal group K, we list the Cartan label, an integer s called the "type", and a super division algebra D. The type is defined mod 2 in (9.24) and mod 8 in (9.25); we choose a convenient integer representative. We use notations $H(s), H^c(s), K(s), D(s)$ when we make the type explicit. The Cartan label is used in the condensed matter literature, where this 10-fold way has many incarnations: see [D, AZ, HHZ, K6, SRFL, FM1, KZ, WS]. In those references the *particle-hole symmetry* determines the internal symmetry group K: in its absence $K=\mathbb{T}$; if particle-hole symmetry is present and squares to $+1$, then $K=\{\pm 1\}$; and if particle-hole symmetry is present and squares to -1, then $K=SU_2$. The existence (and square) of *time-reversal symmetry* in the references above matches that in our account except for the entry AIII, which is usually listed as not having time-reversal symmetry (but see [WS, §III]). The super division algebra D is the unique super division algebra in the Morita class of the Clifford algebra[37] Cliff_s. The groups Spin^c and Pin^c in the first table (9.24) are distinguished as having a central subgroup isomorphic to $\mathbb{T}$, so are called *complex*; the center of the groups in (9.25) is $\{\pm 1\}$, and so they are called *real*.

Remark 9.26. We would have found it more natural from a mathematical point of view in several places to define $H(4) = \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{Spin}_4$ rather than $\mathrm{Spin} \times_{\{\pm 1\}} \mathrm{Spin}_3$, but we lack a physics motivation to do so.

The following embedding allows a uniform treatment of these symmetry groups, and it opens a path to relating this relativistic 10-fold way to other 10-fold ways in the literature. Fix $n \geqslant 0$.

[37]The Clifford algebra $\mathrm{Cliff}_{\pm|s|}$ is generated by $e_1, \dots, e_{|s|}$ subject to $e_a e_b + e_b e_a = \pm 2\delta_{ab}$; see [ABS].

Lemma 9.27. *Fix a real type s as in (9.25), and let $H_n(s)$ denote the n-dimensional version of the group $H(s)$ of type s in Table (9.25). Write $A_n(s) = \mathrm{Cliff}_{+n} \otimes D(s)$. Then there is an embedding*

$$\iota\colon H_n(s) \longrightarrow A_n(s) \tag{9.28}$$

such that the natural map

$$c\colon \mathbb{R}^n \times A_n(s) \longrightarrow A_n(s) \tag{9.29}$$

is $H_n(s)$-equivariant and graded commutes with right multiplication by $A_n(s)$.

Here c is the extension of scalars of Clifford multiplication $\mathbb{R}^n \times \mathrm{Cliff}_{+n} \to \mathrm{Cliff}_{+n}$. (Recall that $\mathbb{R}^n \subset \mathrm{Cliff}_{+n}$.) Note that $A_n(s)$ is Morita equivalent to $\mathrm{Cliff}_{+(n+s)}$; we specify a Morita equivalence in §9.2.2. We regard $H_n(s)$ as an ungraded group, and in fact $\iota(H_n(s))$ is contained in the even part of the superalgebra $A_n(s)$. In the complex case (9.24) there is an embedding $\iota^{\mathbb{C}}\colon \mathrm{Pin}^c_n \hookrightarrow \mathrm{Cliff}^{\mathbb{C}}_n \otimes \mathrm{Cliff}^{\mathbb{C}}_{-1}$ constructed using the same formulas as the real case $s = 1$. Of course, there is also the usual embedding $\iota^{\mathbb{C}}\colon \mathrm{Spin}^c_n \hookrightarrow \mathrm{Cliff}^{\mathbb{C}}_n$.

Proof. The case $s = 0$ requires no comment. For $s = 4$ we use the fact that $SU_2 \cong Sp_1 \subset \mathbb{H}$. The scalar -1 passes between the factors in the real tensor product $\mathrm{Cliff}_{+n} \otimes \mathbb{H}$, which explains the division by $\{\pm 1\}$ in the group H. In the remaining six cases $D(s)$ is a Clifford algebra on $|s|$ generators, and the group $\mathrm{Spin}_{|s|} \subset \mathrm{Cliff}_s$ is isomorphic to $\{\pm 1\}, \mathbb{T}, SU_2$ for $|s| = 1, 2, 3$, respectively. For $|s| = 1, 2$ fix a unit vector $e \in \mathbb{R}^{|s|} \subset D(s)$; for $|s| = 3$ define the volume form $\omega = e_1 e_2 e_3$ as the ordered product of the generators of $\mathrm{Cliff}_{|s|}$. Define ι by

$$\begin{aligned} g &\longmapsto g \otimes 1, && g \in \mathrm{Spin}_n, \\ g &\longmapsto \begin{cases} g \otimes e, & |s| = 1, 2, \\ g \otimes \omega, & |s| = 3, \end{cases} && g \in \mathrm{Pin}^{\pm}_n \setminus \mathrm{Spin}_n, \\ \lambda &\longmapsto 1 \otimes \lambda, && \lambda \in \mathbb{T} \text{ or } SU_2. \end{aligned} \tag{9.30}$$

A case-by-case check completes the proof. To illustrate, we check the equivariance of c for $g \in \mathrm{Pin}_n \setminus \mathrm{Spin}_n$ and $|s| = 1, 2$; it suffices to take $g = e_i$ for some standard basis element $e_i \in \mathbb{R}^n$. For $\xi \in \mathbb{R}^n \subset \mathrm{Cliff}_{+n}$, we have $e_i \cdot (\xi \otimes 1) = -e_i \xi e_i^{-1} \otimes 1$. For $\psi \in \mathrm{Cliff}_{+n}$ homogeneous of parity $|\psi|$ and $x \in D(s)$, we have $e_i \cdot (\psi \otimes x) = (-1)^{|\psi|} e_i \psi \otimes ex$, since e_i acts as left multiplication in $A_n(s)$ by $\iota(e_i)$ and the Koszul sign rule applies in the superalgebra $A_n(s)$. Their Clifford product is

$$-(-1)^{|\psi|} e_i \xi \psi \otimes ex = e_i \cdot (\xi \psi \otimes x), \tag{9.31}$$

which proves the equivariance. We leave the other checks to the reader. ∎

Remark 9.32. In the condensed matter literature free fermion systems are often treated nonrelativistically and so are organized by nonrelativistic symmetry groups. More specifically, they are organized by the subgroup I of internal vector symmetries that fix the points of *space*. (The internal symmetry group K in our account, which starts from a relativistic theory, is the subgroup that fixes the points of *spacetime*.) We can easily compute the group I_n in spacetime dimension n for a general group of symmetries, as in §1. Namely, let $\rho_n\colon H_n \to O_n$ be a Wick-rotated symmetry group. Fix a splitting $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^{n-1}$ of translations of $\mathbb{E}^n$ into Wick-rotated-time translations cross spatial translations. The subgroup $O_1\times O_{n-1}\subset O_n$ preserves that splitting, and $O_1\times\{\mathrm{id}\}\subset O_1\times O_{n-1}$ is the vector subgroup of transformations that fix space pointwise. So for the symmetry group H_n we define the nonrelativistic internal subgroup I_n as the pullback

$$\begin{tikzcd} I_n \arrow[rr, hook] \arrow[d] & & H_n \arrow[d, "\rho_n"] \\ O_1\times\{\mathrm{id}\} \arrow[r, hook] & O_1\times O_{n-1} \arrow[r, hook] & O_n \end{tikzcd} \tag{9.33}$$

The inclusion $H_n \hookrightarrow H_{n+1}$ induces an isomorphism $I_n \xrightarrow{\cong} I_{n+1}$; denote the colimit of these groups as I. We tabulate I for each of the ten fermionic symmetry groups in Tables (9.24) and (9.25):

(9.34)

s	H^c	I		Cartan
0	Spin^c	$\mathbb{T}$	(Spin^c_1)	A
1	Pin^c	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{T}$	(Pin^c_1)	AIII

(9.35)

s	H	I		Cartan
0	Spin	$\{\pm1\}$	(Spin_1)	D
-1	Pin^+	$\mathbb{Z}/2\mathbb{Z}\times\{\pm1\}$	(Pin^+_1)	DIII
-2	$\mathrm{Pin}^+\ltimes_{\{\pm1\}}\mathbb{T}$	$\mathbb{Z}/2\mathbb{Z}\ltimes\mathbb{T}$	(Pin^+_2)	AII
-3	$\mathrm{Pin}^-\times_{\{\pm1\}}SU_2$	$\mathbb{Z}/4\mathbb{Z}\times_{\{\pm1\}}SU_2$	(Pin^+_3)	CII
4	$\mathrm{Spin}\times_{\{\pm1\}}SU_2$	SU_2	(Spin_3)	C
3	$\mathrm{Pin}^+\times_{\{\pm1\}}SU_2$	$\mathbb{Z}/2\mathbb{Z}\times SU_2$	(Pin^-_3)	CI
2	$\mathrm{Pin}^-\ltimes_{\{\pm1\}}\mathbb{T}$	$\mathbb{Z}/4\mathbb{Z}\ltimes_{\{\pm1\}}\mathbb{T}$	(Pin^-_2)	AI
1	Pin^-	$\mathbb{Z}/4\mathbb{Z}$	(Pin^-_1)	BDI

In the physics literature a $\mathbb{Z}/2\mathbb{Z}$ subgroup of I containing a time-reversal symmetry, if it exists, is labeled '$\mathbb{Z}/2\mathbb{Z}_T$'. The $\{\pm1\}$ subgroup is often labeled '$\mathbb{Z}/2\mathbb{Z}_f$' where 'f' means 'fermionic' since the nontrivial element is the center of the spin group. The groups in parentheses are abstractly isomorphic to the group I.

Remark 9.36. In the pullback (9.33) the group I_n has two extra pieces of structure: the canonical central element $k_0\in K\subset I_n$ of order dividing two (Theorem 2.7(2)) and a $\mathbb{Z}/2\mathbb{Z}$-grading $\phi\colon I_n\to O_1=\{\pm1\}$ with $K=\ker\phi$. In condensed matter models we are given (I_n,k_0,ϕ) and part of the determination of the low energy effective field theory is the (re)construction of the symmetry type (H_n, ρ_n). We achieve this as follows. If ϕ is trivial then $I_n = K$, so set $\widetilde{SH}_n \colon = \mathrm{Spin}_n \times I_n$; then define $H_n = SH_n$ by (2.8). If ϕ is surjective, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}_1 \arrow[rr] \arrow[dr] & & \mathrm{Spin}_n \arrow[dr] & & \\
& \widetilde{I}_n \arrow[rr] \arrow[dl] \arrow[dd] & & \widetilde{H}_n \arrow[dl] \arrow[dd] \arrow[dr] & \\
I_n \arrow[rr] \arrow[dd] & & H_n \arrow[dd] & & J \arrow[dd] \\
& \mathrm{Pin}_1^+ \arrow[rr] \arrow[dl] & & \mathrm{Pin}_n^+ \arrow[dr] & \\
O_1 \arrow[rr] & & O_n \arrow[rr] & & \{\pm 1\}
\end{tikzcd} \tag{9.37}$$

in which every parallelogram is a pullback, the kernel of every vertical map is K, and the northeast diagonal composition is exact. Given (I_n, k_0, ϕ) define $\widetilde{I}_n$ by pullback, set $K = \ker \phi$, set $J = \widetilde{I}_n / \mathrm{Spin}_1$, let $\widetilde{H}_n$ be the pullback (2.10), and define H_n using (2.11).

9.2.2. *Twisted Atiyah-Bott-Shapiro map.* Atiyah-Bott-Shapiro [ABS, §11] give a canonical construction of K-theory elements on Thom complexes. The universal incarnation [H, §6.1] is a map of spectra

$$\phi \colon M\mathrm{Spin} \longrightarrow KO. \tag{9.38}$$

Following their arguments we produce similar maps for the group $H(s)$ of type s in Table (9.25). Fix a dimension $n \in \mathbb{Z}^{\geqslant 0}$.

As a first step we stipulate a Morita equivalence

$$A_n(s) \underset{\text{Morita}}{\approx} \mathrm{Cliff}_{+(n+s)}\,. \tag{9.39}$$

There is a sign at stake—for any Clifford algebra A the groupoid of invertible (A, A)-bimodules is equivalent to the groupoid of $\mathbb{Z}/2\mathbb{Z}$-graded lines: the sign is the parity of the line. Define the isomorphism

$$\mathrm{Cliff}_{+n} \otimes \mathrm{Cliff}_{+s} \xrightarrow{\ \cong\ } \mathrm{Cliff}_{+(n+s)} \tag{9.40}$$

as in [ABS, (1.6)], and choose [ABS, (6.9)] a $\mathrm{Cliff}_{\pm 8}$-module $M = M^0 \oplus M^1$ of dimension $8|8$ such that the volume form acts as $+1$ on M^0. There result Morita equivalences (9.39) for all cases except $s = 4$. For that we fix a *quaternionic* $\mathrm{Cliff}_{\pm 4}$-module $N = N^0 \oplus N^1$ of quaternionic dimension $1|1$ such that the volume form acts as $+1$ on N^0.

Now to the twisted ABS construction. Let $\pi\colon V_n \to BH_n(s)$ be the universal bundle associated to $\rho_n\colon H_n(s) \to O_n$. Define the spinor bundle[38]

$$\mathcal{S} := EH_n(s) \times_{H_n(s)} A_n(s)^{\mathrm{op}} \longrightarrow BH_n(s); \tag{9.41}$$

This is a vector bundle of right $A_n(s)^{\mathrm{op}}$-modules, or equivalently of left $A_n(s)$-modules. Left Clifford multiplication (9.29) defines a family of odd skew-adjoint endomorphisms of $\pi^*\mathcal{S} \to V_n$. These operators are invertible off the zero section, and they commute with the left $A_n(s)$-module structure. Therefore, using the Morita equivalence (9.39), they define an element in $KO^{n+s}\bigl(\mathrm{Thom}(BH_n(s); V_n)\bigr)$, where $\mathrm{Thom}(BH_n(s); V_n)$ is the Thom space of the universal bundle $\pi\colon V_n \to BH_n(s)$. Take the limit $n \to \infty$ after subtracting a trivial rank n bundle from V_n to obtain

$$\phi\colon MH(s) \longrightarrow \Sigma^s KO \tag{9.42}$$

out of the Thom spectrum associated to the stable *normal* structure H. For $s = 0$ this is the Atiyah-Bott-Shapiro (ABS) map [H, §6.1]. We rewrite in terms of the stable *tangential* structure H; see the comments following (7.6). That perp maneuver exchanges Pin^+ and Pin^-, which in Table (9.25) exchanges $s \leftrightarrow -s$. Therefore, (9.42) is a generalized ABS map

$$\phi\colon MTH(s) \longrightarrow \Sigma^{-s} KO. \tag{9.43}$$

In the complex case we obtain a generalized ABS map

$$\phi\colon MTH^c(s) \longrightarrow \Sigma^{-s} K. \tag{9.44}$$

9.2.3. *Twisted Dirac operators.* Next, following [LM, §II.7], we define twisted Dirac operators for the structure groups in Table (9.25). Suppose X is an n-dimensional Riemannian manifold equipped with an $H_n(s)$-structure $P \to X$. We assume given a connection on $P \to X$ compatible with the Levi-Civita connection on the orthonormal frame bundle. Use the embedding (9.28) to form the $\mathbb{Z}/2\mathbb{Z}$-graded spinor bundle

$$\mathcal{S}' := P \times_{H_n(s)} A_n(s) \longrightarrow X. \tag{9.45}$$

Clifford multiplication (9.29) defines a vector bundle map $T^*X \otimes \mathcal{S}' \longrightarrow \mathcal{S}'$, and as usual the Dirac operator $D\!\!\!/_X$ acts on smooth sections of $\mathcal{S}'$ as the covariant derivative followed by Clifford multiplication. The Dirac operator is odd and skew-adjoint. (See [38] for our conventions.) It commutes with the right $A_n(s)$-module structure on $\mathcal{S}'$, or equivalently with the left $A_n(s)^{\mathrm{op}}$-module structure.

[38] Our choice of A^{op} in (9.41), rather than A, is essentially a sign choice. We use a geometric model [AS] in which a class in $KO^m(X)$ is represented by a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over X that is a *left* module for Cliff_m equipped with a family of commuting odd *skew*-adjoint (Fredholm) operators.

There are topological and geometric indices of Dirac operators on compact manifolds. The topological index is defined using Fredholm operators [AS]. Namely, if X is closed, then $\not{D}_X$ extends to a Fredholm operator on Sobolev completions of the space of smooth sections of $\mathcal{S}'$. This construction works in families: from a fiber bundle $\mathcal{X} \to S$ of closed Riemannian n-manifolds with $H_n(s)$-structure we obtain a family of odd skew-adjoint Fredholm operators parametrized by S. Recalling that $A_n(s)^{\mathrm{op}}$ is Morita equivalent to $\mathrm{Cliff}_{-(n+s)}$, via (9.39), we deduce that this family of operators has a *topological* index that lies in $KO^{-(n+s)}(S)$. For $s = 0$ this reduces to the usual Clifford-linear Dirac operator definition of the topological index. The Atiyah-Singer index theorem equates this topological index with an analytic index. If S is a smooth manifold and $\mathcal{X} \to S$ a smooth family of Riemannian manifolds with $H_n(s)$-structure, then there is a *geometric* index that lies in the differential cohomology group $\widehat{KO}^{-(n+s)}(S)$; see [FL] for the differential complex K-theory version as well as the Atiyah-Singer theorem in this differential context.

Remark 9.46. For $s = \pm 1$ this discussion specializes to an effective approach to Dirac operators and index theory on unoriented manifolds with a $\mathrm{Pin}^{\pm}$-structure.

Remark 9.47. There is an analogous discussion in the complex case: replace $H \to H^c$ and $KO \to K$.

9.2.4. *Free fermion theories on Minkowski spacetime M^{n-1}.* As before we only treat the eight real fermionic symmetry groups. Fix a type s in Table (9.25). Let $H_{1,n-2}(s)$ be the Lorentz signature anti-Wick rotation of $H_{n-1}(s)$, as in (2.1). If $s = 0$, which is the basic case, then $H_{1,n-2}(s) = \mathrm{Spin}_{1,n-2}$ is the Lorentz spin group. The analog of (9.28) is an embedding (see (A.3) for $\mathrm{Cliff}_{p,q}$ conventions).

$$\iota\colon H_{1,n-2}(s) \longrightarrow \mathrm{Cliff}_{n-2,1} \otimes D(s) =: B_{n-1}(s), \tag{9.48}$$

and there is a Morita equivalence of superalgebras

$$B_{n-1}(s) \underset{\text{Morita}}{\approx} \mathrm{Cliff}_{+(n-3+s)}\,. \tag{9.49}$$

We use the conventions following (9.39) to define the Morita equivalence. The image of ι lies in the even subalgebra $B_{n-1}(s)^0 \subset B_{n-1}(s)$. A free fermionic field is specified by a real spinor representation of $H_{1,n-2}(s)$, which by definition is an *ungraded* real module $\mathbb{S}$ of $B_{n-1}(s)^0$. A spinor field is then a function $\psi\colon M^{n-1} \to \mathbb{S}$.

Remark 9.50. The CRT theorem, which is reviewed in Appendix A, implies that the free fermion theory has a larger Lie group $H_{1,n-2}(s)^{\beta} \supset H_{1,n-2}(s)$ of symmetries; the non-identity component acts antilinearly on the Hilbert space of states. Proposition A.15(3) implies that the embedding (9.48) extends to $H_{1,n-2}(s)^{\beta}$, and so $H_{1,n-2}(s)^{\beta}$ acts on the *real* vector space $\mathbb{S}$, consistent with Proposition A.20(2).

We quickly summarize special facts about a real spinor representation $\mathbb{S}$ of the Lorentz spin group $\mathrm{Spin}_{1,n-2}$; proofs may be found in [De, §6]. Fix a component C of timelike vectors $\xi \in \mathbb{R}^{1,n-2}$ with $|\xi|^2 > 0$. The first special property is the existence of *symmetric* $\mathrm{Spin}_{1,n-2}$-invariant maps

$$\Gamma\colon \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{R}^{1,n-2}. \tag{9.51}$$

If $\mathbb{S}$ is irreducible, then Γ is unique up to a real factor and nonzero Γ are definite. Choose Γ *positive* definite in the sense that $\Gamma(\psi,\psi) \in \overline{C}$ for all $\psi \in \mathbb{S}$. This fixes Γ up to a positive real factor. There are two isomorphism classes of real irreducible representations for $n-1 \equiv 2,6 \pmod 8$ and a unique irreducible in other cases. Let S_1, S_2 be representative irreducibles (in dimensions with a unique irreducible, set $S_2 = 0$); let Z be the commutant of the spin action, so $Z = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$; and fix positive definite Γ for S_1, S_2. A general real spinor representation $\mathbb{S}$ decomposes as

$$\mathbb{S} \cong W_1 \otimes_Z S_1 \;\oplus\; W_2 \otimes_Z S_2 \tag{9.52}$$

for right Z-modules W_1, W_2. Then positive definite pairings Γ in (9.51) correspond to positive definite hermitian forms on W_1, W_2. For each choice there is a unique compatible $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}_{n-2,1}$-module structure on $\mathbb{S}\oplus\mathbb{S}^*$, where $\mathbb{S}$ is in even degree and $\mathbb{S}^*$ in odd degree; in particular, the duality pairing $\mathbb{S}^*\otimes\mathbb{S} \to \mathbb{R}$ is $\mathrm{Spin}_{1,n-2}$-invariant. Conversely, if $\mathbb{S}^0\oplus\mathbb{S}^1$ is a $\mathrm{Cliff}_{n-2,1}$-module, then there is a duality pairing $\mathbb{S}^0\otimes\mathbb{S}^1 \to \mathbb{R}$ that makes the resulting symmetric form (9.51) positive definite. (Deligne proves this for simple modules in [De, (6.1)]; any module is a sum of simples and the argument applies to each summand.) Observe that Γ is a contractible choice.

The group $H_{1,n-2}(s)$ contains the spin group $\mathrm{Spin}_{1,n-2}$ as a subgroup and the quotient $Q_{n-1}(s)$ is compact and independent of n up to isomorphism. An irreducible real representation of $H_{1,n-2}(s)$ decomposes under the subgroup $\mathrm{Spin}_{1,n-2}$ as (9.52), and a central extension $\widehat{Q_{n-1}(s)}$ of $Q_{n-1}(s)$ acts on each W_i. A choice of $\widehat{Q_{n-1}(s)}$-invariant positive definite hermitian form on W_i yields a $H_{1,n-2}(s)$-invariant pairing (9.51), and then a $B_{n-1}(s)$-module $\mathbb{S}\oplus\mathbb{S}^*$. Conversely, every $B_{n-1}(s)$-module has this form.

Definition 9.53. The module $\mathbb{S}$ *admits a mass term* if there is a *nondegenerate* skew-symmetric $H_{1,n-2}(s)$-invariant bilinear form

$$m\colon \mathbb{S}\times\mathbb{S} \longrightarrow \mathbb{R}. \tag{9.54}$$

We call m the *mass form.*

Lemma 9.55. *$\mathbb{S}$ admits a mass term if and only if $\mathbb{S}\oplus\mathbb{S}^*$ extends to a super module of the superalgebra $B_{n-1}(s)[e]$, where e is odd, $e^2 = -1$, and e (graded) commutes with the Clifford generators of $B_{n-1}(s)$.*

If $s = 4$ the hypothesis is that e commutes with $D = \mathbb{H}$. As always, the commutation with Clifford generators obeys the Koszul sign rule.

Proof. Given a $B_{n-1}(s)[e]$-module structure on $\mathbb{S}\oplus\mathbb{S}^*$, define m by

$$m(s_1, s_2) = \langle Es_1, s_2\rangle, \qquad s_1, s_2 \in \mathbb{S}, \tag{9.56}$$

where $E\colon \mathbb{S}\to\mathbb{S}^*$ is part of the action of $e = \left(\begin{smallmatrix} 0 & -E^{-1} \\ E & 0 \end{smallmatrix}\right)$ on $\mathbb{S}\oplus\mathbb{S}^*$. Since $e^2 = -1$, the form m is nondegenerate, and since e (graded) commutes with $B_{n-1}(s)$, the form m is $H_{1,n-2}(s)$-invariant. We must prove that m is skew-symmetric. It suffices to assume that $\mathbb{S}\oplus\mathbb{S}^*$ is a simple $B_{n-1}(s)[e]$-module, since any module is a direct sum of simples. Then m is either symmetric or skew-symmetric.

Let $f \in \mathbb{R}^{1,n-2} \subset \mathrm{Cliff}_{n-2,1} \subset B_{n-1}(s)$ be the Clifford generator with $f^2 = -1$. So f is a timelike vector, and we choose it to lie in C. Write $f = \left(\begin{smallmatrix} 0 & -F^{-1} \\ F & 0 \end{smallmatrix}\right)$ for its action on $\mathbb{S} \oplus \mathbb{S}^*$. The positive definiteness of Γ implies that

$$(s_1, s_2)_{\mathbb{S}} := \langle Fs_1, s_2 \rangle, \qquad s_1, s_2 \in \mathbb{S}, \tag{9.57}$$

is a positive definite inner product on $\mathbb{S}$. The mass form is $m(s_1, s_2) = (F^{-1}Es_1, s_2)_{\mathbb{S}}$. Set $A = F^{-1}E \in \mathrm{End}(\mathbb{S})$. Since m is either symmetric or skew-symmetric, either $A^* = A$ or $A^* = -A$, where $*$ is with respect to the inner product (9.57). But $ef = -fe$ implies $A^2 = -\mathrm{id}_{\mathbb{S}}$, which rules out $A^* = A$ since A^*A is a nonnegative operator.

Conversely, let m be a mass form. Using the inner product (9.57) write

$$m(s_1, s_2) = (Bs_1, s_2)_{\mathbb{S}}, \qquad s_1, s_2 \in \mathbb{S}, \tag{9.58}$$

for an invertible skew-symmetric operator $B\colon \mathbb{S} \to \mathbb{S}$. Define $P = \sqrt{B^*B}$ and $A = P^{-1}B = BP^{-1}$. Then set $E = FA$ and let $e \in B_{n-1}(s)[e]$ act on $\mathbb{S}\oplus\mathbb{S}^*$ via $\left(\begin{smallmatrix} 0 & -E^{-1} \\ E & 0 \end{smallmatrix}\right)$, where as above $f \in B_{n-1}(s)[e]$ acts as $\left(\begin{smallmatrix} 0 & -F^{-1} \\ F & 0 \end{smallmatrix}\right)$. We must check that this determines a well-defined action of $B_{n-1}(s)[e]$. It is easy to verify that $e^2 = -\mathrm{id}_{\mathbb{S}\oplus\mathbb{S}^*}$, and $ef = -fe$ follows from $F^{-1}E = -E^{-1}F$, which in turn follows from $A = -A^{-1}$. For later use we observe the commutation relation $PF^{-1}E = F^{-1}EP$. Let[39] $c \in \mathbb{R}^{1,n-2} \oplus \mathbb{R}^{|s|} \subset B_{n-1}(s)$ be a vector perpendicular to f, and write its action on the module $\mathbb{S} \oplus \mathbb{S}^*$ as $\left(\begin{smallmatrix} 0 & \pm C^{-1} \\ C & 0 \end{smallmatrix}\right)$, the sign determined according as $c^2 = \pm 1$ in $B_{n-1}(s)$. It remains to show that $ec = -ce$ as operators on $\mathbb{S} \oplus \mathbb{S}^*$, or equivalently that

$$(EC^{-1})^2 = \pm\,\mathrm{id}_{\mathbb{S}}\,. \tag{9.59}$$

First, we use (9.56)–(9.58) to write

$$m(s_1, s_2) = \langle FBs_1, s_2 \rangle = \langle EPs_1, s_2 \rangle, \qquad s_1, s_2 \in \mathbb{S}. \tag{9.60}$$

Since $cf = -fc$ in $B_{n-1}(s)$ we have $C^{-1}F = \pm F^{-1}C$. Next, $cf \in H_{1,n-2}(s) \subset B_{n-1}(s)$ preserves the duality pairing $\mathbb{S}^* \otimes \mathbb{S} \to \mathbb{R}$, from which

$$\langle CF^{-1}s^*, C^{-1}Fs \rangle = \mp\langle s^*, s \rangle, \qquad s^* \in \mathbb{S}^*, \quad s \in \mathbb{S}. \tag{9.61}$$

Now since m is $H_{1,n-2}(s)$-invariant,

$$m(C^{-1}Fs_1, C^{-1}Fs_2) = m(s_1, s_2), \qquad s_1, s_2 \in \mathbb{S}. \tag{9.62}$$

Use the first expression in (9.60) together with the previous identities to conclude that $C^{-1}FB = -BC^{-1}F$. It follows that $C^{-1}F$ *commutes* with P. Then rewrite (9.62) using the second expression in (9.60) to deduce $FC^{-1}EPC^{-1}F = \mp EP$. Apply the foregoing to arrive at (9.59). $\square$

[39] We leave the reader to give the appropriate modification for $s = 4$.

There is an abelian group law on free fermion theories: direct sum of Clifford modules $\mathbb{S}$. The relationship [ABS, (11.4)], [A2, p. 383] between Clifford modules and K-theory yields the following.

Theorem 9.63. *The abelian group of relativistic free fermion field theories in dimension $n-1$ with type s, modulo those that admit a mass term, is isomorphic to*

$$KO^{n-3+s}(\mathrm{pt}) \cong \pi_{3-s-n}(KO). \tag{9.64}$$

Massive free fermions are anomaly-free; see [W1, §1.2] for a recent exposition. So the map from a free fermion theory to the isomorphism class of its anomaly factors through the quotient (9.64).

Remark 9.65. The nature of an irreducible real twisted spin representation $\mathbb{S}_0$ depends on the value of $t = n-1+s \pmod 8$. We ask if it is self-conjugate—if $\mathbb{S}_0^* \cong \mathbb{S}_0$—and if so whether the induced nondegenerate bilinear form $\mathbb{S}_0 \otimes \mathbb{S}_0 \to \mathbb{R}$ is symmetric ($\mathbb{S}_0$ orthogonal) or skew-symmetric ($\mathbb{S}_0$ symplectic). Also, the commutant is a real division algebra, so is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We list the types. If $t \equiv 3,4,7$, then $\mathbb{S}_0$ is symplectic, and the commutant is $\mathbb{R},\mathbb{C},\mathbb{H}$, respectively. If $t \equiv 0,1,5$, then $\mathbb{S}_0$ is orthogonal and the commutant is $\mathbb{C},\mathbb{R},\mathbb{H}$, respectively. If $t \equiv 2,6$, then there are two nonisomorphic irreducible spin representations that are each others dual; the commutant is $\mathbb{R},\mathbb{H}$, respectively. For $t \equiv 3,4,7$ the K-group (9.64) vanishes, as it must since there is always a mass term. For $t \equiv 0,1$ the K-group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$—the direct sum of two copies of the irreducible module admits a mass term—and for $t \equiv 5$ it vanishes. For $t \equiv 2,6$ the K-group is isomorphic to $\mathbb{Z}$. These are the cases for which the anomaly theory is not topological.

9.2.5. *The anomaly theory and its deformation class.* Our starting point is the $B_{n-1}(s)^0$-module $\mathbb{S}$ that defines a free fermion theory on Minkowski spacetime M^{n-1} in $(n-1)$ dimensions, as in §9.2.4. In this subsection we sketch the associated n-dimensional anomaly theory, an invertible field theory in n dimensions. (See [F3], [F4, §11] for expositions of anomalies from this viewpoint.) The anomaly theory is not necessarily topological, but it has a deformation class that is topological—or which can be regarded as a continuous invertible topological theory—and we propose a general formula for it. See [W1] for a discussion of many special cases from a more physical viewpoint.

First, the real representation $\mathbb{S}$ of $H_{1,n-2}(s)$ extends to a complex representation $\mathbb{S}_{\mathbb{C}}$ of the complexification $H_{1,n-2}(s)(\mathbb{C})$, which then restricts to a complex representation of $H_{n-1}(s)$. On a curved Riemannian manifold X^{n-1} with differential $H_{n-1}(s)$-structure $P \to X$ there is an associated complex vector bundle $P \times_{H_{n-1}(s)} \mathbb{S}_{\mathbb{C}} \to X$ whose sections are complex spinor fields. There is a Wick-rotated Dirac lagrangian, possibly with mass term, which is a skew-symmetric form on the space of spinor fields. If X is closed, then the fermionic functional integral over the space of spinor fields is the pfaffian of the Dirac operator on X. In a smooth family $\mathfrak{X} \to S$ the pfaffian is not a function, but rather is a section of the *pfaffian line bundle*

$$\mathrm{Pfaff}_{\mathfrak{X}/S} \longrightarrow S. \tag{9.66}$$

The bundle $\mathrm{Pfaff}_{\mathfrak{X}/S} \to S$ carries a canonical hermitian metric and compatible covariant derivative; it is $\mathbb{Z}/2\mathbb{Z}$-graded by the mod 2 index. It is part of the anomaly theory associated to the module $\mathbb{S}$.

We now give a conjectural description of the entire anomaly theory. Fix $k \in \mathbb{Z}^{\geqslant 0}$, which is the codimension in the n-dimensional theory. Let X^{n-k} be a closed $(n-k)$-dimensional Riemannian manifold with differential $H_{n-k}(s)$-structure. The universal Dirac operator (§9.2.3) acts on sections of a real vector bundle $\mathcal{S}' \to X$ of left $A_{n-k}(s)^{\mathrm{op}}$-modules, where $A_{n-k}(s) = \mathrm{Cliff}_{+(n-k)} \otimes D(s)$ is Morita equivalent to $\mathrm{Cliff}_{+(n-k+s)}$; see (9.39). Let $\underline{\mathbb{S} \oplus \mathbb{S}^*} \to X$ be the constant vector bundle with fiber $\mathbb{S} \oplus \mathbb{S}^*$. Then $\mathcal{S}' \otimes_{\mathbb{R}} (\underline{\mathbb{S} \oplus \mathbb{S}^*}) \to X$ is a real vector bundle of $\mathbb{Z}/2\mathbb{Z}$-graded $A_{n-k}(s)^{\mathrm{op}} \otimes B_{n-1}(s)$-modules. Our conventions in §9.2.2 give a definite Morita equivalence $A_{n-k}(s)^{\mathrm{op}} \otimes B_{n-1}(s) \underset{\text{Morita}}{\approx} \mathrm{Cliff}_{-(3-k)}$. For a family $\mathcal{X} \to S$ the geometric index of the Dirac operator[40] with coefficients in $\mathcal{S}' \otimes_{\mathbb{R}} (\underline{\mathbb{S} \oplus \mathbb{S}^*})$ lies in the differential cohomology group $\widehat{KO}^{-(3-k)}(S)$. Notice that it is independent of n and s. The anomaly picks off the lowest piece of the index via the canonical *Pfaffian homomorphism*

$$\mathrm{Pfaff} \colon \widehat{KO}^{-(3-k)}(S) \longrightarrow \widehat{I\mathbb{Z}(1)}^{1+k}(S). \tag{9.67}$$

The invariants in differential $I\mathbb{Z}(1)$ fit together into an invertible field theory; see [HS].

Example 9.68. For $k = 0$, so $\mathcal{X} \to S$ of relative dimension n, there is an isomorphism $\widehat{I\mathbb{Z}(1)}^1(S) \cong \widehat{H}^1(S) \cong \mathrm{Map}(S, \mathbb{T})$. The corresponding lowest piece of the index is the partition function $e^{2\pi i(\xi/2)}$ of the anomaly theory on an n-manifold, where ξ is the Atiyah-Patodi-Singer invariant [APS]. The division by 2 is due to the skew-symmetry of the Dirac form, the same division by 2 that passes from determinant to pfaffian. The equality between the exponentiated ξ-invariant and the integral in differential K-theory has only been proved in a basic case [Klo, O, BuS, FL] as far as we know.

Example 9.69. For $k = 1$, so $\mathcal{X} \to S$ of relative dimension $n-1$, the group $\widehat{I\mathbb{Z}(1)}^2(S)$ is isomorphic to the group of isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded hermitian line bundles $L \to S$ with compatible covariant derivative. For the anomaly theory that element is the pfaffian line bundle $\mathrm{Pfaff}_{\mathcal{X}/S} \to S$. The main theorem in [DF] is the gluing law in the *non-extended* invertible field theory in dimensions $n-1, n$ with partition function the exponentiated ξ-invariant.

The story continues to lower dimensional manifolds, on which the invariants are graded gerbes [Lo, Bu] and higher analogs.

The deformation class of an invertible field theory gotten from integration in differential cohomology is the underlying topological cohomology theory. In the background are techniques from [HS], which lead to the following.

Conjecture 9.70. *Fix a type s in Table (9.25) and a dimension n. Fix an isomorphism class of free fermion theories modulo those that admit a mass term, i.e., an element $[\mathbb{S}] \in \pi_{3-s-n}(KO)$. Then the deformation class of the n-dimensional anomaly theory is the homotopy class of the composition*

$$MTH(s) \xrightarrow{\phi \wedge [\mathbb{S}]} \Sigma^{-s} KO \wedge \Sigma^{-3+s+n} KO \xrightarrow{\mu} \Sigma^{n-3} KO \xrightarrow{\mathrm{Pfaff}} \Sigma^{n+1} I\mathbb{Z}(1), \tag{9.71}$$

where ϕ is the Atiyah-Bott-Shapiro map (9.43), μ is multiplication in the ring spectrum KO, and Pfaff *is the topological version of (9.67).*

[40]Some details of this construction appear in [FH2, Appendix]

There is a similar conjecture in the complex case (9.24) with the usual replacements $H \to H^c$ and $KO \to K$. We hope to address this conjecture in the future. We use it in our computations below.

Remark 9.72. If the group $\pi_{3-s-n}(KO)$ is finite, hence is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, then there is a reflection positive invertible *topological* field theory in the deformation class whose partition function is the mod 2 index. If the group is free cyclic, hence isomorphic to $\mathbb{Z}$, then the deformation class is represented by a reflection positive invertible field theory whose partition function is the exponentiated ξ-invariant of Atiyah-Patodi-Singer, the secondary invariant for a $\mathbb{Z}$-valued topological index in $n+1$ dimensions. This is the case in which there are local anomalies as well as global anomalies, and because of the shift s it happens in both even and odd dimensions.

9.2.6. *Massive free fermion theories.* In §9.2.5 we explained how a free fermion theory in $(n-1)$ dimensions has an associated n-dimensional invertible anomaly theory, and Conjecture 9.70 states its deformation class. Here we show that a second scenario leading to invertible n-dimensional theories has the same starting data. *This* is the scenario we apply in §9.3. Namely, begin with a *massive* free fermion theory in n dimensions. Because the theory has a mass gap its long-range physics is described by a field theory, which naturally is also n-dimensional. As argued in §5.4 we expect that theory to be, at least locally, the product of a topological theory and an invertible theory. But a massive free fermion theory has a unique vacuum on each spatial manifold—the vacuum in the fermionic Fock space—so in fact the long-range effective theory is invertible.

Remark 9.73. One must make choices to define the massive free fermion theory, and they can be summarized as a trivialization of an anomaly; see [F4, §11] for a general discussion. There is a canonical choice for each fixed mass, and it is implicitly used in the discussion below as well as in §9.3. However, when the mass is a not necessarily constant function then there is an anomaly; see [CFLS] for discussion and details.

As in previous sections fix a type s in Table (9.25) and let $H_{1,n-1}(s)$ be the Lorentz signature anti-Wick rotation of the corresponding group $H_n(s)$. In the notation of (9.48) there is an embedding $H_{1,n-1}(s) \hookrightarrow B_{n-1}(s)[e']$, where e' is an extra Clifford generator with $(e')^2 = +1$. By Lemma 9.55 spinor representations of $H_{1,n-1}(s)$ that admit a mass term are in bijection with super modules over the superalgebra $B_{n-1}(s)[e', e]$, where e is an extra Clifford generator with $e^2 = -1$. Observe that $B_{n-1}(s)[e', e]$ is Morita equivalent to $\mathrm{Cliff}_{+(n-3+s)}$. We speculate that

(9.74) (i) the resulting low energy theory is trivial if the $B_{n-1}(s)[e', e]$-module is extended to a module over the algebra $B_{n-1}(s)[e', e, f]$ with $f^2 = -1$.

The group of equivalence classes of $B_{n-1}(s)[e, f]$-modules modulo those that extend is the K-group (9.64). The Morita equivalence to massless theories in dimension $n-1$ and the vanishing of the anomaly for theories that admit a mass term are evidence in favor of (9.74). Furthermore, we speculate that

(9.75) (ii) the low energy theory is invertible and its deformation class is (9.71).

As some evidence supporting (ii) we point out that the partition function in special cases is computed in [W1, §2.1.6, §2.2.3, §3.4, §4.3, §5]. The universal part of the partition function of the low energy theory is an exponentiated ξ-invariant, as in Example 9.68.

9.3. Phases of topological insulators and topological superconductors

We apply Conjecture 8.37 to compute possible topological phases for each of the 10 fermionic symmetry types (9.24) and (9.25). We remind that the fermionic symmetry groups with $K = \mathbb{T}$ pertain to topological insulators; those with $K = \{\pm 1\}$ and $K = SU_2$ pertain to topological superconductors. The abelian group of topological phases—that is, the group of deformation classes of reflection positive invertible topological field theories with symmetry group H in n spacetime dimensions—is

$$TP_n(H) := [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]. \tag{9.76}$$

It may be computed from the homotopy groups[41] $\pi_q MTH$; see the universal property (5.17). As we are only interested in $n \leqslant 5$, we need only compute for $q \leqslant 6$, and for $q = 6$ we only need to know $\pi_6 MTH/\text{torsion}$, since that determines $\mathrm{Hom}(\pi_6 MTH, \mathbb{Z})$. The abelian group $TP_n(H)$ classifies deformation classes of *interacting* theories. The abelian group of deformation classes of massive (gapped) *free* fermion theories in n dimensions modulo those with trivial long-range effective theory is given by Lemma 9.55 and (9.74), at least conjecturally:

$$FF_n(H(s)) := \begin{cases} \pi_{3-s-n}(K) \;\;\;\cong [\Sigma^{-s} K, \Sigma^{n+1} I\mathbb{Z}(1)], & H^c(s) \text{ a \textit{complex} symmetry type;} \\ \pi_{3-s-n}(KO) \cong [\Sigma^{-s} KO, \Sigma^{n+1} I\mathbb{Z}(1)], & H(s) \text{ a \textit{real} symmetry type,} \end{cases} \tag{9.77}$$

where s is the parameter in (9.24) or (9.25). (See Remark 9.65 for an enumeration of the K-theory groups in the real case via the types of spin representation.) According to (9.75) and (9.71) the natural homomorphism

$$\Phi\colon FF_n(H) \longrightarrow TP_n(H) \tag{9.78}$$

from the group of deformation classes of free fermion theories to the group of all theories is the product with the ABS map (9.43). We compute Φ for each symmetry class.

The results are organized by internal symmetry group. Some of the bordism groups appear in the mathematics literature, whereas for the more exotic symmetry groups the computations are new. With the bordism groups in hand, the classification of interacting theories is an immediate consequence of Conjecture 8.37 and the universal property expressed in the short exact sequence (5.17). The free fermion computation is (9.64). The map (9.78) from massive free fermion phases to interacting phases does not follow from the rest—it must also be computed. We give a uniform treatment based on Lemma 9.27 and §9.2.2. Manifold generators and formulas for partition functions in 4 dimensions are worked out in [GPW].

[41] These are Thom's bordism groups, but for the perpendicular tangential structure on the stable normal bundle (see footnote [27]). Note that $\mathrm{Pin}^+/\mathrm{Pin}^-$ and $\mathrm{Pin}^{\tilde c+}/\mathrm{Pin}^{\tilde c-}$ exchange when passing from tangential to normal.

We check our computations against the condensed matter literature, where groups of SPT phases are deduced using very different arguments. There is almost total agreement, and in the few places we differ we use the homotopy computations to predict what should happen in the physics. The computations that we did not find in the physics literature should be considered predictions.

9.3.1. *Internal symmetry group $K = \{\pm 1\}$.* The symmetry groups are classified in Proposition 2.16. The low degree spin and pin bordism groups are described in a geometric way in [KT1]. The general structure of spin bordism is elucidated in [ABP1]. The computation of pin bordism groups in all degrees may be found in [ABP2] and [KT2].

Theorem 9.79. *The low degree bordism groups for $K = \{\pm 1\}$ are:*

(9.80)

q	$\pi_q MT\mathrm{Spin}$	$\pi_q MT\mathrm{Pin}^+$	$\pi_q MT\mathrm{Pin}^-$
6	0	0	$\mathbb{Z}/16\mathbb{Z}$
5	0	0	0
4	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	0
2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$
1	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Corollary 9.81 (Symmetry class D)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* Spin *are isomorphic to:*

(9.82)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Spin}) \xrightarrow{\Phi}$	$TP_n(\mathrm{Spin}) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	0	0	0
4	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
0	0	0	0	0

Literature Note. The groups $TP_1(\mathrm{Spin})$ and $TP_2(\mathrm{Spin})$ were computed by the "group super-cohomology theory" in [GW]; see Table II. That theory is a 2-stage Postnikov truncation of $I\mathbb{Z}(1)$, so in general only computes a subgroup of topological phases; it is the entire group in very low dimensions. The interacting classification $TP_n(\mathrm{Spin})$ appears in [QHZ]: see §IIA for $n = 3$, §IID for $n = 2$, and §IIE for $n = 1$. The group $TP_3(\mathrm{Spin})$ is discussed in [LV, §V A], but their restriction to "non-chiral" phases means that the E_8 phases that generate $TP_3(\mathrm{Spin})$ were not accounted for. All of the groups in the table, but not the map from free fermions to interacting theories, appear in [KTTW]. Those authors conjecture a cobordism classification of interacting fermionic SPT phases.

Proof. That Φ is an isomorphism in low dimensions follows since the ABS map $M\mathrm{Spin} \to KO$ induces an isomorphism on $\pi_{\leqslant 7}$. $\square$

In the next example we meet a nontrivial kernel of Φ, which is to say free fermion phases that become trivial when interactions are allowed.

Corollary 9.83 (Symmetry class DIII)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* Pin^+ *are isomorphic to:*

(9.84)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^+) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^+) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	0	0	0
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Literature Note. There are many arguments in the physics literature that 16 copies of the basic free fermion theory in 4 dimensions has a trivial phase once interactions are allowed, and that this does not occur with fewer copies. (As noted in Remark 8.41, the group $TP_4(\mathrm{Pin}^+)$ is torsion, hence *a priori* some multiple of the free theory necessarily becomes trivial once interactions are allowed.) A sample includes [K1, FCV, WS, MFCV, K4] and [W1, §4]. The interacting case in 3 dimensions is investigated in [W1, §3], and various aspects of the invertible field theory are described explicitly. It is also discussed in [LV, §V B], but the nonzero element is missed within the "K-formalism" as the authors explain. The groups $TP_n(\mathrm{Pin}^+)$ as computed here also appear in [KTTW, Table 2].

Corollary 9.85 (Symmetry class BDI)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* Pin^- *are isomorphic to:*

(9.86)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^-) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^-) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	0	0	0
4	0	0	0	0
3	0	0	0	0
2	$8\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z}$	0
1	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Literature Note. The breaking of the $\mathbb{Z}$ classification of free fermions in 2 spacetime dimensions to the $\mathbb{Z}/8\mathbb{Z}$ classification of interacting fermions is treated in [FK1, FK2, TPB, YWOX], and [W1, §5]. The groups $TP_n(\mathrm{Pin}^-)$, $n = 1, 2$, are computed by the group super-cohomology in [GW, Table II]. The vanishing of $TP_3(\mathrm{Pin}^-)$ is argued in [LV, §V B]. The groups $TP_n(\mathrm{Pin}^-)$ as computed here also appear in [KTTW, Table 2].

9.3.2. *Internal symmetry group $K = \mathbb{T}$.* The symmetry groups are classified in Proposition 9.4. Spin^c bordism groups are computed in [ABP1]; cf. [Sto, Chapter XI]. Pin^c bordism groups are computed in [BG]. The twisted Pin^c bordism computations are new.

Theorem 9.87. *The low degree bordism groups for $K = \mathbb{T}$ are:*

(9.88)

q	$\pi_q MT\mathrm{Spin}^c$	$\pi_q MT\mathrm{Pin}^c$	$\pi_q MT\mathrm{Pin}^{\tilde{c}+}$	$\pi_q MT\mathrm{Pin}^{\tilde{c}-}$
6	$\mathbb{Z}^2$	$\mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}$
5	0	0	0	0
4	$\mathbb{Z}^2$	$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
2	$\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Corollary 9.89 (Symmetry class A)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* Spin^c *are isomorphic to:*

(9.90)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Spin}^c) \xrightarrow{\Phi}$	$TP_n(\mathrm{Spin}^c) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
4	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
2	0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	0	0

Literature Note. The vanishing of the group $TP_4(\mathrm{Spin}^c)$ is mentioned in [WPS] at the end of Appendix F.

Corollary 9.91 (Symmetry class AIII)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* Pin^c *are isomorphic to:*

(9.92)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^c) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^c) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	0	0	0
4	$8\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	0	0
2	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Literature Note. The group $TP_4(\mathrm{Pin}^c)$ and the map from free fermions is discussed in [WS, §III]; see also [SeWi, §A.4] for the map from free fermions. The vanishing of the group $TP_3(\mathrm{Pin}^c)$ is discussed in [LV, §V D] as well as in the last paragraph of [W1, §3.7].

Corollary 9.93 (Symmetry class AII)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* $\mathrm{Pin}^{\tilde c+}$ *are isomorphic to:*

(9.94)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^{\tilde c+}) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^{\tilde c+}) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
4	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Literature Note. The $\mathbb{Z}/2\mathbb{Z}$ invariant of free fermion systems in 3 and 4 spacetime dimensions was introduced by Kane-Mele [KM] and Fu-Kane-Mele [FKM] and has been further studied in many papers. The interacting case in 4 dimensions is investigated in [WPS] and in 3 dimensions in [W1, §3.7]; their results agree with ours. The initial computation in [LV, §V C 2] of $TP_3(\mathrm{Pin}^{\tilde c+}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ was corrected in a subsequent erratum. The original argument in that paper asserts a $\mathbb{Z}/2\mathbb{Z}$ subgroup of bosonic phases, which would have symmetry group $O \ltimes \mathbb{T}$, as in (9.6). We computed that $\pi_3\bigl(M(O \ltimes \mathbb{T})\bigr) \cong \mathbb{Z}/2\mathbb{Z}$ and the natural projection $\mathrm{Pin}^{\tilde c+} \to O \ltimes \mathbb{T}$ induces the zero map on π_3 of the Thom spectra. This implies that the group of bosonic phases is $\mathbb{Z}/2\mathbb{Z}$, as claimed, but that the lift of that bosonic phase to a fermionic phase is trivial. This triviality of the pullback was not noticed initially; our homotopy theoretic methods give a systematic approach, and we encounter this issue again in Literature Note 9.3.2. The physical results in 4 dimensions were recounted in [M] at the end of §VI, where the question of agreement with a bordism computation was raised. This provided strong motivation for the computations in this section. We remark that the description of the partition function of some phases in terms of Stiefel-Whitney classes matches our bordism computations as well. Also, §4.7 of [W1] treats the invertible topological field theory in 4 dimensions defined by the free fermion theory, so only detects the image of Φ in $TP_4(\mathrm{Pin}^{\tilde c+})$.

Corollary 9.95 (Symmetry class AI)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* $\mathrm{Pin}^{\tilde c-}$ *are isomorphic to:*

(9.96)

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^{\tilde c-}) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^{\tilde c-}) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
4	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	0	0
2	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Literature Note. The group $TP_4(\mathrm{Pin}^{\tilde{c}-})$ is discussed in detail in the erratum to [WS]. The group $TP_3(\mathrm{Pin}^{\tilde{c}-})$ is asserted to be cyclic of order two in [LV, §V C 1] generated by a bosonic phase. The bosonic phase is the same one identified for the symmetry class AII—see the Literature Note following (9.94)—and again we compute that its lift to a fermionic phase with symmetry group $\mathrm{Pin}^{\tilde{c}-}$ vanishes, which explains the discrepancy.

9.3.3. *Internal symmetry group $K = SU_2$.* The symmetry groups G^0, G^+, G^- are defined and classified in Proposition 9.16.

Theorem 9.97. *The low degree bordism groups for $K = SU_2$ are:*

(9.98)

q	$\pi_q MTG^0$	$\pi_q MTG^+$	$\pi_q MTG^-$
6	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^4$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/16\mathbb{Z}$
5	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
4	$\mathbb{Z}^2$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$
3	0	0	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
1	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Corollary 9.99 (Symmetry class C)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group $G^0 = \mathrm{Spin} \times_{\{\pm 1\}} SU_2$ are isomorphic to:*

(9.100)

n	$\ker\Phi \longrightarrow$	$FF_n(G^0) \xrightarrow{\ \Phi\ }$	$TP_n(G^0) \longrightarrow$	$\operatorname{coker}\Phi$
5	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	0	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
2	0	0	0	0
1	0	0	0	0
0	0	0	0	0

Literature Note. That $TP_4(G^0) = 0$ was suggested in [WS] in the last paragraph preceding §V A.

Corollary 9.101 (Symmetry class CI)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group $G^+ = \mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$ are* *isomorphic to:*

(9.102)

n	$\ker\Phi$	$\longrightarrow FF_n(G^+)$	$\xrightarrow{\Phi} TP_n(G^+)$	$\longrightarrow \operatorname{coker}\Phi$
5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	0	0
2	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Our computations prove Φ maps the generator of $FF_4(G^+)$ to an element of order 4 in $TP_4(G^+)$.

Literature Note. Wang-Senthil [WS, §V] discusses the $n = 4$ case and conjecture the same group $TP_4(G^+) \cong \mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ that we compute; the map from free fermions also agrees.

Corollary 9.103 (Symmetry class CII)**.** *The groups of deformation classes of free fermion theories and of reflection positive invertible theories with symmetry group* $G^- = \mathrm{Pin}^-\times_{\{\pm1\}}SU_2$ *are isomorphic to:*

(9.104)

n	$\ker\Phi$	$\longrightarrow FF_n(G^-)$	$\xrightarrow{\Phi} TP_n(G^-)$	$\longrightarrow \operatorname{coker}\Phi$
5	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$
4	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$
3	0	0	0	0
2	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Literature Note. The 4-dimensional case is treated in [WS, §VI]; the answer they obtain for $TP_4(G^-)$ is $(\mathbb{Z}/2\mathbb{Z})^5$, which disagrees with the corresponding entry in (9.104), but it may be a different symmetry group they are considering. In any case, in the note following Corollary 9.93, we compute the group of bosonic phases with symmetry group $O\times_{\{\pm1\}}SU_2$ and find $(\mathbb{Z}/2\mathbb{Z})^4$, but the lift to fermionic phases kills a $(\mathbb{Z}/2\mathbb{Z})^2$ subgroup.

10. Computations

The computations in §9.3 involve finitely generated abelian groups having no odd torsion, so it suffices then to make them after completing at 2. This can be done using the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(H^*(MTH),\mathbb{Z}/2)\Rightarrow\pi_{t-s}MTH, \tag{10.1}$$

where A is the mod 2 Steenrod algebra, and, though not indicated in the notation, the homotopy groups have been completed at 2.

What makes this approach tractable is an identification[42] of the spectrum $\Sigma^s MTH(s)$ with

$$\begin{aligned} &M\,\mathrm{Spin} \wedge MTO_{|s|} && -3 \leqslant s \leqslant 0 \\ &M\,\mathrm{Spin} \wedge MO_{|s|} && 0 \leqslant s \leqslant 3 \\ &\Sigma M\,\mathrm{Spin} \wedge MSO_3 && s = 4, \end{aligned} \tag{10.2}$$

and in the complex case, of $\Sigma^s MTH^c(s)$ with

$$M\,\mathrm{Spin}^c \wedge MO_s \approx \Sigma^{-2} M\,\mathrm{Spin} \wedge MU_1 \wedge MO_s. \tag{10.3}$$

Let $A_1 \subset A$ be the sub algebra generated by Sq^1 and Sq^2. Anderson, Brown, and Peterson [ABP1] give an isomorphism

$$H^* M\,\mathrm{Spin} \approx A \bigotimes_{A_1} \{\mathbb{Z}/2 \oplus M\} \tag{10.4}$$

in which M is a graded A_1-module with $M_i = 0$ for $i < 8$. This means that for $t - s < 8$ one can identify the E_2-term of the Adams spectral sequence for[43] $\pi_* MTH(d)$ with

$$\begin{aligned} &\mathrm{Ext}^{s,t}_{A_1}(H^{*-d} MTO_{|d|}, \mathbb{Z}/2) && -3 \leqslant d \leqslant 0 \\ &\mathrm{Ext}^{s,t}_{A_1}(H^{*+d} MO_{|d|}, \mathbb{Z}/2) && -0 \leqslant d \leqslant 3 \\ &\mathrm{Ext}^{s,t}_{A_1}(H^{*+3} MSO_3, \mathbb{Z}/2) && d = 4\ , \end{aligned}$$

and of $\pi_* MTH^c(d)$ with

$$\mathrm{Ext}^{s,t}_{A_1}(H^{*+2+d} MU_1 \wedge MO_d, \mathbb{Z}/2) \qquad d = 0, 1\ .$$

These groups are computed by standard methods, and the computation, as well as the spectral sequences (which collapse) are described Figure 5 and give the results described in tables (9.80), (9.88), and (9.9).

The relationship with the free fermion theories is given by maps of spectra

$$MTH(s) \to \Sigma^{-s} KO \tag{10.5}$$

$$MTH^c(s) \to \Sigma^{-s} K \tag{10.6}$$

[42]Remark: Corollary 2.12 implies that for *any* symmetry type (H, ρ), the spectrum MTH is an MSpin-module.

[43]Here only we use the notation '$H(d)$' in place of '$H(s)$' to avoid the conflict with Adams' homological grading index 's'.

or, under the above identifications, maps

$$
\begin{aligned}
M\,\mathrm{Spin}\wedge MTO_{|s|} &\to KO \qquad -3\leqslant s\leqslant 0\\
M\,\mathrm{Spin}\wedge MO_{|s|} &\to KO \qquad 3\geqslant s\geqslant 0\\
\Sigma M\,\mathrm{Spin}\wedge MSO_3 &\to KO \qquad s=4\\
M\,\mathrm{Spin}^c\wedge MO_s &\to K \qquad s=0,1\ .
\end{aligned}
\tag{10.7}
$$

These are all maps of $M\,\mathrm{Spin}$ (or $M\,\mathrm{Spin}^c$) modules, in which KO and K are into $M\,\mathrm{Spin}$ and $M\,\mathrm{Spin}^c$-modules using the Atiyah-Bott-Shapiro orientation. They are therefore determined by their restrictions

$$
\begin{aligned}
MTO_{|s|} &\to KO \qquad -3\leqslant s\leqslant 0\\
MO_{|s|} &\to KO \qquad 3\geqslant s\geqslant 0\\
\Sigma MSO_3 &\to KO \qquad s=4\\
MO_s &\to K \qquad s=0,1\ .
\end{aligned}
\tag{10.8}
$$

These are described in Propositions 10.24, 10.27, and 10.35 below, and using them, the assertions about the maps in tables (9.82), (9.84), (9.86), (9.90), (9.92), (9.94), (9.96), (9.100), (9.102), and (9.104) can be verified. The details are summarized in the charts in Figure 5. The complex case is easier and left to the reader. See [C, BeC] for a detailed account of the computations.

For the identifications (10.2) and the maps (10.8) we begin with a uniform description of the groups $BH(\pm s)$ (for $s\neq 4$). Write

$$\mathbf{P}=K(\mathbb{Z}/2,1)\times K(\mathbb{Z}/2,2) \tag{10.9}$$

with the group structure

$$(x_1,x_2)*(y_1,y_2)=(x_1+y_1,x_2+y_2+x_1y_1) \tag{10.10}$$

in which $x_i,y_i\in H^i(-,\mathbb{Z}/2)$. With this choice the map

$$BO\xrightarrow{(w_1,w_2)}\mathbf{P} \tag{10.11}$$

is a group homomorphism.

For $s\geqslant 0$ define a map $B\widetilde{H}(s)\to BO$ by the homotopy pullback square

$$
\begin{tikzcd}
B\widetilde{H}(s) \arrow[r] \arrow[d] & BO_s \arrow[d, "{(w_1,w_2)}"] \\
BO \arrow[r, "{(w_1,w_2+w_1^2)}"'] & \mathbf{P}
\end{tikzcd}
\tag{10.12}
$$

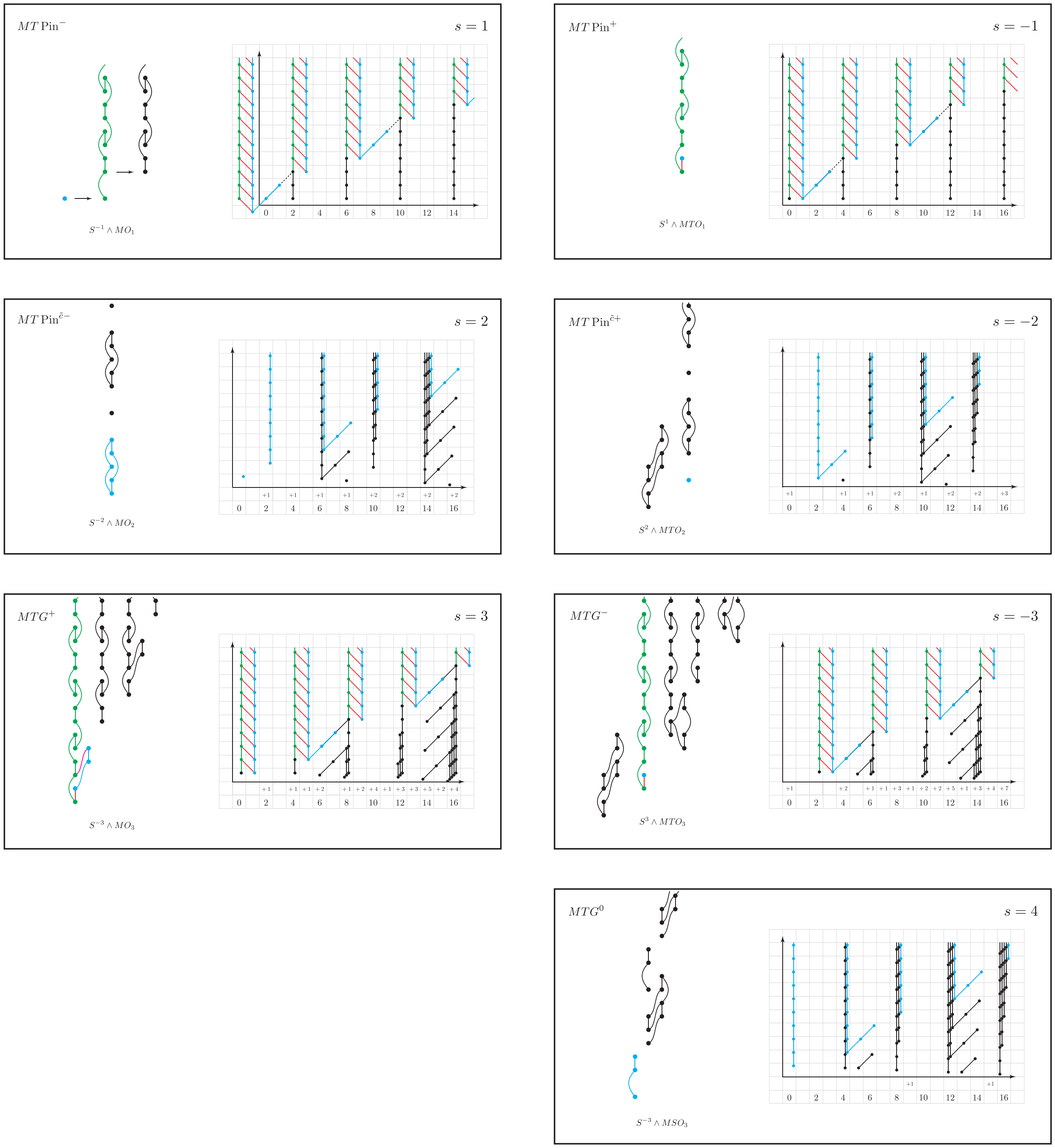

FIGURE 5. The Adams spectral sequences

and set $B\widetilde{H}(-s) \to BO$ to be the composite

$$B\widetilde{H}(s) \to BO \xrightarrow{-\,\mathrm{id}} BO. \tag{10.13}$$

The space $B\widetilde{H}(-s) \to BO$ fits into a homotopy pullback square

$$\begin{tikzcd} B\widetilde{H}(-s) \arrow[r] \arrow[d] & BO_s \arrow[d, "{(w_1,w_2)}"] \\ BO \arrow[r, "{(w_1,w_2)}"'] & \mathbf{P}\ . \end{tikzcd} \tag{10.14}$$

For later reference we note

Remark 10.15. The homotopy fiber of $B\widetilde{H}(\pm s) \to BO$, being the same as the homotopy fiber of $BO_s \to \mathbf{P}$ is

$$\begin{aligned} &B\operatorname{Spin}_s && s \geqslant 1 \\ &\mathbb{Z}/2 \times B\mathbb{Z}/2 && s = 0\ . \end{aligned} \tag{10.16}$$

For $-3 \leqslant s \leqslant 3$ one may identify $B\widetilde{H}(s) \to BO$ with $BH(s) \to BO$. The map $BH(4) \to BO$ fits into a homotopy pullback diagram

$$\begin{tikzcd} BH(4) \arrow[r] \arrow[d] & BSO_3 \arrow[d, "w_2"] \\ BO \arrow[r, "{(w_1,w_2)}"'] & \mathbf{P}\ . \end{tikzcd} \tag{10.17}$$

We leave the verification of these assertions to the reader.

With $s \geqslant 0$, the maps $B\widetilde{H}(\pm s) \to BO$ and $B\widetilde{H}(\pm s) \to BO_s$ can also be expressed in terms of the diagrams of homotopy pullback squares

$$\begin{tikzcd} B\widetilde{H}(s) \arrow[r] \arrow[d] & B\operatorname{Spin} \arrow[d] & \\ BO \times BO_s \arrow[r, "-\operatorname{id} - (V_s - s)"'] & BO \arrow[r, "{(w_1,w_2)}"'] & \mathbf{P} \end{tikzcd} \tag{10.18}$$

and

$$\begin{tikzcd} B\widetilde{H}(-s) \arrow[r] \arrow[d] & B\operatorname{Spin} \arrow[d] & \\ BO \times BO_s \arrow[r, "\operatorname{id} - (V_s - s)"'] & BO \arrow[r, "{(w_1,w_2)}"'] & \mathbf{P}\ . \end{tikzcd} \tag{10.19}$$

A map $X \to B\widetilde{H}(s)$ therefore classifies a pair (V, V_s) consisting of a stable vector bundle V (of virtual dimension 0), a vector bundle V_s of dimension s and a Spin structure on $-V - (V_s - s)$.

Writing $W = -V - (V_s - s)$, so that $V = -W - (V_s - s)$, one sees that $B\widetilde{H}(s)$ classifies pairs (W, V_s) in which W is a stable Spin bundle of virtual dimension zero. Thus $B\widetilde{H}(s) \to BO$ may be identified with the map

$$\begin{aligned} B\,\mathrm{Spin} \times BO_s &\to BO \\ (W, V_s) &\mapsto -W - (V_s - s). \end{aligned}$$

Similarly $B\widetilde{H}(-s) \to BO$ may be identified with

$$\begin{aligned} B\,\mathrm{Spin} \times BO_s &\to BO \\ (W, V_s) &\mapsto -W + (V_s - s), \end{aligned}$$

and $BH(4) \to BO$ with

$$B\,\mathrm{Spin} \times BSO_3 \to BO$$

via either of the maps

$$\begin{aligned} (W, V_3) &\mapsto -W + (V_3 - 3) \quad \text{or} \\ (W, V_3) &\mapsto -W - (V_3 - 3). \end{aligned}$$

This leads to the identifications

$$\begin{aligned} MT\widetilde{H}(s) &\approx \Sigma^{-s} M\,\mathrm{Spin} \wedge MO_s \\ MT\widetilde{H}(-s) &\approx \Sigma^{s} M\,\mathrm{Spin} \wedge MTO_s \\ MTH(4) &\approx \Sigma^{-3} M\,\mathrm{Spin} \wedge MSO(3) \\ &\approx \Sigma^{3} M\,\mathrm{Spin} \wedge MTSO(3). \end{aligned} \tag{10.20}$$

We define $B\widetilde{H}(\pm s)_n \to BO_n$ by the pullback square

$$\begin{tikzcd} B\widetilde{H}(\pm s)_n \arrow[r] \arrow[d] & B\widetilde{H}(\pm s) \arrow[d] \\ BO_n \arrow[r] & BO \end{tikzcd} \tag{10.21}$$

The space $B\widetilde{H}_n(s)$ classifies a pair (V_n, V_s) consisting of vector bundles of dimension n and s and a Spin structure on $-V_n - V_s$ (or, equivalently on $V_n + V_s$), while $B\widetilde{H}(-s)_n$ classifies pairs (V_n, V_s) a Spin structure on $-V_n + V_s$. For $s \geqslant 0$ there is therefore a pullback square

$$\begin{tikzcd} B\widetilde{H}_n(s) \arrow[r] \arrow[d] & B\,\mathrm{Spin}_{n+s} \arrow[d] \\ BO_n \times BO_s \arrow[r] & BO_{n+s} \end{tikzcd} \tag{10.22}$$

Proposition 10.23. *The space $B\widetilde{H}(\pm s)_n$ is the classifying space of a compact Lie group $\widetilde{H}(\pm s)_n$. The group $\widetilde{H}_n(s)$ is the stabilizer in* Spin_{n+s} *of a s-plane in $\mathbb{R}^{n+s}$.*

Proof. The first assertion is a consequence of the pullback square (10.21) and Remark 10.15. The second is immediate from (10.22) □

The construction of §9.2.2 leads to maps

$$MT\widetilde{H}(s) \to \Sigma^{-s}KO$$

and so, by (10.20), to

$$\begin{aligned} M\,\mathrm{Spin} \wedge MTO_s &\to KO \\ M\,\mathrm{Spin} \wedge MO_s &\to KO \\ \Sigma M\,\mathrm{Spin} \wedge MSO_3 &\to KO. \end{aligned}$$

These are maps of $M\,\mathrm{Spin}$-modules, so to describe them it suffices the restricted maps

$$\begin{aligned} MO_s &\to KO \\ MTO_s &\to KO \\ \Sigma MSO_3 &\to KO\ . \end{aligned}$$

Proposition 10.24. *Let $V \to BO_s$ be the universal vector bundle. The map $MO_s \to KO$ corresponds to the element of $KO(V, V-0)$ given by applying the difference bundle construction to*

$$\begin{aligned} V \times \Lambda^*(V) &\to \Lambda^*(V) \\ (v, \omega) &\mapsto v \wedge \omega. \end{aligned}$$

Proof. In the notation of 9.27, the algebra $A(s)$ is $\mathrm{Cliff}_{+s} \otimes \mathrm{Cliff}_{-s}$, so that A^{op} is also $\mathrm{Cliff}_{+s} \otimes \mathrm{Cliff}_{-s}$, but with left Clifford multiplication by $v \in \mathbb{R}^s$ sending $x \otimes y$ to $(-1)^{|x|} x \otimes vy$. The composed embedding $O_s \to H_s \to A^{\mathrm{op}}$ is the map

$$O_s \to \mathrm{Cliff}_{+s} \otimes \mathrm{Cliff}_{-s} \tag{10.25}$$

sending reflection through the hyperplane perpendicular to $v \in \mathbb{R}^s$ to $v \otimes v$.

Let $P \to BO_s$ be the universal principal O_s bundle. The K-theory class described in 9.2.2 is the difference bundle on $(V, V-0)$ associated to the O_s-equivariant "Clifford multiplication" map

$$\mathbb{R}^s \times (A^{\mathrm{op}} \underset{A^{\mathrm{op}}}{\otimes} M) \to (A^{\mathrm{op}} \underset{A^{\mathrm{op}}}{\otimes} M) \tag{10.26}$$

in which $M = \mathrm{Cliff}_s$ is the left A^{op} bimodule specified in §9.2.2, and giving the Morita equivalence of A^{op} with $\mathbb{R}$. Passing to associated bundles, this works out to be

$$\begin{aligned} V \times \mathrm{Cliff}(V) &\to \mathrm{Cliff}(V) \\ (v, \omega) &\mapsto (-1)^{|\omega|}\omega v. \end{aligned}$$

The anti-automorphism of $\mathrm{Cliff}(V)$ extending the identity map of V gives an isomorphism of this with

$$\begin{aligned} V \times \mathrm{Cliff}(V) &\to \mathrm{Cliff}(V) \\ (v, \omega) &\mapsto v\omega. \end{aligned}$$

The claim now follows from the standard way of "wrapping up" the complex $V \times \Lambda(V) \to \Lambda(V)$ using $v \pm \iota_v$ (see [ABS, Proposition 11.6] and the surrounding discussion for the complex case). ☐

Proposition 10.27. *The map $MTO_s \to KO$ factors as*

$$MTO_s \to (BO_s)_+ \to KO \tag{10.28}$$

in which the first map is the map

$$\mathrm{Thom}(BO_s, -V) \to \mathrm{Thom}\big(BO_s, (-V) \oplus V\big) \tag{10.29}$$

and the second corresponds to the trivial line bundle $1 \in KO^0(BO_s)$.

Proof. Write $\mathrm{Gr}_s(\mathbb{R}^{n+s})$ for the Grassmannian of s-planes in $(n+s)$-space, and let V_n and V_s be the universal n-plane and s-plane bundles. These bundles come equipped with a trivialization

$$V_s \oplus V_n \approx \mathrm{Gr}_s(\mathbb{R}^{n+s}) \times \mathbb{R}^{n+s}. \tag{10.30}$$

From the identification $\mathrm{Gr}_s(\mathbb{R}^{n+s}) = \mathrm{Spin}_{n+s}/H_n$ of Proposition 10.23 it follows that the bundle V_n comes equipped with an H_n-structure. The construction of 9.2.2 gives an element $U \in KO^{n+s}(\mathrm{Thom}(\mathrm{Gr}_s(\mathbb{R}^{n+s}), V_n))$. The assertion is that this pulled back from the canonical generator (the suspension of $1 \in KO^0(\mathrm{pt})$) of $\tilde{K}O^{n+s}(S^{n+s})$ along the map

$$\begin{aligned} \mathrm{Thom}(\mathrm{Gr}_s(\mathbb{R}^{n+s}); V_n) &\to \mathrm{Thom}(\mathrm{Gr}_s(\mathbb{R}^{n+s}); V_s \oplus V_n) \\ &\approx S^{n+s} \wedge \mathrm{Gr}_s(\mathbb{R}^{n+s})_+ \to S^{n+s}. \end{aligned}$$

This is immediate from the construction. The algebra $A(s)^{\mathrm{op}}$ is $\mathrm{Cliff}_{-s} \otimes \mathrm{Cliff}_{-n}$. The class U is the complex of left A-modules (which come as right A^{op}-modules) obtained by applying

$$\mathrm{Spin}_{s+n} \underset{H_n}{\times} (-) \tag{10.31}$$

to the the H_n-equivariant Clifford multiplication map

$$\mathbb{R}^n \times \mathrm{Cliff}_{-s} \otimes \mathrm{Cliff}_{-n} \to \mathrm{Cliff}_{-s} \otimes \mathrm{Cliff}_{-n}. \tag{10.32}$$

This map evidently extends to the Spin_{s+n} equivariant Clifford multiplication map

$$\mathbb{R}^s \oplus \mathbb{R}^n \times \mathrm{Cliff}_{-s} \otimes \mathrm{Cliff}_{-n} \to \mathrm{Cliff}_{-s} \otimes \mathrm{Cliff}_{-n} \tag{10.33}$$

so the class U is pulled back from the bundle of left A-modules on $(\mathbb{R}^{s+n}, \mathbb{R}^{s+n} - \{0\})$ obtained by applying

$$\mathrm{Spin}_{n+s} \underset{\mathrm{Spin}_{n+s}}{\times} (-) \tag{10.34}$$

to (10.33). This class represents the suspension of 1. $\square$

For the case $s = 4$ what we require is the following

Proposition 10.35. *The restriction of the map*

$$S^1 \wedge MSO_3 \to KO$$

to $S^4 \to KO$ is the generator of $\tilde{K}O^0(S^4)$.

Proof. From the diagram (10.17) a map to $BH(4)$ can be thought of as consisting of a stable vector bundle V, an oriented 3-plane bundle V_3 and a Spin-structure on $V \oplus V_3$. We map $BSO(4) \to BH(4)$ by taking V to corresponding to the defining representation and V_3 to be one of the two irreducible representations of dimension 3. The construction of §9.2.2 then leads to the bundle on $MSO(4)$ corresponding to the $SO(4)$-equivariant map

$$\mathbb{R}^4 \times N \to N$$

where N is the irreducible quaternionic Cliff_4-module specified in 9.2.2 with $SO(4)$-action from the embedding above. This restricts to the generator of $KO(\mathbb{R}^4, \mathbb{R}^4 - \{0\})$, by [ABS, Theorem 11.5]. $\square$

The two complex cases are handled similarly, using either the pullback squares

$$\begin{tikzcd} BH^c(s) \arrow[rr] \arrow[d] & & BO_s \arrow[d, "{(w_1, \beta w_2)}"] \\ BO \arrow[rr, "{(w_1, \beta w_2)}"'] & & K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \end{tikzcd} \tag{10.36}$$

for the identification

$$MTH^c(s) \approx \Sigma^{-s} M\,\mathrm{Spin}^c \wedge MO_s \tag{10.37}$$

or

$$\begin{tikzcd} HB \arrow[r] \arrow[d] & BO_s \times BU(1) \arrow[d, "(w_1, w_2+c_1)"] \\ BO \arrow[r, "(w_1,w_2)"'] & \mathbf{P} \end{tikzcd} \tag{10.38}$$

for the identification

$$MTH^c(s) \approx \Sigma^{-s-2} M\,\mathrm{Spin} \wedge MU_1 \wedge MO_s. \tag{10.39}$$

11. A topological spin-statistics theorem

In a relativistic quantum field theory the spin-statistics theorem states that the central element of the Lorentz spin group acts on the Hilbert space of the theory as $(-1)^F$, where F is the $\mathbb{Z}/2\mathbb{Z}$-valued grading operator;[44] see [SW, GJ, Kaz] for proofs in the framework of Wightman quantum field theory. In this section we prove the analog for reflection positive *non-extended* invertible topological theories. We do not know a version for fully extended theories. See [J-F] for another account of spin-statistics in topological field theory, but without positivity. A topological version of spin-statistics also enters into [GK] in the context of fermionic lattice models.

To formulate the statement we Wick rotate the central element of the Lorentz spin group to the central element of the Euclidean spin group. On a curved Riemannian spin manifold M, it acts as the *spin flip*: the identity diffeomorphism of M covered by the action of -1 on the spin frames. For a general symmetry group H_n it is the action of the distinguished central element $k_0 \in K$ in the internal symmetry group; see Corollary 2.12. Let $s\mathrm{Vect}_{\mathbb{C}}$ be the symmetric monoidal category of super vector spaces; the symmetry incorporates the Koszul sign rule. Recall the notation (Remark 2.39) for the domain of a not necessarily topological field theory.

Definition 11.1. Let $F\colon \mathrm{Bord}^{\nabla}_{\langle n-1,n\rangle}(H_n) \to s\mathrm{Vect}_{\mathbb{C}}$ be a field theory. We say F *satisfies spin-statistics* if it maps the spin flip on every $(n-1)$-manifold Y to the exponentiated grading operator $(-1)^F$ on the super vector space $F(Y)$.

Example 11.2. The spin-statistics connection fails without reflection positivity. Consider a 1-dimensional invertible topological theory F of spin manifolds with values in the category of $\mathbb{Z}/2\mathbb{Z}$-graded complex lines. There are 4 theories up to isomorphism:[45] $F(\mathrm{pt}_+)$ is either even or odd, the spin flip acts as either $+1$ or -1, and these choices are independent. Half of these theories satisfy spin statistics, and they are precisely the ones for which $F(S^1_{\mathrm{bounding}}) = +1$, which by Theorem 7.22 is the condition for stability, and so for reflection positivity.

[44] F vanishes on bosonic states and is the identity on fermionic states. In a *free* theory there is a dense Fock space of states with a finite number of particles on which F counts the number of fermionic particles modulo two. In any theory $(-1)^F$ is the grading operator on the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space of states.

[45] We compute using Theorem 5.23: $[\Sigma^1 MT\mathrm{Spin}_1, \Sigma^1 I\mathbb{C}^\times] \cong \mathrm{Hom}(\pi_1\Sigma^1 MT\mathrm{Spin}_1, \mathbb{C}^\times)$, the Thom spectrum $\Sigma^1 MT\mathrm{Spin}_1$ is the suspension spectrum of $\mathbb{RP}^\infty_+$, and $\pi_1 \mathbb{RP}^\infty_+ \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By contrast, $\pi_1 MT\mathrm{Spin} \cong \mathbb{Z}/2\mathbb{Z}$, hence $[MT\mathrm{Spin}, \Sigma^1 I\mathbb{C}^\times] \cong \mathbb{Z}/2\mathbb{Z}$, and so by Theorem 1.1 there are only two reflection positive theories.

Theorem 11.3. *Let* $F\colon \mathrm{Bord}_{\langle n-1,n\rangle}(H_n) \to s\mathrm{Line}_{\mathbb{C}}$ *be a reflection positive invertible topological field theory. Then* F *satisfies spin-statistics.*

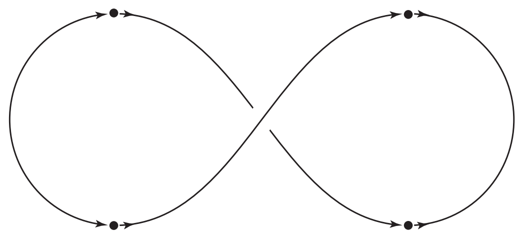

FIGURE 6. The composition $e_Y \circ \tau \circ c_Y$

Proof. We first treat the case $H_n = \mathrm{Spin}_n$. Let Y be a closed H_n-manifold and set $L = F(Y)$. Recall from §4.2 and Definition B.8 the coevaluation $c_Y\colon \varnothing^{n-1} \to Y \amalg Y^\vee$ and the evaluation $e_Y\colon Y^\vee \amalg Y \to \varnothing^{n-1}$. Let $\tau\colon Y \amalg Y^\vee \to Y^\vee \amalg Y$ be the symmetry map. The composition $e_Y \circ \tau \circ c_Y$ is $S^1_{\text{nonbounding}} \times Y$ (see Figure 6), and under F it maps to the composition $\mathbb{C} \to L \otimes L^* \to L^* \otimes L \to \mathbb{C}$. The Koszul sign rule in the symmetry gives

$$F(S^1_{\text{nonbounding}} \times Y) = \mathrm{tr}_s\, \mathrm{id}_L = \mathrm{tr}(-1)^F = \begin{cases} +1, & L \text{ even,} \\ -1, & L \text{ odd,} \end{cases} \tag{11.4}$$

where tr_s is the supertrace. The nonbounding circle is obtained by cutting the bounding circle at two points and regluing using the spin flip diffeomorphism of one of the points and the identity of the other. In other words, it is a triple composition of coevaluation, the indicated diffeomorphism, and evaluation. Take Cartesian product with Y and apply F to conclude that the ratio of (11.4) with $F(S^1_{\text{bounding}} \times Y)$ is the supertrace of the spin flip on Y, and since the spin flip has order two this ratio equals ± 1. But $S^1_{\text{bounding}} \times Y$ is the spin double of c_Y (see Example 4.31), so by reflection positivity we conclude from Proposition 4.26 that $F(S^1_{\text{bounding}} \times Y) = 1$, hence the spin flip acts as $(-1)^F$.

In the general case we use Corollary 2.12 to construct an $H_{k+\ell}$-structure on the Cartesian product of a Spin_k-manifold and an H_ℓ-manifold. Then the argument in the preceding paragraph goes through for Y an H_{n-1}-manifold and the same spin circles. □

Appendix A. The CRT theorem for general symmetry types

In §A.3 we take as our starting point a relativistic quantum field theory in Minkowski spacetime. Positivity of energy gives analytic correlation functions for which the Minkowski correlation functions are boundary values; Euclidean correlation functions are the restriction to a suitable subdomain. This leads to the CRT theorem (Theorem A.23),[46] and we outline Jost's proof [J], extended to general symmetry types. Recall that the symmetry group $H_{1,n-1}$ of a relativistic quantum field theory acts by time-orientation preserving transformations; see (2.1). The CRT theorem asserts that a larger symmetry group, including time-orientation reversing transformations, also acts; the time-reversing elements act antilinearly. There is a subtlety in the Lorentz spin central extensions, flagged in [GT],[47] which we elucidate and generalize to arbitrary symmetry types in §A.2. This subtlety is present even in the spin case without time-reversal symmetry. It implies, for example, that the ten Lorentz signature symmetry groups for free fermion theories (§9) embed in Clifford algebras, a fact which is implicit in §9.2.4. In this appendix we work in the framework of Wightman quantum field theory. One consequence of our discussion (Remark A.42) is a justification of the correspondence between the alternatives

$$\text{pin}^+\text{-structure} \quad \text{vs.} \quad \text{pin}^-\text{-structure} \tag{A.1}$$

in Wick-rotated field theory and the alternatives

$$T^2 = (-1)^F \quad \text{vs.} \quad T^2 = 1 \tag{A.2}$$

for the action of time-reversal T on the Hilbert space $\mathcal{H}$ of states. We begin in §A.1 with a review of pin groups and pin manifolds, which also serves to fix some conventions about Clifford algebras..

We assume the dimension of spacetime is $n \geqslant 3$.

A.1. Pin groups and pin manifolds

References for this section include [ABS, BDGK, KT1]. While we assume the dimension $n \geqslant 3$, with minor modifications the discussion goes through for $n = 1, 2$ as well.

A.1.1. *Pin groups and Clifford algebras.* We take Lorentz signature as our starting point. Let $\mathbb{R}^{1,n-1}$ be the standard vector space with basis $e_0, e_1, \dots, e_{n-1}$ and the standard inner product: $\langle e_0, e_0\rangle = 1$, $\langle e_i, e_i\rangle = -1$, $i = 1, \dots, n-1$, and $\langle e_\mu, e_\nu\rangle = 0$, $\mu \neq \nu$. Its isometry group is the orthogonal group $O_{1,n-1}$. The group of components of $O_{1,n-1}$ is isomorphic to $\{\pm1\} \times \{\pm1\}$; an orthogonal transformation either preserves or exchanges the two components of timelike vectors ξ (vectors with $\langle \xi, \xi\rangle > 0$), and it either preserves or reverses the orientation of any spacelike codimension 1 subspace. In terms of the block matrix $\left(\begin{smallmatrix} a & \alpha \\ \eta & A \end{smallmatrix}\right) \in O_{1,n-1}$ the first question is the sign of the real number a and the second the determinant of the $(n-1)\times(n-1)$ matrix A. The identity component of $O_{1,n-1}$ has a unique (up to isomorphism) connected double covering group $\mathrm{Spin}_{1,n-1}$.

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[46] It is usually called the CPT theorem, but we follow the nomenclature in [W1], which is more appropriate for arbitrary dimensions: the 'P' in 'CPT' is understood to be the parity transformation that acts as -1 on space and so is orientation-preserving if the dimension of spacetime is odd; by contrast, the 'R' in 'CRT' denotes reflection in a single spatial direction and is orientation-reversing in all dimensions. The 'C' is best read as 'complex conjugation'.

[47] The setting of [GT] is "formal field theory" as opposed to that in the Wightman axioms.

It is contained in the even subalgebra of a real Clifford algebra, and there are two equally good choices for the signs:

$$\begin{aligned} &\mathrm{Cliff}_{1,n-1}: \quad e_0^2=+1, \quad e_i^2=-1, \quad i=1,\dots,n-1, \\ &\mathrm{Cliff}_{n-1,1}: \quad e_0^2=-1, \quad e_i^2=+1, \quad i=1,\dots,n-1. \end{aligned} \tag{A.3}$$

The Lorentz orthogonal group $O_{1,n-1}$ has a complexification $O_n(\mathbb{C})$ consisting of complex $n\times n$ orthogonal matrices. This complex group has two components distinguished by the determinant, which is ±1. The identity component $SO_n(\mathbb{C})$ has a subgroup that is the union of the two components of $O_{1,n-1}$ of matrices with determinant 1. Also, $SO_n(\mathbb{C})$ has a unique connected double covering group $\mathrm{Spin}_n(\mathbb{C})$, which contains $\mathrm{Spin}_{1,n-1}$ as a subgroup. The complex Lie group $O_n(\mathbb{C})$ deformation retracts onto its maximal compact subgroup O_n, which is the group of orthogonal symmetries of the real vector space spanned by

$$f_0=i\,e_0,\ f_1=e_1,\ \dots,\ f_{n-1}=e_{n-1} \tag{A.4}$$

with its inherited negative definite inner product. Here i is a choice of complex number with $i^2=-1$. The identity component SO_n has a unique connected double covering group Spin_n, which is the maximal compact subgroup of $\mathrm{Spin}_n(\mathbb{C})$. It is contained in the even subalgebra of a real Clifford algebra, and again there are two equally good choices for the signs:

$$\begin{aligned} &\mathrm{Cliff}_{-n}: \quad f_\mu^2=-1, \quad \mu=0,\dots,n-1, \\ &\mathrm{Cliff}_{+n}: \quad f_\mu^2=+1, \quad \mu=0,\dots,n-1. \end{aligned} \tag{A.5}$$

The four-component orthogonal group $O_{1,n-1}$ has many double cover groups with identity component $\mathrm{Spin}_{1,n-1}$; we discuss two of them in §A.2. In the remainder of this subsection we focus on the two-component compact orthogonal group O_n, which has two double covers $\mathrm{Pin}_n^\pm$ with identity component Spin_n. Each is a subgroup of invertible elements in a real Clifford algebra: $\mathrm{Pin}_n^\pm\subset\mathrm{Cliff}_{\pm n}$. They are group extensions

$$1\longrightarrow\{\pm1\}\longrightarrow\mathrm{Pin}_n^\pm\longrightarrow O_n\longrightarrow 1 \tag{A.6}$$

Observe that $\mathrm{Pin}_1^+\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Pin}_1^-\cong\mathbb{Z}/4\mathbb{Z}$.

A.1.2. *Pin manifolds.* A Riemannian manifold X has a principal O_n-bundle of frames $\mathcal{B}_O(X)\to X$ whose points represent orthonormal bases of the tangent spaces to X. The following is a special case of Definition 2.29.

Definition A.7. A pin$^\pm$-*structure* on X is a pair (P,θ) consisting of a principal $\mathrm{Pin}_n^\pm$-bundle $P\to X$ and an isomorphism $\mathcal{B}_O(X)\xrightarrow{\theta}P/\{\pm1\}$ of principal O_n-bundles.

Pin structures, as spin structures, do not necessarily exist. The obstructions are given by Stiefel-Whitney classes: a pin^+-structure exists on X if and only if[48] $w_2(X) = 0$ and a pin^--structure exists if and only if $(w_1^2 + w_2)(X) = 0$. Double covers of X act on pin structures as follows. If $Q \to X$ is a double cover, viewed as a principal $\{\pm 1\}$-bundle, and (P, θ) is a $\text{pin}^{\pm}$-structure, then $Q \times_X P \to X$ is a principal $(\{\pm 1\} \times \text{Pin}^{\pm}_n)$-bundle. The $\text{Pin}^{\pm}_n$-bundle $(Q \times_X P)/\{\pm 1\} \to X$ associated to the homomorphism $\{\pm 1\} \times \text{Pin}^{\pm}_n \to \text{Pin}^{\pm}_n$ (multiplication in $\text{Pin}^{\pm}_n$ with first argument restricted to the central subgroup in (A.6)), along with a canonical isomorphism of underlying O_n-bundles obtained from θ, is a $\text{pin}^{\pm}$-structure. The set of isomorphism classes of $\text{pin}^{\pm}$-structures, if nonempty, is a torsor over the abelian group $H^1(X; \mathbb{Z}/2\mathbb{Z})$; that is, this group acts freely and transitively on the set of isomorphism classes. There is a canonical double cover of X, the orientation double cover, whose points represent orientations of the tangent spaces to X.

Definition A.8. The w_1-*involution* is the action of the orientation double cover on pin structures.

Recall that the equivalence class of the orientation double cover is $w_1(X) \in H^1(X; \mathbb{Z}/2\mathbb{Z})$.

Remark A.9. Let $\hat{\alpha}$ be the automorphism of $\text{Pin}^{\pm}_n$ that is the identity on Spin_n and multiplication by the central element -1 on the complement; it covers the identity automorphism of O_n. An alternative description of the w_1-transform $(P^{\sim}, \theta)$ of a pin-structure (P, θ) is the same manifold P with the same map θ, but with the $\text{Pin}^{\pm}_n$-action altered by precomposition with $\hat{\alpha}$. (To see this, write the orientation double cover as P/Spin_n and construct the isomorphism of $\text{Pin}^{\pm}_n$-bundles

$$P/\operatorname{Spin}_n \times P \longrightarrow P^{\sim} \tag{A.10}$$

which maps $(\mathfrak{o}, p) \mapsto p$ if $p \in \mathfrak{o}$ and $(\mathfrak{o}, p) \mapsto p \cdot (-1)$ if $p \notin \mathfrak{o}$. Here $\mathfrak{o} \subset P$ is a Spin_n-orbit.)

A.2. Lorentz signature symmetry groups

This section is an exposition and elaboration of ideas in [GT]. We continue with the hypothesis $n \geqslant 3$, largely for convenience of exposition; with minor modifications the discussion goes through for $n = 1, 2$ as well.

A.2.1. *Complex pin groups.* The complex orthogonal group $O_n(\mathbb{C})$ has two components. The identity component $SO_n(\mathbb{C}) \subset O_n(\mathbb{C})$ has a unique isomorphism class of nontrivial double cover groups, any representative of which is called $\text{Spin}_n(\mathbb{C})$.

Proposition A.11. *There are unique complex Lie groups* $\text{Pin}^{\pm}_n(\mathbb{C})$ *with identity component* $\text{Spin}_n(\mathbb{C})$*, which double cover* $O_n(\mathbb{C})$*, and which contain* $\text{Pin}^{\pm}_n$ *as maximal compact subgroups. Furthermore, any complex Lie group that double covers* $O_n(\mathbb{C})$ *and has identity component isomorphic to* $\text{Spin}_n(\mathbb{C})$ *is isomorphic to either* $\text{Pin}^+_n(\mathbb{C})$ *or* $\text{Pin}^-_n(\mathbb{C})$*.*

Remark A.12. We warn that $\text{Pin}^{\pm}_n(\mathbb{C})$ are complex Lie groups, whereas the group 'Pin^c_n', which is defined in [ABS, §3] as a subgroup of the complex Clifford algebra, is a compact real Lie group; it and twisted variants appear in §9.

[48]These are Stiefel-Whitney classes of the *tangent* bundle: $w_q(X) = w_q(TX)$. There is a potential confusion with Stiefel-Whitney classes of the *stable normal* bundle, which is what appears naturally in bordism theory.

Proof. Up to isomorphism there is a unique double covering space $X \to O_n(\mathbb{C})$ whose inverse image over each component of $O_n(\mathbb{C})$ is connected. The restriction over $O_n \subset O_n(\mathbb{C})$ is isomorphic as a double covering space to $\mathrm{Pin}^{\pm}_n \to O_n$. Choose an isomorphism of double covers and transport the group structure, then extend the group structure on the identity component Spin_n to that of $\mathrm{Spin}_n(\mathbb{C})$ on the entire component $X_+ \subset X$ containing Spin_n. Now use covering space theory to extend the group structure to all of X. For example, setting $X_- = X\backslash X_+$, lift the map $X_+ \times X_- \to O_n(\mathbb{C})_-$ to a map $X_+ \times X_- \to X_-$ using basepoints in the compact pin group. In fact, the extension of the group structure is determined by the square of a lift of a single hyperplane reflection, for which there are two choices, and this implies the last assertion. ☐

A.2.2. *Double covers of Lorentz isometry groups.* The two-component group $SO_{1,n-1} \subset O_{1,n-1}$ consists of isometries that preserve the overall orientation of $\mathbb{R}^{1,n-1}$. Let $\mu_m \subset \mathbb{C}^\times$ be the group of m^{th} roots of unity. Using the diagram

$$\begin{tikzcd} \mathrm{Spin}_n(\mathbb{C}) \arrow[rr, hook] \arrow[dr, "\pi_2"'] & & \mathrm{Spin}_n(\mathbb{C}) \times_{\mu_2} \mu_4 \arrow[dl, "\pi_4"] \\ & SO_n(\mathbb{C}) & \end{tikzcd} \tag{A.13}$$

set $\widetilde{SO}^{\alpha}_{1,n-1} = \pi_2^{-1}(SO_{1,n-1})$, and let $\widetilde{SO}^{\beta}_{1,n-1} \subset \mathrm{Spin}_n(\mathbb{C}) \times_{\mu_2} \mu_4$ be the union of $\mathrm{Spin}_{1,n-1}$ and the complement of $\pi_2^{-1}(SO^{\downarrow}_{1,n-1})$ in $\pi_4^{-1}(SO^{\downarrow}_{1,n-1})$, where $SO^{\downarrow}_{1,n-1}$ is the non-identity component of $SO_{1,n-1}$. For the pin groups let $\widetilde{\mathrm{O}}^{\alpha}_{n-1,1}$ and $\widetilde{\mathrm{O}}^{\alpha}_{1,n-1}$ be the inverse image of $O_{1,n-1} \subset O_n(\mathbb{C})$ under the double cover homomorphisms $\mathrm{Pin}^+_n(\mathbb{C}) \to O_n(\mathbb{C})$ and $\mathrm{Pin}^-_n(\mathbb{C}) \to O_n(\mathbb{C})$, respectively. Finally, using the diagram

$$\begin{tikzcd} \mathrm{Pin}^{\pm}_n(\mathbb{C}) \arrow[rr, hook] \arrow[dr, "\pi_2"'] & & \mathrm{Pin}^{\pm}_n(\mathbb{C}) \times_{\mu_2} \mu_4 \arrow[dl, "\pi_4"] \\ & O_n(\mathbb{C}) & \end{tikzcd} \tag{A.14}$$

let $\widetilde{\mathrm{O}}^{\beta}_{n-1,1}$ and $\widetilde{\mathrm{O}}^{\beta}_{1,n-1}$ be the union of $\pi_2^{-1}(O^{\uparrow}_{1,n-1})$ and the complement of $\pi_2^{-1}(O^{\downarrow}_{1,n-1})$ in $\pi_4^{-1}(O^{\downarrow}_{1,n-1})$, where we use the $+$ and $-$ pin groups, respectively. Here $O^{\downarrow}_{1,n-1}$ is the complement of $O^{\uparrow}_{1,n-1} \subset O_{1,n-1}$, the components of time-reversing linear isometries.

Proposition A.15.

(*1*) *Every double cover group of $SO_{1,n-1}$ whose identity component is isomorphic to* $\mathrm{Spin}_{1,n-1}$ *is isomorphic to either $\widetilde{SO}^{\alpha}_{1,n-1}$ or $\widetilde{SO}^{\beta}_{1,n-1}$.*

(*2*) *The double cover group $\widetilde{SO}^{\beta}_{1,n-1}$ of $SO_{1,n-1}$ is a subgroup of the even subalgebras of* $\mathrm{Cliff}_{n-1,1}$ *and* $\mathrm{Cliff}_{1,n-1}$.

(*3*) *The double cover groups $\widetilde{O}^{\beta}_{n-1,1}$ and $\widetilde{O}^{\beta}_{1,n-1}$ of $O_{1,n-1}$ are subgroups of* $\mathrm{Cliff}_{n-1,1}$ *and* $\mathrm{Cliff}_{1,n-1}$, *respectively.*

Summary: the α-double covers are subgroups of complex (s)pin groups; the β-double covers are subgroups of Lorentz signature Clifford algebras.

Proof. For (1), let $g \in SO^{\downarrow}_{1,n-1}$ be the diagonal matrix $\mathrm{diag}(-1,-1,+1,\dots,+1)$. Then the square of a lift of g to a double cover of $SO_{1,n-1}$ has square the identity $+1$ or the central element -1 of $\mathrm{Spin}_{1,n-1}$. By covering space theory, as in the proof of Proposition A.11, we can deduce that this dichotomy determines the group structure on the double cover.

The element e_0e_1 in the Clifford algebra (of either signature $(n-1,1)$ or $(1,n-1)$) acts on $\mathbb{R}^{1,n-1}$ as g and squares to $+1$. On the other hand, g lies in $SO_{1,n-1} \cap SO_n \subset SO_n(\mathbb{C})$, so a lift of g to $\mathrm{Spin}_n(\mathbb{C})$ lies in the compact spin group Spin_n where it squares to -1, as we compute in the Clifford algebra $\mathrm{Cliff}_{\pm n}$. This is the essential point in the proof of (2).

As for (3) there are double covers $\mathrm{Pin}_{n-1,1} \subset \mathrm{Cliff}_{n-1,1}$ and $\mathrm{Pin}_{1,n-1} \subset \mathrm{Cliff}_{1,n-1}$ of $O_{1,n-1}$, as defined in [ABS], [LM, §1.2]. By (2) the restriction over $SO_{1,n-1}$ is isomorphic to $\widetilde{SO}^{\beta}_{1,n-1}$. The element $\mathrm{diag}(-1,+1,\dots,+1) \in O^{\downarrow}_{1,n-1}$ lifts to e_0 in the Clifford algebra, and its square is given in (A.3). Arguing as above with the compact pin groups we deduce that this is opposite the square of a lift in the corresponding complex pin group. This is the new step in proving the isomorphisms

$$\begin{aligned} \mathrm{Pin}_{n-1,1} &\cong \widetilde{\mathrm{O}}^{\beta}_{n-1,1} \\ \mathrm{Pin}_{1,n-1} &\cong \widetilde{\mathrm{O}}^{\beta}_{1,n-1} \end{aligned} \tag{A.16}$$

∎

A.2.3. *General Lorentz signature symmetry groups.* There are analogs of the α and β-extensions of the Lorentz signature vector symmetry group $H_{1,n-1}$ for an arbitrary symmetry type, which as in §2.1 is the quotient of the full symmetry group of a relativistic quantum field theory by translations. It comes equipped with a homomorphism $\rho_n \colon H_{1,n-1} \to O^{\uparrow}_{1,n-1}$. We use the Structure Theorem 2.7, and in particular (2.8), (2.10), and (2.11) to define the α and β-extensions $H^{\alpha/\beta}_{1,n-1}$ of $H_{1,n-1}$ simultaneously. Set

$$SH^{\alpha/\beta}_{1,n-1} \cong \widetilde{SO}^{\alpha/\beta}_{1,n-1} \times K \,/\, \langle(-1,k_0)\rangle. \tag{A.17}$$

If the image of ρ_n is $SO^{\uparrow}_{1,n-1}$, set $H^{\alpha/\beta}_{1,n-1} = SH^{\alpha/\beta}_{1,n-1}$. If ρ_n is surjective, define $\widetilde{H}^{\alpha/\beta}_{1,n-1}$ by pullback

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \widetilde{H}^{\alpha/\beta}_{1,n-1} \arrow[r] \arrow[d, two heads] & \widetilde{\mathrm{O}}^{\alpha/\beta}_{n-1,1} \arrow[r] \arrow[d, two heads] & 1 \\ 1 \arrow[r] & K \arrow[r] & J \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{A.18}$$

where the right vertical map is the determinant homomorphism. Then let

$$H^{\alpha/\beta}_{1,n-1} \cong \widetilde{H}^{\alpha/\beta}_{1,n-1} \,/\, \langle(-1,k_0)\rangle. \tag{A.19}$$

We observe that $H^{\alpha}_{1,n-1}$ is a real subgroup of the complex Lie group $H_n(\mathbb{C})$, the inverse image of $O_{1,n-1}$ under the homomorphism $\rho_n \colon H_n(\mathbb{C}) \to O_n(\mathbb{C})$ in (2.2). Also, our notation is set up so that $\mathrm{Spin}^{\alpha/\beta}_{1,n-1} \cong \widetilde{SO}^{\alpha/\beta}_{1,n-1}$.

A.2.4. *Extensions of real representations.* As just remarked, the α-extension sits as a subgroup of the complex symmetry group. One key feature of the β-extension is the following.

Proposition A.20. *Let $R = R^0 \oplus R^1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded real representation of $H_{1,n-1}$ such that $k_0 \in K \subset H_{1,n-1}$ acts as the grading operator. Let $R_{\mathbb{C}} := R \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification, which carries an action of the complex Lie group $H_n(\mathbb{C})$, hence of the subgroup $H^{\alpha}_{1,n-1}$.*

(1) If $h \in H^{\alpha}_{1,n-1} \backslash H_{1,n-1}$, then $h(R^0) = R^0$ and $h(R^1) = \sqrt{-1}R^1$.

(2) There is a canonical extension of the action of $H_{1,n-1}$ on R to an action of $H^{\beta}_{1,n-1}$.

All Lie groups that appear are ungraded, so act by even transformations of R. The conclusion is that the β-extension acts on *real* representations of $H_{1,n-1}$.

Proof. For (1) it suffices to check for a single element $h \in H^{\alpha}_{1,n-1} \backslash H_{1,n-1}$. By Corollary 2.12, anti-Wick rotated to Lorentz signature, we choose h to be the image in $H^{\alpha}_{1,n-1}$ of a lift of

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SO_{1,1} \cap SO_2 \subset SO_2(\mathbb{C}) \subset SO_n(\mathbb{C}) \tag{A.21}$$

to $\mathrm{Spin}_n(\mathbb{C})$. In the compact spin group $\mathrm{Spin}_2 \subset \mathrm{Spin}_2(\mathbb{C})$ the element h is represented as f_0f_1 and is connected to the identity by the curve $\cos t/2 + \sin t/2\, f_0f_1$, $0 \leqslant t \leqslant \pi$, where we embed $\mathrm{Spin}_2 \subset \mathrm{Cliff}_{-2}$; see (A.5). Complex conjugation, defined so that $\mathrm{Spin}_{1,1} \subset \mathrm{Spin}_2(\mathbb{C})$ is real, takes this curve to the curve $\cos t/2 - \sin t/2\, f_0f_1$, $0 \leqslant t \leqslant \pi$ in $\mathrm{Spin}_2 \subset \mathrm{Spin}_2(\mathbb{C})$. In particular, the complex conjugate of f_0f_1 is $-f_0f_1$. Since -1 maps to k_0 and acts as the grading operator, f_0f_1 is a real operator on $R^0_{\mathbb{C}}$ and a purely imaginary operator on $R^1_{\mathbb{C}}$. This proves (1).

Consider the diagram

$$\begin{tikzcd} H^{\alpha}_{1,n-1} \arrow[rr, hook] \arrow[dr, "\pi_2"'] & & H^{\alpha}_{1,n-1} \times_{\mu_2} \mu_4 \arrow[dl, "\pi_4"] \\ & O_{1,n-1} & \end{tikzcd} \tag{A.22}$$

in which $\mu_2 \subset H^{\alpha}_{1,n-1}$ is generated by k_0. Then $H^{\beta}_{1,n-1} \subset H^{\alpha}_{1,n-1} \times_{\mu_2} \mu_4$ is the union of $H_{1,n-1}$ and the complement of $\pi_2^{-1}(O^{\downarrow}_{1,n-1})$ in $\pi_4^{-1}(O^{\downarrow}_{1,n-1})$. Let $\mu_4 \subset \mathbb{C}^{\times}$ act on $R^1_{\mathbb{C}}$ via scalar multiplication and on $R^0_{\mathbb{C}}$ trivially. Then by (1) the restriction to $H^{\beta}_{1,n-1} \subset H^{\alpha}_{1,n-1} \times_{\mu_2} \mu_4$ is real, i.e., preserves $R \subset R_{\mathbb{C}}$. This proves (2). ∎

A.3. Wick rotation and the CRT theorem

In this section we sketch a rigorous argument for the CRT theorem in relativistic quantum field theory. We use the analytic continuation of correlation functions, working in the framework of Wightman quantum field theory [SW, GJ, Kaz]. Our purpose is to treat general symmetry types. Even for theories with Lorentz symmetry group $H_{1,n-1} = \mathrm{Spin}_{1,n-1}$ there is a subtlety: the group $\widetilde{SO}^{\alpha}_{1,n-1}$ acts on the holomorphic correlation functions, whereas the group $\widetilde{SO}^{\beta}_{1,n-1}$ acts on the Minkowski spacetime correlation functions. (See §A.2.2 for the definitions of these Lie groups.) This argument also demonstrates why only the "Cliffordian" [BDGK] Lorentz signature pin groups $\mathrm{Pin}_{n-1,1}$ and $\mathrm{Pin}_{1,n-1}$ can be symmetries of a relativistic quantum field theory instead of more general possible double covers of $O_{1,n-1}$; see Remark A.42. We assume $n \geqslant 3$.

Recall from §2.1 that Minkowski spacetime M^n is an n-dimensional affine space whose vector space $V = \mathbb{R}^{1,n-1}$ of translations is equipped with an inner product of signature $(1, n-1)$ and a choice of component V_+ of the space $\{\xi : \langle \xi, \xi \rangle > 0\}$ of timelike vectors.[49] To Wick rotate to imaginary time, fix an orthogonal splitting $V = U \oplus U^\perp$ with U a 1-dimensional timelike subspace. Then the Euclidean translation group is $V_E = \sqrt{-1}\,U \oplus U^\perp$ and the corresponding Euclidean space is $E = M \times_V V_E$, an affine space over V_E. Complexified Minkowski spacetime is $M_\mathbb{C} = M \times_V V_\mathbb{C}$, where $V_\mathbb{C}$ is the complexification of V. The symmetry group $H_{1,n-1}$ of a relativistic quantum field theory acts on M^n by time-orientation preserving transformations via a homomorphism $\rho_n \colon H_{1,n-1} \to O^\uparrow_{1,n-1}$, as in (2.1).

Theorem A.23 (CRT Theorem)**.** *Let $\mathcal{Q}$ denote a relativistic quantum field theory with symmetry group $H_{1,n-1}$. Then the symmetry extends to $H^\beta_{1,n-1}$; elements of $H^\beta_{1,n-1} \backslash H_{1,n-1}$ act antilinearly.*

Here $\mathcal{Q}$ is a quantum field theory in the Wightman axiomatic framework. It is determined by its *correlation functions*, called *Wightman functions*; see [Kaz, §1.3]. For simplicity of notation we only discuss 2-point functions in this account. A precise version of Theorem A.23 is (A.41) below.

The *fields* in $\mathcal{Q}$ are defined by a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded real representation

$$\sigma \colon H_{1,n-1} \longrightarrow \mathrm{Aut}(R). \tag{A.24}$$

We write $R = R^0 \oplus R^1$ according to the grading; elements of $H_{1,n-1}$ preserve the grading. The spin-statistics theorem, which we assume in this account, asserts that the special element $k_0 \in K \subset H_{1,n-1}$ defined in Theorem 2.7(2) acts as the grading operator on R. Write $R_\mathbb{C} = R \otimes_\mathbb{R} \mathbb{C}$ for the complexification. Classical fields are functions $M^n \to R$. Quantum fields are R-valued operator-valued distributions $\Phi = \Phi^0 + \Phi^1$ on M^n. The 2-point "function" is a complex distribution whose value on Schwartz functions $f_i \colon M^n \to R^*$ is written

$$\langle \Phi(f_1)\Phi(f_2) \rangle = \int_{M^2} dp_1 dp_2\, f_1(p_1) f_2(p_2) \,\langle \Phi(p_1)\Phi(p_2) \rangle, \tag{A.25}$$

where $\langle \Phi(p_1)\Phi(p_2) \rangle$ denotes the kernel of the $R_\mathbb{C}^{\otimes 2}$-valued distribution on $M^{\times 2}$. The theory $\mathcal{Q}$ has a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ of states, constructed from the correlation functions, and a distinguished vacuum vector $\Omega \in \mathcal{H}^0$. The *field operators* $\Phi(f)$ act as unbounded operators on $\mathcal{H}$, and the 2-point function is the vacuum expectation value of the product of the field operators:

$$\langle \Phi(f_1)\Phi(f_2) \rangle = \langle \Omega\,,\, \Phi(f_1)\Phi(f_2)\Omega \rangle_\mathcal{H}. \tag{A.26}$$

There is a unitary representation of the affine extension of $H_{1,n-1}$ on $\mathcal{H}$—all symmetries preserve the $\mathbb{Z}/2\mathbb{Z}$-grading. The vacuum vector and 2-point function are invariant under that action, in particular under translations. Hence there is an $R_\mathbb{C}^{\otimes 2}$-valued distribution on V with kernel

$$W(\xi) := \langle \Phi(p)\Phi(p+\xi) \rangle, \qquad p \in M^n, \quad \xi \in V, \tag{A.27}$$

[49] The latter choice is required in order to formulate the positivity of energy.

which is independent of p.

The important step in Jost's proof is the construction of holomorphic correlation functions from which the Wightman functions are recovered as boundary values [Kaz, §2.1]. This is a consequence of the positivity of energy and geometric arguments. The holomorphic 2-point function

$$W_{\mathbb{C}} \colon \mathcal{D} \longrightarrow R_{\mathbb{C}}^{\otimes 2} \tag{A.28}$$

has domain $\mathcal{D} \subset V_{\mathbb{C}}$ that is connected and $H_n(\mathbb{C})$-invariant. Define the *backward tube* $\mathcal{T} = V - iV_+ \subset V_{\mathbb{C}}$, where i is a choice of square root of -1. Then[50]

$$\mathcal{D} = SO_n(\mathbb{C})(\mathcal{T}) \cup -SO_n(\mathbb{C})(\mathcal{T}). \tag{A.29}$$

An important feature of $\mathcal{D}$ is that it contains *Jost points*,[51] which in this case of 2-point functions are the real spacelike vectors $\xi \in V \subset V_{\mathbb{C}}$ that satisfy $\langle \xi, \xi \rangle > 0$. From (A.29) we see $\mathcal{T} \subset \mathcal{D}$, and as stated W is a boundary value of $W_{\mathbb{C}}$:

$$W(\xi) = \lim_{\epsilon \to 0^+} W_{\mathbb{C}}(\xi - \epsilon i \eta), \qquad \xi \in V, \quad \eta \in V_+, \tag{A.30}$$

and the limit is independent of η. We also have $V_E \backslash \{0\} \subset \mathcal{D}$, and the Wick-rotated Euclidean 2-point function is the restriction of $W_{\mathbb{C}}$ to $V_E \backslash \{0\}$.

We collect some properties of the holomorphic correlation functions. First, since the inner product on $\mathcal{H}$ is even, it follows that

$$W_{\mathbb{C}} = W_{\mathbb{C}}^0 + W_{\mathbb{C}}^1 \tag{A.31}$$

where $W_{\mathbb{C}}^q$ takes values in $(R_{\mathbb{C}}^q)^{\otimes 2}$, $q = 0, 1$. Note that both $W_{\mathbb{C}}^0$ and $W_{\mathbb{C}}^1$ are even. Next, as already stated, $W_{\mathbb{C}}$ is $H_n(\mathbb{C})$-invariant, hence invariant under the subgroup $H_{1,n-1}^{\alpha} \subset H_n(\mathbb{C})$:

$$W_{\mathbb{C}}(\zeta) = \sigma(h^{\alpha})^{\otimes 2}\, W_{\mathbb{C}}\bigl(\rho_n(h^{\alpha})\zeta\bigr), \qquad h^{\alpha} \in H_{1,n-1}^{\alpha}, \quad \zeta \in \mathcal{D}. \tag{A.32}$$

Now if ξ is real and spacelike, then since field operators at spacelike separated points commute (in the graded sense), and since real spacelike (Jost) points are in the domain $\mathcal{D}$, we have

$$\begin{aligned} W_{\mathbb{C}}^0(-\xi) &= W_{\mathbb{C}}^0(\xi) \\ W_{\mathbb{C}}^1(-\xi) &= -W_{\mathbb{C}}^1(\xi) \end{aligned} \tag{A.33}$$

Continuing with ξ real and spacelike, we claim

$$\begin{aligned} \overline{W_{\mathbb{C}}^0(\xi)} &= W_{\mathbb{C}}^0(\xi) \\ \overline{W_{\mathbb{C}}^1(\xi)} &= -W_{\mathbb{C}}^1(\xi) \end{aligned} \tag{A.34}$$

[50]Note $SO_n(\mathbb{C})(\mathcal{T}) = O_n(\mathbb{C})(\mathcal{T})$.
[51]Here we use $n \geqslant 3$.

Since such ξ lie in $\mathcal{D}$, and $\mathcal{D}$ is connected, we deduce a Schwarz reflection formula valid for all $\zeta \in \mathcal{D}$:

$$\begin{aligned}\overline{W^0_{\mathbb{C}}(\zeta)} &= W^0_{\mathbb{C}}(\bar\zeta)\\ \overline{W^1_{\mathbb{C}}(\zeta)} &= -W^1_{\mathbb{C}}(\bar\zeta)\end{aligned} \tag{A.35}$$

The manipulation that justifies (A.34) is, for any $p \in M^n$ and $\xi \in V$,

$$\overline{W_{\mathbb{C}}(\xi)} = \langle \Phi(p)\Phi(p+\xi)\Omega\,,\,\Omega\rangle = \langle \Omega\,,\,\Phi(p+\xi)\Phi(p)\Omega\rangle = W_{\mathbb{C}}(-\xi); \tag{A.36}$$

then we apply (A.33). The middle step is straightforward in the even case: $\Phi^0(q)$ is self-adjoint for q real. The corresponding manipulation in the odd case uses the adjoint of the odd operator $\Phi^1(q)$, which involves a tricky sign[52] as we explain in the following remark.

Remark A.37. The usual physics conventions are: the norm square of an odd vector in $\mathcal{H}$ is real and positive; for any two operators A, B we have $(AB)^* = B^*A^*$—there is no sign even if both A and B are odd; and the odd field operator $\Phi^1(q)$ is self-adjoint in the usual sense. However, the Koszul sign rule demands that the first two of these be modified to: the norm square of an odd vector in $\mathcal{H}$ is purely imaginary and lies on one of the two rays of nonzero purely imaginary numbers, the choice of which is a convention (Example 6.49); if A, B are operators that have definite parities $|A|, |B|$, then [DM, §4.4]

$$(AB)^* = (-1)^{|A||B|}B^*A^*. \tag{A.38}$$

If we use these conventions, then the odd field operator $\Phi^1(q)$ is not self-adjoint, but rather

$$\Phi^1(q)^* = i\,\Phi^1(q) \tag{A.39}$$

One justification for (A.39) is to consider the $*$-structure on the complex operator algebra, and to note that (A.38) implies that the square of an odd self-adjoint operator is even skew-adjoint, and so if $\Phi^1(q)$ were self-adjoint we would contradict expectations for the quantization of real fields. We remark that the factor i in (A.39) already occurs in quantum mechanics; see [FM1, (4.10)]. The middle step in (A.36) is valid with either the standard physics conventions or the Koszul-compatible notion of adjointness supplemented with (A.39).

Proof of Theorem A.23. Fix $h^\alpha \in H^\alpha_{1,n-1}\backslash H_{1,n-1}$. Then h^α reverses the time orientation, in other words, $H^\alpha(V_+) = -V_+$. Hence for $\xi \in V$ we use (A.30), (A.32), and (A.35) to deduce that for $\xi \in V$ and $q = 0, 1$ we have

$$\begin{aligned}\overline{W^q(\xi)} &= \lim_{\epsilon\to 0^+}\overline{W^q_{\mathbb{C}}(\xi - \epsilon i\eta)}\\ &= \lim_{\epsilon\to 0^+}\overline{\sigma(h^\alpha)^{\otimes 2}\,W^q_{\mathbb{C}}\big(\rho_n(h^\alpha)\xi - \epsilon i\rho_n(h^\alpha)\eta\big)}\\ &= \lim_{\epsilon\to 0^+}(-1)^q\sigma(h^\alpha)^{\otimes 2}\,W_{\mathbb{C}}\big(\rho_n(h^\alpha)\xi + \epsilon i\rho_n(h^\alpha)\eta\big)\\ &= (-1)^q\sigma(h^\alpha)^{\otimes 2}\,W\big(\rho_n(h^\alpha)\xi\big).\end{aligned} \tag{A.40}$$

[52]We thank Greg Moore for help straightening this out.

To pass to the third equation we use the fact that $\sigma(h^\alpha)$ is real on even vectors (Proposition A.20(1)). The construction that proves Proposition A.20(2) combines with (A.40) to yield

$$\overline{W^q(\xi)} = \sigma(h^\beta)^{\otimes 2}\, W\bigl(\rho_n(h^\beta)\xi\bigr), \qquad h^\beta \in H^\beta_{1,n-1}\backslash H_{1,n-1}, \quad \xi \in V. \tag{A.41}$$

This is the precise statement that the Minkowski spacetime 2-point function is antilinear-invariant under elements of $H^\beta_{1,n-1}\backslash H_{1,n-1}$. □

Remark A.42. If $\mathcal{Q}$ is a relativistic quantum field theory with fermionic states and time-reversal symmetry, and no other internal symmetries, then $H_{1,n-1}$ is a double cover of $SO^\uparrow_{1,n-1}$ whose identity component is isomorphic to $\mathrm{Spin}_{1,n-1}$. The complex Lie group $H_n(\mathbb{C})$ is then a double cover of $O_n(\mathbb{C})$ whose identity component is isomorphic to $\mathrm{Spin}_n(\mathbb{C})$. Proposition A.11 implies that $H_n(\mathbb{C})$ is isomorphic to $\mathrm{Pin}^+_n(\mathbb{C})$ or $\mathrm{Pin}^-_n(\mathbb{C})$. The construction with (A.14) and (A.16) tells that the group $H^\beta_{1,n-1}$ is $\mathrm{Pin}_{n-1,1}$ and $\mathrm{Pin}_{1,n-1}$, respectively. Recalling the sign convention (A.3) for Clifford algebras, this proves the correspondence between (A.1) and (A.2) and also limits the possible symmetry groups on relativistic quantum field theories to the Cliffordian pin groups.

Appendix B. Involutions on categories and duality

Definition B.1. Let $\mathcal{C}$ be a category.

(1) An *involution* of $\mathcal{C}$ is a pair (τ,η) of a functor $\tau\colon \mathcal{C}\to\mathcal{C}$ and a natural isomorphism $\eta\colon \mathrm{id}_{\mathcal{C}}\to\tau^2$ such that for any $x\in\mathcal{C}$ we have $\tau\eta_x=\eta_{\tau x}$ as morphisms $\tau x\to\tau^3 x$.

(2) A *fixed point* of τ is a pair (x,θ) of an object $x\in\mathcal{C}$ and an isomorphism $x\xrightarrow{\theta}\tau x$ such that $\tau\theta\circ\theta=\eta_x$ as morphisms $x\to\tau^2 x$.

If $\mathcal{C}$ is a symmetric monoidal category, then the involution τ is required to be a symmetric monoidal functor: for $x,y\in\mathcal{C}$ there is given an isomorphism $\tau x\otimes\tau y\xrightarrow{\cong}\tau(x\otimes y)$ and these isomorphisms are compatible with the symmetry and with η.

Example B.2. Let $\mathcal{C}=\mathrm{Vect}_{\mathbb{C}}$ be the category of complex vector spaces and linear maps. Define $\tau\colon\mathcal{C}\to\mathcal{C}$ to be the functor that takes complex vector spaces and linear maps to their complex conjugates. (The complex conjugate vector space is the same underlying real vector space with the sign of multiplication by $\sqrt{-1}\in\mathbb{C}$ reversed; the complex conjugate of a linear map is the same map of sets.) Then there is a canonical identification of τ^2 with $\mathrm{id}_{\mathcal{C}}$. A fixed point is a complex vector space with a real structure. As a variation, if $\mathcal{C}=s\mathrm{Vect}_{\mathbb{C}}$ is the category of super ($\mathbb{Z}/2\mathbb{Z}$-graded) vector spaces and τ complex conjugation as above, but now η is composed with the exponentiated grading automorphism (denoted '$(-1)^F$' in the physics literature), then a fixed point is a super vector space with a real structure on its even part and a quaternionic structure on its odd part. If we restrict to the subgroupoid $\mathcal{C}^\times$ of super lines and isomorphisms, then all fixed points are even.

Definition B.3. Let (τ,η) be an involution on a category $\mathcal{C}$. The *fixed point category* $\mathcal{C}^\tau$ has as objects fixed points (x,θ), and a morphism $(x,\theta)\to(x',\theta')$ in $\mathcal{C}^\tau$ is a morphism $(x\xrightarrow{f}x')\in\mathcal{C}$ such that the diagram

(B.4)
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\theta"'] & x' \arrow[d, "\theta'"] \\ \tau x \arrow[r, "\tau f"] & \tau x' \end{tikzcd}$$

commutes. There is a *forgetful functor* $\mathcal{C}^\tau \to \mathcal{C}$ that maps $(x, \theta) \mapsto x$.

Example B.5. Let $\mathcal{C}$ be the *groupoid* of $\mathbb{Z}(1)$-torsors:[53] an object T is a set with a simply transitive action of the additive group $\mathbb{Z}(1)$ and a morphism $T \to T'$ is an isomorphism that commutes with the $\mathbb{Z}(1)$-actions. Let τ be the involution that sends a torsor T to its dual $\operatorname{Hom}_{\mathbb{Z}(1)}(T, \mathbb{Z}(1))$ and sends a morphism to its inverse transpose. The dual of T may be identified with T as a set; the dual $\mathbb{Z}(1)$ action by $\zeta \in \mathbb{Z}(1)$ is the original action by $\bar{\zeta}$. The fixed point category $\mathcal{C}^\tau$ is equivalent to the set $\mathbb{Z}/2\mathbb{Z}$: there are two isomorphism classes of objects and no nontrivial automorphisms. The first, which we call 'Type P', is the torsor $\mathbb{Z}(1)$ with complex conjugation θ as a map to the dual torsor. The second, which we call 'Type N', is the torsor $\pi\sqrt{-1} + \mathbb{Z}(1)$ with complex conjugation θ. Observe that in the Type P case the involution θ has a fixed point whereas in the Type N case it does not. Also, $\mathbb{Z}(1)$-torsors form a Picard groupoid, as do torsors for any abelian group, and the fixed point category is a Picard groupoid as well. The Type P torsor is the tensor unit; the square of a Type N torsor has Type P. The names derive from the family $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ of $\mathbb{Z}(1)$-torsors with complex conjugation acting. There are two components $\mathbb{R}^{>0}$ and $\mathbb{R}^{<0}$ of fixed points in the base. The fiber of exp has Type P over positive real numbers and Type N over negative real numbers; the representatives described above are $\exp^{-1}(+1)$ and $\exp^{-1}(-1)$, respectively.

Definition B.6. Let $\mathcal{B}, \mathcal{C}$ be categories with involutions and $F\colon \mathcal{B} \to \mathcal{C}$ a functor. Then *equivariance data* for F is an isomorphism $\phi\colon F\tau_{\mathcal{B}} \xrightarrow{\cong} \tau_{\mathcal{C}} F$ of functors $\mathcal{B} \to \mathcal{C}$ such that for every object $x \in \mathcal{B}$ the diagram

(B.7)
$$\begin{tikzcd} Fx \arrow[r, "F\eta_{\mathcal{B}}"] \arrow[dr, "\eta_{\mathcal{C}}"'] & F\tau_{\mathcal{B}}^2 x \arrow[d, "\phi^2"] \\ & \tau_{\mathcal{C}}^2 F x \end{tikzcd}$$

commutes.

There are additional compatibilities for a symmetric monoidal functor between symmetric monoidal categories; we do not spell them out. We often loosely say that "F is an equivariant functor", but it is important to remember that equivariance is data+condition, not simply a condition.

Next, we review duality in a symmetric monoidal category. Let $\mathcal{C}$ be a symmetric monoidal category and $x \in \mathcal{C}$. Denote the tensor unit by $1 \in \mathcal{C}$. (The tensor unit in $\operatorname{Bord}_{\langle n-1,n\rangle}(H_n)$ is the empty set as an $(n-1)$-dimensional manifold; the tensor unit in $\operatorname{Vect}_{\mathbb{C}}$ is the trivial 1-dimensional vector space $\mathbb{C}$.)

[53] Recall that $\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z} \subset \mathbb{C}$.

Definition B.8. Let x be an object in a symmetric monoidal category $\mathcal{C}$. *Duality data* for x is a triple $(x^\vee, c, e)$ consisting of an object $x^\vee \in \mathcal{C}$ together with morphisms $c\colon 1 \to x \otimes x^\vee$ and $e\colon x^\vee \otimes x \to 1$ such that the compositions

$$\begin{aligned} &x \xrightarrow{c\otimes \mathrm{id}} x \otimes x^\vee \otimes x \xrightarrow{id\otimes e} x \\ &x^\vee \xrightarrow{\mathrm{id}\otimes c} x^\vee \otimes x \otimes x^\vee \xrightarrow{e\otimes \mathrm{id}} x^\vee \end{aligned} \tag{B.9}$$

are identity maps. If $x_0 \xrightarrow{f} x_1$ is a morphism, then the dual morphism is the composition

$$f^\vee : x_1^\vee \xrightarrow{\mathrm{id}\otimes c_{x_0}} x_1^\vee \otimes x_0 \otimes x_0^\vee \xrightarrow{\mathrm{id}\otimes f \otimes \mathrm{id}} x_1^\vee \otimes x_1 \otimes x_0^\vee \xrightarrow{e_{x_1}\otimes \mathrm{id}} x_0^\vee \tag{B.10}$$

The morphism c is called *coevaluation* and e is called *evaluation*. We say that $x^\vee$ is “the” dual to x since any two triples of duality data are uniquely isomorphic. Assuming all objects have duals, we can make choices of duality data for all objects at once and so obtain a duality involution δ on $\mathcal{C}$, but δ does not satisfy Definition B.1 since the direction of morphisms is reversed (B.10); in other words, δ is a functor to the *opposite* category.

Definition B.11. Let $\mathcal{C}$ be a category.

(1) A *twisted involution* of $\mathcal{C}$ is a pair (δ, η) of a functor $\delta\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and a natural isomorphism $\eta\colon \mathrm{id}_{\mathcal{C}} \to \delta^{\mathrm{op}} \circ \delta$ such that for any $x \in \mathcal{C}$ we have $\delta\eta_x \circ \eta_{\delta x} = \mathrm{id}_{\delta x}$.
(2) A *fixed point* of δ is a pair (x, θ) of an object $x \in \mathcal{C}$ and an isomorphism $x \xrightarrow{\theta} \delta x$ such that $\delta\theta \circ \eta_x = \theta$ as morphisms $x \to \delta x$.

Definition B.3 applies with a single change: the direction of the bottom arrow in (B.4) is reversed.

Example B.12. For $\mathcal{C} = f\mathrm{Vect}_{\mathbb{C}}$ the category of finite dimensional complex vector spaces, the duality involution $\delta\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ maps a vector space V to its dual V^* and a linear map $f\colon V \to W$ to $f^*\colon W^* \to V^*$. A fixed point of δ is a vector space V equipped with a nondegenerate symmetric bilinear form; a linear map $f\colon V \to W$ in $\mathcal{C}^\delta$ preserves the bilinear forms. A fixed point for the composite of duality and complex conjugation (Example B.2) is a complex vector space V with a nondegenerate hermitian form; a linear map $f\colon V \to W$ in the fixed point category is a partial isometry—an injective map that preserves the hermitian forms.

Remark B.13. There is a higher categorical context for Definition B.11. Let Cat denote the 2-category of categories. There is an involution $\alpha\colon \mathrm{Cat} \to \mathrm{Cat}$ that sends a category $\mathcal{C}$ to its opposite $\mathcal{C}^{\mathrm{op}}$. (There is an extra categorical layer over Definition B.1: there is a triple (α, η_1, η_2) of data and a single condition.) A twisted involution in the sense of Definition B.11 is fixed point data for α.

Definition B.14. Let (τ, η) be an involution on a symmetric monoidal category $\mathcal{C}$. A *hermitian structure* on an object $x \in \mathcal{C}$ is an isomorphism $h\colon \tau x \to x^\vee$ such that the composition

$$\tau x \cong \tau\big((x^\vee)^\vee\big) \xrightarrow{\tau(h^\vee)} \tau\big((\tau x)^\vee\big) \cong \tau^2(x^\vee) \xrightarrow{\eta^{-1}} x^\vee \tag{B.15}$$

is equal to h.

Proposition 4.8 asserts that every object in a bordism category carries a hermitian structure. Observe that if $F\colon \mathcal{B} \to \mathcal{C}$ is an equivariant symmetric monoidal functor between symmetric monoidal categories with involution, as in Definition B.6, then the image of a hermitian structure on an object $b \in \mathcal{B}$ is a hermitian structure on Fb.

Appendix C. Noncompact Wick-rotated vector symmetry groups

Let (H_n, ρ_n) be a symmetry type, as in Definition 2.4.

Proposition C.1. *Assume $n \geqslant 3$.*

(1) *There exist a canonical noncompact Lie group $\underline{H}_n$, a homomorphism $\underline{H}_n \to GL_n\mathbb{R}$ with kernel K, and an inclusion $H_n \hookrightarrow \underline{H}_n$ such that (i) $H_n \subset \underline{H}_n$ is a maximal compact Lie subgroup, (ii) the inclusion induces an isomorphism on π_0, and (iii) the diagram*

$$\begin{tikzcd} H_n \arrow[r, hook] \arrow[d, "\rho_n"'] & \underline{H}_n \arrow[d] \\ O_n \arrow[r, hook] & GL_n\mathbb{R} \end{tikzcd} \tag{C.2}$$

commutes.

(2) *There exists a canonical Lie group $\widehat{\underline{H}}_n$ that fits into the diagram*

$$\begin{tikzcd} 1 \arrow[r] & H_n \arrow[r, "j_n"] \arrow[d, hook] & \widehat{H}_n \arrow[r] \arrow[d, hook] & \{\pm 1\} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \underline{H}_n \arrow[r] & \widehat{\underline{H}}_n \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{C.3}$$

of group extensions, as well as a canonical homomorphism $\widehat{\underline{H}}_n \to \{\pm 1\} \times GL_n\mathbb{R}$ that fits into a pullback square

$$\begin{tikzcd} \underline{H}_n \arrow[r] \arrow[d] & \widehat{\underline{H}}_n \arrow[d] \\ GL_n\mathbb{R} \arrow[r] & \{\pm 1\} \times GL_n\mathbb{R} \end{tikzcd} \tag{C.4}$$

and a commutative cube built from (3.15) and (C.4).

These noncompact groups are used to define topological bordism categories (§2.2).

Proof. First define $\underline{\mathrm{Spin}}_n$ and $\underline{\mathrm{Pin}}_n^+$ as follows. Choose a lift $P \xrightarrow{\rho} GL_n\mathbb{R} \xrightarrow{\pi} GL_n\mathbb{R}/O_n$ of the homogeneous principal bundle π to a principal Pin_n^+-bundle $\pi \circ \rho$; it is unique up to isomorphism since $GL_n\mathbb{R}/O_n$ is contractible. Define $\underline{\mathrm{Pin}}_n^+$ as the group of automorphism of ρ that cover the action of left multiplication of $GL_n\mathbb{R} = \underline{O}_n$, and $\underline{\mathrm{Spin}}_n \subset \underline{\mathrm{Pin}}_n^+$ the subgroup covering left multiplication by $GL_n^+\mathbb{R} = \underline{SO}_n$. Then set

$$\underline{SH}_n = \underline{\mathrm{Spin}}_n \times K \,/\, \langle(-1, k_0)\rangle, \tag{C.5}$$

analogous to (2.8). If $\rho_n(H_n) = SO_n$, set $\underline{H}_n = \underline{SH}_n$. If ρ_n is surjective, define $\underline{\widetilde{H}}_n$ as the pullback (see (2.10))

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, equal] & \underline{\widetilde{H}}_n \arrow[r] \arrow[d] & \underline{\mathrm{Pin}}_n^+ \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & K \arrow[r] & J \arrow[r] & \{\pm 1\} \arrow[r] & 1 \end{tikzcd} \tag{C.6}$$

and then

$$\underline{H}_n \cong \underline{\widetilde{H}}_n \,/\, \langle(-1, k_0)\rangle. \tag{C.7}$$

It is straightforward to check the properties in (1).

For (2) imitate the proof of Proposition 3.13 with $\underline{\mathrm{Spin}}_n$ and $\underline{\mathrm{Pin}}_n^+$ replacing Spin_n and Pin_n^+, respectively. □

Appendix D. Computations with A_1-modules

The computations described in §10 depend on knowledge of the mod 2 cohomology of the spectra

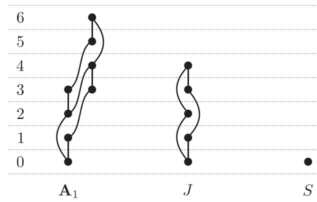
$$\begin{array}{rc} MTO_{|d|} & 0 \leqslant d \leqslant 3 \\ MO_{|d|} & -3 \leqslant d \leqslant 0 \\ MSO_3 & \end{array}$$

as modules over the subalgebra $\mathbf{A}_1$ of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2. The purpose of this appendix is to describe these computations and the methods for arriving at them.

We thank Meng Guo for her careful reading and astute corrections.

D.1. Cell diagrams

It is common practice to depict an $\mathbf{A}_1$ module M as a graph with nodes corresponding to a chosen homogeneous basis for M, at a height corresponding to grading, and with an edge drawn with a straight line between e and e' if the coefficient of e' in $\mathrm{Sq}^1(e)$ is non-zero, and an edge drawn with a curved line if they are analogously related by Sq^2. This works best when a basis can be chosen so that the operations Sq^1 and Sq^2 send basis elements to basis elements. This is the case with all of the $\mathbf{A}_1$ modules needed in this paper. Here are three examples:



For clarity the degrees of the basis elements have been indicated in this example, though we will not usually do this. Topologists call these graphs "cell diagrams." The one on the left is the free $\mathbf{A}_1$ module on one generator (of degree 0) and the one on the right is just $\mathbb{Z}/2 = H^*(S^0)$, concentrated in degree 0. The one in the middle right comes up frequently and was deemed the *Joker* by Adams. It is the cohomology of a spectrum also called J.

As explained in §10 the mod 2 cohomology $H^*M\mathrm{Spin}$ was show by Anderson, Brown and Peterson [ABP1] to have the form

$$A \underset{\mathbf{A}_1}{\otimes} N$$

for some $\mathbf{A}_1$ module N (which they determined). Figure 7 is a cell diagram of N through dimension 28. The modules to the right (in gray) are free, and the modules to the left (in black) are either S or J.

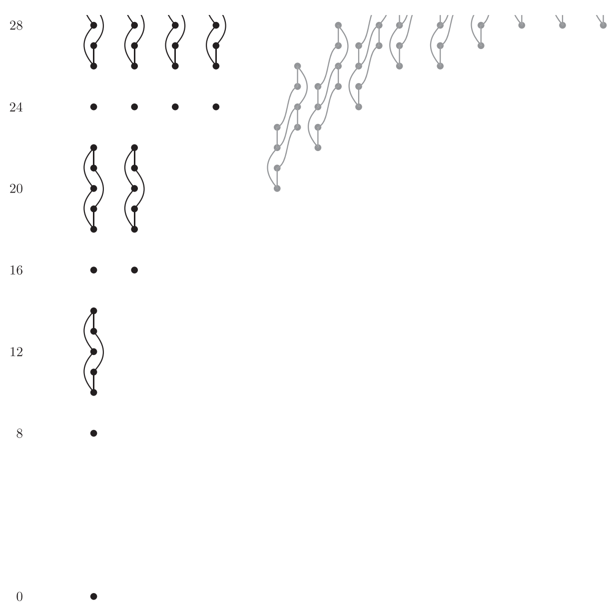


FIGURE 7. The cell diagram for $M\mathrm{Spin}$

How does one use this in practice? Suppose X is a connective spectrum of finite type and one wishes to determine the localization at 2 of $\pi_*M\mathrm{Spin} \wedge X$. One makes three computations, (in which the abutments, though not indicated, have been completed at 2)

$$\begin{aligned}
\mathrm{Ext}^{s,t}_{\mathbf{A}_1}(H^*X, \mathbb{Z}/2) &\Rightarrow \pi_{t-s}\,\mathrm{ko} \wedge X \\
\mathrm{Ext}^{s,t}_{\mathbf{A}_1}(J \otimes H^*X, \mathbb{Z}/2) &\Rightarrow \pi_{t-s}\,\mathrm{ko} \wedge J \wedge X =: M_J(X) \\
\mathrm{Ext}^{s,t}_{\mathbf{A}_1}(\mathbf{A}_1 \otimes H^*X, \mathbb{Z}/2) &= H_*X \ .
\end{aligned}$$

The two spectral sequences often collapse (they do in the cases studied in this paper). Write

$$\begin{aligned} M_S(X) &= \pi_* \,\mathrm{ko} \wedge X \\ M_J(X) &= \pi_* \,\mathrm{ko} \wedge J \wedge X \ . \end{aligned}$$

The result of Anderson-Brown-Peterson [ABP1] is that after localizing at 2, $\pi_* M\mathrm{Spin} \wedge X$ is isomorphic to a sum of copies of $M_S(X)$, $M_J(X)$ and H_*X, shifted according to the location of the corresponding summands in the cell diagram of X:

$$\pi_* M\mathrm{Spin} \wedge X \,=\, M_S(X) \oplus \Sigma^8 M_S(X) \oplus \Sigma^{10} M_J(X) \oplus \cdots \oplus \Sigma^{20} H_* X \oplus \cdots \,.$$

One further comment about the spectral sequences above. If M is a free $\mathbf{A}_1$-module then

$$\begin{aligned} \mathrm{Ext}^{s,t}_{\mathbf{A}_1}(M, \mathbb{Z}/2) &= \mathrm{Ext}^{s,t}_{\mathbf{A}_1}(J \otimes M, \mathbb{Z}/2) = 0 \qquad s > 0 \\ \mathrm{Ext}^{0,t}_{\mathbf{A}_1}(M, \mathbb{Z}/2) &= \mathrm{Hom}_{\mathbf{A}_1}(M, \mathbb{Z}/2) \\ \mathrm{Ext}^{0,t}_{\mathbf{A}_1}(J \otimes M, \mathbb{Z}/2) &= \mathrm{Hom}_{\mathbf{A}_1}(J \otimes M, \mathbb{Z}/2) \end{aligned}$$

In these cases the display of the spectral sequences are all on the line $s = 0$, and the spectral sequences collapse.

More generally if M is of the form $M' \oplus F$ with F a free $\mathbf{A}_1$ module, then

$$\mathrm{Ext}^{s,t}_{\mathbf{A}_1}(M, \mathbb{Z}/2) \approx \mathrm{Ext}^{s,t}_{\mathbf{A}_1}(M', \mathbb{Z}/2) \oplus \mathrm{Ext}^{s,t}_{\mathbf{A}_1}(F, \mathbb{Z}/2)$$

and the spectral sequence is the sum of two spectral sequences, one of which collapses for trivial reasons. The analogous statement holds for the second spectral sequence. For this reason it is useful to omit free summands from the cell diagrams and keep track of them in some other way.

D.2. The charts

We can now explain in more detail what is shown in Figure 5. In each case we are interested in $\pi_* M\mathrm{Spin} \wedge X$ for some appropriate spectrum X. A cell diagram for X, modulo free $\mathbf{A}_1$ summands is shown on the left, with X labeled below it. The chart to the right depicts $\mathrm{Ext}^{s,t}_{\mathbf{A}_1}(H^*(X); \mathbb{Z}/2)$ as a module over $\mathrm{Ext}^{s,t}_{\mathbf{A}_1}(\mathbb{Z}/2, \mathbb{Z}/2)$. Following standard convention the horizontal axis is the $(t-s)$-axis and the vertical axis is the s-axis. Each dot represents a basis element. The contributions from the free summands contribute only to $\mathrm{Ext}^{0,t}$ and to keep the picture uncluttered they are indicated below the table. For example in the case $s = 3$, in dimension $(t - s) = 8$, there is a $\mathbb{Z}/2$ not indicated in graphical notation, but only by the $+1$. The group in that case is the sum of that $\mathbb{Z}/2$ and $\mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/32$.

The color coding allows one to read off the effect of the twisted Dirac operators of §9.2 as described in homotopy theoretic terms in §10. Consider, for example, the case $s = 3$. One needs to know the effect of the map

$$\pi_* M\mathrm{Spin} \wedge S^{-3} \wedge MO_3 \to S^{-3} \wedge KO.$$

The (-1)-connected cover of $S^{-3} \wedge KO$ is equivalent to $\mathrm{ko} \wedge W$, in which W is the finite spectrum whose cell diagram is depicted below

The effect in cohomology of the twisted Dirac operator corresponds to the inclusion of the blue cells, and the cokernel of this map, in the relevant summand, is displayed in green. The Ext charts are correspondingly color coded and the red line indicates the connecting homomorphism in the long exact sequence. The Ext computation of interest is built from the kernel and cokernel of this connecting homomorphism. For example the connecting homomorphism is a monomorphism from the column $(t-s)=1$ to the column $(t-s)=0$, and the only non-zero Ext group in this range is

$$\mathrm{Ext}^{0,0}_{\mathbf{A}_1}(H^*S^{-3}MO_3, \mathbb{Z}/2) = \mathbb{Z}/2.$$

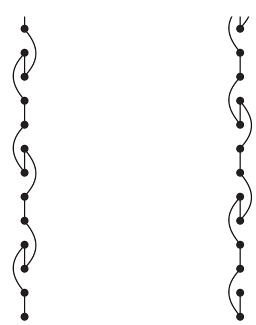
In dimension 6, the group is the sum of $(\mathbb{Z}/2)^2$ (coming from the free summands) and another $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. The fact that the dot in filtration $s=2$ is blue indicates that the corresponding basis element maps non trivially under the map to $\pi_6\Sigma^{-3}KO$.

D.3. The cases $s = \pm 1$

The cell diagrams for $\Sigma^{-1}MO(1)$ and $\Sigma^1 MTO(1)$ are easily derived from the Thom isomorphism and Wu formula

$$\mathrm{Sq}^n(U) = w_n \cdot U$$

for the action of the Steenrod operations on the Thom class of a (virtual) vector bundle. The diagrams work out to be



$S^{-1} \wedge MO_1$ $\qquad$ $S^1 \wedge MTO_1$

and continue infinitely far upward, repeating the evident pattern of Steenrod operations. There are no additional free summands in these cases.

D.4. The case $s = 4$

The next easiest case to understand is the case $s = 4$. To derive it requires a useful technique introduced by Adams and Margolis [AM], and developed considerably further by Margolis [Ma].

The subalgebra $\mathbf{A}_1$ contains two of the Milnor operators

$$\begin{aligned} Q_0 &= \mathrm{Sq}^1 \\ Q_1 &= [\mathrm{Sq}^2, \mathrm{Sq}^1] \end{aligned}$$

and together they generate an exterior algebra

$$E[Q_0, Q_1] \subset \mathbf{A}_1.$$

Definition D.1. Suppose that M is an $\mathbf{A}_1$ module. For $i = 0, 1$ the i^{th} Margolis homology of M is

$$H_*(M; Q_i) = \ker Q_i / \operatorname{image} Q_i.$$

The Margolis homology of a space or spectrum X is the Margolis homology of H^*X

$$H_*(X; Q_i) = H_*(H^*(X); Q_i).$$

Remark D.2. The Milnor elements are primitive, and the Kunneth isomorphism holds:

$$H_*(M \otimes N; Q_i) \approx H_*(M; Q_i) \otimes H_*(N; Q_i).$$

The following theorem of Adams and Margolis [AM, Theorem 3.1] (attributed by Adams and Margolis to Wall, in this particular case) is one reason the Margolis homology groups are important.

Theorem D.3 (Adams-Margolis)**.** *A connected $\mathbf{A}_1$-module M is free if and only if*

$$H_*(M; Q_0) = H_*(M; Q_1) = 0.$$

The action of the Milnor operators on

$$H^*(BSO_3; \mathbb{Z}/2) = \mathbb{Z}/2[w_2, w_3].$$

is given by

$$\begin{aligned} Q_0(w_2) &= w_3 \\ Q_0(w_3) &= 0. \end{aligned}$$

This implies that the Margolis homology with respect to Q_0 is

$$H_*(BSO_3; Q_0) \approx \mathbb{Z}/2[w_2^2].$$

Write U for the Thom class in H^*MO_3. Since $Q_0(U) = w_1U = 0$ the Thom isomorphism commutes with Q_0, and the Margolis homology of MSO_3 with respect to Q_0 is

$$U \cdot \mathbb{Z}/2[w_2^2].$$

For the Q_1 homology note that

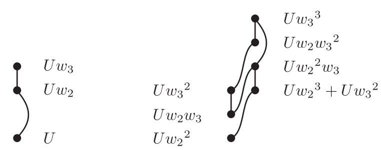
$$\begin{aligned} Q_1(w_2) &= w_2w_3 \\ Q_1(w_3) &= w_3^2 \\ Q_1(U) &= Uw_3. \end{aligned}$$

It follows that $H^*MSO(3)$, as a module over the exterior algebra $E[Q_1]$, is a sum of

$$UF_j = \{Uw_2^j, Uw_2^jw_3, Uw_2^jw_3^2, Uw_2^jw_3^3, \dots\}.$$

Using this one sees that the Margolis homology with respect to Q_1 of $MSO(3)$ has basis $\{Uw_2^{2j+1}\}$.

Now let M and N be the $\mathbf{A}_1$-modules



M $\qquad\qquad$ N

and consider the map

(D.4)
$$(M \oplus N) \otimes \mathbb{Z}/2[w_2^4] \to H^*(MSO_3).$$

The map (D.4) is an inclusion. Together with the Kunneth formula, the computation just described implies that it induces an isomorphism of Margolis homology with respect to both Q_0 and Q_1. By the Theorem of Adams and Margolis its cokernel is free, and there is an isomorphism

$$H^*(MSO_3) \approx (M \oplus N) \otimes \mathbb{Z}/2[w_2^4] \oplus \text{ free modules.}$$

The cell diagram in box $s = 4$ in Figure 5 depicts $(M \oplus N) \otimes \mathbb{Z}/2[w_2^4]$.

One can work out the disposition of the free modules by computing Poincaré series. The Poincaré series for the indecomposables of the free modules (with U placed in degree 0) is the quotient of

$$\frac{1}{(1-t^2)(1-t^3)} - \frac{(1+t^2+t^3+t^4(1+t+2t^2+t^3+t^4+t^5))}{(1-t^8)}$$

by the Poincaré series $(1+t)(1+t^2)(1+t^3)$ of $\mathbf{A}_1$. This works out to be

$$\frac{t^9}{(1-t^6)(1-t^8)} = t^9 + t^{15} + t^{17} + O[t]^{21}.$$

Most of the time this is enough information. However for some purposes it is useful to have a basis for the generators of the free modules. In this case one can work out that the summand of free modules is

$$\mathbf{A}_1[w_3^2, w_2^4] \cdot U w_2^3 w_3,$$

and that

$$(M \oplus N) \otimes Z/2[w_2^4] \oplus \mathbf{A}_1[w_3^2, w_2^4] \otimes U w_2^3 w_3 \to H^*(MSO_3) \tag{D.5}$$

is an isomorphism. We now digress to describe a technique for verifying this. The technique applies to modules over any connected graded Hopf algebra and exploits the fact that such an algebra is a Frobenius algebra. We will describe it explicitly for $\mathbf{A}_1$.

Let $b(x) = \mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2(x)$ (this is the operation that goes from the bottom dot to the top dot in the cell diagram for $\mathbf{A}_1$). If F is a free $\mathbf{A}_1$-module, and $x \in F$ there are elements $a \in \mathbf{A}_1$ and $y \in F$ with $a \cdot x = b(y) \neq 0$. This proved by reducing to the case $F = \mathbf{A}_1$ and either checking directly or appealing to the fact that $\mathbf{A}_1$ is a Frobenius algebra.

Lemma D.6. *Suppose that F and M are $\mathbf{A}_1$ modules and that F is free. A map $F \to M$ is a monomorphism if and only if the induced map $b(F) \to b(M)$ is a monomorphism.*

Proof. The only if statement is clear. For the converse, suppose that $b(F) \to b(M)$ is a monomorphism and $x \in F$. By the above there are $a \in \mathbf{A}_1$ and $y \in F$ with $a \cdot x = b(y) \neq 0$. Since $b(F) \to b(M)$ is a monomorphism the image of $b(y)$ is non-zero, hence so is the image of $a(x)$ and hence so is the image of x. □

Remark D.7. Since $\mathbf{A}_1$ is a finite dimensional Hopf algebra, it is also injective as a module over itself. This means that if $F \subset M$ is a free submodule of finite type (finite rank in each degree) then there is a decomposition $M \approx M' \oplus F$. This leads to a fairly quick way of locating the free summands in an $\mathbf{A}_1$-module M. They are generated by any subset $B \subset M$ with the property that $b(B) \subset b(M)$ is a basis.

Lemma D.8. *For an $\mathbf{A}_1$ module N the following are equivalent*

i) *If F is a free module and $F \subset N$ then $F = 0$.*

ii) *$b(x) = 0$ for all $x \in N$.*

Proof. Suppose that $F \subset N$ is a free submodule. If F is non-zero then there is an $x \in F$ with $b(x) \neq 0$, so $b(N) \neq 0$. Conversely if there is an $x \in N$ with $b(x) \neq 0$ then the map

$$\begin{aligned} \Sigma^{|x|}\mathbf{A}_1 &\to N \\ a &\mapsto a \cdot x \end{aligned}$$

is a monomorphism by Lemma D.6. □

Definition D.9. An $\mathbf{A}_1$ module N *has no free submodules* if it has the equivalent properties above.

By Remark D.7 having a free submodule is equivalent to having a free summand.

Lemma D.10. *Suppose that H is an $\mathbf{A}_1$-module, and $N \subset H$ a summand having no free submodules. If F is a free module and $F \to H$ is a monomorphism, then $F \to H/N$ is a monomorphism.*

Proof. By Lemma D.6 it suffices to show that $b(F) \to b(H/N)$ is a monomorphism. Since $b(N) = 0$ and N is a summand, the map $b(H) \to b(H/N)$ is an isomorphism. □

Returning to the cohomology of MSO_3, we now use these ideas to show that (D.5) is an isomorphism of $\mathbf{A}_1$ modules. Both sides have the same Poincaré series so it suffices to show that the map is a monomorphism, or equivalently that the map

$$\mathbf{A}_1[w_3^2, w_2^4] \otimes Uw_2^3w_3 \to H^*(MSO_3)/\big((M \oplus N) \otimes Z/2[w_2^4]\big)$$

is a monomorphism. Since M and N visibly have no free submodules, neither does $(M \oplus N) \otimes Z/2[w_2^4]$, so by Lemma D.10 it suffices to show that

$$\mathbf{A}_1[w_3^2, w_2^4] \otimes Uw_2^3w_3 \to H^*(MSO_3)$$

is a monomorphism. This is done with the aid of Lemma D.6. Since

$$\begin{aligned}
\mathrm{Sq}^1(w_2^4) &= \mathrm{Sq}^2(w_2^4) = 0\\
\mathrm{Sq}^1(w_3^2) &= \mathrm{Sq}^2(w_3^2) = 0
\end{aligned}$$

and

$$\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2(Uw_2^3w_3) = Uw_3^5$$

the assertion comes down to checking that

$$\{Uw_3^5w_2^{4k}\,w_3^{2\ell}\},$$

is linearly independent, which is easy.

D.5. The case $s = \pm 2$

We begin with the formulas

$$\begin{aligned}
Q_0(w_1) &= w_1^2\\
Q_0(w_2) &= w_1w_2\\
Q_1(w_1) &= w_1^4\\
Q_1(w_2) &= w_1^3w_2 + w_1w_2^2.
\end{aligned}$$

For both MO_2 and MTO_2

$$\begin{aligned} Q_0(U) &= w_1 U \\ Q_1(U) &= (w_1^3 + w_1 w_2)U, \end{aligned}$$

so the Thom isomorphism

$$H^*(MO_2) \approx H^*(MTO_2)$$

induces an isomorphism of Margolis homology.

Restricting attention to MO_2, let

$$F_n \subset H^* MO_2$$

be the subspace with basis

$$\{U\, w_1^i\, w_2^j \mid j \leqslant n\}$$

and $\bar{F}_n$ the subspace with basis

$$\{U\, w_1^i\, w_2^n\},$$

so that there is a vector space isomorphism

$$F_n \approx \bigoplus_{j \leqslant n} \bar{F}_j.$$

The Milnor operator Q_0 preserves the decomposition into the spaces $\bar{F}_j$ and from the formulas above one concludes that

$$H_*(\bar{F}_{2n}; Q_0) = 0$$

and

$$H_*(\bar{F}_{2n+1}; Q_0) = \mathbb{Z}/2\{U\, w_2^{2n+1}\}.$$

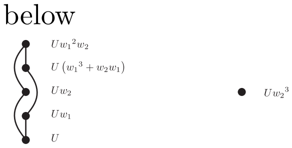
This shows that the Q_0 Margolis homology of $H_* MO_2$ has basis $\{U\, w_2^{2n+1}\}$.

The Milnor operator Q_1 maps F_{n-1} to F_n. We can determine the Margolis homology from the associated spectral sequence. Identifying $F_n/F_{n-1} \approx \bar{F}_n$ and using the formulas above, one easily checks that the first differential in this spectral sequence is the $\mathbb{Z}/2[w_1]$-linear map

$$\begin{aligned} \bar{F}_{2n} &\xrightarrow{\cdot w_1 w_2} \bar{F}_{2n+1} \\ \bar{F}_{2n+1} &\xrightarrow{0} \bar{F}_{2n+2}. \end{aligned}$$

It follows that the Q_1 Margolis homology of $H^*(MO_2)$ also has basis $\{U\, w_2^{2n+1}\}$.

Let M and N be the $\mathbf{A}_1$ modules below



M N

The map

$$\mathbb{Z}/2[w_2^4] \otimes \big(M \oplus N\big) \to H^*(MO_2)$$

is then an inclusion and induces an isomorphism of Margolis homology. If follows that

$$H^*MO_2 \approx \mathbb{Z}/2[w_2^4] \otimes \big(M \oplus N\big) \oplus \text{free}.$$

The location of the free modules can be determined from the Poincaré series. The Poincaré series for the generators is the quotient of

$$\frac{1}{(1-t)(1-t^2)} - \frac{(1+t+t^2+t^3+t^4+t^6)}{(1-t^8)}$$

by the Poincaré series $(1+t)(1+t^2)(1+t^3)$ of $\mathbf{A}_1$. This works out to be

$$\frac{t^2}{(1-t^2)(1-t^8)} = \frac{t^2+t^4}{(1-t^4)(1-t^8)}.$$

In fact the subspace of free modules is a free module over $\mathbf{A}_1[w_1^4, w_2^4]$ and has

$$\{Uw_1^2, Uw_2^2\}$$

as a basis. As before, it suffices from the Poincaré series above to check that the map

$$\mathbf{A}_1[w_1^4, w_2^4]\{Uw_1^2, Uw_2^2\} \to H^*(MO_3)$$

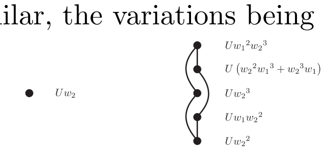
is a monomorphism, and for this to check that the set

$$\{\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2\,\big(Uw_1^2 w_1^{4k} w_2^{4\ell}\big), \mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2\,\big(Uw_2^2 w_1^{4k} w_2^{4\ell}\big)\}$$

is linearly independent. This is easily deduced from the fact that $\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2$ is linear over $\mathbb{Z}/2[w_1^4, w_2^4]$ and

$$\begin{aligned}\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2(Uw_1^2) &= Uw_1^6 w_2\\ \mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2(Uw_2^2) &= Uw_1^4 w_2^3.\end{aligned}$$

The situation with MTO_2 is similar, the variations being the use of the modules



M $\qquad$ N

and the Poincaré series

$$\frac{1+t^6}{(1-t^4)(1-t^8)}$$

for the generators of the free modules, from which one can conclude that the subspace of free modules is the sub $\mathbf{A}_1[w_1^4, w_2^4]$-module with basis

$$\{U, Uw_1^2w_2^2\}$$

on which the operator $\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2$ takes the value

$$Uw_1^4w_2, Uw_1^6w_2^3.$$

D.6. The case $s = \pm 3$

We now turn to the case of MO_3. This is the most complicated of the cases and the specific determination of the free summands was carried out with the aid of Mathematica.

It will be helpful to use the equivalence

$$BO_1 \times BSO_3 \to BO_3$$

classifying the tensor product of the defining vector bundles. Write

$$\begin{aligned} w_i &\in H^i(BO_3) \\ v_i &\in H^iBSO_3 \\ v_1 &\in H^1BO_1 \end{aligned}$$

for the corresponding Stiefel-Whitney classes, so that under the equivalence above we have

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 + v_1^2 \\ w_3 &= v_3 + v_2v_1 + v_1^3. \end{aligned}$$

and

$$\begin{aligned} v_1 &= w_1 \\ v_2 &= w_1^2 + w_2 \\ v_3 &= w_1w_2 + w_3. \end{aligned}$$

Now note that

$$\begin{aligned} Q_0 U &= U(v_1) \\ Q_1 U &= U(v_3 + v_1^3) \end{aligned}$$

so that as far as the Minor operators are concerned there is an isomorphism

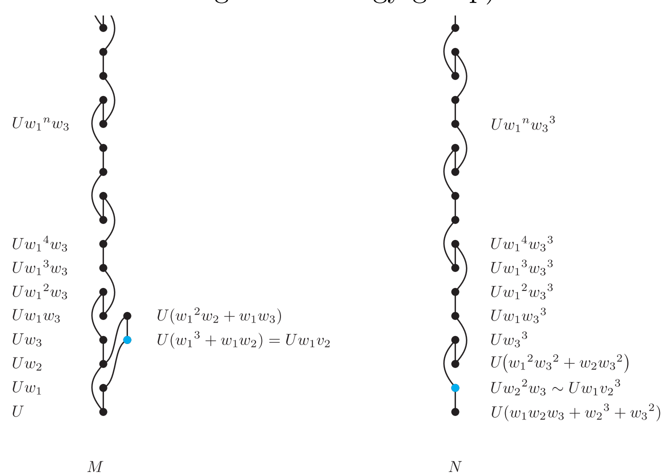
$$H^*(MO(3)) \approx H^*(MSO_3) \otimes H^*(MO_1).$$

From this one concludes that

$$H^*(MO_3; Q_0) = 0$$

and that the Margolis homology $H^*(MO_3; Q_1)$ has basis $\{Uv_1 v_2^{2j+1}\}$.

As in the case of $MSO(3)$ let M and N be the A_1-modules depicted below (in which the blue dot indications the location of the Margolis homology group)



Then the map

$$(M \oplus N) \otimes \mathbb{Z}/2[v_2^4] \to H^*(MO_3)$$

is a monomorphism and induces an isomorphism of Margolis homology groups. It follows that

$$H^*(MO_3) \approx (M \oplus N) \otimes \mathbb{Z}/2[v_2^4] \oplus \text{free}.$$

The Poincaré series for the indecomposables of the free modules (with U placed in degree 0) is the quotient of

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} - \frac{(1-t)^{-1} + t^3 + t^4 + t^6(1-t)^{-1}}{(1-t^8)}$$

by the Poincaré series $(1+t)(1+t^2)(1+t^3)$ of $\mathbf{A}_1$. It works out to be

$$\frac{t^2}{(1-t^4)\,(1-t^8)} + \frac{t^4 + t^5 + t^6 + t^9 + t^{10} + t^{11} + t^{12} + t^{15}}{(1-t^4)\,(1-t^8)\,(1-t^{12})}\ .$$

The free modules correspond to the sum of

$$\mathbf{A}_1[w_1^4, w_2^4]\{Uw_1^2\}$$

and the free $\mathbf{A}_1[w_1^4, w_2^4, w_3^4]$-module on

$$\{Uw_2^2, Uw_2w_3, Uw_3^2, Uw_2^3w_3, Uw_2^2w_3^2, Uw_1^2w_2^3w_3, Uw_1^2w_2^2w_3^2, Uw_2^3w_3^3\}$$

To see that these are linearly independent, one applies $\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2$ to reduce the problem to showing that the union of

$$\left\{U\left(w_1^6w_2 + w_1^5w_3\right)w_1^{4k}w_2^{4\ell}\right\}$$

and the set consisting of the products of $w_1^{4k}w_2^{4\ell}w_3^{4m}$ with the elements of

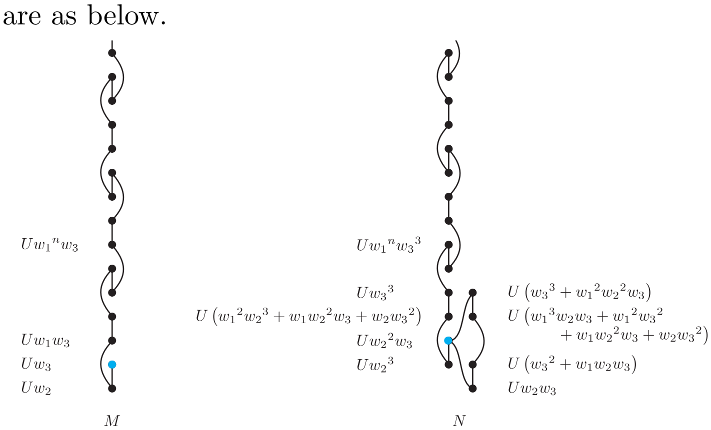
$$\begin{aligned}\Big\{&U\left(w_1^4w_2^3 + w_1^3w_2^2w_3 + w_1^2w_2w_3^2 + w_1w_3^3\right), U\left(w_1^4w_2^2w_3 + w_1^2w_3^3\right),\\ &U\left(w_1^4w_2w_3^2 + w_1^3w_3^3\right), U\left(w_1^2w_2^2w_3^3 + w_3^5\right), U\left(w_1^2w_2w_3^4 + w_1w_3^5\right), U\left(w_1^6w_2^4w_3 + w_1^2w_3^5\right),\\ &U\left(w_1^6w_2^3w_3^2 + w_1^5w_2^2w_3^3 + w_1^4w_2w_3^4 + w_1^3w_3^5\right),\\ &U\left(w_1^4w_2^4w_3^3 + w_3^7\right)\Big\}\end{aligned}$$

is linearly independent. A couple of maneuvers will make this obvious. First of all, let's apply the Thom isomorphism to get rid of the appearance of U. Next regard everything as a module over $\mathbb{Z}/2[w_1^4, w_2^4]$ and look at the associated graded of the increasing filtration by powers of w_3. Doing so reduces the problem to showing that the map from the free $\mathbb{Z}/2[w_1^4, w_2^4]$-module on

$$\{w_1^5w_3, w_1w_3^{3+4k}, w_1^2w_3^{3+4k}, w_1^3w_3^{3+4k}, w_3^{5+4k}, w_1w_3^{5+4k}, w_1^2w_3^{5+4k}, w_1^3w_3^{5+4k}, w_3^{7+4k}\}$$

to $H^*(BO_3)$ is a monomorphism, which is easy.

The analysis is similar for MTO_3. The Margolis homology is the same as that for MO_3 since the ratio of the two Thom classes is w_3^2 which is annihilated by the Milnor operators. The basic modules for MTO_3 are as below.



The Poincaré series for the free modules as the quotient of

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} - \frac{t^2(1-t)^{-1} + t^6(1-t)^{-1} + t^5 + t^6 + t^8 + t^9}{(1-t^8)}$$

by the Poincaré series $(1+t)(1+t^2)(1+t^3)$ of $\mathbf{A}_1$. This can be written as

$$\frac{t^7}{(1-t^4)\,(1-t^8)} + \frac{1 + t^4 + t^6 + t^9 + t^{10} + t^{11} + t^{15} + t^{17}}{(1-t^4)\,(1-t^8)\,(1-t^{12})}$$

The inclusion of the free summands turns out to be the sum of the $\mathbf{A}_1[w_1^4, w_2^4, w_3^4]$ module map

$$\begin{aligned}\mathbf{A}_1[w_1^4, w_2^4, w_3^4]\{U, Uw_2^2, Uw_1^2w_2^2, Uw_2^3w_3, Uw_2^2w_3^2,& \\ Uw_2w_3^3, Uw_2^3w_3^3, Uw_1^2w_2^3w_3^3\} &\to H^*(MTO_3)\end{aligned}$$

and the $\mathbf{A}_1[w_1^4, w_2^4]$-module map

$$\mathbf{A}_1[w_1^4, w_2^4]\{Uw_1^2w_2w_3\} \to H^*(MTO_3).$$

As above, to check this it suffices to apply $\mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^2$ to the generators above and show that the map from the sum of the free $\mathbb{Z}/2[w_1^4, w_2^4, w_3^4]$-module on

$$\begin{aligned}&\big\{U\left(w_1^4w_2 + w_1^3w_3\right), U\left(w_1^2w_2w_3^2 + w_1w_3^3\right), \\ &\quad U\left(w_1^6w_2^3 + w_1^5w_2^2w_3 + w_1^4w_2w_3^2 + w_1^3w_3^3\right), U\left(w_1^4w_2^4w_3 + w_3^5\right), \\ &\quad\quad U\left(w_1^4w_2^3w_3^2 + w_1^3w_2^2w_3^3 + w_1^2w_2w_3^4 + w_1w_3^5\right), \\ &\quad\quad\quad U\left(w_1^4w_2^2w_3^3 + w_1^2w_3^5\right), U\left(w_1^2w_2^2w_3^5 + w_3^7\right), U\left(w_1^6w_2^4w_3^3 + w_1^2w_3^7\right)\big\}\end{aligned}$$

and the free $\mathbb{Z}/2[w_1^4, w_2^4]$-module on

$$U\left(w_1^6w_2^2w_3 + w_1^4w_3^3\right)$$

to $H^*(MTO_3)$ is a monomorphism. Again, by filtering by powers of w_3, using the Thom isomorphism, and looking at the associated graded, it suffices to check that the map from

$$\mathbb{Z}/2[w_1^4, w_2^4]\{w_1^4w_3^3, w_1^3w_3^{1+4k}, w_1w_3^{3+4k}, w_1^3w_3^{3+4k}, w_3^{5+4k}, w_1^2w_3^{5+4k}, w_3^{7+4k}, w_1^2w_3^{7+4k}\}$$

to $H^*(BO_3)$ is a monomorphism, which is obvious.

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