Weil's Conjecture for Function Fields I

Dennis Gaitsgory and Jacob Lurie

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Chapter 1

Introduction

Let K be a global field (for example, the field $\mathbf{Q}$ of rational numbers). To every connected semisimple algebraic group G over K, one can associate a locally compact group $G(\mathbf{A})$, called the *group of adelic points of* G. The group $G(\mathbf{A})$ comes equipped with a canonical left-invariant measure μ_{Tam}, called *Tamagawa measure*, and a discrete subgroup $G(K) \subseteq G(\mathbf{A})$. The Tamagawa measure of the quotient $G(K)\backslash G(\mathbf{A})$ is a nonzero real number $\tau(G)$, called the *Tamagawa number* of the group G. A celebrated conjecture of Weil asserts that if the algebraic group G is simply connected, then the Tamagawa number $\tau(G)$ is equal to 1. In the case where K is a number field, Weil's conjecture was established by Kottwitz (building on earlier work of Langlands and Lai). Our goal in this book (and its sequel) is to show that Weil's conjecture holds also in the case where K is a function field. We begin in this chapter by reviewing the statement of Weil's conjecture, discussing several reformulations that are available in the case of a function field, and outlining the overall strategy of our proof.

The theory of Tamagawa numbers begins with the arithmetic theory of quadratic forms. Let $q = q(x_1, \ldots, x_n)$ and $q' = q'(x_1, \ldots, x_n)$ be positive-definite quadratic forms (that is, homogeneous polynomials of degree 2) with integer coefficients. We say that q and q' are *equivalent* if there is a linear change of coordinates which converts q into q', and that q and q' are *of the same genus* if they are equivalent modulo N, for every positive integer N. Equivalent quadratic forms are always of the same genus, but the converse need not be true. However, one can show that for a fixed nondegenerate quadratic form q, there are only finitely many equivalence classes of quadratic forms of the same genus. Even better, one can say exactly how many there are, counted with multiplicity: this is the subject of the famous *mass formula* of Smith-Minkowski-Siegel (Theorem 1.1.3.5), which we review in §1.1.

To an integral quadratic form q as above, one can associate an algebraic group SO_q over the field $\mathbf{Q}$ of rational numbers (which is connected and semisimple provided that q is nondegenerate). Tamagawa observed that the group of adelic points $\mathrm{SO}_q(\mathbf{A})$ can be equipped with a canonical left-invariant measure μ_{Tam}, and that the Smith-Minkowski-Siegel mass formula is equivalent to the assertion that the Tamagawa number $\tau(\mathrm{SO}_q) = \mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q})\backslash \mathrm{SO}_q(\mathbf{A}))$ is equal to 2. In [39], Weil gave a direct verification of the equality $\tau(\mathrm{SO}_q) = 2$ (thereby reproving the mass formula) and computed Tamagawa numbers in many other examples, observing in each case that *simply connected* groups had Tamagawa number equal to 1. This phenomenon became known as *Weil's conjecture* (Conjecture 1.2.6.4), which we review in §1.2.

In this book, we will study Weil's conjecture over *function fields*: that is, fields K which arise as rational functions on an algebraic curve X over a finite field $\mathbf{F}_q$. In §1.3, we reformulate Weil's conjecture as a mass formula, which counts the number of principal G-bundles over the algebraic curve X (see Conjecture 1.3.3.7). An essential feature of the function field setting is that the objects that we want to count (in this case, principal G-bundles) admit a "geometric" parametrization: they can be identified with $\mathbf{F}_q$-valued points of an algebraic stack $\mathrm{Bun}_G(X)$. In §1.4, we use this observation to reformulate Weil's conjecture yet again: it essentially reduces to a statement about the ℓ-adic cohomology of $\mathrm{Bun}_G(X)$ (Theorem 1.4.4.1), reflecting the heuristic idea that it should admit a "continuous Künneth decomposition"

$$\mathrm{H}^*(\mathrm{Bun}_G(X)) \simeq \bigotimes_{x \in X} \mathrm{H}^*(\mathrm{Bun}_G(\{x\})). \tag{1.1}$$

Our goal in this book is to give a precise formulation of (1.1), and to show that it implies Weil's conjecture (the proof of (1.1) will appear in a sequel to this book). In §1.5, we explain the basic ideas in the simpler setting where X is an algebraic curve over the field $\mathbf{C}$ of complex numbers, where we have the full apparatus of algebraic topology at our disposal. In this case, we formulate a version of (1.1) (see Theorem 1.5.4.10) and show that it is essentially equivalent to a classical result of Atiyah and Bott ([2]), which describes the structure of the rational cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$ (see Theorem 1.5.2.3). We close in §1.6 by giving a more detailed outline of the remainder of this book.

1.1 The Mass Formula

We begin this chapter by reviewing the theory of quadratic forms and the mass formula of Smith-Minkowski-Siegel (Theorem 1.1.3.5).

1.1.1 Quadratic Forms

We begin by introducing some terminology.

Definition 1.1.1.1. Let R be a commutative ring and let $n \geq 0$ be a nonnegative integer. A *quadratic form in n variables* over R is a polynomial

$$q(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$$

which is homogeneous of degree 2.

Given a pair of quadratic forms q and q' (over the same commutative ring R and in the same number of variables), we will say that q and q' are *isomorphic* if there is a linear change of coordinates which transforms q into q'. We can formulate this more precisely as follows:

Definition 1.1.1.2. Let R be a commutative ring, and let $q = q(x_1, \ldots, x_n)$ and $q' = q'(x_1, \ldots, x_n)$ be quadratic forms in n variables over R. An *isomorphism from q to q'* is an invertible matrix $A = (A_{i,j}) \in \mathrm{GL}_n(R)$ satisfying the identity

$$q(x_1, \ldots, x_n) = q'(\sum_{i=1}^{n} A_{1,i} x_i, \sum_{i=1}^{n} A_{2,i} x_i, \cdots, \sum_{i=1}^{n} A_{n,i} x_i).$$

We will say that q and q' are *isomorphic* if there exists an isomorphism from q to q'.

Remark 1.1.1.3 (The Orthogonal Group)**.** Let R be a commutative ring and let q be a quadratic form in n variables over R. The collection of isomorphisms from q to itself forms a subgroup $\mathrm{O}_q(R) \subseteq \mathrm{GL}_n(R)$. We will refer to $\mathrm{O}_q(R)$ as the *orthogonal group* of the quadratic form q.

Example 1.1.1.4. Let $\mathbf{R}$ be the field of real numbers and let $q : \mathbf{R}^n \to \mathbf{R}$ be the standard positive-definite quadratic form, given by $q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Then $\mathrm{O}_q(\mathbf{R})$ can be identified with the usual orthogonal group $\mathrm{O}(n)$. In particular, $\mathrm{O}_q(\mathbf{R})$ is a compact Lie group of dimension $(n^2 - n)/2$.

Remark 1.1.1.5. The theory of quadratic forms admits a "coordinate-free" formulation. Let R be a commutative ring and let M be an R-module. We will say that a function $q : M \to R$ is a *quadratic form* if it satisfies the following pair of identities:

- (a) The symmetric function

  $$b : M \times M \to R \qquad b(x, y) = q(x + y) - q(x) - q(y)$$

  is bilinear: that is, it satisfies the identities $b(x + x', y) = b(x, y) + b(x', y)$ and $b(\lambda x, y) = \lambda b(x, y)$ for $\lambda \in R$.

- (b) For $\lambda \in R$ and $x \in M$, we have $q(\lambda x) = \lambda^2 q(x)$.

In the special case $M = R^n$, a function $q : M \to R$ satisfies conditions (a) and (b) if and only if it is given by a quadratic form in the sense of Definition 1.1.1.1: that is, if and only if $q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} c_{i,j} x_i x_j$ for some scalars $c_{i,j} \in R$. Moreover, the scalars $c_{i,j}$ are uniquely determined by q: we have an identity

$$c_{i,j} = \begin{cases} q(e_i) & \text{if } i = j \\ q(e_i + e_j) - q(e_i) - q(e_j) & \text{if } i \neq j, \end{cases}$$

where $e_1, \ldots, e_n$ denotes the standard basis for $M = R^n$.

Remark 1.1.1.6 (Quadratic Forms and Symmetric Bilinear Forms)**.** Let R be a commutative ring and let M be an R-module. A *symmetric bilinear form* on M is a function $b : M \times M \to R$ satisfying the identities

$$b(x, y) = b(y, x) \quad b(x + x', y) = b(x, y) + b(x', y) \quad b(\lambda x, y) = \lambda b(x, y) \text{ for } \lambda \in R.$$

Every quadratic form $q : M \to R$ determines a symmetric bilinear form $b : M \times M \to R$, given by the formula $b(x, y) = q(x + y) - q(x) - q(y)$. Note that q and b are related by the formula $b(x, x) = 2q(x)$.

If R is a commutative ring in which 2 is invertible (for example, a field of characteristic different from 2), then the construction $q \mapsto b$ is bijective: that is, there is essentially no difference between quadratic forms and symmetric bilinear forms.

If $R = \mathbf{Z}$ is the ring of integers (or, more generally, any commutative ring in which 2 is not a zero-divisor), then the construction $q \mapsto b$ is injective. However, it is not surjective: a symmetric bilinear form $b : M \times M \to R$ can be obtained from a quadratic form on M if and only if it is *even*: that is, if and only if $b(x, x)$ is divisible by 2, for each $x \in M$.

1.1.2 Classification of Quadratic Forms

The most fundamental problem in the theory of quadratic forms can be formulated as follows:

Question 1.1.2.1. Let R be a commutative ring. Can one classify quadratic forms over R up to isomorphism?

The answer to Question 1.1.2.1 depends dramatically on the commutative ring R. As a starting point, let us assume that $R = \kappa$ is a field of characteristic different from 2. In that case, every quadratic form q over R can be diagonalized: that is, q is isomorphic to a quadratic form given by

$$q'(x_1, \ldots, x_n) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$$

for some coefficients $\lambda_i \in R$. These coefficients are not uniquely determined: for example, we are free to multiply each λ_i by the square of an element of $R^\times$, without changing the isomorphism class of the quadratic form q'. If the field R contains many squares, we can say more:

Example 1.1.2.2 (Quadratic Forms over $\mathbf{C}$)**.** Let $R = \mathbf{C}$ be the field of complex numbers (or, more generally, any algebraically closed field of characteristic different from 2). Then every quadratic form over R is isomorphic to a quadratic form given by

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_r^2$$

for some $0 \leq r \leq n$. Moreover, the integer r is uniquely determined: it is an isomorphism-invariant called the *rank* of the quadratic form q.

Example 1.1.2.3 (Quadratic Forms over the Real Numbers)**.** Let $R = \mathbf{R}$ be the field of real numbers. Then every quadratic form over R is isomorphic to a quadratic form given by the formula

$$q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - x_{a+2}^2 - \cdots - x_{a+b}^2$$

for some pair of nonnegative integers a and b satisfying $a + b \leq n$. Moreover, a theorem of Sylvester implies that the integers a and b are uniquely determined. The difference $a - b$ is an isomorphism-invariant of q, called the *signature of* q. We say that a quadratic form is *positive-definite* if it has signature n: that is, if it is isomorphic to the standard Euclidean form $q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Equivalently, a quadratic form q is positive-definite if it satisfies $q(v) > 0$ for every nonzero vector $v \in \mathbf{R}^n$.

Example 1.1.2.4 (Quadratic Forms over p-adic Fields)**.** Let $R = \mathbf{Q}_p$ be the field of p-adic rational numbers, for some prime number p. If $q = q(x_1, \ldots, x_n)$ is a *nondegenerate* quadratic form over R, then one can show that q is determined up to isomorphism by its *discriminant* (an element of the finite group $\mathbf{Q}_p^\times / \mathbf{Q}_p^{\times 2}$) and its *Hasse invariant* (an element of the group $\{\pm 1\}$). In particular, if p is odd and $n \gg 0$, then there are exactly eight isomorphism classes of quadratic forms in n-variables over $\mathbf{Q}_p$. When $p = 2$, there are sixteen isomorphism classes. See [31] for more details.

To address Question 1.1.2.1 for other fields, it is convenient to introduce some terminology.

Notation 1.1.2.5 (Extension of Scalars)**.** Let R be a commutative ring and let $q = \sum_{1 \leq i \leq j \leq n} c_{i,j} x_i x_j$ be a quadratic form in n variables over R. If $\phi : R \to S$ is a homomorphism of commutative rings, we let q_S denote the quadratic form over S given by $q_S(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} \phi(c_{i,j}) x_i x_j$. In this case, we will denote the orthogonal group $\mathrm{O}_{q_S}(S)$ simply by $\mathrm{O}_q(S)$.

Let q and q' be quadratic forms (in the same number of variables) over a commutative ring R. If q and q' are isomorphic, then they remain isomorphic after extending scalars along any ring homomorphism $\phi : R \to S$. In the case where $R = \mathbf{Q}$ is the field of rational numbers, we have the following converse:

Theorem 1.1.2.6 (The Hasse Principle)**.** *Let q and q' be quadratic forms over the field $\mathbf{Q}$ of rational numbers. Then q and q' are isomorphic if and only if the following conditions are satisfied:*

(a) *The quadratic forms $q_{\mathbf{R}}$ and $q'_{\mathbf{R}}$ are isomorphic.*

(b) *For every prime number p, the quadratic forms $q_{\mathbf{Q}_p}$ and $q'_{\mathbf{Q}_p}$ are isomorphic.*

Remark 1.1.2.7. Theorem 1.1.2.6 is known as the *Hasse-Minkowski* theorem: it is originally due to Minkowski, and was later generalized to arbitrary number fields by Hasse.

Remark 1.1.2.8. Theorem 1.1.2.6 asserts that the canonical map

$$\{\text{Quadratic forms over } \mathbf{Q}\}/\sim \to \prod_K \{\text{Quadratic forms over } K\}/\sim$$

is injective, where K ranges over the collection of all completions of $\mathbf{Q}$. It is possible to explicitly describe the image of this map (using the fact that the theory of quadratic forms over real and p-adic fields are well-understood; see Examples 1.1.2.3 and 1.1.2.4). We refer the reader to [31] for a detailed and readable account.

1.1.3 The Smith-Minkowski-Siegel Mass Formula

The Hasse-Minkowski theorem can be regarded as a "local-to-global" principle for quadratic forms over the rational numbers: it asserts that a pair of quadratic forms q and q' are "globally" isomorphic (that is, isomorphic over the field $\mathbf{Q}$) if and only if they are "locally" isomorphic (that is, they become isomorphic after extending scalars to each completion of $\mathbf{Q}$). We now consider the extent to which this principle holds for *integral* quadratic forms.

Definition 1.1.3.1. Let q and q' be quadratic forms over $\mathbf{Z}$. We will say that q and q' *have the same genus* if the following conditions are satisfied:

(a) The quadratic forms $q_{\mathbf{R}}$ and $q'_{\mathbf{R}}$ are isomorphic.

(b) For every positive integer N, the quadratic forms $q_{\mathbf{Z}/N\mathbf{Z}}$ and $q'_{\mathbf{Z}/N\mathbf{Z}}$ are isomorphic.

Remark 1.1.3.2. Let q be a quadratic form over $\mathbf{Z}$. We will say that q is *positive-definite* if the real quadratic form $q_{\mathbf{R}}$ is positive-definite (see Example 1.1.2.3). Equivalently, q is positive-definite if and only if $q(x_1, \ldots, x_n) > 0$ for every nonzero element $(x_1, \ldots, x_n) \in \mathbf{Z}^n$.

Note that if q is a positive-definite quadratic form over $\mathbf{Z}$ and q' is another quadratic form over $\mathbf{Z}$ in the same number of variables, then $q_{\mathbf{R}}$ and $q'_{\mathbf{R}}$ are isomorphic if and only if q' is also positive-definite. For simplicity, we will restrict our attention to positive-definite quadratic forms in what follows.

Remark 1.1.3.3. Let $\Lambda = \mathbf{Z}^n$ be a free abelian group of rank n and let $q : \Lambda \to \mathbf{Z}$ be a quadratic form. Then the associated bilinear form $b(x, y) = q(x + y) - q(x) - q(y)$ determines a group homomorphism $\rho : \Lambda \to \Lambda^\vee$, where $\Lambda^\vee = \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z})$ denotes the dual of Λ. If q is positive-definite, then the map ρ is injective. It follows that the quotient $\Lambda^\vee/\Lambda = \mathrm{coker}(\rho)$ is a finite abelian group.

Remark 1.1.3.4. Let $\Lambda = \mathbf{Z}^n$ be a free abelian group and let $q : \Lambda \to \mathbf{Z}$ be a positive-definite quadratic form. Then, for every integer d, the set $\Lambda_{\leq d} = \{\lambda \in \Lambda : q(\lambda) \leq d\}$ is finite. It follows that the orthogonal group $\mathrm{O}_q(\mathbf{Z})$ is finite. (Alternatively, one can prove this by observing that $\mathrm{O}_q(\mathbf{Z})$ is a discrete subgroup of the compact Lie group $\mathrm{O}_q(\mathbf{R})$).

If two positive-definite quadratic forms q and q' are isomorphic, then they have the same genus. The converse is generally false. However, it is true that each genus contains only *finitely many* quadratic forms, up to isomorphism. Moreover, one has the following:

Theorem 1.1.3.5 (Smith-Minkowski-Siegel Mass Formula)**.** *Let $\Lambda = \mathbf{Z}^n$ be a free abelian group of rank $n \geq 2$ and let $q : \Lambda \to \mathbf{Z}$ be a positive-definite quadratic form. Then*

$$\sum_{q'} \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|} = \frac{2|\Lambda^\vee/\Lambda|^{(n+1)/2}}{\prod_{m=1}^{n} \mathrm{Vol}(S^{m-1})} \prod_p c_p,$$

where the sum on the left hand side is taken over all isomorphism classes of quadratic forms q' in the genus of q, $\mathrm{Vol}(S^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ denotes the volume of the standard $(m-1)$-sphere, and the product on the right ranges over all prime numbers p, with individual factors c_p satisfying $c_p = \frac{2p^{kn(n-1)/2}}{|\mathrm{O}_q(\mathbf{Z}/p^k\mathbf{Z})|}$ for $k \gg 0$.

A version of Theorem 1.1.3.5 appears first in the work of Smith ([34]). It was rediscovered fifteen years later by Minkowski ([27]); Siegel later corrected an error in Minkowski's formulation ([33]) and extended the result.

Example 1.1.3.6. Let K be an imaginary quadratic field, let $\Lambda = \mathcal{O}_K$ be the ring of integers of K, and let $q : \Lambda \to \mathbf{Z}$ be the norm map. Then we can regard q as a positive-definite quadratic form (in two variables). In this case, the mass formula of Theorem 1.1.3.5 reduces to the class number formula for the field K.

Remark 1.1.3.7. In the statement of Theorem 1.1.3.5, if p is a prime number which does not divide $|\Lambda^\vee/\Lambda|$, then the formula $c_p = \frac{2p^{kn(n-1)/2}}{|\operatorname{O}_q(\mathbf{Z}/p^k\mathbf{Z})|}$ is valid for *all* positive integers k (not only for sufficiently large values of k); in particular, we can take $k = 1$ to obtain $c_p = \frac{2p^{n(n-1)/2}}{|\operatorname{O}_q(\mathbf{Z}/p\mathbf{Z})|}$.

1.1.4 The Unimodular Case

To appreciate the content of Theorem 1.1.3.5, it is convenient to consider the simplest case (which is already quite nontrivial).

Definition 1.1.4.1. Let $\Lambda = \mathbf{Z}^n$ be a free abelian group of finite rank and let $q : \Lambda \to \mathbf{Z}$ be a quadratic form. We will say that q is *unimodular* if the quotient group $\Lambda^\vee/\Lambda$ is trivial.

Remark 1.1.4.2. Let $\Lambda = \mathbf{Z}^n$ be a free abelian group of finite rank and let $q : \Lambda \to \mathbf{Z}$ be a quadratic form. Then q is unimodular if and only if it remains nondegenerate after extension of scalars to $\mathbf{Z}/p\mathbf{Z}$, for every prime number p. In particular, if q and q' are positive-definite quadratic forms of the same genus, then q is unimodular if and only if q' is unimodular. In fact, the converse also holds: any two unimodular quadratic forms (in the same number of variables) are of the same genus.

Remark 1.1.4.3. Unimodularity is a very strong condition on a quadratic form $q : \Lambda \to \mathbf{Z}$. For example, the existence of a quadratic form $q : \Lambda \to \mathbf{Z}$ which is both unimodular and positive-definite guarantees that the rank of Λ is divisible by 8.

In the unimodular case, the mass formula of Theorem 1.1.3.5 admits several simplifications:

- The positive-definite unimodular quadratic forms comprise a single genus, so the left hand side of the mass formula is simply a sum over isomorphism classes of unimodular quadratic forms.
- The term $|\Lambda^\vee/\Lambda|$ can be neglected (by virtue of unimodularity).
- Because there are no primes which divide $|\Lambda^\vee/\Lambda|$, the Euler factors c_p appearing in the mass formula are easy to evaluate (Remark 1.1.3.7).

Taking these observations into account, we obtain the following:

Theorem 1.1.4.4 (Mass Formula: Unimodular Case)**.** *Let n be an integer which is a positive multiple of* 8*. Then*

$$\sum_q \frac{1}{|\operatorname{O}_q(\mathbf{Z})|} = \frac{2\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(n/2)}{\operatorname{Vol}(S^0)\operatorname{Vol}(S^1)\cdots\operatorname{Vol}(S^{n-1})}$$
$$= \frac{B_{n/4}}{n}\prod_{1\le j<n/2}\frac{B_j}{4j}.$$

Here ζ denotes the Riemann zeta function, B_j denotes the jth Bernoulli number, and the sum is taken over all isomorphism classes of positive-definite, unimodular quadratic forms q in n variables.

Example 1.1.4.5. Let $n = 8$. The right hand side of Theorem 1.1.4.4 evaluates to $1/696729600$. The integer $696729600 = 2^{14}3^5 5^2 7$ is the order of the Weyl group of the exceptional Lie group E_8, which is also the automorphism group of the root lattice Λ of E_8. Consequently, the fraction $1/696729600$ also appears as one of the summands on the left hand side of the mass formula. It follows from Theorem 1.1.4.4 that no other terms appear on the left hand side: that is, there is a unique positive-definite unimodular quadratic form in eight variables (up to isomorphism), given by the the E_8-lattice Λ.

Remark 1.1.4.6. Theorem 1.1.4.4 allows us to count the number of positive-definite unimodular quadratic forms in a given number of variables, where each quadratic form q is counted with multiplicity $\frac{1}{|\operatorname{O}_q(\mathbf{Z})|}$. Each of the groups $\operatorname{O}_q(\mathbf{Z})$ has at least two elements (since $\operatorname{O}_q(\mathbf{Z})$ contains the group $\langle \pm 1\rangle$), so that the left hand side of Theorem 1.1.4.4 is at most $\frac{C}{2}$, where C is the number of isomorphism classes of positive-definite unimodular quadratic forms in n variables. In particular, Theorem 1.1.4.4 gives an inequality

$$C \ge \frac{4\zeta(2)\zeta(4)\cdots\zeta(n-4)\zeta(n-2)\zeta(n/2)}{\operatorname{Vol}(S^0)\operatorname{Vol}(S^1)\cdots\operatorname{Vol}(S^{n-1})}.$$

The right hand side of this inequality grows very quickly with n. For example, when $n = 32$, we can deduce the existence of more than eighty million pairwise nonisomorphic (positive-definite) unimodular quadratic forms in n variables.

1.2 Adelic Formulation of the Mass Formula

In this section, we sketch a reformulation of the Smith-Minkowski-Siegel mass formula, following ideas of Tamagawa and Weil.

1.2.1 The Adelic Group $\mathrm{O}_q(\mathbf{A})$

Throughout this section, we fix a positive-definite quadratic form $q = q(x_1, \ldots, x_n)$ over the integers $\mathbf{Z}$. Let us attempt to classify quadratic forms q' of the same genus. As a first step, it will be convenient to reformulate Definition 1.1.3.1 in the language of *adeles*.

Notation 1.2.1.1. For each prime number p, we let $\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^k\mathbf{Z}$ denote the ring of p-adic integers. We let $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p \simeq \varprojlim_{N>0} \mathbf{Z}/N\mathbf{Z}$ denote the profinite completion of $\mathbf{Z}$. We let $\mathbf{A}_f$ denote the tensor product $\widehat{\mathbf{Z}} \otimes \mathbf{Q}$, which we refer to as the *ring of finite adeles*. We will generally abuse notation by identifying $\widehat{\mathbf{Z}}$ and $\mathbf{Q}$ with their images in $\mathbf{A}_f$. Let $\mathbf{A}$ denote the Cartesian product $\mathbf{A}_f \times \mathbf{R}$. We refer to $\mathbf{A}$ as the *ring of adeles*.

Proposition 1.2.1.2. *Let $q' = q'(x_1, \ldots, x_n)$ be a quadratic form over $\mathbf{Z}$. Then q and q' have the same genus if and only if they become isomorphic after extending scalars to the product ring $\widehat{\mathbf{Z}} \times \mathbf{R}$.*

Proof. Suppose that q and q' have the same genus; we will show that q and q' become isomorphic after extension of scalars to $\widehat{\mathbf{Z}} \times \mathbf{R}$ (the converse is immediate and left to the reader). Since $q_{\mathbf{R}}$ and $q'_{\mathbf{R}}$ are isomorphic, it will suffice to show that $q_{\widehat{\mathbf{Z}}}$ and $q'_{\widehat{\mathbf{Z}}}$ are isomorphic. Using the product decomposition $\widehat{\mathbf{Z}} \simeq \prod_p \mathbf{Z}_p$, we are reduced to showing that $q_{\mathbf{Z}_p}$ and $q'_{\mathbf{Z}_p}$ are isomorphic for each prime number p.

For each $m > 0$, our assumption that q and q' have the same genus guarantees that we can choose a matrix $A_m \in \mathrm{GL}_n(\mathbf{Z}/p^m\mathbf{Z})$ such that $q = q' \circ A_m$. Choose a matrix $\overline{A}_m \in \mathrm{GL}_n(\mathbf{Z}_p)$ which reduces to A_m modulo p^m (note that the natural map $\mathrm{GL}_n(\mathbf{Z}_p) \to \mathrm{GL}_n(\mathbf{Z}/p^m\mathbf{Z})$ is surjective, since the invertibility of a matrix over $\mathbf{Z}_p$ can be checked after reduction modulo p). Note that the inverse limit topology on $\mathrm{GL}_n(\mathbf{Z}_p)$ is compact, so the sequence $\{\overline{A}_m\}_{m>0}$ has a subsequence which converges to some limit $A \in \mathrm{GL}_n(\mathbf{Z}_p)$. By continuity, we have $q = q' \circ A$, so that the quadratic forms $q_{\mathbf{Z}_p}$ and $q'_{\mathbf{Z}_p}$ are isomorphic. □

Corollary 1.2.1.3. *Let $q' = q'(x_1, \ldots, x_n)$ be a quadratic form over $\mathbf{Z}$ which is of the same genus as q. Then the rational quadratic forms $q_{\mathbf{Q}}$ and $q'_{\mathbf{Q}}$ are isomorphic.*

Proof. By virtue of the Hasse principle (Theorem 1.1.2.6), it will suffice to show that the quadratic forms q and q' become isomorphic after extension of scalars to $\mathbf{R}$ and to $\mathbf{Q}_p$, for each prime number p. In the first case, this is immediate; in the second, it follows from Proposition 1.2.1.2 (since there exists a ring homomorphism $\widehat{\mathbf{Z}} \to \mathbf{Q}_p$). □

Construction 1.2.1.4. Let q' be a quadratic form over $\mathbf{Z}$ in the same genus as q. Then:

- By virtue of Proposition 1.2.1.2, the quadratic forms q and q' become isomorphic after extension of scalars to $\widehat{\mathbf{Z}} \times \mathbf{R}$. That is, we can choose a matrix $A \in \mathrm{GL}_n(\widehat{\mathbf{Z}} \times \mathbf{R})$ satisfying $q = q' \circ A$.

- By virtue of Corollary 1.2.1.3, the quadratic forms q and q' become isomorphic after extension of scalars to $\mathbf{Q}$. That is, we can choose a matrix $B \in \mathrm{GL}_n(\mathbf{Q})$ satisfying $q = q' \circ B$.

Let us abuse notation by identifying $\mathrm{GL}_n(\widehat{\mathbf{Z}} \times \mathbf{R})$ and $\mathrm{GL}_n(\mathbf{Q})$ with their images in $\mathrm{GL}_n(\mathbf{A})$. Then we can consider the product $B^{-1}A \in \mathrm{GL}_n(\mathbf{A})$. We let $[q']$ denote the image of $B^{-1}A$ in the double quotient

$$\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}) / \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}).$$

Note that A is well-defined up to right multiplication by elements of the orthogonal group $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ and that B is well-defined up to right multiplication by elements of the orthogonal group $\mathrm{O}_q(\mathbf{Q})$. It follows that the double coset $[q']$ does not depend on the choice of matrices A and B.

Proposition 1.2.1.5. *Construction 1.2.1.4 determines a bijection*

$$\{\textit{Quadratic forms in the genus of } q\}/\textit{isomorphism} \to \mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A}) / \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}).$$

Proof. Let us sketch the inverse bijection (which is actually easier to define, since it does not depend on the Hasse-Minkowski theorem). For each element $\gamma \in \mathrm{O}_q(\mathbf{A})$, consider the intersection

$$\Lambda(\gamma) = \mathbf{Q}^n \cap \gamma((\widehat{\mathbf{Z}} \times \mathbf{R})^n) \subseteq \mathbf{A}^n .$$

Then $\Lambda(\gamma)$ is a free abelian group of rank n. Moreover, the quadratic form $q_{\mathbf{A}} : \mathbf{A}^n \to \mathbf{A}$ carries $\mathbf{Q}^n$ to $\mathbf{Q}$ and carries $\gamma((\widehat{\mathbf{Z}} \times \mathbf{R})^n)$ into $\widehat{\mathbf{Z}} \times \mathbf{R}$ (since it is invariant under γ), and therefore restricts to a quadratic form $q_\gamma : \Lambda(\gamma) \to \mathbf{Q} \cap (\widehat{\mathbf{Z}} \times \mathbf{R}) = \mathbf{Z}$.

Choose an isomorphism $\mathbf{Z}^n \to \Lambda(\gamma)$, which we can extend to an element $\alpha \in \mathrm{GL}_n(\mathbf{A})$. The condition that $\alpha(\mathbf{Z}^n) = \Lambda(\gamma)$ guarantees that $\alpha \in \mathrm{GL}_n(\mathbf{Q})$ and $\gamma^{-1} \circ \alpha \in \mathrm{GL}_n(\widehat{\mathbf{Z}} \times \mathbf{R})$. It follows that we can take $A = \alpha^{-1} \circ \gamma$ and $B = \alpha^{-1}$ in Construction 1.2.1.3, so that $[q_\gamma]$ is the double coset of $B^{-1}A = \gamma$. This shows that $\gamma \mapsto q_\gamma$ determines a right inverse to Construction 1.2.1.3; we leave it to the reader to verify that it is also a left inverse. □

Let γ be an element of $\mathrm{O}_q(\mathbf{A})$ and let $q_\gamma : \Lambda(\gamma) \to \mathbf{Z}$ be as in the proof of Proposition 1.2.1.5. Then the finite group $\mathrm{O}_{q_\gamma}(\mathbf{Z})$ can be identified with the subgroup of $\mathrm{O}_q(\mathbf{A})$ which preserves the lattice $\Lambda(\gamma)$, or equivalently with the intersection $\gamma^{-1}\,\mathrm{O}_q(\mathbf{Q})\gamma \cap \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. Combining this observation with Proposition 1.2.1.5, we obtain the following approximation to Theorem 1.1.3.5:

Proposition 1.2.1.6. *Let $q = q(x_1, \ldots, x_n)$ be a positive-definite quadratic form in $n \geq 2$ variables over $\mathbf{Z}$. Then*

$$\sum_{q'} \frac{1}{|\operatorname{O}_{q'}(\mathbf{Z})|} = \sum_{\gamma} \frac{1}{|\gamma^{-1} \operatorname{O}_q(\mathbf{Q})\gamma \cap \operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})|}.$$

Here the sum on the left hand side ranges over isomorphism classes of quadratic forms in the genus of q, while the sum on the right hand side ranges over a set of representatives for the double quotient $\operatorname{O}_q(\mathbf{Q})\backslash \operatorname{O}_q(\mathbf{A})/ \operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$.

Warning 1.2.1.7. It is not *a priori* obvious that the sums appearing in Proposition 1.2.1.6 are convergent. However, it is clear that the left hand side converges if and only if the right hand side converges, since the summands can be identified term-by-term.

1.2.2 Adelic Volumes

Let us now regard $\mathbf{A}_f$ as a topological commutative ring, where the sets $\{N\widehat{\mathbf{Z}} \subseteq \mathbf{A}_f\}_{N>0}$ form a neighborhood basis of the identity. We regard the ring of adeles $\mathbf{A} = \mathbf{A}_f \times \mathbf{R}$ as equipped with the product topology (where $\mathbf{R}$ is endowed with the usual Euclidean topology). Then:

- (a) The commutative ring $\mathbf{A}$ is locally compact.
- (b) There is a unique ring homomorphism $\mathbf{Q} \to \mathbf{A}$, which embeds $\mathbf{Q}$ as a discrete subring of $\mathbf{A}$.
- (c) The commutative ring $\mathbf{A}$ contains the product $\widehat{\mathbf{Z}} \otimes \mathbf{R}$ as an open subring.

Now suppose that $q = q(x_1, \ldots, x_n)$ is a positive-definite quadratic form over $\mathbf{Z}$. The topology on $\mathbf{A}$ induces a topology on the general linear group $\mathrm{GL}_n(\mathbf{A})$, which contains the orthogonal group $\operatorname{O}_q(\mathbf{A})$ as a closed subgroup. We then have the following analogues of (a), (b), and (c):

- (a') The orthogonal group $\operatorname{O}_q(\mathbf{A})$ inherits the structure of a locally compact topological group.
- (b') The canonical map $\operatorname{O}_q(\mathbf{Q}) \to \operatorname{O}_q(\mathbf{A})$ embeds $\operatorname{O}_q(\mathbf{Q})$ as a discrete subgroup of $\operatorname{O}_q(\mathbf{A})$.
- (c') The canonical map $\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \to \operatorname{O}_q(\mathbf{A})$ embeds $\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \simeq \operatorname{O}_q(\widehat{\mathbf{Z}}) \times \operatorname{O}_q(\mathbf{R})$ as an open subgroup of $\operatorname{O}_q(\mathbf{A})$. Moreover, the group $\operatorname{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ is also compact: the topological group $\operatorname{O}_q(\widehat{\mathbf{Z}})$ is profinite, and $\operatorname{O}_q(\mathbf{R})$ is a compact Lie group of dimension $n(n-1)/2$ (by virtue of our assumption that q is positive-definite)

Let μ denote a left-invariant measure on the locally compact group $\mathrm{O}_q(\mathbf{A})$ (the theory of Haar measure guarantees that such a measure exists and is unique up to multiplication by a positive scalar). One can show that μ is also right-invariant: that is, the group $\mathrm{O}_q(\mathbf{A})$ is unimodular. It follows that μ induces a measure on the quotient $\mathrm{O}_q(\mathbf{Q})\backslash \mathrm{O}_q(\mathbf{A})$, which is invariant under the right action of $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. We will abuse notation by denoting this measure also by μ. Write $\mathrm{O}_q(\mathbf{Q})\backslash \mathrm{O}_q(\mathbf{A})$ as a union of orbits $\bigcup_{x \in X} O_x$ for the action of the group $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$, where X denotes the set of double cosets $\mathrm{O}_q(\mathbf{Q})\backslash \mathrm{O}_q(\mathbf{A}) / \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$. If $x \in X$ is a double coset represented by an element $\gamma \in \mathrm{O}_q(\mathbf{A})$, then we can identify the orbit O_x with the quotient of $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})$ by the finite subgroup $\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \cap \gamma^{-1} \mathrm{O}_q(\mathbf{Q})\gamma$. We therefore obtain an equality

$$\sum_{\gamma} \frac{1}{|\gamma^{-1} \mathrm{O}_q(\mathbf{Q})\gamma \cap \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})|} = \sum_{x \in X} \frac{\mu(O_x)}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} \tag{1.2}$$

$$= \frac{\mu(\mathrm{O}_q(\mathbf{Q})\backslash \mathrm{O}_q(\mathbf{A}))}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}. \tag{1.3}$$

Combining (1.3) with Proposition 1.2.1.6, we obtain another approximation to Theorem 1.1.3.5:

Proposition 1.2.2.1. *Let $q = q(x_1, \ldots, x_n)$ be a positive-definite quadratic form in $n \geq 2$ variables over $\mathbf{Z}$, and let μ be a left-invariant measure on the locally compact group $\mathrm{O}_q(\mathbf{A})$. Then*

$$\sum_{q'} \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|} = \frac{\mu(\mathrm{O}_q(\mathbf{Q})\backslash \mathrm{O}_q(\mathbf{A}))}{\mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))},$$

where the sum on the left hand side is taken over isomorphism classes of quadratic forms q' in the genus of q.

In what follows, it will be convenient to consider a further reformulation of Proposition 1.2.2.1 in terms of *special* orthogonal groups.

Definition 1.2.2.2. Let q be a quadratic form over $\mathbf{Z}$. For every commutative ring R, we let $\mathrm{SO}_q(R) = \{A \in \mathrm{SL}_n(R) : q = q \circ A\}$. We will refer to $\mathrm{SO}_q(R)$ as the *special orthogonal group of q over R.*

Warning 1.2.2.3. The group $\mathrm{SO}_q(R)$ of Definition 1.2.2.2 can behave strangely when 2 is a zero-divisor in R. For example, if R is a field of characteristic 2 and the quadratic form q_R is nondegenerate, we have $\mathrm{SO}_q(R) = \mathrm{O}_q(R)$. In what follows, this will not concern us: we will consider the groups $\mathrm{SO}_q(R)$ only in the case where R is torsion-free.

Let $G = \{x \in \mathbf{A}^{\times} : x^2 = 1\}$ denote the group of square-roots of unity in $\mathbf{A}$, which we can identify with the Cartesian product $\prod_v \langle \pm 1 \rangle$, where v ranges over all the completions of $\mathbf{Q}$. The group $\mathrm{O}_q(\mathbf{A})$ fits into a short exact sequence

$$0 \to \mathrm{SO}_q(\mathbf{A}) \to \mathrm{O}_q(\mathbf{A}) \xrightarrow{\det} G \to 0.$$

Suppose we are given left-invariant measures μ' and μ'' on $\mathrm{SO}_q(\mathbf{A})$ and G, respectively. We can then use μ' and μ'' to build a left-invariant measure μ on $\mathrm{O}_q(\mathbf{A})$, given by the formula

$$\mu(U) = \int_{x \in G} \mu'(\mathrm{SO}_q(\mathbf{A}) \cap \overline{x}^{-1} U) d\mu'',$$

where $\overline{x}$ denotes any element of $\mathrm{O}_q(\mathbf{A})$ lying over x. An elementary calculation then gives an equality

$$\begin{aligned} \mu(\mathrm{O}_q(\mathbf{Q}) \backslash \mathrm{O}_q(\mathbf{A})) &= \mu'(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A})) \mu''(\langle \pm 1 \rangle \backslash G) \\ &= \frac{\mu'(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A})) \mu''(G)}{2}. \end{aligned}$$

We also have a short exact sequence

$$0 \to \mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \to \mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R}) \xrightarrow{\det} H \to 0,$$

where $H \subseteq G$ is the image of $\det|_{\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})}$. This yields an identity

$$\begin{aligned} \mu(\mathrm{O}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) &= \mu'(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) \mu''(H) \\ &= \frac{\mu'(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) \mu''(G)}{|G/H|}. \end{aligned}$$

Replacing μ by μ' in our notation, we obtain the following:

Proposition 1.2.2.4. *Let $q = q(x_1, \ldots, x_n)$ be a positive-definite quadratic form in $n \geq 2$ variables over $\mathbf{Z}$, and let μ be a left-invariant measure on the locally compact group $\mathrm{SO}_q(\mathbf{A})$. Then*

$$\sum_{q'} \frac{1}{|\mathrm{O}_{q'}(\mathbf{Z})|} = 2^{k-1} \frac{\mu(\mathrm{SO}_q(\mathbf{Q}) \backslash \mathrm{SO}_q(\mathbf{A}))}{\mu(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))},$$

where the sum on the left hand side is taken over isomorphism classes of quadratic forms q' in the genus of q, and k is the number of primes p for which $\mathrm{SO}_q(\mathbf{Z}_p) = \mathrm{O}_q(\mathbf{Z}_p)$.

Warning 1.2.2.5. In the statement of Proposition 1.2.2.4, it is not *a priori* obvious that either the right hand side or the left hand side is finite. However, the above reasoning shows that if one side is infinite, then so is the other.

1.2.3 Digression: Local Fields

In the statement of Proposition 1.2.2.4, the left-invariant measure μ on $\mathrm{SO}_q(\mathbf{A})$ is not unique. However, it is unique up to scalar multiplication, so the quotient

$$\frac{\mu(\mathrm{SO}_q(\mathbf{Q})\backslash \mathrm{SO}_q(\mathbf{A}))}{\mu(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))}$$

is independent of the choice of μ. However, one can do better than this: Tamagawa observed that the group $\mathrm{SO}_q(\mathbf{A})$ admits a *canonical* left-invariant measure, which can be used to evaluate the numerator and denominator independently. The construction of this measure (which we review in §1.2.5) will require some general observations about algebraic varieties over local fields, which we recall for the reader's convenience.

Notation 1.2.3.1. Let K be a local field. Then the additive group $(K, +)$ is locally compact, and therefore admits a translation-invariant measure μ_K, which is unique up to multiplication by a positive real number. In what follows, we will normalize μ_K as follows:

- If K is isomorphic to the field $\mathbf{R}$ of real numbers, we take μ_K to be the standard Lebesgue measure.
- If K is isomorphic to the field $\mathbf{C}$ of complex numbers, then we take μ_K to be *twice* the usual Lebesgue measure (to understand the motivation for this convention, see Example 1.2.3.7 below).
- Suppose that K is a *nonarchimedean* local field: that is, K is the fraction field of a discrete valuation ring $\mathcal{O}_K$ having finite residue field. In this case, we take μ_K to be the unique translation-invariant measure satisfying $\mu_K(\mathcal{O}_K) = 1$.

For every nonzero element $x \in K$, we can define a new translation-invariant measure on K by the construction $U \mapsto \mu_K(xU)$. It follows that there is a unique positive real number $|x|_K$ satisfying $\mu_K(xU) = |x|_K \mu_K(U)$ for every measurable subset $U \subseteq K$. The construction $x \mapsto |x|_K$ determines a group homomorphism $|\bullet|_K : K^\times \to \mathbf{R}_{>0}$. By convention, we extend the definition to *all* elements of K by the formula $|0|_K = 0$.

Example 1.2.3.2. If $K = \mathbf{R}$ is the field of real numbers, then the function $x \mapsto |x|_K$ is the usual absolute value function $\mathbf{R} \mapsto \mathbf{R}_{\geq 0}$. If $K = \mathbf{C}$ is the field of complex numbers, then the function $x \mapsto |x|_K$ is the *square* of the usual absolute value function $\mathbf{C} \mapsto \mathbf{R}_{\geq 0}$.

Example 1.2.3.3. If K is a nonarchimedean local field and $x \in \mathcal{O}_K$, then $|x|_K = |\,\mathcal{O}_K/(x)|^{-1}$ (with the convention that the right hand side vanishes when $x = 0$, so that $\mathcal{O}_K/(x) \simeq \mathcal{O}_K$ is infinite). Equivalently, if the residue field of $\mathcal{O}_K$ is a finite field $\mathbf{F}_q$

with q elements and $\pi \in \mathcal{O}_K$ is a uniformizer, then every nonzero element $x \in K$ can be written as a product $u\pi^k$, where u is an invertible element of $\mathcal{O}_K$. In this case, we have $|x|_K = |u\pi^k|_K = q^{-k}$.

In the special case where $K = \mathbf{Q}_p$ is the field of p-adic rational numbers, we will denote the absolute value $| \bullet |_K$ by $x \mapsto |x|_p$.

Warning 1.2.3.4. If $K \simeq \mathbf{R}$ or K is a nonarchimedean local field, then the function $x \mapsto |x|_K$ is a *norm* on the field K: that is, it satisfies the triangle inequality $|x+y|_K \leq |x|_K + |y|_K$. This is not true in the case $K = \mathbf{C}$ (however, it is not far from being true: the function $x \mapsto |x|_{\mathbf{C}}$ is the square of the usual Euclidean norm).

Construction 1.2.3.5 (The Measure Associated to a Differential Form)**.** Let K be a local field, let X be a smooth algebraic variety of dimension n over K, and let ω be an (algebraic) differential form of degree n on X. Let $X(K)$ denote the set of K-valued points of X, which we regard as a locally compact topological space. For each point $x \in X(K)$, we can choose a Zariski-open subset $U \subseteq X$ which contains x and a system of local coordinates

$$\vec{f} = (f_1, \ldots, f_n) : U \to \mathbf{A}^n,$$

having the property that the differential form $df_1 \wedge df_2 \wedge \cdots \wedge df_n$ is nowhere-vanishing on U. It follows that we can write $\omega|_U = g df_1 \wedge \cdots \wedge df_n$ for some regular function g on U, and that $\vec{f}$ induces a local homeomorphism of topological space $U(K) \to \mathbf{A}^n(K) = K^n$. Let μ_K^n denote the standard measure on K^n (given by the nth power of the measure described in Notation 1.2.3.1), let $\vec{f}^* \mu_K^n$ denote the pullback of μ_K^n to the topological space $U(K)$, and define $\mu_\omega^U = |g|_K \vec{f}^* \mu_K^n$. Then μ_ω^U is a Borel measure on the topological space $U(K)$. It is not difficult to see that μ_ω^U depends only on the open set $U \subseteq X$, and not on the system of coordinates $\vec{f} : U \to \mathbf{A}^n$. Moreover, if V is an open subset of U, we have $\mu_\omega^V = \mu_\omega^U|_{V(X)}$. It follows that there is a unique measure μ_ω on the topological space $X(K)$ satisfying $\mu_\omega|_{U(X)} = \mu_\omega^U$ for every Zariski-open subset $U \subseteq X$ which admits a system of coordinates $\vec{f} : U \to \mathbf{A}^d$. We will refer to μ_ω as *the measure associated to* ω.

Example 1.2.3.6. Let K be the field $\mathbf{R}$ of real numbers. If X is a smooth algebraic variety of dimension n over K, then $X(K)$ is a smooth manifold of dimension n. Moreover, an algebraic differential form of degree n on X determines a smooth differential form of degree n on $X(K)$, and μ_ω the measure obtained by integrating (the absolute value of) ω.

Example 1.2.3.7. Let K be the field $\mathbf{C}$ of complex numbers. If X is a smooth algebraic variety of dimension n over K, then $X(K)$ is a smooth manifold of dimension $2n$. Moreover, an algebraic differential form ω on X determines a smooth $\mathbf{C}$-valued differential form of degree n on $X(K)$, which (by slight abuse of notation) we will also denote by ω. Then the measure μ_ω of Construction 1.2.3.5 is obtained by integrating (the absolute value of) the $\mathbf{R}$-valued $(2n)$-form $\sqrt{-1}^n \omega \wedge \overline{\omega}$. Note that this identification depends on our convention that μ_K is *twice* the usual Lebesgue measure on $\mathbf{C}$ (see Notation 1.2.3.1).

Example 1.2.3.8. Let K be a nonarchimedean local field and let X be a smooth algebraic variety of dimension n over K. Suppose that X is the generic fiber of a scheme $\overline{X}$ which is smooth of dimension n over the ring of integers $\mathcal{O}_K$. Let $\overline{\omega}$ be a nowhere-vanishing n-form on $\overline{X}$, and let $\omega = \overline{\omega}|_X$ be the associated algebraic differential form on X. Then $\overline{X}(\mathcal{O}_K)$ is a compact open subset of $X(K)$, and the measure μ_ω of Construction 1.2.3.5 satisfies the equality $\mu_\omega(\overline{X}(\mathcal{O}_K)) = \frac{|\overline{X}(\kappa)|}{|\kappa|^n}$, where κ denotes the residue field of $\mathcal{O}_K$ (see Variant 1.2.3.10 below).

Remark 1.2.3.9 (Rescaling)**.** In the situation of Construction 1.2.3.5, suppose that we are given a scalar $\lambda \in K$. Then $\mu_{\lambda\omega} = |\lambda|_K \mu_\omega$, where $|\lambda|_K$ is defined as in Notation 1.2.3.1.

In §1.2.5, it will be convenient to consider a slight generalization of Example 1.2.3.8, where the integral model $\overline{X}$ is not assumed to be smooth.

Variant 1.2.3.10. Let K be a nonarchimedean local field and let X be a smooth algebraic variety of dimension n over K. Suppose that X is the generic fiber of a $\mathcal{O}_K$-scheme $\overline{X}$ which fits into a pullback diagram

$$\begin{array}{ccc} \overline{X} & \longrightarrow & \operatorname{Spec}(\mathcal{O}_K) \\ \downarrow & & \downarrow \\ \overline{Y} & \xrightarrow{\overline{f}} & \overline{Z}, \end{array}$$

where $\overline{Y}$ and $\overline{Z}$ are smooth $\mathcal{O}_K$-schemes of dimension d_Y and $d_Z = d_Y - n$, and $\overline{f}$ restricts to a smooth morphism of generic fibers $f : Y \to Z$. Let ω_Y and ω_Z be nowhere-vanishing algebraic differential forms of degree d_Y and d_Z on $\overline{Y}$ and $\overline{Z}$, respectively. Using the canonical isomorphism $\Omega_Y^{d_Y}|_X \simeq \Omega_X^n \otimes (f^* \Omega_Z^{d_Z})|_X$, we see that the ratio $\omega = \frac{\omega_Y}{f^* \omega_Z}|_X$ can be regarded as an algebraic differential form on X. Let π denote a uniformizer for the discrete valuation ring $\mathcal{O}_K$. Then we have an equality

$$\mu_\omega(\overline{X}(\mathcal{O}_K)) = \frac{|\overline{X}(\mathcal{O}_K/\pi^k)|}{|\mathcal{O}_K/\pi^k|^n} \tag{1.4}$$

for all sufficiently large integers k; in particular, we can write

$$\mu_\omega(\overline{X}(\mathcal{O}_K)) = \lim_{k \to \infty} \frac{|\overline{X}(\mathcal{O}_K/\pi^k)|}{|\mathcal{O}_K/\pi^k|^n}.$$

In the special case $\overline{Z} = \mathrm{Spec}(\mathcal{O}_K)$, we have $\overline{X} = \overline{Y}$ and the equality (1.4) holds for *all* $k > 0$; taking $k = 1$, we recover the formula of Example 1.2.3.8.

1.2.4 Tamagawa Measure

In this section, we let K denote a *global* field: either a finite algebraic extension of $\mathbf{Q}$, or the function field of an algebraic curve defined over a finite field. Let M_K denote the set of *places* of K: that is, (equivalence classes of) nontrivial absolute values on K. For each $v \in M_K$, we let K_v denote the local field obtained by completing K with respect to the absolute value v. We let $\mathbf{A}_K$ denote the subset of the product $\prod_{v \in M_K} K_v$ consisting of those elements $(x_v)_{v \in M_K}$ having the property that for almost every element $v \in M_K$, the local field K_v is nonarchimedean and x_v belongs to the ring of integers $\mathcal{O}_{K_v}$. We refer to $\mathbf{A}_K$ as the *ring of adeles of* K. We regard $\mathbf{A}_K$ as a locally compact commutative ring, which contains the field K as a discrete subring.

Remark 1.2.4.1. In the special case $K = \mathbf{Q}$, the ring of adeles $\mathbf{A}_K$ can be identified with the product $\mathbf{A} = (\widehat{\mathbf{Z}} \otimes \mathbf{Q}) \times \mathbf{R}$ of Notation 1.2.1.1. More generally, if K is a finite extension of $\mathbf{Q}$, then we have a canonical isomorphism $\mathbf{A}_K \simeq K \otimes_{\mathbf{Q}} \mathbf{A}$.

Now suppose that G is a linear algebraic group defined over K. Then we can regard the set $G(\mathbf{A}_K)$ of $\mathbf{A}_K$-valued points of G as a locally compact topological group. Tamagawa observed that, in many cases, the group $G(\mathbf{A}_K)$ admits a canonical Haar measure.

Construction 1.2.4.2 (Informal)**.** Let G be a linear algebraic group of dimension n over K, and let Ω denote the collection of left-invariant differential forms of degree n on G (so that Ω is a 1-dimensional vector space over K). Choose a nonzero element $\omega \in \Omega$. For every place $v \in M_K$, the differential form ω determines a (left-invariant) measure $\mu_{\omega,v}$ on the locally compact group $G(K_v)$ (see Construction 1.2.3.5). The *unnormalized Tamagawa measure* is the product measure $\mu_{\mathrm{Tam}}^{\mathrm{un}} = \prod_{v \in M_K} \mu_{\omega,v}$ on the group $G(\mathbf{A}_K) = \prod_{v \in M_K}^{\mathrm{res}} G(K_v)$.

Let us formulate Construction 1.2.4.2 more precisely. Let S be a nonempty finite subset of M_K which contains every archimedean place of K, let $\mathcal{O}_S = \{x \in K | (\forall v \notin S)[x \in \mathcal{O}_{K_v}]\}$ be the ring of S-integers, and suppose that $\overline{G}$ is a smooth group scheme over $\mathcal{O}_S$ with generic fiber G. Then we can regard the Cartesian product

$$\mathbf{A}_K^S = \prod_{v \in M_K} \begin{cases} K_v & \text{if } v \in S \\ \mathcal{O}_{K_v} & \text{if } v \notin S \end{cases}$$

as an open subring of $\mathbf{A}_K$. Set $H = \prod_{v \in M_K - S} \overline{G}(\mathcal{O}_{K_v})$, so that H is a compact topological group and we can regard $H \times \prod_{v \in S} G(K_v)$ as an open subgroup of $G(\mathbf{A}_K)$.

We will say that G *admits a Tamagawa measure* if the infinite product

$$\prod_{v \in M_K - S} \mu_{\omega,v}(\overline{G}(\mathcal{O}(K_v)))$$

converges absolutely to a nonzero real number. In this case, the compact group H admits a unique left-invariant measure μ_H which satisfies the normalization condition $\mu_H(H) = \prod_{v \in M_K - S} \mu_{\omega,v}(\overline{G}(\mathcal{O}(K_v)))$. In this case, we let $\mu_{\mathrm{Tam}}^{\mathrm{un}}$ denote the unique left-invariant measure whose restriction to $H \times \prod_{v \in S} G(K_v)$ coincides with the product measure $\mu_H \times \prod_{v \in S} \mu_{\omega,v}$. It is not difficult to see that this definition does not depend on the chosen subset $S \subseteq M_K$, or on the choice of integral model $\overline{G}$ for the algebraic group G (note that any two choices of integral model become isomorphic after passing to a suitable enlargement of S). Moreover, the measure $\mu_{\mathrm{Tam}}^{\mathrm{un}}$ also does not depend on the differential form ω: this follows from Remark 1.2.3.9 together with the product formula

$$\prod_{v \in M_K} |\lambda|_K = 1$$

for $\lambda \in K^{\times}$.

Remark 1.2.4.3 (Well-Definedness of Tamagawa Measure)**.** Let G be as above. Enlarging the subset $S \subseteq M_K$ if necessary, we can arrange that the group scheme $\overline{G}$ is smooth over $\mathcal{O}_S$, and that ω extends to a nowhere-vanishing differential form of degree n on $\overline{G}$. Using Example 1.2.3.8, we see that the well-definedness of the Tamagawa measure $\mu_{\mathrm{Tam}}^{\mathrm{un}}$ is equivalent to the absolute convergence of the infinite product

$$\prod_{v \in M_K - S} \frac{|\overline{G}(\kappa(v))|}{|\kappa(v)|^n},$$

where $\kappa(v)$ denotes the residue field of the local ring $\mathcal{O}_{K_v}$ for $v \notin S$.

Example 1.2.4.4. Let $G = \mathbf{G}_a$ be the additive group. Then we can canonically extend G to a group scheme $\overline{G}$ over $\mathcal{O}_S$, given by the additive group over $\mathcal{O}_S$. Then each factor appearing in the infinite product

$$\prod_{v \in M_K - S} \frac{|\overline{G}(\kappa(v))|}{|\kappa(v)|^n}$$

is equal to 1, so that G admits a Tamagawa measure.

Example 1.2.4.5. Let $G = \mathbf{G}_m$ be the multiplicative group. Then we can canonically extend G to a group scheme $\overline{G}$ over $\mathcal{O}_S$, given by the multiplicative group over $\mathcal{O}_S$.

Then the infinite product appearing in Remark 1.2.4.3 is given by

$$\prod_{v \in M_K - S} \frac{|\kappa(v)| - 1}{|\kappa(v)|},$$

which does not converge. Consequently, the group G does not admit a Tamagawa measure.

Example 1.2.4.6. Let G be nontrivial finite group, which we regard as a 0-dimensional algebraic group over K. Then we can extend G canonically to a constant group scheme over $\mathcal{O}_S$. In the infinite product of Remark 1.2.4.3, each factor can be identified with the order of G. Consequently, G does not admit a Tamagawa measure.

Remark 1.2.4.7. One can show that a linear algebraic group G admits a Tamagawa measure if and only if G is connected and every character $G \to \mathbf{G}_m$ is trivial. More informally, G admits a Tamagawa measure if and only if it avoids the behaviors described in Examples 1.2.4.5 and 1.2.4.6. In particular, every connected semisimple algebraic group G admits a Tamagawa measure. We refer the reader to [39] for details (we will supply a proof in the function field case later in this book). Moreover, in the semisimple case, Tamagawa measure is also right-invariant (since the left-invariant differential form ω appearing in Construction 1.2.4.2 is also right-invariant).

Remark 1.2.4.8. It is possible to extend the notion of Tamagawa measure to arbitrary linear algebraic groups by modifying Definition 1.2.4.2 to avoid the problems described in Examples 1.2.4.5 and 1.2.4.6. We refer the reader to [28] for details.

Notation 1.2.4.9. Let G be a linear algebraic group over K which admits a Tamagawa measure. The diagonal map $K \hookrightarrow \mathbf{A}_K$ induces a homomorphism $G(K) \to G(\mathbf{A}_K)$, which embeds $G(K)$ as a discrete subgroup of $G(\mathbf{A}_K)$. Since the unnormalized Tamagawa measure $\mu^{\mathrm{un}}_{\mathrm{Tam}}$ is left-invariant, it descends canonically to a measure on the collection of left cosets $G(K)\backslash G(\mathbf{A}_K)$. We will abuse notation by denoting this measure also by $\mu^{\mathrm{un}}_{\mathrm{Tam}}$. We let $\tau^{\mathrm{un}}(G)$ denote the measure $\mu^{\mathrm{un}}_{\mathrm{Tam}}(G(K)\backslash G(\mathbf{A}_K))$. We refer to $\tau^{\mathrm{un}}(G)$ as the *unnormalized Tamagawa number* of G.

Example 1.2.4.10. Let K be a number field and let $G = \mathbf{G}_a$ be the additive group. Then $\tau^{\mathrm{un}}(G) = \sqrt{|\Delta_K|}$, where Δ_K is the discriminant of K.

Example 1.2.4.11. Let K be the function field of an algebraic curve X defined over a finite field $\mathbf{F}_q$, and let $G = \mathbf{G}_a$ be the additive group. Then $\mu^{\mathrm{un}}_{\mathrm{Tam}}$ is the unique translation-invariant measure on $G(\mathbf{A}_K) = \mathbf{A}_K$ having the property that the compact open subgroup $\mathbf{A}^{\circ}_K = \prod_{v \in M_K} \mathcal{O}_{K_v} \subseteq \mathbf{A}_K$ has measure 1. Note that we have an exact sequence of locally compact groups

$$0 \to \mathrm{H}^0(X; \mathcal{O}_X) \to G(\mathbf{A}^{\circ}_K) \to G(K)\backslash G(\mathbf{A}_K) \to \mathrm{H}^1(X; \mathcal{O}_X) \to 0,$$

where the outer terms are finite-dimension vector spaces over the finite field $\mathbf{F}_q$. It follows that the unnormalized Tamagawa number $\tau^{\mathrm{un}}(G)$ is given by $\frac{|\mathrm{H}^1(X;\mathcal{O}_X)|}{|\mathrm{H}^0(X;\mathcal{O}_X)|} = q^{g-1}$, where g is the genus of X.

In what follows, it is useful to consider the following slight modification of Construction 1.2.4.2:

Definition 1.2.4.12. Let G be a linear algebraic group of dimension n over a global field K which admits a Tamagawa measure. Then *Tamagawa measure* is the left invariant measure μ_{Tam} on the locally compact group $G(\mathbf{A})$ given by the formula

$$\mu_{\mathrm{Tam}} = \frac{\mu_{\mathrm{Tam}}^{\mathrm{un}}}{\tau^{\mathrm{un}}(\mathbf{G}_a)^n},$$

where $\mu_{\mathrm{Tam}}^{\mathrm{un}}$ is the unnormalized Tamagawa measure of Construction 1.2.4.2. We will generally abuse notation by not distinguishing between μ_{Tam} and the induced measure on the coset space $G(K)\backslash G(\mathbf{A}_K)$.

The *Tamagawa number* $\tau(G)$ is defined by the formula

$$\tau(G) = \mu_{\mathrm{Tam}}(G(K)\backslash G(\mathbf{A}_K)) = \frac{\tau^{\mathrm{un}}(G)}{\tau^{\mathrm{un}}(\mathbf{G}_a)^n}.$$

Example 1.2.4.13. Let $G = \mathbf{G}_a$ be the additive group. Then the Tamagawa number $\tau(G)$ is equal to 1.

Example 1.2.4.14. In the case $K = \mathbf{Q}$, the normalized and unnormalized versions of Tamagawa measure coincide.

Remark 1.2.4.15. One advantage of working with the normalized Tamagawa measure of Definition 1.2.4.12 (as opposed to the unnormalized Tamagawa measure of Construction 1.2.4.2) is that it in some sense depends only on the group $G(\mathbf{A}_K)$, and not on the choice of global field K. More precisely, suppose that K is a finite extension of a global field $K_0 \subseteq K$, and let G be a linear algebraic group over K which admits a Tamagawa measure. Then the Weil restriction $G_0 = \mathrm{Res}_{K_0}^{K} G$ is a linear algebraic group over K_0 which admits a Tamagawa measure, equipped with a canonical isomorphism $\alpha : G(\mathbf{A}_K) \simeq G_0(\mathbf{A}_{K_0})$. This isomorphism is measure-preserving if we regard both $G(\mathbf{A}_K)$ and $G_0(\mathbf{A}_{K_0})$ as equipped with the Tamagawa measure of Definition 1.2.4.12. This is not true for the unnormalized Tamagawa measure: for example, if K_0 is the field of rational numbers and $G = \mathbf{G}_a$ is the additive group, then we have $\tau^{\mathrm{un}}(G) = \sqrt{|\Delta_K|}$ and $\tau^{\mathrm{un}}(G_0) = 1$.

Warning 1.2.4.16. In the setting of Definition 1.2.4.12, it is not obvious that the Tamagawa number $\tau(G)$ is finite: *a priori*, the quotient space $G(K)\backslash G(\mathbf{A}_K)$ could have infinite measure (note that this phenomenon does not occur when $G = \mathbf{G}_a$, by virtue of Examples 1.2.4.10 and 1.2.4.11), so that the Tamagawa measure μ_{Tam} of Definition 1.2.4.12 is well-defined, and the Tamagawa number $\tau(G)$ is well-defined as an element of $\mathbf{R}_{>0} \cup \{\infty\}$. The finiteness of $\tau(G)$ is established in [6] when K is a number field and in [9] when K is a function field (when G is semisimple, this was proved earlier by Harder).

1.2.5 The Mass Formula and Tamagawa Numbers

Let us now return to the setting of §1.2.1. Let $\Lambda = \mathbf{Z}^n$ be a free abelian group of rank $n \geq 2$ and let $q : \Lambda \to \mathbf{Z}$ be a positive-definite quadratic form. When restricted to $\mathbf{Q}$-algebras, the construction $R \mapsto \mathrm{SO}_q(R)$ is representable by an algebraic group SO_q over the field $\mathbf{Q}$ of rational numbers. One can show that the group SO_q admits a Tamagawa measure μ_{Tam} (in fact, the algebraic group SO_q is connected and semisimple if $n \geq 3$). We can therefore restate Proposition 1.2.2.4 as follows:

Proposition 1.2.5.1. *Let $q = q(x_1, \ldots, x_n)$ be a positive-definite quadratic form in $n \geq 2$ variables over $\mathbf{Z}$. Then*

$$\sum_{q'} \frac{1}{|\,\mathrm{O}_{q'}(\mathbf{Z})|} = 2^{k-1} \frac{\tau(\mathrm{SO}_q)}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))},$$

where the sum on the left hand side is taken over isomorphism classes of quadratic forms q' in the genus of q, k is the number of primes p for which $\mathrm{SO}_q(\mathbf{Z}_p) = \mathrm{O}_q(\mathbf{Z}_p)$, and $\tau(\mathrm{SO}_q)$ denotes the Tamagawa number of the algebraic group SO_q.

Using Proposition 1.2.5.1, we can restate the Smith-Minkowski-Siegel mass formula as an equality

$$2^{k-1} \frac{\tau(\mathrm{SO}_q)}{\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R}))} = \frac{2|\Lambda^\vee/\Lambda|^{(n+1)/2}}{\prod_{m=1}^{n} \mathrm{Vol}(S^{m-1})} \prod_p c_p,$$

where the factors c_p are defined as in Theorem 1.1.3.5. In fact, we can say more: the numerator and denominator of the left hand side can be evaluated independently. Theorem 1.1.3.5 is an immediate consequence of the following pair of assertions:

Theorem 1.2.5.2. *Let $\Lambda = \mathbf{Z}^n$ be a free abelian group of rank $n \geq 2$ and let $q : \Lambda \to \mathbf{Z}$ be a positive-definite quadratic form. Then*

$$\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) = \frac{2^{k-1} \prod_{m=1}^{n} \mathrm{Vol}(S^{m-1})}{|\Lambda^\vee/\Lambda|^{(n+1)/2} \prod_p c_p},$$

where k is the number of primes p for which $\mathrm{SO}_q(\mathbf{Z}_p) = \mathrm{O}_q(\mathbf{Z}_p)$, and the factors c_p are defined as in Theorem 1.1.3.5.

Theorem 1.2.5.3 (Mass Formula, Tamagawa-Weil Version)**.** *Let q be a positive-definite quadratic form in $n \geq 2$ variables over $\mathbf{Z}$. Then the Tamagawa number $\tau(\mathrm{SO}_q)$ is equal to 2.*

The "abstract" mass formula of Theorem 1.2.5.3 has several advantages over the "concrete" mass formula of Theorem 1.1.3.5:

- The equality $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q})\backslash \mathrm{SO}_q(\mathbf{A})) = 2$ continues to hold for nondegenerate quadratic forms q which are not positive-definite, except in the degenerate case where $n = 2$ and q is isotropic (in the latter case, the algebraic group SO_q is isomorphic to the multiplicative group $\mathbf{G}_m$, so the Tamagawa number $\tau(\mathrm{SO}_q)$ is not defined).

- Theorem 1.2.5.3 is really a statement about quadratic forms over $\mathbf{Q}$: note that the measure $\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\mathbf{Q})\backslash \mathrm{SO}_q(\mathbf{A}))$ depends only on the rational quadratic form $q_{\mathbf{Q}}$, and does not change if we rescale $q_{\mathbf{Q}}$ by a nonzero rational number. This invariance is not easily visible in the formulation of Theorem 1.1.3.5, where both sides of the mass formula depend on the choice of an integral quadratic form.

The content of the mass formula is contained primarily in Theorem 1.2.5.3; Theorem 1.2.5.2 is essentially a routine calculation.

Proof Sketch of Theorem 1.2.5.2. Let $\mathfrak{g}$ denote the Lie algebra of the algebraic group SO_q and let $b : \Lambda \times \Lambda \to \mathbf{Z}$ denote the symmetric bilinear form associated to q (given by $b(x, y) = q(x+y)-q(x)-q(y)$). Then there is a canonical isomorphism of rational vector spaces $\rho : \mathbf{Q} \otimes_{\mathbf{Z}} \bigwedge^2(\Lambda) \to \mathfrak{g}$, given concretely by the formula $\rho(x, y)(z) = b(x, z)y - b(y, z)x$. Let Ω be as in Construction 1.2.4.2, which we can identify with the top exterior power of the dual space $\mathfrak{g}^\vee$. It follows that the top exterior power of $\rho^\vee$ supplies an isomorphism $\beta : \Omega \to \mathbf{Q} \otimes \bigwedge^{n(n-1)/2} \bigwedge^2(\Lambda)$. Note that $\bigwedge^{n(n-1)/2} \bigwedge^2(\Lambda)$ is a free abelian group of rank 1, and therefore admits a generator e which is unique up to a sign. Set $\omega = \beta^{-1}(e) \in \Omega$. Then ω is well-defined up to a sign, and therefore determines well-defined measures $\mu_{\omega,\mathbf{R}}$ on $\mathrm{SO}_q(\mathbf{R})$ and $\mu_{\omega,\mathbf{Q}_p}$ on $\mathrm{SO}_q(\mathbf{Q}_p)$ for each prime number p. The definition of Tamagawa measure then yields an identity

$$\mu_{\mathrm{Tam}}(\mathrm{SO}_q(\widehat{\mathbf{Z}} \times \mathbf{R})) = (\prod_p \mu_{\omega,\mathbf{Q}_p}(\mathrm{SO}_q(\mathbf{Z}_p)))\mu_{\omega,\mathbf{R}}(\mathrm{SO}_q(\mathbf{R})).$$

To prove Theorem 1.2.5.2, it will suffice to verify the "local" identities

$$\mu_{\omega,\mathbf{R}}(\mathrm{SO}_q(\mathbf{R})) = \frac{\prod_{m=1}^{n} \mathrm{Vol}(S^{m-1})}{2|\Lambda^\vee/\Lambda|^{(n-1)/2}} \tag{1.5}$$

$$\mu_{\omega,\mathbf{Q}_p}(\mathrm{SO}_q(\mathbf{Z}_p)) = \begin{cases} \frac{2|\Lambda^\vee/\Lambda|_p}{c_p} & \text{if } \mathrm{SO}_q(\mathbf{Z}_p) = \mathrm{O}_q(\mathbf{Z}_p) \\ \frac{|\Lambda^\vee/\Lambda|_p}{c_p} & \text{if } \mathrm{SO}_q(\mathbf{Z}_p) \neq \mathrm{O}_q(\mathbf{Z}_p). \end{cases} \tag{1.6}$$

Let us first prove (1.5). Let V be a finite-dimensional real vector space equipped with a positive-definite bilinear form. Let $\mathfrak{o}(V)$ denote the Lie algebra of $\mathrm{O}(V)$, which we regard as a subspace of $\mathrm{End}(V)$. Then the construction $\mathfrak{o}(V)$ admits a positive-definite bilinear form b', given by $b'(A,B) = \frac{-1}{2}\mathrm{Tr}(AB)$. This symmetric bilinear form determines a bi-invariant Riemannian metric on the compact Lie group $\mathrm{O}(V)$, so that the volumes $\mathrm{Vol}(\mathrm{O}(V))$ and $\mathrm{Vol}(\mathrm{SO}(V))$ are well-defined and depend only on the dimension of V. In the case $V = \mathbf{R} \otimes \Lambda$, we obtain an identity $\mu_{\mathbf{R},\omega}(\mathrm{SO}_q(\mathbf{R})) = \frac{\mathrm{Vol}(\mathrm{SO}(\mathbf{R}^n))}{\sqrt{|D|}} = \frac{\mathrm{Vol}(\mathrm{O}(\mathbf{R}^n))}{2\sqrt{N}}$, where D is the discriminant of the integral bilinear form $b' \circ \rho$ on the lattice $\bigwedge^2(\Lambda)$. An elementary calculation gives $|D| = |\Lambda^\vee/\Lambda|^{(n-1)}$, and a calculation using the fiber sequences $\mathrm{O}(\mathbf{R}^{m-1}) \to \mathrm{O}(\mathbf{R}^m) \to S^{m-1}$ yields

$$\mathrm{Vol}(\mathrm{O}(\mathbf{R}^m)) = \mathrm{Vol}(\mathrm{O}(\mathbf{R}^{m-1}))\,\mathrm{Vol}(S^{m-1}) \qquad \mathrm{Vol}(\mathrm{O}(\mathbf{R}^n)) = \prod_{m=1}^{n} \mathrm{Vol}(S^{m-1}).$$

Combining these identities, we obtain (1.5).

We now prove (1.6). The differential form ω extends uniquely to a left-invariant differential form on the full orthogonal group O_q. Invoking the definition of the constants c_p, we can phrase (1.6) more uniformly as the assertion that the identity

$$\mu_{\omega,\mathbf{Q}_p}(\mathrm{O}_q(\mathbf{Z}_p)) = \frac{|\Lambda^\vee/\Lambda|_p|\,\mathrm{O}_q(\mathbf{Z}/p^k)|}{p^{kn(n-1)/2}}$$

holds for $k \gg 0$. Note that the orthogonal group O_q is representable by a scheme defined over $\mathbf{Z}_p$ (or even over $\mathbf{Z}$), which fits into a pullback diagram

$$\begin{array}{ccc} \mathrm{O}_q & \longrightarrow & \mathrm{Spec}(\mathbf{Z}_p) \\ \downarrow & & \downarrow q \\ \mathrm{GL}_n & \xrightarrow{f} & Q. \end{array}$$

Here Q is the affine space (of dimension $(n^2+n)/2$) which parametrizes quadratic forms on Λ, and $f : \mathrm{GL}_n \to Q$ is the map given by $A \mapsto q \circ A$. The desired equality now follows by combining Remark 1.2.3.9 with Variant 1.2.3.10. □

1.2.6 Weil's Conjecture

The appearance of the number 2 in the statement of Theorem 1.2.5.3 results from the fact that the algebraic group SO_q is not simply connected. Let us assume that q is a nondegenerate quadratic form in at least three variables, so that algebraic group SO_q is semisimple and admits a universal cover $\mathrm{Spin}_q \to \mathrm{SO}_q$. We then have the following more fundamental statement:

Theorem 1.2.6.1 (Mass Formula, Simply Connected Version)**.** *Let $q = q(x_1, \ldots, x_n)$ be a nondegenerate quadratic form in $n \geq 3$ variables over $\mathbf{Q}$, and let Spin_q be the universal cover of the semisimple algebraic group SO_q. Then the Tamagawa number $\tau(\mathrm{Spin}_q)$ is equal to 1.*

Remark 1.2.6.2. In general, there is a simple relationship between the Tamagawa number of a semisimple algebraic group G and the Tamagawa number of the universal cover $\widetilde{G}$; we refer the reader to [29] for details. Using this relationship, it is not difficult to see that Theorem 1.2.6.1 is equivalent to Theorem 1.2.5.3, provided that q is a quadratic form in at least three variables.

Warning 1.2.6.3. Let $q = q(x_1, x_2)$ be a nondegenerate quadratic form in two variables. Then the algebraic group SO_q still admits a canonical double cover $\mathrm{Spin}_q \to \mathrm{SO}_q$. However, it is not true that the Tamagawa number $\tau(\mathrm{Spin}_q)$ is equal to 1. Instead, the group Spin_q is isomorphic to SO_q, and we have an equality of Tamagawa numbers $\tau(\mathrm{Spin}_q) = \tau(\mathrm{SO}_q) = 2$ (provided that q is anisotropic; if q is isotropic, then neither $\tau(\mathrm{Spin}_q)$ or $\tau(\mathrm{SO}_q)$ is well-defined).

Motivated by Theorem 1.2.6.1, Weil proposed the following:

Conjecture 1.2.6.4 (Weil's Conjecture)**.** Let K be a global field and let G be an algebraic group over K which is connected, semisimple, and simply connected. Then the Tamagawa number $\tau(G)$ is equal to 1.

In [39], Weil verified Conjecture 1.2.6.4 in many cases (in particular, he gave a direct proof of Theorem 1.2.6.1, thereby reproving the Smith-Minkowski-Siegel mass formula). When K is a number field, Conjecture 1.2.6.4 was proved in general by Kottwitz in [18] (under the assumption that G satisfies the Hasse principle, which is now known to be automatic), building on earlier work of Langlands ([21]) in the case where G is split and Lai ([19]) in the case where G is quasi-split.

1.3 Weil's Conjecture for Function Fields

In §1.2, we formulated Weil's conjecture for an arbitrary simply connected semisimple algebraic group G over a global field K (Conjecture 1.2.6.4). In the case where $K = \mathbf{Q}$ and $G = \mathrm{Spin}_q$ for a positive-definite quadratic form q, Weil's conjecture is essentially a reformulation of the Smith-Minkowski-Siegel mass formula (Theorem 1.1.3.5). Our goal in this book is to give a proof of Conjecture 1.2.6.4 in the case where K is the function field of an algebraic curve X (defined over a finite field $\mathbf{F}_q$). In this section, we will explain how the function field case of Weil's conjecture can also be regarded as a mass formula: more precisely, it is a (weighted) count for the number of principal G-bundles on X (see Conjecture 1.3.3.7).

1.3.1 Tamagawa Measure in the Function Field Case

We begin by reviewing some terminology. Throughout this section, we let $\mathbf{F}_q$ denote a finite field with q elements and X an algebraic curve over $\mathbf{F}_q$. We assume that the algebraic curve X is smooth, projective, and geometrically connected over $\mathbf{F}_q$ (that is, the unit map $\mathbf{F}_q \to \Gamma(X; \mathcal{O}_X)$ is an isomorphism). We let K_X denote the field of rational functions on X: that is, the residue field of the generic point of X. Then K_X is a *function field*: a global field of positive characteristic.

In what follows, we will write $x \in X$ to mean that x is a closed point of the curve X. For each $x \in X$, we let $\kappa(x)$ denote the residue field of X at the point x. Then $\kappa(x)$ is a finite extension of the finite field $\mathbf{F}_q$. We let $\mathcal{O}_x$ denote the completion of the local ring of X at the point x: this is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to a power series ring $\kappa(x)[[t]]$. We let K_x denote the fraction field of $\mathcal{O}_x$.

The collection of local fields $\{K_x\}_{x\in X}$ can be viewed as the collection of all completions of K_X with respect to nontrivial absolute values: in other words, we can identify the set of closed points of X with the set of *places* M_{K_X} considered in §1.2.4. Let $\mathbf{A}_X = \prod^{\text{res}}_{x \in X} K_x$ denote the ring of adeles of the global field K_X. Then $\mathbf{A}_X$ can be described more precisely as a direct limit $\varinjlim \mathbf{A}_X^S$, where S ranges over all finite sets of closed points of X and $\mathbf{A}_X^S$ denotes the Cartesian product

$$\prod_{x \in X} \begin{cases} K_x & \text{if } x \in S \\ \mathcal{O}_x & \text{if } x \notin S. \end{cases}$$

Here we regard each $\mathbf{A}_X^S$ as equipped with the product topology, and $\mathbf{A}_X$ as equipped with the direct limit topology (so that each $\mathbf{A}_X^S$ is an open subring of $\mathbf{A}_X$). In particular, the ring of adeles $\mathbf{A}_X$ contains a compact open subring $\mathbf{A}_X^{\emptyset} = \prod_{x \in X} \mathcal{O}_x$, which we will refer to as the *ring of integral adeles*.

Let G_0 be a linear algebraic group defined over the function field K_X. For every K_X-algebra R, we let $G_0(R)$ denote the group of R-valued points of G_0. In particular, we can consider the set $G_0(\mathbf{A}_X)$ of adelic points of G_0, which we regard as a locally compact topological group containing $G_0(K_X)$ as a discrete subgroup. If the algebraic group G_0 is connected and semisimple, we let μ_{Tam} denote the Tamagawa measure on $G_0(\mathbf{A}_X)$ of Definition 1.2.4.12, and we let $\tau(G_0) = \mu_{\text{Tam}}(G_0(K_X) \backslash G_0(\mathbf{A}_X))$ denote the Tamagawa number of G_0.

In order to obtain a concrete interpretation of the Tamagawa number $\tau(G_0)$, it will be convenient to choose an *integral model* of G_0: that is, a group scheme $\pi : G \to X$ whose generic fiber is isomorphic to G_0, where the morphism π is smooth and affine (such an integral model can always be found: see for example [8] or §7.1 of [7]). Given such a group scheme, we can associate a group $G(R)$ of *R-valued points* to every commutative ring R equipped with a map $u : \mathrm{Spec}(R) \to X$. In the case where u factors through the generic point of X, we can regard u as equipping R with the structure of K_X-algebra, and $G(R)$ can be identified with the set $G_0(R)$ of R-valued points of G_0. However, a choice of integral model supplies additional structure:

- For each closed point $x \in X$, we can consider the group $G(\mathcal{O}_x)$ of $\mathcal{O}_x$-valued points of G: this is a compact open subgroup of the locally compact group $G(K_x) = G_0(K_x)$.
- For each closed point $x \in X$, we can consider the group $G(\kappa(x))$ of $\kappa(x)$-valued points of G: this is a finite group which appears as a quotient of $G(\mathcal{O}_x)$ (note that the surjectivity of the map $G(\mathcal{O}_x) \to G(\kappa(x))$ follows from the smoothness of G).
- For every finite set S of closed points of X, we can consider the group $G(\mathbf{A}_X^S)$ of $\mathbf{A}_X^S$-valued points of G, which is isomorphic to the direct product

$$\prod_{x \in X} \begin{cases} G(K_x) & \text{if } x \in S \\ G(\mathcal{O}_x) & \text{if } x \notin S. \end{cases}$$

  Here we can view $G(\mathbf{A}_X^S)$ as an open subgroup of $G(\mathbf{A}_X) = G_0(\mathbf{A}_X)$. If $S = \emptyset$, then $G(\mathbf{A}_X^S)$ is a *compact* open subgroup of $G(\mathbf{A}_X)$.

Remark 1.3.1.1. It will often be convenient to assume that the map $\pi : G \to X$ has connected fibers. This can always be arranged by passing to an open subgroup $G^\circ \subseteq G$, given by the union of the connected component of the identity in each fiber of π (note that passage from G to G° does not injure our assumption that the morphism π is affine; the open immersion $G^\circ \hookrightarrow G$ is complementary to a Cartier divisor and is therefore an affine morphism).

Warning 1.3.1.2. Let G_0 be a semisimple algebraic group over the function field K_X. It is generally not possible to choose an integral model $\pi : G \to X$ of G_0 which is semisimple *as a group scheme over* X. It follows from general nonsense that, for all but finitely many closed points $x \in X$, the fiber $G_x = \mathrm{Spec}(\kappa(x)) \times_X G$ is a semisimple algebraic group over $\kappa(x)$. However, we cannot avoid the phenomenon of *bad reduction*: the existence of finitely many closed points $x \in X$ where G_x is not semisimple. At these points, one can use Bruhat-Tits theory to choose *parahoric* models for the algebraic group G_0 (that is, group schemes over $\mathcal{O}_x$ which are not too far from being semisimple). However, we will not need Bruhat-Tits theory in this book.

A choice of integral model $\pi : G \to X$ for the linear algebraic group G_0 allows us to give a very concrete description of the Tamagawa measure μ_{Tam} on $G(\mathbf{A}_X) = G_0(\mathbf{A}_X)$. Let $\Omega_{G/X}$ denote the relative cotangent bundle of the smooth morphism $\pi : G \to X$. Then $\Omega_{G/X}$ is a vector bundle on G of rank $n = \dim(G_0)$. We let $\Omega^n_{G/X}$ denote the top exterior power of $\Omega_{G/X}$, so that $\Omega^n_{G/X}$ is a line bundle on G. Let $\mathcal{L}$ be the pullback of $\Omega^d_{G/X}$ along the identity section $e : X \to G$. Equivalently, we can identify $\mathcal{L}$ with the subbundle of $\pi_* \Omega^n_{G/X}$ consisting of left-invariant sections. Let $\mathcal{L}_0$ be the generic fiber of $\mathcal{L}$, which we regard as a 1-dimensional vector space over K_X, and choose a nonzero element $\omega \in \mathcal{L}_0$. For every closed point $x \in X$, we can apply Construction 1.2.3.5 to the differential form ω to obtain a left-invariant measure $\mu_{\omega,x}$ on the locally compact group $G(K_x)$. Using Remark 1.2.3.9 and Example 1.2.3.8, we see that the measure $\mu_{\omega,x}$ is characterized by the identity

$$\mu_{x,\omega}(G(\mathcal{O}_x)) = \frac{|G(\kappa(x))|}{|\kappa(x)|^{n+v_x(\omega)}},$$

where $v_x(\omega) \in \mathbf{Z}$ denotes the order of vanishing of ω at the point x. It follows that the *unnormalized* Tamagawa measure $\mu^{\mathrm{un}}_{\mathrm{Tam}}$ of Construction 1.2.4.2 is characterized by the formula

$$\mu^{\mathrm{un}}_{\mathrm{Tam}}(G(\mathbf{A}^{\emptyset}_X)) = \prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^{n+v_x(\omega)}}.$$

Using the identity

$$\prod_{x \in X} |\kappa(x)|^{v_x(\omega)} = \prod_{x \in X} q^{\deg(x) v_x(\omega)} = q^{\sum_{x \in X} \deg(x) v_x(\omega)} = q^{\deg(\mathcal{L})} = q^{\deg(\Omega_{G/X})}$$

and the description of $\tau^{\mathrm{un}}(\mathbf{G}_a)$ given in Example 1.2.4.11, we obtain the following:

Proposition 1.3.1.3. *Let X be an algebraic curve of genus g over a finite field $\mathbf{F}_q$ and let $\pi : G \to X$ be a smooth affine group scheme whose generic fiber is connected and semisimple of dimension n. Then the Tamagawa measure μ_{Tam} of Definition 1.2.4.12 is the unique left-invariant measure on the group $G(\mathbf{A}_X)$ which satisfies the identity*

$$\mu_{\mathrm{Tam}}(G(\mathbf{A}^{\emptyset}_X)) = q^{n(1-g)-\deg(\Omega_{G/X})} \prod_{x \in X} \frac{|G(\kappa(x))|}{|\kappa(x)|^n}.$$

Remark 1.3.1.4. For purposes of understanding this book, the reader can feel free dispense with the analytic constructions of §1.2.4 and take Proposition 1.3.1.3 as the *definition* of the Tamagawa measure μ_{Tam}. From this point of view, it is not immediately obvious (but not hard to verify) that the measure μ_{Tam} depends only on the generic fiber $G_0 = \mathrm{Spec}(K_X) \times_X G$, and not on the choice of integral model $G \to X$.

1.3.2 Principal Bundles

Our next goal is to relate measures on adelic groups and their quotients (in the function field case) to more concrete counting problems. First, we review some terminology.

Definition 1.3.2.1. Let X be a scheme and let G be a group scheme over X. For every X-scheme Y, we let $G_Y = G \times_X Y$ denote the associated group scheme over Y. By a *G-bundle on Y*, we will mean a Y-scheme $\mathcal{P}$ equipped with an action

$$G_Y \times_Y \mathcal{P} \simeq G \times_X \mathcal{P} \to \mathcal{P}$$

of G_Y (in the category of Y-schemes) which is locally trivial in the following sense: there exists a faithfully flat map $U \to Y$ and a G_Y-equivariant isomorphism $U \times_Y \mathcal{P} \simeq U \times_Y G_Y \simeq U \times_X G$.

Remark 1.3.2.2. In the situation of Definition 1.3.2.1, suppose that we are given a morphism of X-schemes $f : Y' \to Y$. If $\mathcal{P}$ is a G-bundle on Y, then the fiber product $Y' \times_Y \mathcal{P}$ can be regarded as a G-bundle on Y'. We will refer to $Y' \times_Y \mathcal{P}$ as the *pullback of* $\mathcal{P}$ *along* f, and denote it by $f^* \mathcal{P}$ or by $\mathcal{P}|_{Y'}$ (we employ the latter notation primarily in the case where f is an embedding).

Remark 1.3.2.3. Let G be a group scheme over X. Then G represents a functor h_G from the category of X-schemes to the category of groups, and the functor h_G is a sheaf for the flat topology. Every G-bundle $\mathcal{P}$ on X represents a functor $h_{\mathcal{P}}$, which can be regarded as an h_G-torsor (locally trivial for the flat topology). If G is affine, then every h_G-torsor arises in this way (since affine morphisms satisfy effective descent for the flat topology). For this reason, we will generally use the terminology *G-bundle* and *G-torsor* interchangeably when G is affine (which will be satisfied in all of our applications).

Remark 1.3.2.4. In the special case where G is a smooth over X, any G-bundle $\mathcal{P}$ on an X-scheme Y is smooth as a Y-scheme. It follows that $\mathcal{P}$ can be trivialized over an étale covering $U \to Y$.

Notation 1.3.2.5. Let X be a scheme and let G be an affine group scheme over X. For every X-scheme Y, we let $\mathrm{Tors}_G(Y)$ denote the category whose objects are G-bundles on Y, and whose morphisms are isomorphisms of G-bundles.

We will need the following gluing principle for G-bundles:

Proposition 1.3.2.6 (Beauville-Laszlo)**.** *Let X be a Dedekind scheme (for example, an algebraic curve), let G be a flat affine group scheme over X, and let S be a finite* *set of closed points of X. For each $x \in S$, let $\mathcal{O}_x$ denote the complete local ring of X at x, and let K_x denote the fraction field of $\mathcal{O}_x$. Then the diagram of categories*

$$\begin{array}{ccc} \mathrm{Tors}_G(X) & \longrightarrow & \mathrm{Tors}_G(X - S) \\ \downarrow & & \downarrow \\ \prod_{x \in S} \mathrm{Tors}_G(\mathrm{Spec}(\mathcal{O}_x)) & \longrightarrow & \prod_{x \in S} \mathrm{Tors}_G(\mathrm{Spec}(K_x)) \end{array}$$

is a pullback square.

More informally, Proposition 1.3.2.6 asserts that a G-bundle $\mathcal{P}$ on X can be recovered from its restriction to the open subset $X - S \subseteq X$ and its restriction to a formal neighborhood of each point $x \in S$, provided that we are supplied with "gluing data" over a punctured formal neighborhood $\mathrm{Spec}(K_x)$ of each point $x \in X$. We now exploit this to produce some examples of G-bundles on algebraic curves:

Construction 1.3.2.7 (Regluing)**.** Let X be an algebraic curve over a finite field $\mathbf{F}_q$, let G be a smooth affine group scheme over X, and let γ be an element of the adelic group $G(\mathbf{A}_X)$. Then we can identify γ with a collection of elements $\{\gamma_x \in G(K_x)\}_{x \in X}$, having the property that there exists a finite set S such that $\gamma_x \in G(\mathcal{O}_x)$ whenever $x \notin S$. Using Proposition 1.3.2.6, we can construct a G-bundle $\mathcal{P}_\gamma$ on X with the following features:

- (a) The bundle $\mathcal{P}_\gamma$ is equipped with a trivialization ϕ on the open set $U = X - S$.
- (b) The bundle $\mathcal{P}_\gamma$ is equipped with a trivialization ψ_x over the scheme $\mathrm{Spec}(\mathcal{O}_x)$ for each point $x \in S$.
- (c) For each $x \in S$, the trivializations of $\mathcal{P}_\gamma |_{\mathrm{Spec}(K_x)}$ determined by ϕ and ψ_x differ by the action of $\gamma_x \in G(K_x)$.

It is not difficult to see that the G-bundle $\mathcal{P}_\gamma$ is canonically independent of the choice of S, so long as S contains all points x such that $\gamma_x \notin G(\mathcal{O}_x)$.

In good cases, *all* G-bundles on algebraic curves can be obtained by applying Construction 1.3.2.7. This is a consequence of the following pair of results:

Theorem 1.3.2.8 (Lang)**.** *Let κ be a finite field and let G be a connected algebraic group over κ. Then every G-bundle on* $\mathrm{Spec}(\kappa)$ *is trivial.*

Theorem 1.3.2.9 (Harder)**.** *Let X be an algebraic curve over a finite field $\mathbf{F}_q$, let G be an algebraic group over the fraction field K_X, and let $\mathcal{P}$ be a G-bundle over* $\mathrm{Spec}(K_X)$*. Assume that G is connected, semisimple, and simply connected. Then the following conditions are equivalent:*

(a) *The G-bundle $\mathcal{P}$ is trivial.*

(b) *For every closed point $x \in X$, the fiber product* $\mathrm{Spec}(K_x) \times_{\mathrm{Spec}(K_X)} \mathcal{P}$ *is a trivial G-bundle on* $\mathrm{Spec}(K_x)$.

We refer the reader to [20] and [16] for proofs of Theorem 1.3.2.8 and 1.3.2.9, respectively. Note that Theorem 1.3.2.9 can be regarded as a function field analogue of the Hasse principle for quadratic forms (Theorem 1.1.2.6).

Proposition 1.3.2.10. *Let X be an algebraic curve over a finite field and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then every G-bundle on X can be obtained from Construction 1.3.2.7: that is, it is isomorphic to $\mathcal{P}_\gamma$ for some element $\gamma \in G(\mathbf{A}_X)$.*

Proof. Let $\mathcal{P}$ be a G-bundle on X. For each closed point $x \in X$, the G-bundle $\mathcal{P}$ is trivial when restricted to $\mathrm{Spec}(\kappa(x))$ by virtue of Lang's theorem (Theorem 1.3.2.8). Since G is smooth over X, the G-bundle $\mathcal{P}$ is also smooth over X. Applying Hensel's lemma, we see that any trivialization of $\mathcal{P}$ over $\mathrm{Spec}(\kappa(x))$ can be extended to a trivialization of $\mathcal{P}$ over $\mathrm{Spec}(\mathcal{O}_x)$. In particular, $\mathcal{P}$ is trivial when restricted to each $\mathrm{Spec}(K_x)$. Applying Harder's theorem (Theorem 1.3.2.9), we deduce that $\mathcal{P}$ is trivial over the generic point $\mathrm{Spec}(K_X) \subseteq X$. Using a direct limit argument, we conclude that $\mathcal{P}$ is trivial over some open subset $U \subseteq X$. Let S be the set of closed points of X which are not contained in U. Then $\mathcal{P}$ is trivial over U and over each of the local schemes $\{\mathrm{Spec}(\mathcal{O}_x)\}_{x \in S}$. Using Proposition 1.3.2.6, we conclude that $\mathcal{P}$ is isomorphic to $\mathcal{P}_\gamma$ for some $\gamma \in G(\mathbf{A}_X^S) \subseteq G(\mathbf{A}_X)$. □

Proposition 1.3.2.10 asserts that, under reasonable hypotheses, all G-bundles on X can be obtained by applying the regluing procedure of Construction 1.3.2.7 to an appropriately chosen element $\gamma \in G(\mathbf{A}_X)$. However, the element γ is not uniquely determined: it is possible for different elements of $G(\mathbf{A}_X)$ to give rise to isomorphic G-bundles on X. Let us now analyze exactly how this might occur. Suppose we are given elements $\gamma, \gamma' \in G(\mathbf{A}_X)$. Then the G-bundles $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma'}$ both come equipped with trivializations at the generic point of X. Consequently, the datum of a G-bundle isomorphism

$$\rho_0 : \mathrm{Spec}(K_X) \times_X \mathcal{P}_\gamma \simeq \mathrm{Spec}(K_X) \times_X \mathcal{P}_{\gamma'}$$

is equivalent to the datum of an element $\beta \in G(K_X)$. Unwinding the definitions, we see that ρ_0 can be extended to a G-bundle isomorphism of $\mathcal{P}_\gamma$ with $\mathcal{P}_{\gamma'}$ if and only if the product $\gamma'^{-1}\beta\gamma$ belongs to the subgroup $G(\mathbf{A}_X^\emptyset) \subseteq G(\mathbf{A}_X)$ (moreover, such an extension is automatically unique, since the generic point of X is dense in X). We therefore obtain the following more precise version of Proposition 1.3.2.10:

Proposition 1.3.2.11. *Let X be an algebraic curve over a finite field and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then:*

(a) The construction $\gamma \mapsto \mathcal{P}_\gamma$ induces a bijection from the double quotient

$$G(K_X)\backslash G(\mathbf{A}_X)/G(\mathbf{A}_X^{\emptyset})$$

to the set of isomorphism classes of G-bundles on X.

(b) Let γ be an element of $G(\mathbf{A}_X)$ and let $\overline{\gamma}$ denote its image in $G(K_X)\backslash G(\mathbf{A}_X)$. Then the automorphism group of the G-bundle $\mathcal{P}_\gamma$ can be identified with the subgroup of $G(\mathbf{A}_X^{\emptyset})$ which fixes $\overline{\gamma}$.

1.3.3 Weil's Conjecture as a Mass Formula

We now use Proposition 1.3.2.11 to reformulate Weil's conjecture as a mass formula, analogous to Theorem 1.1.3.5. First, we need a bit of terminology.

Definition 1.3.3.1 (The Mass of a Groupoid)**.** Let $\mathcal{C}$ be a groupoid (that is, a category in which all morphisms are isomorphisms). Assume that, for every object $C \in \mathcal{C}$, the automorphism group $\mathrm{Aut}(C)$ is finite. We let $|\,\mathcal{C}\,|$ denote the sum $\sum_C \frac{1}{|\,\mathrm{Aut}(C)|}$, where C ranges over a set of representatives for all isomorphism classes in $\mathcal{C}$. We will refer to $|\,\mathcal{C}\,|$ as the *mass of the groupoid* $\mathcal{C}$.

Remark 1.3.3.2. In the setting of Definition 1.3.3.1, each term in appearing in the sum $\sum_C \frac{1}{|\,\mathrm{Aut}(C)|}$ is ≤ 1. Consequently, if $\mathcal{C}$ has only n isomorphism classes of objects for some nonnegative integer n, then $|\,\mathcal{C}\,| \leq n$. We can regard $|\,\mathcal{C}\,|$ as a *weighted* count of the number of isomorphism classes in $\mathcal{C}$, where the isomorphism class of an object $C \in \mathcal{C}$ is counted with multiplicity $\frac{1}{|\,\mathrm{Aut}(C)|}$. This is a very natural way to count mathematical objects which admit nontrivial automorphisms.

If the number of isomorphism classes in $\mathcal{C}$ is infinite, then the sum $\sum_C \frac{1}{|\,\mathrm{Aut}(C)|}$ contains infinitely many terms, and may or may not converge. If it does not converge, we will write $|\,\mathcal{C}\,| = \infty$ and say that the *mass of* $\mathcal{C}$ *is infinite*.

Example 1.3.3.3. Let $\mathcal{C}$ be the groupoid whose objects are finite sets and whose morphisms are bijections. Up to isomorphism, $\mathcal{C}$ contains a single object of cardinality n for each $n \geq 0$, whose automorphism group is the symmetric group Σ_n having order $n!$. We therefore have

$$|\,\mathcal{C}\,| = \sum_{n \geq 0} \frac{1}{n!} = e,$$

where e is Euler's constant.

Example 1.3.3.4. Let $q = q(x_1, \ldots, x_n)$ be a positive-definite integral quadratic form. Let $\mathcal{C}$ be the category whose objects are quadratic forms of the same genus as q (Definition 1.1.3.1) and whose morphisms are isomorphisms of quadratic forms (Definition 1.1.1.2). Then $|\,\mathcal{C}\,| = \sum_{q'} \frac{1}{|\,\mathrm{O}_{q'}(\mathbf{Z})|}$ is the sum appearing on the left hand side of the Smith-Minkowski-Siegel mass formula (Theorem 1.1.3.5).

Example 1.3.3.5. Let G be an algebraic group defined over a finite field $\mathbf{F}_q$, and let $\mathrm{Tors}_G(\mathrm{Spec}(\mathbf{F}_q))$ denote the category of principal G-bundles (Notation 1.3.2.5). If G is connected, then Lang's theorem guarantees that every G-bundle on $\mathrm{Spec}(\mathbf{F}_q)$ is trivial (Theorem 1.3.2.8). In this case, the category $\mathrm{Tors}_G(\mathrm{Spec}(\mathbf{F}_q))$ has only a single object (up to isomorphism), whose automorphism group is the finite group $G(\mathbf{F}_q)$. We therefore have $|\,\mathrm{Tors}_G(\mathrm{Spec}(\mathbf{F}_q))| = \frac{1}{|G(\mathbf{F}_q)|}$.

We now return to the setting of Weil's conjecture. Fix an algebraic curve X over a finite field $\mathbf{F}_q$ and a smooth affine group scheme $G \to X$. Then the category $\mathrm{Tors}_G(X)$ of G-bundles on X satisfies the hypotheses of Definition 1.3.3.1, so we can consider the mass

$$|\,\mathrm{Tors}_G(X)| = \sum_{\mathcal{P}} \frac{1}{|\,\mathrm{Aut}(\mathcal{P})|},$$

where the sum is taken over all isomorphism classes of G-bundles on X.

Proposition 1.3.3.6. *Assume that the fibers of G are connected and the generic fiber is semisimple and simply connected, and let μ_{Tam} denote the Tamagawa measure on the group $G(\mathbf{A}_X)$ (and on the quotient space $G(K_X)\backslash G(\mathbf{A}_X)$). Then the Tamagawa number of the generic fiber of G is equal to the product $|\,\mathrm{Tors}_G(X)|\mu_{\mathrm{Tam}}(G(\mathbf{A}_X^{\emptyset}))$.*

Proof. Let Z denote the double quotient $G(K_X)\backslash G(\mathbf{A}_X)/G(\mathbf{A}_X^{\emptyset})$. For each $z \in Z$, let $O_z \subseteq G(K_X)\backslash G(\mathbf{A}_X)$ denote the inverse image of z, so that we have a decomposition

$$G(K_X)\backslash G(\mathbf{A}_X) = \amalg_{z \in Z} O_z$$

into orbits under the right action of $G(\mathbf{A}_X^{\emptyset})$. Using Proposition 1.3.2.11, we can identify Z with the collection of isomorphism classes of G-bundles on X. Under this identification, if $z \in Z$ corresponds to a G-bundle $\mathcal{P}$, then the orbit O_z can be identified with the quotient of $G(\mathbf{A}_X^{\emptyset})$ by a (free) action of the finite group $\mathrm{Aut}(\mathcal{P})$. We therefore compute

$$\begin{aligned} \mu_{\mathrm{Tam}}(G(K_X)\backslash G(\mathbf{A}_X)) &= \sum_{z \in Z} \mu_{\mathrm{Tam}}(O_z) \\ &= \sum_{\mathcal{P}} \frac{\mu_{\mathrm{Tam}}(G(\mathbf{A}_X^{\emptyset}))}{|\,\mathrm{Aut}(\mathcal{P})|}, \end{aligned}$$

where the sum is taken over all isomorphism classes of G-bundles on X. The left hand side of this equality is the Tamagawa number of the generic fiber of G, and the right hand side is $|\operatorname{Tors}_G(X)|\mu_{\text{Tam}}(G(\mathbf{A}_X^{\emptyset}))$. □

Combining Propositions 1.3.1.3 and 1.3.3.6, we obtain the following reformuation of Weil's conjecture:

Conjecture 1.3.3.7 (Weil's Conjecture, Mass Formula Version)**.** Let X be an algebraic curve of genus g over a finite field $\mathbf{F}_q$ and let G be a smooth affine group scheme of dimension n over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected. Then

$$|\operatorname{Tors}_G(X)| = q^{n(g-1)+\deg(\Omega_{G/X})} \prod_{x \in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}.$$

The rest of this book is devoted to giving a proof of Conjecture 1.3.3.7.

Warning 1.3.3.8. In the statement of Conjecture 1.3.3.7, the mass appearing on the left hand side and the product appearing on the right hand side involve infinitely many terms. The convergence of the product on the right hand side is equivalent to the well-definedness of the Tamagawa measure on $G(\mathbf{A}_X)$ (see Remark 1.2.4.3), and the finiteness of the mass $|\operatorname{Tors}_G(X)|$ is equivalent to the finiteness of the Tamagawa number of the generic fiber of G. We regard the convergence of both sides as part of the statement of Conjecure 1.3.3.7, to be established later in this book.

Remark 1.3.3.9. Using Example 1.3.3.5, we can rewrite the equality of Conjecture 1.3.3.7 in the more suggestive form

$$\frac{|\operatorname{Tors}_G(X)|}{q^{n(g-1)+\deg(\Omega_{G/X})}} = \prod_{x \in X} \frac{|\operatorname{Tors}_G(\operatorname{Spec}(\kappa(x)))|}{q^{-n\deg(x)}}.$$

The powers of q which appear in the denominators admit geometric interpretations (in terms of the dimensions of certain algebraic stacks), which we will discuss in §1.4 (see Example 1.4.1.4).

The assertion of Conjecture 1.3.3.7 can be regarded as a function field version of the Smith-Minkowski-Siegel mass formula (Theorem 1.1.3.5). Our perspective is informed by the following table of analogies:

Classical Mass Formula	Conjecture 1.3.3.7
Number field $\mathbf{Q}$	Function field K_X
Quadratic form q over $\mathbf{Q}$	Algebraic Group G_0
Quadratic form q over $\mathbf{Z}$	Integral model G
Quadratic forms in the genus of q	Principal G-bundles $\mathcal{P}$
$\sum_{q'} \frac{1}{\lvert \mathrm{O}_{q'}(\mathbf{Z}) \rvert}$	$\sum_{\mathcal{P}} \frac{1}{\lvert \mathrm{Aut}(\mathcal{P}) \rvert}$.

1.4 Cohomological Formulation of Weil's Conjecture

On general grounds, one would expect Weil's conjecture (Conjecture 1.2.6.4) to be easier to prove in the function field setting: function fields admit an algebro-geometric interpretation (as rational functions on algebraic curves), and can be studied using geometric techniques that have no analogue in the setting of number fields. In more concrete terms, the mass formula of Conjecture 1.3.3.7 has to do with the problem of counting principal bundles on an algebraic curve, while the Smith-Minkowski-Siegel mass formula (Theorem 1.1.3.5) has to do with counting integral quadratic forms within a genus. The former problem has much more structure than the latter, because principal G-bundles on an algebraic curve admit an algebro-geometric parametrization (see Construction 1.4.1.1). In this section, we will exploit this observation to reduce Weil's conjecture to a pair of statements about the ℓ-adic cohomology of a certain algebraic stack (Theorems 1.4.3.3 and 1.4.4.1).

1.4.1 The Moduli Stack of G-Bundles

Let k be a field, let X be an algebraic curve over k (assumed to be smooth, projective, and geometrically connected), and let G be a smooth affine group scheme over X. In §1.3.2, we introduced the notion of a principal G-bundle on X (Definition 1.3.2.1). More generally, for any map of schemes $X' \to X$, we can consider principal G-bundles on X', which form a category $\mathrm{Tors}_G(X')$. In particular, we can take X' to be a product $Y \times_{\mathrm{Spec}(k)} X$, where Y is an arbitrary k-scheme. In this case, we can think of a principal G-bundle on X' as a *family* of principal G-bundles on X, parametrized by the k-scheme Y. This motivates the following:

Construction 1.4.1.1. For every k-scheme S, let $X_S = S \times_{\mathrm{Spec}(k)} X$ denote the S-scheme obtained from X by extension of scalars. We let $\mathrm{Bun}_G(X)(S)$ denote the category $\mathrm{Tors}_G(X_S)$ of principal G-bundles on X_S. The construction $S \mapsto \mathrm{Bun}_G(X)(S)$ determines a contravariant functor $\{$ k-schemes $\} \to \{$ Groupoids $\}$, which we will denote by $\mathrm{Bun}_G(X)$. We refer to $\mathrm{Bun}_G(X)$ as the *moduli stack of G-bundles*.

Warning 1.4.1.2. A careful reader might object that that the construction $S \mapsto \mathrm{Bun}_G(X)(S)$ is not quite a functor, because the pullback operation on G-bundles is well-defined only up to (canonical) isomorphism. This technical point can be addressed in many ways (for example, using the language of fibered categories, which we will review in §3.2.4). For purposes of the present discussion, we will ignore the issue.

If S is a k-scheme, we will refer to objects of the category $\mathrm{Bun}_G(X)(S) = \mathrm{Tors}_G(X_S)$ as *S-valued points* of $\mathrm{Bun}_G(X)$. For many purposes, it is useful to think of $\mathrm{Bun}_G(X)$ as a geometric object, whose S-valued points correspond to maps $S \to \mathrm{Bun}_G(X)$. To make sense of this idea, we are forced to adopt a liberal interpretation of the word "geometric." The functor $\mathrm{Bun}_G(X)$ cannot be represented by an algebraic variety over k, because the categories $\mathrm{Bun}_G(X)(S)$ are not equivalent to sets (principal G-bundles can admit nontrivial automorphisms, and these play an important role in formulating the mass formula of Conjecture 1.3.3.7). However, one does not need to go far beyond the theory of algebraic varieties. The functor $\mathrm{Bun}_G(X)$ is an example of an *algebraic stack*: that is, there exists a scheme U and a map $U \to \mathrm{Bun}_G(X)$ which is representable by smooth surjections (in other words, for every S-valued point of $\mathrm{Bun}_G(X)$, the fiber product $U \times_{\mathrm{Bun}_G(X)} S$ is representable by a smooth S-scheme U_S with nonempty fibers). We will henceforth assume that the reader has some familiarity with the theory of algebraic stacks (for an introduction, we refer the reader to [22]).

Remark 1.4.1.3 (The Tangent Groupoid to a Stack)**.** Let Y be an algebraic variety over k. Suppose that we are given an extension field k' of k and a point $y \in Y(k')$. Recall that the *Zariski tangent space* to Y at the point η is defined as the fiber of the map $Y(k'[\epsilon]/(\epsilon^2)) \to Y(k')$, taken over the point y.

If $\mathcal{Y}$ is an algebraic stack over k equipped with a k'-valued point $y \in \mathcal{Y}(k')$, then we can again consider the fiber of the map $\mathcal{Y}(k'[\epsilon]/(\epsilon^2)) \to \mathcal{Y}(k')$ over the point η. In general, this fiber should be regarded as a groupoid, which we refer to as the *tangent groupoid to $\mathcal{Y}$ at η* and denote by $T_{\mathcal{Y},y}$. We let $\pi_0 T_{\mathcal{Y},y}$ denote the set of isomorphism classes of objects of $T_{\mathcal{Y},y}$, and $\pi_1 T_{\mathcal{Y},y}$ the automorphism group of any choice of object $\overline{y} \in T_{\mathcal{Y},y}$ (this automorphism group is canonically independent of $\overline{y}$). One can regard $\pi_0 T_{\mathcal{Y},y}$ and $\pi_1 T_{\mathcal{Y},y}$ as vector spaces over k'. If $\mathcal{Y}$ is smooth, then the difference $\dim_{k'}(\pi_0 T_{\mathcal{Y},y}) - \dim_{k'}(\pi_1 T_{\mathcal{Y},y})$ is called the *dimension of $\mathcal{Y}$ at the point y*. This dimension is then locally constant on $\mathcal{Y}$: in other words, one can write $\mathcal{Y}$ as a disjoint union of smooth algebraic stacks having constant dimension.

Example 1.4.1.4 (The Tangent Groupoid to $\mathrm{Bun}_G(X)$)**.** Let k' be an extension field of k and let η be a k'-valued point of $\mathrm{Bun}_G(X)$, which we can identify with a G-bundle $\mathcal{P}$ on the curve $X_{k'}$. Let $\mathfrak{g}_{\mathcal{P}}$ denote the vector bundle on $X_{k'}$ obtained by twisting the Lie algebra $\mathfrak{g}$ of G by means of the bundle $\mathcal{P}$. It follows from standard deformation-theoretic arguments that there are canonical isomorphisms

$$\pi_0 T_{\mathrm{Bun}_G(X),\eta} \simeq \mathrm{H}^1(X_{k'}; \mathfrak{g}_{\mathcal{P}}) \qquad \pi_1 T_{\mathrm{Bun}_G(X),\eta} \simeq \mathrm{H}^0(X_{k'}; \mathfrak{g}_{\mathcal{P}}).$$

It follows that the dimension of the algebraic stack $\mathrm{Bun}_G(X)$ at the point η can be identified with the Euler characteristic

$$-\chi(\mathfrak{g}_{\mathcal{P}}) = \dim_k(\mathrm{H}^1(X; \mathfrak{g}_{\mathcal{P}})) - \dim_k(\mathrm{H}^0(X; \mathfrak{g}_{\mathcal{P}})).$$

Let g be the genus of the curve X and let n be the relative dimension of the map $G \to X$. Applying the Riemann-Roch theorem, we deduce that the dimension of $\mathrm{Bun}_G(X)$ is given by $n(g-1) - \deg(\mathfrak{g}_{\mathcal{P}})$.

Note that the action of G on $\bigwedge^n \mathfrak{g}$ determines a morphism of group schemes $G \to \mathbf{G}_m$. If the generic fiber of G is semisimple, then this homomorphism is necessarily trivial, so that the top exterior power $\bigwedge^n \mathfrak{g}_{\mathcal{P}}$ does not depend on the G-bundle $\mathcal{P}$. Extending the above analysis to k'-valued points of $\mathrm{Bun}_G(X)$, where k' is an arbitrary extension field of k, we deduce that the (virtual) dimension of $\mathrm{Bun}_G(X)$ at each point is given by $n(g-1) - \deg(\mathfrak{g}) = n(g-1) + \deg(\Omega_{G/X})$.

Warning 1.4.1.5. If the group scheme G acts nontrivially on the top exterior power $\bigwedge^n \mathfrak{g}$, then $\mathrm{Bun}_G(X)$ need not have constant dimension. In the case where the ground field k is finite, this is related to the failure of unimodularity for the adelic group $G(\mathbf{A}_X)$. This phenomenon arises, for example, if we take G to be a Borel subgroup of a semisimple group scheme.

1.4.2 Counting Points on Algebraic Varieties

Suppose that X is an algebraic curve defined over a finite field $\mathbf{F}_q$, and let G be a smooth affine group scheme over X. To establish Weil's conjecture in the function field case, we need to evaluate the sum $\sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}$ appearing in Conjecture 1.3.3.7. This sum can be regarded as a (weighted) count of principal G-bundles on the curve X, or equivalently as a (weighted) count of $\mathbf{F}_q$-valued points of the moduli stack $\mathrm{Bun}_G(X)$. In this section, we consider the following simpler problem:

Question 1.4.2.1. Let Y be an algebraic variety defined over a finite field $\mathbf{F}_q$, so that the set $Y(\mathbf{F}_q)$ of $\mathbf{F}_q$-valued points of Y is finite. What can one say about the cardinality of $|Y(\mathbf{F}_q)|$ of the set $Y(\mathbf{F}_q)$?

To address Question 1.4.2.1, let us fix an algebraic closure $\overline{\mathbf{F}}_q$ of the finite field $\mathbf{F}_q$. Then the finite set $Y(\mathbf{F}_q)$ can be viewed as a subset of $Y(\overline{\mathbf{F}}_q)$, which we can identify with the set of closed points of an algebraic variety $\overline{Y} = \operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} Y$ defined over $\overline{\mathbf{F}}_q$. We let $\operatorname{Frob} : \overline{Y} \to \overline{Y}$ denote the *geometric Frobenius map*, defined as the product of the identity map $\mathrm{id} : \operatorname{Spec}(\overline{\mathbf{F}}_q) \to \operatorname{Spec}(\overline{\mathbf{F}}_q)$ with the absolute Frobenius map $Y \to Y$. More concretely, if the variety Y is equipped with a projective embedding $i : Y \hookrightarrow \mathbf{P}^n$, then the geometric Frobenius map Frob is given in homogeneous coordinates by the formula

$$\operatorname{Frob}([x_0 : \cdots : x_n]) = [x_0^q : \cdots x_n^q].$$

Note that a point $y \in Y(\overline{\mathbf{F}}_q)$ belongs to the subset $Y(\mathbf{F}_q) \subseteq Y(\overline{\mathbf{F}}_q)$ if and only if it satisfies the equation $\operatorname{Frob}(y) = y$.

In [38], Weil proposed that the description of $Y(\mathbf{F}_q)$ as the fixed-point set of the map $\operatorname{Frob} : \overline{Y} \to \overline{Y}$ could be used to analyze Question 1.4.2.1. His proposal was motivated by the following theorem of topology:

Theorem 1.4.2.2 (Lefschetz Fixed-Point Theorem)**.** *Let M be a compact manifold and let $f : M \to M$ be a smooth map. For each point $x \in M$, let T_x denote the tangent space to M at the point x and let $Df_x : T_x \to T_{f(x)}$ denote the differential of f at x. Let $M^f = \{x \in M : f(x) = x\}$ denote the set of fixed points of f. Assume that, for each $x \in M^f$, the linear map $\mathrm{id}_{T_x} - (Df)_x$ is an isomorphism of the tangent space T_x with itself. Then the set M^f is finite and we have*

$$\sum_{x \in M^f} \epsilon(x) = \sum_{m \geq 0} (-1)^m \operatorname{Tr}(f^* | \operatorname{H}^m(X; \mathbf{Q})),$$

where $\epsilon(x) \in \{\pm 1\}$ denotes the sign of the determinant $\det(\mathrm{id}_{T_x} - (Df)_x)$.

Let us assume for simplicity that the algebraic variety Y of Question 1.4.2.1 is smooth and projective. Weil's insight was that one should be able to apply some version of Theorem 1.4.2.2 to the geometric Frobenius map $\operatorname{Frob} : \overline{Y} \to \overline{Y}$ to obtain a formula for the number $|Y(\mathbf{F}_q)|$ of fixed points of Frob (here each fixed point should appear with multiplicity 1, since the differential of the Frobenius map vanishes). Motivated by this heuristic, Weil made a series of precise conjectures about the numerical behavior of the integers $|Y(\mathbf{F}_q)|$. Moreover, Weil showed that his conjectures would follow from the existence of a sufficiently well-behaved cohomology theory for algebraic varieties, provided that one had a suitable analogue of Theorem 1.4.2.2. These conjectures were eventually proved by constructing such a cohomology theory: Grothendieck's theory of *étale cohomology*.

We will review the theory of étale cohomology (from a slightly unconventional point of view) in §2. For every prime number ℓ which is invertible in $\mathbf{F}_q$, and every algebraic variety V over $\overline{\mathbf{F}}_q$, this theory assigns *ℓ-adic cohomology groups* $\{\operatorname{H}^n(V; \mathbf{Q}_\ell)\}_{n \geq 0}$

and *compactly supported ℓ-adic cohomology groups* $\{\mathrm{H}^n_c(V; \mathbf{Q}_\ell)\}_{n \geq 0}$, which are finite-dimensional vector spaces over $\mathbf{Q}_\ell$. If Y is an algebraic variety over $\mathbf{F}_q$, then the geometric Frobenius map $\mathrm{Frob} : \overline{Y} \to \overline{Y}$ determines a pullback map from $\mathrm{H}^*_c(\overline{Y}; \mathbf{Q}_\ell)$ to itself. We will abuse notation by denoting this map also by Frob. Then Question 1.4.2.1 can be answered by the following algebro-geometric analogue of Theorem 1.4.2.2:

Theorem 1.4.2.3 (Grothendieck-Lefschetz Trace Formula)**.** *Let Y be an algebraic variety over $\mathbf{F}_q$. Then the number of $\mathbf{F}_q$-valued points of Y is given by the formula*

$$|Y(\mathbf{F}_q)| = \sum_{m \geq 0} (-1)^m \operatorname{Tr}(\mathrm{Frob} \mid \mathrm{H}^m_c(\overline{Y}; \mathbf{Q}_\ell)).$$

For our purposes, it will be convenient to write the Grothendieck-Lefschetz trace formula in a slightly different form. Suppose that Y is a smooth variety of dimension n over $\mathbf{F}_q$. Then, from the perspective of ℓ-adic cohomology, Y behaves as if it were a smooth manifold of dimension $2n$. In particular, it satisfies Poincaré duality: that is, there is a perfect pairing

$$\mu : \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) \to \mathbf{Q}_\ell\,.$$

This pairing is not quite Frobenius-equivariant: instead, it fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell \\ \downarrow{\scriptstyle \mathrm{Frob} \otimes \mathrm{Frob}} & & \downarrow{\scriptstyle q^n} \\ \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell) & \xrightarrow{\mu} & \mathbf{Q}_\ell, \end{array}$$

reflecting the fact that the geometric Frobenius map $\mathrm{Frob} : \overline{Y} \to \overline{Y}$ has degree q^n. In particular, pullback along the geometric Frobenius map Frob induces an isomorphism from $\mathrm{H}^*(\overline{Y}; \mathbf{Q}_\ell)$ to itself, and we have the identity

$$q^{-n} \operatorname{Tr}(\mathrm{Frob} \mid \mathrm{H}^i_c(\overline{Y}; \mathbf{Q}_\ell)) \simeq \operatorname{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^{2n-i}(\overline{Y}; \mathbf{Q}_\ell)).$$

We may therefore rewrite Theorem 1.4.2.3 as follows:

Theorem 1.4.2.4 (Grothendieck-Lefschetz Trace Formula, Dual Version)**.** *Let Y be an algebraic variety over $\mathbf{F}_q$ which is smooth of dimension n. Then the number of $\mathbf{F}_q$-points of Y is given by the formula*

$$\frac{|Y(\mathbf{F}_q)|}{q^n} = \sum_{m \geq 0} (-1)^m \operatorname{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^m(\overline{Y}; \mathbf{Q}_\ell)).$$

Remark 1.4.2.5. In the statement of Theorem 1.4.2.4, the denominator q^n appearing on the left hand side can be regarded as a rough estimate for the cardinality $|Y(\mathbf{F}_q)|$, based only on the information that Y is an algebraic variety of dimension n over $\mathbf{F}_q$ (note that this estimate is exactly correct in the case where Y is an affine space over $\mathbf{F}_q$).

1.4.3 The Trace Formula for $\mathrm{Bun}_G(X)$

Let us now return to the setting of Conjecture 1.3.3.7. Let X be an algebraic curve over a finite field $\mathbf{F}_q$, let G be a smooth affine group scheme over X, and let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X (Construction 1.4.1.1). We would like to apply the ideas of §1.4.2 to the problem of counting principal G-bundles on X.

Let $\overline{\mathbf{F}}_q$ denote an algebraic closure of $\mathbf{F}_q$ and set $\overline{X} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$. Let us abuse notation by not distinguishing between the group scheme G and the fiber product $\overline{X} \times_X G$, so that we can consider the moduli stack $\mathrm{Bun}_G(\overline{X})$ of principal G-bundles on $\overline{X}$. Then $\mathrm{Bun}_G(\overline{X})$ can be identified with the fiber product $\mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Bun}_G(X)$.

Construction 1.4.3.1 (The Geometric Frobenius on $\mathrm{Bun}_G(\overline{X})$)**.** Let R be an $\overline{\mathbf{F}}_q$-algebra. Then the Frobenius map $a \mapsto a^q$ determines an $\mathbf{F}_q$-algebra homomorphism $\varphi : R \to R$, and therefore induces a map $\varphi_X : X_R \to X_R$ (which is the identity on X: that is, φ_X is a map of X-schemes). If $\mathcal{P}$ is a principal G-bundle on X_R, then the pullback $\varphi_X^* \mathcal{P}$ is another principal G-bundle on X_R. The construction $\mathcal{P} \mapsto \varphi_R^* \mathcal{P}$ depends functorially on R, and can therefore be regarded as a map of algebraic stacks $\mathrm{Frob} : \mathrm{Bun}_G(\overline{X}) \to \mathrm{Bun}_G(\overline{X})$. We will refer to Frob as the *geometric Frobenius map* of $\mathrm{Bun}_G(X)$.

Let us now fix a prime number ℓ which is invertible in $\mathbf{F}_q$. The theory of ℓ-adic cohomology can be extended to algebraic stacks (we will discuss this extension in §3.2) and supplies cohomology groups $\mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Q}_\ell)$ (for our purposes it will be convenient to take our coefficient ring to be $\mathbf{Z}_\ell$ rather than $\mathbf{Q}_\ell$, though this is ultimately not important). The geometric Frobenius map Frob of Construction 1.4.3.1 induces an automorphism of the graded ring $\mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)$. By abuse of notation, we will denote this automorphism also by Frob. Here, we encounter a slight wrinkle: the cohomology groups $\mathrm{H}^n(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)$ need not vanish for $n \gg 0$, so a correct formulation of the Grothendieck-Lefschetz trace formula requires some care.

Notation 1.4.3.2. Let $\mathbf{C}$ denote the field of complex numbers, and fix an embedding $\iota : \mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Let M be a $\mathbf{Z}_\ell$-module for which $\mathbf{C} \otimes_{\mathbf{Z}_\ell} M$ is a finite-dimensional vector space over $\mathbf{C}$. If ψ is any endomorphism of M as a $\mathbf{Z}_\ell$-module, we let $\mathrm{Tr}(\psi|M) \in \mathbf{C}$ denote the trace of $\mathbf{C}$-linear map $\mathbf{C} \otimes_{\mathbf{Z}_\ell} M \to \mathbf{C} \otimes_{\mathbf{Z}_\ell} M$ determined by ψ. More generally, if ψ is an endomorphism of a graded $\mathbf{Z}_\ell$-module M^*, we let $\mathrm{Tr}(\psi|M^*)$ denote the alternating sum $\sum_{i \geq 0} (-1)^i \mathrm{Tr}(\psi|M^i)$ (provided that this sum is absolutely convergent).

One of our main goals in this book is to prove the following analogue of Theorem 1.4.2.4:

Theorem 1.4.3.3. *[Grothendieck-Lefschetz Trace Formula for* $\mathrm{Bun}_G(X)$*] Assume that the group scheme G has connected fibers and that the generic fiber of G is semisimple. Then we have an equality*

$$q^{-\dim(\mathrm{Bun}_G(X))} |\,\mathrm{Tors}_G(X)| = \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)).$$

Here $\dim(\mathrm{Bun}_G(X))$ *denotes the dimension of the algebraic stack* $\mathrm{Bun}_G(X)$ *(see Example 1.4.1.4),* $|\,\mathrm{Tors}_G(X)| = \sum_{\mathcal{P}} \frac{1}{|\,\mathrm{Aut}(\mathcal{P})|}$ *denotes the mass of the category of principal G-bundles on X, and the trace on the right hand side is defined as in Notation 1.4.3.2.*

Warning 1.4.3.4. Neither the left nor the right hand side of the identity asserted by Theorem 1.4.3.3 is *a priori* well-defined. We should therefore state it more carefully as follows:

(*a*) For each integer i, the tensor product $\mathbf{C} \otimes_{\mathbf{Z}_\ell} \mathrm{H}^i(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)$ is a finite-dimensional vector space over $\mathbf{C}$, so that the trace $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^i(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell))$ is well-defined.

(*b*) The sum

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)) = \sum_{i \geq 0} (-1)^i \,\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^i(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell))$$

is absolutely convergent (beware that, in contrast with the situation of Theorem 1.4.2.4, this sum is generally infinite).

(*c*) The mass $|\,\mathrm{Tors}_G(X)| = \sum_{\mathcal{P}} \frac{1}{|\,\mathrm{Aut}(\mathcal{P})|}$ is finite.

(*d*) We have an equality

$$\frac{|\,\mathrm{Tors}_G(X)|}{q^{\dim(\mathrm{Bun}_G)}} = \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)).$$

Remark 1.4.3.5. In the statement of Theorem 1.4.3.3, the right hand side is *a priori* dependent on a choice of prime number ℓ (which we always assume to be invertible in $\mathbf{F}_q$) and on a choice of embedding $\iota : \mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. However, Theorem 1.4.3.3 shows that this dependence is illusory (since the left hand side is defined without reference to ℓ or ι).

We can regard Theorem 1.4.3.3 as an analogue of Theorem 1.4.2.4, where the smooth $\mathbf{F}_q$-scheme Y is replaced by the algebraic stack $\mathrm{Bun}_G(X)$. The principal difficulty in verifying Theorem 1.4.3.3 comes not from the fact that $\mathrm{Bun}_G(X)$ is a stack, but from the fact that it need not be quasi-compact. For every quasi-compact open substack $U \subseteq \mathrm{Bun}_G(X)$, one can write U as the stack-theoretic quotient of a smooth algebraic variety $\widetilde{U}$ by the action of an algebraic group H over $\mathbf{F}_q$ (for example, we can take $\widetilde{U}$ to be a fiber product $U \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X, D)$, where $\mathrm{Bun}_G(X, D)$ denotes the moduli stack of G-bundles on X which are equipped with a trivialization on some sufficiently large effective divisor $D \subseteq X$). One can then show that U satisfies the Grothendieck-Lefschetz trace formula by applying Theorem 1.4.2.4 to $\widetilde{U}$ and H (for more details, see §5.1). One might hope to prove Theorem 1.4.3.3 by writing $\mathrm{Bun}_G(X)$ as the union of a sequence of well-chosen quasi-compact open substacks

$$U_0 \hookrightarrow U_1 \hookrightarrow U_2 \hookrightarrow \cdots,$$

and making some sort of convergence argument. Using this method, Behrend proved Theorem 1.4.3.3 in many cases (see [4]). In Chapter 5, we will use the same technique to show that Theorem 1.4.3.3 holds in general.

Variant 1.4.3.6 (Steinberg's Formula)**.** Let G be a connected algebraic group of dimension n over a finite field $\mathbf{F}_q$ and let BG denote the classifying stack of G. Then the algebraic stack BG satisfies the Grothendieck-Lefschetz trace formula in the form

$$\frac{|\operatorname{Tors}_G(\operatorname{Spec}(\mathbf{F}_q))|}{q^{\dim(\mathrm{BG})}} = \operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Z}_\ell)). \tag{1.7}$$

Here $\dim(\mathrm{BG}) = -\dim(G) = -n$ and the mass $|\operatorname{Tors}_G(\operatorname{Spec}(\mathbf{F}_q))|$ is equal to $1/|G(\mathbf{F}_q)|$ (Example 1.3.3.5), so the left hand side of (1.7) is given by $\frac{q^n}{|G(\mathbf{F}_q)|}$, and $\overline{\mathrm{BG}}$ denotes the fiber product $\operatorname{Spec}(\overline{\mathbf{F}}_q) \times_{\operatorname{Spec}(\mathbf{F}_q)} \mathrm{BG}$; see Proposition 4.4.4.1. This gives an explicit formula for the order of the finite group $G(\mathbf{F}_q)$, due originally to Steinberg (see [36]).

1.4.4 Weil's Conjecture

Using Theorem 1.4.3.3 (and the description of $\dim(\mathrm{Bun}_G(X))$ supplied in Example 1.4.1.4), we can reformulate Conjecture 1.3.3.7 as follows:

Theorem 1.4.4.1 (Weil's Conjecture, Cohomological Form)**.** *Let X be an algebraic curve over a finite field $\mathbf{F}_q$ and let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber of G is semisimple and simply connected of dimension n. Fix a prime number ℓ which is invertible in $\mathbf{F}_q$ and an embedding $\iota : \mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Then there is an equality*

$$\operatorname{Tr}(\operatorname{Frob}^{-1} \mid \mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)) = \prod_{x \in X} \frac{|\kappa(x)|^n}{|G(\kappa(x))|}.$$

In particular, the trace on the left hand side and the product on the right hand side are both absolutely convergent.

The bulk of this book is devoted to the proof of Theorem 1.4.4.1.

Remark 1.4.4.2. In the statement of Theorem 1.4.4.1, it follows from either Theorem 1.4.4.1 or Theorem 1.4.3.3 that the trace appearing on the left hand side does not depend on the prime number ℓ or the choice of embedding $\iota : \mathbf{Z}_\ell \hookrightarrow \mathbf{C}$.

Remark 1.4.4.3. Let G be as in Theorem 1.4.4.1. For each closed point $x \in X$, let $G_x = \mathrm{Spec}(\kappa(x)) \times_X G$ denote the fiber of G at x, and let BG_x denote the classifying stack of G_x. Set $\overline{\mathrm{BG}}_x = \overline{\mathrm{BG}}_x = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\kappa(x))} \mathrm{BG}_x$, and let Frob_x denote the geometric Frobenius map on $\overline{\mathrm{BG}}_x$. Using Variant 1.4.3.6, we can rewrite the statement of Theorem 1.4.4.1 in the more suggestive form

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)) = \prod_{x \in X} \mathrm{Tr}(\mathrm{Frob}_x^{-1} \,|\, \mathrm{H}^*(\overline{\mathrm{BG}}_x; \mathbf{Z}_\ell)). \tag{1.8}$$

This can be regarded as a cohomological reformulation of the product formula appearing in Remark 1.3.3.9.

1.5 Computing the Cohomology of $\mathrm{Bun}_G(X)$ over $\mathbf{C}$

Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. Most of this book is devoted to understanding the cohomology of the moduli stack $\mathrm{Bun}_G(X)$. For applications to Weil's conjecture, we are ultimately interested in the case where $k = \overline{\mathbf{F}}_q$ is an algebraic closure of a finite field $\mathbf{F}_q$. In this section, we study the simpler situation where $k = \mathbf{C}$ is the field of complex numbers and the group scheme G is assumed to be semisimple (and simply connected) at each point. In this case, calculating the cohomology of $\mathrm{Bun}_G(X)$ can be regarded as a purely topological problem (Proposition 1.5.1.1), which can be attacked by a variety of classical methods. If we additionally assume that the group scheme G is *constant* (that is, it arises from a linear algebraic group over $\mathbf{C}$), then this problem is fairly easy, at least for cohomology with rational coefficients: there is a simple explicit description of the rational cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$ (Theorem 1.5.2.3), due originally to Atiyah and Bott. In this section, we explain the Atiyah-Bott calculation and indicate a mechanism by which it can be extended to the non-constant case (Theorem 1.5.4.10).

1.5.1 Bundles on a Riemann Surface

Let G be a linear algebraic group defined over the field $\mathbf{C}$ of complex numbers. By abuse of notation, we will not distinguish between G and its set $G(\mathbf{C})$ of $\mathbf{C}$-valued points, which we view as a topological group. Throughout this section, we let BG denote a classifying *space* of G, in the sense of algebraic topology: that is, a quotient EG/G, where EG is a contractible space equipped with a free action of G. The classifying space BG enjoys the following universal mapping property: for any reasonably well-behaved topological space Y (for example, any paracompact manifold), the construction

$$(f : Y \to \mathrm{BG}) \mapsto Y \times_{\mathrm{BG}} \mathrm{EG}$$

determines a bijection from the set $[Y, \mathrm{BG}]$ of homotopy classes of continuous maps from Y into BG to the set of isomorphism classes of G-bundles on Y (in the category of topological spaces).

Let M be a manifold, and let $\mathrm{Map}(M, \mathrm{BG})$ denote the collection of all continuous maps from X to BG. We regard $\mathrm{Map}(M, \mathrm{BG})$ as a topological space by equipping it with the compact open topology. Then $\mathrm{Map}(M, \mathrm{BG})$ classifies G-bundles over M in the category of topological space: that is, for any sufficiently nice parameter space Y, we can identify homotopy classes of maps from Y to $\mathrm{Map}(M, \mathrm{BG})$ with isomorphism classes of principal G-bundles on the product $M \times Y$ in the category of topological spaces.

Proposition 1.5.1.1. *Let X be an algebraic curve over $\mathbf{C}$ (which we assume to be smooth and projective), which we identify with the compact Riemann surface $X(\mathbf{C})$ of $\mathbf{C}$-valued points of X. Let $\mathrm{Bun}_G(X) = \mathrm{Bun}_{G \times_{\mathrm{Spec}(\mathbf{C})} X}(X)$ denote the moduli stack of G-bundles on X, in the sense of Construction 1.4.1.1. Then there is a canonical homotopy equivalence $\mathrm{Bun}_G(X) \simeq \mathrm{Map}(X, \mathrm{BG})$.*

Remark 1.5.1.2. In the setting of Proposition 1.5.1.1, the moduli stack $\mathrm{Bun}_G(X)$ and the mapping space $\mathrm{Map}(X, \mathrm{BG})$ both have universal properties: the former classifies G-bundles in the setting of algebraic geometry, and the latter classifies G-bundles in the setting of topological spaces. Roughly speaking, Proposition 1.5.1.1 asserts that there is not much difference between the two.

Proof of Proposition 1.5.1.1. We provide an informal sketch: a rigorous treatment would require us to define the homotopy type of an algebraic stack over $\mathbf{C}$, which would take us too far afield. For simplicity, let us assume that the linear algebraic group G is connected and simply connected (this is the main case we are interested in). Since X is a projective algebraic variety, the category of algebraic G-bundles on X is equivalent to the category of complex-analytic G-bundles on X. Moreover, the simple connectivity of G guarantees that every complex-analytic G-bundle on X is trivial when viewed as a smooth G-bundle. Consequently, we can identify $\mathrm{Bun}_G(X)$ (in a suitable category of differentiable stacks) with the stack-theoretic quotient $\mathcal{A} / \mathrm{Map}_{\mathrm{sm}}(X, G)$, where $\mathcal{A}$ denotes the collection of all complex structures on the trivial G-bundle $\mathcal{P} = G \times X$, and $\mathrm{Map}_{\mathrm{sm}}(X, G)$ denotes the automorphism group of $\mathcal{P}$ (which we identify with the space of smooth maps from X into G).

Since X has complex dimension 1, the space $\mathcal{A}$ can be identified with the collection of all smooth $\overline{\partial}$-connections on $\mathcal{P}$: that is, the collection of all $\mathbf{C}$-antilinear vector bundle maps $T_X \to \mathfrak{g}_X$, where T_X is the tangent bundle of X, $\mathfrak{g}$ is the Lie algebra of G, and $\mathfrak{g}_X$ denotes the constant vector bundle over X associated to $\mathfrak{g}$. In particular, $\mathcal{A}$ is an infinite-dimensional affine space, and therefore contractible. It follows that $\mathrm{Bun}_G(X)$ has the homotopy type of the classifying space of the group $\mathrm{Map}_{\mathrm{sm}}(X, G)$.

Let $\mathrm{Map}(X, G)$ denote the space of all continuous maps from X into G. It follows from standard approximation arguments that the inclusion $\mathrm{Map}_{\mathrm{sm}}(X, G) \hookrightarrow \mathrm{Map}(X, G)$ is a homotopy equivalence. We now complete the proof by observing that $\mathrm{Map}(X, \mathrm{EG})$ is a contractible space equipped with a free action of $\mathrm{Map}(X, G)$ having quotient $\mathrm{Map}(X, \mathrm{EG}) / \mathrm{Map}(X, G) = \mathrm{Map}(X, \mathrm{BG})$ (here we invoke the fact that every map from X to BG factors through EG, by virtue of our assumption that G is simply connected), so that $\mathrm{Map}(X, \mathrm{BG})$ also has the homotopy type of a classifying space of $\mathrm{Map}_{\mathrm{sm}}(X, G)$. □

Warning 1.5.1.3. In the statement of Proposition 1.5.1.1, the assumption that G is simply connected is superfluous. However, the assumption that X is an algebraic curve is essential. If X is a smooth projective variety of higher dimension, then $\overline{\partial}$-connections on a smooth G-bundle need not be integrable. Consequently, the homotopy type of $\mathrm{Bun}_G(X)$ is not so easy to describe in purely homotopy-theoretic terms.

Variant 1.5.1.4. Let X be an algebraic curve over $\mathbf{C}$ and let G be a group scheme over X which is semisimple at each point (but not necessarily constant). In this case, we take classifying spaces fiberwise to obtain a fibration of topological spaces $\pi : \mathrm{BG} \to X$, whose fiber over a point $x \in X$ can be identified with the classifying space BG_x for $G_x = G \times_X \{x\}$. In this situation, one can prove a *relative* version of Proposition 1.5.1.1: the space $\mathrm{Bun}_G(X)$ is homotopy equivalent to the space $\mathrm{Sect}_\pi(X) = \{s : X \to \mathrm{BG} : \pi \circ s = \mathrm{id}_X\}$ of continuous sections of π. We will return to this point in §1.5.4.

1.5.2 The Atiyah-Bott Formula

Let X be an algebraic curve over the field $\mathbf{C}$ of complex numbers and let G be a simply-connected linear algebraic group over $\mathbf{C}$. By virtue of Proposition 1.5.1.1, the cohomology of the moduli stack $\mathrm{Bun}_G(X)$ can be identified with the cohomology of the mapping space $\mathrm{Map}(X, \mathrm{BG})$. In this section, we give an explicit description of the *rational* cohomology of $\mathrm{Map}(X, \mathrm{BG})$. Our starting point is the following well-known description of the rational cohomology of BG itself:

Proposition 1.5.2.1. *Let G be a linear algebraic group over $\mathbf{C}$ which is simply connected, and let* BG *denote the classifying space of G. Then the cohomology ring* $\mathrm{H}^*(\mathrm{BG};\mathbf{Q})$ *is isomorphic to a polynomial algebra on finitely many homogeneous generators $x_1, x_2, \ldots, x_r$ having even degrees $e_1, \ldots, e_r \geq 4$.*

To describe the cohomology of $\mathrm{Map}(X, \mathrm{BG})$ in terms of the cohomology of BG, it will be convenient to introduce some notation.

Notation 1.5.2.2. Let $V = V^*$ be a graded vector space over the rational numbers. We let $\mathrm{Sym}^*(V)$ denote the free graded-commutative algebra generated by V. More explicitly, if we decompose V as a direct sum $V^{\mathrm{even}} \oplus V^{\mathrm{odd}}$ of even and odd degree subspaces, then we have $\mathrm{Sym}^*(V) = \mathrm{Sym}^*(V^{\mathrm{even}}) \otimes_{\mathbf{Q}} \bigwedge^*(V^{\mathrm{odd}})$ where $\mathrm{Sym}^*(V^{\mathrm{even}})$ denotes the usual symmetric algebra on V^{even}, and $\bigwedge^*(V^{\mathrm{odd}})$ denotes the exterior algebra generated by V^{odd}.

In what follows, let us fix an isomorphism $\mathrm{H}^*(\mathrm{BG};\mathbf{Q}) \simeq \mathrm{Sym}^*(V)$, where V is a graded vector space concentrated in even degrees ≥ 4 (the existence of such an isomorphism follows from Proposition 1.5.2.1). If X is a compact Riemann surface, then the evaluation map $e : \mathrm{Map}(X, \mathrm{BG}) \times X \to \mathrm{BG}$ induces a map of graded vector spaces

$$\begin{aligned} V &\to \mathrm{Sym}^*(V) \\ &\simeq \mathrm{H}^*(\mathrm{BG}; Q) \\ &\xrightarrow{e^*} \mathrm{H}^*(\mathrm{Map}(X, \mathrm{BG}) \times X; \mathbf{Q}) \\ &\simeq \mathrm{H}^*(\mathrm{Map}(X, \mathrm{BG}); \mathbf{Q}) \otimes_{\mathbf{Q}} \mathrm{H}^*(X; \mathbf{Q}), \end{aligned}$$

which we can identify with a map $u : \mathrm{H}_*(X;\mathbf{Q}) \otimes_{\mathbf{Q}} V \to \mathrm{H}^*(\mathrm{Map}(X, \mathrm{BG}); \mathbf{Q})$. We then have the following:

Theorem 1.5.2.3 (Atiyah-Bott)**.** *The map u extends to an isomorphism of graded algebras*

$$\mathrm{Sym}^*(\mathrm{H}_*(X;\mathbf{Q}) \otimes_{\mathbf{Q}} V) \to \mathrm{H}^*(\mathrm{Map}(X, \mathrm{BG}); \mathbf{Q}).$$

We will sketch a proof of Theorem 1.5.2.3 in §1.5.6.

Remark 1.5.2.4. To spell out Theorem 1.5.2.3 more explicitly, let g be the genus of the curve X and let $e_1, e_2, \ldots, e_r$ denote the degrees of the polynomial generators of the cohomology ring $\mathrm{H}^*(\mathrm{BG};\mathbf{Q})$. Then Theorem 1.5.2.3 implies that the cohomology ring $\mathrm{H}^*(\mathrm{Map}(X;\mathrm{BG});\mathbf{Q})$ is isomorphic to a tensor product of a polynomial ring on $2r$ generators (of degrees e_i and $e_i - 2$ for $1 \leq i \leq r$) with an exterior algebra on $2gr$ generators (with $2g$ generators of each degree $e_i - 1$).

1.5.3 Digression: Rational Homotopy Theory

For applications to Weil's conjecture, we will need an analogue of Theorem 1.5.2.3 which applies in cases where the group scheme G is not assumed to be constant (Example 1.5.4.15). Before formulating such an analogue, we take a brief excursion through *rational homotopy theory*, following ideas of Sullivan.

Let X be a topological space. For any commutative ring R, the singular cohomology of X is equipped with *cup product maps*

$$\cup : \mathrm{H}^i(X;R) \times \mathrm{H}^j(X;R) \to \mathrm{H}^{i+j}(X;R)$$

which endow $\mathrm{H}^*(X;R)$ with the structure of a graded-commutative ring: that is, the cup product $\cup$ is unital, associative, and satisfies $u \cup v = (-1)^{ij} v \cup u$ for $u \in \mathrm{H}^i(X;R)$ and $v \in \mathrm{H}^j(X;R)$. The cup product operation can be realized at the cochain level: that is, it arises from a map of singular cochain complexes

$$m : C^*(X;R) \otimes_R C^*(X;R) \to C^*(X;R)$$

by passing to cohomology. The map m is unital and associative: that is, it endows the singular cochain complex $C^*(X;R)$ with the structure of a differential graded algebra over R. However, it is not commutative (even in the graded sense): if $\overline{u} \in C^i(X;R)$ and $\overline{v} \in C^j(X;R)$ are cocycles representing cohomology classes $u \in \mathrm{H}^i(X;R)$ and $v \in \mathrm{H}^j(X;R)$, then the identity $u \cup v = (-1)^{ij} v \cup u$ guarantees that we can write $m(\overline{u}, \overline{v}) = (-1)^{ij} m(\overline{v}, \overline{u}) + d\epsilon(u,v)$ for some cochain $\epsilon(u,v) \in C^{i+j-1}(X;R)$, but this cochain is generally nonzero.

The failure of commutativity at the cochain level can sometimes be avoided by computing cohomology in a different way.

Example 1.5.3.1 (de Rham Cohomology)**.** Suppose that X is a smooth manifold and let $\mathbf{R}$ be the field of real numbers. A theorem of de Rham supplies a canonical isomorphism

$$\rho : \mathrm{H}^*_{\mathrm{DR}}(X) \simeq \mathrm{H}^*(X;\mathbf{R}),$$

where $\mathrm{H}^*_{\mathrm{DR}}(X)$ denotes the cohomology of the smooth de Rham complex

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \cdots .$$

Moreover, the map ρ can be regarded as an isomorphism of graded rings, where we equip the de Rham cohomology $\mathrm{H}^*_{\mathrm{DR}}(X)$ with the ring structure arising from the wedge product of differential forms. Note that, unlike the cup product, the wedge product of differential forms satisfies the graded-commutative law $\omega \wedge \omega' = (-1)^{ij} \omega' \wedge \omega$ for $\omega \in \Omega^i(X)$, $\omega' \in \Omega^j(X)$ *before* passing to cohomology.

Let us introduce some terminology which axiomatizes some key features of Example 1.5.3.1.

Definition 1.5.3.2. Let R be a commutative ring. A *graded-commutative R-algebra* is a graded R-algebra A^* whose multiplication satisfies the commutativity law $xy = (-1)^{ij} yx$ for $x \in A^i$ and $y \in A^j$. We let $\mathrm{CAlg}_R^{\mathrm{gr}}$ denote the category whose objects are graded-commutative R-algebras and whose morphisms are homomorphisms of graded R-algebras.

A *commutative differential graded algebra over R* is a graded-commutative R-algebra A^* equipped with an R-linear differential $d : A^* \to A^{*+1}$ satisfying $d^2 = 0$ and the Leibniz rule $d(xy) = (dx)y + (-1)^i x(dy)$ for $x \in A^i$. We let $\mathrm{CAlg}_R^{\mathrm{dg}}$ denote the category whose objects are commutative differential graded algebras over R and whose morphisms are graded R-algebra homomorphisms $f : A^* \to B^*$ satisfying $df = fd$.

Remark 1.5.3.3. Let $A = (A^*, d)$ be a commutative differential graded Lie algebra over R. Then the cohomology $\mathrm{H}^*(A)$ (with respect to the differential d) is a graded-commutative algebra over R.

Example 1.5.3.4. Let X be a topological space. For any commutative ring R, the cohomology ring $\mathrm{H}^*(X; R)$ is a graded-commutative R-algebra.

Example 1.5.3.5. Let X be a smooth manifold. Then the smooth de Rham complex $\Omega^*(X)$ is a commutative differential graded algebra over the field $\mathbf{R}$ of real numbers.

Sullivan observed that there is a variant of Example 1.5.3.5 which makes sense for an *arbitrary* topological space X.

Construction 1.5.3.6 (Sullivan)**.** For each $n \geq 0$, let

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbf{R}_{\geq 0} \,|x_0 + \cdots + x_n = 1\}$$

denote the standard n-simplex and let

$$\Delta^n_+ = \{(x_0, \ldots, x_n) \in \mathbf{R} \,|x_0 + \cdots + x_n = 1\}$$

denote the affine space containing it. We will say that a differential form ω on Δ^n_+ is *polynomial* if it belongs to the subalgebra of $\Omega^*(\Delta^n_+)$ generated (over the rational numbers) by the functions x_i and their differentials dx_i.

Let X be an arbitrary topological space and let $m \geq 0$ be an integer. A *singular m-form* on X is a function

$$\omega : \{\text{Continuous maps } \Delta^n \to X\} \to \{\text{Polynomial } m\text{-forms on } \Delta^n_+\}$$

$$\sigma \mapsto \omega_\sigma$$

which satisfies the following constraint: if $f : \Delta^{n'} \to \Delta^n$ is the map of simplices associated to a nondecreasing function $\{0 < 1 < \ldots < n'\} \to \{0 < 1 < \cdots < n\}$, then $\omega_{\sigma \circ f} = \omega_\sigma|_{\Delta^{n'}_+}$. We let $\Omega^m_{\mathrm{poly}}(X)$ denote the set of all singular m-forms on X.

If ω is a singular m-form on X, then we can define a singular $(m+1)$-form $d\omega$ on X by the formula $(d\omega)_\sigma = d(\omega_\sigma)$. Using this differential, we can regard

$$0 \to \Omega^0_{\mathrm{poly}}(X) \to \Omega^1_{\mathrm{poly}}(X) \to \Omega^2_{\mathrm{poly}}(X) \to \cdots$$

as a chain complex of rational vector spaces, which we will refer to as the *polynomial de Rham complex of* X and denote by $\Omega^*_{\mathrm{poly}}(X)$. We regard $\Omega^*_{\mathrm{poly}}(X)$ as a commutative differential graded algebra over $\mathbf{Q}$, with multiplication given by $(\omega \wedge \omega')_\sigma = \omega_\sigma \wedge \omega'_\sigma$.

Let X be a topological space, let m be a nonnegative integer, and let ω be a singular n-form on X. Then the construction

$$(\sigma : \Delta^n \to X) \mapsto \int_{\Delta^n} \omega_\sigma$$

can be regarded as a singular n-cochain on X with values in $\mathbf{Q}$, which we will denote by $\int \omega$.

Theorem 1.5.3.7 (Sullivan)**.** *For any topological space X, the construction $\omega \mapsto \int \omega$ induces a quasi-isomorphism of chain complexes*

$$\int : \Omega^*_{\mathrm{poly}}(X) \to C^*(X; \mathbf{Q}).$$

Moreover, the induced isomorphism on cohomology $\mathrm{H}^*(\Omega^*_{\mathrm{poly}}(X)) \to \mathrm{H}^*(X; \mathbf{Q})$ is an isomorphism of graded-commutative rings.

By virtue of Theorem 1.5.3.7, one can regard the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ as an "improved" version of the singular cochain complex $C^*(X; \mathbf{Q})$: it has the same cohomology, but has the virtue of seeing the commutativity of the cup product *at the cochain level*. This makes the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(X)$ a much more powerful invariant of X. In fact, Sullivan showed that it is a complete invariant of the *rational* homotopy type of X. To formulate this precisely, let us introduce a bit more terminology (which we will use in §1.5.4).

Definition 1.5.3.8 (The Homotopy Category of $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$)**.** Let $f : A = (A^*, d) \to (B^*, d) = B$ be a morphism of commutative differential graded algebras over $\mathbf{Q}$. We will say that f is a *quasi-isomorphism* if it is a quasi-isomorphism of the underlying chain complexes: that is, if the induced map $\mathrm{H}^*(A) \to \mathrm{H}^*(B)$ is an isomorphism of graded-commutative $\mathbf{Q}$-algebras. We let $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the category obtained from $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ by formally inverting all quasi-isomorphisms. We will refer to $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ as the *homotopy category of commutative differential graded algebras over* $\mathbf{Q}$.

Definition 1.5.3.9 (The Rational Homotopy Category)**.** Let $f : X \to Y$ be a map of simply connected topological spaces. We will say that f is a *rational homotopy equivalence* if the induced map of rational cohomology rings $f^* : \mathrm{H}^*(Y; \mathbf{Q}) \to \mathrm{H}^*(X; \mathbf{Q})$ is an isomorphism. We let $\mathcal{H}_{\mathbf{Q}}$ denote the category obtained from the category of simply connected topological spaces by inverting all rational homotopy equivalences.

It follows immediately from Theorem 1.5.3.7 that any rational homotopy equivalence $f : X \to Y$ induces a quasi-isomorphism of polynomial de Rham complexes $f^* : \Omega^*_{\mathrm{poly}}(Y) \to \Omega^*_{\mathrm{poly}}(X)$. Consequently, the formation of polynomial de Rham complexes determines a functor of homotopy categories

$$\Omega^*_{\mathrm{poly}} : \mathcal{H}_{\mathbf{Q}}^{\mathrm{op}} \to \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}.$$

Theorem 1.5.3.10 (Sullivan)**.** *The functor $\Omega^*_{\mathrm{poly}} : \mathcal{H}_{\mathbf{Q}}^{\mathrm{op}} \to \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ is fully faithful when restricted to the full subcategory of $\mathcal{H}_{\mathbf{Q}}^{\mathrm{op}}$ spanned by those simply connected topological spaces X for which the cohomology $\mathrm{H}^*(X; \mathbf{Q})$ is finite-dimensional in each degree.*

Remark 1.5.3.11. Using Theorem 1.5.3.10, one can reduce topological questions about the rational homotopy category $\mathcal{H}_{\mathbf{Q}}$ to more concrete questions about the structure of commutative differential graded algebras. This is one starting point for the theory of *rational homotopy theory*. Our interests in this book lie in a somewhat orthogonal direction: we will be interested in the theory of commutative differential graded algebras (and its mixed characteristic incarnation as the theory of $\mathbb{E}_\infty$*-algebras*) in its own right, rather than as a tool for capturing topological information.

Warning 1.5.3.12. In the statement of Theorem 1.5.3.7, it is essential that we work over the field $\mathbf{Q}$ of rational numbers (or over some ring which contains $\mathbf{Q}$). The proof of Theorem 1.5.3.7 requires us to integrate polynomial differential forms, and integration of polynomials introduces denominators. When not working rationally, the failure of cup product to be commutative at the cochain level is an unavoidable phenomenon: when $R = \mathbf{Z}/p\mathbf{Z}$, it is responsible for the existence of Steenrod operations on the cohomology $\mathrm{H}^*(X; R)$. In general, we cannot hope to (functorially) replace the cochain complex $C^*(X; R)$ by a commutative differential graded algebra over R (however, we *can* equip $C^*(X; R)$ with the structure of an $\mathbb{E}_\infty$-algebra over R, which is an appropriate replacement in many contexts; see §3.1.4).

1.5.4 The Product Formula

Let us now return to the situation of interest to us. Let X be an algebraic curve over the field $\mathbf{C}$ of complex numbers, and let G be a smooth affine group scheme over X

whose fibers are semisimple and simply connected. In this case, we take classifying spaces fiberwise to obtain a fibration $\pi : \mathrm{BG} \to X$. Moreover, the homotopy type of the moduli stack $\mathrm{Bun}_G(X)$ can be identified with the space $\mathrm{Sect}_\pi(X)$ of continuous sections of π (Variant 1.5.1.4). The problem of computing the cohomology of $\mathrm{Bun}_G(X)$ can therefore be regarded as a special case of the following:

Question 1.5.4.1. Let $\pi : E \to X$ be a fibration of topological spaces and let $\mathrm{Sect}_\pi(X)$ denote the space of sections of π. How can we describe the rational cohomology ring $\mathrm{H}^*(\mathrm{Sect}_\pi(X); \mathbf{Q})$?

To fix ideas, let us assume that X is a manifold (remember that we are primarily interested in the case where X is a Riemann surface). We would like to give an answer to Question 1.5.4.1 which is in the spirit of a Künneth formula. Roughly speaking, we can think of the $\mathrm{Sect}_\pi(X)$ as a "continuous" product $\prod_{x \in X} E_x$, where $E_x = \pi^{-1}\{x\}$ denotes the fiber of π over the point x. We might then expect that the cohomology ring $\mathrm{H}^*(\mathrm{Sect}_\pi(X); \mathbf{Q})$ can be described as a tensor product of the cohomology rings $\{\mathrm{H}^*(E_x; \mathbf{Q})\}_{x \in X}$. Let us begin with a crude attempt to make this precise.

Remark 1.5.4.2. Let A^* and B^* be graded-commutative $\mathbf{Q}$-algebras (Definition 1.5.3.2). Then the tensor product $A^* \otimes_{\mathbf{Q}} B^*$ inherits the structure of a graded-commutative $\mathbf{Q}$-algebra, with multiplication given by

$$(a \otimes b)(a' \otimes b') = (-1)^{ij}(aa' \otimes bb') \quad \text{for } a' \in A^i \text{ and } b \in B^j.$$

Note that we have canonical maps $A^* \to A^* \otimes_{\mathbf{Q}} B^* \leftarrow B^*$, which exhibit $A^* \otimes_{\mathbf{Q}} B^*$ as the coproduct of A^* and B^* in the category $\mathrm{CAlg}^{\mathrm{gr}}_{\mathbf{Q}}$.

Warning 1.5.4.3. Let A^* and B^* be graded-commutative $\mathbf{Q}$-algebras. Then we can equip the tensor product $A^* \otimes_{\mathbf{Q}} B^*$ with the structure of a graded $\mathbf{Q}$-algebra, whose multiplication is given by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'. \tag{1.9}$$

Beware that, with respect to this multiplication, $A^* \otimes_{\mathbf{Q}} B^*$ is generally not graded-commutative. In what follows, we will *always* regard tensor products of graded-commutative algebras as equipped with the multiplication described in Remark 1.5.4.2, rather than the multiplication described in equation (1.9).

Construction 1.5.4.4 (Infinite Tensor Products: Algebraic Version)**.** Let $\{A^*_x\}_{x \in X}$ be a collection of graded-commutative $\mathbf{Q}$-algebras indexed by a set X. We let $\bigotimes^{\mathrm{alg}}_{x \in X} A^*_x$ denote the coproduct of the collection $\{A^*_x\}_{x \in X}$, formed in the category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. By definition, the tensor product $\bigotimes_{x \in X}^{\text{alg}} A_x^*$ is characterized by the following universal property: for any graded-commutative $\mathbf{Q}$-algebra B^*, we have a canonical bijection

$$\mathrm{Hom}(\bigotimes_{x \in X}^{\text{alg}} A_x^*, B^*) \simeq \prod_{x \in X} \mathrm{Hom}(A_x^*, B^*),$$

where the Hom-sets are taken in the category $\mathrm{CAlg}_{\mathbf{Q}}^{\mathrm{dg}}$.

More concretely, the infinite tensor product $\bigotimes_{x \in X}^{\text{alg}} A_x^*$ can be described as the (filtered) direct limit $\varinjlim_{S \subseteq X} \bigotimes_{x \in S} A_x^*$, where S ranges over all finite subsets of X and the finite tensor products $\bigotimes_{x \in S} A_x^*$ can be described by iterating the construction of Example 1.5.4.2.

Let us now return to Question 1.5.4.1. Suppose we are given a fibration of topological spaces $\pi : E \to X$. For each point $x \in X$, evaluation at x determines a continuous map $e_x : \mathrm{Sect}_\pi(X) \to E_x$, which induces a pullback map on cohomology $\rho_x^{\text{alg}} : \mathrm{H}^*(E_x; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{Sect}_\pi(X); \mathbf{Q})$. Amalgamating the homomorphisms ρ_x, we obtain a map

$$\rho_X^{\text{alg}} : \bigotimes_{x \in X}^{\text{alg}} \mathrm{H}^*(E_x; \mathbf{Q}) \to \mathrm{H}^*(\mathrm{Sect}_\pi(X); \mathbf{Q}).$$

If X is a finite set (with the discrete topology) and the rational cohomology of each E_x is finite-dimensional in each degree, then ρ_X^{alg} is an isomorphism: this follows from the Künneth formula. However, the map ρ_X^{alg} is usually very far from being an isomorphism. Note that if $p : [0,1] \to X$ is a path beginning at a point $p(0) = x$ and ending at a point $p(1) = y$, then transport along p determines a homotopy equivalence $\gamma : E_x \to E_y$, and therefore an isomorphism of cohomology rings $\gamma^* : \mathrm{H}^*(E_y; \mathbf{Q}) \to \mathrm{H}^*(E_x; \mathbf{Q})$. This map fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^*(E_y; \mathbf{Q}) & \xrightarrow{\gamma^*} & \mathrm{H}^*(E_x; \mathbf{Q}) \\ & \searrow^{\rho_y^{\text{alg}}} \quad \swarrow_{\rho_x^{\text{alg}}} & \\ & \mathrm{H}^*(\mathrm{Sect}_\pi(X); \mathbf{Q}). & \end{array}$$

This has several unpleasant consequences:

(a) In general, the map ρ_X^{alg} has an enormous kernel. For example, ρ_X^{alg} annihilates $u - \gamma^*(u)$, for any element $u \in \mathrm{H}^*(E_y; \mathbf{Q})$.

(b) If X is connected, then the image of ρ_X^{alg} is the same as the image of ρ_x^{alg}, for any chosen point $x \in X$.

However, it turns out that failure of ρ_X^{alg} to be an isomorphism stems from the fact that our definition of the tensor product

$$\bigotimes_{x \in X}^{\mathrm{alg}} \mathrm{H}^*(E_x; \mathbf{Q}) = \varinjlim_{S \subseteq X} \bigotimes_{x \in S} \mathrm{H}^*(E_x; \mathbf{Q}) \tag{1.10}$$

is too naive: it completely neglects the topology on the space X. We can get a much better approximation to the cohomology ring $\mathrm{H}^*(\mathrm{Sect}_\pi(X); \mathbf{Q})$ by introducing a homotopy-theoretic enhancement of the right hand side of (1.10), which differs from the algebraic tensor product in three (closely related) ways:

(*i*) In place of the cohomology rings $\mathrm{H}^*(E_x; \mathbf{Q})$, we work with polynomial de Rham complexes $\Omega^*_{\mathrm{poly}}(E_x)$ of Construction 1.5.3.6 (before passing to cohomology).

(*ii*) Rather than taking a coproduct indexed by the points of X, we consider instead a *colimit* indexed by the partially ordered set of open disks $U \subseteq X$. Roughly speaking, this has the effect of "turning on" the topology of X, by allowing its points to move.

(*iii*) Rather than working with colimits in the category $\mathrm{CAlg}^{\mathrm{gr}}_{\mathbf{Q}}$ of graded-commutative $\mathbf{Q}$-algebras, we consider *homotopy* colimits in the category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ of commutative differential graded algebras.

We begin by elaborating on (*iii*).

Definition 1.5.4.5 (Homotopy Colimits)**.** Let $\mathcal{J}$ be a small category and consider the category $\mathrm{Fun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$ of functors from $\mathcal{J}$ to the category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ of commutative differential graded algebras over $\mathbf{Q}$ (Definition 1.5.3.2). For every commutative differential graded algebra A, we let $\underline{A} \in \mathrm{Fun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$ denote the constant functor taking the value A.

Let $u : F \to G$ be a morphism in the category $\mathrm{Fun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$, which we regard as a natural transformation between functors $F, G : \mathcal{J} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. We will say that u is a *quasi-isomorphism* if, for every object $J \in \mathcal{J}$, the induced map $F(J) \to G(J)$ is a quasi-isomorphism of commutative differential graded algebras (Definition 1.5.3.8). We let $\mathrm{hFun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$ denote the category obtained from $\mathrm{Fun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$ by formally inverting all quasi-isomorphisms.

Suppose that we are given a functor $F : \mathcal{J} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ and a natural transformation $u : F \to \underline{A}$, for some fixed commutative differential graded algebra $A \in \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. We will say that *$u$ exhibits A as a homotopy colimit of F* if, for every commutative differential graded algebra B, composition with u induces a bijection

$$\mathrm{Hom}_{\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}}(A, B) \to \mathrm{Hom}_{\mathrm{hFun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})}(F, \underline{B}).$$

Remark 1.5.4.6. Let $\mathcal{J}$ be a small category. It follows immediately from the definitions that if a functor $F : \mathcal{J} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ admits a homotopy colimit A, then A is determined up to canonical isomorphism as an object of the homotopy category $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. One can show that every functor $F : \mathcal{J} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ is *quasi-isomorphic* to a functor which admits a homotopy colimit. More precisely, the formation of homotopy colimits determines a functor

$$\mathrm{hocolim} : \mathrm{hFun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}) \to \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$$

which is left adjoint to the diagonal map

$$\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}} \to \mathrm{hFun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}) \qquad \mathrm{A} \mapsto \underline{\mathrm{A}}.$$

Warning 1.5.4.7. For every small category $\mathcal{J}$, there is an evident comparison functor $\phi : \mathrm{hFun}(\mathcal{J}, \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}) \to \mathrm{Fun}(\mathcal{J}, \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}})$. Beware that ϕ is usually very far from being an equivalence of categories (in general it is neither faithful, nor full, nor essentially surjective). Consequently, the notion of *homotopy colimit* (introduced in Definition 1.5.4.5) is not the same as the notion of *colimit in the homotopy category* $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. The latter notion is not very well-behaved. For example, every functor $F : \mathcal{J} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ admits a homotopy colimit (at least after replacing F by a quasi-isomorphic functor), but it is fairly uncommon for there to exist a colimit of the induced functor $\mathcal{J} \to \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$.

We now use the theory of homotopy colimits to introduce a homotopy-theoretic refinement of the algebraic tensor product $\bigotimes^{\mathrm{alg}}_{x \in X} \mathrm{H}^*(E_x; \mathbf{Q})$ considered above.

Construction 1.5.4.8 (Continuous Tensor Product)**.** Let $\pi : E \to X$ be a fibration of topological spaces, where X is a manifold of dimension d, and let $\mathcal{U}_0(X)$ denote the collection of open subsets of X which are homeomorphic to the Euclidean space $\mathbf{R}^d$. For each open set $U \subseteq X$, we let $\mathrm{Sect}_\pi(U)$ denote the space of sections of the projection map $E \times_X U \to U$. Note that an inclusion of open sets $U \subseteq V$ induces a restriction map $\mathrm{Sect}_\pi(V) \to \mathrm{Sect}_\pi(U)$, and therefore a map of polynomial de Rham complexes $\Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U)) \to \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(V))$. We can therefore regard the construction $U \mapsto \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U))$ as a functor $\mathcal{B} : \mathcal{U}_0(X) \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$. We let $\bigotimes_{x \in X} C^*(E_x; \mathbf{Q})$ denote the homotopy colimit of the functor $\mathcal{B}$, in the sense of Remark 1.5.4.6: that is, the image of $\mathcal{B}$ under the left adjoint of the diagonal map $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}} \to \mathrm{hFun}(\mathcal{U}_0(\mathrm{X}), \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$.

Remark 1.5.4.9. Let $\pi : E \to X$ be as in Construction 1.5.4.8. For each open disk $U \in \mathcal{U}_0(X)$ containing a point $x \in U$, we have canonical quasi-isomorphisms of cochain complexes

$$\Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U)) \to C^*(\mathrm{Sect}_\pi(U); \mathbf{Q}) \leftarrow C^*(E_x; \mathbf{Q})$$

(the first by virtue of Theorem 1.5.3.7, and the second because evaluation at x induces a homotopy equivalence $\mathrm{Sect}_\pi(U) \to E_x$). Consequently, we can view Construction 1.5.4.8 as an analogue of the algebraic tensor product $\bigotimes^{\mathrm{alg}}_{x\in X} \mathrm{H}^*(E_x;\mathbf{Q})$, where we replace colimits in the category of graded-commutative algebras (indexed by the collection of points $x \in X$) by homotopy colimits in the category of commutative differential graded algebras (indexed by the collection of open disks in X).

In the situation of Construction 1.5.4.8, the continuous tensor product

$$\bigotimes_{x\in X} C^*(E_x;\mathbf{Q})$$

is well-defined only up to quasi-isomorphism (that is, up to isomorphism in the homotopy category $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$). However, there is an obvious candidate for a representative of $\bigotimes_{x\in X} C^*(E_x;\mathbf{Q})$. For each $U \in \mathcal{U}_0(X)$, the inclusion $U \subseteq X$ induces a map of commutative differential graded algebras

$$\mathcal{B}(U) \to \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U)) \to \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(X)),$$

depending functorially on U. This collection of maps then classifies a comparison map

$$\rho_X : \bigotimes_{x\in X} C^*(E_x;\mathbf{Q}) = \mathrm{hocolim}(B) \to \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(X))$$

in the homotopy category $\mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$, which we can regard as a continuous analogue of the map

$$\rho^{\mathrm{alg}}_X : \bigotimes^{\mathrm{alg}}_{x\in X} \mathrm{H}^*(E_x;\mathbf{Q}) \to \mathrm{H}^*(\mathrm{Sect}_\pi(X);\mathbf{Q})$$

considered above. However, the map ρ_X is much better behaved than its algebraic analogue:

Theorem 1.5.4.10 (The Product Formula)**.** *Let X be a compact manifold of dimension d and $\pi : E \to X$ be a fibration. Assume that for each $x \in X$, the fiber E_x is d-connected and that the cohomology groups $\mathrm{H}^*(E_x;\mathbf{Q})$ are finite-dimensional in each degree. Then the comparison map*

$$\rho_X : \bigotimes_{x\in X} C^*(E_x;\mathbf{Q}) \to \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(X))$$

is a quasi-isomorphism of commutative differential graded algebras. More precisely, the evident natural transformation of functors $\mathcal{B} \to \underline{\Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(X))}$ exhibits $\Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(X))$ as a homotopy colimit of the diagram $\mathcal{B}$, in the sense of Definition 1.5.4.5.

Remark 1.5.4.11. Let $\pi : E \to X$ as in Theorem 1.5.4.10. We can state the product formula more informally by saying that there is a canonical quasi-isomorphism of chain complexes

$$\bigotimes_{x \in X} C^*(E_x; \mathbf{Q}) \to C^*(\mathrm{Sect}_\pi(X); \mathbf{Q})$$

(by virtue of Theorem 1.5.3.7).

Remark 1.5.4.12. In the statement of Theorem 1.5.4.10, the hypothesis that X is a compact manifold is not very important: one can formulate a similar statement for any topological space X which is homotopy equivalent to a finite cell complex of dimension $\leq d$ (see Remark 3.1.7.4).

Example 1.5.4.13. In the special case $d = 0$, Theorem 1.5.4.10 reduces to the Künneth formula for rational cohomology.

Remark 1.5.4.14. In §3.1.7, we will formulate an analogue of Theorem 1.5.4.10 for cohomology with coefficients in any commutative ring Λ (Theorem 3.1.7.3).

Example 1.5.4.15. Let X be an algebraic curve over $\mathbf{C}$ (which we identify with the Riemann surface $X(\mathbf{C})$), let G be a semisimple group scheme over X whose fibers are simply connected, and let $\pi : E \to X$ be the fibration whose fibers are the classifying spaces of the fibers of G. Combining Theorem 1.5.4.10 with a Variant 1.5.1.4, we obtain a canonical quasi-isomorphism of differential graded algebras

$$\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q}) \to C^*(\mathrm{Bun}_G(X); \mathbf{Q}).$$

Warning 1.5.4.16. In the statement of Theorem 1.5.4.10, the connectivity assumption on the fibers of the map $\pi : E \to X$ is essential. For example, suppose that π is the fibration of Example 1.5.4.15, where G is a semisimple group scheme over X which is *not* simply connected. In this case, replacing G by its universal cover does not change the continuous tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$, but can change the number of connected components of the moduli stack $\mathrm{Bun}_G(X)$.

1.5.5 Proof of the Product Formula

We now turn to the proof of Theorem 1.5.4.10. Since neither Theorem 1.5.4.10 nor its proof plays any logical role in our proof of Weil's conjecture, we will provide only an informal sketch. Let $\pi : E \to X$ be a fibration of topological spaces, where X is a compact manifold of dimension d and the fibers $E_x = E \times_X \{x\}$ are d-connected topological spaces whose rational cohomology groups $\mathrm{H}^*(E_x; \mathbf{Q})$ are finite-dimensional in each degree. We will assume for simplicity that the manifold X is smooth and connected.

For each open set $U \subseteq X$, we let $\bigotimes_{x \in U} C^*(E_x; \mathbf{Q})$ denote the homotopy colimit

$$\mathrm{hocolim}_{V \in \mathcal{U}_0(U)} \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(V)),$$

which we regard as a commutative differential graded algebra over $\mathbf{Q}$. Using the universal property of the homotopy colimit, we obtain a tautological comparison map

$$\rho_U : \bigotimes_{x \in U} C^*(E_x; \mathbf{Q}) \to \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U)).$$

Let us say that the open set U is *good* if the cohomology groups $\mathrm{H}^*(\mathrm{Sect}_\pi(U); \mathbf{Q})$ are finite-dimensional in each degree and the map ρ_U is a quasi-isomorphism. To prove Theorem 1.5.4.10, we must show that the manifold X is good (when regarded as an open subset of itself).

Note that if $U \subseteq X$ is an open disk, then the collection of open sets $\mathcal{U}_0(U)$ contains U itself (as a largest element). From this, we immediately deduce that every open disk $U \subseteq X$ is good. Moreover, it is easy to see that the empty set $\emptyset \subseteq X$ is good. Using standard covering arguments, we can reduce to proving the following:

$(*)$ If U and V are good open subsets of X for which the intersection $U \cap V$ is good, then the union $U \cup V$ is also good.

We now observe that if U and V are as in $(*)$, then functoriality determines a commutative diagram σ :

$$\begin{array}{ccc} \bigotimes_{x \in U \cap V} C^*(E_x; \mathbf{Q}) & \longrightarrow & \bigotimes_{x \in U} C^*(E_x; \mathbf{Q}) \\ \downarrow & & \downarrow \\ \bigotimes_{x \in V} C^*(E_x; \mathbf{Q}) & \longrightarrow & \bigotimes_{x \in U \cup V} C^*(E_x; \mathbf{Q}), \end{array}$$

which is well-defined up to quasi-isomorphism. It follows by relatively formal arguments that σ is a homotopy pushout square: that is, it exhibits the continuous tensor product $\bigotimes_{x \in U \cup V} C^*(E_x; \mathbf{Q})$ as a homotopy colimit of the diagram

$$\bigotimes_{x \in U} C^*(E_x; \mathbf{Q}) \leftarrow \bigotimes_{x \in U \cap V} C^*(E_x; \mathbf{Q}) \to \bigotimes_{x \in V} C^*(E_x; \mathbf{Q})$$

(in the sense of Definition 1.5.4.5). Consequently, to deduce that $\rho_{U \cup V}$ is good, it will suffice to show that the diagram

$$\begin{array}{ccc} \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U \cap V)) & \longrightarrow & \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U)) \\ \downarrow & & \downarrow \\ \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(V)) & \longrightarrow & \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U \cup V) \end{array}$$

is also a homotopy pushout square. This (and the finite-dimensionality of the cohomology groups $\mathrm{H}^*(\mathrm{Sect}_\pi(U \cup V); \mathbf{Q})$) follow from the convergence of the cohomological Eilenberg-Moore spectral sequence for the homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{Sect}_\pi(U \cup V) & \longrightarrow & \mathrm{Sect}_\pi(U) \\ \downarrow & & \downarrow \\ \mathrm{Sect}_\pi(V) & \longrightarrow & \mathrm{Sect}_\pi(U \cap V). \end{array}$$

Here the space $\mathrm{Sect}_\pi(U \cap V)$ is simply connected, because the fibers of π are assumed to be d-connected and $U \cap V$ is homotopy equivalent to a cell complex of dimension $< d$ (except in the trivial case $U = V = X$, in which case there is nothing to prove).

1.5.6 Proof of the Atiyah-Bott Formula

Let X be an algebraic curve over the field $\mathbf{C}$ of complex numbers and let G be a smooth affine group scheme over X, whose fibers are semisimple and simply connected. Then the product formula of Theorem 1.5.4.10 supplies a canonical quasi-isomorphism

$$\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q}) \quad \simeq \quad C^*(\mathrm{Bun}_G(X); \mathbf{Q}) \tag{1.11}$$

(see Example 1.5.4.15). In theory, this quasi-isomorphism gives a complete description of the cohomology ring $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$ in terms of data which is "local" on the curve X. Our goal in this section is to translate theory into practice by explaining that, in the case where the group scheme G is constant, (1.11) is essentially equivalent to the classical Atiyah-Bott formula (Theorem 1.5.2.3). If the group scheme G is not constant, then it is more difficult to extract concrete information about $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$ from (1.11). Nevertheless, we will show in Chapter 4 that the analysis of this section can be adapted to describe the cohomology groups of the successive quotients in an appropriate filtration of $C^*(\mathrm{Bun}_G(X); \mathbf{Q})$ (which comprise the second page of a spectral sequence converging to $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Q})$).

We begin with some general remarks.

Notation 1.5.6.1 (Homotopy Colimits of Chain Complexes)**.** Let $\mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$ denote the category of cochain complexes of rational vector spaces. We let $\mathrm{hVect}_{\mathbf{Q}}^{\mathrm{dg}}$ denote the homotopy category of $\mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$ (obtained from $\mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$ by formally adjoining inverses to quasi-isomorphisms).

For every small category $\mathcal{J}$, we let $\mathrm{Fun}(\mathcal{J}, \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}})$ denote the category of functors from $\mathcal{J}$ to $\mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$, and let $\mathrm{hFun}(\mathcal{J}, \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}})$ be the homotopy category obtained from $\mathrm{Fun}(\mathcal{J}, \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}})$ by formally inverting all quasi-isomorphisms. One can show that the diagonal map $\mathrm{hVect}_{\mathbf{Q}}^{\mathrm{dg}} \to \mathrm{hFun}(\mathcal{J}, \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}})$ admits a left adjoint, which we will refer to as the *homotopy colimit* functor and denote by

$$\mathrm{hocolim} : \mathrm{hFun}(\mathcal{J}, \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}) \to \mathrm{hVect}_{\mathbf{Q}}^{\mathrm{dg}}.$$

Remark 1.5.6.2. Every cochain complex $V = (V^*, d)$ in $\mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$ is (noncanonically) quasi-isomorphic to its cohomology $\mathrm{H}^*(V)$, regarded as a chain complex with trivial differential. Using this observation, one can show that the homotopy category $\mathrm{hVect}_{\mathbf{Q}}^{\mathrm{dg}}$ is equivalent to the category of graded vector spaces over $\mathbf{Q}$.

Example 1.5.6.3 (The Homotopy Colimit of a Constant Functor)**.** Let $\mathcal{J}$ be a small category and let $F : \mathcal{J} \to \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$ be the constant functor taking the value $\mathbf{Q}$. In this case, one can identify the homotopy colimit $\mathrm{hocolim}(F)$ with the rational chain complex $C_*(\mathrm{N}(\mathcal{J}); \mathbf{Q})$, where $\mathrm{N}(\mathcal{J})$ is the *nerve* of the category $\mathcal{J}$ (see Example 2.1.2.3). More generally, if $F : \mathcal{J} \to \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$ is the constant functor taking any value $V \in \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}$, then we have a canonical quasi-isomorphism $\mathrm{hocolim}(F) \simeq C_*(\mathrm{N}(\mathcal{J}); \mathbf{Q}) \otimes_{\mathbf{Q}} V$.

Remark 1.5.6.4. In the situation of Example 1.5.6.3, suppose that $\mathcal{J} = \mathcal{U}_0(X)$ is the partially ordered set of open disks in a manifold X. Then the nerve $\mathrm{N}(\mathcal{U}_0(X))$ is canonically homotopy equivalent to X (this follows formally from the fact that every point of X has a neighborhood basis of open disks). Consequently, the homotopy colimit $\mathrm{hocolim}(F)$ appearing in Example 1.5.6.3 can be identified with the rational chain complex $C_*(X; \mathbf{Q})$.

Construction 1.5.6.5 (Symmetric Algebras)**.** Let $V = (V^*, d)$ be a cochain complex of vector spaces over $\mathbf{Q}$. Then the symmetric algebra $\mathrm{Sym}^*(V)$ inherits the structure of a commutative differential graded algebra over $\mathbf{Q}$. Moreover, the functor $V \mapsto \mathrm{Sym}^*(V)$ carries quasi-isomorphisms to quasi-isomorphisms, and therefore induces a functor on homotopy categories

$$\mathrm{Sym}^* : \mathrm{hVect}_{\mathbf{Q}}^{\mathrm{dg}} \to \mathrm{hCAlg}_{\mathbf{Q}}^{\mathrm{dg}}.$$

Remark 1.5.6.6. The symmetric algebra functor $\mathrm{Sym}^* : \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}} \to \mathrm{CAlg}_{\mathbf{Q}}^{\mathrm{dg}}$ is an example of a *left Quillen functor* between model categories. It follows that Sym^* commutes with homotopy colimits. More precisely, for any small category $\mathcal{J}$, the diagram

$$\begin{array}{ccc} \mathrm{hFun}(\mathcal{J}, \mathrm{Vect}_{\mathbf{Q}}^{\mathrm{dg}}) & \xrightarrow{\mathrm{hocolim}} & \mathrm{hVect}_{\mathbf{Q}}^{\mathrm{dg}} \\ \downarrow{\scriptstyle \mathrm{Sym}^*} & & \downarrow{\scriptstyle \mathrm{Sym}^*} \\ \mathrm{hFun}(\mathcal{J}, \mathrm{CAlg}_{\mathbf{Q}}^{\mathrm{dg}}) & \xrightarrow{\mathrm{hocolim}} & \mathrm{hCAlg}_{\mathbf{Q}}^{\mathrm{dg}} \end{array}$$

commutes up to canonical isomorphism, where the upper horizontal map is defined in Notation 1.5.6.1 and the lower horizontal map is defined in Remark 1.5.4.6.

Proposition 1.5.6.7. *Let $A = (A^*, d)$ be a commutative differential graded algebra over $\mathbf{Q}$. The following conditions are equivalent:*

- (a) *There exists a graded vector space V and an isomorphism of graded rings $\alpha : \mathrm{Sym}^*(V) \to \mathrm{H}^*(A)$.*
- (b) *There exists a graded vector space V and a quasi-isomorphism of commutative differential graded algebras $\beta : \mathrm{Sym}^*(V) \to A$ (where the domain has trivial differential).*

Proof. The implication $(b) \Rightarrow (a)$ is obvious. Conversely, suppose that there exists a graded vector space V and an isomorphism $\alpha : \mathrm{Sym}^*(V) \to \mathrm{H}^*(A)$. Let $\{v_i\}_{i \in I}$ be a basis of V, where each v_i is homogeneous of degree d_i for some integer d_i. Then each $\alpha(v_i)$ can be represented by some cocycle $a_i \in A^{d_i}$. There is a unique morphism of differential graded algebras $\beta : \mathrm{Sym}^*(V) \to A$ satisfying $\beta(v_i) = a_i$. By construction, the map β induces α after passing to cohomology, and is therefore a quasi-isomorphism. □

Example 1.5.6.8. Let G be a connected linear algebraic group over $\mathbf{C}$. Then the rational cohomology $\mathrm{H}^*(\mathrm{BG}; \mathbf{Q})$ of the classifying space BG is isomorphic to a symmetric algebra $\mathrm{Sym}^*(V)$ (Proposition 1.5.2.1). Applying Proposition 1.5.6.7, we deduce that the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(\mathrm{BG})$ is quasi-isomorphic to $\mathrm{Sym}^*(V)$.

Proof of Theorem 1.5.2.3. Let X be an algebraic curve over $\mathbf{C}$ (which we identify with its underlying Riemann surface $X(\mathbf{C})$), let G be a simply connected linear algebraic group over $\mathbf{C}$, and let $\pi : X \times \mathrm{BG} \to X$ be the projection map. For every open set $U \subseteq X$, we have a diagonal embedding $\delta : \mathrm{BG} \to \mathrm{Sect}_\pi(U)$, which is a homotopy equivalence if U is contractible. Using Example 1.5.6.8, we can choose a graded vector space V and a quasi-isomorphism of differential graded algebras $\alpha : \mathrm{Sym}^*(V) \to \Omega^*_{\mathrm{poly}}(\mathrm{BG})$. Let $\underline{V}$ denote the constant functor $\mathcal{U}_0(X) \to \mathrm{Vect}^{\mathrm{dg}}_{\mathbf{Q}}$ with the value V and let $\mathcal{B} : \mathcal{U}_0(X) \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the functor given by $\mathcal{B}(U) = \Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U))$ (as in Construction 1.5.4.8), so that α induces a quasi-isomorphism $\mathrm{Sym}^*(\underline{V}) \to \mathcal{B}$ in the category $\mathrm{Fun}(\mathcal{U}_0(X), \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})$. Combining Remark 1.5.6.4, Remark 1.5.6.6, and Theorem 1.5.4.10, we obtain canonical quasi-isomorphisms

$$\begin{aligned} \mathrm{Sym}^*(C_*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} V) &\simeq \mathrm{Sym}^*(\mathrm{hocolim}\, \underline{V}) \\ &\leftarrow \mathrm{hocolim}(\mathrm{Sym}^*(\underline{V})) \\ &\rightarrow \mathrm{hocolim}(\mathcal{B}) \\ &= \bigotimes_{x \in X} C^*(\mathrm{BG}; \mathbf{Q}) \\ &\rightarrow \Omega^*_{\mathrm{poly}}(\mathrm{Map}(X, \mathrm{BG})). \end{aligned}$$

Passing to cohomology, we obtain an isomorphism

$$\mathrm{Sym}^*(\mathrm{H}_*(X;\mathbf{Q}) \otimes_{\mathbf{Q}} V) \simeq \mathrm{H}^*(\mathrm{Map}(X,\mathrm{BG});\mathbf{Q}),$$

which is easily seen to coincide with the map appearing in the statement of Theorem 1.5.2.3. □

1.6 Summary of This Book

Our goal in this book is to prove Weil's conjecture for function fields as articulated in Conjecture 1.3.3.7: if X is an algebraic curve over a finite field $\mathbf{F}_q$ and G is a smooth affine group scheme over X with connected fibers whose generic fiber is semisimple and simply connected, then we have an equality

$$\frac{|\operatorname{Tors}_G(X)|}{q^{\dim(\mathrm{Bun}_G(X))}} = \prod_{x\in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}. \tag{1.12}$$

As explained in §1.4, we will achieve this by comparing both sides with the trace of the arithmetic Frobenius on the ℓ-adic cohomology of $\mathrm{Bun}_G(\overline{X})$, where $\overline{X} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$. Consequently, our proof can be broken naturally into two steps:

(a) Showing that the trace $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X});\mathbf{Z}_\ell))$ is equal to the Euler product $\prod_{x\in X} \frac{|\kappa(x)|^n}{|G(\kappa(x))|}$ (Theorem 1.4.4.1).

(b) Showing that the moduli stack $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula $\frac{|\operatorname{Tors}_G(X)|}{q^{\dim(\mathrm{Bun}_G(X))}} = \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X});\mathbf{Z}_\ell))$ (Theorem 1.4.3.3).

The majority of this book is devoted to step (a). Let us consider a more general situation, where $\overline{X}$ is an algebraic curve over an arbitrary algebraically closed field k. In the case where $k = \mathbf{C}$ is the field of complex numbers and the group scheme G is everywhere semisimple, Theorem 1.5.4.10 asserts that the rational cochain complex $C^*(\mathrm{Bun}_G(\overline{X});\mathbf{Q})$ can be realized as a continuous tensor product $\bigotimes_{x\in\overline{X}} C^*(\mathrm{BG}_x;\mathbf{Q})$ (in the sense of Construction 1.5.4.8). The main ingredient in our proof of Weil's conjecture is a purely algebro-geometric version of this result, which makes sense over any algebraically closed field k (where we replace singular cohomology with ℓ-adic cohomology, for any prime number ℓ which does not vanish in k). Heuristically, this result asserts that there is a canonical quasi-isomorphism of ℓ-adic cochain complexes

$$\bigotimes_{x\in\overline{X}} C^*(\mathrm{BG}_x;\mathbf{Z}_\ell) \xrightarrow{\sim} C^*(\mathrm{Bun}_G(\overline{X});\mathbf{Z}_\ell). \tag{1.13}$$

Our first objective is to give a rigorous definition of both sides appearing in (1.13). In Chapter 3, we associate to each algebraic stack $\mathcal{Y}$ over k a chain complex of $\mathbf{Z}_\ell$-modules $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$, which we refer to as the *ℓ-adic cochain complex of* $\mathcal{Y}$ (Construction 3.2.5.1). In particular, this allows us to contemplate the ℓ-adic cochain complex $C^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)$ as well as each individual factor $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ appearing on the left hand side of (1.13). To make precise sense of the tensor product $\bigotimes_{x \in \overline{X}} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$, we will need to exploit some additional structure. Recall that Construction 1.5.4.8 made use of the observation that for any topological space Y, the rational cochain complex $C^*(Y; \mathbf{Q})$ is quasi-isomorphic to the commutative differential graded algebra $\Omega^*_{\mathrm{poly}}(Y)$. This has a parallel in the ℓ-adic setting: for any algebraic stack $\mathcal{Y}$ over k, the ℓ-adic cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ can be viewed as an $\mathbb{E}_\infty$-*algebra* over $\mathbf{Z}_\ell$ (see §3.1 for a review of the theory of $\mathbb{E}_\infty$-algebras). We can therefore view $\{C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)\}_{x \in \overline{X}}$ as a family of $\mathbb{E}_\infty$-algebras, parametrized by the points of $\overline{X}$. In §3.4, we show that this family can itself be regarded as an $\mathbb{E}_\infty$-algebra in the setting of ℓ-adic sheaves on $\overline{X}$ (Theorem 3.4.0.3), which we denote by $[\mathrm{BG}]_{\overline{X}}$. Using this observation, we can make sense of the tensor product $\bigotimes_{x \in \overline{X}} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$: it is defined to be an $\mathbb{E}_\infty$-algebra A over $\mathbf{Z}_\ell$ which is universal among those which admit a map of commutative algebras $[\mathrm{BG}]_{\overline{X}} \to \omega_{\overline{X}} \otimes \underline{A}$ (where $\underline{A}$ denotes the constant ℓ-adic sheaf on $\overline{X}$ with the value A). We will denote this universal $\mathbb{E}_\infty$-algebra A by $\int_{\overline{X}} [\mathrm{BG}]_{\overline{X}}$ and refer to it as the *factorization homology* of $[\mathrm{BG}]_{\overline{X}}$ (Definition 4.1.1.3).

In the preceding discussion, we have indulged in an abuse of terminology which will be ubiquitous in this book: though we refer to the object $[\mathrm{BG}]_{\overline{X}}$ as an ℓ-adic sheaf, it really belongs to a suitable *derived* category of ℓ-adic sheaves on $\overline{X}$. This forces us to walk a delicate line. On the one hand, we will need to make use of many sheaf-theoretic constructions that are really well-defined only up to quasi-isomorphism. For example, the algebra structure on $[\mathrm{BG}]_{\overline{X}}$ is encoded by a map

$$m : \Delta^!([\mathrm{BG}]_{\overline{X}} \boxtimes [\mathrm{BG}]_{\overline{X}}) \to [\mathrm{BG}]_{\overline{X}},$$

where $\Delta^!$ is the exceptional inverse image functor associated to the diagonal embedding $\Delta : \overline{X} \to \overline{X} \times_{\mathrm{Spec}(k)} \overline{X}$. This functor is well-defined at the level of derived categories, but is not t-exact (and is therefore difficult to work with at the level of individual sheaves). On the other hand, passage to the derived category involves a loss of information that we cannot afford. To construct the factorization homology $\int_{\overline{X}} [\mathrm{BG}]_{\overline{X}}$, it is not enough to view $[\mathrm{BG}]_{\overline{X}}$ as a commutative algebra in the derived category of ℓ-adic sheaves on $\overline{X}$: we need to use the fact that the multiplication map m above is commutative and associative up to *coherent* homotopy, rather than merely up to homotopy. We will reconcile these requirements by systematically using the theory of ∞-categories, as developed in [25] and [23]. For the reader's convenience, we review this theory in §2.1. In addition, in §2.2 and §2.3, we review the theory of étale and ℓ-adic sheaves, emphasizing the ∞-categorical point of view. In particular, we introduce in §2.3 an ∞-category of constructible ℓ-adic sheaves, which does not seem to have appeared explicitly in the existing literature.

In Chapter 4, we return to the proof of Theorem 1.4.4.1. Using the formalism developed in Chapters 2 and 3, we construct a canonical map

$$\int_{\overline{X}} [\mathrm{BG}]_{\overline{X}} \xrightarrow{\rho} C^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell) \tag{1.14}$$

and assert (without proof) that it is a quasi-isomorphism (Theorem 4.1.2.1). The remainder of Chapter 4 is devoted to showing that, in the special case where $\overline{X}$ and G are defined over a finite field $\mathbf{F}_q$, we can then deduce the numerical equality

$$\prod_{x \in X} \frac{|\kappa(x)|^n}{|G(\kappa(x))|} = \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(\overline{X}); \mathbf{Z}_\ell)). \tag{1.15}$$

Heuristically, one can deduce (1.15) from the quasi-isomorphism (1.14) by passing to cohomology groups and taking the trace of Frobenius. This heuristic does not translate directly into a proof, because one encounters certain convergence issues when rearranging infinite sums. We proceed instead by using the theory of Koszul duality to translate the quasi-isomorphism of (1.14) to a statement concerning the *motive G relative to X* (see Construction 4.5.1.1), from which we deduce (1.15) by applying the Grothendieck-Lefschetz trace formula.

Chapter 5 of this book is devoted to the proof of the Grothendieck-Lefschetz trace formula for the moduli stack $\mathrm{Bun}_G(X)$ (Theorem 1.4.3.3), and is mostly independent of the rest of this book. Roughly speaking, the strategy is to choose a suitable stratification of $\mathrm{Bun}_G(X)$ by locally closed substacks $\{\mathfrak{X}_\alpha\}_{\alpha \in A}$, where each $\mathfrak{X}_\alpha$ can be realized as a global quotient stack Y_α / H_α. By applying the classical Grothendieck-Lefschetz trace formula to Y_α and H_α, one can deduce that each $\mathfrak{X}_\alpha$ satisfies the Grothendieck-Lefschetz trace formula (Proposition 5.1.0.1). This formally implies that $\mathrm{Bun}_G(X)$ also satisfies the Grothendieck-Lefschetz trace formula, provided that the eigenvalues of Frobenius on the cohomology of the stacks $\mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathfrak{X}_\alpha$ decay quickly as α varies (see Proposition 5.2.2.3). In the case where the group scheme G is everywhere reductive, Behrend proved the Grothendieck-Lefschetz trace formula for $\mathrm{Bun}_G(X)$ by applying this strategy to the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$ (see Theorem 5.3.2.2). We generalize this argument to the case where G fails to be semisimple at finitely many points (Theorem 5.0.0.3) by exploiting the relationship between G-bundles on X and on finite (possibly ramified) covers of X.

Warning 1.6.0.1. The preceding discussion can be summarized as follows: the goal of this book is to give a precise formulation of a geometric product formula (Theorem

4.1.2.1, which asserts that the map ρ of (1.15) is a quasi-isomorphism), and to show that it implies the function field case of Weil's conjecture. The proof of Theorem 4.1.2.1 is not given here, but will appear in a sequel volume. Note that Theorem 4.1.2.1 is an algebro-geometric analogue of Theorem 1.5.4.10, whose proof was sketched in §1.5.5. Our argument made essential use of the local contractibility of the analytic topology of a Riemann surface, which has no obvious analogue in the setting of algebraic geometry. Our proof of Theorem 4.1.2.1 is much more indirect, and relies on some relatively sophisticated geometric ideas (such as the use of Verdier duality on the Ran space $\mathrm{Ran}(X)$ of the algebraic curve X) which are outside the scope of the present volume.

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Chapter 2

The Formalism of ℓ-adic Sheaves

Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. For each closed point $x \in X$, let G_x denote the fiber of G at x and let BG_x denote its classifying stack. One of our principal aims in this book is to make sense of the idea that (under mild hypotheses) the cohomology of $\mathrm{Bun}_G(X)$ should admit a "continuous" Künneth decomposition

$$\bigotimes_{x \in X} \mathrm{H}^*(\mathrm{BG}_x) \simeq \mathrm{H}^*(\mathrm{Bun}_G(X)). \tag{2.1}$$

In §1.5, we gave a precise formulation of this heuristic (see Example 1.5.4.15) in the case where $k = \mathbf{C}$ is the field of complex numbers, the fibers of G are semisimple and simply connected, and cohomology is taken to mean *singular* cohomology (since the algebraic stacks $\mathrm{Bun}_G(X)$ and BG_x have underlying homotopy types). Let us recall several key features of our approach:

(*a*) To make sense of the continuous tensor product appearing on the left hand side of (2.1), it was important to work at the level of cochain complexes, rather than at the level of cohomology. Consequently, it would be more accurate to say that §1.5 outlined the construction of a quasi-isomorphism

$$\alpha : \bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q}) \to C^*(\mathrm{Bun}_G(X); \mathbf{Q}). \tag{2.2}$$

(*b*) Our definition of the continuous tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$ made use of the fact that the cochain complexes $C^*(\mathrm{BG}_x; \mathbf{Q})$ can be equipped with additional structures, which are *cochain level* refinements of the cup product on cohomology. In the setting of §1.5, this was articulated by replacing the rational cochain complexes $C^*(Y; \mathbf{Q})$ by the polynomial de Rham complexes $\Omega^*_{\mathrm{poly}}(Y)$ of Construction 1.5.3.6.

(c) To define the continuous tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$, it was important to regard the individual factors $C^*(\mathrm{BG}_x; \mathbf{Q})$ as depending "continuously" on the parameter x. In the setting of §1.5, this continuity was encoded by the functor $\mathcal{B}$ of Construction 1.5.4.8.

For applications to Weil's conjecture, we would like to make sense of (2.1) in the case where k is the algebraic closure of a finite field $\mathbf{F}_q$ (and the curve X and group scheme G are both defined over $\mathbf{F}_q$). In this situation, the language of singular cohomology is not available and we instead work with ℓ-adic cohomology, where ℓ is some prime number which is invertible in k. In Chapter 4, we will formulate a version of (2.1) in the ℓ-adic setting (Theorem 4.1.2.1). This result will share the essential features of its classical avatar:

(a') To make sense of the left hand side of (2.1) in the ℓ-adic setting, we will need to understand the theory of ℓ-adic cohomology *at the cochain level*. In other words, we need to refine the construction $(x \in X) \mapsto \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$ to a construction $(x \in X) \mapsto C^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$, where $C^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$ is some cochain complex of $\mathbf{Q}_\ell$-modules whose cohomology can be identified with $\mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$.

(b') We will need to exploit the existence of algebraic structures on the ℓ-adic cochain complexes $C^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$, which refine the cup product maps $\cup : \mathrm{H}^i(\mathrm{BG}_x; \mathbf{Q}_\ell) \times \mathrm{H}^j(\mathrm{BG}_x; \mathbf{Q}_\ell) \to \mathrm{H}^{i+j}(\mathrm{BG}_x; \mathbf{Q}_\ell)$.

(c') We will need to regard the ℓ-adic cochain complexes $C^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$ as depending continuously on the point $x \in X$, in some sense.

Let us begin by outlining an approach which partially achieves these goals. Assume for simplicity that every fiber of G is semisimple and simply connected, and let BG denote the classifying stack of G (so that BG is an algebraic stack equipped with a projection map $\pi : \mathrm{BG} \to X$, whose fibers are the classifying stacks BG_x). To every algebraic stack Y over k, one can associate a triangulated category $\mathcal{D}_\ell(Y)$ whose objects can be understood as "complexes of ℓ-adic sheaves on Y" (we will give a precise definition of $\mathcal{D}_\ell(Y)$ in §2.3, at least in the special case where Y is a quasi-projective k-scheme). The projection map π determines a (derived) pushforward functor $\pi_* : \mathcal{D}_\ell(\mathrm{BG}) \to \mathcal{D}_\ell(X)$. Let $\underline{\mathbf{Q}_\ell}$ denote the constant sheaf on BG with value $\mathbf{Q}_\ell$, and set $\mathcal{B} = \pi_* \underline{\mathbf{Q}_\ell} \in \mathcal{D}_\ell(X)$. For each point $x \in X$, the stalk $\mathcal{B}_x \in \mathcal{D}_\ell(\{x\})$ can be identified with a chain complex of vector spaces over $\mathbf{Q}_\ell$, whose cohomology groups are given by $\mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$. Consequently, the ℓ-adic complex $\mathcal{B}$ satisfies the requirements of (a') and (c').

Warning 2.0.0.1. If the group scheme G fails to be semisimple at some point $x \in X$, then the cohomology of the stalk $\mathcal{B}_x$ need not be isomorphic to $\mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$. To handle points of bad reduction, it is convenient to twist $\mathcal{B}$ by the dualizing sheaf ω_X of the curve X and to consider costalks in place of stalks; we will return to this point in Chapter 3.

Let us now consider point (b'). For any algebraic stack Y, the formation of tensor products of ℓ-adic sheaves endows the category $\mathcal{D}_\ell(Y)$ with the structure of a symmetric monoidal category. In particular, it makes sense to consider commutative algebra objects of $\mathcal{D}_\ell(Y)$. Moreover, the direct image functor $\pi_* : \mathcal{D}_\ell(\mathrm{BG}) \to \mathcal{D}_\ell(X)$ carries commutative algebras to commutative algebras, so that $\mathcal{B}$ can be regarded as a commutative algebra object of $\mathcal{D}_\ell(X)$. For each point $x \in X$, the stalk $\mathcal{B}_x$ inherits the structure of a commutative algebra object of the category $\mathcal{D}_\ell(\{x\})$, which determines a multiplication map

$$m : C^*(\mathrm{BG}_x; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} C^*(\mathrm{BG}_x; \mathbf{Q}_\ell) \to C^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$$

which is well-defined (as well as commutative and associative) up to chain homotopy. Unfortunately, this is not good enough for our ultimate application: since $\mathbf{Q}_\ell$ is a field, specifying the multiplication map m up to chain homotopy is equivalent to specifying the cup product $\cup : \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell) \to \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Q}_\ell)$. Our formulation of the product formula in §1.5 made essential use of the polynomial de Rham complex, which witnesses the commutativity and associativity of the cup products *at the cochain level* (and thereby captures much more information than the cup product alone; see Theorem 1.5.3.7). The ℓ-adic product formula we discuss in Chapter 4 will need to make use of analogous structures, which are simply not visible at the level of the triangulated category $\mathcal{D}_\ell(X)$.

Our goal in this chapter is to remedy the situation described above by introducing a mathematical object $\mathrm{Shv}_\ell(X)$ which refines the triangulated category $\mathcal{D}_\ell(X)$. This object is not itself a category but instead is an example of an ∞-*category*, which we will refer to as the ∞-*category of ℓ-adic sheaves on* X (Definition 2.3.4.1). The triangulated category $\mathcal{D}_\ell(X)$ can be identified with the homotopy category of $\mathrm{Shv}_\ell(X)$; in particular, the objects of $\mathcal{D}_\ell(X)$ and $\mathrm{Shv}_\ell(X)$ are the same. However, there is a large difference between commutative algebra objects of $\mathcal{D}_\ell(X)$ (which can be viewed as chain complexes of ℓ-adic sheaves $\mathcal{F}$ equipped with a multiplication which is commutative and associative up to homotopy) and commutative algebra objects of the ∞-category $\mathrm{Shv}_\ell(X)$ (where we require commutativity and associativity up to *coherent* homotopy). We can achieve (b') by viewing the complex $\mathcal{B}$ as a commutative algebra of the latter sort: we will return to this point in Chapter 3.

We begin in §2.1 with a brief introduction to the language of ∞-categories, emphasizing some examples which are particularly relevant for our applications (such as the ∞-category Mod_Λ of chain complexes of Λ modules; see Example 2.1.4.8). The remainder of this chapter is devoted to giving an exposition of the theory of ℓ-adic cohomology, placing an emphasis on the ∞-categorical perspective. We begin in §2.2 by reviewing the theory of étale sheaves. To every scheme Y and every commutative ring Λ, one can associate a stable ∞-category $\mathrm{Shv}(Y;\Lambda)$ of Mod_Λ-valued étale sheaves on Y (Definition 2.2.1.2). This can be regarded as an "enhancement" of the derived category of the abelian category of sheaves of Λ-modules on Y, whose objects are cochain complexes

$$\cdots \to \mathcal{F}^{-2} \to \mathcal{F}^{-1} \to \mathcal{F}^{0} \to \mathcal{F}^{1} \to \mathcal{F}^{2} \to \cdots.$$

The ∞-category of étale sheaves $\mathrm{Shv}(Y;\Lambda)$ contains a full subcategory $\mathrm{Shv}^c(Y;\Lambda)$ of constructible perfect complexes, which we will discuss in §2.2.6. However, the ∞-category $\mathrm{Shv}^c(Y;\Lambda)$ is too small for many of our purposes: it fails to contain many of the objects we are interested in (for example, the cochain complex $C^*(\mathrm{Bun}_G(X);\mathbf{Z}/\ell\mathbf{Z})$ typically has cohomology in infinitely many degrees), and does not have good closure properties under various categorical constructions we will need to use (such as the formation of infinite direct limits). On the other hand, allowing arbitrary chain complexes (in particular, chain complexes which are not bounded below) raises some technical convergence issues. We will avoid these issues by restricting our attention to the case where the scheme Y is quasi-projective over an algebraically closed field k. In this case, the étale site of Y has finite cohomological dimension, which implies that $\mathrm{Shv}(Y;\Lambda)$ is compactly generated by the subcategory $\mathrm{Shv}^c(Y;\Lambda)$.

The construction $Y \mapsto \mathrm{Shv}(Y;\Lambda)$ depends functorially on Λ: every map of commutative rings $\Lambda \to \Lambda'$ induces base change functors

$$\mathrm{Shv}(Y;\Lambda) \to \mathrm{Shv}(Y;\Lambda') \qquad \mathrm{Shv}^c(Y;\Lambda) \to \mathrm{Shv}^c(Y;\Lambda')$$

(see §2.2.8). In particular, we have a tower of ∞-categories

$$\cdots \to \mathrm{Shv}^c(Y;\mathbf{Z}/\ell^3\mathbf{Z}) \to \mathrm{Shv}^c(Y;\mathbf{Z}/\ell^2\mathbf{Z}) \to \mathrm{Shv}^c(Y;\mathbf{Z}/\ell\mathbf{Z}).$$

We will denote the (homotopy) inverse limit of this tower by $\mathrm{Shv}^c_\ell(Y)$, and refer to it as the *∞-category of constructible ℓ-adic sheaves on Y*. In §2.3, we define the ∞-category $\mathrm{Shv}_\ell(Y)$ of *ℓ-adic sheaves on Y* to be the Ind-completion of $\mathrm{Shv}^c_\ell(Y)$. These ∞-categories provide a convenient formal setting for formulating most of the constructions of this book: the ∞-category $\mathrm{Shv}_\ell(Y)$ contains all constructible ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$ as well as other objects obtained by limiting procedures (such as localizations of the form $\mathcal{F}[\ell^{-1}]$). Many important foundational results in the theory of étale cohomology (such as the smooth and proper base change theorems) can be extended to the setting of ℓ-adic sheaves in a purely formal way; we will review the situation in §2.4.

2.1 Higher Category Theory

In this section, we give a brief introduction to the theory of ∞-*categories* (also known in the literature as *quasi-categories* or *weak Kan complexes*). The formalism of ∞-categories supplies an efficient language for discussing homotopy coherent constructions in mathematics (such as the formation of homotopy colimits; see Definition 1.5.4.5), which we will make use of throughout this book.

Warning 2.1.0.1. A comprehensive account of the theory of ∞-categories would take us far afield of our goals. In this section, we will content ourselves with explaining the basic definitions and their motivation. For a more detailed treatment, we refer the reader to [25] and [23].

2.1.1 Motivation: Deficiencies of the Derived Category

Let Λ be a commutative ring. Throughout this section, we let $\mathrm{Chain}(\Lambda)$ denote the abelian category whose objects are chain complexes

$$\cdots \to V_2 \to V_1 \to V_0 \to V_{-1} \to V_{-2} \to \cdots$$

of Λ-modules. We will always employ *homological* conventions when discussing chain complexes (so that the differential on a chain complex lowers degree). If V_* is a chain complex, then its *homology* $\mathrm{H}_*(V_*)$ is given by

$$\mathrm{H}_n(V_*) = \{x \in V_n : dx = 0\}/\{x \in V_n : (\exists y \in V_{n+1})[x = dy]\}.$$

Any map of chain complexes $\alpha : V_* \to W_*$ induces a map $\mathrm{H}_*(V_*) \to \mathrm{H}_*(W_*)$. We say that α is a *quasi-isomorphism* if it induces an isomorphism on homology.

For many purposes, it is convenient to treat quasi-isomorphisms as if they are isomorphisms (emphasizing the idea that a chain complex is just a vessel for carrying information about its homology). One can make this idea explicit using Verdier's theory of *derived categories*. The derived category $\mathcal{D}(\Lambda)$ can be defined as the category obtained from $\mathrm{Chain}(\Lambda)$ by formally inverting all quasi-isomorphisms.

The theory of derived categories is a very useful tool in homological algebra, but has a number of limitations. Many of these stem from the fact that $\mathcal{D}(\Lambda)$ is not very well-behaved from a categorical point of view. The category $\mathcal{D}(\Lambda)$ does not generally have limits or colimits, even of very simple types. For example, a morphism $f : X \to Y$ in $\mathcal{D}(\Lambda)$ generally does not have a cokernel in $\mathcal{D}(\Lambda)$. However, there is a substitute: every morphism f in $\mathcal{D}(\Lambda)$ fits into a "distinguished triangle"

$$X \xrightarrow{f} Y \to \mathrm{Cn}(f) \to \Sigma X.$$

Here the object $\mathrm{Cn}(f)$ is called the *cone* of f, and it behaves in some respects like a cokernel: every map $g : Y \to Z$ such that $g \circ f = 0$ factors through $\mathrm{Cn}(f)$, though the factorization is generally not unique. The object $\mathrm{Cn}(f) \in \mathcal{D}(\Lambda)$ (and, in fact, the entire diagram above) is well-defined up to isomorphism, but *not* up to canonical isomorphism: there is no functorial procedure for constructing the cone $\mathrm{Cn}(f)$ from the datum of a morphism f in the category $\mathcal{D}(\Lambda)$. And this is only a very simple example: for other types of limits and colimits (such as taking invariants or coinvariants with respect to the action of a group), the situation is even worse.

Let $f, g : V_* \to W_*$ be maps of chain complexes. Recall that a *chain homotopy* from f_* to g_* is a collection of maps $h_n : V_n \to W_{n+1}$ such that $f_n - g_n = d \circ h_n + h_{n-1} \circ d$. We say that f_* and g_* are *chain-homotopic* if there exists a chain homotopy from f_* to g_*. Chain-homotopic maps induce the same map from $\mathrm{H}_*(V_*)$ to $\mathrm{H}_*(W_*)$, and have the same image in the derived category $\mathcal{D}(\Lambda)$. In fact, there is an alternative description of the derived category $\mathcal{D}(\Lambda)$, which places an emphasis on the notion of chain-homotopy rather than quasi-isomorphism:

Definition 2.1.1.1. Let Λ be a commutative ring. We define a category $\mathcal{D}'(\Lambda)$ as follows:

- The objects of $\mathcal{D}'(\Lambda)$ are the *K-projective* chain complexes of Λ-modules, in the sense of [35]. A chain complex V_* is K-projective if, for every surjective quasi-isomorphism $W'_* \to W_*$ of chain complexes, every chain map $f : V_* \to W_*$ can be lifted to a map $f' : V_* \to W'_*$.
- A morphism from V_* to W_* in $\mathcal{D}'(\Lambda)$ is a chain-homotopy equivalence class of chain maps from V_* to W_*.

Remark 2.1.1.2. Chain homotopic morphisms $f, g : V_* \to W_*$ have the same image in the derived category $\mathcal{D}(\Lambda)$. Consequently, the construction $V_* \mapsto V_*$ determines a functor $\mathcal{D}'(\Lambda) \to \mathcal{D}(\Lambda)$, which can be shown to be an equivalence of categories.

Remark 2.1.1.3. If $V_* \in \mathrm{Chain}(\Lambda)$ is K-projective, then each V_n is a projective Λ-module. The converse holds if $V_n \simeq 0$ for $n \gg 0$ or if the commutative ring Λ has finite projective dimension (for example, if $\Lambda = \mathbf{Z}$), but not in general. For example, the chain complex of $\mathbf{Z}/4\mathbf{Z}$-modules

$$\cdots \to \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \xrightarrow{2} \mathbf{Z}/4\mathbf{Z} \to \cdots$$

is not K-projective.

From the perspective of Definition 2.1.1.1, categorical issues with the derived category stem from the fact that we are identifying chain-homotopic morphisms in $\mathcal{D}'(\Lambda)$

without remembering *how* they are chain-homotopic. For example, suppose that we wish to construct the cone of a morphism $[f] : V_* \to W_*$ in $\mathcal{D}'(\Lambda)$. By definition, $[f]$ is an equivalence class of chain maps from V_* to W_*. If we choose a representative f for the equivalence class $[f]$, then we can construct the mapping cone $\mathrm{Cn}([f])$ by equipping the direct sum $W_* \oplus V_{*-1}$ with a differential which depends on f. If h is a chain-homotopy from f to g, we can use h to construct an isomorphism of chain complexes $\alpha_h : \mathrm{Cn}(f) \simeq \mathrm{Cn}(g)$. However, the isomorphism α_h *depends on* h: different choices of chain homotopy can lead to different isomorphisms, even up to chain-homotopy.

2.1.2 The Differential Graded Nerve

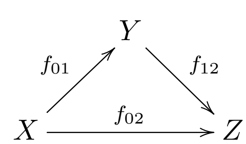
It is possible to correct many of the deficiencies of the derived category by keeping track of more information. To do so, it is useful to work with mathematical structures which are a bit more elaborate than categories, where the primitive notions include not only "object" and "morphism" but also a notion of "homotopy between morphisms." Before giving a general definition, let us spell out the structure that is visible in the theory of chain complexes over a commutative ring.

Construction 2.1.2.1. Let Λ be a commutative ring. We define a sequence of sets $S_0, S_1, S_2, \ldots$ as follows:

- Let S_0 denote the set of *objects* under consideration: in our case, these are K-projective chain complexes of Λ-modules (strictly speaking, this is not a set but a proper class, because we are trying to describe a "large" category).

- Let S_1 denote the set of *morphisms* under consideration. That is, S_1 is the collection of all chain maps $f : X \to Y$, where X and Y are K-projective chain complexes of Λ-modules.

- Let S_2 denote the set of all pairs consisting of a (not necessarily commuting) diagram



  together with a chain homotopy f_{012} from f_{02} to $f_{12} \circ f_{01}$. Here X, Y, and Z are K-projective chain complexes of Λ-modules.

- More generally, we let S_n denote the collection of all $(n+1)$-tuples

$$\{X(0), X(1), \ldots, X(n)\}$$

of K-projective chain complexes, together with chain maps $f_{ij} : X(i) \to X(j)$ which are compatible with composition *up to coherent homotopy*. More precisely, this means that for every subset $I = \{i_- < i_m < \ldots < i_1 < i_+\} \subseteq \{0, \ldots, n\}$, we supply a collection of maps $f_I : X(i_-)_k \to X(i_+)_{k+m}$ satisfying the identities

$$d(f_I(x)) = (-1)^m f_I(dx) + \sum_{1 \leq j \leq m} (-1)^j (f_{I-\{i_j\}}(x) - (f_{\{i_j, \ldots, i_1, i_+\}} \circ f_{\{i_-, i_m, \ldots i_j\}})(x)).$$

Suppose we are given an element $(\{X(i)\}_{0 \leq i \leq n}, \{f_I\})$ of S_n. Then for $0 \leq i \leq n$, we can regard $X(i)$ as an element of S_0. If we are given a pair of integers $0 \leq i < j \leq n$, then $f_{\{i,j\}}$ is a chain map from $X(i)$ to $X(j)$, which we can regard as an element of S_1. More generally, given any nondecreasing map $\alpha : \{0, \ldots, m\} \to \{0, \ldots, n\}$, we can define a map $\alpha^* : S_n \to S_m$ by the formula

$$\alpha^*(\{X(j)\}_{0 \leq j \leq n}, \{f_I\}) = (\{X(\alpha(j))\}_{0 \leq j \leq m}, \{g_J\}),$$

where

$$g_J(x) = \begin{cases} f_{\alpha(J)}(x) & \text{if } \alpha|_J \text{ is injective} \\ x & \text{if } J = \{j, j'\} \text{ and } \alpha(j) = \alpha(j') \\ 0 & \text{otherwise.} \end{cases}$$

This motivates the following:

Definition 2.1.2.2. A *simplicial set* $X_\bullet$ consists of the following data:

- For every integer $n \geq 0$, a set X_n (called the *set of n-simplices of $X_\bullet$*).
- For every nondecreasing map of finite sets $\alpha : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$, a map of sets $\alpha^* : X_n \to X_m$.

This data is required to be be compatible with composition: that is, we have

$$\mathrm{id}^*(x) = x \qquad (\alpha \circ \beta)^*(x) = \beta^*(\alpha^*(x))$$

whenever α and β are composable nondecreasing maps.

Example 2.1.2.3 (The Nerve of a Category)**.** Let $\mathcal{C}$ be a category. We can associate to $\mathcal{C}$ a simplicial set $\mathrm{N}(\mathcal{C})_\bullet$, whose n-simplices are given by chains of composable morphisms

$$C_0 \to C_1 \to \cdots \to C_n$$

in $\mathcal{C}$. We refer to $\mathrm{N}(\mathcal{C})_\bullet$ as the *nerve* of the category $\mathcal{C}$.

Example 2.1.2.4. Let Λ be a commutative ring and let $\mathrm{Chain}'(\Lambda)$ denote the full subcategory of $\mathrm{Chain}(\Lambda)$ spanned by the K-projective chain complexes of Λ-modules. Construction 2.1.2.1 yields a simplicial set $\{S_n\}_{n \geq 0}$, which we will denote by Mod_Λ. The simplicial set Mod_Λ can be regarded as an enlargement of the nerve $\mathrm{N}(\mathrm{Chain}'(\Lambda))_\bullet$ (more precisely, we can identify $\mathrm{N}(\mathrm{Chain}'(\Lambda))_\bullet$ with the simplicial subset of Mod_Λ whose n-simplices are pairs $(\{X(i)\}_{0 \leq i \leq n}, \{f_I\})$ for which $f_I = 0$ whenever I has cardinality > 2).

The construction $\mathrm{Chain}'(\Lambda) \mapsto \mathrm{Mod}_\Lambda$ can be regarded as a variant of Example 2.1.2.3 which takes into account the structure of $\mathrm{Chain}'(\Lambda)$ as a *differential graded* category. We refer to §[23].1.3.1 for more details.

2.1.3 The Weak Kan Condition

Let $\mathcal{C}$ be a category. Then the simplicial set $\mathrm{N}(\mathcal{C})_\bullet$ determines $\mathcal{C}$ up to isomorphism. For example, the objects of $\mathcal{C}$ are just the 0-simplices of $\mathrm{N}(\mathcal{C})_\bullet$ and the morphisms of $\mathcal{C}$ are just the 1-simplices of $\mathrm{N}(\mathcal{C})_\bullet$. Moreover, given a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, the composition $h = g \circ f$ is the unique 1-morphism in $\mathcal{C}$ for which there exists a 2-simplex $\sigma \in \mathrm{N}(\mathcal{C})_2$ satisfying

$$\alpha_0^*(\sigma) = g \qquad \alpha_1^*(\sigma) = h \qquad \alpha_2^*(\sigma) = f,$$

where $\alpha_i : \{0, 1\} \to \{0, 1, 2\}$ denotes the unique injective map whose image does not contain i.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, then there is a bijective correspondence between functors $F : \mathcal{C} \to \mathcal{D}$ and maps of simplicial sets $\mathrm{N}(\mathcal{C})_\bullet \to \mathrm{N}(\mathcal{D})_\bullet$. We can summarize the situation as follows: the construction $\mathcal{C} \mapsto \mathrm{N}(\mathcal{C})_\bullet$ furnishes a fully faithful embedding from the category of (small) categories to the category of simplicial sets. It is therefore natural to ask about the essential image of this construction: which simplicial sets arise as the nerves of categories? To answer this question, we need a bit of terminology:

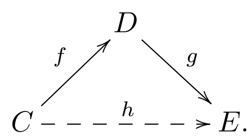
Notation 2.1.3.1. Let $X_\bullet$ be a simplicial set. For $0 \leq i \leq n$, we define a set $\Lambda_i^n(X_\bullet)$ as follows:

- To give an element of $\Lambda_i^n(X_\bullet)$, one must give an element $\sigma_J \in X_m$ for every subset $J = \{j_0 < \cdots < j_m\} \subseteq \{0, \ldots, n\}$ which does not contain $\{0, 1, \ldots, i-1, i+1, \ldots, n\}$. These elements are subject to the compatibility condition $\sigma_I = \alpha^* \sigma_J$ whenever $I = \{i_0 < \cdots < i_\ell\} \subseteq \{j_0 < \cdots < j_m\}$ and α satisfies $i_k = j_{\alpha(k)}$.

More informally, $\Lambda_i^n(X_\bullet)$ is the set of "partially defined" n-simplices of $X_\bullet$, which are missing their interior and a single face. There is an evident restriction map $X_n \to \Lambda_i^n(X_\bullet)$.

Proposition 2.1.3.2. *Let $X_\bullet$ be a simplicial set. Then $X_\bullet$ is isomorphic to the nerve of a category if and only if, for each $0 < i < n$, the restriction map $X_n \to \Lambda^n_i(X_\bullet)$ is bijective.*

For example, the bijectivity of the map $X_2 \to \Lambda^2_1(X_\bullet)$ encodes the existence and uniqueness of composition: it says that every pair of composable morphisms $f : C \to D$ and $g : D \to E$ can be completed uniquely to a commutative diagram



Example 2.1.3.3. Let Z be a topological space. We can associate to Z a simplicial set $\mathrm{Sing}(Z)_\bullet$, whose n-simplices are continuous maps $\Delta^n \to Z$ (here Δ^n denotes the standard topological n-simplex). The simplicial set $\mathrm{Sing}(Z)_\bullet$ is called the *singular simplicial set* of Z.

From the perspective of homotopy theory, the singular simplicial set $\mathrm{Sing}(Z)_\bullet$ is a complete invariant of X. More precisely, from $\mathrm{Sing}(Z)_\bullet$ one can functorially construct a topological space which is (weakly) homotopy equivalent to Z. Consequently, the simplicial set $\mathrm{Sing}(Z)_\bullet$ can often serve as a surrogate for Z. For example, there is a combinatorial recipe for extracting the homotopy groups of Z directly from $\mathrm{Sing}(Z)_\bullet$. However, this recipe works only for a special class of simplicial sets:

Definition 2.1.3.4. Let $X_\bullet$ be a simplicial set. We say that X is a *Kan complex* if, for $0 \leq i \leq n$, the map $X_n \to \Lambda^n_i(X_\bullet)$ is surjective.

Example 2.1.3.5. For any topological space Z, the singular simplicial set $\mathrm{Sing}(Z)_\bullet$ is a Kan complex. To see this, let H denote the topological space obtained from the standard n-simplex Δ^n by removing the interior and the ith face. Then $\Lambda^n_i(\mathrm{Sing}(Z)_\bullet)$ can be identified with the set of continuous maps from H into Z. Any continuous map from H into Z can be extended to a map from Δ^n into Z, since H is a retract of Δ^n.

The converse of Example 2.1.3.5 fails: not every Kan complex is isomorphic to the singular simplicial set of a topological space. However, every Kan complex $X_\bullet$ is *homotopy equivalent* to the singular simplicial set of a topological space, which can be constructed explicitly from $X_\bullet$. In fact, something stronger is true: the construction $Z \mapsto \mathrm{Sing}(Z)_\bullet$ induces an equivalence from the homotopy category of nice spaces (say, CW complexes) to the homotopy category of Kan complexes (which can be defined in a purely combinatorial way).

Example 2.1.3.6. Let Λ be a commutative ring. A *simplicial* Λ*-module* is a simplicial set $X_\bullet$ for which each of the sets X_n is equipped with the structure of a Λ-module, and each of the maps $\alpha^* : X_n \to X_m$ is a Λ-module homomorphism homomorphism. One can show that every simplicial Λ-module is a Kan complex, so that one has homotopy groups $\{\pi_n X_\bullet\}_{n \geq 0}$. According to the classical *Dold-Kan correspondence*, the category of simplicial Λ-modules is equivalent to the category $\mathrm{Chain}_{\geq 0}(\Lambda) \subseteq \mathrm{Chain}(\Lambda)$ of nonnegatively graded chain complexes of Λ-modules. Under this equivalence, the homotopy groups of a simplicial Λ-module $X_\bullet$ can be identified with the homology groups of the corresponding chain complex.

The hypothesis of Proposition 2.1.3.2 resembles the definition of a Kan complex, but is different in two important respects. Definition 2.1.3.4 requires that every element of $\Lambda^n_i(X_\bullet)$ can be extended to an n-simplex of X. Proposition 2.1.3.2 requires this condition only in the case $0 < i < n$, but demands that the extension be unique. Neither condition implies the other, but they admit a common generalization:

Definition 2.1.3.7. A simplicial set $X_\bullet$ is an ∞*-category* if, for each $0 < i < n$, the map $X_n \to \Lambda^n_i(X_\bullet)$ is surjective.

Remark 2.1.3.8. A simplicial set $X_\bullet$ satisfying the requirement of Definition 2.1.3.7 is also referred to as a *quasi-category* or a *weak Kan complex* in the literature.

Example 2.1.3.9. Any Kan complex is an ∞-category. In particular, for any topological space Z, the singular simplicial set $\mathrm{Sing}(Z)_\bullet$ is an ∞-category.

Example 2.1.3.10. For any category $\mathcal{C}$, the nerve $\mathrm{N}(\mathcal{C})_\bullet$ is an ∞-category.

Example 2.1.3.11 (The ∞-Category Associated to a 2-Category)**.** Let $\mathcal{C}$ be a *strict* 2-category (that is, a 2-category in which composition is strictly associative, rather than associative up to isomorphism). For each integer $n \geq 0$, we let $\mathrm{N}(\mathcal{C})_n$ denote the set of all triples $(\{C_i\}_{0 \leq i \leq n}, \{f_{ij}\}_{0 \leq i \leq j \leq n}, \{\alpha_{ijk}\}_{0 \leq i \leq j \leq k \leq n})$, where:

- For $0 \leq i \leq n$, C_i is an object of $\mathcal{C}$.
- For $0 \leq i \leq j \leq n$, f_{ij} is a 1-morphism from C_i to C_j in the 2-category $\mathcal{C}$, which is required to be the identity in the case $i = j$.
- For $0 \leq i \leq j \leq k \leq n$, α_{ijk} is a 2-morphism from f_{ik} to $f_{jk} \circ f_{ij}$ in the 2-category $\mathcal{C}$, which is required to be the identity map if $i = j$ or $j = k$.
- For $0 \leq i \leq j \leq k \leq l \leq n$, the diagram

$$\begin{array}{ccc} f_{il} & \xrightarrow{\alpha_{ijl}} & f_{jl} \circ f_{ij} \\ \downarrow{\scriptstyle \alpha_{ikl}} & & \downarrow{\scriptstyle \alpha_{jkl}} \\ f_{kl} \circ f_{ik} & \xrightarrow{\alpha_{ikl}} & f_{kl} \circ f_{jk} \circ f_{ij} \end{array}$$

commutes (in the category of 1-morphisms from C_i to C_l.

The construction $n \mapsto \mathrm{N}(\mathcal{C})_n$ determines a simplicial set $\mathrm{N}(\mathcal{C})_\bullet$, which we will refer to as the *nerve of the* 2*-category* $\mathcal{C}$. One can show that $\mathrm{N}(\mathcal{C})_\bullet$ is an ∞-category (in the sense of Definition 2.1.3.7) if and only if every 2-morphism in $\mathcal{C}$ is invertible.

Remark 2.1.3.12. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ can be regarded as a (strict) 2-category, having no 2-morphisms other than the identity maps. In this case, the nerve $\mathrm{N}(\mathcal{C})_\bullet$ of Example 2.1.2.3 agrees with the nerve $\mathrm{N}(\mathcal{C})_\bullet$ of Example 2.1.3.11.

Remark 2.1.3.13. Using a variant of Example 2.1.3.11, one can assign a nerve $\mathrm{N}(\mathcal{C})_\bullet$ to a non-strict 2-category $\mathcal{C}$ (also known as a *bicategory* or *weak* 2*-category* in the literature), which will again be an ∞-category provided that every 2-morphism in $\mathcal{C}$ is invertible.

2.1.4 The Language of Higher Category Theory

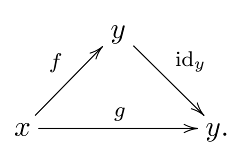
By virtue of the discussion following Example 2.1.2.3, no information is lost by identifying a category $\mathcal{C}$ with the simplicial set $\mathrm{N}(\mathcal{C})_\bullet$. It is often convenient to abuse notation by identifying $\mathcal{C}$ with its nerve, thereby viewing a category as a special type of ∞-category. We will generally use category-theoretic notation and terminology when discussing ∞-categories. Here is a brief sampler; for a more detailed discussion of how the basic notions of category theory can be generalized to this setting, we refer the reader to the first chapter of [25].

- Let $\mathcal{C} = \mathcal{C}_\bullet$ be an ∞-category. An *object* of $\mathcal{C}$ is an element of the set $\mathcal{C}_0$ of 0-simplices of $\mathcal{C}$. We will indicate that x is an object of $\mathcal{C}$ by writing $x \in \mathcal{C}$.

- A *morphism* of $\mathcal{C}$ is an element f of the set $\mathcal{C}_1$ of 1-simplices of $\mathcal{C}$. More precisely, we will say that f is a *morphism from x to y* if $\alpha_0^*(f) = x$ and $\alpha_1^*(f) = y$, where $\alpha_i : \{0\} \hookrightarrow \{0,1\}$ denotes the map given by $\alpha_i(0) = i$. We will often indicate that f is a morphism from x to y by writing $f : x \to y$.

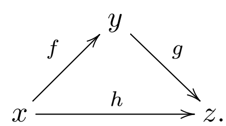
- For any object $x \in \mathcal{C}$, there is an *identity morphism* id_x, given by $\beta^*(x)$ where $\beta : \{0,1\} \to \{0\}$ is the unique map.

- Given a pair of morphisms $f, g : x \to y$ in $\mathcal{C}$, we say that f and g are *homotopic* if there exists a 2-simplex $\sigma \in \mathcal{C}_2$ whose faces are as indicated in the diagram



In this case, we will write $f \simeq g$, and we will say that σ *is a homotopy from* f *to* g. One can show that homotopy is an equivalence relation on the collection of morphisms from x to y.

- Given a pair of morphisms $f : x \to y$ and $g : y \to z$, it follows from Definition 2.1.3.7 that there exists a 2-simplex with boundary as indicated in the diagram



Definition 2.1.3.7 does *not* guarantee that the morphism h is unique. However, one can show that h is unique up to homotopy. We will generally abuse terminology and refer to h as the composition of f and g, and write $h = g \circ f$.

- Composition of morphisms in $\mathcal{C}$ is associative up to homotopy. Consequently, we can define an ordinary category $\mathrm{h}\mathcal{C}$ as follows:
  - The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
  - Given objects $x, y \in \mathcal{C}$, the set of morphisms from x to y in $\mathrm{h}\mathcal{C}$ is the set of equivalence classes (under the relation of homotopy) of morphisms from x to y in $\mathcal{C}$.
  - Given morphisms $[f] : x \to y$ and $[g] : y \to z$ in $\mathrm{h}\mathcal{C}$ represented by morphisms f and g in $\mathcal{C}$, we define $[g] \circ [f]$ to be the morphism from x to z in $\mathrm{h}\mathcal{C}$ given by the homotopy class of $g \circ f$.

  We refer to $\mathrm{h}\mathcal{C}$ as the *homotopy category* of $\mathcal{C}$.

- We will say that a morphism f in $\mathcal{C}$ is an *equivalence* if its image $[f]$ is an isomorphism in $\mathrm{h}\mathcal{C}$ (in other words, f is an equivalence if it admits an inverse up to homotopy). We say that two objects $x, y \in \mathcal{C}$ are *equivalent* if there exists an equivalence $f : x \to y$.

- An ∞-category $\mathcal{C}$ is a Kan complex if and only if every morphism in $\mathcal{C}$ is an equivalence. More generally, if $\mathcal{C}$ is an arbitrary ∞-category, then there is a largest Kan complex $\mathcal{C}^{\simeq}$ which is contained in $\mathcal{C}$; it consists of all simplices of $\mathcal{C}$ whose 1-dimensional facets are equivalences.

The theory of ∞-categories allows us to treat topological spaces (via their singular simplicial sets) and ordinary categories (via the nerves) as examples of the same type of object. This is often very convenient.

Definition 2.1.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. A *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a map of simplicial sets from $\mathcal{C}$ to $\mathcal{D}$.

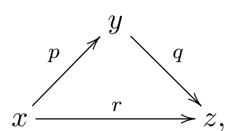
Remark 2.1.4.2. Let $\mathcal{C}$ be an ∞-category. The homotopy category of $\mathcal{C}$ admits another characterization: it is universal among ordinary categories for which there exists a functor from $\mathcal{C}$ to (the nerve of) $\mathrm{h}\mathcal{C}$.

Example 2.1.4.3. Let Z be a topological space and let $\mathcal{C}$ be a category. Unwinding the definitions, we see that a functor from $\mathrm{Sing}(Z)_\bullet$ to $\mathrm{N}(\mathcal{C})_\bullet$ consists of the following data:

(1) For each point $z \in Z$, an object $C_z \in \mathcal{C}$.

(2) For every path $p : [0,1] \to Z$, a morphism $\alpha_p : C_{p(0)} \to C_{p(1)}$, which is an identity morphism if the map p is constant.

(3) For every continuous map $\Delta^2 \to Z$, which we write informally as



we have $\alpha_r = \alpha_q \circ \alpha_p$ (an equality of morphisms from C_x to C_z).

Here condition (3) encodes simultaneously the assumption that the map α_p depends only on the homotopy class of p, and that the construction $p \mapsto \alpha_p$ is compatible with concatenation of paths. Moreover, it follows from condition (3) that each of the maps α_p is an isomorphism (since every path is invertible up to homotopy). Consequently, we see that the data of a functor from $\mathrm{Sing}(Z)_\bullet$ into $\mathrm{N}(\mathcal{C})_\bullet$ recovers the classical notion of a *local system on Z with values in* $\mathcal{C}$.

One of the main advantages of working in the setting of ∞-categories is that the collection of functors from one ∞-category to another can easily be organized into a third ∞-category.

Notation 2.1.4.4. For every integer $n \geq 0$, we let Δ^n denote the simplicial set given by the nerve of the linearly ordered set $\{0 < 1 < \cdots < n\}$. We refer to Δ^n as the *standard n-simplex*. By definition, an m-simplex of Δ^n is given by a nondecreasing map $\{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$.

Let X and Y be simplicial sets. We let $\mathrm{Fun}(X, Y)$ denote the simplicial of maps from X to Y. More precisely, $\mathrm{Fun}(X, Y)$ is the simplicial set whose n-simplices are maps $\Delta^n \times X \to Y$ (more generally, giving a map of simplicial sets $Z \to \mathrm{Fun}(X, Y)$ is equivalent to giving a map $Z \times X \to Y$).

One can show that if the simplicial set Y is an ∞-category, then $\mathrm{Fun}(X, Y)$ is also an ∞-category (for any simplicial set X). Note that the objects of $\mathrm{Fun}(X, Y)$ are functors from X to Y, in the sense of Definition 2.1.4.1. We will refer to $\mathrm{Fun}(X, Y)$ as the *∞-category of functors from X to Y*.

Example 2.1.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. Then the simplicial set

$$\mathrm{Fun}(\mathrm{N}(\mathcal{C})_\bullet, \mathrm{N}(\mathcal{D})_\bullet)$$

is isomorphic to the nerve of the category of functors from $\mathcal{C}$ to $\mathcal{D}$. In particular, there is a bijection between the set of functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of classical category theory) and the set of functors from $\mathrm{N}(\mathcal{C})_\bullet$ to $\mathrm{N}(\mathcal{D})_\bullet$ (in the sense of Definition 2.1.4.1).

Remark 2.1.4.6. It follows from Example 2.1.4.5 that no information is lost by passing from a category $\mathcal{C}$ to the associated ∞-category $\mathrm{N}(\mathcal{C})$. For the remainder of this book, we will generally abuse notation by identifying each category $\mathcal{C}$ with its nerve.

Remark 2.1.4.7 (Equivalences of ∞-Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of ∞-categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ with the following properties:

- The composition $F \circ G$ is equivalent to the identity functor $\mathrm{id}_{\mathcal{D}}$ (where we view both $F \circ G$ and $\mathrm{id}_{\mathcal{D}}$ as objects of the functor ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$).
- The composition $G \circ F$ is equivalent to the identity functor $\mathrm{id}_{\mathcal{C}}$ (where we view both $G \circ F$ and $\mathrm{id}_{\mathcal{C}}$ as objects of the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$).

When specialized to (the nerves of) ordinary categories, this recovers the usual definition of an equivalence of categories (see Remark 2.1.4.5).

Example 2.1.4.8. Let Λ be a commutative ring and let $\mathrm{Mod}_\Lambda = \{S_n\}_{n \geq 0}$ denote the simplicial set introduced in Construction 2.1.2.1. Then Mod_Λ is an ∞-category, which we will refer to as the *derived ∞-category of Λ-modules*. It can be regarded as an enhancement of the usual derived category $\mathcal{D}(\Lambda)$ of Λ-modules, in the sense that the homotopy category of Mod_Λ is equivalent to $\mathcal{D}(\Lambda)$ (in fact, the homotopy category of Mod_Λ is *isomorphic* to the category $\mathcal{D}'(\Lambda)$ of Definition 2.1.1.1).

Notation 2.1.4.9. Let Λ be a commutative ring. For every integer n, the construction $M_* \mapsto \mathrm{H}_n(M_*)$ determines a functor from the ∞-category Mod_Λ to the ordinary abelian category of Λ-modules. We will say that an object $M_* \in \mathrm{Mod}_\Lambda$ is *discrete* if $\mathrm{H}_n(M_*) \simeq 0$ for $n \neq 0$. One can show that the construction $M_* \mapsto \mathrm{H}_0(M_*)$ induces an equivalence from the ∞-category of discrete objects of Mod_Λ to the ordinary category of Λ-modules. We will generally abuse notation by identifying the abelian category of Λ-modules with its inverse image under this equivalence. We will sometimes refer to Λ-modules as as *discrete* Λ*-modules* or *ordinary* Λ*-modules*, to distinguish them from more general objects of Mod_Λ.

Remark 2.1.4.10. The ∞-category Mod_Λ is, in many respects, easier to work with than the usual derived category $\mathcal{D}(\Lambda)$. For example, we have already mentioned that there is no functorial way to construct the cone of a morphism in $\mathcal{D}(\Lambda)$. However, Mod_Λ does not suffer from the same problem: there is a functor $\mathrm{Fun}(\Delta^1, \mathrm{Mod}_\Lambda) \to \mathrm{Mod}_\Lambda$, given on objects by $f \mapsto \mathrm{Cn}(f)$.

We close this section with an alternative description of the ∞-category Mod_Λ.

Construction 2.1.4.11. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms in $\mathcal{C}$. Then we can form a new ∞-category $\mathcal{C}[W^{-1}]$ equipped with a functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ which enjoys the following universal property: for every ∞-category $\mathcal{D}$, composition with F induces an equivalence from $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D})$ to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors $G : \mathcal{C} \to \mathcal{D}$ with the following property: for each morphism $w \in W$, the image $G(w)$ is an equivalence in $\mathcal{D}$. This universal property characterizes the ∞-category $\mathcal{C}[W^{-1}]$ up to equivalence.

In the situation of Construction 2.1.4.11, the ∞-category $\mathcal{C}[W^{-1}]$ will generally not be (equivalent to the nerve of) an ordinary category, even if $\mathcal{C}$ is an ordinary category to begin with. In fact, one can show that *every* ∞-category has the form $\mathcal{C}[W^{-1}]$, where $\mathcal{C}$ is an ordinary category (and W is some collection of morphisms in $\mathcal{C}$).

Example 2.1.4.12. Let Λ be a commutative ring, let $\mathrm{Chain}(\Lambda)$ be the category of chain complexes of Λ modules, and let W be the collection of all quasi-isomorphisms in $\mathrm{Chain}(\Lambda)$. Then $\mathrm{Chain}(\Lambda)[W^{-1}]$ is equivalent to the ∞-category Mod_Λ of Example 2.1.4.8. More precisely, the canonical maps $\mathrm{Chain}(\Lambda)[W^{-1}] \leftarrow \mathrm{Chain}'(\Lambda)[W'^{-1}] \to \mathrm{Mod}_\Lambda$ are equivalences, where $\mathrm{Chain}'(\Lambda)$ is the full subcategory of $\mathrm{Chain}(\Lambda)$ spanned by the K-projective chain complexes, and W' is the collection of all quasi-isomorphisms between objects of $\mathrm{Chain}'(\Lambda)$. This can be regarded as an ∞-categorical generalization of the equivalence $\mathcal{D}(\Lambda) \simeq \mathcal{D}'(\Lambda)$ of Remark 2.1.1.2.

2.1.5 Example: Limits and Colimits

The theory of ∞-categories is a robust generalization of ordinary category theory. In particular, most of the important notions of ordinary category theory (adjoint functors, Kan extensions, Pro-objects and Ind-objects, ...) can be generalized to the setting of ∞-categories in a natural way. For a detailed introduction (including complete definitions and proofs of the basic categorical facts we will need), we refer the reader to [25]. For the reader's convenience, we briefly sketch how this generalization plays out for the categorical constructs which will appear most frequently in this book: limits and colimits.

Construction 2.1.5.1 (Overcategories and Undercategories)**.** Let $\mathcal{C}$ be an ∞-category containing an object C. We define simplicial sets $\mathcal{C}_{C/}$ and $\mathcal{C}_{/C}$ as follows:

- An n-simplex of $\mathcal{C}_{C/}$ is an $(n+1)$-simplex of $\mathcal{C}$ whose first vertex is C.
- An n-simplex of $\mathcal{C}_{/C}$ is an $(n+1)$-simplex of $\mathcal{C}$ whose last vertex is C.

One can show that $\mathcal{C}_{C/}$ and $\mathcal{C}_{/C}$ are also ∞-categories. We refer to $\mathcal{C}_{/C}$ as the *∞-category of objects of $\mathcal{C}$ over C*, and to $\mathcal{C}_{C/}$ as the *∞-category of objects of $\mathcal{C}$ under C*.

Example 2.1.5.2. Let $\mathcal{C}$ be a category and let $\mathrm{N}(\mathcal{C})$ be its nerve, which we regard as an ∞-category. For each object $C \in \mathcal{C}$, the simplicial sets $\mathrm{N}(\mathcal{C})_{C/}$ and $\mathrm{N}(\mathcal{C})_{/C}$ are isomorphic to the nerves of categories $\mathcal{C}_{C/}$ and $\mathcal{C}_{/C}$, which can be described as follows:

- The objects of $\mathcal{C}_{C/}$ are morphisms $\alpha : C \to D$ in $\mathcal{C}$, and the morphisms in $\mathcal{C}_{C/}$ are commutative diagrams

$$\begin{array}{ccc} & C & \\ {}^{\alpha}\swarrow & & \searrow^{\alpha'} \\ D & \longrightarrow & D'. \end{array}$$

- The objects of $\mathcal{C}_{/C}$ are morphisms $\beta : E \to C$ in $\mathcal{C}$, and the morphisms in $\mathcal{C}_{/C}$ are commutative diagrams

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {}_{\beta}\searrow & & \swarrow_{\beta'} \\ & C. & \end{array}$$

For every object C of an ∞-category $\mathcal{C}$, we have evident (forgetful) functors $\mathcal{C}_{/C} \to \mathcal{C} \leftarrow \mathcal{C}_{C/}$.

Definition 2.1.5.3. Let $\mathcal{C}$ be an ∞-category. We say that an object $C \in \mathcal{C}$ is *initial* if the forgetful functor $\mathcal{C}_{C/} \to \mathcal{C}$ is an equivalence of ∞-categories (Remark 2.1.4.7). We say that $C \in \mathcal{C}$ is *final* if the forgetful functor $\mathcal{C}_{/C} \to \mathcal{C}$ is an equivalence of ∞-categories.

Remark 2.1.5.4. Let $\mathcal{C}$ be an ∞-category. One can show that an initial (final) object $C \in \mathcal{C}$ is unique up to equivalence, if it exists.

Example 2.1.5.5. Let $\mathcal{C}$ be a category. Then an object $C \in \mathcal{C}$ is initial or final (in the sense of classical category theory) if and only if C is initial or final when regarded as an object of the ∞-category $\mathrm{N}(\mathcal{C})$ (in the sense of Definition 2.1.5.3).

Construction 2.1.5.6 (Limits and Colimits)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. There is an evident diagonal functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, which assigns to each object $D \in \mathcal{D}$ the constant functor $c_D : \mathcal{C} \to \mathcal{D}$ taking the value D. Let $F : \mathcal{C} \to \mathcal{D}$ be an arbitrary functor, which we regard as an object of the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. One can show that the fiber products

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})_{F/} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \mathcal{D} \qquad \mathrm{Fun}(\mathcal{C}, \mathcal{D})_{/F} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \mathcal{D}$$

are also ∞-categories. Then:

- A *limit of* F is a final object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})_{/F} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \mathcal{D}$.
- A *colimit of* F is an initial object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})_{F/} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \mathcal{D}$.

We will generally abuse terminology by identifying a limit or colimit of F with its image in the ∞-category $\mathcal{D}$.

More informally, if $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then a limit of F is an object $D \in \mathcal{D}$ which is universal among those for which there exists a natural transformation of functors $c_D \to F$, while a colimit of F is an object $D \in \mathcal{D}$ which is universal among those for which there exists a natural transformation $F \to c_D$.

Remark 2.1.5.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then a limit or colimit of F is uniquely determined up to equivalence, if it exists (see Remark 2.1.5.4. We will denote a limit of a functor F by $\varprojlim_{C \in \mathcal{C}} F(C)$, and a colimit of F by $\varinjlim_{C \in \mathcal{C}} F(C)$. One can show that $\varprojlim(F)$ and $\varinjlim(F)$ depend functorially on F (provided that we restrict our attention to those functors which admit limits or colimits).

Example 2.1.5.8. When $\mathcal{C}$ is empty, then the notions of limit and colimit of a functor $F : \mathcal{C} \to \mathcal{D}$ reduce to the notions of final and initial object of $\mathcal{D}$, respectively.

Example 2.1.5.9. Let $F : \mathcal{C} \to \mathcal{D}$ be ordinary categories. Then we can identify limits (colimits) of the induced map $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathcal{D})$ (in the sense of Construction 2.1.5.6) with limits and colimits of F, in the usual sense of category theory.

Example 2.1.5.10. Let $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ denote the category of commutative differential graded algebras over $\mathbf{Q}$, let W be the collection of all quasi-isomorphisms in $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$, and let $\mathrm{CAlg}_{\mathbf{Q}}$ denote the ∞-category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}[W^{-1}]$ obtained from $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ by formally inverting every morphism in W (see Example 2.1.4.11). Let $\mathcal{J}$ be a small category.

Then every functor between ordinary categories $F : \mathcal{J} \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ determines a functor of ∞-categories $F' : \mathcal{J} \to \mathrm{CAlg}_{\mathbf{Q}}$. One can show that the functor $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}} \to \mathrm{CAlg}_{\mathbf{Q}}$ carries homotopy colimits of F (in the sense of Definition 1.5.4.5) to colimits of the functor F' (in the sense of Construction 2.1.5.6). One of the main virtues of the ∞-categorical framework is that it provides a language to express the idea that homotopy limits and colimits are solutions to universal mapping problems.

We now briefly review some notions which are useful for computing with limits and colimits.

Definition 2.1.5.11 (Cofinality)**.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *weakly contractible* if, for every Kan complex $\mathcal{D}$, the diagonal map $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an equivalence of ∞-categories.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We will say that F is *left cofinal* if, for every object $D \in \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible. We say that F is *right cofinal* if, for every object $D \in \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible.

In this book, we will primarily be interested in the special case of Definition 2.1.5.11 where $\mathcal{C}$ and $\mathcal{D}$ are (nerves of) ordinary categories.

Definition 2.1.5.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories (or ∞-categories). Then F admits a right adjoint G if and only if, for each object $D \in \mathcal{D}$, the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ has a final object $(\overline{D}, v : F(\overline{D}) \to D)$ (the functor G is then given on objects by $G(D) = \overline{D}$). In this case, the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is automatically weakly contractible. It follows that if F admits a right adjoint, then F is right cofinal. Similarly, if F admits a left adjoint, then it is left cofinal.

We refer the reader to §[25].4.1 for a proof of the following:

Proposition 2.1.5.13. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Then:*

- *If F is left cofinal, then G admits a colimit if and only if $(G \circ F)$ admits a colimit. In this case, there is a canonical equivalence $\varinjlim_{C \in \mathcal{C}} (G \circ F)(C) \to \varinjlim_{D \in \mathcal{D}} G(D)$ in the ∞-category $\mathcal{E}$.*
- *If F is right cofinal, then G admits a limit if and only if $(G \circ F)$ admits a limit. In this case, there is a canonical equivalence $\varprojlim_{D \in \mathcal{D}} G(D) \to \varprojlim_{C \in \mathcal{C}} (G \circ F)(C)$ in the ∞-category $\mathcal{E}$.*

Remark 2.1.5.14. Let $\mathcal{D}$ and $\mathcal{E}$ be ∞-categories and suppose we are given a functor $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$. We will sometimes abuse notation by denoting a limit of G by $\varprojlim_{D \in \mathcal{D}} G(D)$

and a colimit of G by $\varinjlim_{D \in \mathcal{D}} G(D)$. Note that in this case, to identify the limit of G with the limit of $G \circ F^{\mathrm{op}}$ for some functor $F : \mathcal{C} \to \mathcal{D}$, it suffices to show that F is *left* cofinal. Dually, to identify the colimit of G with the colimit of $G \circ F^{\mathrm{op}}$, it suffices to show that F is *right* cofinal.

2.1.6 Stable ∞-Categories

We now describe a special feature of the ∞-category Mod_Λ of Example 2.1.4.8 called *stability* (Definition 2.1.6.5); roughly speaking, stability articulates the idea that Mod_Λ is of a "linear" nature (for example, it is possible to add morphisms). The homotopy category of a stable ∞-category inherits additional structure: it is a *triangulated category* in the sense of Verdier (Remark 2.1.6.7).

Definition 2.1.6.1. Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is an object which is both initial and final. We will say that $\mathcal{C}$ is *pointed* if it admits a zero object.

Definition 2.1.6.2. Let $\mathcal{C}$ be a pointed ∞-category, and suppose that $\mathcal{C}$ admits finite colimits. For any morphism $f : C \to D$ in $\mathcal{C}$, we let $\mathrm{cofib}(f)$ denote a pushout $0 \amalg_C D$, where 0 denotes a zero object of $\mathcal{C}$. We will refer to $\mathrm{cofib}(f)$ as the *cofiber of f*. For any object $C \in \mathcal{C}$, we let $\Sigma(C)$ denote the cofiber of the (essentially unique) map $C \to 0$; we refer to $\Sigma(C)$ as the *suspension of C*.

Example 2.1.6.3. Let $\mathcal{C} = \mathrm{Mod}_\Lambda$ be the derived ∞-category of Λ-modules, and let f be a morphism in $\mathcal{C}$ which we can identify with a map of chain complexes $M_* \to N_*$. Then the cofiber $\mathrm{cofib}(f)$ can be realized explicitly as the mapping cone $\mathrm{Cn}(f)$. In particular, the suspension $\Sigma(M_*)$ of a chain complex M_* can be identified with the shifted complex M_{*-1}.

Variant 2.1.6.4. Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits. For any morphism $f : C \to D$ in $\mathcal{C}$, we let $\mathrm{fib}(f)$ denote the pullback $0 \times_D C$. We will refer to $\mathrm{fib}(f)$ as the *fiber of f*. For any object $C \in \mathcal{C}$, we let $\Omega(C)$ denote the fiber of the (essentially unique) map $0 \to C$; we refer to $\Omega(C)$ as the *loop object of C*.

Definition 2.1.6.5. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *stable* if it is pointed, admits finite limits and colimits, and the construction $C \mapsto \Sigma(C)$ induces an equivalence of ∞-categories $\Sigma : \mathcal{C} \to \mathcal{C}$ (in this case, an inverse equivalence is given by the construction $C \mapsto \Omega(C)$).

Example 2.1.6.6. Let Λ be a commutative ring. Then the ∞-category Mod_Λ of Example 2.1.4.8 is stable.

Remark 2.1.6.7. Let $\mathcal{C}$ be a stable ∞-category. Then $\mathrm{h}\mathcal{C}$ admits the structure of a triangulated category, where the shift functor $X \mapsto X[1]$ on $\mathrm{h}\mathcal{C}$ is induced by the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$. See Theorem [23].1.1.2.14 for more details.

2.2 Étale Sheaves

Let X be a scheme and let Λ be a commutative ring. One can associate to X an abelian category $\mathcal{A}$ of *étale sheaves of* Λ*-modules* on X. The derived category $\mathcal{D}(\mathcal{A})$ provides a useful setting for performing a wide variety of sheaf-theoretic constructions. However, there are other basic constructions (such as the formation of mapping cones) which cannot be carried out functorially at the level of derived categories. One way to remedy the situation is to introduce an ∞-category $\mathrm{Shv}(X;\Lambda)$ whose homotopy category is equivalent to the derived category $\mathcal{D}(\mathcal{A})$. It is possible to produce such an ∞-category by applying a purely formal procedure to the abelian category $\mathcal{A}$ (see §[23].1.3.2 and §[23].1.3.5). However, it will be more convenient for us to define $\mathrm{Shv}(X;\Lambda)$ directly as the ∞-category of (hypercomplete) Mod_{Λ}*-valued* sheaves on X. Our goal in this section is to give a brief introduction to this point of view, and to review some of the basic properties of étale sheaves which will be needed in the later sections of this book. We will confine our attention here to the most formal aspects of the theory, where the coefficient ring Λ can be taken to be arbitrary; base change and finiteness theorems for étale cohomology, which require additional hypotheses on Λ, will be discussed in §2.3 and §2.4.

Remark 2.2.0.1. Since the apparatus of étale cohomology is treated exhaustively in other sources (such as [1] and [10]; see also [13] for an expository account), we will be content to summarize the relevant definitions and give brief indications of proofs.

Warning 2.2.0.2. To simplify the exposition, we will confine our attention to the discussion of étale sheaves on quasi-projective k-schemes, where k is an algebraically closed field. This restriction is largely unnecessary: many of the constructions we describe make sense for more general schemes. Beware, however, that some of our results (such as Lemma 2.2.4.1) depend on the boundedness of the cohomological dimension of finitely presented k-schemes, and would require modification if we were to work with arbitrary schemes.

2.2.1 Sheaves of Λ-Modules

Throughout this section, we let k denote an algebraically closed field and Sch_k the category of quasi-projective k-schemes.

Notation 2.2.1.1. Let X be a quasi-projective k-scheme. We let $\mathrm{Sch}_X^{\mathrm{et}}$ denote the category whose objects are étale maps $U \to X$ (where U is also a quasi-projective

k-scheme). Morphisms in $\mathrm{Sch}_X^{\mathrm{et}}$ are given by commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

We will say that a collection of morphisms $\{f_\alpha : U_\alpha \to V\}$ in $\mathrm{Sch}_X^{\mathrm{et}}$ is an *étale covering* if the induced map $\amalg U_\alpha \to V$ is surjective. The collection of étale coverings determines a Grothendieck topology on the category $\mathrm{Sch}_X^{\mathrm{et}}$, which we refer to as the *étale topology.*

Definition 2.2.1.2. Let Λ be a commutative ring, and let Mod_Λ be the ∞-category of chain complexes over Λ (see Example 2.1.4.8). A Mod_Λ*-valued presheaf on* X is a functor of ∞-categories

$$\mathcal{F} : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda \,.$$

If $\mathcal{F}$ is a Mod_Λ-valued presheaf on X and $U \in \mathrm{Sch}_X^{\mathrm{et}}$, then we can regard $\mathcal{F}(U)$ as a chain complex of Λ-modules. For each integer n, the construction $U \mapsto \mathrm{H}_n(\mathcal{F}(U))$ determines a presheaf of abelian groups on X. We let $\pi_n \mathcal{F}$ denote the étale sheaf of abelian groups on X obtained by sheafifying the presheaf $U \mapsto \mathrm{H}_n(\mathcal{F}(U))$. We will say that $\mathcal{F}$ is *locally acyclic* if, for every integer n, the sheaf $\pi_n \mathcal{F}$ vanishes.

We let $\mathrm{Shv}(X;\Lambda)$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ spanned by those Mod_Λ-valued presheaves $\mathcal{F}$ which have the following property: for every locally acyclic object $\mathcal{F}' \in \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$, every morphism $\alpha : \mathcal{F}' \to \mathcal{F}$ is nullhomotopic.

Remark 2.2.1.3. Let $\mathcal{F} : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda$ be a Mod_Λ-valued presheaf on a quasi-projective k-scheme X. Then $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ if and only if the following conditions are satisfied:

(1) The presheaf $\mathcal{F}$ is a sheaf with respect to the étale topology on $\mathrm{Sch}_X^{\mathrm{et}}$. That is, for every covering $\{f_\alpha : U_\alpha \to V\}$, the canonical map $\mathcal{F}(V) \to \varprojlim \mathcal{F}(U)$ is an equivalence in Mod_Λ, where the limit is taken over all objects $U \in \mathrm{Sch}_V^{\mathrm{et}}$ for which the map $U \to V$ factors through some f_α.

(2) The Mod_Λ-valued sheaf $\mathcal{F}$ is *hypercomplete* (see Definition [24].I.1.1.15). This is a technical hypothesis which is necessary only because we consider potentially unbounded complexes, where descent for Čech coverings does not necessarily imply descent for arbitrary hypercoverings.

Remark 2.2.1.4 (Sheafification)**.** Let X be a quasi-projective k-scheme and let Λ be a commutative ring. Then the inclusion functor $\mathrm{Shv}(X;\Lambda) \hookrightarrow \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ admits a left adjoint $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$. If $\mathcal{F}$ is an arbitrary Mod_Λ-valued presheaf on X, we will refer to $\widetilde{\mathcal{F}}$ as the *sheafification of* $\mathcal{F}$.

Example 2.2.1.5. If $X = \mathrm{Spec}(k)$, then the ∞-category $\mathrm{Shv}(X;\Lambda)$ is equivalent to Mod_Λ. Concretely, this equivalence is implemented by the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X) \in \mathrm{Mod}_\Lambda$.

2.2.2 The t-Structure on $\mathrm{Shv}(X;\Lambda)$

Let k be an algebraically closed field and let Λ be a commutative ring. For every quasi-projective k-scheme X, the ∞-category $\mathrm{Shv}(X;\Lambda)$ is stable (Definition 2.1.6.5). We now construct a t-structure on the ∞-category $\mathrm{Shv}(X;\Lambda)$ (or equivalently on the triangulated category $\mathrm{hShv(X;\Lambda)}$).

Warning 2.2.2.1. In this book, we will use homological indexing conventions when working with t-structures on triangulated categories, rather than the cohomological conventions which can be found (for example) in [5]. One can translate between conventions using the formulae

$$\mathcal{C}^{\leq n} = \mathcal{C}_{\geq -n} \qquad \mathcal{C}^{\geq n} = \mathcal{C}_{\leq -n}\,.$$

Notation 2.2.2.2. Let X be a quasi-projective k-scheme and let Λ be a commutative ring. For each integer $n \in \mathbf{Z}$, we let $\mathrm{Shv}(X;\Lambda)_{\leq n}$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by those objects $\mathcal{F}$ for which $\pi_m \mathcal{F} \simeq 0$ for $m > 0$, and we let $\mathrm{Shv}(X;\Lambda)_{\geq n}$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by those objects $\mathcal{F}$ for which $\pi_m \mathcal{F} \simeq 0$ for $m < 0$.

Proposition 2.2.2.3. *Let X be a quasi-projective k-scheme. Then the full subcategories $(\mathrm{Shv}(X;\Lambda)_{\geq 0}, \mathrm{Shv}(X;\Lambda)_{\leq 0})$ determine a t-structure on $\mathrm{Shv}(X;\Lambda)$. Moreover, the construction $\mathcal{F} \mapsto \pi_0 \mathcal{F}$ determines an equivalence of categories from the heart*

$$\mathrm{Shv}(X;\Lambda)^{\heartsuit} = \mathrm{Shv}(X;\Lambda)_{\geq 0} \cap \mathrm{Shv}(X;\Lambda)_{\leq 0}$$

of $\mathrm{Shv}(X;\Lambda)$ to the abelian category of étale sheaves of Λ-modules on X.

Proof. See Theorem [24].I.2.1.9. □

Remark 2.2.2.4. Let X be a quasi-projective k-scheme. The ∞-category

$$\mathcal{C} = \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$$

of *all* Mod_Λ-valued presheaves admits a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$, where a presheaf $\mathcal{F}$ belongs to $\mathcal{C}_{\geq 0}$ (respectively $\mathcal{C}_{\leq 0}$) if and only if the homology groups of the chain complex $\mathcal{F}(U)$ are concentrated in degrees ≥ 0 (respectively ≤ 0). With respect to this t-structure, the sheafification functor

$$\mathcal{C} \to \mathrm{Shv}(X;\Lambda) \qquad \mathcal{F} \mapsto \widetilde{\mathcal{F}}$$

of Remark 2.2.1.4 is t-exact. It follows that the inclusion functor $\mathrm{Shv}(X;\Lambda) \hookrightarrow \mathcal{C}$ is left t-exact: in fact, we have $\mathrm{Shv}(X;\Lambda)_{\leq 0} = \mathcal{C}_{\leq 0} \cap \mathrm{Shv}(X;\Lambda)$. Beware that the inclusion functor $\mathrm{Shv}(X;\Lambda) \hookrightarrow \mathcal{C}$ is not t-exact: for example, it does not carry the heart of $\mathrm{Shv}(X;\Lambda)$ to the heart of $\mathcal{C}$ (see Warning 2.2.2.5 below).

Throughout this book, we will use this equivalence of Proposition 2.2.2.3 to identify the abelian category of sheaves of Λ-modules on X with the full subcategory $\mathrm{Shv}(X;\Lambda)^{\heartsuit} \subseteq \mathrm{Shv}(X;\Lambda)$. In particular, if $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$, we will generally identify the sheaves $\pi_n \mathcal{F}$ with the corresponding objects of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$.

Warning 2.2.2.5. Let X be a quasi-projective k-scheme, let Λ be a commutative ring, and let $\mathcal{F}$ be an object of the abelian category $\mathcal{A}$ of étale sheaves of Λ-modules on X. Then there are two *different* ways in which $\mathcal{F}$ can be interpreted as a Mod_{Λ}-valued presheaf on X:

(*a*) One can view $\mathcal{F}$ as a presheaf with values in the abelian category $\mathrm{Mod}_{\Lambda}^{\heartsuit}$ of (discrete) Λ-modules, which determines a functor

$$\mathcal{F}_0 : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_{\Lambda}^{\heartsuit} \subseteq \mathrm{Mod}_{\Lambda}.$$

(*b*) Using the equivalence of abelian categories $\mathcal{A} \simeq \mathrm{Shv}(X;\Lambda)^{\heartsuit}$, one can identify $\mathcal{F}$ with an object

$$\mathcal{F}_1 \in \mathcal{A} \simeq \mathrm{Shv}(X;\Lambda)^{\heartsuit} \subseteq \mathrm{Shv}(X;\Lambda) \subseteq \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_{\Lambda}).$$

The functors $\mathcal{F}_0$ and $\mathcal{F}_1$ are generally *not* the same. By construction, the functor $\mathcal{F}_0$ has the property that for every étale X-scheme U, the chain complex $\mathcal{F}_0(U) \in \mathrm{Mod}_{\Lambda}$ has homology concentrated in degree zero, but the homologies of $\mathcal{F}_1(U)$ are given by the formula

$$\mathrm{H}_n(\mathcal{F}_1(U)) \simeq \mathrm{H}_{\mathrm{et}}^{-n}(U;\mathcal{F}|_U).$$

Note also that $\mathcal{F}_1$ is a Mod_{Λ}-valued sheaf with respect to the étale topology on $\mathrm{Sch}_X^{\mathrm{et}}$ but $\mathcal{F}_0$ is not (in fact, $\mathcal{F}_1$ can be identified with the sheafification of $\mathcal{F}_0$, in the sense of Remark 2.2.1.4).

To any Grothendieck abelian category $\mathcal{A}$, one can associate a stable ∞-category $\mathcal{D}(\mathcal{A})$ called the *(unbounded) derived ∞-category of* $\mathcal{A}$, whose homotopy category is the classical derived category of $\mathcal{A}$; see §[23].1.3.5 for details.

Proposition 2.2.2.6. *Let k be an algebraically closed field, let X be a quasi-projective k-scheme and let Λ be a commutative ring. Then the inclusion* $\mathrm{Shv}(X;\Lambda)^{\heartsuit} \hookrightarrow \mathrm{Shv}(X;\Lambda)$ *extends uniquely to a (t-exact) equivalence of ∞-categories* $\theta : \mathcal{D}(\mathrm{Shv}(X;\Lambda)^{\heartsuit}) \simeq \mathrm{Shv}(X;\Lambda)$*. In particular, the homotopy category of* $\mathrm{Shv}(X;\Lambda)$ *is equivalent to the unbounded derived category of* $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$.

Proof. This follows from the fact that the Grothendieck site $\mathrm{Sch}_X^{\mathrm{et}}$ is an ordinary category (rather than an ∞-category) and that Λ is an ordinary ring (rather than a ring spectrum); see Theorem [24].I.2.1.9. □

2.2.3 Functoriality

Throughout this section, we fix an algebraically closed field k and a commutative ring Λ. We now investigate the dependence of the ∞-category $\mathrm{Shv}(X;\Lambda)$ on the choice of quasi-projective k-scheme X.

Remark 2.2.3.1 (Functoriality)**.** Let $f : X \to Y$ be morphism of quasi-projective k-schemes. Then f determines a base-change functor $\mathrm{Sch}_Y^{\mathrm{et}} \to \mathrm{Sch}_X^{\mathrm{et}}$, given by $U \mapsto U \times_Y X$. Composition with this base-change functor induces a map $\mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(Y;\Lambda)$, which we will denote by f_* and refer to as *pushforward along* f. The functor f_* admits a left adjoint, which we will denote by f^* and refer to as *pullback along* f. If $\mathcal{F} \in \mathrm{Shv}(Y;\Lambda)$, we will sometimes denote the pullback $f^* \mathcal{F}$ by $\mathcal{F}|_X$, particularly in those cases when f exhibits X as a subscheme of Y.

Proposition 2.2.3.2. *Let X be a quasi-projective k-scheme, and let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ vanishes.*

(2) *For every k-valued point $\eta : \mathrm{Spec}(k) \to X$, the stalk $\eta^* \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}(k);\Lambda) \simeq \mathrm{Mod}_\Lambda$ vanishes.*

Proof. The implication (1) ⇒ (2) is trivial. Suppose that $\mathcal{F}$ satisfies (2); we will show that $\mathcal{F} \simeq 0$ by proving that the identity map $\mathrm{id} : \mathcal{F} \to \mathcal{F}$ is nullhomotopic. For this, it will suffice to show that $\mathcal{F}$ is locally acyclic: that is, each of the sheaves of abelian groups $\pi_n \mathcal{F}$ vanishes. We may therefore assume without loss of generality that $\mathcal{F}$ belongs to the heart of $\mathrm{Shv}(X;\Lambda)$. We will abuse notation by identifying $\mathcal{F}$ with the corresponding sheaf of abelian groups on $\mathrm{Sch}_X^{\mathrm{et}}$. Choose an object $U \in \mathrm{Sch}_X^{\mathrm{et}}$ and a section $s \in \mathcal{F}(U)$; we wish to show that $s = 0$. Let $V \subseteq U$ be the largest open subset for which $s|_V = 0$. Suppose for a contradiction that $V \neq U$. Then we can choose a point $\eta_U : \mathrm{Spec}(k) \to U$ which does not factor through V. Let η denote the composition of η_U with the map $U \to X$, so that $\eta^* \mathcal{F} \simeq 0$ by virtue of (2). It follows that the map η_U factors as a composition $\mathrm{Spec}(k) \to \widetilde{U} \to U$, where $s|_{\widetilde{U}} = 0$. We conclude that s vanishes on the open subset of U given by the union of V with the image of $\widetilde{U}$, contradicting the maximality of V. □

Remark 2.2.3.3. Let k be an algebraically closed field, let X be a quasi-projective k-scheme, and let Λ be a commutative ring. Then an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ belongs to $\mathrm{Shv}(X;\Lambda)_{\geq 0}$ if and only if, for every point $\eta : \mathrm{Spec}(k) \to X$, the stalk $\eta^* \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}(k);\Lambda) \simeq \mathrm{Mod}_\Lambda$ belongs to $(\mathrm{Mod}_\Lambda)_{\geq 0}$. Similarly, $\mathcal{F}$ belongs to $\mathrm{Shv}(\mathrm{Spec}(k);\Lambda)_{\leq 0}$ if and only if each stalk $\eta^* \mathcal{F}$ belongs to $(\mathrm{Mod}_\Lambda)_{\leq 0}$.

Construction 2.2.3.4. Let $f : X \to Y$ be an étale morphism between quasi-projective k-schemes. Then composition with f induces a forgetful functor $u : \mathrm{Sch}_X^{\mathrm{et}} \to \mathrm{Sch}_Y^{\mathrm{et}}$. The pullback functor $f^* : \mathrm{Shv}(Y;\Lambda) \to \mathrm{Shv}(X;\Lambda)$ is then given by composition with u. From this description, we immediately deduce that f^* preserves limits and colimits. Using Corollary [25].5.5.2.9, we deduce that f^* admits a left adjoint which we will denote by $f_!$. In the special case where f is an open immersion, we will refer to $f_!$ as the functor of *extension by zero along f*.

Proposition 2.2.3.5. *Suppose we are given a diagram of quasi-projective k-schemes σ :*

$$\begin{array}{ccc} U_X & \xrightarrow{f'} & U_Y \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{f} & Y, \end{array}$$

where j' and j are étale. If σ is a pullback diagram, then the associated diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}(U_X;\Lambda) & \longleftarrow & \mathrm{Shv}(U_Y;\Lambda) \\ \uparrow & & \uparrow \\ \mathrm{Shv}(X;\Lambda) & \longleftarrow & \mathrm{Shv}(Y;\Lambda) \end{array}$$

satisfies the Beck-Chevalley property: that is, the induced natural transformation $j'_! f'^* \to f^* j_!$ is an equivalence of functors from $\mathrm{Shv}(U_Y;\Lambda)$ to $\mathrm{Shv}(X;\Lambda)$ (see §2.4 for a more detailed discussion).

Proof. Passing to right adjoints, we are reduced to proving that the canonical map $j^* f_* \to f'_* j'^*$ is an equivalence. Let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$. Using the descriptions of the pullback and pushforward functors supplied by Remark 2.2.3.1 and Construction 2.2.3.4, we must show that for every object $V \in \mathrm{Sch}_{U_Y}^{\mathrm{et}}$, the restriction map $\mathcal{F}(U_X \times_{U_Y} V) \to \mathcal{F}(X \times_Y V)$ is an equivalence. This is evidently satisfied whenever σ is a pullback square. $\square$

2.2.4 Compact Generation of $\mathrm{Shv}(X;\Lambda)$

Let k be an algebraically closed field and let Λ be a commutative ring, which we regard as fixed throughout this section. Our goal is to show that, for every quasi-projective k-scheme X, the ∞-category $\mathrm{Shv}(X;\Lambda)$ is compactly generated (we will give an explicit description of the compact objects of $\mathrm{Shv}(X;\Lambda)$ in §2.2.6). We begin with the following standard observation:

Lemma 2.2.4.1. *Let k be an algebraically closed field, let X be a quasi-projective k-scheme of Krull dimension d, and let $\mathcal{F}$ be an étale sheaf of abelian groups on X. Then the cohomology groups $\mathrm{H}^n(X;\mathcal{F})$ vanish for $n > 2d+1$.*

Proof. Let $\mathrm{Shv}_{\mathrm{Nis}}(X;\mathbf{Z})$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}})$ spanned by those functors which are sheaves with respect to the Nisnevich topology, and let $\iota : \mathrm{Shv}(X;\mathbf{Z}) \hookrightarrow \mathrm{Shv}_{\mathrm{Nis}}(X;\mathbf{Z})$ denote the inclusion map. Let $\mathcal{F}'$ denote the object of the heart $\mathrm{Shv}(X;\mathbf{Z})^{\heartsuit}$ corresponding to $\mathcal{F}$, so that we have a canonical isomorphism $\mathrm{H}^n(X;\mathcal{F}) \simeq \mathrm{H}_{-n}\,\mathcal{F}'(X)$. Since the ∞-topos of Nisnevich sheaves on X has homotopy dimension $\leq d$ (see [24].B.5), it will suffice to show that $\mathcal{F}'$ belongs to $\mathrm{Shv}_{\mathrm{Nis}}(X;\Lambda)_{\geq -d-1}$. To prove this, it will suffice to show that for every map $\eta : \mathrm{Spec}(R) \to X$ which exhibits R as the Henselization of X with respect to some finite extension of some residue field of X, the cohomology groups $\mathrm{H}^m(\mathrm{Spec}(R);\eta^*\mathcal{F})$ vanish for $m > d+1$. Let κ' denote the residue field of R, and let $\eta_0 : \mathrm{Spec}(\kappa') \to X$ be the restriction of η. Then κ' is an extension of k having trancendence degree $\leq d$, and is therefore a field of cohomological dimension $\leq d$ (see [32]). Since the ring R is Henselian, the canonical map $\mathrm{H}^m(\mathrm{Spec}(R);\eta^*\mathcal{F}) \to \mathrm{H}^m(\mathrm{Spec}(\kappa');\eta_0^*\mathcal{F})$ is an isomorphism so that that $\mathrm{H}^m(\mathrm{Spec}(R);\eta^*\mathcal{F})$ vanishes for $m > d+1$ as desired. □

Proposition 2.2.4.2. *Let X be a quasi-projective k-scheme. Then the full subcategory $\mathrm{Shv}(X;\Lambda) \subseteq \mathrm{Fun}((\mathrm{Sch}_X^{et})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ is closed under colimits.*

Proof. The inclusion $\mathrm{Shv}(X;\Lambda) \hookrightarrow \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ is a left exact functor between stable ∞-categories and therefore preserves finite colimits. It will therefore suffice to show that it preserves filtered colimits. Let $\{\mathcal{F}_\alpha\}$ be a filtered diagram of objects of $\mathrm{Shv}(X;\Lambda)$ having colimit $\mathcal{F}$. We wish to prove that for each $U \in \mathrm{Sch}_X^{\mathrm{et}}$, the canonical map $\varinjlim \mathcal{F}_\alpha(U) \to \mathcal{F}(U)$ is an equivalence. In other words, we want to show that for each integer n, the induced map $\varinjlim \pi_n\,\mathcal{F}_\alpha(U) \to \pi_n\,\mathcal{F}(U)$ is an isomorphism of abelian groups. Shifting if necessary, we may suppose that $n = 0$. Replacing each $\mathcal{F}_\alpha$ by a truncation if necessary, we may suppose that each $\mathcal{F}_\alpha$ belongs to $\mathrm{Shv}(X;\Lambda)_{\geq 0}$. Using Lemma 2.2.4.1, one can show that there exists an integer $N \gg 0$ such that the canonical map $\pi_0\,\mathcal{G}(U) \to \pi_0(\tau_{\leq N}\,\mathcal{G})(U)$ is an isomorphism, for each $\mathcal{G} \in \mathrm{Shv}(X;\Lambda)$. Replacing each $\mathcal{F}_\alpha$ by $\tau_{\leq N}\,\mathcal{F}_\alpha$, we may assume that $\{\mathcal{F}_\alpha\}$ is a diagram in $\mathrm{Shv}(X;\Lambda)_{\leq N}$ for some integer N. The desired result now follows formally from the fact that the Grothendieck topology on $\mathrm{Sch}_X^{\mathrm{et}}$ is *finitary* (that is, every covering admits a finite refinement); see Corollary [24].A.2.18 for more details. □

Corollary 2.2.4.3. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the pushforward functor $f_* : \mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(Y;\Lambda)$ preserves colimits.*

Notation 2.2.4.4. Let X be a quasi-projective k-scheme, so that there is a unique morphism of k-schemes $f : X \to \mathrm{Spec}(k)$. Pullback along f determines a functor

$$\mathrm{Mod}_\Lambda \simeq \mathrm{Shv}(\mathrm{Spec}(k); \Lambda) \xrightarrow{f^*} \mathrm{Shv}(X; \Lambda),$$

which we will denote by $M \mapsto \underline{M}_X$. For each $M \in \mathrm{Mod}_\Lambda$, we will refer to $\underline{M}_X$ as the *constant sheaf on X with value M*. By construction, the functor $M \mapsto \underline{M}_X$ is left adjoint to the global sections functor $\mathcal{F} \mapsto \mathcal{F}(X) \in \mathrm{Mod}_\Lambda$. Equivalently, $\underline{M}_X$ can be described as the sheafification (in the sense of Remark 2.2.1.4) of the constant functor $(\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda$ taking the value M.

Proposition 2.2.4.5. *Let X be a quasi-projective k-scheme. Then the ∞-category $\mathrm{Shv}(X; \Lambda)$ is compactly generated. Moreover, the full subcategory*

$$\mathrm{Shv}^c(X; \Lambda) \subseteq \mathrm{Shv}(X; \Lambda)$$

spanned by the compact objects is the smallest stable subcategory of $\mathrm{Shv}(X; \Lambda)$ which is closed under retracts and contains every object of the form $j_!\underline{\Lambda}_U$, where $j : U \to X$ is an object of the category Sch_X^{et}.

Proof. We first show that for each $j : U \to X$ in $\mathrm{Sch}_X^{\mathrm{et}}$, the sheaf $j_!\underline{\Lambda}_U$ is a compact object of $\mathrm{Shv}(X; \Lambda)$. To prove this, it suffices to show that the functor

$$\mathcal{F} \mapsto \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(j_!\underline{\Lambda}_U, \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Mod}_\Lambda}(\Lambda, \mathcal{F}(U))$$

commutes with filtered colimits, which follows immediately from Proposition 2.2.4.2.

Let $\mathcal{C} \subseteq \mathrm{Shv}(X; \Lambda)$ be the smallest full subcategory which contains every object of the form $j_!\underline{\Lambda}_U$ and is closed under retracts. Since $\mathcal{C}$ consists of compact objects of $\mathrm{Shv}(X; \Lambda)$, the inclusion $\mathcal{C} \hookrightarrow \mathrm{Shv}(X; \Lambda)$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Shv}(X; \Lambda)$ which commutes with filtered colimits (Proposition [25].5.3.5.10). Moreover, since $\mathcal{C}$ is closed under retracts, we can identify $\mathcal{C}$ with the full subcategory of $\mathrm{Ind}(\mathcal{C})$ spanned by the compact objects. To complete the proof that $\mathrm{Shv}(X; \Lambda)$ is a compactly generated ∞-category and that $\mathcal{C}$ is the ∞-category of compact objects of $\mathrm{Shv}(X; \Lambda)$, it will suffice to show that F is an equivalence of ∞-categories. Using Corollary [25].5.5.2.9, we deduce that F has a right adjoint G. We wish to show that F and G are mutually inverse equivalences. Since F is fully faithful, it will suffice to show that G is conservative. Since G is an exact functor between stable ∞-categories, it will suffice to show that if $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ satisfies $G(\mathcal{F}) \simeq 0$, then $\mathcal{F} \simeq 0$. This is clear, since $G(\mathcal{F}) \simeq 0$ implies that

$$\pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^n j_!\underline{\Lambda}_U, G(\mathcal{F})) \simeq \pi_0 \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\Sigma^n j_!\underline{\Lambda}, \mathcal{F}) \simeq \mathrm{H}_n(\mathcal{F}(U))$$

vanishes for each $U \in \mathrm{Sch}_X^{\mathrm{et}}$. □

2.2.5 The Exceptional Inverse Image

Let k be an algebraically closed field and let Λ be a commutative ring. If $f : X \to Y$ is any morphism between quasi-projective k-schemes, then Corollary 2.2.4.3 guarantees that the direct image functor $f_* : \mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(Y;\Lambda)$ preserves small colimits. It follows from the adjoint functor theorem (see Corollary [25].5.5.2.9) that the functor f_* admits a right adjoint.

Notation 2.2.5.1. If $f : X \to Y$ is a *proper* morphism between quasi-projective k-schemes, then we let $f^! : \mathrm{Shv}(Y;\Lambda) \to \mathrm{Shv}(X;\Lambda)$ denote a right adjoint to the direct image functor $f_* : \mathrm{Shv}(X : \Lambda) \to \mathrm{Shv}(Y;\Lambda)$. We will refer to $f^!$ as the *exceptional inverse image functor*.

Warning 2.2.5.2. When the coefficient ring Λ is finite, one can extend the definition of the inverse image functor $f^!$ to the case where $f : X \to Y$ is an *arbitrary* morphism of quasi-projective k-schemes. However, the functor $f^!$ is defined as the right adjoint to the compactly supported direct image functor $f_!$, rather than the usual direct image functor f_*. Since we do not wish to address the homotopy coherence issues which arise in setting up an "enhanced" six-functor formalism, we will not consider this additional generality: that is, we consider the functor $f^!$ as defined only when f is proper, and the functor $f_!$ as defined only when f is étale (a special case of the relationship between $f^!$ and $f_!$ is articulated in Example 2.4.4.6). For our applications, we will primarily be interested in the functor $f^!$ in the special case where f is a closed immersion.

Example 2.2.5.3. Let n be a positive integer which is invertible in k and let $\Lambda = \mathbf{Z}/n\mathbf{Z}$. For every integer d, we let $\Lambda(d)$ denote the free Λ-module of rank 1 which is given by the dth tensor power of $\mu_n(k) = \{x \in k : x^n = 1\}$. If X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$, we let $\mathcal{F}(d)$ denote the object of $\mathrm{Shv}(X;\Lambda)$ given by the formula $\mathcal{F}(d)(U) = \mathcal{F}(U) \otimes_\Lambda \Lambda(d)$ (so that $\mathcal{F}(d)$ is noncanonically isomorphic to $\mathcal{F}$). We will refer to $\mathcal{F}(d)$ as the *d-fold Tate twist of $\mathcal{F}$*.

If $f : X \to Y$ is a proper smooth morphism of relative dimension d, then the main result of [37] supplies an equivalence

$$f^! \mathcal{F} \simeq \Sigma^{2d} f^* \mathcal{F}(d).$$

Remark 2.2.5.4. Let X be a quasi-projective k-scheme which is smooth of dimension d, let n be a positive integer which is invertible in k, and let $\eta : \mathrm{Spec}(k) \to X$ be a point of X. Then there is an equivalence $\eta^! \underline{\mathbf{Z}/n\mathbf{Z}} \simeq \Sigma^{-2d} \underline{\mathbf{Z}/n\mathbf{Z}}(-d)$. To prove this, we can work locally with respect to the étale topology on X, and thereby reduce to the case where $X = \mathbf{P}^n$ so that there exists a proper morphism $\pi : X \to \mathrm{Spec}(k)$. In this case, Example 2.2.5.3 supplies an equivalence

$$\begin{aligned} \eta^! \underline{\mathbf{Z}/n\mathbf{Z}} &\simeq \eta^!(\pi^! \Sigma^{-2d} \underline{\mathbf{Z}/n\mathbf{Z}}(-d)) \\ &\simeq (\pi \circ \eta)^! \Sigma^{-2d} \underline{\mathbf{Z}/n\mathbf{Z}}(-d) \\ &\simeq \Sigma^{-2d} \underline{\mathbf{Z}/n\mathbf{Z}}(-d). \end{aligned}$$

Remark 2.2.5.5. Let $i : Y \to X$ be a closed immersion of quasi-projective k-schemes, set $U = X - Y$, and let $j : U \to X$ be the complementary open immersion. Then the pushforward functor $i_* : \mathrm{Shv}(Y; \Lambda) \to \mathrm{Shv}(X; \Lambda)$ is a fully faithful embedding, whose essential image is the full subcategory of $\mathrm{Shv}(X; \Lambda)$ spanned by those objects $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ such that $j^* \mathcal{F} \simeq 0$ (see Proposition [24].I.3.1.18).

Let $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$. Then the fiber $\mathcal{K}$ of the canonical map $\mathcal{F} \to j_* j^* \mathcal{F}$ satisfies $j^* \mathcal{K} \simeq 0$, so we can write $\mathcal{K} \simeq i_* \mathcal{K}_0$ for some $\mathcal{K}_0 \in \mathrm{Shv}(Y; \Lambda)$. For each $\mathcal{G} \in \mathrm{Shv}(Y; \Lambda)$, we have canonical homotopy equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Shv}(Y;\Lambda)}(\mathcal{G}, \mathcal{K}_0) &\simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, i_* \mathcal{K}_0) \\ &\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, j_* j^* \mathcal{F})) \\ &\simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, \mathcal{F}) \to \mathrm{Map}_{\mathrm{Shv}(U;\Lambda)}(j^* i_* \mathcal{G}, j^* \mathcal{F})) \\ &\simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(i_* \mathcal{G}, \mathcal{F}). \end{aligned}$$

so that $\mathcal{K}_0$ can be identified with the sheaf $i^! \mathcal{F}$. In other words, we have a canonical fiber sequence $i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$. Using similar reasoning, we obtain a canonical fiber sequence $j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}$.

Remark 2.2.5.6. If $i : X \to Y$ is a closed immersion of quasi-projective k-schemes, then Remark 2.2.5.5 gives an explicit construction of the functor $i^!$ (which does not depend on Corollary 2.2.4.3): namely, for each object $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$, we can identify $i^!$ with a preimage (under the functor i_*) of the fiber of the unit map $\mathcal{F} \to j_* j^* \mathcal{F}$.

Proposition 2.2.5.7. *Let $i : Y \hookrightarrow X$ be a closed immersion of quasi-projective k-schemes. Then:*

(1) *The functor $i^!$ preserves filtered colimits.*

(2) *The functor i_* preserves compact objects.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition [25].5.5.7.2. We will prove (1). Since the functor i_* is a fully faithful embedding which preserves colimits (Corollary 2.2.4.3), it will suffice to show that the composite functor $\mathcal{F} \mapsto i_* i^! \mathcal{F}$ preserves filtered colimits. Using the existence of a fiber sequence $i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$, we are reduced to proving that the functor $\mathcal{F} \mapsto j_* j^* \mathcal{F}$ preserves filtered colimits, which follows from Corollary 2.2.4.3. $\square$

Proposition 2.2.5.8. *Let X be a quasi-projective k-scheme. Then there exists an integer n with the following property: for every closed immersion $i : Y \hookrightarrow X$ and every commutative ring Λ, the functor $i^!$ carries $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ into $\mathrm{Shv}(Y;\Lambda)_{\geq n}$.*

Proof. Let d be the Krull dimension of X. We will prove that $n = -2d$ has the desired property. Let $i : Y \hookrightarrow X$ be a closed immersion, and let $j : U \to X$ be the complementary open immersion. To prove that the functor $i^!$ carries $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ into $\mathrm{Shv}(Y;\Lambda)_{\geq n}$, it will suffice to show that the composite functor $i_* i^!$ carries $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$ to $\mathrm{Shv}(X;\Lambda)_{\geq n}$. Using the fiber sequence of functors

$$\Sigma j_* j^* \to i_* i^! \to \mathrm{id},$$

we are reduced to proving that the functor j_* carries $\mathrm{Shv}(U;\Lambda)^{\heartsuit}$ into $\mathrm{Shv}(X;\Lambda)_{\geq n-1}$. Let $\mathcal{F} \in \mathrm{Shv}(U;\Lambda)^{\heartsuit}$. We will prove that $j_* \mathcal{F} \in \mathrm{Shv}(X;\Lambda)_{\geq n-1}$ by proving that $(j_* \mathcal{F})(V) \in (\mathrm{Mod}_{\Lambda})_{\geq n-1}$ for every étale map $V \to X$. Equivalently, we must show that the cohomology groups $\mathrm{H}^i(U \times_X V; \mathcal{F}|_{U \times_X V})$ vanish for $i > 2d+1$, which follows from Lemma 2.2.4.1. □

2.2.6 Constructible Sheaves

Let k be an algebraically closed field and let Λ be a commutative ring. In §2.2.4, we proved that the ∞-category $\mathrm{Shv}(X;\Lambda)$ is compactly generated, for every quasi-projective k-scheme X (Proposition 2.2.4.5). In this section, we will show that the compact objects of $\mathrm{Shv}(X;\Lambda)$ can be identified with the (perfect) constructible complexes on X (Proposition 2.2.6.2).

Definition 2.2.6.1. Let X be a quasi-projective k-scheme. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is *constant* if it is equivalent to $\underline{M}_X$, for some $M \in \mathrm{Mod}_{\Lambda}$. We will say that $\mathcal{F}$ is *locally constant* if there is an étale covering $\{f_\alpha : U_\alpha \to X\}$ for which each pullback $f_\alpha^* \mathcal{F} \in \mathrm{Shv}(U_\alpha;\Lambda)$ is constant.

Proposition 2.2.6.2. *Let X be a quasi-projective k-scheme. Then an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is compact if and only if the following conditions are satisfied:*

(1) *There exists a finite sequence of quasi-compact open subsets*

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$$

such that, for $1 \leq i \leq n$, if Y_i denotes the locally closed reduced subscheme of X with support $U_i - U_{i-1}$, then each restriction $\mathcal{F}|_{Y_i}$ is locally constant.

(2) *For every k-valued point $\eta : \mathrm{Spec}(k) \to X$, the stalk $\eta^* \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}(k);\Lambda) \simeq \mathrm{Mod}_{\Lambda}$ is perfect (that is, it is a compact object of Mod_{Λ}).*

Definition 2.2.6.3. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is *constructible* if it satisfies conditions (1) and (2) of Proposition 2.2.6.2 (equivalently, if it is a compact object of $\mathrm{Shv}(X;\Lambda)$). We let $\mathrm{Shv}^c(X;\Lambda)$ denote the full subcategory of $\mathrm{Shv}(X;\Lambda)$ spanned by the constructible objects.

Warning 2.2.6.4. Some authors use the term *constructible* to refer to sheaves which are required to satisfy some weaker version of condition (2), such as the finiteness of the graded abelian group $\mathrm{H}_*(\eta^* \mathcal{F})$ for each point $\eta : \mathrm{Spec}(k) \to X$ (when $\Lambda = \mathbf{Z}$ or Λ is finite).

Proof of Proposition 2.2.6.2. We begin by showing that every compact object $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ satisfies conditions (1) and (2). Using Proposition 2.2.4.5, we may reduce to the case where $\mathcal{F} = j_!\underline{\Lambda}_U$ for some étale map $j : U \to X$. We first show that $\mathcal{F}$ satisfies (1). We may assume that $X \neq \emptyset$, otherwise the result is vacuous. Using Noetherian induction on X (and Proposition 2.2.3.5), we may suppose that the restriction $\mathcal{F}|_Y$ satisfies (1) for every nonempty closed subscheme $Y \subseteq X$. It will therefore suffice to show that $\mathcal{F}|_V$ satisfies (1) for some nonempty open subscheme $V \subseteq X$. Passing to an open subscheme, we may suppose that $j : U \to X$ is finite étale of some fixed rank r. In this case, we claim that $j_!\underline{\Lambda}$ is locally constant. Choose a finite étale surjection $\widetilde{X} \to X$ such that the fiber product $U \times_X \widetilde{X}$ is isomorphic to a disjoint union of r copies of $\widetilde{X}$. Using Proposition 2.2.3.5, we may replace X by $\widetilde{X}$. In this case, the sheaf $j_!\underline{\Lambda}_U \simeq \underline{\Lambda}_X^r$ is constant.

We now show that for every étale map $j : U \to X$, the sheaf $j_!\underline{\Lambda}_U$ satisfies condition (2). Using Proposition 2.2.3.5, we may replace X by $\mathrm{Spec}(k)$ and thereby reduce to the case where X is the spectrum of an algebraically closed field. In this case, U is a disjoint union of finitely many copies of X, so that $j_!\underline{\Lambda}_U$ can be identified with a free module Λ^r as an object of $\mathrm{Shv}(X;\Lambda) \simeq \mathrm{Mod}_\Lambda$.

Now suppose that $\mathcal{F}$ is a sheaf satisfying conditions (1) and (2); we wish to show that $\mathcal{F}$ is a compact object of $\mathrm{Shv}(X;\Lambda)$. Without loss of generality we may suppose that X is nonempty. Using Noetherian induction on X, we may assume that for every closed immersion $i : Y \to X$ whose image is a proper closed subset of X, the pullback $i^* \mathcal{F}$ is a compact object of $\mathrm{Shv}(Y;\Lambda)$. Using Proposition 2.2.5.7 we deduce that $i_* i^* \mathcal{F}$ is a compact object of $\mathrm{Shv}(Y;\Lambda)$. Let $j : U \to X$ denote the complementary open immersion, so that we have a fiber sequence

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

It will therefore suffice to show that there exists a nonempty open subset $U \subseteq X$ such that $j_! j^* \mathcal{F}$ is a compact object of $\mathrm{Shv}(X;\Lambda)$. Since the functor j^* preserves colimits, $j_!$ preserves compact objects; it will therefore suffice to show that we can choose U such that $j^* \mathcal{F}$ is a compact object of $\mathrm{Shv}(U;\Lambda)$. Since $\mathcal{F}$ satisfies (1), we may pass to a nonempty open subscheme of X and thereby reduce to the case where $\mathcal{F}$ is locally constant.

For each object $\mathcal{G} \in \mathrm{Shv}(X; \Lambda)$, let $\underline{\mathrm{Map}}_{\Lambda}(\mathcal{F}, \mathcal{G})$ denote the sheaf of (Λ-linear) maps from $\mathcal{F}$ to $\mathcal{G}$, so that we have a canonical equivalence

$$\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\underline{\Lambda}_X, \underline{\mathrm{Map}}_{\Lambda}(\mathcal{F}, \mathcal{G})).$$

It follows from Proposition 2.2.4.2 that $\underline{\Lambda}_X$ is a compact object of $\mathrm{Shv}(X; \Lambda)$. Consequently, to show that $\mathcal{F}$ is a compact object of $\mathrm{Shv}(X; \Lambda)$, it will suffice to show that the functor $\mathcal{G} \mapsto \underline{\mathrm{Map}}_{\Lambda}(\mathcal{F}, \mathcal{G})$ commutes with filtered colimits. This assertion can be tested locally with respect to the étale topology on X. We may therefore assume without loss of generality that the sheaf $\mathcal{F} = \underline{M}_X$ is constant. In this case, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Mod}_{\Lambda}}(M, \mathcal{G}(X)).$$

Since M is a compact object of Mod_{Λ} and the functor $\mathcal{G} \mapsto \mathcal{G}(X)$ commutes with filtered colimits (Proposition 2.2.4.2), we conclude that $\mathcal{F}$ is a compact object of $\mathrm{Shv}(X; \Lambda)$, as desired. □

Remark 2.2.6.5 (Extension by Zero)**.** Let $i : X \to Y$ be a locally closed immersion between quasi-projective k-schemes. Then i factors as a composition $X \xrightarrow{i'} \overline{X} \xrightarrow{i''} Y$ where $\overline{X}$ denotes the scheme-theoretic closure of X in Y, i'' is a closed immersion, and i' is an open immersion. We let $i_!$ denote the composite functor

$$\mathrm{Shv}(X; \Lambda) \xrightarrow{i'_!} \mathrm{Shv}(\overline{X}; \Lambda) \xrightarrow{i''_*} \mathrm{Shv}(Y; \Lambda),$$

which we will refer to as the functor of *extension by zero* from X to Y.

Remark 2.2.6.6. It follows from Proposition 2.2.6.2 that for every compact object $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$, there exists a finite stratification of X by locally closed subschemes Y_α and a finite filtration of $\mathcal{F}$ whose successive quotients have the form $i_{\alpha!}\mathcal{F}_\alpha$, where $\mathcal{F}_\alpha \in \mathrm{Shv}(Y_\alpha; \Lambda)$ is a locally constant sheaf with perfect stalks, and $i_\alpha : Y_\alpha \to X$ denotes the inclusion map.

2.2.7 Sheaves of Vector Spaces

Let k be an algebraically closed field. In this section, we summarize some special features enjoyed by the ∞-categories $\mathrm{Shv}(X; \Lambda)$ in the case where the commutative ring Λ is a field.

Remark 2.2.7.1. The results of this section are valid more generally if Λ is a commutative ring of finite projective dimension, such as the ring $\mathbf{Z}$ of integers. However, we are primarily interested in the case where $\Lambda = \mathbf{Z}/\ell\mathbf{Z}$ for some prime number ℓ which is invertible in k.

Proposition 2.2.7.2. *Let X be a quasi-projective k-scheme and let Λ be a field. If $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)$ is compact, then each truncation $\tau_{\geq n}\mathcal{F}$ and $\tau_{\leq n}\mathcal{F}$ is also a compact object of* $\mathrm{Shv}(X;\Lambda)$.

Proof. Using Proposition 2.2.6.2, we can reduce to the case where $\mathcal{F} = \underline{M}_X$, where M is a perfect object of Mod_Λ. We now observe that our assumption that Λ is a field guarantees that the truncations $\tau_{\geq n}M$ and $\tau_{\leq n}M$ are also perfect objects of Mod_Λ. □

Proposition 2.2.7.3. *Let X be a quasi-projective k-scheme, and let Λ be a field. Then there exists an integer n with the following property: for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X;\Lambda)^{\heartsuit}$, if $\mathcal{F}$ is compact object of $\mathrm{Shv}(X;\Lambda)$, then $\mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{G}) \simeq 0$ for $m > n$.*

Proof. Using Proposition 2.2.5.8, we can choose an integer n' such that, for every closed immersion $i : Y \hookrightarrow X$, the sheaf $i^!\,\mathcal{G}$ belongs to $\mathrm{Shv}(Y;\Lambda)_{\geq n'}$. Let d be the Krull dimension of X. We will prove that $n = n' + 2d + 1$ has the desired property. For this, it will suffice to prove the following:

$(*)$ Let $\mathcal{H} \in \mathrm{Shv}(X;\Lambda)$ have the property that $i^!\,\mathcal{H} \in \mathrm{Shv}(Y;\Lambda)_{\geq n'}$ for every closed immersion $i : Y \to X$. Then $\mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{H}) \simeq 0$ for $m > n$.

We prove $(*)$ using Noetherian induction on X. Using Proposition 2.2.6.2, we can choose an open immersion $j : U \to X$ such that $j^*\mathcal{F}$ is locally constant, hence a dualizable object of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$. Let $i : Y \hookrightarrow X$ be a complementary closed immersion, so that we have a fiber sequence $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F}$. We therefore obtain an exact sequence

$$\mathrm{Ext}^m_{\mathrm{Shv}(U;\Lambda)}(j^*\mathcal{F}, j^*\mathcal{H}) \to \mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{H}) \to \mathrm{Ext}^m_{\mathrm{Shv}(Y;\Lambda)}(i^*\mathcal{F}, i^!\,\mathcal{H}).$$

The first group can be identified with $\mathrm{H}^m(U; (j^*\mathcal{F})^\vee \otimes_\Lambda j^*\mathcal{H}))$, which vanishes for $m > n$ by virtue of Lemma 2.2.4.1. The third group vanishes for $m > n$ by the inductive hypothesis, so that $\mathrm{Ext}^m_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{H})$ also vanishes for $m > n$. □

Proposition 2.2.7.4. *Let X be a quasi-projective k-scheme, let Λ be a field, and let $\mathcal{F}$ be an object of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$. If $\mathcal{F}$ is constructible, then $\mathcal{F}$ is a Noetherian object of the abelian category $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$.*

Proof. Proceeding by Noetherian induction, we may suppose that for each proper closed subscheme $Y \subsetneq X$, each constructible object $\mathcal{G} \in \mathrm{Shv}(Y;\Lambda)^{\heartsuit}$ is Noetherian. We will prove the following:

($*_n$) Let $\mathcal{F} \in \mathrm{Shv}(X;\Lambda)^{\heartsuit}$ be constructible. Suppose there exists a nonempty connected open subset $U \subseteq X$ containing a point x such that $\mathcal{F}|_U$ is locally constant and the stalk $\mathcal{F}_x$ has dimension $\leq n$ (when regarded as a vector space over Λ). Then $\mathcal{F}$ is a Noetherian object of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$.

The proof proceeds by induction on n. Let U and $\mathcal{F}$ satisfy the hypotheses of $(*_n)$. We will abuse notation by identifying $\mathcal{F}$ with a sheaf of Λ-vector spaces on X. Suppose we are given an ascending chain of subobjects $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \cdots$ of $\mathcal{F}$; we wish to show that it is eventually constant. If each restriction $\mathcal{F}_m|_U$ vanishes, then we have $\mathcal{F}_m \simeq i_* i^* \mathcal{F}_m$. We are therefore reduced to proving that the sequence of inclusions

$$i^* \mathcal{F}_0 \subseteq i^* \mathcal{F}_1 \subseteq i^* \mathcal{F}_2 \subseteq \cdots$$

stabilizes, which follows from our inductive hypothesis. We may therefore assume that some $\mathcal{F}_m|_U \neq 0$ for some integer m. Using Proposition 2.2.4.5 we can write $\mathcal{F}_m$ as the colimit of a filtered diagram $\{\mathcal{F}_\alpha\}$ of constructible objects of $\mathrm{Shv}(X;\Lambda)$. Using Proposition 2.2.7.2, we can assume that each $\mathcal{F}_\alpha$ belongs to $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$. Choose an index α for which the map $\mathcal{F}_\alpha|_U \to \mathcal{F}_m|_U$ is nonzero. Using Proposition 2.2.6.2, we can choose a nonempty open subset $U' \subseteq U$ such that $\mathcal{F}_\alpha|_{U'}$ is locally constant. Choose an étale U-scheme V such that the map $\mathcal{F}_\alpha(V) \to \mathcal{F}_m(V) \subseteq \mathcal{F}(V)$ is nonzero (as a map of vector spaces over Λ), and let $V' = U' \times_U V$. Then V' is dense in V, so that the map $\mathcal{F}(V) \to \mathcal{F}(V')$ is injective. Using the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F}_\alpha(V) & \longrightarrow & \mathcal{F}_\alpha(V') \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V'), \end{array}$$

we deduce that the map of vector spaces $\mathcal{F}_\alpha(V') \to \mathcal{F}(V')$ is nonzero, so that the map of sheaves $\mathcal{F}_\alpha|_{U'} \to \mathcal{F}|_{U'}$ is nonzero. Replacing U by U', we may reduce to the case where $\mathcal{F}_\alpha|_U$ is locally constant. Note that the cofiber of the map $\mathcal{F}_\alpha \to \mathcal{F}$ is constructible, so that (by virtue of Proposition 2.2.7.2) the cokernel $\mathcal{G} = \mathrm{coker}(\mathcal{F}_\alpha \to \mathcal{F})$ is a constructible object of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$. For any point $x \in U$, we have $\dim \mathcal{G}_x < \dim \mathcal{F}_x$, so that our inductive hypothesis implies that $\mathcal{G}$ is a Noetherian object of $\mathrm{Shv}(X;\Lambda)^{\heartsuit}$. The sheaf $\mathcal{F}/\mathcal{F}_m$ is a quotient of $\mathcal{G}$, and therefore also Noetherian. It follows that the sequence of subobjects $\{\mathcal{F}_{m'}/\mathcal{F}_m \subseteq \mathcal{F}/\mathcal{F}_m\}_{m' \geq m}$ is eventually constant, so that the sequence $\{\mathcal{F}_{m'} \subseteq \mathcal{F}\}_{m' \geq m}$ is eventually constant. □

2.2.8 Extension of Scalars

Let k be an algebraically closed field and let X be a quasi-projective k-scheme. We now study the dependence of the ∞-category $\mathrm{Shv}(X;\Lambda)$ on the commutative ring Λ. Note that every ring homomorphism $f : \Lambda \to \Lambda'$ determines a forgetful functor $\mathrm{Mod}_{\Lambda'} \to \mathrm{Mod}_{\Lambda}$. This functor preserves inverse limits, and therefore induces a functor $\mathrm{Shv}(X;\Lambda') \to \mathrm{Shv}(X;\Lambda)$. We will refer to this functor as *restriction of scalars along f*.

Proposition 2.2.8.1. *Let X be a quasi-projective k-scheme and let $f : \Lambda \to \Lambda'$ be a morphism of commutative rings. Then the restriction of scalars functor $\mathrm{Shv}(X;\Lambda') \to \mathrm{Shv}(X;\Lambda)$ admits a left adjoint.*

Proof. This is a formal consequence of the adjoint functor theorem (see Corollary [25].5.5.2.9). For a more explicit description of the left adjoint functor, see Remark 2.2.8.3 below. □

Construction 2.2.8.2. In the situation of Proposition 2.2.8.1, we will denote the left adjoint to the restriction of scalars functor by

$$(\mathcal{F} \in \mathrm{Shv}(X;\Lambda)) \mapsto (\Lambda' \otimes_{\Lambda} \mathcal{F} \in \mathrm{Shv}(X;\Lambda')),$$

and refer to it as the functor of *extension of scalars along f*.

Remark 2.2.8.3. In the situation of Construction 2.2.8.2, the extension of scalars functor is given more explicitly by the formula

$$(\Lambda' \otimes_{\Lambda} \mathcal{F})(U) = \Lambda' \otimes_{\Lambda} \mathcal{F}(U),$$

where the expression on the right hand side indicates the *left derived* tensor product of $\mathcal{F}(U)$ (which we regard as a chain complex of Λ-modules) with Λ' (in other words, it denotes the image of $\mathcal{F}(U)$ under the left adjoint to the forgetful functor $\mathrm{Mod}_{\Lambda'} \to \mathrm{Mod}_{\Lambda}$). Note that the presheaf $U \mapsto \Lambda' \otimes_{\Lambda} \mathcal{F}(U)$ belongs to $\mathrm{Shv}_{\Lambda'}(X)$: this follows from Proposition 2.2.4.2, since we can write Λ' as a filtered colimit of perfect Λ-modules.

Proposition 2.2.8.4. *Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for $d \geq 1$. Then $\mathcal{F}$ is constructible if and only if the object $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})$ is constructible.*

Proof. It follows from Proposition 2.2.4.2 that the forgetful functor $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})$ preserves colimits and therefore the left adjoint $\mathcal{F} \mapsto (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F}$ preserves compact objects; this proves the "only if" direction. For the converse, suppose that $\mathcal{F} \in \mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})$ has the property that $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})$ is constructible. Then the functor

$$\begin{aligned} \mathcal{G} &\mapsto \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}((\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{F}, (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{G}) \\ &\simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{G}) \end{aligned}$$

preserves filtered colimits. It then follows by induction that for $1 \leq i \leq d$, the functor

$$\mathcal{G} \mapsto \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, (\mathbf{Z}/\ell^i\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^d\mathbf{Z}} \mathcal{G})$$

preserves filtered colimits. Taking $i = d$, we deduce that $\mathcal{F}$ is a compact object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$, hence constructible. □

2.2.9 Stability Properties of Constructible Sheaves

Let k be an algebraically closed field. The theory of étale sheaves is particularly well-behaved when the coefficient ring Λ has the form $\mathbf{Z}/\ell^d\mathbf{Z}$, where ℓ is a prime number which is invertible in k. We now recall some of the special features of this situation, which will play an important role in our discussion of ℓ-adic sheaves in §2.3.

Proposition 2.2.9.1 (Persistence of Constructibility)**.** *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes, let ℓ be a prime number which is invertible in k, and let $d \geq 0$. Then:*

(1) *The pushforward functor $f_* : \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ carries the ∞-category $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ into $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$.*

(2) *The pullback functor $f^* : \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ carries the ∞-category $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ into $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$.*

(3) *If f is proper, then the exceptional inverse image functor*

$$f^! : \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$$

carries the ∞-category $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ into $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$.

(4) *If f is étale, then the functor*

$$f_! : \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$$

carries the ∞-category $\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ into $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$.

Remark 2.2.9.2. Assertions (2) and (4) of Proposition 2.2.9.1 follow immediately from the fact that the functors f_* and f^* preserve filtered colimits (and remain valid when $\mathbf{Z}/\ell^d\mathbf{Z}$ is replaced by an arbitrary commutative ring).

Proof of Proposition 2.2.9.1. By virtue of Proposition 2.2.8.4, we can assume without loss of generality that $d = 1$. In this case, the desired result is proved as Corollaire 1.5 ("Théorème de finitude") on page 234 of [10] (note that our definition of constructibility is different from the notion of constructibility considered in [10], but the two notions agree in the case $d = 1$; see Warning 2.2.6.4). □

Remark 2.2.9.3. Let X be a quasi-projective k-scheme, let ℓ be a prime number which is invertible in k, and let $\mathcal{F}$ be a compact object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. Using Propositions 2.2.9.1 and 2.2.6.2, we see that that there exists a finite collection of locally closed immersions $i_\alpha : Y_\alpha \hookrightarrow X$ (having disjoint images) such that $\mathcal{F}$ admits a filtration with successive quotients of the form $i_{\alpha *} \mathcal{F}_\alpha$, where each $\mathcal{F}_\alpha$ is a locally constant sheaf on Y_α with perfect stalks (compare with Remark 2.2.6.6).

Corollary 2.2.9.4. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes, let ℓ be a prime number which is invertible in k, and let $d \geq 0$. Then the functor $f^! : \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ preserves filtered colimits.*

Proof. This is a reformulation of assertion (1) of Proposition 2.2.9.1. □

Corollary 2.2.9.5. *Let X be a quasi-projective k-scheme, let ℓ be a prime number which is invertible in k, and let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for some integer $d \geq 0$. Then the groups $\mathrm{Ext}^i_{\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, \mathcal{G})$ are finite.*

Proof. Using Proposition 2.2.4.5, we may reduce to the case where $\mathcal{F} = j_! \underline{\mathbf{Z}/\ell^d\mathbf{Z}}_U$ for some étale morphism $j : U \to X$. In this case, we have

$$\begin{aligned} \mathrm{Ext}^i_{\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}, \mathcal{G}) &\simeq \mathrm{Ext}^i_{\mathrm{Shv}(U;\mathbf{Z}/\ell^d\mathbf{Z})}(\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_U, j^* \mathcal{G}) \\ &\simeq \mathrm{H}^i(\pi_* j^* \mathcal{G}), \end{aligned}$$

where $\pi : U \to \mathrm{Spec}(k)$ denotes the projection map. The desired result now follows from Proposition 2.2.9.1. □

Proposition 2.2.9.6. *Let X be a quasi-projective k-scheme, let ℓ be a prime number which is invertible in k, and let $\mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for some $d \geq 0$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ vanishes.*

(2) *For every point $\eta : \mathrm{Spec}(k) \to X$, the stalk $\mathcal{F}_\eta = \eta^* \mathcal{F}$ vanishes.*

(3) *For every point $\eta : \mathrm{Spec}(k) \to X$, the costalk $\eta^! \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}(k); \mathbf{Z}/\ell^d\mathbf{Z})$ vanishes.*

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious, and the implication (2) $\Rightarrow$ (1) is Proposition 2.2.3.2 (which is valid for any coefficient ring Λ). Assume that $\mathcal{F}$ satisfies (3); we will prove that $\mathcal{F} \simeq 0$ using Noetherian induction on X. Using Proposition 2.2.6.2, we can choose a nonempty open subset U such that $\mathcal{F}|_U$ is locally constant. Shrinking U if necessary, we may suppose that U is smooth of dimension $n \geq 0$. Let $i : Y \to X$ be a closed immersion complementary to U. Then $i^! \mathcal{F} \in$

$\mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$ satisfies condition (3), so that $i^! \mathcal{F} \simeq 0$ by the inductive hypothesis. We may therefore replace X by U, and thereby reduce to the case where $\mathcal{F}$ is locally constant. The assertion that $\mathcal{F}$ vanishes is local on X; we may therefore suppose further that X is smooth and $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ has the form $\underline{M}_X$ for some perfect object $M \in \mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$. Arguing as in Remark 2.2.5.4 (and choosing a primitive ℓ^dth root of unity in k), we see that for any point $\eta : \mathrm{Spec}(k) \to X$, the pullback $\eta^! \mathcal{F}$ is equivalent to $\Sigma^{-2n} M$. It then follows from (3) that $M \simeq 0$, so that $\mathcal{F} \simeq 0$ as desired. □

2.3 ℓ-adic Sheaves

Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Let X be a quasi-projective k-scheme. For every commutative ring Λ, the theory outlined in §2.2 associates an ∞-category $\mathrm{Shv}(X; \Lambda)$ of (hypercomplete) Mod_Λ-valued étale sheaves on X. This theory is very well-behaved when the commutative ring Λ has the form $\mathbf{Z}/\ell^d\mathbf{Z}$ for some $d \geq 0$, but badly behaved when $\Lambda = \mathbf{Z}$ or $\Lambda = \mathbf{Q}$. To remedy the situation, it is convenient to introduce the formalism of *ℓ-adic sheaves*: roughly speaking, a (constructible) ℓ-adic sheaf on X is a compatible system $\{\mathcal{F}_d\}_{d \geq 0}$, where each $\mathcal{F}_d$ is a (constructible) object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. The collection of ℓ-adic sheaves on X can be organized into an ∞-category which we will denote by $\mathrm{Shv}_\ell(X)$. Our goal in this section is to review the definition of the ∞-categories $\mathrm{Shv}_\ell(X)$ and summarize some of the properties which we will need later in this book.

2.3.1 ℓ-Completeness

We begin by reviewing some homological algebra.

Definition 2.3.1.1. Let Λ be a commutative ring, and let M be an object of Mod_Λ. We will say that M is *ℓ-complete* if the limit of the diagram

$$\cdots \to M \xrightarrow{\ell} M \xrightarrow{\ell} M$$

vanishes in the ∞-category Mod_Λ.

Remark 2.3.1.2. In the situation of Definition 2.3.1.1, let $\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} M$ denote the cofiber of the map $\ell^d : M \to M$. We then have a tower of fiber sequences

$$\{M \xrightarrow{\ell^d} M \to \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} M\}_{d \geq 0}.$$

Passing to the limit, we see that M is ℓ-complete if and only if the canonical map

$$M \to \varprojlim((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} M)$$

is an equivalence in the ∞-category Mod_Λ.

Remark 2.3.1.3. Let Λ be a commutative ring. Then an object $M \in \mathrm{Mod}_\Lambda$ is ℓ-complete if and only if each homology group $\mathrm{H}_n(M)$ is ℓ-complete, when regarded as a discrete object of Mod_Λ. To prove this, we may assume without loss of generality that $\Lambda = \mathbf{Z}$, in which case M is noncanonically equivalent to the product $\prod_{n \in \mathbf{Z}} \Sigma^n \mathrm{H}_n(M)$.

If Λ is Noetherian and each homology group $\mathrm{H}_n(M)$ is finitely generated as a Λ-module, then M is ℓ-complete if and only if each of the homology groups $\mathrm{H}_n(M)$ is isomorphic to its ℓ-adic completion $\varprojlim \mathrm{H}_n(M)/\ell^d \mathrm{H}_n(M)$, where the limit is taken in the abelian category of Λ-modules.

Remark 2.3.1.4. Let Λ be a commutative ring, let $M_\bullet$ be a simplicial object of Mod_Λ, and let $|M_\bullet| \in \mathrm{Mod}_\Lambda$ denote its geometric realization. Suppose that there exists an integer $n \in \mathbf{Z}$ such that the simplicial abelian groups $\mathrm{H}_m(M_\bullet)$ vanish for $m < n$. Then if each M_q is ℓ-complete, the geometric realization $|M_\bullet|$ is ℓ-complete. To prove this, it will suffice to show that each homology group $\mathrm{H}_i(|M_\bullet|)$ is ℓ-complete (Remark 2.3.1.3). We may therefore replace $M_\bullet$ by a sufficiently large skeleton, in which case $|M_\bullet|$ is a finite colimit of ℓ-complete objects of Mod_Λ.

Definition 2.3.1.5. Let X be a quasi-projective k-scheme and let Λ be a commutative ring. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ is *ℓ-complete* if, for every object $U \in \mathrm{Sch}_X^{\mathrm{et}}$, the object $\mathcal{F}(U) \in \mathrm{Mod}_\Lambda$ is ℓ-complete.

Remark 2.3.1.6. Let $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$. The following conditions are equivalent:

(1) The sheaf $\mathcal{F}$ is ℓ-complete.

(2) The limit of the tower
$$\cdots \to \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}$$
vanishes.

(3) The canonical map $\mathcal{F} \to \varprojlim(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$ is an equivalence in $\mathrm{Shv}(X; \Lambda)$.

Remark 2.3.1.7. Let X be a quasi-projective k-scheme, let Λ a commutative ring, and let $\mathcal{C} \subseteq \mathrm{Shv}(X; \Lambda)$ be the full subcategory spanned by the ℓ-complete objects. Then the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Shv}(X; \Lambda)$ admits a left adjoint L, given by the formula
$$L\,\mathcal{F} = \varprojlim(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}).$$
We will refer to L as the *ℓ-adic completion functor*. Note that an object $\mathcal{F} \in \mathrm{Shv}(X; \Lambda)$ is annihilated by the functor L if and only if the map $\ell : \mathcal{F} \to \mathcal{F}$ is an equivalence.

Proposition 2.3.1.8. *Let Λ be a commutative ring and suppose that ℓ is not a zero-divisor in Λ. Let X be a quasi-projective k-scheme and let $\mathcal{C} \subseteq \mathrm{Shv}(X;\Lambda)$ be the full subcategory spanned by the ℓ-complete objects. Then the composite functor*

$$\mathcal{C} \hookrightarrow \mathrm{Shv}(X;\Lambda) \xrightarrow{\theta} \varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$$

is an equivalence of ∞-categories (here θ is obtained from the extension of scalars functors described in §2.2.8).

Proof. We first prove that θ is fully faithful when restricted to $\mathcal{C}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be objects of $\mathrm{Shv}(X;\Lambda)$. We compute

$$\begin{aligned} \mathrm{Map}(\theta(\mathcal{F}),\theta(\mathcal{F}')) &\simeq \varprojlim_{d\geq 0} \mathrm{Map}_{\mathrm{Shv}(X;\Lambda/\ell^d\Lambda)}((\mathbf{Z}/\ell^d\mathbf{Z})\otimes_{\mathbf{Z}}\mathcal{F},(\mathbf{Z}/\ell^d\mathbf{Z})\otimes_{\mathbf{Z}}\mathcal{F}') \\ &\simeq \varprojlim_{d\geq 0} \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},(\mathbf{Z}/\ell^d\mathbf{Z})\otimes_{\mathbf{Z}}\mathcal{F}') \\ &\simeq \mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},L\,\mathcal{F}'), \end{aligned}$$

where L is defined as in Remark 2.3.1.7. If $\mathcal{F}'$ is ℓ-complete, then the canonical map

$$\mathrm{Map}_{\mathrm{Shv}(X;\Lambda)}(\mathcal{F},\mathcal{F}') \to \mathrm{Map}(\theta(\mathcal{F}),\theta(\mathcal{F}'))$$

is a homotopy equivalence.

It remains to prove essential surjectivity. Suppose we are given an object of the inverse limit $\varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$, which we can identify with a compatible sequence of objects

$$\{\mathcal{F}_d \in \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)\}_{d\geq 0}.$$

Let us abuse notation by identifying each $\mathcal{F}_d$ with its image in $\mathrm{Shv}(X;\Lambda)$, and set $\mathcal{F} = \varprojlim \mathcal{F}_d \in \mathrm{Shv}(X;\Lambda)$. Since each $\mathcal{F}_d$ is ℓ-complete, it follows that $\mathcal{F}$ is also ℓ-complete. Moreover, we have a canonical map $\theta(\mathcal{F}) \to \{\mathcal{F}_d\}_{d\geq 0}$ in the ∞-category $\varprojlim_{d\geq 0} \mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$. To prove that this map is an equivalence, it will suffice to show that for each integer $d \geq 0$, the canonical map

$$(\Lambda/\ell^d\Lambda)\otimes_\Lambda \varprojlim_{e\geq d} \mathcal{F}_e \to \mathcal{F}_d$$

is an equivalence in $\mathrm{Shv}(X;\Lambda/\ell^d\Lambda)$. Since $\Lambda/\ell^d\Lambda$ is a perfect Λ-module, we can identify this with the natural map

$$\varprojlim_{e\geq d}(\Lambda/\ell^d\Lambda)\otimes_\Lambda \mathcal{F}_e \simeq \varprojlim_{e\geq d}((\Lambda/\ell^d\Lambda)\otimes_\Lambda(\Lambda/\ell^e\Lambda))\otimes_{\Lambda/\ell^e\Lambda}\mathcal{F}_e \to (\Lambda/\ell^d\Lambda)\otimes_{\Lambda/\ell^e\Lambda}\mathcal{F}_e\,.$$

This map is an equivalence, since the inverse system $\{(\Lambda/\ell^d\Lambda)\otimes_\Lambda(\Lambda/\ell^e\Lambda)\}_{e\geq d}$ is equivalent to $\Lambda/\ell^d\Lambda$ as a Pro-object of the ∞-category Mod_Λ. □

2.3.2 Constructible ℓ-adic Sheaves

We now study a variant of Definition 2.2.6.3.

Definition 2.3.2.1. Let X be a quasi-projective k-scheme. We will say that an object $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z})$ is a *constructible ℓ-adic sheaf* if it satisfies the following conditions:

(1) The sheaf $\mathcal{F}$ is ℓ-complete.

(2) For each integer $d \geq 0$, the sheaf $(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ is constructible (see §2.2.8).

We let $\mathrm{Shv}_\ell^c(X)$ denote the full subcategory of $\mathrm{Shv}(X; \mathbf{Z})$ spanned by the constructible ℓ-adic sheaves.

Remark 2.3.2.2. In the situation of Definition 2.3.2.1, it suffices to verify condition (2) in the case $d = 1$, by virtue of Proposition 2.2.8.4.

Remark 2.3.2.3. Let $\mathcal{F}$ be as in Definition 2.3.2.1. Then $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ is constructible as an object of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ if and only if it is constructible as an object of $\mathrm{Shv}(X; \mathbf{Z})$. Consequently, condition (2) can be rephrased as follows:

(2′) The cofiber of the map $\ell : \mathcal{F} \to \mathcal{F}$ is a constructible object of $\mathrm{Shv}(X; \mathbf{Z})$.

Remark 2.3.2.4. It follows from Proposition 2.3.1.8 that the forgetful functor

$$\mathrm{Shv}(X; \mathbf{Z}_\ell) \to \mathrm{Shv}(X; \mathbf{Z})$$

is an equivalence when restricted to ℓ-complete objects. Consequently, we can replace $\mathrm{Shv}(X; \mathbf{Z})$ by $\mathrm{Shv}(X; \mathbf{Z}_\ell)$ in Definition 2.3.2.1 without changing the notion of constructible ℓ-adic sheaf.

Warning 2.3.2.5. Let X be a quasi-projective k-scheme. Neither of the full subcategories $\mathrm{Shv}_\ell^c(X), \mathrm{Shv}^c(X; \mathbf{Z}) \subseteq \mathrm{Shv}(X; \mathbf{Z})$ contains the other. Objects of $\mathrm{Shv}^c(X; \mathbf{Z})$ are generally not ℓ-complete (this is true even if we replace $\mathbf{Z}$ by $\mathbf{Z}_\ell$), and objects of $\mathrm{Shv}_\ell^c(X)$ need not be locally constant when restricted to any nonempty open subset of X.

Remark 2.3.2.6. Let X be a quasi-projective k-scheme. Using Proposition 2.3.1.8, we can identify $\mathrm{Shv}_\ell^c(X)$ with a (homotopy) inverse limit of the tower of ∞-categories

$$\cdots \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell^3\mathbf{Z}) \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell^2\mathbf{Z}) \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z}).$$

Proposition 2.3.2.7. *Let X be a quasi-projective k-scheme. Then the equivalence of ∞-categories $\mathrm{Shv}_\ell^c(X) \simeq \varprojlim \{\mathrm{Shv}_\ell^c(X; \mathbf{Z}/\ell^d\mathbf{Z})\}_{d \geq 0}$ induces an equivalence of homotopy categories*

$$\theta : \mathrm{hShv}_\ell^c(\mathrm{X}) \to \varprojlim \{\mathrm{hShv}_\ell^c(\mathrm{X}; \mathbf{Z}/\ell^{\mathrm{d}}\mathbf{Z})\}_{\mathrm{d} \geq 0}.$$

Remark 2.3.2.8. Proposition 2.3.2.7 implies that the homotopy category of $\mathrm{Shv}_\ell^c(X)$ can be identified with the constructible derived category of $\mathbf{Z}_\ell$-sheaves considered elsewhere in the literature (see, for example, [5]).

Proof of Proposition 2.3.2.7. It follows immediately from the definitions that θ is essentially surjective; we will show that θ is fully faithful. For every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell^c(X)$ having images $\mathcal{F}_d, \mathcal{G}_d \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$, we have a Milnor exact sequence

$$0 \to \lim^1\{\mathrm{Ext}^{n-1}(\mathcal{F}_d, \mathcal{G}_d)\} \to \mathrm{Ext}^n(\mathcal{F}, \mathcal{G}) \to \lim^0\{\mathrm{Ext}^n(\mathcal{F}_d, \mathcal{G}_d)\} \to 0.$$

Since each of the groups $\mathrm{Ext}^{n-1}(\mathcal{F}_d, \mathcal{G}_d)$ is finite (Corollary 2.2.9.5), the first term of this sequence vanishes. It follows that the canonical map

$$\mathrm{Map}_{\mathrm{hShv}_\ell^c(X)}(\mathcal{F}, \mathcal{G}) \to \varprojlim \mathrm{Map}_{\mathrm{hShv}(X; \mathbf{Z}/\ell^d\mathbf{Z})}(\mathcal{F}_d, \mathcal{G}_d)$$

is bijective. □

Notation 2.3.2.9 (Tate Twists)**.** Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ be a constructible ℓ-adic sheaf, which we can identify with an inverse system $\{\mathcal{F}_m \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^m\mathbf{Z})\}_{m \geq 0}$. For every integer d, the inverse system $\{\mathcal{F}_m(d) \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^m\mathbf{Z})\}_{m \geq 0}$ determines another constructible ℓ-adic sheaf on X, which we will denote by $\mathcal{F}(d)$ and refer to as the *d-fold Tate twist of* $\mathcal{F}$ (see Example 2.2.5.3). Note that $\mathcal{F}(d)$ is noncanonically equivalent to $\mathcal{F}$ (one can choose an equivalence by choosing a trivialization of the Tate module $\varprojlim \mu_{p^m}(k)$).

2.3.3 Direct and Inverse Images

We now analyze the dependence of the ∞-category $\mathrm{Shv}_\ell^c(X)$ on the choice of quasi-projective k-scheme X.

Proposition 2.3.3.1. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then:*

(1) *The pushforward functor $f_* : \mathrm{Shv}(X; \mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z})$ carries constructible ℓ-adic sheaves to constructible ℓ-adic sheaves.*

(2) *The resulting functor from $\mathrm{Shv}_\ell^c(X)$ to $\mathrm{Shv}_\ell^c(Y)$ admits a left adjoint $f^*_\wedge$, which carries an object $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ to the ℓ-completion of $f^* \mathcal{F}$.*

Proof. The functor f_* preserves limits, and therefore carries ℓ-complete objects to ℓ-complete objects. Assertion (1) is now a consequence of Proposition 2.2.9.1. To prove

(2), let $L : \mathrm{Shv}(X; \mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z})$ denote the ℓ-completion functor. If $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$, then we have a natural homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}(Y;\mathbf{Z})}(\mathcal{F}, f_* \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(Lf^* \mathcal{F}, \mathcal{G})$$

whenever $\mathcal{G} \in \mathrm{Shv}(X; \mathbf{Z}_\ell)$ is ℓ-complete. It will therefore suffice to show that $Lf^* \mathcal{F}$ is constructible. By construction, $Lf^* \mathcal{F}$ is ℓ-complete. It will therefore suffice to show that each tensor product

$$(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} Lf^* \mathcal{F} \simeq (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} f^* \mathcal{F} \simeq f^*(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$$

is a constructible object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$ for each $d \geq 0$, which follows immediately from Proposition 2.2.9.1. □

Warning 2.3.3.2. In the situation of Proposition 2.3.3.1, the pullback functor $f^* : \mathrm{Shv}(Y; \mathbf{Z}_\ell) \to \mathrm{Shv}(X; \mathbf{Z}_\ell)$ does not preserve ℓ-constructibility. For example, if $Y = \mathrm{Spec}(k)$ and $\mathcal{F} \in \mathrm{Shv}(Y; \mathbf{Z}_\ell)$ is the constant sheaf with value $\mathbf{Z}_\ell$, then the chain complex $(f^* \mathcal{F})(X)$ computes the étale cohomology of X with coefficients in the constant sheaf associated to $\mathbf{Z}_\ell$, while the chain complex $(f^*_\wedge \mathcal{F})(X)$ computes the ℓ-adic cohomology of X (see §3.2.1).

Proposition 2.3.3.3. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Then the functor $f^! : \mathrm{Shv}(Y; \mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z})$ carries $\mathrm{Shv}^c_\ell(Y)$ into $\mathrm{Shv}^c_\ell(X)$.*

Proof. The functor $f^!$ preserves limits and therefore carries ℓ-complete objects to ℓ-complete objects. It will therefore suffice to show that if $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$, then

$$(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} f^! \mathcal{F} \simeq f^!((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F})$$

is constructible for each $d \geq 0$, which follows from Proposition 2.2.9.1. □

Remark 2.3.3.4. In the situation of Proposition 2.3.3.3, the functor $f^! : \mathrm{Shv}^c_\ell(Y) \to \mathrm{Shv}^c_\ell(X)$ can be identified with the inverse limit of the tower of exceptional inverse image functors $f^! : \mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$.

Example 2.3.3.5. If $f : X \to Y$ is a smooth morphism of relative dimension d, then Example 2.2.5.3 supplies an equivalence $f^! \mathcal{F} \simeq f^* \mathcal{F}(d)$, which depends functorially on $\mathcal{F} \in \mathrm{Shv}^c_\ell(Y)$.

Proposition 2.3.3.6. *Let $f : X \to Y$ be an étale morphism between quasi-projective k-schemes. Then:*

(1) *The pullback functor $f^* : \mathrm{Shv}(X; \mathbf{Z}) \to \mathrm{Shv}(Y; \mathbf{Z})$ carries $\mathrm{Shv}^c_\ell(X)$ into $\mathrm{Shv}^c_\ell(Y)$.*

(2) *When regarded as a functor from* $\mathrm{Shv}_\ell^c(X)$ *to* $\mathrm{Shv}_\ell^c(Y)$*, the functor* f^* *admits a left adjoint* $f_!^\wedge$*, which carries an object* $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$ *to the* ℓ*-completion of* $f_! \mathcal{F}$*.*

Proof. We first prove (1). If $\mathcal{F} \in \mathrm{Shv}(Y; \mathbf{Z})$ is a constructible ℓ-adic sheaf, then $\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}$ belongs to $\mathrm{Shv}^c(Y; \mathbf{Z}/\ell^d\mathbf{Z})$, so that Proposition 2.2.9.1 shows that

$$\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} f^* \mathcal{F} \simeq f^*(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$$

for each $d \geq 0$. Since f is étale, the pullback functor f^* preserves limits, and therefore carries ℓ-complete objects to ℓ-complete objects.

We now prove (2). Let $f_!$ denote the left adjoint to the pullback functor $f^* : \mathrm{Shv}(Y; \mathbf{Z}) \to \mathrm{Shv}(X; \mathbf{Z})$ (see Construction 2.2.3.4), and let $f_!^\wedge$ denote the composition of $f_!$ with the ℓ-completion functor. It follows immediately from the definitions that for every object $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z})$ and every ℓ-complete object $\mathcal{G} \in \mathrm{Shv}(Y; \mathbf{Z})$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}(Y;\mathbf{Z})}(f_!^\wedge \mathcal{F}, \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, f^* \mathcal{G}).$$

It will therefore suffice to show that if $\mathcal{F}$ is an ℓ-adic constructible sheaf, then $f_!^\wedge \mathcal{F}$ is an ℓ-adic constructible sheaf. Since $f_!^\wedge \mathcal{F}$ is ℓ-complete by construction, we are reduced to proving that each tensor product

$$\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} f_!^\wedge \mathcal{F} \simeq f_!(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$$

is a compact object of $\mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$, which follows from Proposition 2.2.9.1. □

Proposition 2.3.3.7. *Let* X *be a quasi-projective* k*-scheme and let* $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)$*. The following conditions are equivalent:*

(1) *The sheaf* $\mathcal{F}$ *vanishes.*

(2) *For every point* $\eta : \mathrm{Spec}(k) \to X$*, the stalk* $\mathcal{F}_\eta = \eta^* \mathcal{F}$ *vanishes.*

(3) *For every point* $\eta : \mathrm{Spec}(k) \to X$*, the costalk* $\eta^! \mathcal{F} \in \mathrm{Shv}_\ell(\mathrm{Spec}(k)) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$ *vanishes.*

Proof. Note that since $\mathcal{F}$ is ℓ-complete, it vanishes if and only if $\mathcal{F}_1 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ vanishes. Similarly, the stalk (costalk) of $\mathcal{F}$ at a point $\eta \in X(k)$ vanishes if and only if the stalk (costalk) of $\mathcal{F}_1$ vanishes at η. The desired result now follows from the corresponding assertion for $\mathcal{F}_1 \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ (Proposition 2.2.9.6). □

2.3.4 General ℓ-adic Sheaves

For our applications in this book, the setting of constructible ℓ-adic sheaves will be too restrictive: we will encounter many examples of sheaves which are not constructible. To accommodate these examples, we introduce the following enlargement of $\mathrm{Shv}_\ell^c(X)$:

Definition 2.3.4.1. Let X be a quasi-projective k-scheme. We let $\mathrm{Shv}_\ell(X)$ denote the ∞-category $\mathrm{Ind}(\mathrm{Shv}_\ell^c(X))$ of Ind-objects of $\mathrm{Shv}_\ell^c(X)$ (see §[25].5.3.5). We will refer to $\mathrm{Shv}_\ell(X)$ as the *∞-category of ℓ-adic sheaves on X*.

Remark 2.3.4.2. Let X be a quasi-projective k-scheme. By abstract nonsense, the fully faithful embedding $\mathrm{Shv}_\ell^c(X) \hookrightarrow \mathrm{Shv}(X; \mathbf{Z}_\ell)$ extends to a colimit-preserving functor $\theta : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}(X; \mathbf{Z}_\ell)$. However, this functor need not be fully faithful, since the objects of $\mathrm{Shv}_\ell^c(X)$ need not be compact in $\mathrm{Shv}(X; \mathbf{Z}_\ell)$.

Example 2.3.4.3. If $X = \mathrm{Spec}(k)$, then the essential image of the inclusion $\mathrm{Shv}_\ell^c(X) \hookrightarrow \mathrm{Shv}(X; \mathbf{Z}_\ell) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$ consists precisely of the compact objects of $\mathrm{Mod}_{\mathbf{Z}_\ell}$. It follows that the forgetful functor of Remark 2.3.4.2 induces an equivalence $\mathrm{Shv}_\ell(X) \to \mathrm{Shv}(X; \mathbf{Z}_\ell) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$.

Remark 2.3.4.4. Let X be a quasi-projective k-scheme. Then there is a fully faithful exact functor $\mathrm{Shv}_\ell^c(X) \to \mathrm{Shv}_\ell(X)$. We will generally abuse notation by identifying $\mathrm{Shv}_\ell^c(X)$ with its essential image under this embedding.

Notation 2.3.4.5. Let $f : X \to Y$ be a morphism between quasi-projective k-schemes. Then the adjoint functors

$$f_* : \mathrm{Shv}_\ell^c(X) \to \mathrm{Shv}_\ell^c(Y) \qquad f_\wedge^* : \mathrm{Shv}_\ell^c(Y) \to \mathrm{Shv}_\ell^c(X)$$

extend (in an essentially unique way) to a pair of adjoint functors relating the ∞-categories $\mathrm{Shv}_\ell(X)$ and $\mathrm{Shv}_\ell(Y)$, which we will denote by

$$f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y) \qquad f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X).$$

If f is proper, then the functor $f^! : \mathrm{Shv}_\ell^c(Y) \to \mathrm{Shv}_\ell^c(X)$ admits an essentially unique extension to a functor $\mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ which commutes with filtered colimits. This extension is a right adjoint to the pushforward functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$, and will be denoted by $f^!$.

If f is étale, then the functor $f_!^\wedge : \mathrm{Shv}_\ell^c(X) \to \mathrm{Shv}_\ell^c(Y)$ admits an essentially unique extension to a functor $\mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ which commutes with filtered colimits. This extension is left adjoint to the pullback functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$, and will be denoted by $f_!$.

Warning 2.3.4.6. There is some potential for confusion, because the operations introduced in Notation 2.3.4.5 need not be compatible with the corresponding operations on étale sheaves studied in §2.2. That is, the diagrams of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(Y) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(X) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(Y;\mathbf{Z}_\ell) & \xrightarrow{f^*} & \mathrm{Shv}(X;\mathbf{Z}_\ell) \end{array} \qquad \begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f_!} & \mathrm{Shv}_\ell(Y) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(X;\mathbf{Z}_\ell) & \xrightarrow{f_!} & \mathrm{Shv}(Y;\mathbf{Z}_\ell) \end{array}$$

$$\begin{array}{ccc} \mathrm{Shv}_\ell(Y) & \xrightarrow{f^!} & \mathrm{Shv}_\ell(X) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(Y;\mathbf{Z}_\ell) & \xrightarrow{f^!} & \mathrm{Shv}(X;\mathbf{Z}_\ell), \end{array}$$

where the vertical maps are given by the forgetful functors of Remark 2.3.4.2, need not commute. In the first two cases, this is because the definition of f^* and $f_!$ on ℓ-adic sheaves involves the process of ℓ-completion; in the third, it is because the functor $f^! : \mathrm{Shv}(Y;\mathbf{Z}_\ell) \to \mathrm{Shv}(X;\mathbf{Z}_\ell)$ need not preserve colimits (the functor $f^!$ preserves filtered colimits if and only if the direct image functor f_* carries constructible sheaves to constructible sheaves; this is generally false when working with sheaves of $\mathbf{Z}_\ell$-modules). However, the analogous diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f_*} & \mathrm{Shv}_\ell(Y) \\ \downarrow & & \downarrow \\ \mathrm{Shv}(X;\mathbf{Z}_\ell) & \xrightarrow{f_*} & \mathrm{Shv}(Y;\mathbf{Z}_\ell) \end{array}$$

does commute (up to canonical homotopy).

The existence of the adjunction $(f_!, f^*)$ when $f : X \to Y$ is an étale morphism has the following consequence:

Proposition 2.3.4.7. *Let $f : X \to Y$ be an étale morphism between quasi-projective k-schemes. Then the pullback functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ preserves limits.*

In fact, we have the following stronger assertion:

Proposition 2.3.4.8. *Let $f : X \to Y$ be a smooth morphism between quasi-projective k-schemes. Then the functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ preserves limits.*

Proof. Using Corollary 2.3.5.2 and Proposition 2.3.4.7, we see that the result is local with respect to the étale topology on X. We may therefore assume without loss of generality that f factors as a composition $X \xrightarrow{f'} \mathbf{P}^n \times Y \xrightarrow{f''} Y$, where the map f' is étale. Since f'^* preserves limits (Proposition 2.3.4.7), we may replace f by f'' and thereby reduce to the case where f is smooth and proper. In this case, the functor f^* is equivalent to a shift of the functor $f^!$ (Example 2.3.3.5) and therefore admits a left adjoint (given by a shift of f_*). □

Remark 2.3.4.9. Let X be a quasi-projective k-scheme and let $\eta : \mathrm{Spec}(k) \to X$ be a k-valued point of X. Then the pullback functor $\eta^* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(\mathrm{Spec}(k))$ carries each ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ to an object of $\mathrm{Shv}_\ell(\mathrm{Spec}(k)) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$. We will denote this object by $\mathcal{F}_\eta$ and refer to it as the *stalk* of $\mathcal{F}$ at the point η.

Warning 2.3.4.10. Proposition 2.3.3.7 does not extend to non-constructible ℓ-adic sheaves. It is possible to have a nonzero object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ whose stalk $\mathcal{F}_\eta$ vanishes for every k-valued point $\eta \in X(k)$.

The pathology of Warning 2.3.4.10 can be avoided by restricting our attention to ℓ-adic sheaves which are *ℓ-complete*, in the following sense:

Definition 2.3.4.11. Let X be a quasi-projective k-scheme. We say that an object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ is *ℓ-complete* if the inverse limit of the tower

$$\cdots \to \mathcal{F} \xrightarrow{\ell} \mathcal{F} \xrightarrow{\ell} \mathcal{F}$$

vanishes in the ∞-category $\mathrm{Shv}_\ell(X)$.

Remark 2.3.4.12. Let X be a quasi-projective k-scheme. An object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ is ℓ-complete if and only if, for every object $\mathcal{F}' \in \mathrm{Shv}_\ell(X)$, the inverse limit of mapping spaces

$$\cdots \to \mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}', \mathcal{F}) \xrightarrow{\ell} \mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}', \mathcal{F}) \xrightarrow{\ell} \mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}', \mathcal{F})$$

is contractible. Moreover, it suffices to verify this condition when $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)$ is constructible.

Remark 2.3.4.13. Let X be a quasi-projective k-scheme. Then every constructible ℓ-adic sheaf $\mathcal{F}$ on X is ℓ-complete (this follows from Remark 2.3.4.12, since $\mathcal{F}$ is ℓ-complete when viewed as an object of $\mathrm{Shv}(X; \mathbf{Z})$).

Proposition 2.3.4.14. *Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ be ℓ-complete. The following conditions are equivalent:*

(1) *The sheaf* $\mathcal{F}$ *vanishes.*

(2) *For every étale morphism* $f : U \to X$*, the object* $C^*(U; f^* \mathcal{F}) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$ *(see Construction 3.2.1.1) vanishes.*

(3) *For every* k*-valued point* $x \in X(k)$*, the stalk* $x^* \mathcal{F}$ *vanishes.*

Proof. The implication (1) ⇒ (2) is trivial. Conversely, suppose that (2) is satisfied. Write $\mathcal{F}$ as the colimit of a filtered diagram $\{\mathcal{F}_\alpha\}$ in $\mathrm{Shv}_\ell^c(X)$. For each integer $d \geq 0$, let $\mathcal{F}_d$ denote the cofiber of the canonical map $\ell^d : \mathcal{F} \to \mathcal{F}$, so that $\mathcal{F}_d$ can be written as a colimit $\varinjlim_\alpha \mathcal{F}_{\alpha,d}$ where $\mathcal{F}_{\alpha,d} = \mathrm{cofib}(\ell^d : \mathcal{F}_\alpha \to \mathcal{F}_\alpha)$. Note that we can identify the diagram $\{\mathcal{F}_{\alpha,d}\}$ with an object of the Ind-category $\mathrm{Ind}(\mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})) \simeq \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. Using condition (2) we see that this Ind-object vanishes, so that $\mathcal{F}_d \simeq 0$. Since $\mathcal{F}$ is ℓ-complete, it follows that $\mathcal{F} \simeq \varprojlim \mathcal{F}_d \simeq 0$. This proves that (2) ⇒ (1). The proof that (1) and (3) are equivalent is similar (using Proposition 2.2.3.2). □

2.3.5 Cohomological Descent

Let X be a quasi-projective k-scheme. For every commutative ring Λ, the theory of Mod_Λ-valued sheaves on X satisfies effective descent for the étale topology: that is, the construction

$$(U \in \mathrm{Sch}_X^{\mathrm{et}}) \mapsto \mathrm{Shv}(U; \Lambda)$$

is a sheaf of ∞-categories with respect to the étale topology. We now establish an analogous statement for ℓ-adic sheaves.

Proposition 2.3.5.1 (Effective Cohomological Descent)**.** *Let* $f : U \to X$ *be a surjective étale morphism between quasi-projective* k*-schemes, and let* $U_\bullet$ *denote the simplicial scheme given by the nerve of the map* f *(so that* U_m *is the* $(m+1)$*st fiber power of* U *over* X*). Then the canonical map*

$$\psi : \mathrm{Shv}_\ell(X) \to \varprojlim \mathrm{Shv}_\ell(U_\bullet)$$

is an equivalence of ∞-categories.

Corollary 2.3.5.2. *Let* $f : X \to Y$ *be a smooth surjection between quasi-projective* k*-schemes. Then the functor* $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ *is conservative.*

Proof. Since f is a smooth surjection, there exists a map $g : X' \to X$ such that the composite map $f \circ g$ is an étale surjection. Replacing X by X', we may suppose that f is étale. In this case, the desired result follows immediately from Proposition 2.3.5.1. □

The proof of Proposition 2.3.5.1 depends on the following result:

Lemma 2.3.5.3. *Let X be a quasi-projective k-scheme. Suppose that $\mathcal{F}_\bullet$ is an augmented simplicial object of $\mathrm{Shv}_\ell^c(X)$ satisfying the following conditions:*

- (a) *There exists an integer n such that $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}_\bullet$ is an augmented simplicial object of $\mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq n}$.*

- (b) *The image of $\mathcal{F}_\bullet$ in $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ is a colimit diagram (that is, it exhibits $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{-1}$ as a geometric realization of the simplicial object $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_\bullet$).*

Then $\mathcal{F}_\bullet$ is a colimit diagram in both $\mathrm{Shv}(X; \mathbf{Z})$ and $\mathrm{Shv}_\ell(X)$.

Proof of Proposition 2.3.5.1. It follows from the Beck-Chevalley property of Variant 2.4.3.1 (and Corollary [23].4.7.5.3) that the functor ψ admits a fully faithful left adjoint

$$\phi : \varprojlim \mathrm{Shv}_\ell(U_\bullet) \to \mathrm{Shv}_\ell(X).$$

To complete the proof, it will suffice to show that for each object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the counit map $v : (\phi \circ \psi)(\mathcal{F}) \to \mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(X)$. For each $n \geq 0$, let $f_n : U_n \to X$ denote the projection map. Unwinding the definitions, we can identify v with the natural map $|f_{\bullet !} f_\bullet^* \mathcal{F}| \to \mathcal{F}$. Writing $\mathcal{F}$ as a colimit of constructible ℓ-adic sheaves, we may assume without loss of generality that $\mathcal{F}$ is constructible. By virtue of Lemma 2.3.5.3, it will suffice to prove this after tensoring with $\mathbf{Z}/\ell\mathbf{Z}$, in which case the desired result follows from the fact that the construction $U \mapsto \mathrm{Shv}(U; \mathbf{Z}/\ell\mathbf{Z})$ satisfies étale descent. □

Proof of Lemma 2.3.5.3. We first prove that $\mathcal{F}_\bullet$ is a colimit diagram in $\mathrm{Shv}(X; \mathbf{Z})$: that is, that the canonical map $\alpha : |\mathcal{F}_\bullet| \to \mathcal{F}_{-1}$ is an equivalence in $\mathrm{Shv}(X; \mathbf{Z})$. Condition (b) implies that α is an equivalence after tensoring with $\mathbf{Z}/\ell\mathbf{Z}$. Since the codomain of α is ℓ-complete, it will suffice to show that the domain of α is also ℓ-complete. For each integer m, let $\mathcal{F}(m)$ denote the colimit of the m-skeleton of $\mathcal{F}_\bullet$ (formed in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$). Then each $\mathcal{F}(m)$ belongs to $\mathrm{Shv}_\ell^c(X)$ and is therefore ℓ-complete, and we have an equivalence $|\mathcal{F}_\bullet| \simeq \varinjlim \mathcal{F}(m)$. Fix an étale map $V \to X$; we wish to prove that

$$|\mathcal{F}_\bullet|(V) \simeq \varinjlim \mathcal{F}(m)(V) \in \mathrm{Mod}_{\mathbf{Z}_\ell}$$

is ℓ-complete. According to Remark 2.3.1.3, this is equivalent to the assertion that for every integer i, the abelian group $\varinjlim \mathrm{H}_i(\mathcal{F}(m)(V))$ is ℓ-complete (in the derived sense). To prove this, it will suffice to show that the direct system of abelian groups $\{\mathrm{H}_i(\mathcal{F}(m)(V))\}$ is eventually constant. Let $\mathcal{K}(m)$ denote the fiber of the map $\mathcal{F}(m) \to \mathcal{F}(m+1)$, so that we have an exact sequence

$$\mathrm{H}_i(\mathcal{K}(m)(V)) \to \mathrm{H}_i(\mathcal{F}(m)) \to \mathrm{H}_i(\mathcal{F}(m+1)(V)) \to \mathrm{H}_{i-1}(\mathcal{K}(m)(V)).$$

It will therefore suffice to show that the groups $\mathrm{H}_i(\mathcal{K}(m)(V))$ vanish for $m \gg i$. Since $\mathcal{K}(m)$ is ℓ-complete, we have a Milnor exact sequence

$$\lim^1\{\mathrm{H}_{i+1}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m)(V)) \to \mathrm{H}_i(\mathcal{K}(m)(V)) \to \varprojlim\{\mathrm{H}_i(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m)(V))\}.$$

Corollary 2.2.9.5 implies that the left term vanishes. We are therefore reduced to proving that $\mathrm{H}_i(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m)(V)) \simeq 0$ for $m \gg i$. Using induction on d, we can reduce to the case $d = 1$. Using Lemma 2.2.4.1, we are reduced to the problem of showing that $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{K}(m) \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq i}$ for $m \gg i$. This follows easily from assumption (a). This completes the proof that α is an equivalence in $\mathrm{Shv}(X; \mathbf{Z})$.

Note that each $\mathcal{F}(m)$ is a constructible ℓ-adic sheaf, and is a colimit of the m-skeleton of $\mathcal{F}_\bullet$ in both $\mathrm{Shv}(X; \mathbf{Z})$ and $\mathrm{Shv}_\ell(X)$. The sheaf $\mathcal{F}_{-1}$ can be identified with the colimit of the sequence

$$\mathcal{F}(0) \to \mathcal{F}(1) \to \cdots$$

in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$; we wish to show that $\mathcal{F}$ is also a colimit of this sequence in $\mathrm{Shv}_\ell(X)$. Equivalently, we wish to show that for every object $\mathcal{G} \in \mathrm{Shv}^c_\ell(X)$, the canonical map

$$\varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}(m)) \to \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F})$$

is a homotopy equivalence; here we write $\mathrm{Map}_{\mathcal{C}}(C, D)$ for the space of maps from C to D in an ∞-category $\mathcal{C}$ (see [25]). For each $m \geq 0$, let $\mathcal{F}'(m)$ denote the cofiber of the canonical map $\mathcal{F}(m) \to \mathcal{F}$, so that we have a fiber sequence

$$\varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}(m)) \to \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}) \to \varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}'(m)).$$

It will therefore suffice to show that the space $\varinjlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}'(m))$ is contractible. We will prove the following more precise statement: for every integer q, the mapping space $\mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}'(m))$ is q-connective for m sufficiently large (depending on q). Since $\mathcal{F}'(m)$ is ℓ-complete, we can identify $\mathrm{Map}_{\mathrm{Shv}^c(X; \mathbf{Z}_\ell)}(\mathcal{G}, \mathcal{F}'(m))$ with the limit of a tower of spaces $\mathrm{Map}_{\mathrm{Shv}^c(X; \mathbf{Z}_\ell)}(\mathcal{G}, \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m))$. It will therefore suffice to show that each of these spaces is $(q+1)$-connective. Using the existence of a fiber sequence

$$\begin{array}{c} \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathbf{Z}/\ell^{d+1}\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)), \end{array}$$

we can reduce to the case $d = 1$. That is, we are reduced to proving that the mapping spaces $\mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}, \mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m))$ are $(q+1)$-connective for $q \gg m$. This follows from Proposition 2.2.7.3, since condition (a) guarantees that the sheaves $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}'(m)$ are highly connected for $m \gg 0$. □

2.3.6 The t-Structure on ℓ-Constructible Sheaves

Let X be a quasi-projective k-scheme. In §2.2.2, we introduced a t-structure on the stable ∞-category $\mathrm{Shv}(X;\Lambda)$, for every commutative ring Λ. We now describe an analogous t-structure on the stable ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves. Our starting point is the following result:

Proposition 2.3.6.1. *Let X be a quasi-projective k-scheme. Then there exists a t-structure $(\mathrm{Shv}^c_\ell(X)_{\geq 0}, \mathrm{Shv}^c_\ell(X)_{\leq 0})$ on the ∞-category $\mathrm{Shv}^c_\ell(X)$ of constructible ℓ-adic sheaves on X which is uniquely characterized by the following property:*

- *A constructible ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$ belongs to $\mathrm{Shv}^c_\ell(X)_{\geq 0}$ if and only if $\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\geq 0}$.*

Warning 2.3.6.2. In the situation of Proposition 2.3.6.1, we can regard $\mathrm{Shv}^c_\ell(X)$ as a full subcategory of $\mathrm{Shv}(X;\mathbf{Z})$, which is also equipped with a t-structure by virtue of Proposition 2.2.2.3. Beware that the inclusion $\mathrm{Shv}^c_\ell(X) \hookrightarrow \mathrm{Shv}(X;\mathbf{Z})$ is *not* t-exact. However, it is left t-exact: see Remark 2.3.6.5 below.

Example 2.3.6.3. Let $X = \mathrm{Spec}(k)$, so that $\mathrm{Shv}^c_\ell(X)$ can be identified with the ∞-category $\mathrm{Mod}^{\mathrm{pf}}_{\mathbf{Z}_\ell}$ of perfect $\mathbf{Z}_\ell$-modules. Under this identification, the t-structure of Proposition 2.3.6.1 agrees with the usual t-structure of $\mathrm{Mod}^{\mathrm{pf}}_{\mathbf{Z}_\ell}$.

Lemma 2.3.6.4. *Let $\mathcal{A}$ be an abelian category. For each object $M \in \mathcal{A}$, let $M/\ell^d M$ and $M[\ell^d]$ denote the cokernel and kernel of the map $\ell^d : M \to M$. Suppose we are given a tower of objects*

$$\cdots \to M_3 \to M_2 \to M_1 \to M_0$$

satisfying the following conditions:

(a) Each of the maps $M_{d+1} \to M_d$ induces an equivalence $M_{d+1}/\ell^d M_{d+1} \simeq M_d$.

(b) The object M_1 is Noetherian.

Then, for each integer $m \geq 0$, the tower $\{M_d[\ell^m]\}_{d\geq 0}$ is equivalent to a constant Pro-*object of $\mathcal{A}$.*

Proof. For each $d \geq 0$, let N_d denote the image of the natural map $M_{d+m}[\ell^m] \to M_d[\ell^m]$. If $d \geq m$, multiplication by ℓ^{d-m} induces a map $\theta_d : M_m \to M_d[\ell^m]$. Let N'_d denote the fiber product $M_m \times_{M_d} N_d$, which we regard as a subobject of M_m. Assumption (a) implies that M_m admits a finite filtration by quotients of M_1, so that M_m is Noetherian by virtue of (b). Note that $N'_d = \ker(\theta_{d+m}) \subseteq N'_{d+m}$, so that the subobjects $N'_d \subseteq M_m$ form an ascending chain

$$N'_m \subseteq N'_{2m} \subseteq N'_{3m} \subseteq \cdots .$$

Since M_m is Noetherian, this chain must eventually stabilize. We may therefore choose an integer a_0 such that $N'_{am} = N'_{(a-1)m} = \ker(\theta_{am})$ for $a \geq a_0$. Using the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N'_{(a-1)m} & \longrightarrow & M_m & \xrightarrow{\theta_{am}} & \mathrm{Im}(\theta_{am}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \mathrm{id} & & \downarrow & & \\ 0 & \longrightarrow & N'_{am} & \longrightarrow & M_m & \xrightarrow{\theta_{(a+1)m}} & \mathrm{Im}(\theta_{(a+1)m} & \longrightarrow & 0, \end{array}$$

we see that multiplication by ℓ^m induces an isomorphism from $\mathrm{Im}(\theta_{am})$ to $\mathrm{Im}(\theta_{(a+1)m})$ for $a \geq a_0$. This isomorphism factors as a composition

$$\mathrm{Im}(\theta_{am}) \hookrightarrow M_{am}[\ell^m] \xrightarrow{\ell^m} \mathrm{Im}(\theta_{(a+1)m}),$$

so that for $a \geq a_0$ the object $M_{am}[\ell^m]$ splits as a direct sum $\mathrm{Im}(\theta_{am}) \oplus N_{am}$. Note that the restriction map $M_{(a+1)m}[\ell^m] \to M_{am}[\ell^m]$ has image N_{am} and kernel $\mathrm{Im}(\theta_{(a+1)m})$, and therefore restricts to an isomorphism $N_{(a+1)m} \to N_{am}$ for $a \geq a_0$. It follows that the tower $\{M_{am}[\ell^m]\}_{a \geq a_0}$ is isomorphic to the direct sum of a constant tower $\{N_{am}\}_{a \geq a_0}$ and a tower $\{\mathrm{Im}(\theta_{am})\}_{a \geq a_0}$ with vanishing transition maps, and is therefore equivalent to a constant Pro-object of $\mathcal{A}$. □

Proof of Proposition 2.3.6.1. For each integer n, let $\mathrm{Shv}^c_\ell(X)_{\leq n}$ denote the full subcategory of $\mathrm{Shv}^c_\ell(X)$ spanned by those objects $\mathcal{F}$ such that, for each object $\mathcal{G} \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$, the mapping space $\mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \Sigma^{-m} \mathcal{F})$ is contractible for $m > n$. To prove Proposition 2.3.6.1, it will suffice to show that for each object $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$, there exists a fiber sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ where $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$ and $\mathcal{F}'' \in \mathrm{Shv}^c_\ell(X)_{\leq -1}$.

For each integer $d \geq 0$, let $\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ denote the image of $\mathcal{F}$ in $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$, so that $\mathcal{F} \simeq \varprojlim \{\mathcal{F}_d\}_{d \geq 0}$. Set $\mathcal{F}' = \varprojlim \{\tau_{\geq 0} \mathcal{F}_d\}_{d \geq 0}$ and $\mathcal{F}'' = \varprojlim \{\tau_{\leq -1} \mathcal{F}_d\}_{d \geq 0}$, where the limits are formed in $\mathrm{Shv}(X; \mathbf{Z})$. We will prove that $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$. Assuming this, it follows that $\mathcal{F}'' \in \mathrm{Shv}^c_\ell(X)$. Note that for $\mathcal{G} \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$, the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \mathcal{F}) &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}^c_\ell(X)}(\mathcal{G}, \tau_{\leq -1} \mathcal{F}_d) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}_\ell(X; \mathbf{Z}/\ell^d\mathbf{Z})}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}, \tau_{\leq -1} \mathcal{F}_d) \end{aligned}$$

is contractible, since each tensor product $\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\geq 0}$. It follows that $\mathcal{F}''$ belongs to $\mathrm{Shv}^c_\ell(X)_{\leq -1}$, as desired.

It remains to prove that $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)_{\geq 0}$. For this, we must establish three things:

- (a) The object $\mathcal{F}' \in \mathrm{Shv}(X; \mathbf{Z})$ is ℓ-complete.
- (b) The tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}'$ is a compact object of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$.
- (c) The tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}'$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0}$.

Assertion (a) is obvious (since the collection of ℓ-complete objects of $\mathrm{Shv}(X; \mathbf{Z})$ is closed under limits). We will deduce (b) and (c) from the following:

- $(*)$ The tower $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \tau_{\geq 0} \mathcal{F}_d\}_{d \geq 0}$ is a constant Pro-object of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$.

Note that if a tower $\{C_d\}_{d \geq 0}$ in some ∞-category $\mathcal{C}$ is Pro-equivalent to an object $C \in \mathcal{C}$, then C can be identified with a retract of C_d for $d \gg 0$. In particular, using assertion $(*)$ (and the fact that the construction $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \bullet$ preserves limits), we can identify $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}'$ with a retract of some $\mathcal{G} = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \tau_{\geq 0} \mathcal{F}_d$ for some $d \geq 0$. From this, assertion (c) is obvious and assertion (b) follows from Proposition 2.2.6.2.

Note that the tower $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/\ell^d\mathbf{Z})\}$ determines a constant Pro-object of $\mathrm{Mod}_{\mathbf{Z}}$, so that the Pro-objects $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_d\}_{d \geq 0}$ and $\{\tau_{\geq 0}(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}_d)\}_{d \geq 0}$ are likewise constant. For each $d \geq 0$, form a fiber sequence

$$(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \tau_{\geq 0} \mathcal{F}_d \to \tau_{\geq 0}((\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_d) \to \mathcal{G}_d .$$

To prove $(*)$, it will suffice to show that the tower $\{\mathcal{G}_d\}_{d \geq 0}$ is constant. Unwinding the definitions, we see that each $\mathcal{G}_d$ belongs to the heart $\mathrm{Shv}(X; \mathbf{Z})^{\heartsuit}$, where it can be identified with the kernel of the map $\pi_{-1} \mathcal{F}_d \to \pi_{-1} \mathcal{F}_d$ given by multiplication by ℓ. For each integer m, let us regard $\pi_m \mathcal{F}$ as an object of $\mathrm{Shv}(X; \mathbf{Z})^{\heartsuit}$, and let $(\pi_m \mathcal{F})/\ell^d$ and $(\pi_m \mathcal{F})[\ell^d]$ denote the cokernel and kernel of the multiplication map $\ell^d : \pi_m \mathcal{F} \to \pi_m \mathcal{F}$, so that we have exact sequences

$$0 \to (\pi_{-1} \mathcal{F})/\ell^d \to \pi_{-1} \mathcal{F}_d \to (\pi_{-2} \mathcal{F})[\ell^d] \to 0$$

which determine an exact sequence of Pro-objects

$$0 \to \{(\pi_{-1} \mathcal{F})/\ell^d)[\ell]\}_{d \geq 0} \to \{\mathcal{G}_d\}_{d \geq 0} \to \{(\pi_{-2} \mathcal{F})[\ell]\}_{d \geq 0}.$$

The last of these Pro-objects is trivial (it has vanishing transition maps), so we are reduced to proving that the Pro-object $\{(\pi_{-1} \mathcal{F})/\ell^d)[\ell]\}_{d \geq 0}$ is constant. Note that $(\pi_{-1} \mathcal{F})/\ell$ is a subobject of $\pi_{-1} \mathcal{F}_1$, and is therefore a Noetherian object of the abelian category $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})^{\heartsuit}$ by virtue of Proposition 2.2.7.4. The desired result now follows from Lemma 2.3.6.4. □

Remark 2.3.6.5. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)$. The proof of Proposition 2.3.6.1 shows that $\mathcal{F}$ belongs to $\mathrm{Shv}_{\ell}^{c}(X)_{\leq 0}$ if and only if the canonical map $\mathcal{F} \to \varprojlim \tau_{\leq 0}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$ is an equivalence in $\mathrm{Shv}(X;\mathbf{Z})$. In particular, every object of $\mathrm{Shv}_{\ell}^{c}(X)_{\leq 0}$ belongs to $\mathrm{Shv}(X;\mathbf{Z})_{\leq 0}$. In other words, the inclusion $\mathrm{Shv}_{\ell}^{c}(X) \hookrightarrow \mathrm{Shv}(X;\mathbf{Z})$ is left t-exact.

Remark 2.3.6.6. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)$. For each integer $d \geq 0$, let $\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}^c(X;\mathbf{Z}/\ell^d\mathbf{Z})$. If $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)_{\leq n}$, then Remark 2.3.6.5 implies that $\mathcal{F} \in \mathrm{Shv}(X;\mathbf{Z})_{\leq n}$ so that each of the sheaves $\mathcal{F}_d$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})_{\leq n+1}$. Conversely, if $\mathcal{F}_1$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\leq n+1}$, then it follows by induction on d that each $\mathcal{F}_d \in \mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})_{\leq n+1}$, so that the proof of Proposition 2.3.6.1 shows that $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)_{\leq n+1}$.

Proposition 2.3.6.7. *Let X be a quasi-projective k-scheme. Then the t-structure on $\mathrm{Shv}_{\ell}^{c}(X)$ is right and left bounded: that is, we have*

$$\mathrm{Shv}_{\ell}^{c}(X) = \bigcup_{n} \mathrm{Shv}_{\ell}^{c}(X)_{\leq n} = \bigcup_{n} \mathrm{Shv}_{\ell}^{c}(X)_{\geq -n}.$$

Proof. Let $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)$. For each integer $d \geq 0$, let $\mathcal{F}_d = \mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}^c(X;\mathbf{Z}/\ell^d\mathbf{Z})$. The characterization of constructibility given by Proposition 2.2.6.2 shows that there exists an integer $n \geq 0$ such that $\mathcal{F}_1 \in \mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\geq -n} \cap \mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\leq n}$. It follows by induction on d that each $\mathcal{F}_d$ belongs to $\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})_{\geq -n} \cap \mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})_{\geq n}$, so that $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)_{\geq -n} \cap \mathrm{Shv}_{\ell}^{c}(X)_{\leq n}$. □

2.3.7 Exactness of Direct and Inverse Images

Let $f : X \to Y$ be a map of quasi-projective k-schemes. Then f determines a pair of adjoint functors

$$\mathrm{Shv}_{\ell}^{c}(Y) \underset{f_*}{\overset{f^*_{\wedge}}{\rightleftarrows}} \mathrm{Shv}_{\ell}^{c}(X).$$

In this section, we study the exactness properties of these functors with respect to the t-structure of Proposition 2.3.6.1.

Proposition 2.3.7.1. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the pullback functor $f^*_{\wedge} : \mathrm{Shv}_{\ell}^{c}(Y) \to \mathrm{Shv}_{\ell}^{c}(X)$ is t-exact.*

Proof. If $\mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(Y)_{\geq 0}$, then

$$\begin{aligned} (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} f^*_{\wedge} \mathcal{F} &\simeq f^*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \\ &\in f^* \mathrm{Shv}(Y;\mathbf{Z}/\ell\mathbf{Z})_{\geq 0} \\ &\subseteq \mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})_{\geq 0}. \end{aligned}$$

This proves that the functor f^* is right t-exact.

To prove left exactness, we must work a little bit harder. Assume that $\mathcal{F} \in \mathrm{Shv}_\ell^c(Y)_{\leq 0}$, and for $d \geq 0$ set $\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \in \mathrm{Shv}(Y; \mathbf{Z}/\ell^d\mathbf{Z})$. We have

$$\begin{aligned} \tau_{\geq 1} f^*_\wedge \mathcal{F} &\simeq \varprojlim \tau_{\geq 1}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} f^* \mathcal{F}) \\ &\simeq \varprojlim \tau_{\geq 1} f^*(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \\ &= \varprojlim \tau_{\geq 1} f^* \mathcal{F}_d \\ &\simeq \varprojlim f^* \tau_{\geq 1} \mathcal{F}_d . \end{aligned}$$

It will therefore suffice to show that $\varprojlim f^* \tau_{\geq 1} \mathcal{F}_d$ vanishes in $\mathrm{Shv}(X; \mathbf{Z})$. Since the limit is ℓ-complete, we are reduced to proving that the limit

$$\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \varprojlim f^* \tau_{\geq 1} \mathcal{F}_d \simeq \varprojlim f^*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d)$$

vanishes. Using the characterization of $\mathrm{Shv}_\ell^c(Y)_{\leq 0}$ obtained in the proof of Proposition 2.3.6.1, we see that the limit $\varprojlim(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d)$ vanishes in $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$. It will therefore suffice to show that the natural map

$$f^* \varprojlim(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d) \to \varprojlim f^*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \tau_{\geq 1} \mathcal{F}_d)$$

is an equivalence in $\mathrm{Shv}(X; \mathbf{Z})$. This follows from assertion $(*)$ from the proof of Proposition 2.3.6.1. □

Corollary 2.3.7.2. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the direct image functor $f_* : \mathrm{Shv}_\ell^c(X) \to \mathrm{Shv}_\ell^c(Y)$ is left t-exact. If f is a finite morphism, then f_* is t-exact.*

Proof. The left t-exactness of f_* follows immediately from the right t-exactness of the adjoint functor f^* (Proposition 2.3.7.1). If f is a finite morphism, then for $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)_{\geq 0}$ we have

$$\begin{aligned} \mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} f_* \mathcal{F} &\simeq f_*(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}) \\ &\in f_* \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0} \\ &\subseteq \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0}, \end{aligned}$$

so that f_* is right t-exact. □

Corollary 2.3.7.3. *Let $f : X \to Y$ be a finite morphism of quasi-projective k-schemes. Then the exceptional inverse image functor $f^! : \mathrm{Shv}_\ell^c(Y) \to \mathrm{Shv}_\ell^c(X)$ is left t-exact.*

Corollary 2.3.7.4. *Let X be a quasi-projective k-scheme. Then:*

(a) *An object* $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$ *belongs to* $\mathrm{Shv}^c_\ell(X)_{\geq 0}$ *if and only if, for each point* $\eta : \mathrm{Spec}(k) \to X$*, the stalk* $\eta^*_\wedge \mathcal{F} \in \mathrm{Shv}^c_\ell(\mathrm{Spec}(k)) \simeq \mathrm{Mod}^{\mathrm{pf}}_{\mathbf{Z}_\ell}$ *belongs to* $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\geq 0}$.

(b) *An object* $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$ *belongs to* $\mathrm{Shv}^c_\ell(X)_{\leq 0}$ *if and only if, for each point* $\eta : \mathrm{Spec}(k) \to X$*, the stalk* $\eta^* \mathcal{F} \in \mathrm{Shv}^c_\ell(\mathrm{Spec}(k)) \simeq \mathrm{Mod}^{\mathrm{pf}}_{\mathbf{Z}_\ell}$ *belongs to* $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$.

Proof. We will prove (b); the proof of (a) is similar. The "only if" direction follows immediately from Proposition 2.3.7.1 and Example 2.3.6.3. Conversely, suppose that $\eta^* \mathcal{F}$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$ for each point $\eta : \mathrm{Spec}(k) \to X$. Since the functor $\eta^*_\wedge$ is t-exact (Proposition 2.3.7.1), it follows that the canonical map $\alpha : \mathcal{F} \to \tau_{\leq 0} \mathcal{F}$ induces an equivalence after passing to the stalk at each point, so that α is an equivalence by virtue of Proposition 2.3.3.7. □

2.3.8 The Heart of $\mathrm{Shv}^c_\ell(X)$

Let X be a quasi-projective k-scheme. In this section, we study the relationship between the ∞-category $\mathrm{Shv}^c_\ell(X)$ of ℓ-constructible sheaves on X (introduced in Definition 2.3.2.1) and the classical abelian category of ℓ-adic sheaves (introduced in [15]). We begin by recalling the definition of the latter.

Definition 2.3.8.1. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^\heartsuit$. We will say that the sheaf $\mathcal{F}$ is *imperfect constructible* if it satisfies the following conditions:

(1) There exists a finite sequence of quasi-compact open subsets

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = X$$

such that, for $1 \leq i \leq n$, if Y_i denotes the locally closed reduced subscheme of X with support $U_i - U_{i-1}$, then each restriction $\mathcal{F}|_{Y_i}$ is locally constant.

(2) For every k-valued point $\eta : \mathrm{Spec}(k) \to X$, the pullback $\eta^* \mathcal{F} \in \mathrm{Shv}(\mathrm{Spec}(k); \mathbf{Z}/\ell^d\mathbf{Z}) \simeq \mathrm{Mod}^\heartsuit_{\mathbf{Z}/\ell^d\mathbf{Z}}$ is finite (when regarded as an abelian group).

We let $\mathrm{Shv}^\circ(X; \mathbf{Z}/\ell^d\mathbf{Z})$ denote the full subcategory of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^\heartsuit$ spanned by the imperfect constructible objects.

Example 2.3.8.2. Let $\mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell^d\mathbf{Z})$. Then each of the cohomology sheaves $\pi_i \mathcal{F}$ is imperfect constructible.

Remark 2.3.8.3. If X is a quasi-projective k-scheme, then the full subcategory

$$\mathrm{Shv}^\circ(X; \mathbf{Z}/\ell^d\mathbf{Z}) \subseteq \mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^\heartsuit$$

is closed under the formation of kernels, cokernels, and extensions. Consequently, it forms an abelian category.

Remark 2.3.8.4. For every pair of integers $d' \geq d \geq 0$, the construction

$$\mathcal{F} \mapsto \tau_{\leq 0}((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}/\ell^{d'}\mathbf{Z}} \mathcal{F})$$

carries $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^{d'}\mathbf{Z})$ into $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^d\mathbf{Z})$. We therefore have a tower of (abelian) categories and right-exact functors

$$\cdots \to \mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^3\mathbf{Z}) \to \mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^2\mathbf{Z}) \to \mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell\mathbf{Z}).$$

We will denote the homotopy inverse limit of this tower by $\mathrm{Shv}^{\circ}(X)$.

Proposition 2.3.8.5. *Let X be a quasi-projective k-scheme, and let $\phi : \mathrm{Shv}^c_{\ell}(X)_{\geq 0} \to \mathrm{Shv}^{\circ}(X)$ denote the functor given on objects by the formula*

$$\phi(\mathcal{F}) = \{\tau_{\leq 0}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})\}_{d \geq 0}.$$

Then θ induces an equivalence of categories $\mathrm{Shv}^c_{\ell}(X)^{\heartsuit} \simeq \mathrm{Shv}^{\circ}(X)$. In particular, $\mathrm{Shv}^{\circ}(X)$ is an abelian category.

Proof. Let $\psi : \mathrm{Shv}^{\circ}(X) \to \mathrm{Shv}(X; \mathbf{Z})$ be the functor given by $\psi\{\mathcal{F}_d\}_{d \geq 0} = \varprojlim \mathcal{F}_d$ (where the limit is formed in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$). The proof of Proposition 2.3.6.1 shows that the composite functor $\psi \circ \phi : \mathrm{Shv}^c_{\ell}(X)_{\geq 0} \to \mathrm{Shv}(X; \mathbf{Z})$ is given by $\mathcal{F} \mapsto \tau_{\leq 0} \mathcal{F}$ (where the truncation is formed with respect to the t-structure of Proposition 2.3.6.1). Consequently, ψ is a left homotopy inverse of the restriction $\phi|_{\mathrm{Shv}^c_{\ell}(X)^{\heartsuit}}$. To complete the proof, it will suffice to show that ψ factors through the full subcategory $\mathrm{Shv}^c_{\ell}(X)^{\heartsuit} \subseteq \mathrm{Shv}(X; \mathbf{Z})$ and that it is a right homotopy inverse to $\phi|_{\mathrm{Shv}^c_{\ell}(X)^{\heartsuit}}$. To prove this, we must show that every object $\{\mathcal{F}_d\}_{d \geq 0}$ of $\mathrm{Shv}^{\circ}(X)$ has the following properties:

- (a) The inverse limit $\mathcal{F} = \varprojlim \mathcal{F}_d$ (formed in the ∞-category $\mathrm{Shv}(X; \mathbf{Z})$) is ℓ-complete.
- (b) The tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ is a constructible object of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$.
- (c) The limit $\mathcal{F} = \varprojlim \mathcal{F}_d$ belongs to $\mathrm{Shv}^c_{\ell}(X)_{\geq 0}$: in other words, the tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\geq 0}$.
- (d) The limit $\mathcal{F} = \varprojlim \mathcal{F}_d$ belongs to $\mathrm{Shv}^c_{\ell}(X)_{\leq 0}$.
- (e) For each integer $d \geq 0$, the canonical map $\tau_{\leq 0}(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}) \to \mathcal{F}_d$ is an equivalence in $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})^{\heartsuit}$.

Assertion (a) is clear. Note that the tensor product $(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ can be identified with the limit of the diagram $\{(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_d\}_{d \geq 0}$. For each $d \geq 1$, we have

$$\pi_i(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F}_d) \simeq \begin{cases} \mathcal{F}_1 & \text{if } i = 0 \\ \ker(\ell : \mathcal{F}_d \to \mathcal{F}_d) & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 2.3.6.4 and Proposition 2.2.7.4, we see that the tower $\{\ker(\ell : \mathcal{F}_d \to \mathcal{F}_d)\}_{d \geq 1}$ is constant as a Pro-object of $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell\mathbf{Z})$. It follows that $\pi_i(\mathbf{Z}/\ell\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{F})$ is $\mathcal{F}_1$ when $i = 0$, a retract of some $\ker(\ell : \mathcal{F}_d \to \mathcal{F}_d)$ if $i = 1$, and vanishes otherwise. This proves (b) and (c). To prove (d), we note that for $\mathcal{G} \in \mathrm{Shv}^c_{\ell}(X)_{\geq 0}$, the mapping space

$$\begin{aligned} \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{G}, \Sigma^{-1}\mathcal{F}) &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{G}, \Sigma^{-1}\mathcal{F}_d) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})}(\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}} \mathcal{G}, \Sigma^{-1}\mathcal{F}_d) \end{aligned}$$

is contractible because each $(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\geq 0}$. To prove (e), we first observe that for $d' \geq d$, we have

$$\pi_i((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}) \simeq \begin{cases} \mathcal{F}_d & \text{if } i = 0 \\ \ker(\ell^d : \mathcal{F}_{d'} \to \mathcal{F}_{d'}) & \text{if } i = 1 \\ 0. & \text{otherwise} \end{cases}$$

Using Lemma 2.3.6.4 and Proposition 2.2.7.4, we see that the tower $\{\ker(\ell^d : \mathcal{F}_{d'} \to \mathcal{F}_{d'})\}_{d' \geq d}$ is equivalent to a constant Pro-object of $\mathrm{Shv}^{\circ}(X; \mathbf{Z}/\ell^d\mathbf{Z})$, so that the tower $\{((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}\}_{d' \geq d}$ is constant and we obtain an equivalence

$$\tau_{\leq 0} \varprojlim_{d'} ((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}) \to \varprojlim_{d'} \tau_{\leq 0}((\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}_{d'}) \simeq \mathcal{F}_d \, .$$

□

2.3.9 The t-Structure on $\mathrm{Shv}_{\ell}(X)$

We now extend the constructions of §2.3.6 to the setting of ℓ-adic sheaves which are not constructible.

Notation 2.3.9.1. Let X be a quasi-projective k-scheme. We let $\mathrm{Shv}_{\ell}(X)_{\geq 0}$ and $\mathrm{Shv}_{\ell}(X)_{\leq 0}$ denote the essential images of the fully faithful embeddings

$$\mathrm{Ind}(\mathrm{Shv}^c_{\ell}(X)_{\geq 0}) \hookrightarrow \mathrm{Ind}(\mathrm{Shv}^c_{\ell}(X)) = \mathrm{Shv}_{\ell}(X)$$

$$\mathrm{Ind}(\mathrm{Shv}^c_{\ell}(X)_{\leq 0}) \hookrightarrow \mathrm{Ind}(\mathrm{Shv}^c_{\ell}(X)) = \mathrm{Shv}_{\ell}(X).$$

It follows from Proposition 2.3.6.1 that the full subcategories $(\mathrm{Shv}_{\ell}(X)_{\geq 0}, \mathrm{Shv}_{\ell}(X)_{\leq 0})$ determine a t-structure on the ∞-category $\mathrm{Shv}_{\ell}(X)$.

Remark 2.3.9.2. Let $f : X \to Y$ be a morphism of quasi-projective k-schemes. Then the pullback functor $f^* : \mathrm{Shv}_{\ell}(Y) \to \mathrm{Shv}_{\ell}(X)$ is t-exact, and the pushforward functor

$f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ is left t-exact. If f is finite, then $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ is t-exact and $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ is left t-exact. These assertions follow immediately from Proposition 2.3.7.1, Corollary 2.3.7.2, and Corollary 2.3.7.3. Beware that the analogue of Corollary 2.3.7.4 for non-constructible ℓ-adic sheaves is generally false: for example, one can find nonzero objects of $\mathrm{Shv}_\ell(X)$ with vanishing stalks (or costalks) at every point.

Proposition 2.3.9.3. *Let X be a quasi-projective k-scheme. Then the t-structure on $\mathrm{Shv}_\ell(X)$ is right complete (that is, the canonical map $\mathrm{Shv}_\ell(X) \to \varprojlim_n \mathrm{Shv}_\ell(X)_{\geq -n}$ is an equivalence of ∞-categories). Moreover, the canonical map $\mathrm{Shv}_\ell(X) \to \varprojlim_n \mathrm{Shv}_\ell(X)_{\leq n}$ is fully faithful.*

Remark 2.3.9.4. We do not know if the t-structure on $\mathrm{Shv}_\ell(X)$ is left complete.

Lemma 2.3.9.5. *Let X be a quasi-projective k-scheme. Then there exists an integer q with the following property: if $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)_{\leq 0}$ and $\mathcal{G} \in \mathrm{Shv}_\ell(X)_{\geq q}$, then every morphism $\mathcal{F} \to \mathcal{G}$ is nullhomotopic.*

Proof. By virtue of Proposition 2.2.7.3, we can choose an integer n for which the groups $\mathrm{Ext}^m_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}(\mathcal{F}', \mathcal{G}')$ vanish whenever $\mathcal{F}' \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z})^\heartsuit$, $\mathcal{G}' \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})^\heartsuit$, and $m > n$. We will show that $q = n+3$ has the desired property. Let $\mathcal{F} \in \mathrm{Shv}_\ell^c(X)_{\leq 0}$ and $\mathcal{G} \in \mathrm{Shv}_\ell(X)_{\geq n+3}$; we wish to prove that $\mathrm{Ext}^0_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \mathcal{G}) \simeq 0$. Writing $\mathcal{G}$ as a filtered colimit of objects of $\mathrm{Shv}_\ell^c(X)_{\geq n+3}$, we may assume that $\mathcal{G}$ is constructible. For each $d \geq 0$, set

$$\mathcal{F}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \qquad \mathcal{G}_d = (\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}.$$

Note that $\mathcal{F}$ can be regarded as an object of $\mathrm{Shv}(X; \mathbf{Z})_{\leq 0}$ (Remark 2.3.6.5), so that each $\mathcal{F}_d$ belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})_{\leq 1}$. We have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim \mathrm{Map}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)$$

which gives rise to Milnor exact sequences

$$0 \to \lim{}^1\{\mathrm{Ext}^{-1}_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)\} \to \mathrm{Ext}^0_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \mathcal{G}) \to \lim{}^0\{\mathrm{Ext}^0_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)\}.$$

It will therefore suffice to show that the groups $\mathrm{Ext}^i_{\mathrm{Shv}(X;\mathbf{Z})}(\mathcal{F}, \mathcal{G}_d)$ vanish for $i \in \{0, -1\}$. Writing $\mathcal{G}_d$ as a successive extension of finitely many copies of $\mathcal{G}_1$, we may reduce to the case $d = 1$. We are therefore reduced to showing that the groups $\mathrm{Ext}^i_{\mathrm{Shv}(X;\mathbf{Z}/\ell\mathbf{Z})}(\mathcal{F}_1, \mathcal{G}_1)$ vanish for $i \in \{0, -1\}$. The desired result now follows by writing $\mathcal{F}_1$ and $\mathcal{G}_1$ as successive extensions of objects belonging to the heart $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})^\heartsuit$. □

Proof of Proposition 2.3.9.3. The right completeness of the $\mathrm{Shv}_\ell(X)$ follows formally from the right boundedness of $\mathrm{Shv}_\ell^c(X)$ (Proposition 2.3.9.3). To see this, we first observe that the full subcategory $\mathrm{Shv}_\ell(X)_{\leq 0}$ is closed under infinite direct sums. To show that $\mathrm{Shv}_\ell(X)$ is right complete, it will suffice to show that the intersection $\bigcap \mathrm{Shv}_\ell(X)_{\leq -n}$ consists only of zero objects (Proposition [23].1.2.1.19). To prove this, let $\mathcal{F} \in \bigcap \mathrm{Shv}_\ell(X)_{\leq -n}$. If $\mathcal{F} \neq 0$, then there exists an object $\mathcal{F}' \in \mathrm{Shv}^c_\ell(X)$ and a nonzero map $\mathcal{F}' \to \mathcal{F}$. This is impossible, since $\mathcal{F}'$ belongs to $\mathrm{Shv}_\ell(X)_{\geq m}$ for some integer m (by virtue of the right boundedeness of the t-structure on $\mathrm{Shv}^c_\ell(X)$).

To complete the proof, it will suffice to show that for every object $\mathcal{G} \in \mathrm{Shv}_\ell(X)$, the canonical map $\mathcal{G} \to \varprojlim \tau_{\leq n} \mathcal{G}$ is an equivalence. Equivalently, we must show that the object $\varprojlim \tau_{\geq n} \mathcal{G}$ vanishes. To prove this, we argue that for each constructible object $\mathcal{F} \in \mathrm{Shv}^c_\ell(X)$, the mapping space $\mathrm{Map}_{\mathrm{Shv}_\ell(X)}(\mathcal{F}, \varprojlim \tau_{\geq n} \mathcal{G})$ is contractible. We have Milnor exact sequences

$$\lim{}^1 \{\mathrm{Ext}^{m-1}(\mathcal{F}, \tau_{\geq n} \mathcal{G})\}_{n \geq 0} \to \mathrm{Ext}^m(\mathcal{F}, \varprojlim \tau_{\geq n} \mathcal{G}) \to \lim{}^0 \{\mathrm{Ext}^m(\mathcal{F}, \tau_{\geq n} \mathcal{G})\}_{n \geq 0},$$

where all Ext-groups are formed in the stable ∞-category $\mathrm{Shv}_\ell(X)$. The desired result now follows from Lemma 2.3.9.5 (and the left boundedness of $\mathrm{Shv}^c_\ell(X)$), which guarantees that the groups $\mathrm{Ext}^{m-1}(\mathcal{F}, \tau_{\geq n} \mathcal{G})$ and $\mathrm{Ext}^m(\mathcal{F}, \tau_{\geq n} \mathcal{G})$ are trivial for $n \gg 0$. □

2.4 Base Change Theorems

Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Our goal is to review some nontrivial results in the theory of étale cohomology which will be applied in later chapters of this book.

2.4.1 Digression: The Beck-Chevalley Property

We begin with some general categorical remarks.

Notation 2.4.1.1. Suppose we are given a diagram of ∞-categories and functors σ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathcal{C}' & \xrightarrow{f'} & \mathcal{D}' \end{array}$$

which commutes up to specified homotopy: that is, we are given an equivalence of functors $\alpha : g' \circ f \simeq f' \circ g$. Suppose that f and f' admit left adjoints f^L and f'^L, respectively. Then σ determines a map $\beta : f'^L \circ g' \to g \circ f^L$, given by the composition

$$f'^L \circ g' \to f'^L \circ g' \circ f \circ f^L \xrightarrow{\alpha} f'^L \circ f' \circ g \circ f^L \to g \circ f^L$$

where the first and third maps are given by composition with the unit and counit for the adjunctions between the pairs (f^L, f) and (f'^L, f'), respectively. We will refer to

β as the *left Beck-Chevalley transformation* determined by α. We will say that the diagram σ is *left adjointable* if the functors f and f' admit left adjoints and the natural transformation β is an equivalence. If f and f' admit right adjoints f^R and f'^R, then a dual construction yields a natural transformation

$$\gamma : g \circ f^R \to f'^R \circ g',$$

which we will refer to as the *right Beck-Chevalley transformation* determined by σ. We will say that σ is *right adjointable* if the functors f and f' admit right adjoints and the natural transformation γ is an equivalence.

Remark 2.4.1.2. In the situation of Notation 2.4.1.1, suppose that the functors f, f', g, and g' all admit left adjoints. We then obtain a diagram σ^L:

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{g'^L} & \mathcal{D} \\ \downarrow{\scriptstyle f'^L} & & \downarrow{\scriptstyle f^L} \\ \mathcal{C}' & \xrightarrow{g^L} & \mathcal{C} \end{array}$$

which commutes up to (preferred) homotopy, and the vertical maps admit right adjoints g' and g. We therefore obtain a right Beck-Chevalley transformation $f'^L \circ g' \to g \circ f^L$ for σ^L, which agrees (up to canonical homotopy) with the left Beck-Chevalley transformation for σ.

Remark 2.4.1.3. Suppose we are given a diagram of ∞-categories σ :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathcal{C}' & \xrightarrow{f'} & \mathcal{D}', \end{array}$$

where the functors f and f' admit left adjoints f^L and f'^L, and the functors g and g' admit right adjoints g^R and g'^R. Applying the Construction of Notation 2.4.1.1 to σ and to the transposed diagram σ^t :

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{g} & \mathcal{C}' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathcal{D} & \xrightarrow{g'} & \mathcal{D}', \end{array}$$

we obtain left and right Beck-Chevalley transformations

$$\beta : f'^L \circ g' \to g \circ f^L \qquad \gamma : f \circ g^R \to g'^R \circ f'.$$

Unwinding the definitions, we see that γ is the natural transformation obtained from β by passing to right adjoints. In particular, under the assumption that the relevant adjoints exist, the diagram σ is left adjointable if and only if the diagram σ^t is right adjointable.

2.4.2 Smooth and Proper Base Change

We now specialize to the setting of algebraic geometry. Suppose we are given a commutative diagram σ :

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

of quasi-projective k-schemes. Then σ determines a diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'^*} \\ \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X'). \end{array}$$

Each functor in this diagram admits a right adjoint, so we obtain a right Beck-Chevalley transformation $\beta : f^* p_* \to p'_* f'^*$. The following statement summarizes some of the main foundational results in the theory of étale cohomology:

Theorem 2.4.2.1 (Smooth and Proper Base Change)**.** *Suppose we are given a pullback diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S. \end{array}$$

If either p is proper or f is smooth, then the Beck-Chevalley morphism $\beta : f^* p_* \to p'_* f'^*$ is an equivalence of functors from $\mathrm{Shv}_\ell(X)$ to $\mathrm{Shv}_\ell(S')$.

Proof. Let $\mathcal{F} \in \mathrm{Shv}_\ell(X')$; we wish to prove that the canonical map $\beta_{\mathcal{F}} : f^* p_* \mathcal{F} \to p'_* f'^* \mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(S')$. Writing $\mathcal{F}$ as a filtered colimit of constructible ℓ-adic sheaves (and using the fact that the functors f^*, p_*, p'_*, and f'^* commute with filtered colimits), we can reduce to the case where $\mathcal{F}$ is constructible. In this case, the domain and codomain of $\beta_{\mathcal{F}}$ are constructible ℓ-adic sheaves (see Notation 2.3.4.5). We may therefore identify $\beta_{\mathcal{F}}$ with a morphism in the ∞-category $\mathrm{Shv}^c_\ell(S') \subseteq \mathrm{Shv}(S'; \mathbf{Z}_\ell)$.

Since the domain and codomain of $\beta_{\mathcal{F}}$ are ℓ-complete, it will suffice to show that the induced map

$$(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} f^* p_* \mathcal{F} \to (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} p'_* f'^* \mathcal{F}$$

is an equivalence in $\mathrm{Shv}(S'; \mathbf{Z}/\ell\mathbf{Z})$. For this, it suffices to show that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}^c(S; \mathbf{Z}/\ell\mathbf{Z}) & \xrightarrow{p^*} & \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z}) \\ \downarrow f^* & & \downarrow f'^* \\ \mathrm{Shv}^c(S'; \mathbf{Z}/\ell\mathbf{Z}) & \xrightarrow{p'^*} & \mathrm{Shv}^c(X'; \mathbf{Z}/\ell\mathbf{Z}) \end{array}$$

is right adjointable: that is, that the canonical map $f^* p_* \mathcal{G} \to p'_* f'^* \mathcal{G}$ is an equivalence for each constructible object $\mathcal{G} \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$. The constructibility of $\mathcal{G}$ implies that it can be written as a finite extension of suspensions of objects belonging to the heart $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ (which we can identify with the abelian category étale sheaves of $\mathbf{Z}/\ell\mathbf{Z}$ on X). The desired result now follows from the usual smooth and proper base change theorems for étale cohomology (see [10]). □

2.4.3 Direct Images and Extension by Zero

In the situation of Theorem 2.4.2.1, suppose that the map f is étale. Then the pullback functors f^* and f'^* admit left adjoints $f_!$ and $f'_!$. Invoking the dual of Remark 2.4.1.3, we obtain the following version of Proposition 2.2.3.5:

Variant 2.4.3.1. Suppose we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow p' & & \downarrow p \\ S' & \xrightarrow{f} & S, \end{array}$$

where f is étale. Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \\ \downarrow p^* & & \downarrow p'^* \\ \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X') \end{array}$$

is left adjointable. In other words, the associated Beck-Chevalley transformation $\beta' : f'_! p'^* \to p^* f_!$ is an equivalence of functors from $\mathrm{Shv}_\ell(S')$ to $\mathrm{Shv}_\ell(X)$.

Remark 2.4.3.2. It is easy to deduce Variant 2.4.3.1 directly from Proposition 2.2.3.5; the full force of the smooth base change theorem is not required.

Construction 2.4.3.3. Suppose that we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where f is étale, so that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p'_*} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \end{array}$$

commutes up to canonical homotopy (Theorem 2.4.2.1). Note that the horizontal maps admit left adjoints $f'_!$ and $f_!$, respectively, so that there is an associated left Beck-Chevalley transformation $\gamma : f_! p'_* \to p_* f'_!$. By virtue of Remark 2.4.1.2, we can also identify γ with the right Beck-Chevalley transformation associated to the diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f'_!} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \end{array}$$

of Variant 2.4.3.1.

Proposition 2.4.3.4. *Suppose that we are given a pullback diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S, \end{array}$$

where f is étale. If p is proper, then the natural transformation $\gamma : f_! p'_* \to p_* f'_!$ of Construction 2.4.3.3 is an equivalence. In other words, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p'_*} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \end{array}$$

is left adjointable, and the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X') \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f'_!} \\ \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \end{array}$$

is right adjointable.

Proof. Let $\mathcal{F} \in \mathrm{Shv}_\ell(X')$; we wish to show that the map $\gamma_{\mathcal{F}} : f_! p'_* \mathcal{F} \to p_* f'_! \mathcal{F}$ is an equivalence in $\mathrm{Shv}_\ell(S)$. Writing $\mathcal{F}$ as a filtered colimit of constructible ℓ-adic sheaves (and using the fact that the functors $f_!$, p_*, $f'_!$, and p'_* commute with filtered colimits), we can reduce to the case where $\mathcal{F}$ is constructible. In this case, the domain and codomain of $\gamma_{\mathcal{F}}$ are also constructible ℓ-adic sheaves. Using Proposition 2.3.3.7, we are reduced to showing that $\beta_{\mathcal{F}}$ induces an equivalence $\eta^* f_! p'_* \mathcal{F} \to \eta^* p_* f'_! \mathcal{F}$ for every point $\eta : \mathrm{Spec}(k) \to S$. Using Theorem 2.4.2.1 and Variant 2.4.3.1, we can replace S by $\mathrm{Spec}(k)$. In this case, S' is isomorphic to a disjoint union of finitely many copies of $\mathrm{Spec}(k)$ and the result is easy. □

2.4.4 Base Change for Exceptional Inverse Images

Let us now return to the setting of Theorem 2.4.2.1. Suppose we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S. \end{array}$$

Note that the right adjointability of the diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S) & \xrightarrow{p^*} & \mathrm{Shv}_\ell(X) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f'^*} \\ \mathrm{Shv}_\ell(S') & \xrightarrow{p'^*} & \mathrm{Shv}_\ell(X'). \end{array}$$

is equivalent to the left adjointability of the diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X') & \xrightarrow{f'_*} & \mathrm{Shv}_\ell(X) \\ \downarrow{\scriptstyle p'_*} & & \downarrow{\scriptstyle p_*} \\ \mathrm{Shv}_\ell(S') & \xrightarrow{f_*} & \mathrm{Shv}_\ell(S) \end{array}$$

(see Remark 2.4.1.2). If p is proper, then the vertical maps admit right adjoints given by $p^!$ and $p'^!$, respectively. Invoking Remark 2.4.1.3, we obtain:

Variant 2.4.4.1. Suppose we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S, \end{array}$$

where p is proper. Then the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \end{array}$$

is right adjointable. In other words, the right Beck-Chevalley transformation

$$\beta'' : f'_* p'^! \to p^! f_*$$

is an equivalence of functors from $\mathrm{Shv}_\ell(S')$ to $\mathrm{Shv}_\ell(X)$.

We have the following dual version of Construction 2.4.3.3:

Construction 2.4.4.2. Suppose that we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper, so that the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \end{array}$$

commutes up to canonical homotopy (Theorem 2.4.2.1). Note that the horizontal maps admit right adjoints $p^!$ and $p'^!$, so that there is an associated right Beck-Chevalley transformation $\gamma' : f'^* p^! \to p'^! f^*$ of functors from $\mathrm{Shv}_\ell(S)$ to $\mathrm{Shv}_\ell(X')$. Using Remark

2.4.1.2, we can also identify γ' with the left Beck-Chevalley transformation associated to the diagram

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{f_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle p'^!} & & \downarrow{\scriptstyle p^!} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{f'_*} & \mathrm{Shv}_\ell(X) \end{array}$$

of Variant 2.4.4.1.

Proposition 2.4.4.3. *Suppose we are given a commutative diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper. If f is smooth, then the natural transformation $\gamma' : f'^* p^! \to p'^! f^*$ of Construction 2.4.4.2 is an equivalence. In other words, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle f'^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \end{array}$$

is right adjointable and the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S') & \xrightarrow{f_*} & \mathrm{Shv}_\ell(S) \\ \downarrow{\scriptstyle p'^!} & & \downarrow{\scriptstyle p^!} \\ \mathrm{Shv}_\ell(X') & \xrightarrow{f'_*} & \mathrm{Shv}_\ell(X) \end{array}$$

is left adjointable.

Remark 2.4.4.4. In the situation of Proposition 2.4.4.3, suppose that p is proper and f is étale. In this case, the natural transformation $\gamma' : f'^* p^! \to p'^! f^*$ is obtained from the natural transformation $\gamma : f_! p'_* \to p_* f'_!$ of Construction 2.4.3.3 by passing to right adjoints. In this case, Proposition 2.4.4.3 reduces to Proposition 2.4.3.4.

Proof of Proposition 2.4.4.3. Fix an object $\mathcal{F} \in \mathrm{Shv}_\ell(S)$; we wish to show that the map $\gamma'_{\mathcal{F}} : f'^* p^! \mathcal{F} \to p'^! f^* \mathcal{F}$ is an equivalence. Since the construction $\mathcal{F} \mapsto \gamma'_{\mathcal{F}}$ preserves filtered colimits, we may assume without loss of generality that $\mathcal{F}$ is a constructible ℓ-adic sheaf. For every point $\eta : \mathrm{Spec}(k) \to X$, let i_η denote the inclusion of the fiber product $X' \times_X \operatorname{Spec}(k)$ into X'. By virtue of Proposition 2.3.3.7, it will suffice to show that $i_\eta^! \gamma'_{\mathcal{F}}$ is an equivalence for each η. Let $f'' : X' \times_X \operatorname{Spec}(k) \to \operatorname{Spec}(k)$ denote the projection map, so that $i_\eta^! \gamma_{\mathcal{F}}$ fits into a commutative diagram

$$\begin{array}{ccccc} & & i_\eta^! f'^* p^! & & \\ & \overset{\beta'}{\nearrow} & & \overset{\eta^! \gamma'_{\mathcal{F}}}{\searrow} & \\ f''^* \eta^! p^! & & \xrightarrow{\beta''} & & i_\eta^! p'^! f^*. \end{array}$$

It will therefore suffice to show that β' and β'' are equivalences. We may therefore replace the map p by either η or $p \circ \eta$, and thereby reduce to the case where p is a closed immersion.

Let $j : U \to S$ be an open immersion complementary to p, let U' denote the fiber product $U \times_S S'$, and let $j' : U' \to S'$ denote the projection onto the second factor. If p is a closed immersion, then the pushforward functor p'_* is fully faithful. It will therefore suffice to show that $p'_* \gamma'_{\mathcal{F}}$ is an equivalence. Identifying $p'_* f'^* p^! \mathcal{F}$ with $f^* p_* p^! \mathcal{F}$, we see that $p'_* \gamma'_{\mathcal{F}}$ fits into a commutative diagram of fiber sequences

$$\begin{array}{ccccc} f^* p_* p^! \mathcal{F} & \longrightarrow & f^* \mathcal{F} & \longrightarrow & f^* j_* j^* \mathcal{F} \\ \downarrow{\scriptstyle p'_* \gamma'_{\mathcal{F}}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \rho} \\ p'_* p'^! f^* \mathcal{F} & \longrightarrow & f^* \mathcal{F} & \longrightarrow & j'_* j'^* f^* \mathcal{F}. \end{array}$$

It will therefore suffice to show that ρ is an equivalence. This follows from Theorem 2.4.2.1, since f is smooth. □

Example 2.4.4.5. Let X be a quasi-projective k-scheme, and let $j : U \to X$ be an open immersion whose image is also closed in X. Then j is a proper map, and the diagram

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{id}} & U \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} \\ U & \xrightarrow{j} & X \end{array}$$

is a pullback square. Then Proposition 2.4.4.3 supplies a canonical equivalence

$$j^! \simeq \mathrm{id}^* j^! \simeq \mathrm{id}^! j^* \simeq j^*.$$

Example 2.4.4.6. Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Let $U \subseteq X$ be the locus over which f is étale, let f_0 be the restriction of f to U, let $j : U \hookrightarrow X$ be the inclusion map, and let $\delta : U \to U \times_Y X$ be the diagonal map. Then δ exhibits U as a direct summand of $U \times_Y X$, so that Example 2.4.4.5 supplies an equivalence $\delta^! \simeq \delta^*$. Applying Proposition 2.4.4.3 to the pullback square

$$\begin{array}{ccc} U \times_Y X & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{f_0} & Y, \end{array}$$

we obtain a natural equivalence

$$j^* f^! \simeq \delta^* \pi_2^* f^! \simeq \delta^* \pi_1^! f_0^* \simeq \delta^! \pi_1^! f_0^* \simeq f_0^*.$$

In particular, if f is both étale and proper, then the functors $f^!$ and f^* are canonically equivalent to one another (one can show that this equivalence agrees with the one supplied by Example 2.3.3.5).

Variant 2.4.4.7. Suppose we are given a commutative diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array}$$

where p is proper. Let $U \subseteq X$ be an open subset for which the restriction $p|_U$ is smooth, and let $U' \subseteq X'$ denote the inverse image of U. Then the natural transformation $\gamma' : f'^* p^! \to p'^! f^*$ of Construction 2.4.4.2 induces an equivalence

$$(f'^* p^! \mathcal{F})|_{U'} \to (p'^! f^* \mathcal{F})|_{U'}$$

for each object $\mathcal{F} \in \mathrm{Shv}_\ell(S)$. In particular, if p is smooth, then γ' is an equivalence.

Proof. The assertion is local on U. We may therefore assume without loss of generality that there exists an étale map of S-schemes $g : U \to \mathbf{P}^n \times_{\mathrm{Spec}(k)} S$. Let $\Gamma \subseteq U \times \mathbf{P}^n$ denote the graph of g, let $\overline{\Gamma} \subseteq X \times_{\mathrm{Spec}(k)} \mathbf{P}^n$ be the closure of Γ, and let $q : \overline{\Gamma} \to X$ be the projection onto the first factor. Then q is a proper morphism which restricts to an isomorphism over the open set U. Using Example 2.4.4.6, we can replace X by $\overline{\Gamma}$ and thereby reduce to the case where g extends to a map $\overline{g} : \mathbf{P}^n \times_{\mathrm{Spec}(k)} S$. Using Example 2.4.4.6 again, we can replace X by $\mathbf{P}^n \times_{\mathrm{Spec}(k)} S$, and thereby reduce to the case where p is smooth. In this case, the desired result follows from the description of the functors $p^!$ and $p'^!$ supplied by Example 2.3.3.5 (and the fact that this description is compatible with base change). $\square$

Construction 2.4.4.8. Suppose that we are given a pullback diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S, \end{array}$$

where p is proper and f is étale. Then Proposition 2.4.4.3 supplies a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(S) & \xrightarrow{f^*} & \mathrm{Shv}_\ell(S') \\ \downarrow{\scriptstyle p^!} & & \downarrow{\scriptstyle p'^!} \\ \mathrm{Shv}_\ell(X) & \xrightarrow{f'^*} & \mathrm{Shv}_\ell(X'). \end{array}$$

Note that the horizontal maps admit left adjoints $f_!$ and $f'_!$, so that there is an associated left Beck-Chevalley transformation $\delta : f'_! p'^! \to p^! f_!$ of functors from $\mathrm{Shv}_\ell(S')$ to $\mathrm{Shv}_\ell(X)$. Using Remarks 2.4.1.2 and 2.4.4.4, we see that δ can also be identified with the right Beck-Chevalley transformation associated to the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell(X') & \xrightarrow{p'_*} & \mathrm{Shv}_\ell(S') \\ \downarrow{\scriptstyle f'_!} & & \downarrow{\scriptstyle f_!} \\ \mathrm{Shv}_\ell(X) & \xrightarrow{p_*} & \mathrm{Shv}_\ell(S) \end{array}$$

given by Proposition 2.4.3.4.

Proposition 2.4.4.9. *Suppose we are given a commutative diagram of quasi-projective k-schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S, \end{array}$$

where p is proper and f is étale, and let $\delta : f'_! p'^! \to p^! f_!$ be the natural transformation of Construction 2.4.4.8. If U is an open subset of X such that $p|_U$ is smooth, then δ induces an equivalence

$$(f'_! p'^! \mathcal{F})|_U \to (p^! f_! \mathcal{F})|_U$$

for each object $\mathcal{F} \in \mathrm{Shv}_\ell(S')$. In particular, if p is smooth, then δ is an equivalence, so *that the diagrams of ∞-categories*

$$\begin{array}{ccc} \mathrm{Shv}_{\ell}(S) & \xrightarrow{f^*} & \mathrm{Shv}_{\ell}(S') \\ \downarrow{\scriptstyle p^!} & & \downarrow{\scriptstyle p'^!} \\ \mathrm{Shv}_{\ell}(X) & \xrightarrow{f'^*} & \mathrm{Shv}_{\ell}(X') \end{array} \qquad \begin{array}{ccc} \mathrm{Shv}_{\ell}(X') & \xrightarrow{p'_*} & \mathrm{Shv}_{\ell}(S') \\ \downarrow{\scriptstyle f'_!} & & \downarrow{\scriptstyle f_!} \\ \mathrm{Shv}_{\ell}(X) & \xrightarrow{p_*} & \mathrm{Shv}_{\ell}(S) \end{array}$$

are left and right adjointable, respectively.

Proof. Arguing as in the proof of Variant 2.4.4.7, we may reduce to the case where p is smooth, in which case the desired result follows from the desciption of the functors $p^!$ and $p'^!$ supplied by Example 2.3.3.5. □

Chapter 3

$\mathbb{E}_\infty$-Structures on ℓ-Adic Cohomology

Let M be a compact manifold and let $\pi : E \to M$ be a fibration of topological spaces. In §1.5.4, we introduced the *continuous tensor product* $\bigotimes_{y \in M} C^*(E_y; \mathbf{Q})$ (Construction 1.5.4.8) and proved that, under some mild hypotheses on π, there is a canonical quasi-isomorphism

$$\bigotimes_{y \in M} C^*(E_y; \mathbf{Q}) \quad \simeq \quad C^*(\mathrm{Sect}_\pi(M); \mathbf{Q}) \tag{3.1}$$

(Theorem 1.5.4.10). Recall that the continuous tensor product $\bigotimes_{y \in M} C^*(E_y; \mathbf{Q})$ was defined as the homotopy colimit of a diagram $\mathcal{B}$ taking values in the category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ of differential graded algebras over $\mathbf{Q}$ (see Construction 1.5.4.8).

For applications to Weil's conjecture, we would like to formulate a version of (3.1) in the setting of algebraic geometry, where we replace M by an algebraic curve X (defined over an algebraically closed field k) and E by the classifying stack BG of a smooth affine group scheme over X. Our goal in this chapter is to lay the groundwork by constructing an analogue of the functor $\mathcal{B}$. Recall that the value of $\mathcal{B}$ on an open disk $U \subseteq M$ is given by the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(\mathrm{Sect}_\pi(U))$, which is canonically quasi-isomorphic to the singular cochain complex $C^*(E_y; \mathbf{Q})$ for any choice of base point $y \in U$. In §3.2, we associate to every algebraic stack $\mathcal{Y}$ over k its *ℓ-adic cochain complex* $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ (here ℓ is some prime number which is invertible in k), which is an algebro-geometric analogue of the singular cochain complex of a topological space. In order to regard these ℓ-adic cochain complexes as an adequate replacement for the diagram $\mathcal{B}$, we will need to address the following questions:

(*a*) How does the ℓ-adic cochain complex $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ change as the point $x \in X$

varies? (In other words, what is the algebro-geometric analogue of the statement that the construction $U \mapsto \mathcal{B}(U)$ is a functor?)

(*b*) In what sense can we regard the ℓ-adic cochain complex $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ as an algebra? (In other words, what is the algebro-geometric analogue of the statement that each $\mathcal{B}(U)$ is a commutative differential graded algebra over $\mathbf{Q}$?)

Let us begin by addressing (*a*). Note that, for each open disk $U \subseteq M$, the chain complex $\mathcal{B}(U)$ is quasi-isomorphic to the hypercohomology of U with coefficients in the (derived) direct image $\pi_* \underline{\mathbf{Q}}$, where $\underline{\mathbf{Q}}$ denotes the constant sheaf on E with values in $\mathbf{Q}$. This has an obvious algebro-geometric counterpart: if G is a smooth affine group scheme over an algebraic curve X and $\pi : \mathrm{BG} \to X$ denotes the projection map, then we can regard the direct image $\pi_* \underline{\mathbf{Z}_\ell}$ as an ℓ-adic sheaf on X. If the fibers of G are reductive, then the map π is a locally trivial fiber bundle in the étale topology, and the stalk of $\pi_* \underline{\mathbf{Z}_\ell}$ at any closed point $x \in X$ can be identified with the ℓ-adic cochain complex $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$. However, the assumption that G is *everywhere* reductive is unreasonably strong. The group schemes G of interest to us will be reductive (even semisimple) at the generic point of X, but we cannot avoid the possibility that G might have "bad reduction" at finitely many closed points $x \in X$. At such a point, the stalk $x^*(\pi_* \underline{\mathbf{Z}_\ell})$ need not be quasi-isomorphic to the ℓ-adic cochain complex $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$. However, using the smoothness of BG over X (and the smooth base change theorem), we can always compute the *costalk* $x^!(\pi_* \underline{\mathbf{Z}_\ell})$: it agrees with the shifted (and Tate-twisted) ℓ-adic cochain complex $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell[-2](-1))$. For our purposes, it will be useful to compensate for this shift by replacing the direct image $\pi_* \underline{\mathbf{Z}_\ell}$ by the tensor product $[\mathrm{BG}]_X = \omega_X \otimes (\pi_* \underline{\mathbf{Z}_\ell})$, which we will refer to as *the cohomology sheaf of the morphism* $\mathrm{BG} \to X$. We will regard the ℓ-adic sheaf $[\mathrm{BG}]_X \in \mathrm{Shv}_\ell(X)$ as an algebro-geometric incarnation of the functor $\mathcal{B}$ of Construction 1.5.4.8, and the identifications $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell) \simeq x^! [\mathrm{BG}]_X$ as an answer to question (*a*).

We now consider (*b*). For every algebraic stack $\mathcal{Y}$, the ℓ-adic cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ is a cochain complex of $\mathbf{Z}_\ell$-modules, whose cohomology groups form a graded-commutative algebra over $\mathbf{Z}_\ell$ with respect to the cup product

$$\cup : \mathrm{H}^i(\mathcal{Y}; \mathbf{Z}_\ell) \times \mathrm{H}^j(\mathcal{Y} : \mathbf{Z}_\ell) \to \mathrm{H}^{i+j}(\mathcal{Y}; \mathbf{Z}_\ell).$$

It is generally impossible to witness the commutativity of the cup product at the cochain level: that is, to equip $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ with the structure of a differential graded algebra over $\mathbf{Z}_\ell$ (in the sense of Definition 1.5.3.2) which depends functorially on $\mathcal{Y}$. However, we can achieve something almost as good: the cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ can always be regarded as an $\mathbb{E}_\infty$-*algebra over* $\mathbf{Z}_\ell$: that is, it admits a multiplication which is commutative and associative up to *coherent* homotopy. Put differently, we cannot arrange that $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ has the structure of a commutative algebra in the ordinary category Chain($\mathbf{Z}_\ell$) of chain complexes over $\mathbf{Z}_\ell$, but we *can* arrange that it has the structure of a commutative algebra in the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$ of Example 2.1.4.8. We will prove this (thereby answering question (*b*)) in §3.2, after first reviewing the theory of $\mathbb{E}_\infty$-algebras (and, more generally, commutative algebra objects of ∞-categories) in §3.1.

For our applications, it will be important to address questions (*a*) and (*b*) simultaneously. In other words, we need to understand not only that each of the ℓ-adic cochain complexes $C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ has the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$, but the sense in which these $\mathbb{E}_\infty$-algebra structures depend "continuously" on the point $x \in X$. We will address this point in §3.4 by showing that the relative cohomology sheaf $[\mathrm{BG}]_X$ can be regarded as a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$ (Theorem 3.4.0.3). However, there is a slight wrinkle: the algebra structure on the cohomology sheaf $[\mathrm{BG}]_X$ is given by a map

$$m : [\mathrm{BG}]_X \otimes^! [\mathrm{BG}]_X \to [\mathrm{BG}]_X$$

whose domain is not the usual tensor product of ℓ-adic sheaves, but is instead defined by the formula $\mathcal{F} \otimes^! \mathcal{G} = \delta^!(\mathcal{F} \boxtimes \mathcal{G})$ where $\delta : X \to X \times X$ is the diagonal map (see Construction 3.3.0.2). We will study the functor $\otimes^!$ and its coherence properties in §3.3.

Remark 3.0.0.1. In Chapter 4, we will formulate an algebro-geometric analogue of (3.1) which relates the ℓ-adic cochain complexes $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ and $\{C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)\}_{x \in X}$ (see Theorem 4.1.2.1). In order to deduce Weil's conjecture, we do not need to know that this statement holds at the integral level: it is sufficient to establish an analogous result relating $C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$ to $\{C^*(\mathrm{BG}_x; \mathbf{Q}_\ell)\}_{x \in X}$. With rational coefficients, the distinction between $\mathbb{E}_\infty$-algebras and commutative differential graded algebras disappears: for any algebraic stack $\mathcal{Y}$, it is possible to replace $C^*(\mathcal{Y}; \mathbf{Q}_\ell)$ by a commutative differential graded algebra over $\mathbf{Q}_\ell$, analogous to the polynomial de Rham complex of Construction 1.5.3.6 (see Proposition 3.1.5.4). However, this observation does not lead to any simplifications of our overall strategy: in order to formulate Theorem 4.1.2.1, we will need to regard the cohomology sheaf $[\mathrm{BG}]_X$ as a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$ (with respect to the !-tensor product of §3.3). Even with coefficients in $\mathbf{Q}_\ell$, such algebras do not admit obvious "concrete" models when X is not a point. Consequently, we do not know how to avoid appealing to the ∞-categorical theory of commutative algebras described in §3.1.

3.1 Commutative Algebras

Let X be a compact manifold and let $\pi : E \to X$ be a fibration of topological spaces. Under some mild hypotheses, Theorem 1.5.4.10 supplies a canonical quasi-isomorphism

$$\bigotimes_{x \in X} C^*(E_x; \mathbf{Q}) \xrightarrow{\sim} C^*(\mathrm{Sect}_\pi(X); \mathbf{Q}).$$

Our goal in this section is to address the following:

Question 3.1.0.1. Let Λ be a commutative ring. Can we make sense of a continuous tensor product $\bigotimes_{x \in X} C^*(E_x; \Lambda)$? If so, do we have a continuous Künneth decomposition $\bigotimes_{x \in X} C^*(E_x; \Lambda) \xrightarrow{\sim} C^*(\mathrm{Sect}_\pi(X); \Lambda)$?

Remark 3.1.0.2. As stated, Question 3.1.0.1 is a bit orthogonal to the ultimate aims of this book. What we would *really* like to do is to prove an analogue of Theorem 1.5.4.10 in the setting of algebraic geometry, where singular cohomology is replaced by ℓ-adic cohomology. For applications to Weil's conjecture, it would suffice to know that such a result held with coefficients in $\mathbf{Q}_\ell$. However, our answer to Question 3.1.0.1 will introduce some ideas which play an essential role in the formulation of the algebro-geometric product formula of Chapter 4.

To appreciate the difficulties raised by Question 3.1.0.1, let us revisit the construction of the continuous tensor product $\bigotimes_{x \in X} C^*(E_x; \Lambda)$ in the special case $\Lambda = \mathbf{Q}$. Our definition made essential use of Sullivan's *polynomial de Rham complex* $Y \mapsto \Omega^*_{\mathrm{poly}}(Y)$ (Definition 1.5.3.6) to construct a diagram $\mathcal{B} : \mathcal{U}_0(X) \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$, which assigns to each open disk $U \subseteq X$ a commutative differential graded algebra $\mathcal{B}(U)$ which is (canonically) quasi-isomorphic to $C^*(E_x; \mathbf{Q})$ for any point $x \in U$; the continuous tensor product $\bigotimes_{x \in X} C^*(E_x; \Lambda)$ was then defined to as the homotopy colimit of $\mathcal{B}$ (Construction 1.5.4.8).

If Y is an arbitrary topological space and Λ is an arbitrary commutative ring, then we can regard the cohomology $\mathrm{H}^*(Y; \Lambda)$ as a graded-commutative algebra over Λ. The multiplication on $\mathrm{H}^*(Y; \Lambda)$ is given at the level of cochains by a cup product map

$$\cup : C^*(Y; \Lambda) \otimes_\Lambda C^*(Y; \Lambda) \to C^*(Y; \Lambda).$$

Unfortunately, this structure is not a good replacement for the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(Y)$:

(a) The map $\cup$ exhibits $C^*(Y; \Lambda)$ as a differential graded algebra over Λ: that is, as an associative algebra object of the category $\mathrm{Chain}(\Lambda)$ of chain complexes over Λ (see §2.1.1). However, $C^*(Y; \Lambda)$ is not a *commutative* differential graded algebra:

the cup product operation is not commutative at the level of cochains. In the setting of Construction 1.5.4.8, commutativity is essential: note that if X is a finite set and $\{A_x\}_{x \in X}$ is a collection of commutative differential graded algebras over $\mathbf{Q}$, then the tensor product $\bigotimes_{x \in X} A_x$ can be identified with the coproduct of $\{A_x\}_{x \in X}$ in the category of commutative differential graded algebras, but *not* in the category of *all* differential graded algebras.

(*b*) Though the cup product is not commutative on the nose, it is commutative up to homotopy: that is, it exhibits $C^*(Y; \Lambda)$ as a commutative algebra object in the derived category $\mathcal{D}(\Lambda)$. However, this is not good enough for our purposes. For example, if Λ is a field, then the homotopy class of the cup product map $\cup : C^*(Y; \Lambda) \otimes_\Lambda C^*(Y; \Lambda) \to C^*(Y; \Lambda)$ is completely determined by the ring structure on $\mathrm{H}^*(Y; \Lambda)$. We could exploit this structure to define an *algebraic* tensor product $\bigotimes_{x \in X}^{\mathrm{alg}} \mathrm{H}^*(E_x; \Lambda)$ (see Construction 1.5.4.4). However, this construction is far too crude to be useful to us (for the reasons articulated in §1.5.4).

In the definition of the polynomial de Rham complex $\Omega^*_{\mathrm{poly}}(Y)$, it is possible to replace the field $\mathbf{Q}$ of rational numbers by an arbitrary commutative ring Λ. However, the resulting object is badly behaved unless Λ is an algebra over $\mathbf{Q}$. More precisely, the proof that the integration map $\int : \Omega^*_{\mathrm{poly}}(Y) \to C^*(Y; \mathbf{Q})$ is a quasi-isomorphism (Theorem 1.5.3.7) relies on Poincaré's lemma (for algebraic differential forms on a simplex), which is valid only in characteristic zero. Moreover, this difficulty turns out to be essential: one can show that if there exists *any* functor from topological spaces to commutative differential graded Λ-algebras which is (functorially) quasi-isomorphic to $C^*(\bullet; \Lambda)$, then Λ must be an algebra over $\mathbf{Q}$.

In general, we cannot hope to replace $C^*(Y; \Lambda)$ by a commutative algebra in the category $\mathrm{Chain}(\Lambda)$ of chain complexes (that is, by a commutative differential graded algebra over Λ). On the other hand, it is not enough to observe that $C^*(Y; \Lambda)$ is a commutative algebra in the derived category $\mathcal{D}(\Lambda)$ (that is, as a chain complex with a multiplication which is commutative and associative up to homotopy). Our goal in this section is to show that Question 3.1.0.1 can be addressed in general by working between these extremes: by regarding $C^*(Y; \Lambda)$ as a commutative algebra object of the ∞-category Mod_Λ of Example 2.1.4.8 (that is, as a chain complex equipped with a multiplication which is commutative and associative up to *coherent* homotopy). To explain this point, we need to review the theory of *symmetric monoidal* structures on ∞-categories, which will play an important role throughout the rest of this chapter (and in Chapter 4).

3.1.1 Commutative Monoids

Recall that a *commutative monoid* is a set M equipped with a multiplication map $M \times M \to M$ and a unit object $1 \in M$ satisfying the identities

$$1x = x \quad xy = yx \quad x(yz) = (xy)z$$

for all $x, y, z \in M$. This notion admits the following generalization:

Definition 3.1.1.1 (Commutative Monoids in a Category)**.** Let $\mathcal{C}$ be a category which admits finite products and let $\mathbf{1}$ denote the final object of $\mathcal{C}$. A *commutative monoid object of* $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with a multiplication map $m : M \times M \to M$ and a unit map $e : \mathbf{1} \to M$ for which the diagrams

$$\begin{array}{ccc} \mathbf{1} \times M & \xrightarrow{e \times \mathrm{id}} & M \times M \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle m} & \\ & M & \end{array} \qquad \begin{array}{ccc} M \times M & \xrightarrow{\sigma} & M \times M \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & M & \end{array}$$

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times \mathrm{id}} & M \times M \\ \Big\downarrow{\scriptstyle \mathrm{id} \times m} & & \Big\downarrow{\scriptstyle m} \\ M \times M & \xrightarrow{m} & M \end{array}$$

commute; here σ denotes the automorphism of $M \times M$ given by permuting its factors.

Example 3.1.1.2. If $\mathcal{C}$ is the category of sets, then commutative monoid objects of $\mathcal{C}$ are simply commutative monoids.

We now generalize Definition 3.1.1.1 to the case where $\mathcal{C}$ is an ∞-category which admits finite products. Roughly speaking, we would like to say that a commutative algebra object of $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with a unital multiplication $m : M \times M \to M$ satisfying suitable commutative and associative laws, articulated by the commutative diagrams which appear in Definition 3.1.1.1. However, we should not ask only that these diagrams commute up to homotopy: that would amount to considering commutative monoid objects of the homotopy category $\mathrm{h}\mathcal{C}$. Instead, we should demand *witnesses* in $\mathcal{C}$ to the commutativity of the above diagrams (encoded by higher-dimensional simplices of $\mathcal{C}$). These witnesses should be taken as part of the data of a commutative monoid, and should be required to satisfy "higher-order" commutativity and associativity properties of their own. In the absence of a good organizing principle, it is prohibitively difficult to give a precise (or useful) definition which follows this line of thought. Instead, we consider a reformulation of Definition 3.1.1.1 which is more amenable to generalization, following some ideas introduced by Segal.

Construction 3.1.1.3. Let Fin_* denote the category of pointed finite sets: the objects of Fin_* are pairs $(I, *)$, where I is a finite set and $*$ is an element of I, and morphisms from $(I, *)$ to $(I', *')$ are given by functions $f : I \to I'$ satisfying $f(*) = *'$.

Let $\mathcal{C}$ be a category which admits finite products and let M be a commutative monoid object of $\mathcal{C}$. We define a functor $X_M : \mathrm{Fin}_* \to \mathcal{C}$ as follows:

- For each object $(I, *) \in \mathrm{Fin}_*$, we have $X_M(I, *) = \prod_{i \in I - \{*\}} M \in \mathcal{C}$.
- For each morphism $f : (I, *) \to (I', *')$ in Fin_*, the induced map
$$X_M(I, *) \to X_M(I', *')$$
is given by the product (taken over elements $i' \in I' - \{*'\}$) of maps
$$\prod_{f(i)=i'} M \to M$$
determined by the multiplication on M.

In the situation of Construction 3.1.1.3, the functor X_M completely encodes the monoid structure on M. More precisely, we have the following fact, whose proof we leave to the reader:

Proposition 3.1.1.4. *Let $\mathcal{C}$ be a category which admits finite products. Then the construction $M \mapsto X_M$ determines a fully faithful embedding*
$$\{\textit{Commutative monoid objects of } \mathcal{C}\} \to \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C}).$$
The essential image of this embedding consists of those functors $X : \mathrm{Fin}_* \to \mathcal{C}$ with the following property:

$(*)$ *Let $(I, *)$ be a pointed finite set. For each element $i \in I - \{*\}$, let $\rho_i : I \to \{i, *\}$ be given by*
$$\rho_i(j) = \begin{cases} i & \text{if } i = j \\ * & \text{otherwise.} \end{cases}$$
Then the maps ρ_i induce an isomorphism
$$X(I, *) \to \prod_{i \in I - \{*\}} X(\{i, *\}, *)$$
in the category $\mathcal{C}$.

Motivated by Proposition 3.1.1.4, we introduce the following:

Definition 3.1.1.5. Let $\mathcal{C}$ be an ∞-category which admits finite products. A *commutative monoid object of* $\mathcal{C}$ is a functor $X : \mathrm{Fin}_* \to \mathcal{C}$ which satisfies condition $(*)$ of Proposition 3.1.1.4. We let $\mathrm{CMon}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathcal{C})$ spanned by the commutative monoid objects of $\mathcal{C}$. We refer to $\mathrm{CMon}(\mathcal{C})$ as *the ∞-category of commutative monoid objects of* $\mathcal{C}$.

Example 3.1.1.6. Let $\mathcal{C}$ be a category which admits finite products. Then the ∞-category of commutative monoid objects of $\mathcal{C}$ (in the sense of Definition 3.1.1.5) is equivalent to the category of commutative monoid objects of $\mathcal{C}$ (in the sense of Definition 3.1.1.1): this is the content of Proposition 3.1.1.4.

Remark 3.1.1.7. Let $\mathcal{C}$ be an ∞-category which admits finite products. Evaluation on the pointed finite set $(\{0, *\}, *)$ determines a forgetful functor $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$. We will generally abuse notation by not distinguishing between a commutative monoid object $X \in \mathrm{CMon}(\mathcal{C})$ and its image $X(\{0, *\}, *) \in \mathcal{C}$.

Remark 3.1.1.8 (Functoriality)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite products, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite products. Then composition with F determines a functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{D})$.

Example 3.1.1.9. Let $\mathcal{C}$ be an ∞-category which admits finite products and let $\mathrm{h}\mathcal{C}$ be its homotopy category. Applying Remark 3.1.1.8 to the canonical map $\mathcal{C} \to \mathrm{h}\mathcal{C}$, we obtain a forgetful functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathrm{h}\mathcal{C})$. In particular, if X is a commutative monoid object of $\mathcal{C}$, then we can also regard X as a commutative monoid object of the homotopy category $\mathrm{h}\mathcal{C}$: in particular, there is a multiplication map $m : X \times X \to X$ which is commutative and associative up to homotopy.

3.1.2 Symmetric Monoidal ∞-Categories

Let $\mathcal{E}$ be an ∞-category. Roughly speaking, a *symmetric monoidal structure* on $\mathcal{C}$ is a tensor product functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which is commutative, associative, and unital up to coherent homotopy. To make this idea precise, it is convenient to regard symmetric monoidal ∞-categories as commutative monoids (in the sense of Definition 3.1.1.5) in a suitable context.

Construction 3.1.2.1 (The ∞-Category of ∞-Categories)**.** The collection of all (small) ∞-categories can be organized into a category $\mathcal{C}\mathrm{at}_\infty^\Delta$, whose morphisms are given by functors of ∞-categories (that is, maps of simplicial sets). Note that if $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then $\mathrm{Hom}_{\mathcal{C}\mathrm{at}_\infty^\Delta}(\mathcal{C}, \mathcal{D})$ can be identified with the set of 0-simplices of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$, where $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ is the largest Kan complex contained in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$

(more concretely, $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$ is obtained from $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ by discarding noninvertible natural transformations between functors). Note that composition of functors determines a strictly associative multiplication

$$\mathrm{Fun}(\mathcal{D},\mathcal{E})^{\simeq} \times \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq} \to \mathrm{Fun}(\mathcal{C},\mathcal{E})^{\simeq}.$$

Consequently, we can regard $\mathcal{C}\mathrm{at}_{\infty}^{\Delta}$ as enriched over the category of Kan complexes. We let $\mathcal{C}\mathrm{at}_{\infty}$ denote the homotopy coherent nerve of $\mathcal{C}\mathrm{at}_{\infty}^{\Delta}$ (see Definition [25].1.1.5.5). We refer to $\mathcal{C}\mathrm{at}_{\infty}$ as *the ∞-category of ∞-categories.*

Definition 3.1.2.2. A *symmetric monoidal ∞-category* is a commutative monoid object of the ∞-category $\mathcal{C}\mathrm{at}_{\infty}$.

In what follows, we will generally abuse notation by identifying a symmetric monoidal ∞-category with its image under the forgetful functor $\mathrm{CMon}(\mathcal{C}\mathrm{at}_{\infty}) \to \mathcal{C}\mathrm{at}_{\infty}$ (Remark 3.1.1.7). Note that if $\mathcal{C}$ is a symmetric monoidal ∞-category, then $\mathcal{C}$ inherits the structure of a commutative monoid object of the homotopy category $\mathrm{h}\mathcal{C}\mathrm{at}_{\infty}$. In particular, we can regard $\mathcal{C}$ as equipped with a tensor product operation

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which is unital, associative, and commutative up to homotopy.

Warning 3.1.2.3. Definition 3.1.2.2 really captures the notion of a *small* symmetric monoidal ∞-category (since the objects of $\mathcal{C}\mathrm{at}_{\infty}$ are themselves small ∞-categories). In practice, we will also want to consider *large* symmetric monoidal ∞-categories, like the ∞-category Mod_R of chain complexes over a commutative ring R (see §3.1.4). To avoid burdening the exposition with irrelevant technicalities, we will generally ignore the distinction in what follows.

Warning 3.1.2.4. Our definition of symmetric monoidal ∞-category differs from the definition which appears in [23], but is essentially equivalent: see Example [23].2.4.2.4.

Example 3.1.2.5. Let $\mathcal{C}$ be an ∞-category which admits finite products. Then we can regard $\mathcal{C}$ as a symmetric monoidal ∞-category, whose underlying tensor product is the Cartesian product functor

$$\times : (\mathcal{C} \times \mathcal{C}) \to \mathcal{C}\,.$$

Moreover, the resulting symmetric monoidal structure on $\mathcal{C}$ is essentially unique (see Proposition [23].2.4.1.5 and Corollary [23].2.4.1.9). Similarly, if $\mathcal{C}$ admits finite coproducts, then the coproduct functor

$$\amalg : (\mathcal{C} \times \mathcal{C}) \to \mathcal{C}$$

determines a symmetric monoidal structure on $\mathcal{C}$.

Example 3.1.2.6. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathcal{J}$ be an arbitrary ∞-category. Then the ∞-category of functors $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ inherits the structure of a symmetric monoidal ∞-category, whose underlying tensor product is computed levelwise (this follows from the observation that the construction $\mathrm{Fun}(\mathcal{J}, \bullet)$ determines a product-preserving functor from $\mathcal{C}\mathrm{at}_\infty$ to itself).

3.1.3 Commutative Algebra Objects

For any symmetric monoidal ∞-category $\mathcal{C}$, there is an associated notion of *commutative algebra object* of $\mathcal{C}$: that is, an object $A \in \mathcal{C}$ equipped with a multiplication $m : A \otimes A \to A$ which is unital, commutative, and associative up to coherent homotopy. This notion is a special case of the following:

Definition 3.1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. A *symmetric monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism from $\mathcal{C}$ to $\mathcal{D}$ in the ∞-category $\mathrm{CMon}(\mathcal{C}\mathrm{at}_\infty)$. The collection of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ can be regarded as objects of an ∞-category $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$, whose k-simplices are given by symmetric monoidal functors from $\mathcal{C}$ to the symmetric monoidal ∞-category $\mathrm{Fun}(\Delta^k, \mathcal{D})$ (see Example 3.1.2.6). We will refer to $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ as *the ∞-category of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$*.

More informally, a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is a functor which commutes with tensor products and is compatible with the commutativity and associativity constraints of $\mathcal{C}$ and $\mathcal{D}$.

Definition 3.1.3.2 (Commutative Algebras)**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category. A *commutative algebra object of* $\mathcal{C}$ is a symmetric monoidal functor $A : \mathrm{Fin} \to \mathcal{C}$. Here Fin denotes the category of finite sets, which we regard as a symmetric monoidal ∞-category via the formation of disjoint unions $\amalg : \mathrm{Fin} \times \mathrm{Fin} \to \mathrm{Fin}$ (see Example 3.1.2.5). We let $\mathrm{CAlg}(\mathcal{C}) = \mathrm{Fun}^{\otimes}(\mathrm{Fin}, \mathcal{C})$ denote the ∞-category of commutative algebra objects of $\mathcal{C}$.

Remark 3.1.3.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $A : \mathrm{Fin} \to \mathcal{C}$ be a commutative algebra object of $\mathcal{C}$. Let $A(*)$ denote the value of the functor A on the singleton set $\{*\}$. The construction $A \mapsto A(*)$ determines a forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$. We will generally abuse notation by identifying A with its image under this forgetful functor. Note that for every finite set I, the compatibility of A with tensor products supplies a canonical equivalence

$$A(I) = A(\amalg_{i \in I} \{i\}) \simeq \bigotimes_{i \in I} A(\{i\}) = A(*)^{\otimes I}.$$

Consequently, the projection map $I \to *$ induces an "I-fold multiplication map" $A(*)^{\otimes I} \to A(*)$. Taking I to be a two-element set, we obtain a multiplication $A(*) \otimes A(*) \to A(*)$. It is not difficult to see that this multiplication is commutative and associative up to homotopy: that is, it exhibits $A(*)$ as a commutative algebra object of the homotopy category $\mathrm{h}\mathcal{C}$ (which we can regard as a symmetric monoidal category in the usual sense).

Remark 3.1.3.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. Then there is an essentially unique commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$ for which the unit map $\mathbf{1} \to A$ is an equivalence. We will refer to this commutative algebra as the *unit algebra* in $\mathcal{C}$ and denote it by $\mathbf{1}$. It is an initial object of the ∞-category $\mathrm{CAlg}(\mathcal{C})$ (see Proposition [23].3.2.1.8).

Remark 3.1.3.5. Let $\mathcal{C}$ be an ∞-category which admits limits of some particular type. Then the ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits limits of the same type, which are preserved by the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$; see Corollary [23].3.2.2.5.

Example 3.1.3.6. Let $\mathcal{C}$ be an ∞-category which admits finite products, and regard $\mathcal{C}$ as equipped with the symmetric monoidal structure given by the Cartesian product (see Example 3.1.2.5). Then there is a canonical equivalence of ∞-categories $\mathrm{CAlg}(\mathcal{C}) \simeq \mathrm{CMon}(\mathcal{C})$. For a proof, see Proposition [23].2.4.2.5.

Example 3.1.3.7. Let $\mathcal{C}$ be an ∞-category which admits finite coproducts, and regard $\mathcal{C}$ as equipped with the symmetric monoidal structure given by the formation of coproducts (see Example 3.1.2.5). Then the forgetful functor $\mathrm{CAlg}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories. In other words, every object $C \in \mathcal{C}$ admits an essentially unique commutative algebra structure, whose underlying multiplication is given by the codiagonal $C \amalg C \to C$. For a proof, we refer the reader to Proposition [23].2.4.3.9.

We close by describing a slight modification of Definition 3.1.3.2 which will be useful in §3.3.5:

Variant 3.1.3.8. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. A *nonunital commutative algebra* is a symmetric monoidal functor $A : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$, where $\mathrm{Fin}^{\mathrm{s}}$ denotes the category whose objects are finite sets and whose morphisms are surjective functions (which we regard as a symmetric monoidal subcategory of the category Fin of finite sets).

3.1.4 Tensor Products of Chain Complexes

Let Λ be a commutative ring. In §2.1.2, we defined the ∞-category Mod_Λ, whose objects are chain complexes of Λ-modules. We now show that the tensor product of chain complexes determines a symmetric monoidal structure on the ∞-category Mod_Λ. This is a special case of the following general fact (for a proof, see Proposition [23].4.1.7.4):

Proposition 3.1.4.1. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let W be a collection of morphisms in $\mathcal{C}$ with the following property:*

($*$) *If $f : C \to D$ is a morphism belonging to W and E is any object of $\mathcal{C}$, then the induced map $(f \otimes \mathrm{id}) : C \otimes E \to D \otimes E$ belongs to W.*

Let $\mathcal{C}[W^{-1}]$ be the ∞-category obtained from $\mathcal{C}$ by inverting the morphisms of W (see Construction 2.1.4.11). Then there is an essentially unique symmetric monoidal structure on the ∞-category $\mathcal{C}[W^{-1}]$ for which the functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is symmetric monoidal. Moreover, if $\mathcal{D}$ is an arbitrary symmetric monoidal ∞-category, then composition with F induces a fully faithful embedding

$$\mathrm{Fun}^{\otimes}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}),$$

whose essential image consists of those symmetric monoidal functors $G : \mathcal{C} \to \mathcal{D}$ which carry each morphism of W to an equivalence in $\mathcal{D}$.

If Λ is any commutative ring, then the hypothesis of Proposition 3.1.4.1 is satisfied if we take $\mathcal{C} = \mathrm{Chain}'(\Lambda)$ to be the category of K-projective chain complexes of Λ modules and W to be the collection of all quasi-isomorphisms in $\mathrm{Chain}'(\Lambda)$. We therefore obtain the following result:

Corollary 3.1.4.2. *Let Λ be a commutative ring. Then there is an essentially unique symmetric monoidal structure on the ∞-category Mod_Λ for which the canonical map $\mathrm{Chain}'(\Lambda) \to \mathrm{Mod}_\Lambda$ is symmetric monoidal.*

Notation 3.1.4.3. If Λ is a commutative ring, we will denote the underlying tensor product for the symmetric monoidal structure of Corollary 3.1.4.2 by

$$\otimes_\Lambda : \mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda .$$

Warning 3.1.4.4. For any commutative ring Λ, Example 2.1.4.12 supplies a canonical functor $\theta : \mathrm{Chain}(\Lambda) \to \mathrm{Mod}_\Lambda$. Beware that this functor is generally *not* symmetric monoidal (though it becomes symmetric monoidal when restricted to the full subcategory $\mathrm{Chain}'(\Lambda) \subseteq \mathrm{Chain}(\Lambda)$ of K-projective chain complexes). If M_* and N_* are chain complexes of Λ-modules, then we have a canonical equivalence

$$\theta(M_*) \otimes_\Lambda \theta(N_*) \simeq \theta(M_* \otimes^L_\Lambda N_*),$$

where $M_* \otimes^L_\Lambda N_*$ denotes the *left derived* tensor product of M_* with N_* over Λ.

Remark 3.1.4.5. If Λ is a field, then every chain complex of Λ-modules is K-projective. In this case, the functor $\theta : \mathrm{Chain}(\Lambda) \to \mathrm{Mod}_\Lambda$ of Warning 3.1.4.4 is symmetric monoidal.

Warning 3.1.4.6. Let Λ be a commutative ring. Then the usual abelian category of Λ-modules can be identified with the full subcategory of the ∞-category Mod_Λ spanned by the discrete objects (see Notation 2.1.4.9). Beware that the tensor product on Mod_Λ is generally not compatible with this inclusion: if M and N are discrete Λ-modules, then the tensor product $M \otimes_\Lambda N$ (formed in the ∞-category Mod_Λ) is given by the left derived tensor product $M \otimes_\Lambda^L N$, whose homology groups are given by

$$\mathrm{H}_i(M \otimes_\Lambda N) = \mathrm{Tor}_i^\Lambda(M, N).$$

In particular $M \otimes_\Lambda N$ is discrete if and only if the groups $\mathrm{Tor}_i^\Lambda(M, N) \simeq 0$ for $i > 0$ (this is automatic, for example, if M or N is flat over Λ). *Unless otherwise specified*, we will always use the notation $\otimes_\Lambda$ to indicate the tensor product in the ∞-category Mod_Λ, rather than the abelian category of discrete Λ-modules.

3.1.5 $\mathbb{E}_\infty$-Algebras

Let Λ be a commutative ring, and regard the ∞-category of chain complexes Mod_Λ as equipped with the symmetric monoidal structure of Corollary 3.1.4.2.

Definition 3.1.5.1. An *$\mathbb{E}_\infty$-algebra over* Λ is a commutative algebra object of the ∞-category Mod_Λ (see Definition 3.1.3.2). We let CAlg_Λ denote the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_\Lambda)$ whose objects are $\mathbb{E}_\infty$-algebras over Λ.

Warning 3.1.5.2. In the case where $\Lambda = \mathbf{Q}$ is the field of rational numbers, we gave an *a priori* different definition of the ∞-category CAlg_Λ in Example 2.1.5.10. We will see in a moment that these definitions are actually equivalent (Proposition 3.1.5.4).

Remark 3.1.5.3. Let A be an $\mathbb{E}_\infty$-algebra over Λ. Then we can regard A as a chain complex of Λ-modules (which, without loss of generality, we can assume to be K-projective) equipped with a multiplication $m : A \otimes_\Lambda A \to A$ which is commutative, associative, and unital up to *coherent* homotopy. It follows that the cohomology ring $\mathrm{H}^*(A)$ inherits the structure of a graded-commutative algebra over Λ.

Assume for simplicity that Λ is a field, so that Corollary 3.1.4.2 supplies a symmetric monoidal functor $\theta : \mathrm{Chain}(\Lambda) \to \mathrm{Mod}_\Lambda$. It follows that θ carries commutative algebra objects of $\mathrm{Chain}(\Lambda)$ (that is, commutative differential graded algebras over Λ) to commutative algebra objects of Mod_Λ (that is, $\mathbb{E}_\infty$-algebras over Λ). We therefore obtain a functor $\mathrm{CAlg}(\theta) : \mathrm{CAlg}_\Lambda^{\mathrm{dg}} \to \mathrm{CAlg}_\Lambda$. Note that this functor carries quasi-isomorphisms of differential graded algebras to equivalences in Mod_Λ.

Proposition 3.1.5.4. *Let Λ be a field of characteristic zero, and let W be the collection of all quasi-isomorphisms in* $\mathrm{CAlg}_\Lambda^{\mathrm{dg}}$*. Then the construction above induces an equivalence of ∞-categories* $\mathrm{CAlg}_\Lambda^{\mathrm{dg}}[W^{-1}] \to \mathrm{CAlg}_\Lambda$.

Proof. This is a special case of Proposition [23].7.1.4.11. □

Remark 3.1.5.5. In the construction of the functor $\mathrm{CAlg}_\Lambda^{\mathrm{dg}} \to \mathrm{CAlg}_\Lambda$, one does not need to require Λ to be a field: the functor $\theta : \mathrm{Chain}(\Lambda) \to \mathrm{Mod}_\Lambda$ is always *lax* symmetric monoidal (see Warning 3.2.3.2), and therefore carries commutative algebras to commutative algebras. Similarly, Proposition 3.1.5.4 is valid for *any* commutative ring Λ which is an algebra over $\mathbf{Q}$.

Warning 3.1.5.6. Proposition 3.1.5.4 is false if we take Λ to be a field of positive characteristic. In this case, the theory of $\mathbb{E}_\infty$-algebras over Λ is very different (and much better behaved) than the theory of commutative differential graded algebras over Λ.

Remark 3.1.5.7. Proposition 3.1.5.4 has an analogue for associative algebras, which is valid for *any* commutative ring Λ; see Proposition [23].7.1.4.6.

3.1.6 $\mathbb{E}_\infty$-Structures on Cochain Complexes

Let Λ be a commutative ring. Our next goal is construct an abundant supply of examples of $\mathbb{E}_\infty$-algebras over Λ, given by the cochain complexes $C^*(Y;\Lambda)$ where Y is a topological space. We begin with a few general remarks.

Definition 3.1.6.1. Let $\mathcal{C}\mathrm{at}_\infty$ denote the ∞-category of (small) ∞-categories (Construction 3.1.2.1). We let $\mathcal{S}$ denote the full subcategory of $\mathcal{C}\mathrm{at}_\infty$ whose objects are Kan complexes. We will refer to $\mathcal{S}$ as *the ∞-category of spaces*.

Remark 3.1.6.2. For every topological space Y, the singular simplicial set $\mathrm{Sing}(Y)_\bullet$ of Example 2.1.3.3 is a Kan complex. The construction $Y \mapsto \mathrm{Sing}(Y)_\bullet$ determines a functor $\mathrm{Top} \to \mathcal{S}$, where Top is the ordinary category of topological spaces (with morphisms given by continuous maps). One can show that this functor induces an equivalence of ∞-categories $\mathrm{Top}[W^{-1}] \to \mathcal{S}$, where W is the collection of all weak homotopy equivalences between topological spaces (that is, those maps which induce isomorphisms at the level of homotopy groups). In other words, we can regard the ∞-category $\mathcal{S}$ as obtained from the ordinary category of topological spaces by inverting all weak homotopy equivalences.

The ∞-category $\mathcal{S}$ can be described by a universal mapping property:

Proposition 3.1.6.3. *The ∞-category $\mathcal{S}$ admits small colimits. Moreover, if $\mathcal{C}$ is any other ∞-category which admits small colimits, then the evaluation functor $F \mapsto F(*)$ induces an equivalence of ∞-categories $\mathrm{Fun}^L(\mathcal{S},\mathcal{C}) \to \mathcal{C}$, where $*$ denotes the one-point space and $\mathrm{Fun}^L(\mathcal{S},\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S},\mathcal{C})$ is the full subcategory spanned by those functors $F : \mathcal{S} \to \mathcal{C}$ which preserve small colimits.*

Proof. This is a special case of Theorem [25].5.1.5.6. □

More informally, Proposition 3.1.6.3 asserts that the ∞-category $\mathcal{S}$ of spaces is freely generated by the one-point space $*$ under small colimits. Given any ∞-category $\mathcal{C}$ which admits small colimits and any object $C \in \mathcal{C}$, there is an essentially unique colimit-preserving functor $F : \mathcal{S} \to \mathcal{C}$ satisfying $F(*) = C$; this functor carries a Kan complex $X \in \mathcal{C}$ to the colimit of the constant functor $X \to \mathcal{C}$ taking the value C (note that the Kan complex X can itself be regarded as an ∞-category). We are interested in the following special cases:

Corollary 3.1.6.4. *Let Λ be a commutative ring. Then:*

(a) *There is an essentially unique functor $F : \mathcal{S} \to \mathrm{Mod}_{\Lambda}^{\mathrm{op}}$ which preserves small limits and satisfies $F(*) = \Lambda$.*

(b) *There is an essentially unique functor $G : \mathcal{S} \to \mathrm{CAlg}_{\Lambda}^{\mathrm{op}}$ which preserves small limits and satisfies $G(*) = \Lambda$.*

(c) *The diagram of ∞-categories*

$$\begin{array}{ccc} & \mathcal{S} & \\ {}^{F}\swarrow & & \searrow^{G} \\ \mathrm{CAlg}_{\Lambda}^{\mathrm{op}} & \longrightarrow & \mathrm{Mod}_{\Lambda}^{\mathrm{op}} \end{array}$$

commutes (up to essentially unique homotopy); here the horizontal map is the forgetful functor $\mathrm{CAlg}_{\Lambda} \to \mathrm{Mod}_{\Lambda}$.

Proof. Assertions (a) and (b) follow immediately from Proposition 3.1.6.3; assertion (c) follows from the uniqueness asserted by (b) (note that the forgetful functor $\mathrm{CAlg}_{\Lambda} \to \mathrm{Mod}_{\Lambda}$ preserves small limits; see Remark 3.1.3.5). □

We now identify the functor F appearing in Corollary 3.1.6.4. Note that the formation of singular cochain complexes determines a functor

$$\mathrm{Top} \to \mathrm{Chain}(\Lambda)^{\mathrm{op}} \qquad Y \mapsto C^*(Y; \Lambda).$$

Let W be the collection of weak homotopy equivalences in Top, and let W' be the collection of quasi-isomorphisms in $\mathrm{Chain}(\Lambda)$. The functor $Y \mapsto C^*(Y; \Lambda)$ carries W to W', and therefore induces a functor of ∞-categories

$$\mathcal{S} \simeq \mathrm{Top}[W^{-1}] \to \mathrm{Chain}(\Lambda)[W'^{-1}] \simeq \mathrm{Mod}_{\Lambda}.$$

Let us abuse notation by denoting this functor also by $C^*(\bullet;\Lambda)$, so that we have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Top} & \xrightarrow{C^*(\bullet;\Lambda)} & \mathrm{Chain}(\Lambda)^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathrm{Sing}} & & \downarrow \\ \mathcal{S} & \xrightarrow{C^*(\bullet;\Lambda)} & \mathrm{Mod}_\Lambda^{\mathrm{op}} . \end{array}$$

Proposition 3.1.6.5. *Let Λ be a commutative ring and let $F : \mathcal{S} \to \mathrm{Mod}_\Lambda^{\mathrm{op}}$ be as in Corollary 3.1.6.4. Then F is equivalent to the functor $C^*(\bullet;\Lambda) : \mathcal{S} \to \mathrm{Mod}_\Lambda^{\mathrm{op}}$ described above.*

Proof. It follows immediately from the definitions that $C^*(*;\Lambda)$ is equivalent to Λ. Consequently, we are reduced to showing that the functor $Y \mapsto C^*(Y;\Lambda)$ carries colimits in the ∞-category $\mathcal{S}$ to limits in the ∞-category Mod_Λ. This is essentially equivalent to excision in singular cohomology. Alternatively, it can be deduced from the fact that the functor $C^*(\bullet;\Lambda)$ admits a right adjoint $\mathrm{Mod}_\Lambda^{\mathrm{op}} \to \mathcal{S}$, which can be described concretely using the Dold-Kan correspondence. □

Combining Corollary 3.1.6.4 with Proposition 3.1.6.5, we obtain the following:

Corollary 3.1.6.6. *Let Λ be a commutative ring. Then the composite functor*

$$\mathrm{Top} \xrightarrow{C^*(\bullet;\Lambda)} \mathrm{Chain}(\Lambda)^{\mathrm{op}} \to \mathrm{Mod}_\Lambda^{\mathrm{op}}$$

admits an essentially unique lift to a functor $\mathrm{Top} \to \mathrm{CAlg}_\Lambda^{\mathrm{op}}$.

More informally, Corollary 3.1.6.6 asserts that there is an essentially unique way to endow the singular cochain complex $C^*(Y;\Lambda)$ of every topological space Y with the structure of an $\mathbb{E}_\infty$-algebra over Λ which depends *functorially* on Y (more precisely, this lift is uniquely determined by a choice of quasi-isomorphism $\Lambda \simeq C^*(*;\Lambda)$, which is ambiguous up to multiplication by invertible elements of Λ).

Example 3.1.6.7. The polynomial de Rham complex $Y \mapsto \Omega^*_{\mathrm{poly}}(Y)$ of Construction 1.5.3.6 determines a functor from the category Top of topological spaces to the opposite of the category $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ of commutative differential graded algebras over $\mathbf{Q}$. Composing with the functor $\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}} \to \mathrm{CAlg}_{\mathbf{Q}}$ of Proposition 3.1.5.4, we obtain a map $G : \mathrm{Top} \to \mathrm{CAlg}^{\mathrm{op}}_{\mathbf{Q}}$. It follows from Theorem 1.5.3.7 that the composition

$$\mathrm{Top} \xrightarrow{\Omega^*_{\mathrm{poly}}} (\mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}})^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{op}}_{\mathbf{Q}} \to \mathrm{Mod}^{\mathrm{op}}_{\mathbf{Q}}$$

is quasi-isomorphic to the singular cochain functor $Y \mapsto C^*(Y; \mathbf{Q})$. It follows that F is equivalent to the functor $\mathrm{Top} \to \mathrm{CAlg}_{\mathbf{Q}}^{\mathrm{op}}$ whose existence (and uniqueness) is asserted by Corollary 3.1.6.6.

3.1.7 The Topological Product Formula with General Coefficients

We now apply the formalism of $\mathbb{E}_\infty$-algebras to give an affirmative answer to Question 3.1.0.1 for an arbitrary commutative ring Λ. In what follows, let us regard the construction $Y \mapsto C^*(Y; \Lambda)$ as a (contravariant) functor from the category of topological spaces to the ∞-category CAlg_Λ of $\mathbb{E}_\infty$-algebras over Λ. We begin by adapting Construction 1.5.4.8:

Construction 3.1.7.1 (Continuous Tensor Product)**.** Let $\pi : E \to X$ be a fibration of topological spaces, where X is a manifold of dimension d, and let $\mathcal{U}_0(X)$ denote the collection of open subsets of X which are homeomorphic to the Euclidean space $\mathbf{R}^d$. For each open set $U \subseteq X$, we let $\mathrm{Sect}_\pi(U)$ denote the space of sections of the projection map $E \times_X U \to U$. We define a functor of ∞-categories $\mathcal{B} : \mathcal{U}_0(X) \to \mathrm{CAlg}_\Lambda$ by the formula $\mathcal{B}(U) = C^*(\mathrm{Sect}_\pi(U); \Lambda)$. We now define

$$\bigotimes_{x \in X} C^*(E_x; \mathbf{Q}) = \varinjlim(\mathcal{B}),$$

where the colimit is formed in the ∞-category CAlg_Λ.

Remark 3.1.7.2. Let $\Lambda = \mathbf{Q}$ be the field of rational numbers. The functor of ∞-categories $\mathcal{B} : \mathcal{U}_0(X) \to \mathrm{CAlg}_{\mathbf{Q}}$ of Construction 3.1.7.1 is closely related to the functor which appears in Construction 1.5.4.8 (which, to avoid confusion, we will denote by $\mathcal{B}^{\mathrm{dg}} : \mathcal{U}_0(X) \to \mathrm{CAlg}_{\mathbf{Q}}^{\mathrm{dg}}$). More precisely, the functor $\mathcal{B}$ is equivalent the composition of $\mathcal{B}^{\mathrm{dg}}$ with the functor $\mathrm{CAlg}_{\mathbf{Q}}^{\mathrm{dg}} \to \mathrm{CAlg}_{\mathbf{Q}}$ of Proposition 3.1.5.4: this follows immediately from Example 3.1.6.7. Moreover, it follows from Example 2.1.5.10 that we can identify the homotopy colimit $\mathcal{B}^{\mathrm{dg}}$ with a colimit of the functor $\mathcal{B}$ (both are well-defined up to canonical isomorphism in the homotopy category $\mathrm{hCAlg}_{\mathbf{Q}}^{\mathrm{dg}} \simeq \mathrm{hCAlg}_{\mathbf{Q}}$). It follows that the continuous tensor product $\bigotimes_{x \in X} C^*(E_x; \Lambda)$ of Construction 3.1.7.1 specializes, in the case $\Lambda = \mathbf{Q}$, to the continuous tensor product $\bigotimes_{x \in X} C^*(E_x; \mathbf{Q})$ of Construction 1.5.4.8.

In the situation of Construction 3.1.7.1, we have an evident natural transformation from $\mathcal{B}$ to the constant functor taking the value $C^*(\mathrm{Sect}_\pi(X; \Lambda))$, which is classified by a morphism of $\mathbb{E}_\infty$-algebras

$$\rho_X : \bigotimes_{x \in X} C^*(E_x; \mathbf{Q}) = \varinjlim(\mathcal{B}) \to C^*(\mathrm{Sect}_\pi(X); \Lambda).$$

We have the following analogue of Theorem 1.5.4.10:

Theorem 3.1.7.3 (The Product Formula)**.** *Let X be a compact manifold of dimension d and $\pi : E \to X$ be a fibration. Assume that for each $x \in X$, the fiber E_x is d-connected and that the homology groups $\mathrm{H}_*(E_x; \mathbf{Z})$ are finitely generated in each degree. Then the comparison map*

$$\rho_X : \bigotimes_{x \in X} C^*(E_x; \Lambda) \to C^*(\mathrm{Sect}_\pi(X); \Lambda)$$

is an equivalence of $\mathbb{E}_\infty$-algebras over Λ.

Sketch. Using universal coefficient arguments, one can reduce to the case where $\Lambda = \mathbf{Q}$ or $\Lambda = \mathbf{F}_p$ for some prime p. In the special case $\Lambda = \mathbf{Q}$, the desired result follows from Theorem 1.5.4.10 (see Remark 3.1.7.2). When $\Lambda = \mathbf{F}_p$, we can repeat the proof of Theorem 1.5.4.10 given in §1.5.5 without essential change; the only caveat is that we need finiteness properties of the $\mathbf{F}_p$-cohomology of the fibers E_x, rather than the rational cohomology of the fibers E_x (however, the requisite finiteness properties follow from our assumption that the integral homology groups of the fibers E_x are finitely generated). □

Remark 3.1.7.4. In the statement of Theorem 3.1.7.3, the hypothesis that X is a compact manifold can be relaxed. For any fibration of topological spaces $\pi : E \to X$, the construction $(x \in X) \mapsto C^*(E_x; \Lambda)$ can be promoted to a functor of ∞-categories $\mathcal{A} : \mathrm{Sing}(X)_\bullet \to \mathrm{CAlg}_\Lambda$, which is essentially equivalent to the datum of the functor $\mathcal{B}$ in the special case where X is a manifold. In this case, there is canonical map $\varinjlim(\mathcal{A}) \to C^*(\mathrm{Sect}_\pi(X); \Lambda)$, which is an equivalence provided that the fibers E_x satisfy the hypotheses of Theorem 3.1.7.3 and the space X is homotopy equivalent to a finite cell complex of dimension $\leq d$.

3.2 Cohomology of Algebraic Stacks

Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. Suppose we are given an algebraic curve X over k and a smooth affine group scheme G over X. Our principal aim in this book is to describe a "local-to-global" mechanism which relates ℓ-adic cohomology of the moduli stack $\mathrm{Bun}_G(X)$ to the ℓ-adic cohomologies of the classifying stacks $\{\mathrm{BG}_x\}_{x \in X}$. To describe this mechanism precisely, we will need to work at the *cochain* level: that is, to regard the ℓ-adic cohomology groups of an algebraic stack $\mathcal{Y}$ over k as the cohomology of a cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$. Our goal in this section is to sketch the construction of the cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$, and to explain that it admits the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$.

3.2.1 ℓ-Adic Cohomology of Algebraic Varieties

We begin with a discussion of ℓ-adic cohomology for schemes which are quasi-projective over k.

Construction 3.2.1.1. Let X be a quasi-projective k-scheme and let $\pi : X \to \mathrm{Spec}(k)$ be the projection map. Composing the direct image functor $\pi_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(\mathrm{Spec}(k))$ with the equivalence $\mathrm{Shv}_\ell(\mathrm{Spec}(k)) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$ of Example 2.3.4.3, we obtain a functor $\mathrm{Shv}_\ell(X) \to \mathrm{Mod}_{\mathbf{Z}_\ell}$, which we will denote by $\mathcal{F} \mapsto C^*(X; \mathcal{F})$. For each object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we will refer to $C^*(X; \mathcal{F})$ as the *complex of cochains on* X *with values in* $\mathcal{F}$. We will denote the nth cohomology group of the cochain complex $C^*(X; \mathcal{F})$ by $\mathrm{H}^n(X; \mathcal{F})$, and refer to it as the *nth (hyper)cohomology group of* X *with values in* $\mathcal{F}$.

Remark 3.2.1.2. In the situation of Construction 3.2.1.1, the direct image functor $\pi_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(\mathrm{Spec}(k))$ preserves constructible sheaves (see Proposition 2.3.3.1). It follows that the ℓ-adic cochain complex $C^*(X; \mathbf{Z}_\ell)$ is perfect: that is, the ℓ-adic cohomology groups $\mathrm{H}^n(X; \mathbf{Z}_\ell)$ are finitely generated modules over $\mathbf{Z}_\ell$, which vanish for all but finitely many integers n.

Variant 3.2.1.3 (Cohomology with Finite Coefficients)**.** Let X be a quasi-projective k-scheme, let $d \geq 0$ be a nonnegative integer, and let $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$ denote the constant ℓ-adic sheaf on X with value $\mathbf{Z}/\ell^d\mathbf{Z}$ (that is, the cofiber of the map $\ell^d : \underline{\mathbf{Z}_\ell}_X \to \underline{\mathbf{Z}_\ell}_X$). We let $C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$ denote the complex $C^*(X; \underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X)$ of cochains on X with values in $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$, and for each integer n we let $\mathrm{H}^n(X; \mathbf{Z}/\ell^d\mathbf{Z})$ denote the nth cohomology group of $C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$.

Remark 3.2.1.4. In the situation of Variant 3.2.1.3, the short exact sequence of abelian groups

$$0 \to \mathbf{Z}_\ell \xrightarrow{\ell^d} \mathbf{Z}_\ell \to \mathbf{Z}/\ell^d\mathbf{Z} \to 0$$

induces a fiber sequence of cochain complexes

$$C^*(X; \mathbf{Z}_\ell) \xrightarrow{\ell^d} C^*(X; \mathbf{Z}_\ell) \to C^*(X; \mathbf{Z}/\ell^d\mathbf{Z}),$$

hence a long exact sequence of homology groups

$$\cdots \to \mathrm{H}^*(X; \mathbf{Z}_\ell) \xrightarrow{\ell^d} \mathrm{H}^*(X; \mathbf{Z}_\ell) \to \mathrm{H}^*(X; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{H}^{*+1}(X; \mathbf{Z}_\ell) \to \cdots.$$

Warning 3.2.1.5. Let X be a quasi-projective k-scheme and let d be a nonnegative integer. We have now assigned two different meanings to the notation $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$:

(1) In Notation 2.2.4.4, we defined $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$ to be an object of $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z})$: namely, the sheafification of the constant functor $(\mathrm{Shv}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}/\ell^d\mathbf{Z}}$ taking the value $\mathbf{Z}/\ell^d\mathbf{Z}$.

(2) In Variant 3.2.1.3, we defined $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$ as an object of the ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves on X.

However, these two definitions are compatible: the second can be obtained as the image of the first under the forgetful functor $\mathrm{Shv}(X; \mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{Shv}_\ell(X)$. It follows that the cochain complex $C^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$ can be identified with $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X(X)$, where $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$ is interpreted in the sense of Notation 2.2.4.4.

Remark 3.2.1.6. Let X be a quasi-projective k-scheme. For every nonnegative integer d, the cohomology groups $\mathrm{H}^*(X; \mathbf{Z}/\ell^d\mathbf{Z})$ are finite. This follows from Remarks 3.2.1.2 and 3.2.1.4 (or more directly from Proposition 2.2.9.1).

Remark 3.2.1.7 (Comparison with Étale Cohomology)**.** Let X be a quasi-projective k-scheme, let $d \geq 0$ be a nonnegative integer, and set $\Lambda = \mathbf{Z}/\ell^d\mathbf{Z}$. Let Mod_Λ denote the ∞-category of chain complexes of Λ-modules, and let $\mathrm{Mod}_\Lambda^\heartsuit$ denote the usual abelian category of Λ-modules (which we regard as a full subcategory of Mod_Λ). Let $\mathrm{Sch}_X^{\mathrm{et}}$ denote the category of étale X-schemes (see Notation 2.2.1.1), and let $c : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda$ denote the constant functor taking the value Λ. We can then sheafify the functor c in (at least) two different senses:

(i) We can consider c as a presheaf on X with values in the abelian category $\mathrm{Mod}_\Lambda^\heartsuit$, and form its sheafification $\widetilde{c} : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda^\heartsuit$ in the 1-categorical sense. The functor $\widetilde{c}$ can be regarded as an object of the abelian category $\mathcal{A}$ of $\mathrm{Mod}_\Lambda^\heartsuit$-valued sheaves on X.

(ii) We can consider c as a presheaf on X with values in the stable ∞-category Mod_Λ, and form its sheafification $\underline{\Lambda}_X$ in the sense of Remark 2.2.1.4. The functor $\underline{\Lambda}_X$ is an object of the stable ∞-category $\mathrm{Shv}(X; \Lambda)$.

Beware that we can regard both $\widetilde{c}$ and $\underline{\Lambda}_X$ as functors from $(\mathrm{Shv}_X^{\mathrm{et}})^{\mathrm{op}}$ to the ∞-category Mod_Λ, and that these functors generally do not coincide. However, they are closely related. Note that the abelian category $\mathcal{A}$ has enough injectives, so we can choose an injective resolution

$$0 \to \widetilde{c} \to I^0 \to I^1 \to \cdots .$$

The construction $U \mapsto I^*(U)$ determines a functor from $(\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}$ to the category $\mathrm{Chain}(\Lambda)$ of chain complexes of Λ-modules, hence also a functor $(\mathrm{Shv}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_\Lambda$. It follows from the injectivity of the abelian sheaves $I^\bullet$ that the resulting functor is an object of the ∞-category $\mathrm{Shv}(X;\Lambda)$, and from the acyclicity of the chain complex $I^\bullet$ that the composite map $c \to \widetilde{c} \to I^\bullet$ exhibits $I^\bullet$ as a sheafification of c (in the presheaf ∞-category $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$). Consequently, we can identify $\underline{\Lambda}_X$ with the functor

$$(U \in \mathrm{Sch}_X^{\mathrm{et}}) \mapsto (I^0(U) \to I^1(U) \to I^2(U) \to \cdots).$$

Taking $U = X$ and passing to cohomology, we obtain a canonical isomorphism

$$\mathrm{H}^*(X;\Lambda) \simeq \mathrm{H}^*(\underline{\Lambda}_X(X)) \simeq \mathrm{H}^*_{\mathrm{et}}(X;\Lambda),$$

where the left hand side is defined as in Variant 3.2.1.3 and the right hand side is the usual étale cohomology of X with coefficients in Λ.

Remark 3.2.1.8 (Passage to the Inverse Limit)**.** Let X be a quasi-projective k-scheme. Then we can identify the constant sheaf $\underline{\mathbf{Z}_\ell}_X$ with the inverse limit of the tower $\{\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X\}_{d\geq 0}$ in the ∞-category $\mathrm{Shv}_\ell(X)$. The construction $\mathcal{F} \mapsto C^*(X;\mathcal{F})$ commutes with inverse limits (since it is a right adjoint), so we can identify the ℓ-adic cochain complex $C^*(X;\mathbf{Z}_\ell)$ with the limit of the tower $\{C^*(X;\mathbf{Z}/\ell^d\mathbf{Z})\}_{d\geq 0}$ in the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$. It follows that, for every integer n, we have a Milnor short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(X;\mathbf{Z}/\ell^d\mathbf{Z}) \to \mathrm{H}^n(X;\mathbf{Z}_\ell) \to \varprojlim \mathrm{H}^n(X;\mathbf{Z}/\ell^d\mathbf{Z}) \to 0.$$

However, the term on the left automatically vanishes (since the étale cohomology groups $\mathrm{H}^{n-1}(X;\mathbf{Z}/\ell^d\mathbf{Z})$ are finite by virtue of Remark 3.2.1.6). We therefore have canonical isomorphisms

$$\mathrm{H}^n(X;\mathbf{Z}_\ell) \simeq \varprojlim \mathrm{H}^n(X;\mathbf{Z}/\ell^d\mathbf{Z}) \simeq \varprojlim \mathrm{H}^n_{\mathrm{et}}(X;\mathbf{Z}/\ell^d\mathbf{Z}).$$

In other words, Construction 3.2.1.1 recovers the classical theory of ℓ-adic cohomology.

Warning 3.2.1.9. Let X be a quasi-projective k-scheme. The construction $U \mapsto C^*(U;\mathbf{Z}_\ell)$ determines a functor $\mathcal{F} : (\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$. This functor satisfies étale descent, and can therefore be regarded as an object of the ∞-category $\mathrm{Shv}(X;\mathbf{Z}_\ell)$ of Definition 2.2.1.2. However, the functor $\mathcal{F}$ is generally *not* the sheafification of the constant functor taking the value $\mathbf{Z}_\ell$. Consequently, the ℓ-adic cohomology groups $\mathrm{H}^*(X;\mathbf{Z}_\ell)$ of Construction 3.2.1.1 need not be isomorphic to the étale cohomology groups $\mathrm{H}^*_{\mathrm{et}}(X;\mathbf{Z}_\ell)$ (defined by regarding $\mathbf{Z}_\ell$ as a constant sheaf on the étale site of X). The sheaf $\mathcal{F}$ can instead be regarded as the ℓ-adic completion of the sheafification of the constant sheaf with value $\mathbf{Z}_\ell$.

Variant 3.2.1.10 (Cohomology with Rational Coefficients)**.** Let X be a quasi-projective k-scheme and let $\underline{\mathbf{Q}_\ell}_X = \underline{\mathbf{Z}_\ell}_X[\ell^{-1}]$ denote the ℓ-adic sheaf on X obtained from $\underline{\mathbf{Z}_\ell}_X$ by inverting ℓ. We let $C^*(X;\mathbf{Q}_\ell)$ denote the complex $C^*(X;\underline{\mathbf{Q}_\ell}_X)$ of cochains on X with values in $\underline{\mathbf{Q}_\ell}_X$, and for each integer n we let $\mathrm{H}^n(X;\mathbf{Q}_\ell)$ denote the nth cohomology group of $C^*(X;\mathbf{Q}_\ell)$.

Remark 3.2.1.11. Let X be a quasi-projective k-scheme. Since the construction $\mathcal{F} \mapsto C^*(X; \mathcal{F})$ commutes with filtered colimits, we have canonical equivalences

$$C^*(X; \mathbf{Q}_\ell) \simeq C^*(X; \mathbf{Z}_\ell)[\ell^{-1}] \qquad \mathrm{H}^*(X; \mathbf{Q}_\ell) \simeq \mathrm{H}^*(X; \mathbf{Z}_\ell)[\ell^{-1}].$$

Remark 3.2.1.12 (Functoriality)**.** Let $f : X \to Y$ be a map of quasi-projective k-schemes. Then the constant sheaf $\underline{\mathbf{Z}_\ell}_X$ can be regarded as the pullback of the constant sheaf $\underline{\mathbf{Z}_\ell}_Y$ along f. Consequently, we have a unit map $\underline{\mathbf{Z}_\ell}_Y \to f_* \underline{\mathbf{Z}_\ell}_X$ in the ∞-category $\mathrm{Shv}_\ell(Y)$, which induces a map

$$C^*(Y; \mathbf{Z}_\ell) \to C^*(Y; f_* \underline{\mathbf{Z}_\ell}_X) \simeq C^*(X; \mathbf{Z}_\ell).$$

Elaborating on this reasoning, one sees that the construction $X \mapsto C^*(X; \mathbf{Z}_\ell)$ determines a functor of ∞-categories $\mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}_\ell}$. Similarly, the constructions $X \mapsto C^*(X; \mathbf{Z}/\ell^d \mathbf{Z})$ and $X \mapsto C^*(X; \mathbf{Q}_\ell)$ determine functors

$$C^*(\bullet; \mathbf{Z}/\ell^d \mathbf{Z}) : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}/\ell^d \mathbf{Z}} \qquad C^*(\bullet; \mathbf{Q}_\ell) : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Q}_\ell}.$$

3.2.2 Digression: Tensor Products of ℓ-Adic Sheaves

Let X be a quasi-projective k-scheme. In §2.3, we introduced the ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves on X (Definition 2.3.4.1). In this section, we explain how to endow the ∞-category $\mathrm{Shv}_\ell(X)$ with a symmetric monoidal structure, in the sense of Definition 3.1.2.2.

Construction 3.2.2.1 (Tensor Products of Presheaves)**.** Let Λ be a commutative ring, and regard the ∞-category Mod_Λ as equipped with the symmetric monoidal structure described in §3.1.4. For every ∞-category $\mathcal{J}$, the functor ∞-category $\mathrm{Fun}(\mathcal{J}, \mathrm{Mod}_\Lambda)$ inherits a symmetric monoidal structure. We will denote the underlying tensor product functor by

$$\odot : \mathrm{Fun}(\mathcal{J}, \mathrm{Mod}_\Lambda) \times \mathrm{Fun}(\mathcal{J}, \mathrm{Mod}_\Lambda) \to \mathrm{Fun}(\mathcal{J}, \mathrm{Mod}_\Lambda);$$

it is given levelwise by the formula $(F \odot G)(J) = F(J) \otimes_\Lambda G(J)$.

In particular, if X is a quasi-projective k-scheme, then we can regard the ∞-category $\mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$ as a symmetric monoidal ∞-category via the levelwise tensor product.

Warning 3.2.2.2. Let X be a quasi-projective k-scheme and suppose we are given objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X; \Lambda)$, where Λ is some commutative ring. We can then form the levelwise tensor product $\mathcal{F} \odot \mathcal{G}$ of Construction 3.2.2.1, which we regard as a functor from $(\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}$ to the ∞-category Mod_Λ. However, the levelwise tensor product $\mathcal{F} \odot \mathcal{G}$ is usually *not* an object of $\mathrm{Shv}(X; \Lambda)$.

In the situation of Warning 3.2.2.2, to obtain a good tensor product on the ∞-category $\mathrm{Shv}(X;\Lambda)$, we need to sheafify the levelwise tensor product of Construction 3.2.2.1.

Definition 3.2.2.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$. We will say that L is *compatible with the symmetric monoidal structure on* $\mathcal{C}$ if the following condition is satisfied: for every morphism $f : C \to D$ in $\mathcal{C}$ and every object $E \in \mathcal{C}$, if L carries f to an equivalence in $\mathcal{C}_0$, then L carries the induced map $(f \otimes \mathrm{id}_E) : C \otimes E \to D \otimes E$ to an equivalence in $\mathcal{C}_0$.

Proposition 3.2.2.4. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory, and suppose that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$ which is compatible with the symmetric monoidal structure on $\mathcal{C}$. Then there is an essentially unique symmetric monoidal structure on the ∞-category $\mathcal{C}_0$ for which L admits the structure of a symmetric monoidal functor.*

Proof. Concretely, we equip $\mathcal{C}_0$ with the tensor product given by $(C, D) \mapsto L(C \otimes D)$. For a proof that this construction yields a symmetric monoidal structure on $\mathcal{C}_0$, we refer the reader to Proposition [23].2.2.1.9. □

Remark 3.2.2.5. In the situation of Proposition 3.2.2.4, the resulting symmetric monoidal structure on $\mathcal{C}_0$ is characterized by the following universal property: for every symmetric monoidal ∞-category $\mathcal{D}$, composition with L induces a fully faithful functor

$$\mathrm{Fun}^{\otimes}(\mathcal{C}_0, \mathcal{D}) \to \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathcal{C} \to \mathcal{D}$ with the following property: for every morphism $f : C \to C'$ in $\mathcal{C}$ for which Lf is an equivalence in $\mathcal{C}_0$, the morphism Ff is also an equivalence in $\mathcal{D}$.

Example 3.2.2.6 (Tensor Products of Sheaves)**.** Let X be a quasi-projective k-scheme and let Λ be a commutative ring. Then the inclusion functor

$$\mathrm{Shv}(X;\Lambda) \hookrightarrow \mathrm{Fun}((\mathrm{Sch}_X^{\mathrm{et}})^{\mathrm{op}}, \mathrm{Mod}_\Lambda)$$

admits a left adjoint L (given by the sheafification functor of Remark 2.2.1.4). One can show that the left adjoint L is compatible with the pointwise tensor product of Construction 3.2.2.1. Applying Proposition 3.2.2.4, we deduce that the ∞-category $\mathrm{Shv}(X;\Lambda)$ admits a symmetric monoidal structure. We will denote the underlying tensor product functor by $\otimes_\Lambda : \mathrm{Shv}(X;\Lambda) \times \mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(X;\Lambda)$, given concretely by the formula $\mathcal{F} \otimes \mathcal{G} = L(\mathcal{F} \odot \mathcal{G})$.

Example 3.2.2.7. Let X be a quasi-projective k-scheme, let Λ be a commutative ring, and let $\mathrm{Shv}(X;\Lambda)$ be the ∞-category introduced in Definition 2.2.1.2. Let $\mathcal{C} \subseteq \mathrm{Shv}(X;\Lambda)$ denote the full subcategory spanned by the ℓ-complete objects (see §2.3.1). Then the inclusion $\mathcal{C} \hookrightarrow \mathrm{Shv}(X;\Lambda)$ admits a left adjoint (given by the formation of ℓ-adic completions), and this left adjoint is compatible with the symmetric monoidal structure of Example 3.2.2.6. It follows that $\mathcal{C}$ admits a symmetric monoidal structure, whose underlying tensor product $\widehat{\otimes}_\Lambda : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the ℓ-adic completion of the tensor product $\otimes_\Lambda$ of Example 3.2.2.6.

Proposition 3.2.2.8. *Let X be a quasi-projective k-scheme and let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}(X;\mathbf{Z})$ be constructible ℓ-adic sheaves on X (in the sense of Definition 2.3.2.1). Then the completed tensor product $\mathcal{F} \widehat{\otimes}_{\mathbf{Z}} \mathcal{G}$ is also a constructible ℓ-adic sheaf on X.*

Proof. The tensor product $\mathcal{F} \widehat{\otimes}_{\mathbf{Z}} \mathcal{G}$ is ℓ-adically complete by construction. It will therefore suffice to show that, for every integer $d \geq 0$, the sheaf

$$(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F} \widehat{\otimes}_{\mathbf{Z}} \mathcal{G} \in \mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})$$

is constructible. Unwinding the definitions, we can identify this sheaf with the tensor product of the constructible sheaves $(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ and $(\mathbf{Z}/\ell^d\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}$, formed in the ∞-category $\mathrm{Shv}(X;\mathbf{Z}/\ell^d\mathbf{Z})$. The desired result now follows easily from the characterization of constructible sheaves supplied by Proposition 2.2.6.2. □

It follows from Proposition 3.2.2.8 (together with the observation that the ℓ-adic completion of the constant sheaf $\underline{\mathbf{Z}}_X$ is a constructible ℓ-adic sheaf) that the symmetric monoidal structure described in Example 3.2.2.7 restricts to a symmetric monoidal structure on the ∞-category $\mathrm{Shv}_\ell^c(X)$ of constructible ℓ-adic sheaves on X.

Remark 3.2.2.9. Let X be a quasi-projective k-scheme. According to Proposition 2.3.2.7, the ∞-category $\mathrm{Shv}_\ell^c(X)$ of constructible ℓ-adic sheaves on X can be identified with the inverse limit of the tower

$$\cdots \to \mathrm{Shv}^c(X;\mathbf{Z}/\ell^3\mathbf{Z}) \to \mathrm{Shv}^c(X;\mathbf{Z}/\ell^2\mathbf{Z}) \to \mathrm{Shv}^c(X;\mathbf{Z}/\ell\mathbf{Z}).$$

Each ∞-category in this tower admits a symmetric monoidal structure (obtained by restricting the symmetric monoidal structure of Example 3.2.2.6), and each of the transition functors can be regarded as a symmetric monoidal functor. It follows that the inverse limit $\mathrm{Shv}_\ell^c(X) \simeq \varprojlim \mathrm{Shv}^c(X;\mathbf{Z}/\ell^d\mathbf{Z})$ inherits a symmetric monoidal structure, which coincides with the symmetric monoidal structure obtained from Proposition 3.2.2.8 and Example 3.2.2.7.

Proposition 3.2.2.10. *Let X be a quasi-projective k-scheme and let $\mathrm{Shv}_\ell(X)$ denote the ∞-category of ℓ-adic sheaves on X (Definition 2.3.4.1). Then $\mathrm{Shv}_\ell(X)$ admits an essentially unique symmetric monoidal structure with the following properties:*

(a) *The inclusion* $\mathrm{Shv}_\ell^c(X) \hookrightarrow \mathrm{Shv}_\ell(X)$ *is symmetric monoidal (where we endow* $\mathrm{Shv}_\ell^c(X)$ *with the symmetric monoidal structure given by the completed tensor product* $\widehat{\otimes}_{\mathbf{Z}}$ *of Example 3.2.2.7).*

(b) *The tensor product functor* $\mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$ *preserves small colimits separately in each variable.*

Proof. We begin by observing that the tensor product $\widehat{\otimes}_{\mathbf{Z}} : \mathrm{Shv}_\ell^c(X) \times \mathrm{Shv}_\ell^c(X) \to \mathrm{Shv}_\ell^c(X)$ is exact in each variable (this follows from the analogous assertion for the tensor product $\otimes_{\mathbf{Z}} : \mathrm{Mod}_{\mathbf{Z}} \times \mathrm{Mod}_{\mathbf{Z}} \to \mathrm{Mod}_{\mathbf{Z}}$). Proposition 3.2.2.10 is now a formal consequence of Corollary [23].4.8.1.14. □

Notation 3.2.2.11. Let X be a quasi-projective k-scheme. We let $\otimes : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$ denote the tensor product functor underlying the symmetric monoidal structure of Proposition 3.2.2.10. If $\mathcal{F}$ and $\mathcal{G}$ are ℓ-adic sheaves on X, we will refer to $\mathcal{F} \otimes \mathcal{G} \in \mathrm{Shv}_\ell(X)$ as the *tensor product of* $\mathcal{F}$ *and* $\mathcal{G}$.

Example 3.2.2.12. When $X = \mathrm{Spec}(k)$, the equivalence $\mathrm{Shv}_\ell(X) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$ of Example 2.3.4.3 can be promoted to an equivalence of symmetric monoidal ∞-categories: that is, it carries tensor products of ℓ-adic sheaves on X to tensor products of chain complexes over Λ.

Remark 3.2.2.13 (Functoriality)**.** All of the constructions outlined in this section depend functorially on the quasi-projective k-scheme X. If $f : X \to Y$ is a morphism of quasi-projective k-schemes, then we can regard the pullback $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ as a symmetric monoidal functor, where $\mathrm{Shv}_\ell(Y)$ and $\mathrm{Shv}_\ell(X)$ are equipped with the symmetric monoidal structure described in Proposition 3.2.2.10.

3.2.3 $\mathbb{E}_\infty$-Structures on ℓ-adic Cochain Complexes

Let Y be a topological space and let Λ be a commutative ring. In §3.1.6, we saw that the singular cochain complex $C^*(Y;\Lambda)$ admits the structure of an $\mathbb{E}_\infty$-algebra over Λ. In this section, we establish an analogue for ℓ-adic cochain complexes of quasi-projective k-schemes. The proof is based on the following general categorical fact:

Proposition 3.2.3.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be symmetric monoidal* ∞*-categories and let* $f : \mathcal{C} \to \mathcal{D}$ *be a symmetric monoidal functor, so that composition with* f *induces a functor* $F : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{D})$*. Suppose that the functor* f *admits a right adjoint* g*. Then the commutative diagram of* ∞*-categories*

$$\begin{array}{ccc} \mathrm{CAlg}(\mathcal{C}) & \xrightarrow{F} & \mathrm{CAlg}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

is right adjointable, in the sense of Notation 2.4.1.1 (here the vertical maps are the forgetful functors). In other words, the functor F admits a right adjoint $G : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C})$, which is computed (after forgetting commutative algebra structures) by applying the functor g.

Warning 3.2.3.2. In the situation of Proposition 3.2.3.1, the functor $g : \mathcal{D} \to \mathcal{C}$ need not be symmetric monoidal. However, it can always be regarded as a *lax* symmetric monoidal functor: that is, for every pair of objects $D, D' \in \mathcal{D}$, there is a comparison map $g(D) \otimes g(D') \to g(D \otimes D')$ which is compatible with the commutativity and associativity constraints on $\mathcal{C}$ and $\mathcal{D}$, but need not be an equivalence. Any lax symmetric monoidal functor $g : \mathcal{D} \to \mathcal{C}$ carries commutative algebras to commutative algebras, and therefore induces a functor $G : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C})$.

Construction 3.2.3.3. Let X be a quasi-projective k-scheme and let $\pi : X \to \mathrm{Spec}(k)$ be the projection map. Then the pullback functor $\pi^* : \mathrm{Shv}_\ell(\mathrm{Spec}(k)) \to \mathrm{Shv}_\ell(X)$ is symmetric monoidal, and therefore induces a functor $\mathrm{CAlg}_{\mathbf{Z}_\ell} \simeq \mathrm{CAlg}(\mathrm{Shv}_\ell(\mathrm{Spec}(k))) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(X))$ which we will denote by $M \mapsto \underline{M}_X$. Applying Proposition 3.2.3.1, we deduce that the functor F admits a right adjoint $\mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}_{\mathbf{Z}_\ell}$, which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) & \longrightarrow & \mathrm{CAlg}_{\mathbf{Z}_\ell} \\ \downarrow & & \downarrow \\ \mathrm{Shv}_\ell(X) & \xrightarrow{C^*(X;\bullet)} & \mathrm{Mod}_{\mathbf{Z}_\ell}. \end{array}$$

In what follows, we will abuse notation by denoting the upper vertical map also by $\mathcal{F} \mapsto C^*(X;\mathcal{F})$. We can informally summarize the situation as follows: if $\mathcal{F}$ admits the structure of a commutative algebra object of $\mathrm{Shv}_\ell(X)$, then the cochain complex $C^*(X;\mathcal{F})$ admits the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$.

Let us regard the constant sheaf $\underline{\mathbf{Z}_\ell}_X$ as a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$ as in Remark 3.1.3.4. Applying the above reasoning to $\underline{\mathbf{Z}_\ell}_X$, we deduce that the ℓ-adic cochain complex $C^*(X;\mathbf{Z}_\ell)$ admits the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$. This $\mathbb{E}_\infty$-algebra is characterized by the following universal property: for every $\mathbb{E}_\infty$-algebra $A \in \mathrm{CAlg}_{\mathbf{Z}_\ell}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathbf{Z}_\ell}}(A, C^*(X;\mathbf{Z}_\ell)) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}(\underline{A}_X, \underline{\mathbf{Z}_\ell}_X).$$

Variant 3.2.3.4. Let X be a quasi-projective k-scheme. Applying the logic of Construction 3.2.3.3 to the constant sheaves $\underline{\mathbf{Z}/\ell^d\mathbf{Z}}_X$ and $\underline{\mathbf{Q}_\ell}_X$, we see that the cochain complexes $C^*(X;\mathbf{Z}/\ell^d\mathbf{Z})$ and $C^*(X;\mathbf{Q}_\ell)$ can be regarded as $\mathbb{E}_\infty$-algebras over $\mathbf{Z}_\ell$. In fact, we can say more: they admit the structure of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}/\ell^d\mathbf{Z}$ and $\mathbf{Q}_\ell$, respectively.

Remark 3.2.3.5 (Cup Products)**.** Let X be a quasi-projective k-scheme and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$ be some coefficient ring. Then the $\mathbb{E}_\infty$-algebra structure on the ℓ-adic cochain complex $C^*(X;\Lambda)$ endows the cohomology $\mathrm{H}^*(X;\Lambda)$ with the structure of a graded-commutative ring (see Remark 3.1.5.3). The associated multiplication is given by the classical cup product map

$$\cup : \mathrm{H}^i(X;\Lambda) \times \mathrm{H}^j(X;\Lambda) \to \mathrm{H}^{i+j}(X;\Lambda).$$

Remark 3.2.3.6. Let X be a quasi-projective k-scheme and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$ be some coefficient ring. Then we can identify $C^*(X;\Lambda)$ with the tensor product $C^*(X;\mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \Lambda$ (which is the coproduct of $C^*(X;\mathbf{Z}_\ell)$ with Λ in the ∞-category $\mathrm{CAlg}_{\mathbf{Z}_\ell}$ of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}_\ell$).

Remark 3.2.3.7 (Functoriality)**.** Let $f : X \to Y$ be a morphism of quasi-projective k-schemes and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$ be some coefficient ring. Then the pullback map $C^*(Y;\Lambda) \to C^*(X;\Lambda)$ of Remark 3.2.1.12 can be regarded as a morphism of $\mathbb{E}_\infty$-algebras over Λ. In fact, we can regard the construction $X \mapsto C^*(X;\Lambda)$ as a functor from the category $\mathrm{Sch}_k^{\mathrm{op}}$ of quasi-projective k-schemes to the ∞-category CAlg_Λ of $\mathbb{E}_\infty$-algebras over Λ.

3.2.4 Algebraic Stacks and Fibered Categories

Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. In §1.4.1, we gave an informal description of the moduli stack $\mathrm{Bun}_G(X)$ (see Construction 1.4.1.1) as a contravariant functor

$$\{\ k\text{-schemes}\ \} \to \{\ \text{Groupoids}\ \}.$$

This functor assigns to each k-scheme S the category $\mathrm{Tors}_G(X_S)$ of principal G-bundles on the relative curve $X_S = S \times_{\mathrm{Spec}(k)} X$, and to each morphism of k-schemes $\phi : S \to S'$ the pullback functor $\phi^* : \mathrm{Bun}_G(X)(S') \to \mathrm{Bun}_G(X)(S)$, given on objects by the formula $\phi^* \mathcal{P} = S' \times_S \mathcal{P}$. Beware that this construction is not strictly functorial: given another k-scheme homomorphism $\psi : S'' \to S'$, the iterated pullback

$$\psi^*(\phi^* \mathcal{P}) = S'' \times_{S'} (S' \times_S \mathcal{P})$$

is canonically isomorphic to $S'' \times_S \mathcal{P}$, but not literally identical. One can address this technical point by regarding the collection of groupoids as forming a 2-category (or ∞-category), rather than an ordinary category, and generalizing the notion of functor appropriately. In this section, we will review a more traditional solution to the same problem, using the language of *fibered categories*.

Construction 3.2.4.1 (The Category of Points of $\mathrm{Bun}_G(X)$)**.** Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. We define a category $\mathrm{Pt}(\mathrm{Bun}_G(X))$ as follows:

- The objects of $\mathrm{Pt}(\mathrm{Bun}_G(X))$ are pairs $(S, \mathcal{P})$, where S is a quasi-projective k-scheme and $\mathcal{P}$ is a G-bundle on the relative curve $X_S = S \times_{\mathrm{Spec}(k)} X$.
- A morphism from $(S, \mathcal{P})$ to $(S', \mathcal{P}')$ in the category $\mathrm{Pt}(\mathrm{Bun}_G(X))$ is a commutative diagram of k-schemes

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{P}' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S', \end{array}$$

  where the upper horizontal map is G-equivariant. Equivalently, a morphism from $(S, \mathcal{P})$ to $(S', \mathcal{P}')$ is given by a morphism of k-schemes $f : S \to S'$, together with a G-bundle isomorphism $\mathcal{P} \simeq f^* \mathcal{P}' = S \times_{S'} \mathcal{P}'$.

Remark 3.2.4.2. In the situation of Construction 3.2.4.1, the restriction to quasi-projective k-schemes S is not important: one can capture essentially the same information by allowing a larger class of test objects (such as arbitrary k-schemes) or a smaller class of test objects (such as affine k-schemes of finite type).

Let Sch_k denote the category of quasi-projective k-schemes. In the situation of Construction 3.2.4.1, the assignment $(S, \mathcal{P}) \mapsto S$ determines a forgetful functor $\pi : \mathrm{Pt}(\mathrm{Bun}_G(X)) \to \mathrm{Sch}_k$. For every quasi-projective k-scheme X, we can recover the category of $\mathrm{Tors}_G(X_S)$ of principal G-bundles on X_S as the fiber product $\mathrm{Pt}(\mathrm{Bun}_G(X)) \times_{\mathrm{Sch}_k} \{S\}$. Moreover, the map π also encodes the functoriality of the construction $S \mapsto \mathrm{Tors}_G(X_S)$: given an object $(S, \mathcal{P}) \in \mathrm{Pt}(\mathrm{Bun}_G(X))$ and a k-scheme morphism $f : S' \to S$, we can choose any lift of f to a morphism $\overline{f} : (S', \mathcal{P}') \to (S, \mathcal{P})$ in the category $\mathrm{Pt}(\mathrm{Bun}_G(X))$. Such a lift then exhibits $\mathcal{P}'$ as a fiber product $S' \times_S \mathcal{P}$.

More generally, for any functor of categories $\pi : \mathcal{C} \to \mathcal{D}$ and any object $D \in \mathcal{D}$, let $\mathcal{C}_D$ denote the fiber product $\mathcal{C} \times_{\mathcal{D}} \{D\}$. We might then ask if $\mathcal{C}_D$ depends functorially on D, in some sense. This requires an assumption on the functor π.

Definition 3.2.4.3. Let $\pi : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We say that a morphism $\alpha : C' \to C$ in $\mathcal{C}$ is *π-Cartesian* if, for every object $C'' \in \mathcal{C}$, composition with α induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(C'', C') \to \mathrm{Hom}_{\mathcal{C}}(C'', C) \times_{\mathrm{Hom}_{\mathcal{D}}(\pi C'', \pi C)} \mathrm{Hom}_{\mathcal{D}}(\pi C'', \pi C').$$

We will say that π is a *Cartesian fibration* if, for every object $C \in \mathcal{C}$ and every morphism $\alpha_0 : D \to \pi(C)$ in the category $\mathcal{D}$, there exists a π-Cartesian morphism $\alpha : \overline{D} \to C$ with $\alpha_0 = \pi(\alpha)$.

Warning 3.2.4.4. A functor $\pi : \mathcal{C} \to \mathcal{D}$ satisfying the requirements of Definition 3.2.4.3 is more often referred to as an *fibration* or a *Grothendieck fibration* between categories. We use the term *Cartesian fibration* to remain consistent with the conventions of [25].

Example 3.2.4.5. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. Then the forgetful functor

$$\pi : \mathrm{Pt}(\mathrm{Bun}_G(X)) \to \mathrm{Sch}_k \qquad \pi(S, \mathcal{P}) = S$$

is a Cartesian fibration of categories. Moreover, every morphism in the category $\mathrm{Pt}(\mathrm{Bun}_G(X))$ is π-Cartesian (in other words, the category $\mathrm{Pt}(\mathrm{Bun}_G(X))$ is *fibered in groupoids* over the category Sch_k).

Example 3.2.4.6 (The Category of Points of BG)**.** Let X be a quasi-projective k-scheme and let G be a smooth affine group scheme over X. We define a category Pt(BG) as follows:

- The objects of Pt(BG) are triples $(S, f, \mathcal{P})$, where S is a quasi-projective k-scheme, $f : S \to X$ is a morphism of k-schemes, and $\mathcal{P}$ is a G-bundle on S.

- A morphism from $(S, f, \mathcal{P})$ to $(S', f', \mathcal{P}')$ in the category Pt(BG) is a commutative diagram of k-schemes

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{P}' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X, & \end{array}$$

  where the upper horizontal map is G-equivariant. Equivalently, a morphism from $(S, f, \mathcal{P})$ to $(S', f', \mathcal{P}')$ is given by a morphism of X-schemes $g : S \to S'$, together with a G-bundle isomorphism $\mathcal{P} \simeq g^* \mathcal{P}' = S \times_{S'} \mathcal{P}'$.

The construction $(S, f, \mathcal{P}) \mapsto S$ determines a Cartesian fibration of categories $\pi : \mathrm{Pt}(\mathrm{BG}) \to \mathrm{Sch}_k$. Moreover, every morphism in Pt(BG) is π-Cartesian.

Example 3.2.4.7 (The Category of Points of a k-Scheme)**.** Let X be a quasi-projective k-scheme. We can associate to X a category $\mathrm{Pt}(X)$, which we call the *category of points* of X. By definition, an object of $\mathrm{Pt}(X)$ is a pair (S, ϕ), where S is a quasi-projective

k-scheme and $\phi : S \to X$ is a morphism of k-schemes. A morphism from (S, ϕ) to (S', ϕ') is given by a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & {\scriptstyle \phi}\searrow \quad \swarrow{\scriptstyle \phi'} & \\ & X. & \end{array}$$

The construction $(S, \phi) \mapsto S$ determines a Cartesian fibration of categories $\pi : \mathrm{Pt}(X) \to \mathrm{Sch}_k$, where every morphism in $\mathrm{Pt}(X)$ is π-Cartesian.

Examples 3.2.4.5, 3.2.4.6, and 3.2.4.7 can all be encapsulated by the following:

Example 3.2.4.8. Let $\mathcal{X}$ be an algebraic stack which is locally of finite type over the field k. We define a category $\mathrm{Pt}(\mathcal{X})$ as follows:

- The objects of $\mathrm{Pt}(\mathcal{X})$ are pairs (S, f), where S is quasi-projective k-scheme and f is an S-valued point of $\mathcal{X}$ (that is, a morphism from S to $\mathcal{X}$ of algebraic stacks over k).

- If (S, f) and (S', f') are objects of $\mathrm{Pt}(\mathcal{X})$, then a morphism from (S, f) to (S', f') in $\mathrm{Pt}(\mathcal{X})$ is a pair (g, α), where $g : S \to S'$ is a morphism of k-schemes and α is an isomorphism of f with $f' \circ g$ in the category of S-valued points of $\mathcal{X}$. Put another way, a morphism from (S, f) to (S', f') is a diagram of algebraic stacks over k

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & \mathcal{X} & \end{array}$$

  which commutes up to a *specified* 2-isomorphism.

We will refer to $\mathrm{Pt}(\mathcal{X})$ as *the category of points of* $\mathcal{X}$. The construction $(S, f) \mapsto S$ determines a Cartesian fibration of categories $\pi : \mathrm{Pt}(\mathcal{X}) \to \mathrm{Sch}_k$. Moreover, every morphism in $\mathrm{Pt}(\mathcal{X})$ is π-Cartesian.

Remark 3.2.4.9. In many treatments of the theory of algebraic stacks, an algebraic stack $\mathcal{X}$ over k is identified with the Cartesian fibration $\pi : \mathrm{Pt}(\mathcal{X}) \to \mathrm{Sch}_k$ (or some variant thereof, where we replace Sch_k by a suitable category of test objects). In this book, we will implicitly follow this convention: the cohomological invariants of an algebraic stack $\mathcal{X}$ will be defined using the category $\mathrm{Pt}(\mathcal{X})$. To avoid confusion, our notation will maintain a (technically irrelevant) distinction between the algebraic stack $\mathcal{X}$ itself (which we think of as an algebro-geometric object) and its category of points $\mathrm{Pt}(\mathcal{X})$.

Example 3.2.4.10 (Grothendieck Construction)**.** Let $\mathcal{D}$ be a category, and let U be a functor from $\mathcal{D}^{\mathrm{op}}$ to the category $\mathcal{C}\mathrm{at}$ of categories. We can define a new category $\mathcal{D}_U$ as follows:

(1) The objects of $\mathcal{D}_U$ are pairs (D, u) where D is an object of $\mathcal{D}$ and u is an object of the category $U(D)$.

(2) A morphism from (D, u) to (D', u') consists of a pair (ϕ, α), where $\phi : D \to D'$ is a morphism in $\mathcal{D}$ and $\alpha : u \to U(\phi)(u')$ is a morphism in the category $U(D)$.

The construction $(D, u) \mapsto D$ determines a Cartesian fibration of categories $\pi : \mathcal{D}_U \to \mathcal{D}$, where a morphism (ϕ, α) in $\mathcal{D}_U$ is π-Cartesian if and only if α is an isomorphism. The passage from the functor U to the category $\mathcal{D}_U$ is often called the *Grothendieck construction*.

For any Cartesian fibration $F : \mathcal{C} \to \mathcal{D}$, the category $\mathcal{C}$ is equivalent to $\mathcal{D}_U$, for some functor $U : \mathcal{D} \to \mathcal{C}\mathrm{at}$. Moreover, the datum of the Cartesian fibration F and the datum of the functor U are essentially equivalent to one another (in a suitable 2-categorical sense); see Proposition 3.2.6.4.

3.2.5 ℓ-Adic Cohomology of Algebraic Stacks

In Chapter 2, we introduced the ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves on a scheme X, under the assumption that X is quasi-projective over an algebraically closed field k. In this chapter, we applied the theory of ℓ-adic sheaves to construct the ℓ-adic cochain complex $C^*(X; \mathbf{Z}_\ell)$ (Construction 3.2.1.1) and to equip it with an $\mathbb{E}_\infty$-algebra structure (Construction 3.2.3.3). In each of these constructions, the quasi-projectivity assumption on X is largely superfluous: the theory of ℓ-adic sheaves (and ℓ-adic cohomology) can be extended to a much larger class of geometric objects. In this section, we consider a generalization to algebraic stacks (assumed for simplicity to be locally of finite type over k) that will be needed in the proof of Weil's conjecture.

Construction 3.2.5.1. Let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}\}$ be a coefficient ring, let $\mathcal{X}$ be an algebraic stack which is locally of finite type over k, and let $\mathrm{Pt}(\mathcal{X})$ denote the category of points of $\mathcal{X}$ (Example 3.2.4.8), whose objects are given by pairs (S, f) where S is a quasi-projective k-scheme and $f : S \to \mathcal{X}$ is a morphism of algebraic stacks over k. Using Remark 3.2.3.7, we can regard the construction $(S, f) \mapsto C^*(S; \Lambda)$ as a functor from the category $\mathrm{Pt}(\mathcal{X})^{\mathrm{op}}$ to the ∞-category CAlg_Λ of $\mathbb{E}_\infty$-algebras over Λ. We let $C^*(\mathcal{X}; \Lambda)$ denote the inverse limit $\varprojlim_{(S,f) \in \mathrm{Pt}(\mathcal{X})^{\mathrm{op}}} C^*(S; \Lambda)$, formed in the ∞-category CAlg_Λ. We will refer to $C^*(\mathcal{X}; \Lambda)$ as the *complex of* Λ*-valued cochains on* $\mathcal{X}$. For every integer n, we let $\mathrm{H}^n(\mathcal{X}; \Lambda)$ denote the nth cohomology of the chain complex $C^*(\mathcal{X}; \Lambda)$, so that $\mathrm{H}^*(\mathcal{X}; \Lambda)$ can be regarded as a graded-commutative algebra over Λ.

Example 3.2.5.2. Let X be a quasi-projective k-scheme. Then X can be regarded as an algebraic stack (which is of finite type over k). For each coefficient ring $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$, we have supplied two different definitions for the cochain complex $C^*(X;\Lambda)$: one given by Construction 3.2.1.1 (or Variants 3.2.1.3 and 3.2.1.10), and one given by Construction 3.2.5.1. However, the resulting $\mathbb{E}_\infty$-algebras are canonically equivalent to one another: this follows from the observation that the identity map $\mathrm{id} : X \to X$ is a final object of the category $\mathrm{Pt}(X)$.

Remark 3.2.5.3. For every integer $d \geq 0$, the extension of scalars functor $\mathrm{CAlg}_{\mathbf{Z}_\ell} \to \mathrm{CAlg}_{\mathbf{Z}/\ell^d\mathbf{Z}}$ commutes with limits. It follows that, for every algebraic stack $\mathcal{X}$ of finite type over k, we have a canonical equivalence

$$\mathbf{Z}/\ell^d\mathbf{Z} \otimes_{\mathbf{Z}_\ell} C^*(\mathcal{X};\mathbf{Z}_\ell) \to C^*(\mathcal{X};\mathbf{Z}/\ell^d\mathbf{Z})$$

of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}/\ell^d\mathbf{Z}$. Similarly, we can identify $C^*(\mathcal{X};\mathbf{Z}_\ell)$ with the limit of the tower

$$\cdots \to C^*(\mathcal{X};\mathbf{Z}/\ell^3\mathbf{Z}) \to C^*(\mathcal{X};\mathbf{Z}/\ell^2\mathbf{Z}) \to C^*(\mathcal{X};\mathbf{Z}/\ell\mathbf{Z}).$$

Proposition 3.2.5.4. *Let $\mathcal{X}$ be an algebraic stack which is of finite type over k. Then the canonical map $C^*(\mathcal{X};\mathbf{Z}_\ell)[\ell^{-1}] \to C^*(\mathcal{X};\mathbf{Q}_\ell)$ is an equivalence of $\mathbb{E}_\infty$-algebras over $\mathbf{Q}_\ell$.*

Proof. Assume for simplicity that the algebraic stack $\mathcal{X}$ has quasi-projective diagonal (this condition is satisfied for all algebraic stacks of interest to us in this book). Choose a smooth surjective map $U_0 \to \mathcal{X}$, where U_0 is an affine scheme. For each $n \geq 0$, let U_n denote the $(n+1)$-fold fiber product of U_0 over $\mathcal{X}$, so that each U_n is a quasi-projective k-scheme. Then $U_\bullet$ is a simplicial scheme. For each coefficient ring $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$, we can identify $C^*(\mathcal{X};\Lambda)$ with the totalization of the cosimplicial $\mathbb{E}_\infty$-algebra $C^*(U_\bullet;\Lambda)$. The desired result now follows from Remark 3.2.1.11, since the functor

$$\mathrm{Mod}_{\mathbf{Z}_\ell} \to \mathrm{Mod}_{\mathbf{Q}_\ell} \qquad M \mapsto M[\ell^{-1}]$$

commutes with totalizations of cosimpicial objects when restricted to the full subcategory $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$. □

Warning 3.2.5.5. The conclusion of Proposition 3.2.5.4 generally does not hold if $\mathcal{X}$ is only assumed to be locally of finite type over k. For example, suppose that $\mathcal{X}$ is a disjoint union of infinitely many copies of $\mathrm{Spec}(k)$. In this case, the cochain complex $C^*(\mathcal{X};\mathbf{Z}_\ell)$ is equivalent to a product of infinitely many copies of $\mathbf{Z}_\ell$, while $C^*(\mathcal{X};\mathbf{Q}_\ell)$ is equivalent to a product of infinitely many copies of $\mathbf{Q}_\ell$. In this case, the canonical map $C^*(\mathcal{X};\mathbf{Z}_\ell)[\ell^{-1}] \to C^*(\mathcal{X};\mathbf{Q}_\ell)$ is not a quasi-isomorphism.

In this book, we are primarily interested in the ℓ-adic cohomology of the moduli stack $\mathrm{Bun}_G(X)$, where X is an algebraic curve over k and G is a smooth affine group scheme over X. The moduli stack $\mathrm{Bun}_G(X)$ is never quasi-compact (except in trivial cases), so we cannot apply Proposition 3.2.5.4 directly. Nevertheless, if the generic fiber of G is semisimple, then one can show that the canonical map

$$C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)[\ell^{-1}] \to C^*(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)$$

is a quasi-isomorphism. This can be proved using the methods of Chapter 5. However, this compatibility will not be needed in this book.

Remark 3.2.5.6 (Functoriality)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism between algebraic stacks which are locally of finite type over k. Then f determines a functor $\mathrm{Pt}(\mathcal{X}) \to \mathrm{Pt}(\mathcal{Y})$, which in turn determines a map

$$f^* : C^*(\mathcal{Y}; \Lambda) = \varprojlim_{(S,f) \in \mathrm{Pt}(\mathcal{Y})} C^*(S; \Lambda) \to \varprojlim_{(S',f') \in \mathrm{Pt}(\mathcal{X})} C^*(S'; \Lambda) = C^*(\mathcal{X}; \Lambda)$$

for every coefficient ring $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$. Elaborating on this argument, we can view the construction $\mathcal{X} \mapsto C^*(\mathcal{X}; \Lambda)$ as a contravariant functor from (the nerve of) the 2-category of algebraic stacks (which are locally of finite type over k) to the ∞-category CAlg_Λ of $\mathbb{E}_\infty$-algebras over Λ.

3.2.6 Digression: Fibered ∞-Categories

In §3.2.4, we introduced the notion of a Cartesian fibration between categories (Definition 3.2.4.3). Our discussion placed emphasis on examples of geometric origin: an algebraic stack $\mathcal{X}$ (assumed to be locally of finite type over a field k) can be encoded by its category of points $\mathrm{Pt}(\mathcal{X})$ (Example 3.2.4.8), together with a Cartesian fibration from $\mathrm{Pt}(\mathcal{X})$ to the category Sch_k of quasi-projective k-schemes. However, the theory of Cartesian fibrations is applicable more broadly: for any category $\mathcal{D}$, the Grothendieck construction of Example 3.2.4.10 supplies a useful dictionary

$$\{ \text{ Functors } \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at} \ \} \simeq \{ \text{ Cartesian fibrations } \mathcal{C} \to \mathcal{D} \ .\} \tag{3.2}$$

In this section, we describe an ∞-categorical generalization of the theory of Cartesian fibrations and of the equivalence (3.2), which will be needed in both §3.3 and §3.4. Our presentation will be somewhat terse: for a detailed discussion (and proofs of the results stated here), we refer the reader to [25] (particularly §[25].2.4 and §[25].3.2).

Definition 3.2.6.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories (or, more generally, any map of simplicial sets). We will say that F is an *inner fibration* if, for all $0 < i < n$, the induced map $\mathcal{C}_n \to \Lambda^n_i(\mathcal{C}) \times_{\Lambda^n_i(\mathcal{D})} \mathcal{D}_n$ is surjective (see Notation 2.1.3.1).

Assume that F is an inner fibration, and let $\alpha : C \to C'$ be a morphism in $\mathcal{C}$ (which we regard as a 1-simplex of $\mathcal{C}$). We will say that α is *F-Cartesian* if, for each $n \geq 2$

and each element $\sigma \in \Lambda^n_n(\mathcal{C})$ satisfying $\sigma_{\{n-1,n\}} = \alpha$ (see Notation 2.1.3.1), the induced map

$$\mathcal{C}_n \times_{\Lambda^n_n(\mathcal{C})} \{\sigma\} \to \mathcal{D}_n \times_{\Lambda^n_n(\mathcal{D})} \{\sigma\}$$

is surjective.

We will say that F is a *Cartesian fibration* if it is an inner fibration and, for every object $C \in \mathcal{C}$ and every morphism $\alpha : D \to F(C)$ in the ∞-category $\mathcal{D}$, there exists an F-Cartesian morphism $\overline{\alpha} : \overline{D} \to C$ in $\mathcal{C}$ satisfying $F(\overline{\alpha}) = \alpha$.

Example 3.2.6.2. Let $\mathcal{C}$ be a simplicial set and let $F : \mathcal{C} \to \Delta^0$ be the projection map. Then:

- The map F is an inner fibration if and only if $\mathcal{C}$ is an ∞-category (this follows immediately from the definitions).
- If $\mathcal{C}$ is an ∞-category, then a morphism of $\mathcal{C}$ is F-Cartesian if and only if it is an equivalence (Proposition [25].1.2.4.3).
- If $\mathcal{C}$ is an ∞-category, then F is automatically a Cartesian fibration.

Example 3.2.6.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories, which we regard (via passing to nerves) as a functor of ∞-categories. Then:

- The map F is automatically an inner fibration.
- A morphism of $\mathcal{C}$ is F-Cartesian in the sense of Definition 3.2.6.1 if and only if it is F-Cartesian in the sense of Definition 3.2.4.3.
- The functor F is a Cartesian fibration in the sense of Definition 3.2.6.1 if and only if it is a Cartesian fibration in the sense of Definition 3.2.4.3.

Example 3.2.4.10 admits the following ∞-categorical generalization:

Proposition 3.2.6.4. *[∞-Categorical Grothendieck Construction] Let $\mathcal{C}\mathrm{at}_\infty$ be the ∞-category of ∞-categories (see Construction 3.1.2.1) and let $\mathcal{D}$ be an arbitrary ∞-category. There is an explicit construction which associates to each functor $U : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ a new ∞-category $\mathcal{D}_U$ equipped with a Cartesian fibration $\pi : \mathcal{D}_U \to \mathcal{D}$. Moreover, this construction induces a bijection*

$$\{\textit{Functors } \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty\}/\textit{homotopy} \simeq \{\textit{Cartesian fibrations } \mathcal{C} \to \mathcal{D}\}/\textit{equivalence}.$$

Proof. The construction $U \mapsto \mathcal{D}_U$ is given by the (marked) unstraightening functor of Corollary [25].3.2.1.5. The asserted bijectivity is the main content of Theorem [25].3.2.0.1. □

Remark 3.2.6.5. In the special case where $\mathcal{D}$ is an ordinary category and the functor $U : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ factors through the subcategory $\mathcal{C}\mathrm{at} \subseteq \mathcal{C}\mathrm{at}_\infty$, the Cartesian fibration $\mathcal{D}_U \to \mathcal{D}$ of Proposition 3.2.6.4 agrees with the classical Grothendieck construction described in Example 3.2.4.10.

Remark 3.2.6.6. Let $\pi : \mathcal{C} \to \mathcal{D}$ be a Cartesian fibration of ∞-categories and let $U : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ be a functor which corresponds to π under the bijection of Proposition 3.2.6.4. In this situation, we will say that the functor U *classifies* the Cartesian fibration π, or that π *is classified by* the functor U. In this case, the functors π and U determine one another up to equivalence. The functor U can be described informally by the formula $U(D) = \mathcal{C} \times_{\mathcal{D}} \{D\}$.

Each of the notions defined above can be dualized:

Variant 3.2.6.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ denote the induced functor of opposite ∞-categories. We will say that a morphism of $\mathcal{C}$ is F*-coCartesian* if it is F^{op}-Cartesian (when regarded as a morphism in the ∞-category $\mathcal{C}^{\mathrm{op}}$). We will say that F is a *coCartesian fibration* if the functor F^{op} is a Cartesian fibration.

Applying Proposition 3.2.6.4 to the ∞-category $\mathcal{D}^{\mathrm{op}}$ (and composing with the autoequivalence of $\mathcal{C}\mathrm{at}_\infty$ given by $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{op}}$), we obtain a bijective correspondence

$$\{\text{Functors } \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty\}/\text{homotopy} \simeq \{\text{coCartesian fibrations } \mathcal{C} \to \mathcal{D}\}/\text{equivalence}.$$

If $\pi : \mathcal{C} \to \mathcal{D}$ is a coCartesian fibration which corresponds to a functor $U : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ under this bijection, then we will say that the functor U *classifies* the coCartesian fibration π, or that π *is classified by* the functor U.

3.3 The !-Tensor Product

Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible over k. For every quasi-projective k-scheme X, the ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves on X can be endowed with the symmetric monoidal structure of Proposition 3.2.2.10, given by the formation of tensor products of ℓ-adic sheaves discussed in §3.2.2. This gives rise to an an *external* tensor product operation:

Construction 3.3.0.1. Let X and Y be quasi-projective k-schemes, and let $X \times Y = X \times_{\mathrm{Spec}(k)} Y$ denote their product. For every pair of ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Y)$, we define

$$\mathcal{F} \boxtimes \mathcal{G} = (\pi_X^* \mathcal{F}) \otimes (\pi_Y^* \mathcal{G}) \in \mathrm{Shv}_\ell(X \times Y).$$

Here $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projection maps. We will refer to $\mathcal{F} \boxtimes \mathcal{G}$ as the *external tensor product* of $\mathcal{F}$ and $\mathcal{G}$.

The external tensor product $\mathcal{F} \boxtimes \mathcal{G}$ of Construction 3.3.0.1 is defined in terms of the usual tensor product of ℓ-adic sheaves (on the product $X \times Y$) studied in §3.2.2. Conversely, we can recover the tensor product of §3.2.2 from the external tensor product of Construction 3.3.0.1: if $\mathcal{F}$ and $\mathcal{G}$ are ℓ-adic sheaves on a fixed quasi-projective k-scheme X, then we have a canonical equivalence $\mathcal{F} \otimes \mathcal{G} \simeq \delta^*(\mathcal{F} \boxtimes \mathcal{G})$, where $\delta : X \to X \times X$ denotes the diagonal map. This perspective suggests a variant:

Construction 3.3.0.2. Let X be a quasi-projective k-scheme and let $\delta : X \to X \times X$ denote the diagonal map. For every pair of ℓ-adic sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)$, we define

$$\mathcal{F} \otimes^! \mathcal{G} = \delta^!(\mathcal{F} \boxtimes \mathcal{G}) \in \mathrm{Shv}_\ell(X).$$

The construction $(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \otimes^! \mathcal{G}$ determines a functor $\otimes^! : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$, which we will refer to as the *!-tensor product* functor.

Our goal in this section is to prove the following result:

Theorem 3.3.0.3. *For every quasi-projective k-scheme X, the ∞-category* $\mathrm{Shv}_\ell(X)$ *admits a symmetric monoidal structure whose underlying tensor product is the !-tensor product*

$$\otimes^! : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$$

of Construction 3.3.0.2. Moreover, this symmetric monoidal structure depends functorially on X: for every proper morphism $f : X \to Y$, the exceptional inverse image functor $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ can be regarded as a symmetric monoidal functor from $\mathrm{Shv}_\ell(Y)$ *to* $\mathrm{Shv}_\ell(X)$.

Remark 3.3.0.4. In the statement of Theorem 3.3.0.3, the properness assumption on $f : X \to Y$ can be removed, given an appropriate definition of the exceptional inverse image for non-proper morphisms. However, a rigorous proof would require substantial modifications of the ideas presented here. For our purposes in this book, Theorem 3.3.0.3 will be sufficient.

Let us now briefly outline our approach to Theorem 3.3.0.3. Fix a quasi-projective k-scheme X. Roughly speaking, Theorem 3.3.0.3 asserts that the !-tensor product

$$\otimes^! : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$$

is commutative, associative, and unital, up to coherent homotopy. In particular, for ℓ-adic sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Shv}_\ell(X)$, we have canonical equivalences

$$\alpha : \mathcal{F} \otimes^! \mathcal{G} \simeq \mathcal{G} \otimes^! \mathcal{F} \qquad \beta : \mathcal{F} \otimes^! (\mathcal{G} \otimes^! \mathcal{H}) \simeq (\mathcal{F} \otimes^! \mathcal{G}) \otimes^! \mathcal{H}. \tag{3.3}$$

The existence of the equivalence α follows easily from the definitions (since the tensor product of §3.2.2 is commutative up to equivalence), but the existence of β is less obvious: it will be constructed in §3.3.2, using some formal properties of the external tensor product which we establish in §3.3.1.

To establish the unitality of the !-tensor product, we need to specify an ℓ-adic sheaf $\omega_X \in \mathrm{Shv}_\ell(X)$ for which there are equivalences $\omega_X \otimes^! \mathcal{F} \simeq \mathcal{F}$, depending functorially on $\mathcal{F} \in \mathrm{Shv}_\ell(X)$. It follows from general considerations that the object ω_X is uniquely determined up to equivalence: we will refer to it as the *dualizing sheaf* of X. The construction of the dualizing sheaf ω_X (and the verification of its universal property) will be carried out in §3.3.3.

The commutativity and associativity equivalences of (3.3) can be used to equip the homotopy category $\mathrm{hShv}_\ell(\mathrm{X})$ with a symmetric monoidal structure. However, this is not good enough for our applications: we will need to study the ∞-category of commutative algebra objects of $\mathrm{Shv}_\ell(X)$, whose definition requires a symmetric monoidal structure on the ∞-category $\mathrm{Shv}_\ell(X)$ itself. More informally, we need to know that the !-tensor product $\otimes^!$ is commutative and associative not only up to homotopy, but up to *coherent* homotopy. The proof of this will require somewhat elaborate (but completely formal) categorical constructions, which we carry out in §3.3.4 and 3.3.5.

3.3.1 The Künneth Formula

Let X and Y be quasi-projective k-schemes and let $X \times Y = X \times_{\mathrm{Spec}(k)} Y$ denote their product. For any coefficient ring $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$, the multiplication on $C^*(X \times Y; \Lambda)$ induces a map

$$\begin{aligned} C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) &\to C^*(X \times Y;\Lambda) \otimes_\Lambda C^*(X \times Y;\Lambda) \\ &\to C^*(X \times Y;\Lambda). \end{aligned}$$

We then have the following result:

Theorem 3.3.1.1 (Künneth Formula)**.** *For every pair of quasi-projective k-schemes X and Y and every coefficient ring $\Lambda \in \{\mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Z}_\ell, \mathbf{Q}_\ell\}$, the canonical map*

$$C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) \to C^*(X \times Y;\Lambda)$$

is an equivalence.

Remark 3.3.1.2. When $\Lambda \in \{\mathbf{Z}/\ell\mathbf{Z}, \mathbf{Q}_\ell\}$ is a field, Theorem 3.3.1.1 asserts that we have a canonical isomorphism

$$\mathrm{H}^*(X \times Y;\Lambda) \simeq \mathrm{H}^*(X;\Lambda) \otimes_\Lambda \mathrm{H}^*(Y;\Lambda).$$

Theorem 3.3.1.1 is a special case of a more general result (Corollary 3.3.1.6), which we will prove below by studying the relationship between the external tensor product

$\boxtimes$ of Construction 3.3.0.1 with direct images. Note that if $f : X \to Y$ is a morphism of quasi-projective k-schemes and Z is another quasi-projective k-scheme, then we have a canonical equivalence

$$(f \times \mathrm{id}_Z)^*(\mathcal{H} \boxtimes \mathcal{G}) \simeq f^* \mathcal{H} \boxtimes \mathcal{G}$$

for $\mathcal{H} \in \mathrm{Shv}_\ell(Y)$, $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$. Taking $\mathcal{H} = f_* \mathcal{F}$ for $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ (and composing with the counit map $f^* \mathcal{H} \to \mathcal{F}$), we obtain a map

$$\theta_{\mathcal{F},\mathcal{G}} : f_* \mathcal{F} \boxtimes \mathcal{G} \to (f \times \mathrm{id}_Z)_*(\mathcal{F} \boxtimes \mathcal{G}).$$

Proposition 3.3.1.3. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes and let Z be a quasi-projective k-scheme. Then for every pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Z)$, the canonical map*

$$\theta_{\mathcal{F},\mathcal{G}} : (f_* \mathcal{F}) \boxtimes \mathcal{G} \to (f \times \mathrm{id}_Z)_*(\mathcal{F} \boxtimes \mathcal{G})$$

is an equivalence in $\mathrm{Shv}_\ell(Y \times Z)$.

Before giving the proof of Proposition 3.3.1.3, let us collect some consequences.

Corollary 3.3.1.4. *Let $f : X \to Y$ be a morphism of quasi-projective k-schemes and let Z be another quasi-projective k-scheme, so that we have a pullback square*

$$\begin{array}{ccc} X \times Z & \xrightarrow{f'} & Y \times Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X & \xrightarrow{f} & Y. \end{array}$$

For each sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the canonical map $g'^* f_* \mathcal{F} \to f'_* g^* \mathcal{F}$ is an equivalence.

Proof. Apply Proposition 3.3.1.3 in the special case $\mathcal{G} = \underline{\mathbf{Z}_\ell}_Z \in \mathrm{Shv}_\ell(Z)$. $\square$

Corollary 3.3.1.5. *Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of quasi-projective k-schemes. For every pair of ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{F}' \in \mathrm{Shv}_\ell(X')$, the canonical map $(f_* \mathcal{F}) \boxtimes (f'_* \mathcal{F}') \to (f \times f')_*(\mathcal{F} \boxtimes \mathcal{F}')$ is an equivalence in $\mathrm{Shv}_\ell(Y \times Y')$.*

Corollary 3.3.1.6. *Let X and X' be quasi-projective k-schemes. For every pair of ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, $\mathcal{F}' \in \mathrm{Shv}_\ell(X')$, the canonical map*

$$C^*(X; \mathcal{F}) \otimes_{\mathbf{Z}_\ell} C^*(X'; \mathcal{F}') \to C^*(X \times X'; \mathcal{F} \boxtimes \mathcal{F}')$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof of Theorem 3.3.1.1. Let X and Y be quasi-projective k-schemes and let $\Lambda \in \{\mathbf{Z}_\ell, \mathbf{Z}/\ell^d\mathbf{Z}, \mathbf{Q}_\ell\}$; we wish to show that the tautological map $C^*(X;\Lambda) \otimes_\Lambda C^*(Y;\Lambda) \to C^*(X \times Y;\Lambda)$ is an equivalence. Using Remarks 3.2.1.4 and 3.2.1.11, we can reduce to the case $\Lambda = \mathbf{Z}_\ell$, in which case the desired result is a special case of Corollary 3.3.1.6. □

The proof of Proposition 3.3.1.3 will require some preliminaries. Let $f : X \to Y$ be a morphism of quasi-projective k-schemes, and suppose we are given ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Y)$. Since the pullback functor f^* commutes with tensor products (Remark 3.2.2.13), the counit $f^* f_* \mathcal{F} \to \mathcal{F}$ induces a map

$$f^*((f_* \mathcal{F}) \otimes \mathcal{G}) \simeq (f^* f_* \mathcal{F}) \otimes f^* \mathcal{G} \to \mathcal{F} \otimes f^* \mathcal{G},$$

which we can identify with a map $\beta_{\mathcal{F},\mathcal{G}} : (f_* \mathcal{F}) \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^* \mathcal{G})$ in the ∞-category $\mathrm{Shv}_\ell(Y)$.

Proposition 3.3.1.7 (Projection Formula)**.** *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Then for every pair of objects $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ and $\mathcal{G} \in \mathrm{Shv}_\ell(Y)$, the preceding construction induces an equivalence*

$$\beta_{\mathcal{F},\mathcal{G}} : (f_* \mathcal{F}) \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^* \mathcal{G}).$$

Proof. The construction $(\mathcal{F}, \mathcal{G}) \mapsto \beta_{\mathcal{F},\mathcal{G}}$ commutes with filtered colimits separately in each variable. We may therefore assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves. In this case, $\beta_{\mathcal{F},\mathcal{G}}$ is a morphism between constructible ℓ-adic sheaves. Consequently, to prove that $\beta_{\mathcal{F},\mathcal{G}}$ is an equivalence, it will suffice to show that the image of $\beta_{\mathcal{F},\mathcal{G}}$ in $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence. In other words, it suffices to prove the analogue of Proposition 3.3.1.7 when $\mathcal{F}$ and $\mathcal{G}$ are constructible objects of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ and $\mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$, respectively.

Let us regard $\mathcal{F}$ as fixed. Using Corollary 2.2.4.3, we see that the construction $\mathcal{G} \mapsto \beta_{\mathcal{F},\mathcal{G}}$ preserves colimits. It follows that the collection of those objects $\mathcal{G} \in \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$ for which $\beta_{\mathcal{F},\mathcal{G}}$ is an equivalence is closed under colimits. Using Proposition 2.2.4.5, we may suppose that $\mathcal{G} = j_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_U$ for some étale map $j : U \to Y$. Form a pullback diagram

$$\begin{array}{ccc} U_X & \xrightarrow{j'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & Y. \end{array}$$

Unwinding the definitions, we can identify $(f_* \mathcal{F}) \otimes \mathcal{G}$ with the object $j_! j^* f_* \mathcal{F}$, and $\mathcal{F} \otimes f^* \mathcal{G}$ with $j'_! j'^* \mathcal{F}$. Under these identifications, the map $\beta_{\mathcal{F},\mathcal{G}}$ factors as a composition

$$j_! j^* f_* \mathcal{F} \xrightarrow{\beta'} j_! f'_* j'^* \xrightarrow{\beta''} f_* j'_! j'^* \mathcal{F},$$

where β' is an equivalence by Theorem 2.4.2.1 (since j is étale) and β'' is an equivalence by Proposition 2.4.3.4 (since f is proper). □

Proof of Proposition 3.3.1.3. The construction $(\mathcal{F}, \mathcal{G}) \mapsto \theta_{\mathcal{F},\mathcal{G}}$ preserves filtered colimits in $\mathcal{F}$ and $\mathcal{G}$. We may therefore assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves. In this case, $\theta_{\mathcal{F},\mathcal{G}}$ is a morphism of constructible ℓ-adic sheaves on $Y \times Z$. Consequently, to prove that $\theta_{\mathcal{F},\mathcal{G}}$ is an equivalence, it will suffice to show that the image of $\theta_{\mathcal{F},\mathcal{G}}$ in $\mathrm{Shv}(Y \times Z; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence. It will therefore suffice to prove the analogue of Proposition 3.3.1.3 where $\mathcal{F}$ and $\mathcal{G}$ are compact objects of $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})$ and $\mathrm{Shv}(Z; \mathbf{Z}/\ell\mathbf{Z})$, respectively.

We first consider two special cases:

(a) If the map f is proper, then the desired result follows immediately from the projection formula (Proposition 3.3.1.7).

(b) Suppose that Z is smooth and that $\mathcal{G}$ is locally constant. In this case, we can assume that $\mathcal{G}$ is the constant sheaf $\underline{M}_Z$, where $M \in \mathrm{Mod}_{\mathbf{Z}/\ell\mathbf{Z}}$ is perfect (since the assertion is local with respect to the étale topology on Z). The collection of those M for which $\theta_{\mathcal{F},\mathcal{G}}$ is an equivalence is closed under shifts, retracts, and finite colimits; we may therefore assume that $M = \mathbf{Z}/\ell\mathbf{Z}$. In this case, the desired result follows from the smooth base change theorem (Theorem 2.4.2.1).

We now treat the general case. For the remainder of the proof, we will regard $f : X \to Y$ and $\mathcal{F} \in \mathrm{Shv}^c(X; \mathbf{Z}/\ell\mathbf{Z})$ as fixed. Let d denote the dimension of Z; we will proceed by induction on d. It follows from case (a) that if we are given a proper map $g : Z \to Z'$, then we can identify $\theta_{\mathcal{F},g_* \mathcal{G}}$ with the image of $\theta_{\mathcal{F},\mathcal{G}}$ under the pushforward functor $(\mathrm{id} \times g)_* : \mathrm{Shv}(Y \times Z; \mathbf{Z}/\ell\mathbf{Z}) \to \mathrm{Shv}(Y \times Z'; \mathbf{Z}/\ell\mathbf{Z})$.

Since the desired conclusion can be tested locally on Z, we may assume without loss of generality that Z is affine. In this case, we can use Noether normalization to choose a finite map $g : Z \to \mathbf{A}^d$. Then $\mathrm{cofib}(\theta_{\mathcal{F},\mathcal{G}})$ vanishes if and only if $g_* \mathrm{cofib}(\theta_{\mathcal{F},\mathcal{G}}) \simeq \mathrm{cofib}(\theta_{\mathcal{F},g_* \mathcal{G}})$ vanishes. We may therefore replace $\mathcal{G}$ by $g_* \mathcal{G}$, and thereby reduce to the case where $Z = \mathbf{A}^d$ is an affine space.

Using Proposition 2.2.6.2, we can choose a nonempty open subset $U \subseteq Z$ such that $\mathcal{G}|_U$ is locally constant. Applying a translation if necessary, we may suppose that U contains the origin $0 \in \mathbf{A}^d = Z$. Set $\mathcal{H} = \mathrm{cofib}(\theta_{\mathcal{F},\mathcal{G}})$, so that $\mathcal{H} \in \mathrm{Shv}^c(Y \times Z; \mathbf{Z}/\ell\mathbf{Z})$. Using (b), we see that $\mathcal{H}$ vanishes on the open set $Y \times U$. We wish to prove that $\mathcal{H} \simeq 0$. Suppose otherwise: then $\mathcal{H}$ has a nonvanishing stalk at some closed point (y, z) of $Y \times Z$. Since $\mathcal{H}$ vanishes on $Y \times U$, z is not the origin of $Z \simeq \mathbf{A}^d$. Applying a linear change of coordinates, we may assume without loss of generality that $z = (1, 0, \ldots, 0)$. Let $\overline{Z} = \mathbf{P}^1 \times \mathbf{A}^{d-1}$, let $j : Z \to \overline{Z}$ denote the inclusion map, let $\overline{\mathcal{G}} = j_! \mathcal{G}$, and set $\overline{\mathcal{H}} = \theta_{\mathcal{F},\overline{\mathcal{G}}} \in \mathrm{Shv}^c(Y \times \overline{Z}; \Lambda)$. Let $g : \overline{Z} \to \mathbf{A}^{d-1}$ denote the projection map onto the second factor. Since $\overline{\mathcal{H}}$ vanishes on $Y \times_{\operatorname{Spec}(k)} U$, the support of $\overline{\mathcal{H}}$ has finite intersection with the fiber $(\operatorname{id} \times g)^{-1}\{(y,0)\}$. Using the proper base change theorem (Theorem 2.4.2.1), we see that the stalk of $\mathcal{H}$ at (y,z) can be identified with a direct summand of the stalk of $(\operatorname{id} \times g)_* \overline{\mathcal{H}}$ at the point $(y,0)$. In particular, we have $0 \neq (\operatorname{id} \times g)_* \overline{\mathcal{H}} \simeq \operatorname{cofib}(\theta_{\mathcal{F}, g_* \overline{\mathcal{G}}})$, contradicting our inductive hypothesis. □

3.3.2 Associativity of the !-Tensor Product

Let X be a quasi-projective k-scheme. Our goal in this section is to verify the associativity of the !-tensor product

$$\otimes^! : \operatorname{Shv}_\ell(X) \times \operatorname{Shv}_\ell(X) \to \operatorname{Shv}_\ell(X)$$

described in Construction 3.3.0.2. To this end, suppose we are given ℓ-adic sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \operatorname{Shv}_\ell(X)$; we wish to construct a canonical equivalence

$$(\mathcal{F} \otimes^! \mathcal{G}) \otimes^! \mathcal{H} \quad \simeq \quad \mathcal{F} \otimes^! (\mathcal{G} \otimes^! \mathcal{H}). \tag{3.4}$$

Note that the associativity of the usual tensor product on $\operatorname{Shv}_\ell(X \times X \times X)$ guarantees that the external tensor products $(\mathcal{F} \boxtimes \mathcal{G}) \boxtimes \mathcal{H}$ and $\mathcal{F} \boxtimes (\mathcal{G} \boxtimes \mathcal{H})$ are canonically equivalent; we will abuse notation by denoting both of these objects by $\mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{H} \in \operatorname{Shv}_\ell(X \times X \times X)$. We will verify the associativity of $\otimes^!$ by showing that both sides of (3.4) can be identified with $\delta^{(3)!}(\mathcal{F} \boxtimes \mathcal{G} \boxtimes \mathcal{H})$, where $\delta^{(3)} : X \to X \times X \times X$ is the diagonal embedding. This is a consequence of the following:

Proposition 3.3.2.1. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes, let Z be a quasi-projective k-scheme, and let $f' = f \times \operatorname{id}_Z : X \times Z \to Y \times Z$. Then, for every pair of objects $\mathcal{F} \in \operatorname{Shv}_\ell(Y)$ and $\mathcal{G} \in \operatorname{Shv}_\ell(Z)$, the canonical map*

$$\mu_{\mathcal{F},\mathcal{G}} : (f^! \mathcal{F} \boxtimes \mathcal{G}) \to f'^! f'_* (f^! \mathcal{F} \boxtimes \mathcal{G}) \xrightarrow{\theta^{-1}} f'^! (f_* f^! \mathcal{F} \boxtimes \mathcal{G}) \to f'^! (\mathcal{F} \boxtimes \mathcal{G})$$

is an equivalence; here $f' = (f \times \operatorname{id}_Z) : X \times Z \to Y \times Z$ and $\theta = \theta_{f^! \mathcal{F}, \mathcal{G}}$ is the equivalence of Proposition 3.3.1.3.

Before giving the proof of Proposition 3.3.2.1, we make some auxiliary remarks.

Construction 3.3.2.2. Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes and suppose we are given objects $\mathcal{F}, \mathcal{G} \in \operatorname{Shv}_\ell(Y)$. Tensoring the counit map $f_* f^! \mathcal{G} \to \mathcal{G}$ with $\mathcal{F}$ and applying Proposition 3.3.1.7, we obtain a map

$$f_*(f^* \mathcal{F} \otimes f^! \mathcal{G}) \simeq \mathcal{F} \otimes f_* f^! \mathcal{G} \to \mathcal{F} \otimes \mathcal{G},$$

which in turn classifies a map $\rho_{\mathcal{F},\mathcal{G}} : f^* \mathcal{F} \otimes f^! \mathcal{G} \to f^! (\mathcal{F} \otimes \mathcal{G})$ in $\operatorname{Shv}_\ell(X)$.

Proposition 3.3.2.3. *Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes. Let $U \subseteq X$ be an open subset for which $f|_U$ is smooth. Then the natural map $\rho_{\mathcal{F},\mathcal{G}} : f^* \mathcal{F} \otimes f^! \mathcal{G} \to f^!(\mathcal{F} \otimes \mathcal{G})$ induces an equivalence $(f^* \mathcal{F} \otimes f^! \mathcal{G})|_U \to f^!(\mathcal{F} \otimes \mathcal{G})|_U$ for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(Y)$.*

Proof. Let $\underline{\mathbf{Z}_{\ell}}_Y$ denote the unit object of $\mathrm{Shv}_\ell(Y)$. Note that we have a commutative diagram

$$\begin{array}{ccccc} & & f^* \mathcal{F} \otimes f^! \mathcal{G} & & \\ & \nearrow^{\rho_{\mathcal{G},\underline{\mathbf{Z}_{\ell}}_Y}} & & \searrow & \\ f^* \mathcal{F} \otimes f^* \mathcal{G} \otimes f^! \underline{\mathbf{Z}_{\ell}}_Y & & \xrightarrow{\rho_{\mathcal{F}\otimes\mathcal{G},\underline{\mathbf{Z}_{\ell}}_Y}} & & f^!(\mathcal{F} \otimes \mathcal{G}). \end{array}$$

It will therefore suffice to show that the maps $\rho_{\mathcal{G},\underline{\mathbf{Z}_{\ell}}_Y}$ and $\rho_{\mathcal{F}\otimes\mathcal{G},\underline{\mathbf{Z}_{\ell}}_Y}$ are equivalences over the open set U. We may therefore reduce to the case where $\mathcal{G} = \underline{\mathbf{Z}_{\ell}}_Y$. Since the construction $\mathcal{F} \mapsto \rho_{\mathcal{F},\underline{\mathbf{Z}_{\ell}}_Y}$ preserves filtered colimits, we may assume without loss of generality that $\mathcal{F}$ is a constructible ℓ-adic sheaf. In this case, $\rho_{\mathcal{F},\underline{\mathbf{Z}_{\ell}}_Y}$ is a morphism of constructible ℓ-adic sheaves. To prove that it is an equivalence over U, it will suffice to show that its image in $\mathrm{Shv}(U; \mathbf{Z}/\ell\mathbf{Z})$ is an equivalence. In other words, we are reduced to proving that for each constructible object $\mathcal{F}_1 \in \mathrm{Shv}(Y; \mathbf{Z}/\ell\mathbf{Z})$, the canonical map $(f^* \mathcal{F}_1 \otimes f^! \underline{\mathbf{Z}/\ell\mathbf{Z}}_Y)|_U \to (f^! \mathcal{F}_1)|_U$ is an equivalence. Using Proposition 2.2.4.5, we may assume without loss of generality that $\mathcal{F}_1 = g_! \underline{\mathbf{Z}/\ell\mathbf{Z}}_V$ where $g : V \to Y$ is an étale map. In this case, the desired result follows from Proposition 2.4.4.9. □

Proof of Proposition 3.3.2.1. We first treat the case where $f : X \to Y$ is a closed immersion. In this case, $f_*(\mu_{\mathcal{F},\mathcal{G}})$ can be identified with a homotopy inverse to the equivalence $\theta_{f^! \mathcal{F},\mathcal{G}}$ of Proposition 3.3.1.3, and the desired result follows from the fact that the functor f_* is fully faithful (and, in particular, conservative).

To treat the general case, we first choose an immersion $i : X \hookrightarrow \mathbf{P}^n$. Then f factors as a composition $X \xrightarrow{(i,f)} \mathbf{P}^n \times Y \xrightarrow{\pi} Y$, where π denotes the projection onto the second factor. Since f is proper, the map $(i, f) : X \to \mathbf{P}^n \times Y$ is a closed immersion. Using the first part of the proof, we can replace f by the projection map $\pi : \mathbf{P}^n \times Y \to Y$. In this case, the desired result follows from Proposition 3.3.2.3. □

For later use, we note the following more symmetric version of Proposition 3.3.2.1:

Corollary 3.3.2.4. *Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of quasi-projective k-schemes. For every pair of ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$ and $\mathcal{F}' \in \mathrm{Shv}_\ell(Y')$, there is a canonical equivalence $(f^! \mathcal{F}) \boxtimes (f'^! \mathcal{F}') \simeq (f \times f')^!(\mathcal{F} \boxtimes \mathcal{F}')$ of ℓ-adic sheaves on $X \times X'$.*

3.3.3 Dualizing Sheaves

Our next goal is to show that there exists a unit with respect to the !-tensor product on $\mathrm{Shv}_\ell(X)$, where X is a quasi-projective k-scheme. We can state this result more precisely as follows:

Proposition 3.3.3.1. *Let X be a quasi-projective k-scheme. Then there exists an object $\mathcal{E} \in \mathrm{Shv}_\ell(X)$ for which the construction $\mathcal{F} \mapsto \mathcal{E} \otimes^! \mathcal{F}$ is equivalent to the identity functor from $\mathrm{Shv}_\ell(X)$ to itself.*

Remark 3.3.3.2. In the situation of Proposition 3.3.3.1, the object $\mathcal{E}$ is unique up to equivalence: note that if $\mathcal{E}'$ is another object of $\mathrm{Shv}_\ell(X)$ which is a unit with respect to $\otimes^!$, then we have the equivalences

$$\mathcal{E} \simeq \mathcal{E} \otimes^! \mathcal{E}' \simeq \mathcal{E}'.$$

Our goal in this section is to prove Proposition 3.3.3.1 by constructing an ℓ-adic sheaf $\omega_X \in \mathrm{Shv}_\ell(X)$ called the *dualizing sheaf of* X, and showing that it has the appropriate universal property. We begin with the case where X is projective: in this case, the dualizing sheaf ω_X can be characterized by a universal property.

Notation 3.3.3.3. Let $f : X \to Y$ is a proper morphism of quasi-projective k-schemes. We let $\omega_{X/Y}$ denote the ℓ-adic sheaf given by $f^! \underline{\mathbf{Z}_\ell}_Y$. We will refer to $\omega_{X/Y}$ as the *relative dualizing sheaf* of the morphism f. In the special case where $Y = \mathrm{Spec}(k)$, we will denote $\omega_{X/Y}$ by ω_X, and refer to it as the *dualizing sheaf* of X.

Example 3.3.3.4. If $f : X \to Y$ is a proper smooth morphism of relative dimension d, then Example 2.3.3.5 supplies an equivalence $\omega_{X/Y} \simeq \Sigma^{2d} \underline{\mathbf{Z}_\ell(d)}_X$. More generally, one can show that if $U \subseteq X$ is an open subset for which $f|_U$ is a smooth morphism of relative dimension d, then $\omega_{X/Y}|_U$ is equivalent to $\Sigma^{2d} \underline{\mathbf{Z}_\ell(d)}_X$.

Remark 3.3.3.5. Suppose we are given a commutative diagram of quasi-projective k-schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y' & \longrightarrow & Y, \end{array}$$

where the vertical maps are proper. Then Construction 2.4.4.2 supplies a canonical map $f^* \omega_{X/Y} \to \omega_{X'/Y'}$, which is an equivalence over the inverse image of the smooth locus of p (see Variant 2.4.4.7).

If the morphism $f : X \to Y$ is proper and smooth, then the exceptional inverse image functor $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ differs from the usual inverse image functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ by tensor product with the relative dualizing sheaf $\omega_{X/Y}$. More precisely, we have the following result (which is a special case of Proposition 3.3.2.3):

Proposition 3.3.3.6. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. For each object $\mathcal{F} \in \mathrm{Shv}_\ell(Y)$, Construction 3.3.2.2 supplies a map*

$$f^* \mathcal{F} \otimes \omega_{X/Y} \to f^! \mathcal{F}$$

which is an equivalence over the smooth locus of f.

Proof of Proposition 3.3.3.1. Let X be a quasi-projective k-scheme; we wish to show that there exists a unit for the !-tensor product on $\mathrm{Shv}_\ell(X)$. Choose an open embedding $j : X \to \overline{X}$, where $\overline{X}$ is a projective k-scheme. Let $\pi : \overline{X} \to \mathrm{Spec}(k)$ be the projection map. Let $\omega_{\overline{X}} = \pi^! \mathbf{Z}_\ell$ denote the dualizing sheaf on $\overline{X}$, and set $\omega_X = j^* \omega_{\overline{X}}$. We will complete the proof by showing that ω_X has the desired property. Let

$$\delta_X : X \to X \times X \qquad \delta_{\overline{X}} : \overline{X} \to \overline{X} \times \overline{X}$$

denote the diagonal maps, and let $\pi_1 : \overline{X} \times \overline{X} \to \overline{X}$ be the projection onto the first factor. Using Proposition 3.3.2.1, we obtain a canonical equivalence $\pi_2^! \mathcal{G} \simeq \omega_{\overline{X}} \boxtimes \mathcal{G}$ for each object $\mathcal{G} \in \mathrm{Shv}_\ell(\overline{X})$. Applying the functor $\delta_{\overline{X}}^!$, we obtain an equivalence

$$\omega_{\overline{X}} \otimes^! \mathcal{G} \simeq \delta_{\overline{X}}^! (\omega_{\overline{X}} \boxtimes \mathcal{G}) \simeq \delta_{\overline{X}}^! \pi_2^! \mathcal{G} \simeq \mathcal{G}.$$

For any object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we have canonical equivalences

$$\begin{aligned}
\mathcal{F} &\simeq j^* j_* \mathcal{F} \\
&\simeq j^* (\omega_{\overline{X}} \otimes^! j_* \mathcal{F}) \\
&\simeq j^* \delta_{\overline{X}}^! (\omega_{\overline{X}} \boxtimes j_* \mathcal{F}) \\
&\overset{\alpha}{\simeq} \delta_X^! (j \times j)^* (\omega_{\overline{X}} \boxtimes j_* \mathcal{F}) \\
&\simeq \delta_X^! (j^* \omega_{\overline{X}} \boxtimes j^* j_* \mathcal{F}) \\
&\simeq \delta_X^! (\omega_X \boxtimes \mathcal{F}) \\
&= \omega_X \otimes^! \mathcal{F},
\end{aligned}$$

where the equivalence α is obtained by applying Proposition 2.4.4.3 to the pullback square

$$\begin{array}{ccc}
X & \xrightarrow{\delta_X} & X \times X \\
\downarrow & & \downarrow {\scriptstyle j \times j} \\
\overline{X} & \xrightarrow{\delta_{\overline{X}}} & \overline{X} \times \overline{X}.
\end{array}$$

□

Remark 3.3.3.7. In the proof of Proposition 3.3.3.1, it follows from Remark 3.3.3.2 that the sheaf $\omega_X = \omega_{\overline{X}}|_X$ depends only on the k-scheme X, and not on the choice of compactification $\overline{X}$. We will refer to ω_X as the *dualizing sheaf of* X. Note that this terminology is compatible with that of Notation 3.3.3.3 (if X is already projective, we can take $\overline{X} = X$).

3.3.4 The ∞-Category $\mathrm{Shv}_\ell^\star$

The results of §3.3.2 and §3.3.3 show that for every quasi-projective k-scheme X, the !-tensor product functor $\otimes^! : \mathrm{Shv}_\ell(X) \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$ is commutative, associative, and unital up to equivalence. However, the statement of Theorem 3.3.0.3 is stronger: it asserts that the !-tensor product underlies a symmetric monoidal structure on $\mathrm{Shv}_\ell(X)$, in the sense of Definition 3.1.2.2. In this section, we prove an analogous statement for the external tensor product of Construction 3.3.0.1 (Proposition 3.3.4.4). First, we need to introduce a bit of notation.

Construction 3.3.4.1 (The ∞-Category $\mathrm{Shv}_\ell^\star$)**.** According to Notation 2.3.4.5, every morphism $f : X \to Y$ of quasi-projective k-schemes determines a pullback functor on ℓ-adic sheaves $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$. Elaborating on this construction, we can regard the construction $X \mapsto \mathrm{Shv}_\ell(X)$ as defining a functor $\chi : \mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, where $\mathcal{C}\mathrm{at}_\infty$ denotes the ∞-category of ∞-categories (see Construction 3.1.2.1). We let $\mathrm{Shv}_\ell^\star$ denote the ∞-category obtained by applying the dual of Proposition 3.2.6.4 to the functor χ, so that we have a coCartesian fibration of ∞-categories $\mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ (classified by χ in the sense of Variant 3.2.6.7).

Remark 3.3.4.2. We can describe the ∞-category ∞-category $\mathrm{Shv}_\ell^\star$ more informally as follows:

- (a) The objects of the ∞-category $\mathrm{Shv}_\ell^\star$ are pairs $(X, \mathcal{F})$, where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$.
- (b) Given a pair of objects $(X, \mathcal{F}), (Y, \mathcal{G}) \in \mathrm{Shv}_\ell^\star$, a morphism from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ is given by a pair (f, α), where $f : Y \to X$ is a morphism of k-schemes and $\alpha : f^* \mathcal{F} \to \mathcal{G}$ is a morphism of ℓ-adic sheaves on Y.

Warning 3.3.4.3. Strictly speaking, the construction $X \mapsto \mathrm{Shv}_\ell(X)$ does not define a functor with codomain $\mathcal{C}\mathrm{at}_\infty$ because the objects of $\mathcal{C}\mathrm{at}_\infty$ are *small* ∞-categories, and the ∞-categories $\mathrm{Shv}_\ell(X)$ are not small. Of course, this has no real impact on the discussion: the ∞-category $\mathrm{Shv}_\ell^\star$ is still perfectly sensible (albeit large). To avoid burdening the exposition with irrelevant technicalities, we will ignore the distinction between large and small ∞-categories in the discussion which follows.

The main result of this section can be formulated as follows:

Proposition 3.3.4.4. *The ∞-category $\mathrm{Shv}_\ell^\star$ admits a symmetric monoidal structure, whose underlying tensor product is given by $(X, \mathcal{F}) \otimes (Y, \mathcal{G}) = (X \times Y, \mathcal{F} \boxtimes \mathcal{G})$.*

Remark 3.3.4.5. In practice, we are interested not so much in the statement of Proposition 3.3.4.4 (which asserts the existence of a symmetric monoidal structure) but in the proof given below (which supplies a particular symmetric monoidal structure).

The proof of Proposition 3.3.4.4 will require some categorical preliminaries.

Definition 3.3.4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. We will say that F is a *symmetric monoidal Cartesian fibration* if it satisfies the following conditions:

(1) As a functor of ∞-categories, F is a Cartesian fibration (Definition 3.2.6.1).

(2) If $u : X \to X'$ and $v : Y \to Y'$ are F-Cartesian morphisms in $\mathcal{C}$, then the tensor product $(u \otimes v) : X \otimes Y \to X' \otimes Y'$ is also an F-Cartesian morphism in $\mathcal{C}$.

Dually, we say that F is a *symmetric monoidal coCartesian fibration* if the induced map $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a symmetric monoidal Cartesian fibration. In other words, F is a symmetric monoidal coCartesian fibration if it is a coCartesian fibration, and the collection of coCartesian morphisms in $\mathcal{C}$ is closed under tensor products.

Remark 3.3.4.7. Let Fin denote the category of finite sets, equipped with the symmetric monoidal structure given by the formation of disjoint unions. Suppose we are given a symmetric monoidal ∞-category $\mathcal{C}$ equipped with a symmetric monoidal coCartesian fibration $F : \mathcal{C} \to \mathrm{Fin}$. Fix a one-element set $\{*\}$, and let $\mathcal{C}_*$ denote the fiber $\mathcal{C} \times_{\mathrm{Fun}} \{*\}$. Using the fact that F is a coCartesian fibration and that the singleton $\{*\}$ is a final object of the category Fin, we deduce that the inclusion functor $\mathcal{C}_* \hookrightarrow \mathcal{C}$ is fully faithful and admits a left adjoint L: concretely, the functor L carries an object C to the codomain of an F-coCartesian morphism $C \to LC$ covering the projection map $F(C) \to \{*\}$.

Using the assumption that the collection of F-coCartesian morphisms is closed under tensor products, we deduce that the functor L is compatible with the symmetric monoidal structure on $\mathcal{C}$ (in the sense of Definition 3.2.2.3). Consequently, there is an essentially unique symmetric monoidal structure on the ∞-category $\mathcal{C}_*$ for which the functor L is symmetric monoidal.

Remark 3.3.4.8. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories, and suppose we are given a symmetric monoidal coCartesian fibration $F : \mathcal{C} \to \mathcal{D}$. Let A be a commutative algebra object of $\mathcal{D}$, which we regard as a symmetric monoidal functor $\mathrm{Fin} \to \mathcal{D}$.

Applying the construction of Remark 3.3.4.7 to the projection map $\mathcal{C} \times_{\mathcal{D}} \mathrm{Fin} \to \mathrm{Fin}$, we obtain a symmetric monoidal structure on the ∞-category

$$\mathcal{C}_A = \mathcal{C} \times_{\mathcal{D}} \{A\} \simeq (\mathcal{C} \times_{\mathcal{D}} \mathrm{Fin}) \times_{\mathrm{Fin}} \{*\}.$$

It is not hard to see that this symmetric monoidal structure depends functorially on A: in other words, the construction $A \mapsto \mathcal{C}_A$ determines a functor $\mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$.

Variant 3.3.4.9. In the situation of Remark 3.3.4.8, suppose that A is a *nonunital* commutative algebra object of the ∞-category $\mathcal{D}$ (see Variant 3.1.3.8). In this case, we can apply the construction of Remark 3.3.4.8 to equip the fiber $\mathcal{C}_A$ with the structure of a *nonunital symmetric monoidal ∞-category* (that is, a nonunital commutative algebra object of the ∞-category $\mathcal{C}\mathrm{at}_\infty$).

Example 3.3.4.10. In the situation of Remark 3.3.4.8, suppose that the ∞-category $\mathcal{D}$ admits finite coproducts, and that the symmetric monoidal structure on $\mathcal{D}$ is given by the formation of coproducts (see Example 3.1.2.5). In this case, every object $D \in \mathcal{D}$ admits an essentially unique commutative algebra structure, whose multiplication is given by the codiagonal $D \amalg D \to D$ (Example 3.1.3.7). Applying Remark 3.3.4.8, we deduce that the construction $D \mapsto \mathcal{C}_D = \mathcal{C} \times_{\mathcal{D}} \{D\}$ determines a functor $\mathcal{D} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$. In other words, each fiber $\mathcal{C}_D$ of F inherits a symmetric monoidal structure.

Concretely, if $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the tensor product underlying the symmetric monoidal structure on $\mathcal{C}$, then each fiber $\mathcal{C}_D$ inherits a symmetric monoidal structure whose underlying tensor product $\otimes_D$ can be characterized as follows: for every pair of objects $C, C' \in \mathcal{C}_D$, the tensor product $C \otimes_D C'$ is the codomain of an F-coCartesian morphism $C \otimes C' \to C \otimes_D C'$, lying over the codiagonal $D \amalg D \to D$.

We will deduce Proposition 3.3.4.4 from the following general categorical principle:

Proposition 3.3.4.11 (Symmetric Monoidal Grothendieck Construction)**.** *Let $\mathcal{D}$ be an ∞-category which admits finite coproducts, which we regard as endowed with the symmetric monoidal structure given by the formation of coproducts. Then the construction*

$$(F : \mathcal{C} \to \mathcal{D}) \mapsto \{\mathcal{C}_D\}_{D \in \mathcal{D}}$$

of Example 3.3.4.10 induces a bijection from the collection of equivalence classes of symmetric monoidal coCartesian fibrations $F : \mathcal{C} \to \mathcal{D}$ and the collection of equivalence classes of functors $\chi : \mathcal{D} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$.

Remark 3.3.4.12. In the situation of Proposition 3.3.4.11, suppose we are given a functor $\chi : \mathcal{D} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$. Applying the Grothendieck construction to the underlying functor $\mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$, we obtain a coCartesian fibration $F : \mathcal{C} \to \mathcal{D}$. The functor

χ endows each fiber $\mathcal{C}_D$ of F with a symmetric monoidal structure, whose underlying tensor product we will denote by $\otimes_D$. Proposition 3.3.4.11 implies that $\mathcal{C}$ inherits a symmetric monoidal structure (and that F inherits the structure of a symmetric monoidal coCartesian fibration). For every pair of objects $\overline{X}, \overline{Y} \in \mathcal{C}$ having images $X, Y \in \mathcal{D}$, we can lift the canonical maps $X \xrightarrow{f} X \amalg Y \xleftarrow{g} Y$ to F-coCartesian morphisms $\overline{f} : \overline{X} \to \overline{X}'$ and $\overline{g} : \overline{Y} \to \overline{Y}'$ in the ∞-category $\mathcal{C}$. Unwinding the definitions, one sees that the tensor product on $\mathcal{C}$ is given concretely by the construction $(\overline{X}, \overline{Y}) \mapsto (\overline{X}' \otimes_{X \amalg Y} \overline{Y}')$. In the case where $\mathcal{D} = \mathrm{Sch}_k^{\mathrm{op}}$ and the functor χ is given by $X \mapsto \mathrm{Shv}_\ell(X)$, this recovers the external tensor product of Construction 3.3.0.1.

Proof of Proposition 3.3.4.4 from Proposition 3.3.4.11. According to Remark 3.2.2.13, we can regard the construction $X \mapsto \mathrm{Shv}_\ell(X)$ as a functor χ from the category $\mathrm{Sch}_k^{\mathrm{op}}$ to the ∞-category $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ of symmetric monoidal ∞-categories. Note that the category $\mathrm{Sch}_k^{\mathrm{op}}$ admits finite coproducts (which are given by the formation of Cartesian products in the category of k-schemes). Applying Proposition 3.3.4.11, we see that the functor χ classifies a symmetric monoidal coCartesian fibration $F : \mathcal{C} \to \mathrm{Sch}_k^{\mathrm{op}}$. By construction, the underlying ∞-category of $\mathcal{C}$ can be identified with $\mathrm{Shv}_\ell^\star$. Using Remark 3.3.4.12 below, we see that the tensor product underlying the symmetric monoidal structure on $\mathcal{C}$ is given by the construction $(X, \mathcal{F}) \otimes (Y, \mathcal{G}) = (X \times Y, \mathcal{F} \boxtimes \mathcal{G})$. □

Proof of Proposition 3.3.4.11. According to Proposition [23].2.4.3.16, the datum of a symmetric monoidal coCartesian fibration $F : \mathcal{C} \to \mathcal{D}$ is equivalent to the datum of a *lax* symmetric monoidal functor $\rho : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$. If the symmetric monoidal structure on $\mathcal{D}$ is given by the formation of coproducts, then Theorem [23].2.4.3.18 implies that the datum of a lax symmetric monoidal functor from $\mathcal{D}$ to $\mathcal{C}\mathrm{at}_\infty$ is equivalent to the datum of an arbitrary functor $\mathcal{D} \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$. The composition of these equivalences is implemented by the construction of Example 3.3.4.10. □

3.3.5 The ∞-Category $\mathrm{Shv}_\ell^!$

In §3.3.4 we introduced an ∞-category $\mathrm{Shv}_\ell^\star$, whose objects are pairs $(X, \mathcal{F})$ where X is a quasi-projective k-scheme and $\mathcal{F}$ is an ℓ-adic sheaf on X (Construction 3.3.4.1). Moreover, we showed that the ∞-category $\mathrm{Shv}_\ell^\star$ admits a symmetric monoidal structure (Proposition 3.3.4.4) which simultaneously encodes the symmetric monoidal structure on each of the ∞-categories $\mathrm{Shv}_\ell(X)$ (given by the usual tensor product of ℓ-adic sheaves); see Proposition 3.3.4.11.

Our goal in this section is to construct a closely related ∞-category, which we will denote by $\mathrm{Shv}_\ell^!$. The objects of $\mathrm{Shv}_\ell^!$ are the same as the objects of $\mathrm{Shv}_\ell^\star$: they are pairs $(X, \mathcal{F})$, where X is a quasi-projective k-scheme and $\mathcal{F}$ is an ℓ-adic sheaf on X. However, the morphisms are different: roughly speaking, the datum of a morphism from $(X, \mathcal{F})$

to $(Y, \mathcal{G})$ in $\mathrm{Shv}_\ell^!$ is a *proper* map of k-schemes $f : Y \to X$, together with a map of ℓ-adic sheaves $f^! \mathcal{F} \to \mathcal{G}$. We will show that the ∞-category $\mathrm{Shv}_\ell^!$ admits a symmetric monoidal structure (Proposition 3.3.5.13), whose underlying tensor product is given by

$$((X, \mathcal{F}), (Y, \mathcal{G})) \mapsto (X \times Y, \mathcal{F} \boxtimes \mathcal{G}).$$

Using Proposition 3.3.4.11, we can regard the ∞-category $\mathrm{Shv}_\ell^!$ as encoding a family of symmetric monoidal ∞-categories $\{\mathrm{Shv}_\ell(X)\}$, depending functorially on a quasi-projective k-scheme X. However, the resulting tensor product on each $\mathrm{Shv}_\ell(X)$ is not the usual tensor product, but the !-tensor product of Construction 3.3.0.2.

We begin with some general categorical observations.

Construction 3.3.5.1 (The Cartesian Dual)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a Cartesian fibration of ∞-categories (see Definition 3.2.6.1). Then F is classified by a functor $\chi_F : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ (see Remark 3.2.6.6), given informally by the construction $\chi_F(D) = \mathcal{C} \times_{\mathcal{D}} \{D\}$. The construction $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{op}}$ determines an equivalence from the ∞-category $\mathcal{C}\mathrm{at}_\infty$ to itself. Consequently, we can also consider the functor $\chi_F^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$, given by the formula $\chi_F^{\mathrm{op}}(D) = (\mathcal{C} \times_{\mathcal{D}} \{D\})^{\mathrm{op}}$. Applying the Grothendieck construction to the functor χ_F^{op}, we obtain a new Cartesian fibration of ∞-categories $F' : \mathcal{C}' \to \mathcal{D}$, whose fibers are characterized (up to equivalence) by the formula

$$(\mathcal{C}' \times_{\mathcal{D}} \{D\}) \simeq (\mathcal{C} \times_{\mathcal{D}} \{D\})^{\mathrm{op}}.$$

Remark 3.3.5.2. If $F : \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration, then the Cartesian dual $F' : \mathcal{C}' \to \mathcal{D}$ should be regarded as well-defined only up to equivalence. For explicit constructions, we refer the reader to §[24].IV.3.4.2 and [3].

Remark 3.3.5.3. Cartesian duality is a symmetric relation. If $F : \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration with Cartesian dual $F' : \mathcal{C}' \to \mathcal{D}$, then F is also the Cartesian dual of F'.

Remark 3.3.5.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a Cartesian fibration of ∞-categories and let $F' : \mathcal{C}' \to \mathcal{D}$ be the dual Cartesian fibration. We can describe the ∞-category $\mathcal{C}'$ more informally as follows:

- Objects of $\mathcal{C}'$ are objects of $\mathcal{C}$.
- If C and C' are objects of $\mathcal{C}'$ having images $D, D' \in \mathcal{D}$, then a morphism from C to C' in $\mathcal{C}'$ is given by a diagram

$$C \xleftarrow{f} E \xrightarrow{g} C'$$

in the ∞-category $\mathcal{C}$, where $F(f) = \mathrm{id}_D$ and the morphism g is F-Cartesian.

Variant 3.3.5.5 (The Dual of a coCartesian Fibration)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration of ∞-categories, classified by a functor $\chi_F : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ (see Variant 3.2.6.7). Then we can also consider the functor

$$\chi_F^{\mathrm{op}} : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty \qquad \chi_F^{\mathrm{op}}(D) = (\mathcal{C} \times_{\mathcal{D}} \{D\})^{\mathrm{op}},$$

which classifies a coCartesian fibration of ∞-categories $F'' : \mathcal{C}'' \to \mathcal{D}$. We will refer to F'' as the *coCartesian dual of* F.

Warning 3.3.5.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories which is both a Cartesian fibration and a coCartesian fibration. Then we can consider *both* the coCartesian dual $F' : \mathcal{C}' \to \mathcal{D}$ and the Cartesian dual $F'' : \mathcal{C}'' \to \mathcal{D}$ of F. For each object $D \in \mathcal{D}$, we have a canonical equivalence

$$\mathcal{C}' \times_{\mathcal{D}} \{D\} \simeq (\mathcal{C} \times_{\mathcal{D}} \{D\})^{\mathrm{op}} \simeq \mathcal{C}'' \times_{\mathcal{D}} \{D\}.$$

However, the ∞-categories $\mathcal{C}'$ and $\mathcal{C}''$ are usually *not* equivalent to one another (we will see in a moment that this phenomenon occurs in the case of primary interest to us: see Remark 3.3.5.11).

Remark 3.3.5.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a Cartesian fibration of ∞-categories. Then every morphism $\alpha : D \to D'$ in $\mathcal{D}$ determines a functor $\alpha^* : (\mathcal{C} \times_{\mathcal{D}} \{D'\}) \to (\mathcal{C} \times_{\mathcal{D}} \{D\})$. The functor F is a coCartesian fibration if and only if each of the functors α^* admits a left adjoint.

Similarly, a coCartesian fibration $F : \mathcal{C} \to \mathcal{D}$ is a Cartesian fibration if and only if, for each morphism $\alpha : D \to D'$, the induced functor $(\mathcal{C} \times_{\mathcal{D}} \{D\}) \to (\mathcal{C} \times_{\mathcal{D}} \{D'\})$ admits a right adjoint (which is then given by the functor α^*). See Corollary [25].5.2.2.5.

We now specialize to the case of interest to us.

Lemma 3.3.5.8. *Let $F : \mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ be the coCartesian fibration of ∞-categories appearing in Construction 3.3.4.1. Then:*

(1) *The functor F is also Cartesian fibration. Consequently, F admits a Cartesian dual $F' : \mathcal{C} \to \mathrm{Sch}_k^{\mathrm{op}}$.*

(2) *The functor F' is a coCartesian fibration. Consequently, F' admits a coCartesian dual $F'' : \mathcal{D} \to \mathrm{Sch}_k^{\mathrm{op}}$.*

Proof. By virtue of Remark 3.3.5.7, assertion (1) is equivalent to the statement that for every morphism $f : X \to Y$ of quasi-projective k-schemes, the pullback functor $f^* : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ admits a right adjoint f_*. Note that the fibers of $F' : \mathcal{C} \to \mathrm{Sch}_k^{\mathrm{op}}$

over an object $X \in \mathrm{Sch}_k^{\mathrm{op}}$ can be identified with the ∞-category $\mathrm{Shv}_\ell(X)^{\mathrm{op}}$. Moreover, if $f : X \to Y$ is a morphism of quasi-projective k-schemes, then the induced map

$$\mathrm{Shv}_\ell(X)^{\mathrm{op}} \simeq \mathcal{C} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \{X\} \to \mathcal{C} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \{Y\} \simeq \mathrm{Shv}_\ell(Y)^{\mathrm{op}}$$

is given by (the opposite of) the functor f_*. Since the functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ preserves small colimits, it admits a right adjoint (Corollary [25].5.5.2.9). Consequently, the opposite functor $f_* : \mathrm{Shv}_\ell(X)^{\mathrm{op}} \to \mathrm{Shv}_\ell(Y)^{\mathrm{op}}$ admits a left adjoint. Assertion (2) now follows from Remark 3.3.5.7. □

The coCartesian fibration $F'' : \mathcal{D} \to \mathrm{Sch}_k^{\mathrm{op}}$ is classified by a functor $\chi : \mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ (in the sense of Variant 3.2.6.7), which can be described as follows:

- To each quasi-projective k-scheme X, the functor χ assigns the ∞-category $\mathrm{Shv}_\ell(X)$ of ℓ-adic sheaves on X.
- To each morphism of quasi-projective k-schemes $f : X \to Y$, the functor χ assigns the right adjoint of the direct image functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$.

The right adjoint of the direct image functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$ is usually not a well-behaved construct when the morphism f is not proper. We therefore restrict our attention to the following variant:

Construction 3.3.5.9 (The ∞-Category $\mathrm{Shv}_\ell^!$)**.** Let $\mathrm{Sch}_k^{\mathrm{pr}}$ denote the category whose objects are quasi-projective k-schemes and whose morphisms are proper maps (which we regard as a non-full subcategory of Sch_k). We let $\mathrm{Shv}_\ell^!$ denote the fiber product

$$(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathcal{D},$$

where $\mathcal{D}$ is the ∞-category appearing in conclusion (2) of Lemma 3.3.5.8. By construction, the ∞-category $\mathrm{Shv}_\ell^!$ is equipped with a coCartesian fibration $F : \mathrm{Shv}_\ell^! \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$, which is the coCartesian dual of the Cartesian dual of the projection map

$$(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}.$$

Remark 3.3.5.10. The ∞-category $\mathrm{Shv}_\ell^!$ can be described more informally as follows:

- The objects of $\mathrm{Shv}_\ell^!$ are pairs $(X, \mathcal{F})$, where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$.
- A morphism from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ in the ∞-category $\mathrm{Shv}_\ell^!$ is given by a proper map of k-schemes $f : Y \to X$, together with a map $f^! \mathcal{F} \to \mathcal{G}$ in the ∞-category $\mathrm{Shv}_\ell(Y)$ of ℓ-adic sheaves on Y.

Remark 3.3.5.11. The ∞-categories $(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$ and $\mathrm{Shv}_\ell^!$ illustrate the phenomenon described in Warning 3.3.5.6: note that they can be described respectively as the Cartesian and coCartesian duals of the same (Cartesian and coCartesian) fibration $\mathcal{C} \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$.

Warning 3.3.5.12. By extending the definition of exceptional inverse images to non-proper morphisms $f : X \to Y$, one can construct an enlargement $\overline{\mathrm{Shv}}_\ell^!$ of the ∞-category $\mathrm{Shv}_\ell^!$ which fits into a pullback diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{Shv}_\ell^! & \longrightarrow & \overline{\mathrm{Shv}}_\ell^! \\ \downarrow & & \downarrow{\scriptstyle G} \\ (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} & \longrightarrow & \mathrm{Sch}_k^{\mathrm{op}}, \end{array}$$

where the vertical maps are coCartesian fibrations. Beware, however, that the coCartesian fibration $G : \overline{\mathrm{Shv}}_\ell^! \to \mathrm{Sch}_k^{\mathrm{op}}$ does *not* agree with the coCartesian fibration $F'' : \mathcal{D} \to \mathrm{Sch}_k^{\mathrm{op}}$ appearing in Lemma 3.3.5.8 (because the functors $f^!$ and f_* are generally not adjoint if $f : X \to Y$ is a not a proper morphism of k-schemes).

We can now formulate the main result of this section:

Proposition 3.3.5.13. *The ∞-category $\mathrm{Shv}_\ell^!$ of Construction 3.3.5.9 admits a symmetric monoidal structure, whose underlying tensor product is given by the construction*

$$((X, \mathcal{F}), (Y, \mathcal{G})) \mapsto (X \times Y, \mathcal{F} \boxtimes \mathcal{G}).$$

Moreover, the forgetful functor $\mathrm{Shv}_\ell^! \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ can be regarded as a symmetric monoidal coCartesian fibration; here $(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ is equipped with the symmetric monoidal structure given by the formation of Cartesian products in the category of k-schemes.

Remark 3.3.5.14. As with Proposition 3.3.4.4, we are interested not so much in the statement of Proposition 3.3.5.13 but in its proof, which will supply a *particular* symmetric monoidal structure on the ∞-category $\mathrm{Shv}_\ell^!$ which we will need to work with later.

Before giving the proof of Proposition 3.3.5.13, let us see that it resolves the coherence issues for the !-tensor product of Construction 3.3.0.2.

Proof of Theorem 3.3.0.3 from Proposition 3.3.5.13. Let X be a quasi-projective k-scheme. Then the diagonal map $\delta : X \to X \times X$ exhibits X as a *nonunital* commutative algebra object of the ∞-category $(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ (see Variant 3.1.3.8); beware that this algebra admits a unit only when X is proper over $\mathrm{Spec}(k)$. Applying Variant 3.3.4.9 to the symmetric monoidal coCartesian fibration $F : \mathrm{Shv}_\ell^! \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$, we obtain a *nonunital* symmetric monoidal structure on the ∞-category $F^{-1}\{X\} \simeq \mathrm{Shv}_\ell(X)$. Using the description of the underlying tensor product of the symmetric monoidal structure on $\mathrm{Shv}_\ell^!$ supplied by Proposition 3.3.5.13, we see that the underlying tensor product of the symmetric monoidal on $\mathrm{Shv}_\ell(X)$ is given by the construction

$$(\mathcal{F}, \mathcal{G}) \mapsto \delta^!(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \otimes^! \mathcal{G}.$$

We may therefore regard $\mathrm{Shv}_\ell(X)$ as a *nonunital* commutative algebra object of the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories, whose underlying multiplication is given by the !-tensor product $\otimes^!$. According to Theorem [23].5.4.4.5, we can promote $\mathrm{Shv}_\ell(X)$ to a symmetric monoidal ∞-category (in an essentially unique way) if and only if the multiplication $\otimes^!$ is quasi-unital: that is, if and only if there exists an object $\mathcal{E} \in \mathrm{Shv}_\ell(X)$ for which the functor $\mathcal{F} \mapsto \mathcal{E} \otimes^! \mathcal{F}$ is equivalent to the identity functor from $\mathrm{Shv}_\ell(X)$ to itself. This follows from Proposition 3.3.3.1.

It follows formally that the construction $X \mapsto \mathrm{Shv}_\ell(X)$ determines a functor from the ∞-category $(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ to the ∞-category $\mathrm{Fun}^{\otimes}(\mathrm{Fin}^{\mathrm{s}}, \mathcal{C}\mathrm{at}_\infty)$ of *nonunital* symmetric monoidal ∞-categories (here $\mathrm{Fin}^{\mathrm{s}}$ denotes the category whose objects are finite sets and whose morphisms are surjections; see Variant 3.1.3.8). In particular, for every proper map $f : X \to Y$, the exceptional inverse image functor $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ commutes with the formation of !-tensor products. To complete the proof of Theorem 3.3.0.3, it will suffice to show that each $f^!$ can be promoted to a symmetric monoidal functor. According to Theorem [23].5.4.4.5, this is equivalent to the assertion that the functor $f^!$ carries the dualizing sheaf ω_Y (which is the unit object of $\mathrm{Shv}_\ell(Y)$) to the dualizing sheaf ω_X (which is the unit object of $\mathrm{Shv}_\ell(X)$). To prove this, choose compatible compactifications $\overline{X}$ and $\overline{Y}$, so that we have a pullback diagram of k-schemes

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ Y & \longrightarrow & \overline{Y}, \end{array}$$

where the horizontal maps are open immersions. We then have $\omega_X = \omega_{\overline{X}}|_X$ and $\omega_Y = \omega_{\overline{Y}}|_Y$ (see the proof of Proposition 3.3.3.1). Using Proposition 2.4.4.3, we obtain an equivalence $f^!\omega_Y \simeq (\overline{f}^!\omega_{\overline{Y}})|_X$. We are therefore reduced to proving that $\overline{f}^!\omega_{\overline{Y}}$ is equivalent to $\omega_{\overline{X}}$, which follows immediately from the construction. □

Remark 3.3.5.15 (Functoriality of the Dualizing Sheaf)**.** Regard the construction $X \mapsto \mathrm{Shv}_\ell(X)$ as a functor from the ∞-category $(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ to the ∞-category $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty)$ of symmetric monoidal ∞-categories. In particular, we can regard the unit map

$\{\omega_X\} \to \mathrm{Shv}_\ell(X)$ as a natural transformation of (symmetric monoidal) functors. Applying the Grothendieck construction, we deduce that the projection map $\mathrm{Shv}_\ell^! \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ admits a (symmetric monoidal) section, given on objects by the construction $X \mapsto (X, \omega_X)$.

We now turn to the proof of Proposition 3.3.5.13. We begin by introducing some terminology.

Notation 3.3.5.16 (∞-Categories of Cartesian and coCartesian Fibrations)**.** Let $\mathcal{C}\mathrm{at}_\infty$ denote the ∞-category of ∞-categories (see Construction 3.1.2.1). Let $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ denote the ∞-category of arrows in $\mathcal{C}\mathrm{at}_\infty$: that is, the ∞-category whose objects are functors $F : \mathcal{C} \to \mathcal{D}$. We define (non-full) subcategories

$$\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty) \supseteq \mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$$

as follows:

- Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, regarded as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$. Then F belongs to $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ if and only if F is a Cartesian fibration, and to $\mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ if and only if F is a coCartesian fibration.

- Let $\alpha : F \to F'$ be a morphism in $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$, which we can identify with a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{C}' \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F'} \\ \mathcal{D} & \longrightarrow & \mathcal{D}' . \end{array}$$

  Then the morphism α belongs to $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ if and only if F and F' are Cartesian fibrations, and the functor G carries F-Cartesian morphisms to F'-Cartesian morphisms. The morphism α belongs to $\mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ if and only if F and F' are coCartesian fibrations, and the functor G carries F-coCartesian morphisms to F'-coCartesian morphisms.

Remark 3.3.5.17. Each of the ∞-categories

$$\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty) \supseteq \mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$$

admits finite products (which are computed pointwise). Consequently, we can contemplate commutative monoid objects (in the sense of Definition 3.1.1.5) in $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$, $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$, and $\mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$. Unwinding the definitions, we see that:

- Commutative monoid objects of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ can be identified with symmetric monoidal functors $F : \mathcal{C} \to \mathcal{D}$.
- Commutative monoid objects of the ∞-category $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ can be identified with symmetric monoidal Cartesian fibrations $F : \mathcal{C} \to \mathcal{D}$ (in the sense of Definition 3.3.4.6).
- Commutative monoid objects of the ∞-category $\mathrm{Fun}^{\mathrm{coCart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ can be identified with symmetric monoidal coCartesian fibrations $F : \mathcal{C} \to \mathcal{D}$ (in the sense of Definition 3.3.4.6).

Remark 3.3.5.18. The formation of Cartesian duals determines a functor from the ∞-category $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ to itself. This functor is homotopy inverse to itself (Remark 3.3.5.3), and is therefore an equivalence of ∞-categories. In particular, it commutes with finite products, and therefore carries commutative monoid objects of $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$ to commutative monoid objects of $\mathrm{Fun}^{\mathrm{Cart}}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$. Combining this observation with Remark 3.3.5.17, we deduce that if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal Cartesian fibration, then the Cartesian dual $F' : \mathcal{C}' \to \mathcal{D}$ inherits the structure of a symmetric monoidal Cartesian fibration (in particular, the ∞-category $\mathcal{C}'$ inherits a symmetric monoidal structure).

Similarly, if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal coCartesian fibration, then its coCartesian dual $F'' : \mathcal{C}'' \to \mathcal{D}$ inherits the structure of a symmetric monoidal coCartesian fibration.

Proof of Proposition 3.3.5.13. The proof of Proposition 3.3.4.4 shows that we can regard the forgetful functor $F : \mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ as a symmetric monoidal coCartesian fibration. Note that the functor F is also a Cartesian fibration (see Lemma 3.3.5.8). We claim that it is a symmetric monoidal Cartesian fibration: that is, the collection of F-Cartesian morphisms is stable under tensor product. This is a reformulation of Proposition 3.3.1.3.

Let $F' : \mathcal{C} \to \mathrm{Sch}_k^{\mathrm{op}}$ be the Cartesian dual of F, as in the proof of Lemma 3.3.5.8. Applying Remark 3.3.5.18, we see that F' inherits the structure of a symmetric monoidal Cartesian fibration. Moreover, Lemma 3.3.5.8 implies that F' is also a coCartesian fibration. Beware that F' is *not* a symmetric monoidal coCartesian fibration (that is, the collection of F'-coCartesian morphisms is not closed under tensor products). However, we can remedy the situation by restricting our attention to proper morphisms of k-schemes. Let $\mathcal{C}_0$ denote the fiber product $(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathcal{C}$ and let $F'_0 : \mathcal{C}_0 \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ be the projection map. Then F'_0 is a coCartesian fibration (Lemma 3.3.5.8), whose coCartesian dual is the forgetful functor $\mathrm{Shv}_\ell^! \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$. By virtue of Remark 3.3.5.18, it will suffice to show that F'_0 is a *symmetric monoidal* coCartesian fibration: that is, that the collection of F'_0-coCartesian morphisms is closed under tensor products in $\mathcal{C}'_0$. This is a reformulation of Corollary 3.3.2.4. □

Remark 3.3.5.19. The proof of Proposition 3.3.5.13 supplies a specific symmetric monoidal structure on the ∞-category $\mathrm{Shv}^!_\ell$ (well-defined up to equivalence), which in turn supplies a specific symmetric monoidal structure on the ∞-category $\mathrm{Shv}_\ell(X)$ for every quasi-projective k-scheme X. This symmetric monoidal structure determines an associativity constraint on the !-tensor product $\otimes^!$: that is, a collection of equivalences

$$\mathcal{F} \otimes^! (\mathcal{G} \otimes^! \mathcal{H}) \simeq (\mathcal{F} \otimes^! \mathcal{G}) \otimes^! \mathcal{H}$$

depending functorially on $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Shv}_\ell(X)$. We leave it to the reader to verify that these equivalences agree with those constructed more explicitly in §3.3.2.

3.4 The Cohomology Sheaf of a Morphism

Throughout this section, we fix an algebraically closed field k and a prime number ℓ which is invertible in k. In §3.2.5, we defined the ℓ-adic cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ of an algebraic stack $\mathcal{Y}$ (assumed for simplicity to be locally of finite type over k) and saw that it admits the structure of an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$. Our goal in this section is to describe a relative version of this construction, which can be applied to an algebraic stack $\mathcal{Y}$ equipped with a map $\pi : \mathcal{Y} \rightarrow X$ where X is a quasi-projective k-scheme. In this case, the fiber $\mathcal{Y}_x = \mathcal{Y} \times_X \{x\}$ of π over every closed point $x \in X$ can be regarded as an algebraic stack over k, so we can contemplate the ℓ-adic cochain complex $C^*(\mathcal{Y}_x; \mathbf{Z}_\ell)$. Our goal is to address the following:

Question 3.4.0.1. Let X be a quasi-projective k-scheme and let $\pi : \mathcal{Y} \rightarrow X$ be a morphism of algebraic stacks which is locally of finite type. What can one say about the collection of $\mathbb{E}_\infty$-algebras $\{C^*(\mathcal{Y}_x; \mathbf{Z}_\ell)\}_{x \in X}$?

Example 3.4.0.2. Let $\pi : Y \rightarrow X$ be a proper morphism of quasi-projective k-schemes. Proposition 3.2.3.1 implies that the functor $\pi_* : \mathrm{Shv}_\ell(Y) \rightarrow \mathrm{Shv}_\ell(X)$ carries commutative algebras (with respect to the standard tensor product on the ∞-category $\mathrm{Shv}_\ell(Y)$) to commutative algebras (with respect to the standard tensor product on $\mathrm{Shv}_\ell(X)$). In particular, $\mathcal{A} = \pi_* \underline{\mathbf{Z}_\ell}_Y$ can be regarded as a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$. The proper base change theorem (Theorem 2.4.2.1) implies that the stalk of $\mathcal{A}$ at a closed point $x \in X$ can be identified with the ℓ-adic cochain complex $C^*(Y_x; \mathbf{Z}_\ell)$. Moreover, these identifications are compatible with commutative algebra structures. We can therefore answer Question 3.4.0.1 (in this special case) as follows: the collection of $\mathbb{E}_\infty$-algebras $\{C^*(\mathcal{Y}_x; \mathbf{Z}_\ell)\}_{x \in X}$ can be identified with the stalks of an ℓ-adic sheaf $\mathcal{A}$, which is a commutative algebra object of the ∞-category $\mathrm{Shv}_\ell(X)$ (with respect to the usual tensor product).

For applications to Weil's conjecture, we would like to address Question 3.4.0.1 in the case where X is an algebraic curve and $\mathcal{Y} = \mathrm{BG}$ is the classifying stack of a smooth affine group scheme over X. In this case, the map $\pi : \mathrm{BG} \to X$ is not proper, so the cochain complexes $\{C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)\}_{x \in X}$ need not arise as the stalks of an ℓ-adic sheaf on X (we can still define $\mathcal{A} \in \mathrm{Shv}_\ell(X)$ as in Example 3.4.0.2, but the comparison maps $x^* \mathcal{A} \to C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)$ are generally not equivalences unless G is reductive in a neighborhood of x). However, all is not lost: using the smoothness of the map $\pi : \mathrm{BG} \to X$, one can show that the $\mathbb{E}_\infty$-algebras $\{C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)\}_{x \in X}$ arise as the *costalks* of an ℓ-adic sheaf $[\mathrm{BG}]_X$, which is a commutative algebra with respect to the !-tensor product of §3.3. This is a special case of the following:

Theorem 3.4.0.3. *Let X be a quasi-projective k-scheme and let $\pi : \mathcal{Y} \to X$ be a smooth morphism of algebraic stacks. Then there exists an ℓ-adic sheaf $[\mathcal{Y}]_X$ which is a commutative algebra with respect to the !-tensor product on $\mathrm{Shv}_\ell(X)$ with the following property: for each point $x \in X$, the costalk $x^![\mathcal{Y}]_X$ can be identified with the ℓ-adic cochain complex $C^*(\mathcal{Y}_x; \mathbf{Z}_\ell)$ (as an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$). Moreover, the construction $\mathcal{Y} \mapsto [\mathcal{Y}]_X$ is (contravariantly) functorial in $\mathcal{Y}$.*

As with the main results of §3.3, we are interested not in the statement of Theorem 3.4.0.3 so much as in its proof, which provides an explicit construction of the sheaf $[\mathcal{Y}]_X$ (this sheaf will play a central role in Chapter 4). We begin in §3.4.1 by defining the sheaf $[\mathcal{Y}]_X$: roughly speaking, it can be described as the direct image $\pi_* \pi^* \omega_X$, where ω_X is the dualizing sheaf of X (see Construction 3.4.1.2 for a description which avoids the language of ℓ-adic sheaves on algebraic stacks). The bulk of this section is devoted to studying the naturality properties of the construction $\mathcal{Y} \mapsto [X]_{\mathcal{Y}}$. The two main properties we need can be stated as follows:

(a) If $\pi : \mathcal{Y} \to X$ is smooth, then the formation of the sheaf $[\mathcal{Y}]_X$ is compatible with the formation of exceptional inverse images. More precisely, for every pullback diagram

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

where f is proper, we have a canonical equivalence $[\mathcal{Y}']_{X'} \simeq f^![\mathcal{Y}]_X$ of ℓ-adic sheaves on X'. We will prove this result in §3.4.3 (see Proposition 3.4.3.2).

(b) Let X and X' be quasi-projective k-schemes, and suppose we are given maps of algebraic stacks $\mathcal{Y} \to X$ and $\mathcal{Y}' \to X'$ which are of finite type. Then the sheaf $[\mathcal{Y} \times \mathcal{Y}']_{X \times X'}$ can be identified with the external tensor product $[\mathcal{Y}]_X \boxtimes [\mathcal{Y}']_{X'}$ of Construction 3.3.0.1 (see Theorem 3.4.5.1; here all products are formed over the base scheme $\mathrm{Spec}(k)$).

Assuming (a) and (b), we can informally sketch the proof of Theorem 3.4.0.3. Assume for simplicity that the map $\pi : \mathcal{Y} \to X$ is smooth and that $\mathcal{Y}$ is quasi-compact (we can reduce to this case by exhausting $\mathcal{Y}$ by quasi-compact open substacks). Let $\delta : X \to X \times X$ be the diagonal map. Combining (a) and (b), we obtain an equivalence of ℓ-adic sheaves

$$\begin{aligned} [\mathcal{Y} \times_X \mathcal{Y}]_X &\simeq \delta^! [\mathcal{Y} \times \mathcal{Y}]_{X \times X} \\ &\simeq \delta^! ([\mathcal{Y}]_X \boxtimes [\mathcal{Y}]_X) \\ &\simeq [\mathcal{Y}]_X \otimes^! [\mathcal{Y}]_X. \end{aligned}$$

The relative diagonal $\mathcal{Y} \to \mathcal{Y} \times_X \mathcal{Y}$ then induces a multiplication map

$$m : [\mathcal{Y}]_X \otimes^! [\mathcal{Y}]_X \simeq [\mathcal{Y} \times_X \mathcal{Y}]_X \to [\mathcal{Y}]_X.$$

The description of the costalks of $[\mathcal{Y}]_X$ follows from (a), applied to the inclusion maps $\{x\} \hookrightarrow X$.

To turn the preceding sketch into a rigorous proof, we need to show that the multiplication m is commutative and associative up to coherent homotopy, depends functorially on $\mathcal{Y}$, and is compatible with the formation of exceptional inverse images. This will require us to formulate and prove more elaborate versions of (a) and (b), which articulate homotopy-coherent aspects of the construction $\mathcal{Y} \mapsto [\mathcal{Y}]_X$. We carry this out in §3.4.4 and 3.4.6, respectively (following some groundwork that we lay out in §3.4.2). We then combine these analyses in §3.4.7 to prove Theorem 3.4.0.3.

3.4.1 The Sheaf $[\mathcal{Y}]_{\mathcal{F}}$

Let X be a quasi-projective k-scheme and let ω_X denote the dualizing sheaf of X (see Remark 3.3.3.7). Given any morphism $f : Y \to X$ of quasi-projective k-schemes, we let $[Y]_X \in \mathrm{Shv}_\ell(X)$ denote the ℓ-adic sheaf given by $f_* f^* \omega_X$. We will refer to the ℓ-adic sheaf $[Y]_X$ as the *cohomology sheaf of the morphism f*.

Remark 3.4.1.1. We will primarily be interested in the construction $Y \mapsto [Y]_X$ in the special case where Y is *smooth* over X. In this case, for any proper morphism of quasi-projective k-schemes $g : X' \to X$, Variant 2.4.3.1 and Proposition 2.4.4.3 supply an equivalence of ℓ-adic sheaves

$$[Y \times_X X']_{X'} \simeq g^! [Y]_X.$$

Taking $X' = \mathrm{Spec}(k)$, we obtain the following informal description of $[Y]_X$: it is the ℓ-adic sheaf whose costalk at a point $x \in X(k)$ can be identified with the cochain complex $C^*(Y_x; \mathbf{Z}_\ell)$, where Y_x denotes the fiber $Y \times_X \{x\}$.

Our goal in this section is to generalize the construction $Y \mapsto [Y]_X$ to the case where Y is an algebraic stack. For later use, it will be convenient to consider a further generalization which depends on a choice of ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$.

Construction 3.4.1.2. Let X be a quasi-projective k-scheme, let $\mathcal{Y}$ be an algebraic stack which is locally of finite type over k, and let $\pi : \mathcal{Y} \to X$ be a morphism. Let $\mathrm{Pt}(\mathcal{Y})$ denote the category of points of $\mathcal{Y}$ (see Notation 3.2.4.8), whose objects are quasi-projective k-schemes Y equipped with a map $f : Y \to \mathcal{Y}$. For each object $(Y, f) \in \mathrm{Pt}(\mathcal{Y})$, the map π determines a morphism of quasi-projective k-schemes $(\pi \circ f) : Y \to X$. For each ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we let $[\mathcal{Y}]_{\mathcal{F}}$ denote the inverse limit

$$\varprojlim_{(Y,f)\in \mathrm{Pt}(\mathcal{Y})} (\pi \circ f)_*(\pi \circ f)^* \mathcal{F} \in \mathrm{Shv}_\ell(X).$$

Example 3.4.1.3. Let $\pi : Y \to X$ be a morphism of quasi-projective k-schemes. By abuse of terminology, we can also regard Y as an algebraic stack over k. In this case, the category $\mathrm{Pt}(Y)$ has a final object (given by the identity map $\mathrm{id} : Y \to Y$), so we can identify $[Y]_{\mathcal{F}}$ with the pushforward $\pi_* \pi^* \mathcal{F}$. In particular, if $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{\leq n}$ for some integer n, then $[Y]_{\mathcal{F}} \in \mathrm{Shv}_\ell(X)_{\leq n}$.

Notation 3.4.1.4. Let $\pi : \mathcal{Y} \to X$ be as in Construction 3.4.1.2. We let $[\mathcal{Y}]_X$ denote the sheaf $[\mathcal{Y}]_{\omega_X} \in \mathrm{Shv}_\ell(X)$, where ω_X is the dualizing sheaf of X.

Example 3.4.1.5. Let $\mathcal{Y}$ be an algebraic stack which is locally of finite type over k. Then we have a canonical equivalence $[\mathcal{Y}]_{\mathrm{Spec}(k)} \simeq C^*(\mathcal{Y}; \mathbf{Z}_\ell)$. Here we abuse notation by identifying the ∞-category $\mathrm{Shv}_\ell(\mathrm{Spec}(k))$ with $\mathrm{Mod}_{\mathbf{Z}_\ell}$ and the ℓ-adic cochain complex $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ is defined in §3.2.

Example 3.4.1.6. Let X be a quasi-projective k-scheme. For every ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, we have a canonical equivalence $[X]_{\mathcal{F}} \simeq \mathcal{F}$. In particular, the cohomology sheaf $[X]_X$ is the dualizing sheaf ω_X.

Remark 3.4.1.7. Let $\pi : \mathcal{Y} \to X$ be as in Construction 3.4.1.2. Then the ℓ-adic sheaf $[\mathcal{Y}]_X$ is ℓ-complete. When $\mathcal{Y}$ is a quasi-projective k-scheme this follows from Remark 2.3.4.12 (since $[\mathcal{Y}]_X$ is constructible), and the general case follows from the observation that the collection of ℓ-complete objects of $\mathrm{Shv}_\ell(X)$ is closed under limits.

Remark 3.4.1.8 (Functoriality in $\mathcal{Y}$)**.** Let X be a quasi-projective k-scheme, and suppose we are given a commutative diagram of algebraic stacks

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Y}' \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

For every ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the morphism f induces a pullback map $f^* : [\mathcal{Y}']_{\mathcal{F}} \to [\mathcal{Y}]_{\mathcal{F}}$. We can summarize the situation informally by saying that the ℓ-adic sheaf $[\mathcal{Y}]_{\mathcal{F}}$ depends functorially on $\mathcal{Y}$. We will discuss this point in more detail in §3.4.2 and 3.4.4.

Remark 3.4.1.9. Let X be a quasi-projective k-scheme and let $\mathcal{F} \in \mathrm{Shv}_\ell(X)$. Let $\mathcal{Y}$ be an algebraic stack which is locally of finite type over k, equipped with a map $\pi : \mathcal{Y} \to X$. Suppose that $\mathcal{Y}$ can be realized as a filtered union of open substacks $\{\mathcal{U}_\alpha\}$. Then the tautological map $[\mathcal{Y}]_{\mathcal{F}} \to \varprojlim_\alpha [\mathcal{U}_\alpha]_{\mathcal{F}}$ is an equivalence in $\mathrm{Shv}_\ell(X)$.

In practice, one does not need to use the *entire* fibered category $\mathcal{Y}$ to compute the cohomology sheaf $[\mathcal{Y}]_{\mathcal{F}}$ of Construction 3.4.1.2.

Proposition 3.4.1.10. *Let X be a quasi-projective k-scheme and let $\mathcal{Y}$ be an algebraic stack which is locally of finite type over k equipped with a map $\mathcal{Y} \to X$. Let U_0 be a quasi-projective k-scheme equipped with a map $\rho : U_0 \to \mathcal{Y}$, and let $U_\bullet$ be the simplicial scheme given by the iterated fiber powers of U_0 over $\mathcal{Y}$. Suppose that ρ is locally surjective with respect to the étale topology. Then, for every object $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the canonical map*

$$[\mathcal{Y}]_{\mathcal{F}} \simeq \mathrm{Tot}[U_\bullet]_{\mathcal{F}} = \varprojlim_{[n] \in \mathbf{\Delta}} [U_n]_{\mathcal{F}}$$

is an equivalence of ℓ-adic sheaves on X.

Proof. Assume for simplicity that the diagonal map $\mathcal{Y} \to \mathcal{Y} \times_X \mathcal{Y}$ is affine (this assumption is not needed, but is satisfied in all cases of interest to us). Let $\mathrm{Pt}(\mathcal{Y})$ be the category of points of $\mathcal{Y}$, so that the simplicial scheme $U_\bullet$ determines a functor $\rho : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Pt}(\mathcal{Y})$. Let $\mathrm{Pt}_0(\mathcal{Y})$ denote the full subcategory of $\mathrm{Pt}(\mathcal{Y})$ spanned by those maps $f : Y \to \mathcal{Y}$ which factor through ρ. Note for each object $(Y, f) \in \mathrm{Pt}(\mathcal{Y})$, the fiber product $\mathbf{\Delta}^{\mathrm{op}} \times_{\mathrm{Pt}(\mathcal{Y})} \mathrm{Pt}(\mathcal{Y})_{(Y,f)/}$ is empty if $(Y, f) \notin \mathrm{Pt}_0(\mathcal{Y})$, and weakly contractible otherwise. It follows that ρ induces a left cofinal map $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Pt}_0(\mathcal{Y})$, hence an equivalence $\varprojlim_{(Y,f) \in \mathrm{Pt}_0(\mathcal{Y})} (\pi \circ f)_* (\pi \circ f)^* \mathcal{F} = \mathrm{Tot}[U_\bullet]_{\mathcal{F}}$. To complete the proof, it suffices to observe that the functor

$$\mathrm{Pt}(\mathcal{Y})^{\mathrm{op}} \to \mathrm{Shv}_\ell(X) \qquad (Y, f) \mapsto (\pi \circ f)_* (\pi \circ f)^* \mathcal{F}$$

is a right Kan extension of its restriction to $\mathrm{Pt}_0(\mathcal{Y})^{\mathrm{op}}$. This follows from our assumption that ρ is locally surjective in the étale topology. □

3.4.2 Functoriality

Let X be a quasi-projective k-scheme. The ℓ-adic sheaf $[\mathcal{Y}]_{\mathcal{F}} \in \mathrm{Shv}_\ell(X)$ of Construction 3.4.1.2 can be regarded as a *contravariant* functor of the algebraic stack $\mathcal{Y}$, and a *covariant* functor of the ℓ-adic sheaf $\mathcal{F}$. Our first goal in this section is to articulate the dependence of $[\mathcal{Y}]_{\mathcal{F}}$ on the triple $(X, \mathcal{Y}, \mathcal{F})$ more precisely (Proposition 3.4.2.2). First, we need some notation.

Notation 3.4.2.1 (The 2-Category RelStk)**.** We define a 2-category RelStk as follows:

- An object of RelStk consists of a quasi-projective k-scheme X together with a morphism of algebraic stacks $\pi : \mathcal{Y} \to X$ which is locally of finite type.
- A morphism from $\pi : \mathcal{Y} \to X$ to $\pi' : \mathcal{Y}' \to X'$ in the category RelStk is a commutative diagram of algebraic stacks over k

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\phi} & \mathcal{Y}' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow{f} & X'. \end{array}$$

  We regard the collection of morphisms from π to π' as a category, where the set $\mathrm{Hom}((\phi, f), (\phi', f'))$ is empty unless $f = f'$, in which case it is the set of all isomorphisms of ϕ with ϕ' which are compatible with π.

In what follows, we will abuse notation by identifying RelStk with its associated ∞-category (given by the nerve construction of Example 2.1.3.11). Note that the construction $(\pi : \mathcal{Y} \to X) \mapsto X$ determines a forgetful functor $\mathrm{RelStk} \to \mathrm{Sch}_k$ which is both a Cartesian and coCartesian fibration. For every quasi-projective k-scheme X, we let RelStk_X denote the fiber $\mathrm{RelStk} \times_{\mathrm{Sch}_k} \{X\}$: that is, the 2-category of algebraic stacks which are locally of finite type over X.

Let $\mathrm{Shv}_\ell^\star$ be the ∞-category appearing in Construction 3.3.4.1. In what follows, we will study the fiber product $\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$. This fiber product can be described more informally as follows:

- The objects of $\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$ are triples $(X, \mathcal{Y}, \mathcal{F})$ where X is a quasi-projective k-scheme, $\mathcal{Y}$ is an algebraic stack equipped with a morphism $\pi : \mathcal{Y} \to X$ which is locally of finite type, and $\mathcal{F}$ is an ℓ-adic sheaf on X.
- A morphism from $(X, \mathcal{Y}, \mathcal{F})$ to $(X', \mathcal{Y}', \mathcal{F}')$ in $\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$ is a commutative diagram of algebraic stacks

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\phi} & \mathcal{Y} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X, \end{array}$$

  together with a morphism of ℓ-adic sheaves $f^* \mathcal{F} \to \mathcal{F}'$.

Proposition 3.4.2.2. *There exists a functor of ∞-categories*

$$\Phi^\star : \mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star \to \mathrm{Shv}_\ell^\star$$

with the following properties:

(1) *The diagram of ∞-categories*

$$\begin{array}{ccc} \mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star & \xrightarrow{\Phi^\star} & \mathrm{Shv}_\ell^\star \\ & \searrow \quad \swarrow & \\ & \mathrm{Sch}_k^{\mathrm{op}} & \end{array}$$

commutes. In particular, for every quasi-projective k-scheme X, $\Phi^\star$ restricts to a functor $\Phi^\star_X : \mathrm{RelStk}_X^{\mathrm{op}} \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$.

(2) *For every quasi-projective k-scheme X, the functor $\Phi^\star_X$ is given on objects by the formula $\Phi^\star_X(\mathcal{Y}, \mathcal{F}) = [\mathcal{Y}]_{\mathcal{F}}$.*

(3) *Let $\mathcal{C}$ be the full subcategory of $\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$ spanned by those triples $(X, \mathcal{Y}, \mathcal{F})$ for which the projection map $\mathcal{Y} \to X$ is an isomorphism. Then $\Phi^\star|_{\mathcal{C}} : \mathcal{C} \to \mathrm{Shv}_\ell^\star$ is given by the projection onto the second factor.*

(4) *Let $q : \mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ denote the forgetful functor. Then $\Phi^\star|_{\mathcal{C}}$ is a q-right Kan extension of its restriction to $\mathcal{C}$ (see §[25].4.3.2 for a discussion of relative Kan extensions).*

Remark 3.4.2.3. It follows from Proposition [25].4.3.2.15 that the functor $\Phi^\star$ is determined (up to a contractible space of choices) by properties (1), (3) and (4). A reader who finds Construction 3.4.1.2 too informal can take property (2) as the *definition* of the relative cohomology sheaves $[\mathcal{Y}]_{\mathcal{F}}$.

Proof of Proposition 3.4.2.2. We first show that there exists a functor

$$\Phi^\star : \mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star \to \mathrm{Shv}_\ell^\star$$

satisfying conditions (1), (3), and (4) by verifying the criterion of Proposition [25].4.3.2.15. Let $C = (X, \mathcal{Y}, \mathcal{F})$ be an object of the fiber product $\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$ and define

$$\mathcal{C}_{C/} = \mathcal{C} \times_{\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star} (\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star)_{C/}.$$

Unwinding the definitions, we can identify objects of $\mathcal{C}_{C/}$ with pairs $(f : Y \to \mathcal{Y}, \alpha)$ where Y is a quasi-projective k-scheme, $f : Y \to \mathcal{Y}$ is a morphism of algebraic stacks over k, and $\alpha : (\pi \circ f)^* \mathcal{F} \to \mathcal{G}$ is a morphism of ℓ-adic sheaves on Y. The construction $(f : Y \to \mathcal{Y}, \alpha) \mapsto (\pi \circ f)_* \mathcal{G}$ determines a functor $F : \mathcal{C}_{C/} \to \mathrm{Shv}_\ell(X)$. Using the criteria of Propositions [25].4.3.1.10 and [25].4.3.1.9, we are reduced to showing that F admits a limit in the ∞-category $\mathrm{Shv}_\ell(X)$ (in which case the functor $\Phi^\star$ is given by $\Phi(^\star C) = (X, \varprojlim(F)))$. To see this, let $\mathcal{C}^0_{C/}$ be the full subcategory of $\mathcal{C}_{C/}$ spanned by those pairs $(f : Y \to \mathcal{F}, \alpha)$ where α is an equivalence. Then the inclusion functor $\mathcal{C}^0_{C/} \hookrightarrow \mathcal{C}$ is right cofinal (since it admits a right adjoint). We conclude by observing that the inverse limit $\varprojlim(F|_{\mathcal{C}^0_{C/}})$ coincides with $[\mathcal{Y}]_{\mathcal{F}}$ by construction. □

3.4.3 Compatibility with Exceptional Inverse Images

Let X be a quasi-projective k-scheme and let $\pi : \mathcal{Y} \to X$ be a morphism of algebraic stacks which is locally of finite type. Our goal in this section is to show that, if π is smooth, then the costalks of the cohomology sheaf $[\mathcal{Y}]_X$ can be identified with the ℓ-adic cochain complexes $C^*(\mathcal{Y}_x; \mathbf{Z}_\ell)$ (Corollary 3.4.3.4). We will deduce this from a more general assertion about the compatibility of Construction 3.4.1.2 with exceptional inverse images (Proposition 3.4.3.2).

We begin by considering a special case of the functoriality articulated by Proposition 3.4.2.2. Let $f : X' \to X$ be a morphism of quasi-projective k-schemes, let $\pi : \mathcal{Y} \to X$ be a morphism of algebraic stacks which is locally of finite type, and let $\mathcal{F}$ be an ℓ-adic sheaf on X'. We then have a canonical map

$$(X, \mathcal{Y}, f_* \mathcal{F}) \to (X', X' \times_X \mathcal{Y}, \mathcal{F})$$

in the ∞-category $\mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star$. Applying the functor $\Phi^\star$ of Proposition 3.4.2.2, we obtain a map $(X, [\mathcal{Y}]_{f_* \mathcal{F}}) \to (X', [X' \times_X \mathcal{Y}]_{\mathcal{F}})$ in the ∞-category $\mathrm{Shv}_\ell^\star$, which we can identify with a map $[\mathcal{Y}]_{f_* \mathcal{F}} \to f_*[X' \times_X \mathcal{Y}]_{\mathcal{F}}$ in the ∞-category $\mathrm{Shv}_\ell(X)$.

Proposition 3.4.3.1 (Compatibility with Direct Images)**.** *Let $f : X' \to X$ be a morphism of quasi-projective k-schemes, let $\pi : \mathcal{Y} \to X$ be a morphism of algebraic stacks which is locally of finite type, and let $\mathcal{F} \in \mathrm{Shv}_\ell(X')$. Then the comparison map $[\mathcal{Y}]_{f_* \mathcal{F}} \to f_*[X' \times_X \mathcal{Y}]_{\mathcal{F}}$ is an equivalence if either f is proper or π is smooth.*

Proof. Using Remark 3.4.1.9, we can reduce to the case where $\mathcal{Y}$ is quasi-compact, so that there exists a smooth surjection $U_0 \to \mathcal{Y}$ for some quasi-projective k-scheme U_0. Let $U_\bullet$ be the simplicial scheme given by the iterated fiber powers of U_0 over $\mathcal{Y}$. Using Proposition 3.4.1.10, are reduced to proving that the conclusion of Proposition 3.4.3.1 holds for each of the projection maps $U_n \to X$. We may therefore assume without loss of generality that $\mathcal{Y}$ is a k-scheme. Repeating the above arguments, we can assume that $\mathcal{Y}$ is a quasi-compact k-scheme and therefore choose a surjection $U_0 \to \mathcal{Y}$ as above, so that each U_n admits a monomorphism $U_n \to (U_0)^{n+1}$ and is therefore quasi-projective.

Replacing $\mathcal{Y}$ by U_n once more, we are reduced to the special case where $\mathcal{Y}$ is a quasi-projective k-scheme. In this case, the desired result follows from the smooth and proper base change theorems (Theorem 2.4.2.1). □

Let $\pi : \mathcal{Y} \to X$ be as in Proposition 3.4.3.1, let $f : X' \to X$ be a *proper* morphism of quasi-projective k-schemes, and let $\mathcal{F} \in \mathrm{Shv}_\ell(X)$ be an ℓ-adic sheaf on X. Applying Proposition 3.4.3.1 to the exceptional inverse image $f^! \mathcal{F} \in \mathrm{Shv}_\ell(X')$, we obtain an equivalence

$$f_*([X' \times_X \mathcal{Y}]_{f^! \mathcal{F}}) \simeq [\mathcal{Y}]_{f_* f^! \mathcal{F}}.$$

Composing with the counit map $[\mathcal{Y}]_{f_* f^! \mathcal{F}} \to [\mathcal{Y}]_{\mathcal{F}}$ and using the adjunction between f_* and $f^!$, we obtain a comparison map $[X' \times_X \mathcal{Y}]_{f^! \mathcal{F}} \to f^!([\mathcal{Y}]_{\mathcal{F}})$.

Proposition 3.4.3.2. *[Compatibility with Exceptional Inverse Images] Let $f : X' \to X$ be a proper morphism of quasi-projective k-schemes and let $\pi : \mathcal{Y} \to X$ be a smooth morphism of algebraic stacks. Then, for every ℓ-adic sheaf $\mathcal{F} \in \mathrm{Shv}_\ell(X)$, the construction above yields an equivalence $[X' \times_X \mathcal{Y}]_{f^! \mathcal{F}} \to f^!([\mathcal{Y}]_{\mathcal{F}})$ of ℓ-adic sheaves on X'. In particular (taking $\mathcal{F} = \omega_X$), we obtain a canonical equivalence $[\mathcal{Y}']_{X'} \simeq f^!([\mathcal{Y}]_X)$.*

Warning 3.4.3.3. The conclusion of Proposition 3.4.3.2 does not necessarily hold if we drop the assumption that π is smooth.

Proof of Proposition 3.4.3.2. Arguing as in the proof of Proposition 3.4.3.1, we can use Remark 3.4.1.9 and Proposition 3.4.1.10 to reduce to the case where $\mathcal{Y}$ is a quasi-projective k-scheme. In this case, the desired result follows from Proposition 2.4.4.3. □

Corollary 3.4.3.4. *Let X be a quasi-projective k-scheme and let $\pi : \mathcal{Y} \to X$ be a smooth morphism of algebraic stacks. Then, for each closed point $x \in X$, the costalk $x^![\mathcal{Y}]_X$ can be identified with the ℓ-adic cochain complex $C^*(\mathcal{Y}_x; \mathbf{Z}_\ell)$.*

Proof. Combine Proposition 3.4.3.2 with Example 3.4.1.5. □

3.4.4 Functoriality Revisited

In §3.4.2, we showed that the construction $(X, \mathcal{Y}, \mathcal{F}) \mapsto [\mathcal{Y}]_{\mathcal{F}}$ of §3.4.1 determines a functor of ∞-categories

$$\Phi^\star : \mathrm{RelStk}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star \to \mathrm{Shv}_\ell^\star$$

(see Proposition 3.4.2.2). The functor $\Phi^\star$ *directly* encodes the data of the comparison maps $[\mathcal{Y}]_{f_* \mathcal{F}} \to f_*[X' \times_X \mathcal{Y}]_{\mathcal{F}}$ of Proposition 3.4.3.1, and therefore *indirectly* encodes the data of the comparison maps $[X' \times_X \mathcal{Y}]_{f^! \mathcal{F}} \to f^![\mathcal{Y}]_{\mathcal{F}}$ of Proposition 3.4.3.2. For our purposes, we are primarily interested in the latter comparison. Our goal in this section is to construct another functor $\Phi^!$ which encodes (most of) the same information as the functor $\Phi^\star$, but in a format which is better suited to our applications.

Notation 3.4.4.1 (The 2-Category $\mathrm{RelStk}^{\mathrm{sm}}$)**.** Let RelStk be the ∞-category introduced in Notation 3.4.2.1, whose objects are given by maps $\pi : \mathcal{Y} \to X$ where X is a quasi-projective k-scheme and $\pi : \mathcal{Y} \to X$ is a morphism of algebraic stacks which is locally of finite type. We let $\mathrm{RelStk}^{\mathrm{sm}}$ denote the full subcategory of RelStk spanned by those objects $\pi : \mathcal{Y} \to X$ for which the morphism π is smooth.

Note that the construction $(\pi : \mathcal{Y} \to X) \mapsto X$ determines a forgetful functor $\mathrm{RelStk}^{\mathrm{sm}} \to \mathrm{Sch}_k$ which is a Cartesian fibration. For every quasi-projective k-scheme X, we let $\mathrm{RelStk}^{\mathrm{sm}}_X$ denote the fiber product $\mathrm{RelStk}^{\mathrm{sm}} \times_{\mathrm{Sch}_k} \{X\}$ (that is, the 2-category of algebraic stacks equipped with a smooth map to X).

The main result of this section can be stated as follows:

Proposition 3.4.4.2. *There exists a functor of ∞-categories*

$$\Phi^! : (\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^! \to \mathrm{Shv}_\ell^!$$

with the following properties:

(1) *The diagram of ∞-categories*

$$\begin{array}{ccc} (\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^! & \xrightarrow{\Phi^!} & \mathrm{Shv}_\ell^! \\ & \searrow \quad \swarrow & \\ & \mathrm{Sch}_k^{\mathrm{op}} & \end{array}$$

commutes. In particular, for every quasi-projective k-scheme X, $\Phi^!$ restricts to a functor $\Phi^!_X : (\mathrm{RelStk}^{\mathrm{sm}}_X)^{\mathrm{op}} \times \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(X)$.

(2) *For every quasi-projective k-scheme X, the functor $\Phi^!_X$ coincides with the functor $\Phi^\star_X$ appearing in Proposition 3.4.2.2 (given on objects by $(\mathcal{Y}, \mathcal{F}) \mapsto [\mathcal{Y}]_{\mathcal{F}}$).*

Remark 3.4.4.3. The ∞-category $(\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^!$ can be described more informally as follows:

(i) The objects of $(\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^!$ are triples $(X, \mathcal{Y}, \mathcal{F})$ where X is a quasi-projective k-scheme, $\mathcal{Y}$ is an algebraic stack equipped with a smooth map $\mathcal{Y} \to X$, and $\mathcal{F}$ is an ℓ-adic sheaf on X.

(ii) A morphism from $(X, \mathcal{Y}, \mathcal{F})$ to $(X', \mathcal{Y}', \mathcal{F}')$ is given by a commutative diagram σ :

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

where f is proper, together with a morphism of ℓ-adic sheaves $\alpha : f^! \mathcal{F} \to \mathcal{F}'$.

The essential content of Proposition 3.4.4.2 is that the data described in (ii) gives rise to a morphism of relative cohomology sheaves $\beta : f^![\mathcal{Y}]_{\mathcal{F}} \to [\mathcal{Y}']_{\mathcal{F}'}$. Concretely, the map β is given by the composition

$$\begin{array}{rcl} f^![\mathcal{Y}]_{\mathcal{F}} & \xleftarrow{\sim} & [\mathcal{Y} \times_X X']_{f^! \mathcal{F}} \\ & \to & [\mathcal{Y}']_{f^! \mathcal{F}} \\ & \xrightarrow{\alpha} & [\mathcal{Y}']_{\mathcal{F}'}, \end{array}$$

where the first map is supplied by the comparison equivalence of Proposition 3.4.3.2, and the second is given by pullback along the map $\mathcal{Y}' \to \mathcal{Y} \times_X X'$ determined by the diagram σ.

Corollary 3.4.4.4. *The construction* $(\mathcal{Y} \to X) \mapsto (X, [\mathcal{Y}]_X)$ *determines a functor of* ∞*-categories* $\Psi : (\mathrm{RelStk}^{\mathrm{sm}} \times_{\mathrm{Sch}_k} \mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \to \mathrm{Shv}_\ell^!$.

Proof. According to Remark 3.3.5.15, the construction $X \mapsto (X, \omega_X)$ determines a functor of ∞-categories $\chi : (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \to \mathrm{Shv}_\ell^!$. We now define Ψ to be the composition

$$(\mathrm{RelStk}^{\mathrm{sm}} \times_{\mathrm{Sch}_k} \mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \xrightarrow{\mathrm{id} \times \chi} (\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^! \xrightarrow{\Phi^!} \mathrm{Shv}_\ell^! .$$

□

Remark 3.4.4.5. In practice, we are interested not only in the statements of Proposition 3.4.4.2 and Corollary 3.4.4.4, but also in their proofs (which produce particular functors that we will use later).

We now turn to the proof of Proposition 3.4.4.2. Our strategy is an elaboration of Remark 3.4.4.3: we would like to argue that the desired functor $\Phi^!$ can be formally extracted from the functor $\Phi^\star$ of Proposition 3.4.2.2, using the comparison results established in §3.4.3. To make this precise, we need a few categorical remarks.

Definition 3.4.4.6. Let $q : \mathcal{A} \to \mathcal{C}$ be a coCartesian fibration of ∞-categories, let $r : \mathcal{B} \to \mathcal{C}$ and $s : \mathcal{D} \to \mathcal{C}$ be Cartesian fibrations of ∞-categories, and let $\lambda : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{D}$ be a functor for which the diagram

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \xrightarrow{\lambda} & \mathcal{D} \\ & \searrow \quad \swarrow_{s} & \\ & \mathcal{C} & \end{array}$$

commutes. We will say that λ is *balanced* if, for every morphism (α, β) in the fiber product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$, if α is a q-coCartesian morphism in $\mathcal{A}$ and β is an r-Cartesian morphism in $\mathcal{B}$, then $\lambda(\alpha, \beta)$ is an s-Cartesian morphism in $\mathcal{D}$.

Example 3.4.4.7. The functor $\Phi^\star$ of Proposition 3.4.2.2 determines a balanced functor

$$(\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star \to \mathrm{Shv}_\ell^\star .$$

This follows from Proposition 3.4.3.1 (note that the forgetful functor $q : \mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$ is a Cartesian fibration, where a morphism $(X, \mathcal{F}) \to (X', \mathcal{F}')$ is q-Cartesian if it exhibits $\mathcal{F}$ as the direct image of $\mathcal{F}'$ along the underlying map $X' \to X$).

Construction 3.4.4.8. [The Balanced Dual] Let $q : \mathcal{A} \to \mathcal{C}$, $r : \mathcal{B} \to \mathcal{C}$, and $s : \mathcal{D} \to \mathcal{C}$ be as in Definition 3.4.4.6, and suppose that q, r, and s are classified by functors $\chi_q : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$, $\chi_r, \chi_s : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. Using Corollary [25].3.2.2.12, one can show that the data of a balanced functor $\lambda : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{D}$ is equivalent to the data of a natural transformation $\chi_r \to U$, where $U : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ is given by the formula $U(C) = \mathrm{Fun}(\chi_q(C), \chi_s(C))$.

Let $\sigma : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ denote the functor which assigns to each ∞-category its opposite. Any natural transformation from χ_r to U determines a natural transformation from $\sigma \circ \chi_r$ to the functor $\sigma \circ U$ given by

$$(\sigma \circ U)(C) = \mathrm{Fun}(\chi_q(C), \chi_s(C))^{\mathrm{op}} = \mathrm{Fun}(\chi_q(C)^{\mathrm{op}}, \chi_s(C)^{\mathrm{op}}).$$

It follows that the datum of a balanced functor $\lambda : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{D}$ is equivalent to the data of a balanced functor $\mathcal{A}'' \times_{\mathcal{C}} \mathcal{B}' \to \mathcal{D}'$, where $\mathcal{A}'' \to \mathcal{C}$ denotes the coCartesian dual of q, $\mathcal{B}' \to \mathcal{C}$ denotes the Cartesian dual of $\mathcal{B}$, and $\mathcal{D}' \to \mathcal{C}$ denotes the Cartesian dual of $\mathcal{D}$ (see Construction 3.3.5.1 and Variant 3.3.5.5). We will refer to λ' as the *balanced dual* of λ.

Proof of Proposition 3.4.4.2. Let $\mathcal{A}''$ be the coCartesian dual of the coCartesian fibration $(\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \to \mathrm{Sch}_k^{\mathrm{op}}$. The ∞-category $\mathcal{A}''$ can be described more informally as follows:

- The objects of $\mathcal{A}''$ are smooth morphisms of algebraic stacks $\pi : \mathcal{Y} \to X$, where X is a quasi-projective k-scheme.
- A morphism from $(\pi : \mathcal{Y} \to X)$ to $(\pi' : \mathcal{Y}' \to X')$ in $\mathcal{A}''$ is given by a morphism of quasi-projective k-schemes $X' \to X$ together with a morphism of algebraic stacks $f : \mathcal{Y} \times_X X' \to \mathcal{Y}'$ for which the diagram

$$\begin{array}{ccc} \mathcal{Y} \times_X X' & \xrightarrow{f} & \mathcal{Y}' \\ & \searrow \quad \swarrow_{\pi'} & \\ & X' & \end{array}$$

commutes.

Let $\mathcal{B}'$ be the Cartesian dual of the Cartesian fibration $\mathrm{Shv}_\ell^\star \to \mathrm{Sch}_k^{\mathrm{op}}$. Using Example 3.4.4.7, we see that the functor $\Phi^\star$ of Proposition 3.4.2.2 determines a balanced functor

$$(\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^\star \to \mathrm{Shv}_\ell^\star,$$

which admits a balanced dual $F : \mathcal{A}'' \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathcal{B}' \to \mathcal{B}'$. Set $\mathcal{A}_0'' = \mathcal{A}'' \times_{\mathrm{Sch}_k^{\mathrm{op}}} (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$ and $\mathcal{B}_0' = \mathcal{B}' \times_{\mathrm{Sch}_k^{\mathrm{op}}} (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$, so that F induces a functor $F_0 : \mathcal{A}_0'' \times_{(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}} \mathcal{B}_0' \to \mathcal{B}_0'$ which fits into a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{A}_0'' \times_{(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}} \mathcal{B}_0' & \xrightarrow{F_0} & \mathcal{B}_0' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}. & \end{array}$$

Note that the projection maps

$$\mathcal{A}_0'' \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \qquad q : \mathcal{B}_0' \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$$

are coCartesian fibrations (see Lemma 3.3.5.8), whose coCartesian duals are given by the projection maps

$$(\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} \qquad \mathrm{Shv}_\ell^! \to (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}.$$

It follows that p is also a coCartesian fibration. Moreover, the functor F_0 carries p-coCartesian morphisms to q-coCartesian morphisms (this is a reformulation of Proposition 3.4.3.2). It follows that F_0 induces a functor from the coCartesian dual of p to the coCartesian dual of q, which we write as $\Phi^! : (\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_\ell^! \to \mathrm{Shv}_\ell^!$. It is now easy to check that $\Phi^!$ has the desired properties. □

3.4.5 Compatibility with External Tensor Products

Let Y and Y' be quasi-projective k-schemes. According to Theorem 3.3.1.1, the ℓ-adic cochain complex $C^*(Y \times Y'; \mathbf{Z}_\ell)$ can be identified with the tensor product $C^*(Y; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(Y'; \mathbf{Z}_\ell)$. Our goal in this section is to prove a generalization of this result, where we replace the ℓ-adic cochain complex $C^*(Y; \mathbf{Z}_\ell)$ by the cohomology sheaf $[\mathcal{Y}]_{\mathcal{F}}$ of Construction 3.4.1.2:

Theorem 3.4.5.1. *Let X and X' be quasi-projective k-schemes, let $\pi : \mathcal{Y} \to X$ and $\pi' : \mathcal{Y}' \to X'$ be finite type morphisms of algebraic stacks, and suppose we are given ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{<\infty}$ and $\mathcal{F}' \in \mathrm{Shv}_\ell(X')_{<\infty}$. Then there is a canonical equivalence*

$$[\mathcal{Y} \times \mathcal{Y}']_{\mathcal{F} \boxtimes \mathcal{F}'} \simeq [\mathcal{Y}]_{\mathcal{F}} \boxtimes [\mathcal{Y}']_{\mathcal{F}'}$$

in the ∞*-category* $\mathrm{Shv}_\ell(X \times X')$ *(here all products are formed relative to the base scheme* $\mathrm{Spec}(k)$*).*

Example 3.4.5.2 (Relative Künneth Formula)**.** Let $\pi : \mathcal{Y} \to X$ and $\pi' : \mathcal{Y}' \to X'$ be as in Theorem 3.4.5.1. Taking $\mathcal{F} = \omega_X$ and $\mathcal{F}' = \omega_{X'}$, we obtain an equivalence of ℓ-adic sheaves

$$[\mathcal{Y} \times_{\mathrm{Spec}(k)} \mathcal{Y}']_{X \times_{\mathrm{Spec}(k)} X'} \simeq [\mathcal{Y}]_X \boxtimes [\mathcal{Y}']_{X'}.$$

Example 3.4.5.3 (Absolute Künneth Formula)**.** Let $\mathcal{Y}$ and $\mathcal{Y}'$ be algebraic stacks of finite type over k. Applying Example 3.4.5.2 (in the special case $X = X' = \mathrm{Spec}(k)$), we obtain an equivalence of ℓ-adic cochain complexes $C^*(\mathcal{Y} \times \mathcal{Y}'; \mathbf{Z}_\ell) \simeq C^*(\mathcal{Y}; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(\mathcal{Y}'; \mathbf{Z}_\ell)$. This equivalence is induced by the multiplication on $C^*(\mathcal{Y} \times \mathcal{Y}'; \mathbf{Z}_\ell)$ described in Construction 3.2.3.3.

Warning 3.4.5.4. In the statement of Theorem 3.4.5.1 (and Examples 3.4.5.2 and 3.4.5.3), we require that the algebraic stacks $\mathcal{Y}$ and $\mathcal{Y}'$ are of finite type over k, rather than merely locally of finite type. Note that the assertion of Example 3.4.5.3 fails in the case where both $\mathcal{Y}$ and $\mathcal{Y}'$ are disjoint unions of infinitely many copies of $\mathrm{Spec}(k)$. The assertion of Theorem 3.4.5.1 can also fail if we drop the boundedness assumptions on the sheaves $\mathcal{F}$ and $\mathcal{F}'$.

Variant 3.4.5.5. In the situation of Example 3.4.5.3, suppose that $\mathcal{Y} = Y$ is a quasi-projective k-scheme. In this case, the canonical map

$$\theta : C^*(\mathcal{Y}; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(\mathcal{Y}'; \mathbf{Z}_\ell) \to C^*(\mathcal{Y} \times \mathcal{Y}'; \mathbf{Z}_\ell)$$

is an equivalence assuming only that $\mathcal{Y}'$ is *locally* of finite type over k. To see this, we observe that $C^*(\mathcal{Y}; \mathbf{Z}_\ell)$ is a perfect object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$, so that θ can be realized as an inverse limit of maps

$$\theta_\alpha : C^*(\mathcal{Y}; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(\mathcal{U}_\alpha; \mathbf{Z}_\ell) \to C^*(\mathcal{Y} \times \mathcal{U}_\alpha; \mathbf{Z}_\ell)$$

where $\mathcal{U}_\alpha$ ranges over all quasi-compact open substacks of $\mathcal{Y}'$. Moreover, each of the maps θ_α is a quasi-isomorphism by virtue of Example 3.4.5.3.

The main ingredient in the proof of Theorem 3.4.5.1 is the following:

Proposition 3.4.5.6. *Let X and Y be quasi-projective k-schemes. Suppose that $\mathcal{F}^\bullet$ is a cosimplicial object of $\mathrm{Shv}_\ell(X)_{\leq 0}$ and that $\mathcal{G}^\bullet$ is a cosimplicial object of $\mathrm{Shv}_\ell(Y)_{\leq 0}$. Then the canonical map $\mathrm{Tot}(\mathcal{F}^\bullet) \boxtimes \mathrm{Tot}(\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)$ is an equivalence in $\mathrm{Shv}_\ell(X \times Y)$.*

Before giving the proof of Proposition 3.4.5.6, let us apply it to the situation of Theorem 3.4.5.1.

Proof of Theorem 3.4.5.1. Let $\pi : \mathcal{Y} \to X$ and $\pi' : \mathcal{Y}' \to X'$ be as in the statement of Theorem 3.4.5.1 and suppose we are given ℓ-adic sheaves $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{<\infty}$ and $\mathcal{F}' \in \mathrm{Shv}_\ell(X')_{<\infty}$. We wish to construct an equivalence

$$[\mathcal{Y} \times \mathcal{Y}']_{\mathcal{F} \boxtimes \mathcal{F}'} \simeq [\mathcal{Y}]_{\mathcal{F}} \boxtimes [\mathcal{Y}']_{\mathcal{F}'}.$$

Let us assume for simplicity that $\mathcal{Y}$ and $\mathcal{Y}'$ have quasi-projective diagonals (this condition is satisfied in all cases of interest to us) and choose quasi-projective k-schemes U_0 and U_0' equipped with smooth surjections $\rho : U_0 \to \mathcal{Y}$ and $\rho' : U_0' \to \mathcal{Y}'$. Let $U_\bullet$ denote the simplicial k-scheme given by the iterated fiber product of U_0 with itself over $\mathcal{Y}$, define $U'_\bullet$ similarly, and consider the natural maps $\phi_\bullet : U_\bullet \to X$ and $\phi'_\bullet : U'_\bullet \to X'$. Using Proposition 3.4.1.10, Proposition 3.4.5.6, and Corollary 3.3.1.5, we obtain equivalences

$$\begin{aligned}
[\mathcal{Y} \times \mathcal{Y}']_{\mathcal{F} \boxtimes \mathcal{F}'} &\simeq \mathrm{Tot}[U_\bullet \times U'_\bullet)]_{\mathcal{F} \boxtimes \mathcal{F}'} \\
&\simeq \mathrm{Tot}((\phi_\bullet \boxtimes \phi'_\bullet)_*(\phi_\bullet \boxtimes \phi'_\bullet)^*(\mathcal{F} \boxtimes \mathcal{F}')) \\
&\simeq \mathrm{Tot}((\phi_{\bullet *}\phi_\bullet^* \mathcal{F}) \boxtimes (\phi'_{\bullet *}\phi'^*_\bullet \mathcal{F}')) \\
&\simeq \mathrm{Tot}(\phi_{\bullet *}\phi_\bullet^* \mathcal{F}) \boxtimes \mathrm{Tot}(\phi'_{\bullet *}\phi'^*_\bullet \mathcal{F}) \\
&\simeq [\mathcal{Y}]_{\mathcal{F}} \boxtimes [\mathcal{Y}']_{\mathcal{F}'}.
\end{aligned}$$

□

Remark 3.4.5.7. At this point, the reader might reasonably object that Theorem 3.4.5.1 asserts the existence of a *canonical* equivalence, but our proof constructs an equivalence which depends *a priori* on a choice of smooth atlases $U_0 \to \mathcal{Y}$ and $U_0' \to \mathcal{Y}'$. This point will be addressed in §3.4.6.

We now turn to the proof of Proposition 3.4.5.6.

Lemma 3.4.5.8. *Let X be a quasi-projective k-scheme and let $\mathcal{F}, \mathcal{G} \in \mathrm{Shv}_\ell(X)_{\leq 0}$. Then $\mathcal{F} \otimes \mathcal{G} \in \mathrm{Shv}_\ell(X)_{\leq 2}$.*

Remark 3.4.5.9. With more effort, one can show that the tensor product functor carries $\mathrm{Shv}_\ell(X)_{\leq 0} \times \mathrm{Shv}_\ell(X)_{\leq 0}$ into $\mathrm{Shv}_\ell(X)_{\leq 1}$, but Lemma 3.4.5.8 will be sufficient for our purposes.

Remark 3.4.5.10. Let X be a quasi-projective k-scheme and let Λ be a field. Then the tensor product functor $\otimes : \mathrm{Shv}(X;\Lambda) \times \mathrm{Shv}(X;\Lambda) \to \mathrm{Shv}(X;\Lambda)$ is left t-exact: that is, it carries $\mathrm{Shv}(X;\Lambda)_{\leq 0} \times \mathrm{Shv}(X;\Lambda)_{\leq 0}$ into $\mathrm{Shv}(X;\Lambda)_{\leq 0}$. This follows from Remark 2.2.3.3, since the tensor product $\otimes_\Lambda : \mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$ carries $(\mathrm{Mod}_\Lambda)_{\leq 0} \times (\mathrm{Mod}_\Lambda)_{\leq 0}$ into $(\mathrm{Mod}_\Lambda)_{\leq 0}$.

Proof of Lemma 3.4.5.8. Since $\mathrm{Shv}_\ell(X)_{\leq 1}$ is closed under filtered colimits and the tensor product $\otimes$ preserves filtered colimits separately in each variable, we may assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are constructible, so that $\mathcal{F} \otimes \mathcal{G}$ is likewise constructible. Set $\mathcal{F}_1 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{F}$ and $\mathcal{G}_1 = (\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} \mathcal{G}$. Using Remark 2.3.6.6 we see that $\mathcal{F}_1, \mathcal{G}_1 \in \mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq 1}$. Using Remark 3.4.5.10, we conclude that the tensor product

$$(\mathbf{Z}/\ell\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{F}_1 \otimes_{\mathbf{Z}/\ell\mathbf{Z}} \mathcal{G}_1$$

belongs to $\mathrm{Shv}(X; \mathbf{Z}/\ell\mathbf{Z})_{\leq 2}$, so that $\mathcal{F} \otimes \mathcal{G}$ belongs to $\mathrm{Shv}^c_\ell(X)_{\leq 2}$ by Remark 2.3.6.6. □

Lemma 3.4.5.11. *Let X be a quasi-projective k-scheme, let $\mathcal{F} \in \mathrm{Shv}_\ell(X)_{\leq 0}$, and let $\mathcal{G}^\bullet$ be a cosimplicial object of $\mathrm{Shv}_\ell(X)_{\leq 0}$. Then the canonical map $\theta : \mathcal{F} \otimes \mathrm{Tot}(\mathcal{G}^\bullet) \to \mathrm{Tot}(\mathcal{F} \otimes \mathcal{G}^\bullet)$ is an equivalence in $\mathrm{Shv}_\ell(X)$.*

Proof. For each $n \geq 0$, let $\mathrm{Tot}^n(\mathcal{G}^\bullet)$ denote the nth stage of the Tot-tower of $\mathcal{G}^\bullet$ (that is, the limit of the restriction of $\mathcal{G}$ to the category $\mathbf{\Delta}_{\leq n}$ of simplices of dimension $\leq n$). The construction $\mathcal{G}^\bullet \mapsto \mathrm{Tot}^n(\mathcal{G}^\bullet)$ is given by a finite limit, and therefore commutes with any exact functor. It follows that θ can be identified with the composition

$$\begin{aligned} \mathcal{F} \otimes \mathrm{Tot}(\mathcal{G}^\bullet) &\simeq \mathcal{F} \otimes \varprojlim \mathrm{Tot}^n(\mathcal{G}^\bullet) \\ &\xrightarrow{\theta'} \varprojlim (\mathcal{F} \otimes \mathrm{Tot}^n(\mathcal{G}^\bullet)) \\ &\simeq \varprojlim \mathrm{Tot}^n(\mathcal{F} \otimes \mathcal{G}^\bullet) \\ &\simeq \mathrm{Tot}(\mathcal{F} \otimes \mathcal{G}^\bullet). \end{aligned}$$

We are therefore reduced to proving that θ' is an equivalence. Since $\mathrm{Shv}_\ell(X)$ is right complete, it will suffice to show that the fiber of θ' belongs to $\mathrm{Shv}_\ell(X)_{\leq -m}$ for each integer m. For $n \geq m+2$, let $\mathcal{H}_n$ denote the cofiber of the natural map $\mathrm{Tot}^n(\mathcal{G}^\bullet) \to \mathrm{Tot}^{m+2}(\mathcal{G}^\bullet)$, so that we have a pushout square

$$\begin{array}{ccc} \mathcal{F} \otimes \varprojlim \mathrm{Tot}^n(\mathcal{G}^\bullet) & \xrightarrow{\theta'} & \varprojlim_{n' \geq n} (\mathcal{F} \otimes \mathrm{Tot}^n(\mathcal{G}^\bullet)) \\ \downarrow & & \downarrow \\ \mathcal{F} \otimes \varprojlim \mathcal{H}_n & \xrightarrow{\theta''} & \varprojlim \mathcal{F} \otimes \mathcal{H}_n . \end{array}$$

Since each $\mathcal{G}^q$ belongs to $\mathrm{Shv}_\ell(X)_{\leq 0}$, the cofibers $\mathcal{H}_n$ belong to $\mathrm{Shv}_\ell(X)_{\leq -m-2}$. Using Lemma 3.4.5.8, we deduce that the domain and codomain of θ'' belong to $\mathrm{Shv}_\ell(X)_{\leq -m}$, so that $\mathrm{fib}(\theta') \simeq \mathrm{fib}(\theta'')$ belongs to $\mathrm{Shv}_\ell(X)_{\leq -m}$ as desired. □

Proof of Proposition 3.4.5.6. Embedding X and Y into projective space, we may assume without loss of generality that X and Y are smooth. Let $p : X \times Y \to X$ and

$q : X \times Y \to Y$ denote the projection maps onto the first and second factor, respectively. Unwinding the definitions, we wish to show that the composite map

$$\begin{aligned}
\mathrm{Tot}(\mathcal{F}^\bullet) \boxtimes \mathrm{Tot}(\mathcal{G}^\bullet) &\simeq p^* \mathrm{Tot}(\mathcal{F}^\bullet) \otimes q^* \mathrm{Tot}(\mathcal{G}^\bullet) \\
&\xrightarrow{\theta} \mathrm{Tot}(p^* \mathcal{F}^\bullet) \otimes \mathrm{Tot}(q^* \mathcal{G}^\bullet) \\
&\xrightarrow{\theta'} \mathrm{Tot}(p^* \mathcal{F}^\bullet \otimes \mathrm{Tot}(q^* \mathcal{G}^\bullet)) \\
&\xrightarrow{\theta''} \mathrm{Tot}(\mathrm{Tot}(p^* \mathcal{F}^\bullet \otimes q^* \mathcal{G}^\bullet)) \\
&\simeq \mathrm{Tot}(\mathcal{F}^\bullet \boxtimes \mathcal{G}^\bullet)
\end{aligned}$$

is an equivalence. The map θ is an equivalence by Proposition 2.3.4.8, and the maps θ' and θ'' are equivalences by virtue of Lemma 3.4.5.11. □

3.4.6 Tensor Functoriality

In §3.4.5, we proved that (under some mild hypotheses) the construction $\mathcal{F} \mapsto [\mathcal{Y}]_{\mathcal{F}}$ is compatible with the formation of external tensor products (Theorem 3.4.5.1). However, this is not sufficient for our applications: we will need to know not only that there exist equivalences

$$[\mathcal{Y} \times \mathcal{Y}']_{\mathcal{F} \boxtimes \mathcal{F}'} \simeq [\mathcal{Y}]_{\mathcal{F}} \boxtimes [\mathcal{Y}']_{\mathcal{F}'},$$

but also that they can be made compatible with the commutativity and associativity constraints on the external tensor product $\boxtimes$, up to coherent homotopy. Before we can formulate this statement precisely, we need a bit of notation.

Notation 3.4.6.1. Let RelStk denote the 2-category of Notation 3.4.2.1, whose objects are maps of algebraic stacks $\pi : \mathcal{Y} \to X$ where X is a quasi-projective k-scheme and π is locally of finite type. We let $\mathrm{RelStk}_{\mathrm{ft}}$ denote the full subcategory of RelStk spanned by those objects $\pi : \mathcal{Y} \to X$ where π is of finite type. Similarly, we let $\mathrm{RelStk}^{\mathrm{sm}}_{\mathrm{ft}} = \mathrm{RelStk}^{\mathrm{sm}} \cap \mathrm{RelStk}_{\mathrm{ft}}$ denote the full subcategory spanned by those objects $\pi : \mathcal{Y} \to X$ where π is smooth and of finite type.

Let $\mathrm{Shv}^\star_\ell$ and $\mathrm{Shv}^!_\ell$ be the ∞-categories introduced in Constructions 3.3.4.1 and 3.3.5.9, whose objects can be identified with pairs $(X, \mathcal{F})$ where X is a quasi-projective k-scheme and $\mathcal{F} \in \mathrm{Shv}_\ell(X)$. We let $\mathrm{Shv}^\star_c \subseteq \mathrm{Shv}^\star_\ell$ and $\mathrm{Shv}^!_c \subseteq \mathrm{Shv}^!_\ell$ denote the full subcategories spanned by those objects $(X, \mathcal{F})$ where $\mathcal{F}$ is a constructible ℓ-adic sheaf on X.

The functors $\Phi^\star$ and $\Phi^!$ of Propositions 3.4.2.2 and 3.4.4.2 can be restricted to functors

$$\Phi^\star_c : \mathrm{RelStk}^{\mathrm{op}}_{\mathrm{ft}} \times_{\mathrm{Sch}^{\mathrm{op}}_k} \mathrm{Shv}^\star_c \to \mathrm{Shv}^\star_\ell$$

$$\Phi^!_c : (\mathrm{RelStk}^{\mathrm{sm}}_{\mathrm{ft}})^{\mathrm{op}} \times_{\mathrm{Sch}^{\mathrm{op}}_k} \mathrm{Shv}^!_c \to \mathrm{Shv}^!_\ell .$$

In what follows, we will regard the 2-categories $\mathrm{RelStk}_{\mathrm{ft}}$ and $\mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{sm}}$ as equipped with the symmetric monoidal structure given by the formation of Cartesian products (see Example 3.1.2.5), and the ∞-categories $\mathrm{Shv}_{\ell}^{\star}$ and $\mathrm{Shv}_{\ell}^{!}$ as equipped with the symmetric monoidal structures given by the formation of external tensor products (see Propositions 3.3.4.4 and 3.3.5.13 and their proofs). Our goal in this section is to prove the following variant of Theorem 3.4.5.1:

Theorem 3.4.6.2. *The functors*

$$\Phi_c^{\star} : \mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^{\star} \to \mathrm{Shv}_{\ell}^{\star}$$

$$\Phi_c^{!} : (\mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^{!} \to \mathrm{Shv}_{\ell}^{!}$$

of Notation 3.4.6.1 can be promoted to symmetric monoidal functors.

Remark 3.4.6.3. As usual, our interest in Theorem 3.4.6.2 is not in the statement, but in the proof (which will provide natural symmetric monoidal structures on the functors $\Phi_c^{\star}$ and $\Phi_c^{!}$).

We begin by analyzing the functor $\Phi_c^{\star}$ of Theorem 3.4.6.2.

Proposition 3.4.6.4. *The functor*

$$\Phi_c^{\star} : \mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^{\star} \to \mathrm{Shv}_{\ell}^{\star}$$

admits a symmetric monoidal strucutre.

Proof. We proceed in two steps. First, let RelSch denote the full subcategory of $\mathrm{RelStk}_{\mathrm{ft}}$ spanned by those objects $(\mathcal{Y} \to X)$ where the algebraic stack $\mathcal{Y}$ is itself a quasi-projective k-scheme. Then the constructions $(Y \to X) \mapsto Y$ and $(Y \to X) \mapsto X$ determine functors $e_0, e_1 : \mathrm{RelSch} \to \mathrm{Sch}_k$. For $i \in \{0, 1\}$, we let $\mathrm{RelSch}_{(i)}^{\mathrm{op}}$ denote the fiber product $\mathrm{RelSch}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^{\star}$. There is an evident symmetric monoidal functor $u : \mathrm{RelSch}_{(1)}^{\mathrm{op}} \to \mathrm{RelSch}_{(0)}^{\mathrm{op}}$, given by the construction

$$(f : Y \to X, \mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(X)) \mapsto (f : Y \to X, f^* \mathcal{F} \in \mathrm{Shv}_{\ell}^{c}(Y)).$$

This functor has a right adjoint $v : \mathrm{RelSch}_{(0)}^{\mathrm{op}} \to \mathrm{RelSch}_{(1)}^{\mathrm{op}}$, given by the construction

$$(f : Y \to X, \mathcal{G} \in \mathrm{Shv}_{\ell}^{c}(Y)) \mapsto (f : Y \to X, f_* \mathcal{G} \in \mathrm{Shv}_{\ell}^{c}(X)).$$

Using the symmetric monoidal structure on the functor u, we obtain a lax symmetric structure on the functor v, which is actually symmetric monoidal by virtue of Proposition 3.3.1.3.

Let us regard $\mathrm{RelSch}^{\mathrm{op}}_{(1)}$ as a full subcategory of $\mathrm{RelStk}_{\mathrm{ft}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^\star$, and let $\Phi_0^\star : \mathrm{RelSch}^{\mathrm{op}}_{(1)} \to \mathrm{Shv}_\ell^\star$ denote the composite functor

$$\mathrm{RelSch}^{\mathrm{op}}_{(1)} \xrightarrow{u} \mathrm{RelSch}^{\mathrm{op}}_{(0)} \xrightarrow{v} \mathrm{RelSch}^{\mathrm{op}}_{(1)} = \mathrm{RelSch}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^\star \to \mathrm{Shv}_c^\star \subseteq \mathrm{Shv}_\ell^\star .$$

It follows immediately from the definitions that $\Phi_0^\star$ is equivalent to the restriction $\Phi_c^\star|_{\mathrm{RelSch}^{\mathrm{op}}_{(1)}}$ after neglecting symmetric monoidal structures.

To complete the construction, we will use the formalism of Day convolution in the setting of ∞-operads, as developed in §[23].2.2.6. Suppose we are given a pair of symmetric monoidal coCartesian fibrations of ∞-categories $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$ satisfying the following conditions:

(*i*) For each object $C \in \mathcal{C}$, the fiber $\mathcal{A}_C = \mathcal{A} \times_{\mathcal{C}} \{C\}$ is essentially small.

(*ii*) For each object $C \in \mathcal{C}$, the fiber $\mathcal{B}_C = \mathcal{B} \times_{\mathcal{C}} \{C\}$ admits small colimits.

(*iii*) For every morphism $C \to C'$ in $\mathcal{C}$, the transport functor $\mathcal{B}_C \to \mathcal{B}_{C'}$ preserves small colimits.

(*iv*) For every pair of objects $C, C' \in \mathcal{C}$, the tensor product on $\mathcal{B}$ induces a functor $\mathcal{B}_C \times \mathcal{B}_{C'} \to \mathcal{B}_{C \otimes C'}$, which preserves small colimits separately in each variable.

In this case, we can construct a new symmetric monoidal coCartesian fibration $\rho : \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ with the following features:

(*a*) For each object $C \in \mathcal{C}$, we have a canonical equivalence $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})_C \simeq \mathrm{Fun}(\mathcal{A}_C, \mathcal{B}_C)$.

(*b*) For every pair of objects $F \in \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})_C \simeq \mathrm{Fun}(\mathcal{A}_C, \mathcal{B}_C)$ and $F' \in \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})_{C'} \simeq \mathrm{Fun}(\mathcal{A}_{C'}, \mathcal{B}_{C'})$, the tensor product $F \otimes F' \in \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})_{C \otimes C'} \simeq \mathrm{Fun}(\mathcal{A}_{C \otimes C'}, \mathcal{B}_{C \otimes C'})$ is given on objects by the formula

$$(F \otimes F')(X) = \varinjlim_{A \otimes A' \to X} F(A) \otimes F'(A'),$$

where the colimit is taken over the ∞-category $(\mathcal{A}_C \times \mathcal{A}_{C'}) \times_{\mathcal{A}_{C \otimes C'}} (\mathcal{A}_{C \otimes C'})_{/X}$.

(*c*) For each section $s : \mathcal{C} \to \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ of ρ, we can associate a functor $F_s : \mathcal{A} \to \mathcal{B}$ by the formula $F_s(A) = s(C)(A)$, where C denotes the image of A in the ∞-category $\mathcal{C}$. This construction induces an equivalence from the ∞-category of lax symmetric monoidal sections of ρ to the ∞-category of lax symmetric monoidal functors $F : \mathcal{A} \to \mathcal{B}$ for which the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

commutes.

For a proof, we refer the reader to Proposition [23].2.2.6.16. We will apply this formalism in our situation by taking $\mathcal{A} = \mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^{\star}$, $\mathcal{B} = \mathrm{Shv}_{\ell}^{\star}$, and $\mathcal{C} = \mathrm{Sch}_k^{\mathrm{op}}$ (and also after replacing $\mathcal{A}$ by the full subcategory $\mathcal{A}' = \mathrm{RelSch}_{(1)}^{\mathrm{op}} \subseteq \mathcal{A}$), we obtain symmetric monoidal ∞-categories $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}', \mathcal{B})$ which can be described informally as follows:

- For each quasi-projective k-scheme X, let Sch_X denote the category of quasi-projective X-schemes and let $\mathrm{RelStk}_{X\,\mathrm{ft}}$ denote the category $\mathrm{RelStk}_{\mathrm{ft}} \times_{\mathrm{Sch}_k} \{X\}$ of algebraic X-stacks of finite type. The objects of $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ are given by pairs (X, F), where X is a quasi-projective k-scheme and

  $$F : \mathrm{RelStk}_{X\,\mathrm{ft}}^{\mathrm{op}} \times \mathrm{Shv}_{\ell}^{c}(X) \to \mathrm{Shv}_{\ell}(X)$$

  is a functor; the objects of $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}', \mathcal{B})$ are pairs (X, F_0) where X is a quasi-projective k-scheme and $F_0 : \mathrm{Sch}_X^{\mathrm{op}} \times \mathrm{Shv}_{\ell}^{c}(X) \to \mathrm{Shv}_{\ell}(X)$ is a functor.

- The tensor product on the symmetric monoidal category $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})$ is given concretely by the formula

  $$(X, F) \otimes (X', F') = (X \times_{\mathrm{Spec}(k)} X', F \star F')$$
  $$(F \star F')(\mathcal{Y}, \mathcal{G}) = \varinjlim_{\mathcal{F} \boxtimes \mathcal{F}' \to \mathcal{G}} F(\mathcal{Y}, \mathcal{F}) \boxtimes F(\mathcal{Y}, \mathcal{F}'),$$

  and the tensor product on $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}', \mathcal{B})$ is described similarly.

From the above description, it is not hard to see that the inclusion functor $i : \mathrm{RelSch}_{(1)}^{\mathrm{op}} \hookrightarrow \mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{op}}$ induces a symmetric monoidal functor $i^* : \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B}) \to \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}', \mathcal{B})$ (and, moreover, that this functor preserves the class of morphisms which are coCartesian relative to $\mathcal{C}$). The functor i^* admits a right adjoint $i_* : \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}', \mathcal{B}) \to \mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B})$, given concretely by the formula

$$i_*(X, F_0) = (X, F) \qquad F(\mathcal{Y}, \mathcal{F}) = \varprojlim_{Y \to \mathcal{Y}} F_0(Y, \mathcal{F}),$$

where the limit is taken over all quasi-projective X-schemes Y equipped with a map $Y \to \mathcal{Y}$ in $\mathrm{RelStk}_{X\,\mathrm{ft}}$.

Using (c), we can identify the symmetric monoidal functor $\Phi_0^{\star} : \mathrm{RelSch}_{(1)}^{\mathrm{op}} \to \mathrm{Shv}_{\ell}^{\star}$ with a lax symmetric monoidal section s of the projection map $\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}', \mathcal{B}) \to \mathcal{C}$. The composition $i_* \circ s$ is then a lax symmetric monoidal section of the projection map

$\mathrm{Fun}^{\mathcal{C}}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}$, which we can identify (using (c) again) with a lax symmetric monoidal functor

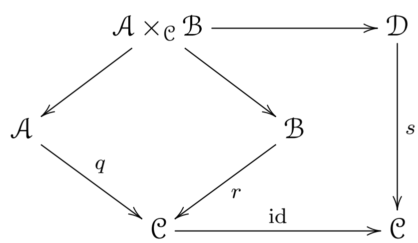
$$\mathrm{RelStk}^{\mathrm{op}}_{\mathrm{ft}} \times_{\mathrm{Sch}^{\mathrm{op}}_k} \mathrm{Shv}^{\star}_c = \mathcal{A} \to \mathcal{B} = \mathrm{Shv}^{\star}_{\ell}\,.$$

Using the explicit description of i_* given above, it is not hard to see that this lax symmetric monoidal functor agrees, as a functor, with the map $\Phi^{\star}_c$ of Notation 3.4.6.1. We therefore obtain a *lax* symmetric monoidal structure on the functor $\Phi^{\star}_c$. To complete the proof of Proposition 3.4.6.4, it will suffice to show that this lax symmetric monoidal structure is actually a symmetric monoidal structure: that is, for any pair of objects $(X, \mathcal{Y}, \mathcal{F}), (X', \mathcal{Y}', \mathcal{F}') \in \mathcal{A}$, the resulting map

$$[\mathcal{Y}]_{\mathcal{F}} \boxtimes [\mathcal{Y}']_{\mathcal{F}'} \to [\mathcal{Y} \times \mathcal{Y}']_{\mathcal{F} \boxtimes \mathcal{F}'}$$

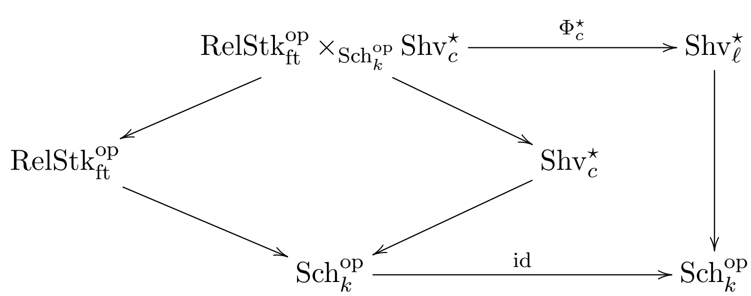
is an equivalence of ℓ-adic sheaves on $X \times_{\mathrm{Spec}(k)} X'$. This is a special case of Theorem 3.4.5.1. □

Proof of Theorem 3.4.6.2. Proposition 3.4.6.4 determines a symmetric monoidal structure on the functor $\Phi^{\star}_c$. We will use this to construct a symmetric monoidal structure on the functor $\Phi^{!}_c$ using some formal categorical constructions. Let $\mathcal{C}\mathrm{at}^{\mathrm{Bal}}_{\infty}$ denote the ∞-category whose objects are commutative diagrams of ∞-categories



where q is a coCartesian fibration, r is a Cartesian fibration, s is a Cartesian fibration, and λ is a balanced functor (Definition 3.4.4.6), which we regard as a subcategory of $\mathrm{Fun}(P, \mathcal{C}\mathrm{at}_{\infty})$ for a suitable partially ordered set P (morphisms in $\mathcal{C}\mathrm{at}^{\mathrm{Bal}}_{\infty}$ are required to preserve q-coCartesian, r-Cartesian, and s-Cartesian morphisms).

Using Proposition 3.4.6.4 and Example 3.4.4.7, we see that the diagram σ :



can be regarded as a commutative monoid object of the ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Bal}}$. The formation of balanced duals (Construction 3.4.4.8) determines a functor from the ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Bal}}$ to itself whose square is equivalent to the identity. In particular, this functor is an equivalence of ∞-categories, and therefore carries commutative monoid objects to commutative monoid objects. Applying this functor to σ, we obtain a diagram of ∞-categories σ':

$$\begin{array}{ccccc}
 & \mathcal{A} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathcal{B}_c & \xrightarrow{F} & & \mathcal{B} \\
\swarrow & & \searrow & & \\
\mathcal{A} & & & \mathcal{B}_c & \big\downarrow s \\
\searrow & & \swarrow & & \\
 & \mathrm{Sch}_k^{\mathrm{op}} & \xrightarrow{\mathrm{id}} & & \mathrm{Sch}_k^{\mathrm{op}}
\end{array}$$

which is equipped with the structure of a commutative monoid object of $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{Bal}}$. Here we can identify the objects of $\mathcal{A}$ with maps $\pi : \mathcal{Y} \to X$, where X is a quasi-projective k-scheme and $\mathcal{Y}$ is an algebraic stack of finite type over X. Let $\mathcal{A}^{\mathrm{sm}}$ be the full subcategory of $\mathcal{A}$ spanned by those objects $\pi : \mathcal{Y} \to X$ where π is smooth, set $\mathcal{B}^{\mathrm{pr}} = \mathcal{B} \times_{\mathrm{Sch}_k^{\mathrm{op}}} (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}$, and define $\mathcal{B}_c^{\mathrm{pr}} \subseteq \mathcal{B}^{\mathrm{pr}}$ similarly. Let F_0 be the restriction of F to the product $\mathcal{A}^{\mathrm{sm}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathcal{B}_c^{\mathrm{pr}}$, so that F_0 fits into a commutative diagram of symmetric monoidal ∞-categories τ :

$$\begin{array}{ccc}
\mathcal{A}^{\mathrm{sm}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathcal{B}_c^{\mathrm{pr}} & \xrightarrow{F_0} & \mathcal{B}^{\mathrm{pr}} \\
\big\downarrow & & \big\downarrow \\
(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} & \xrightarrow{\mathrm{id}} & (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}.
\end{array}$$

As in the proof of Proposition 3.3.5.13, we observe that the vertical maps in this diagram are symmetric monoidal coCartesian fibrations. We may therefore regard τ as a morphism between commutative algebra objects of the ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{coCart}}$ of Notation 3.3.5.16. Applying the coCartesian duality of Variant 3.3.5.5, we obtain a new diagram of symmetric monoidal ∞-categories

$$\begin{array}{ccc}
(\mathrm{RelStk}_{\mathrm{ft}}^{\mathrm{sm}})^{\mathrm{op}} \times_{\mathrm{Sch}_k^{\mathrm{op}}} \mathrm{Shv}_c^{!} & \longrightarrow & \mathrm{Shv}_{\ell}^{!} \\
\big\downarrow & & \big\downarrow \\
(\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}} & \xrightarrow{\mathrm{id}} & (\mathrm{Sch}_k^{\mathrm{pr}})^{\mathrm{op}}.
\end{array}$$

We complete the proof by observing that the upper horizontal map agrees, after neglecting symmetric monoidal structures, with the functor $\Phi_c^{!}$ of Notation 3.4.6.1. $\square$

3.4.7 The Algebra Structure on $[\mathcal{Y}]_X$

We now specialize our study of cohomology sheaves $[\mathcal{Y}]_{\mathcal{F}}$ to the case where $\mathcal{F} = \omega_X$ is the dualizing sheaf of the base scheme X.

Proposition 3.4.7.1. *The construction $(\pi : \mathcal{Y} \to X) \mapsto (X, [\mathcal{Y}]_X)$ determines a symmetric monoidal functor*

$$\Psi^{\mathrm{ft}} : (\mathrm{RelStk}^{\mathrm{sm}}_{\mathrm{ft}} \times_{\mathrm{Sch}_k} Sch^{\mathrm{pr}}_k)^{\mathrm{op}} \to \mathrm{Shv}^!_\ell .$$

Proof. According to Remark 3.3.5.15, the construction $X \mapsto (X, \omega_X)$ determines a symmetric monoidal functor of ∞-categories $\chi : (\mathrm{Sch}^{\mathrm{pr}}_k)^{\mathrm{op}} \to \mathrm{Shv}^!_\ell$. Note that this functor factors through the full subcategory $\mathrm{Shv}^!_c \subseteq \mathrm{Shv}^!_\ell$ (since the dualizing sheaf ω_X is constructible for each $X \in \mathrm{Sch}^{\mathrm{pr}}_k$). We now define Ψ^{ft} to be the composition

$$(\mathrm{RelStk}^{\mathrm{sm}}_{\mathrm{ft}} \times_{\mathrm{Sch}_k} \mathrm{Sch}^{\mathrm{pr}}_k)^{\mathrm{op}} \xrightarrow{\mathrm{id} \times \chi} (\mathrm{RelStk}^{\mathrm{sm}})^{\mathrm{op}}_{\mathrm{ft}} \times_{\mathrm{Sch}^{\mathrm{op}}_k} \mathrm{Shv}^!_c \xrightarrow{\Phi^!_c} \mathrm{Shv}^!_\ell .$$

□

Remark 3.4.7.2. Neglecting symmetric monoidal structures, the functor Ψ^{ft} of Proposition 3.4.7.1 is given by restricting the functor Ψ of Corollary 3.4.4.4 to those maps $\pi : \mathcal{Y} \to X$ which are of finite type. Beware that the functor Ψ itself is not symmetric monoidal (see Warning 3.4.5.4).

Note that the functor Ψ^{ft} of Proposition 3.4.7.1 fits into a commutative diagram of symmetric monoidal ∞-categories

$$\begin{array}{ccc} (\mathrm{RelStk}^{\mathrm{sm}}_{\mathrm{ft}} \times_{\mathrm{Sch}_k} \mathrm{Sch}^{\mathrm{pr}}_k)^{\mathrm{op}} & \xrightarrow{\Psi^{\mathrm{ft}}} & \mathrm{Shv}^!_\ell \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & (\mathrm{Sch}^{\mathrm{pr}}_k)^{\mathrm{op}}, & \end{array}$$

where the vertical maps are symmetric monoidal coCartesian fibrations and the functor Ψ^{ft} carries p-coCartesian morphisms to q-coCartesian morphisms. For every quasi-projective k-scheme X, the diagonal map $\delta : X \to X \times X$ exhibits X as a *nonunital* commutative algebra object of $(\mathrm{Sch}^{\mathrm{pr}}_k)^{\mathrm{op}}$. Using the construction of Variant 3.3.4.9, we see that Ψ^{ft} restricts to a nonunital symmetric monoidal functor

$$\Psi^{\mathrm{ft}}_X : (\mathrm{RelStk}^{\mathrm{sm}}_{X\,\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Shv}_\ell(X).$$

Here $\mathrm{RelStk}^{\mathrm{sm}}_{X\,\mathrm{ft}}$ denotes the 2-category of algebraic stacks $\mathcal{Y}$ equipped with a smooth morphism $\pi : \mathcal{Y} \to X$ of finite type (where the nonunital symmetric monoidal structure is given by fiber products over X and therefore has a unit, given by the identity map $\mathrm{id} : X \to X$), while $\mathrm{Shv}_\ell(X)$ is equipped with the nonunital symmetric monoidal structure given by the !-tensor product (which also has a unit, given by the dualizing sheaf ω_X). The functor Ψ_X^{ft} carries the identity map $\mathrm{id} : X \to X$ to the cohomology sheaf $[X]_X = \omega_X$ (see Example 3.4.1.6). Applying Theorem [23].5.4.4.5, we deduce that Ψ_X^{ft} can be promoted (in an essentially unique way) to a symmetric monoidal functor. This proves the following:

Proposition 3.4.7.3. *For every quasi-projective k-scheme X, the construction $\mathcal{Y} \mapsto [\mathcal{Y}]_X$ determines a symmetric monoidal functor*

$$\Psi_X^{\mathrm{ft}} : (\mathrm{RelStk}^{\mathrm{sm}}_{X\,\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Shv}_\ell(X),$$

where we regard $\mathrm{Shv}_\ell(X)$ as equipped with the symmetric monoidal structure of Theorem 3.3.0.3 (given by the !-tensor product).

Remark 3.4.7.4. The functor Ψ_X^{ft} of Proposition 3.4.7.3 depends functorially on the quasi-projective k-scheme X. In particular, for any proper map $f : X' \to X$, the diagram of ∞-categories

$$\begin{array}{ccc} (\mathrm{RelStk}^{\mathrm{sm}}_{X\,\mathrm{ft}})^{\mathrm{op}} & \xrightarrow{\Psi_X^{\mathrm{ft}}} & \mathrm{Shv}_\ell(X) \\ \downarrow{\scriptstyle X' \times_X \bullet} & & \downarrow{\scriptstyle f^!} \\ (\mathrm{RelStk}^{\mathrm{sm}}_{X'\,\mathrm{ft}})^{\mathrm{op}} & \xrightarrow{\Psi_{X'}^{\mathrm{ft}}} & \mathrm{Shv}_\ell(X') \end{array}$$

commutes up to (canonical) homotopy.

Proof of Theorem 3.4.0.3. Let X be a quasi-projective k-scheme. If $\pi : \mathcal{Y} \to X$ is a morphism of algebraic stacks which is smooth and of finite type, then the relative diagonal $\delta : \mathcal{Y} \to \mathcal{Y} \times_X \mathcal{Y}$ exhibits $\mathcal{Y}$ as a commutative algebra object of the ∞-category $(\mathrm{RelStk}^{\mathrm{sm}}_{X\,\mathrm{ft}})^{\mathrm{op}}$ (see Example 3.1.3.7). Applying the symmetric monoidal functor Ψ_X^{ft} of Proposition 3.4.7.3, we see that the cohomology sheaf $[\mathcal{Y}]_X = \Psi_X^{\mathrm{ft}}(\mathcal{Y})$ inherits the structure of a commutative algebra in the ∞-category $\mathrm{Shv}_\ell(X)$ (with respect to the !-tensor product).

If $\mathcal{Y}$ is an arbitrary algebraic stack equipped with a smooth map $\pi : \mathcal{Y} \to X$, then we can write $\mathcal{Y}$ as a filtered union $\bigcup \mathcal{U}_\alpha$ of open substacks of finite type over X. Remark 3.4.1.9 then supplies an equivalence of $[\mathcal{Y}]_X$ with the inverse limit $\varprojlim [\mathcal{U}_\alpha]_X$ of a diagram of commutative algebra objects of $\mathrm{Shv}_\ell(X)$, and therefore inherits the structure of a commutative algebra object of $\mathrm{Shv}_\ell(X)$.

Using Remark 3.4.7.4, we see that the commutative algebra structure on $[\mathcal{Y}]_X$ depends functorially on X. More precisely, if $f : X' \to X$ is a proper morphism of quasi-projective k-schemes, then the equivalence $[\mathcal{Y} \times_X X']_{X'} \simeq f^! [\mathcal{Y}]_X$ of Proposition 3.4.3.2

is an equivalence of commutative algebra objects of $\mathrm{Shv}_\ell(X)$. Applying this observation in the special case where f is the inclusion of a closed point $x \in X$, we see that the costalk of $[\mathcal{Y}]_X$ can be identified with the ℓ-adic cochain complex $C^*(\mathcal{Y}_x; \mathbf{Z}_\ell) \simeq [\mathcal{Y}_x]_{\mathrm{Spec}(k)}$ as an $\mathbb{E}_\infty$-algebra over $\mathbf{Z}_\ell$. $\square$

Chapter 4

Computing the Trace of Frobenius

Let X_0 be an algebraic curve defined over a finite field $\mathbf{F}_q$, and let G_0 be a smooth affine group scheme over X_0 with connected fibers, whose generic fiber is semisimple and simply connected. Fix an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$ and let $X = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} X_0$ and $G = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} G_0$ denote the $\overline{\mathbf{F}}_q$-schemes associated to X_0 and G_0, respectively. Our goal in this chapter is to compute the trace $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$, where ℓ is a prime number which is invertible in $\mathbf{F}_q$.

We will follow the strategy outlined in Chapter 1. If X is an algebraic curve over the field $\mathbf{C}$ of complex numbers and G is a smooth affine group scheme over X whose fibers are semisimple and simply connected, then Theorem 1.5.4.10 (and Example 1.5.4.15) supply a quasi-isomorphism

$$C^*(\mathrm{Bun}_G(X); \mathbf{Q}) \simeq \bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q}), \tag{4.1}$$

whose right hand side is the *continuous tensor product* of Construction 1.5.4.8. In §4.1, we will formulate an ℓ-adic version of (4.1) (Theorem 4.1.2.1) which makes sense over an arbitrary algebraically closed ground field k (of characteristic different from ℓ) and for group schemes G which have bad reduction at finitely many points. We do not prove Theorem 4.1.2.1 here: a proof will appear in a sequel to this book.

The remainder of this chapter is devoted to explaining how Theorem 4.1.2.1 can be used to compute the trace of the arithmetic Frobenius automorphism on the ℓ-adic cohomology of $\mathrm{Bun}_G(X)$. In the case where G and X are defined over a finite field $\mathbf{F}_q \subseteq k$, we can use Theorem 4.1.2.1 to rewrite the cohomological version of Weil's conjecture (Theorem 1.4.4.1) as an equality

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell))) = \prod_{x \in X_0} \mathrm{Tr}(\mathrm{Frob}_x^{-1} \,|\, \mathrm{H}^*(\mathrm{BG}_x; \mathbf{Z}_\ell)). \tag{4.2}$$

Here equation (4.2) can be viewed as a *multiplicative* version of the Grothendieck-Lefschetz trace formula

$$\mathrm{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^*_c(X; \mathcal{F})) = \sum_{x \in X_0} \mathrm{Tr}(\mathrm{Frob}_x \,|\, \mathcal{F}_x), \tag{4.3}$$

where $\mathcal{F}$ denotes an ℓ-adic sheaf on the curve X_0.

We will eventually justify (4.2) by showing that it reduces to (4.3) for an appropriately chosen ℓ-adic sheaf $\mathcal{F} = \mathcal{M}(G)$, which we refer to as the *motive of G relative to X* (Construction 4.5.1.1). The definition of $\mathcal{M}(G)$ involves a version of Koszul duality, which we formalize in §4.2 using the notion of *cotangent fiber* $\cot(A)$ of an augmented commutative algebra (Definition 4.2.1.5). In §4.5, we combine this theory with Theorem 4.1.2.1 to obtain a quasi-isomorphism $(\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))[\ell^{-1}] \simeq C^*(X; \mathcal{M}(G))$ (Proposition 4.5.1.3).

One difficulty we will need to grapple with is that the trace

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)) = \sum (-1)^i \, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^i(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$$

is not *a priori* well-defined, since the cohomology groups $\mathrm{H}^i(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ are generally nonzero for infinitely many values of i. In §4.3 we explain how to circumvent this obstacle using Koszul duality: under appropriate hypotheses, we show that the trace of an automorphism of an augmented commutative algebra A can be determined from the trace of the induced automorphism (and its iterates) on the cotangent fiber $\cot(A)$ (Proposition 4.3.2.1). In §4.4, we combine this result with Proposition 4.5.1.3 and the Grothendieck-Lefschetz trace formula to complete the proof of Theorem 1.4.4.1.

4.1 The Product Formula

Throughout this section, we let k denote an algebraically closed field and ℓ a prime number which is invertible in k. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. Our goal is to formulate an ℓ-adic analogue of the product formula (Theorem 1.5.4.10), which asserts that (under suitable hypotheses) there is a canonical quasi-isomorphism

$$\rho : \bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Z}_\ell) \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell),$$

whose domain is a "continuous" tensor product of chain complexes over $\mathbf{Z}_\ell$. The main difficulty lies in finding a reasonable definition for this tensor product.

We begin by recalling how this difficulty was resolved in Chapter 1 in the classical setting, when X is defined over the field $\mathbf{C}$ of complex numbers and the group scheme G is everywhere semisimple. Let us abuse notation by identifying X with the Riemann surface $X(\mathbf{C})$, and let $\mathcal{U}_0(X)$ denote the collection of open subsets of X which are homeomorphic to $\mathbf{R}^2$. In this case, we studied a functor $\mathcal{B} : \mathcal{U}_0(X) \to \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}$, whose value on an open disk U is given by the polynomial de Rham complex of a classifying space for G-bundles on U. The functor $\mathcal{B}$ can be regarded as a kind of local system on X, taking values in the ∞-category $\mathrm{CAlg}_{\mathbf{Q}}$ of $\mathbb{E}_\infty$-algebras over $\mathbf{Q}$ (this can be formalized using Example 2.1.4.3). There is a pair of adjoint functors

$$\mathrm{hFun}(\mathcal{U}_0(\mathrm{X}), \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}) \underset{\delta}{\overset{\mathrm{hocolim}}{\rightleftarrows}} \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}, \tag{4.4}$$

where δ denotes the diagonal map, and the continuous tensor product $\bigotimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$ was defined as the image of $\mathcal{B}$ under the left adjoint $\mathrm{hocolim} : \mathrm{hFun}(\mathcal{U}_0(\mathrm{X}), \mathrm{CAlg}^{\mathrm{dg}}_{\mathbf{Q}}) \to \mathrm{hCAlg}^{\mathrm{dg}}_{\mathbf{Q}}$ (Construction 1.5.4.8).

Let us now return to working over an arbitrary algebraically closed field k. To any smooth affine group scheme G over X, we can associate a classifying stack BG, which is an algebraic stack equipped with a smooth map $\mathrm{BG} \to X$. In §3.4, we showed that the ℓ-adic cochain complexes $\{C^*(\mathrm{BG}_x; \mathbf{Z}_\ell)\}_{x \in X}$ can be identified with the costalks of a certain ℓ-adic sheaf $[\mathrm{BG}]_X$, which can be regarded as a commutative algebra with respect to the !-tensor product on $\mathrm{Shv}_\ell(X)$ (Theorem 3.4.0.3). Let $\pi : X \to \mathrm{Spec}(k)$ denote the projection map. Then the exceptional inverse image functor

$$\pi^! : \mathrm{Mod}_{\mathbf{Z}_\ell} \simeq \mathrm{Shv}_\ell(\mathrm{Spec}(k)) \to \mathrm{Shv}_\ell(X)$$

is symmetric monoidal (if we regard $\mathrm{Shv}_\ell(X)$ as equipped with the !-tensor product), and therefore carries commutative algebra objects to commutative algebra objects. In §4.1.1, we show that $\pi^!$ participates in an adjunction

$$\mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \underset{\pi^!}{\overset{\int_X}{\rightleftarrows}} \mathrm{CAlg}_{\mathbf{Z}_\ell},$$

which we can regard as an algebro-geometric analogue of (4.4). Here the functor $\int_X : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}_{\mathbf{Z}_\ell}$ is an incarnation of *factorization homology* (of a particularly simple type: it is the factorization homology in the setting of *commutative* factorization algebras), which we will discuss in detail in the sequel to this book.

Once all of our definitions are in place, it will be easy to formulate an ℓ-adic analogue of Theorem 1.5.4.10. For any smooth affine group scheme $G \to X$, the factorization homology $\int_X [\mathrm{BG}]_X$ can be regarded as an ℓ-adic incarnation of the continuous tensor product $\otimes_{x \in X} C^*(\mathrm{BG}_x; \mathbf{Q})$ of Construction 1.5.4.8. The universal property of $\int_X [\mathrm{BG}]_X$ then provides a comparison map $\rho : \int_X [\mathrm{BG}]_X \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$. In §4.1.2, we will describe some hypotheses which guarantee that ρ is a quasi-isomorphism (Theorem 4.1.2.1).

4.1.1 Factorization Homology

We begin with some general remarks. Let $f : X \to Y$ be a proper morphism between quasi-projective k-schemes, and let $f^! : \mathrm{Shv}_\ell(Y) \to \mathrm{Shv}_\ell(X)$ denote the exceptional inverse image functor of Notation 2.3.4.5. Then we can regard $f^!$ as a symmetric monoidal functor, where we regard $\mathrm{Shv}_\ell(X)$ and $\mathrm{Shv}_\ell(Y)$ as equipped with the symmetric monoidal structures given by the !-tensor product (see Theorem 3.3.0.3). It follows that $f^!$ determines a functor on commutative algebra objects

$$f^! : \mathrm{CAlg}(\mathrm{Shv}_\ell(Y)) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(X)).$$

Proposition 4.1.1.1. *Let $f : X \to Y$ be a proper morphism of quasi-projective k-schemes. Then the functor $f^! : \mathrm{CAlg}(\mathrm{Shv}_\ell(Y)) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(X))$ admits a left adjoint $f_+ : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(Y))$.*

Warning 4.1.1.2. When regarded as a functor from $\mathrm{Shv}_\ell(Y)$ to $\mathrm{Shv}_\ell(X)$, the functor $f^!$ is *defined* as a right adjoint to the direct image functor $f_* : \mathrm{Shv}_\ell(X) \to \mathrm{Shv}_\ell(Y)$. However, the functor f_* does not carry commutative algebras of $\mathrm{Shv}_\ell(X)$ (with respect to the !-tensor product) to commutative algebras of $\mathrm{Shv}_\ell(Y)$. Consequently, the functor f_* is quite different from the functor f_+ appearing in Proposition 4.1.1.1. For example, suppose that $X = Y \amalg Y$ is a disjoint union of two copies of Y, so that $\mathrm{Shv}_\ell(X)$ can be identified with a product $\mathrm{Shv}_\ell(Y) \times \mathrm{Shv}_\ell(Y)$. In this case, the functor f_* is given by the formula $f_*(\mathcal{F}, \mathcal{G}) = \mathcal{F} \oplus \mathcal{G}$, while f_+ is given by $f_+(\mathcal{F}, \mathcal{G}) = \mathcal{F} \otimes^! \mathcal{G}$.

Proof of Proposition 4.1.1.1. As a functor from $\mathrm{Shv}_\ell(Y)$ to $\mathrm{Shv}_\ell(X)$, the functor $f^!$ can be defined as the right adjoint to f_*, and therefore preserves small limits. Moreover, it also preserves filtered colimits (see Notation 2.3.4.5). Since the forgetful functors

$$\mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{Shv}_\ell(X) \qquad \mathrm{CAlg}(\mathrm{Shv}_\ell(Y)) \to \mathrm{Shv}_\ell(Y)$$

preserve small limits and filtered colimits, it follows that the functor

$$f^! : \mathrm{CAlg}(\mathrm{Shv}_\ell(Y)) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(X))$$

also preserves small limits and filtered colimits. The existence of the left adjoint f_+ now follows from the ∞-categorical version of the adjoint functor theorem (Corollary [25].5.5.2.9). □

Definition 4.1.1.3 (Factorization Homology)**.** Let X be a projective k-scheme with structural morphism $f : X \to \mathrm{Spec}(k)$, and regard the ∞-category $\mathrm{Shv}_\ell(X)$ as endowed with the !-tensor product. We let

$$\int_X : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}_{\mathbf{Z}_\ell}$$

denote the composition of the functor $f_+ : \mathrm{CAlg}(\mathrm{Shv}_\ell(X)) \to \mathrm{CAlg}(\mathrm{Shv}_\ell(\mathrm{Spec}(k)))$ with the equivalence $\mathrm{CAlg}(\mathrm{Shv}_\ell(\mathrm{Spec}(k))) \simeq \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell}) = \mathrm{CAlg}_{\mathbf{Z}_\ell}$. If $\mathcal{B}$ is a commutative algebra object of $\mathrm{Shv}_\ell(X)$, we will refer to $\int_X \mathcal{B}$ as the *factorization homology of* $\mathcal{B}$.

Remark 4.1.1.4. Our presentation of Definition 4.1.1.3 is rather unsatisfying: the factorization homology of a commutative algebra $\mathcal{A} \in \mathrm{CAlg}(\mathrm{Shv}_\ell(X))$ is defined using the functor f_+ of Proposition 4.1.1.1, which we proved by invoking an abstract existence result. However, the factorization homology functor $\int_X$ (and, more generally, the functor f_+ associated to a proper morphism) also admits a concrete description, given by *integration over the Ran space* $\mathrm{Ran}(X)$. This concrete description will play an essential role in our proof of the product formula (Theorem 4.1.2.1), but not in our application of it. We therefore postpone a discussion to the sequel to this book.

Remark 4.1.1.5. Let X be a projective k-scheme, let $\mathcal{B}$ be a commutative algebra object of $\mathrm{Shv}_\ell(X)$, and let $f : X \to \mathrm{Spec}(k)$ be the projection map. We then have a unit map $u : \mathcal{B} \to f^! \int_X \mathcal{B}$ (of commutative algebra objects of $\mathrm{Shv}_\ell(X)$). For each closed point $x \in X$, passing to costalks at x yields a map $u_x : x^! \mathcal{B} \to \int_X \mathcal{B}$ of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}_\ell$. Heuristically, we can think of the maps u_x as exhibiting $\int_X \mathcal{B}$ as a "continuous tensor product" $\bigotimes_{x \in X} x^! \mathcal{B}$ of the costalks of $\mathcal{B}$ (compare with Construction 1.5.4.8).

4.1.2 Formulation of the Product Formula

We now return to the setting of Weil's conjecture. Let X be an algebraic curve over k and let G be a smooth affine group scheme over X. To the group scheme G, we can associate two algebraic stacks:

- The classifying stack BG, which is equipped with a smooth map $\pi : \mathrm{BG} \to X$.
- The moduli stack $\mathrm{Bun}_G(X)$ of Construction 1.4.1.1.

Moreover, we have a tautological evaluation map $e : X \times_{\mathrm{Spec}(k)} \mathrm{Bun}_G(X) \to \mathrm{BG}$. Let $\pi : X \to \mathrm{Spec}(k)$ be the projection map. Using Theorem 3.4.0.3, we obtain a canonical map

$$[\mathrm{BG}]_X \xrightarrow{e^*} [X \times_{\mathrm{Spec}(k)} \mathrm{Bun}_G(X)]_X \simeq \pi^! C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

of commutative algebra objects of $\mathrm{Shv}_\ell(X)$ (with respect to the !-tensor product). This determines a comparison map

$$\rho : \int_X [\mathrm{BG}]_X \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell).$$

The main ingredient in our proof of Weil's conjecture is the following theorem:

Theorem 4.1.2.1. *Let G be a smooth affine group scheme over X with connected fibers, whose generic fiber is semisimple and simply connected. Then the comparison map*

$$\rho : \int_X [\mathrm{BG}]_X \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

is an equivalence in the ∞-category $\mathrm{Mod}_{\mathbf{Z}_\ell}$. In particular, the ℓ-adic cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ can be identified with the homology of the chain complex $\int_X [\mathrm{BG}]_X$.

The proof of Theorem 4.1.2.1 will appear in a sequel to this book.

Remark 4.1.2.2. Theorem 4.1.2.1 can be regarded as an algebro-geometric analogue of Theorem 1.5.4.10 (or, more precisely, of the special case of Theorem 1.5.4.10 described in Example 1.5.4.15). However, it is in some respects more general than Theorem 1.5.4.10: here we allow the group scheme G to have bad reduction at finitely many points.

Remark 4.1.2.3. Theorem 4.1.2.1 can be regarded as a characterization of the ℓ-adic cochain complex $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ as an object of the ∞-category $\mathrm{CAlg}_{\mathbf{Z}_\ell}$ of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}_\ell$. It can be stated more concretely as follows: for every $\mathbb{E}_\infty$-algebra A over $\mathbf{Z}_\ell$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathbf{Z}_\ell}}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), A) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X \otimes \underline{A}),$$

where $\omega_X \otimes \underline{A} \simeq \pi^! A$ denotes the constant commutative algebra object of $\mathrm{Shv}_\ell(X)$ associated to A.

4.2 The Cotangent Fiber

Let $X = \mathrm{Spec}(A)$ be an affine algebraic variety over a field k and let $x \in X(k)$ be a k-valued point of X, corresponding to a k-algebra homomorphism $\epsilon : A \to k$. We let $\mathfrak{m}_x = \ker(\epsilon)$, so that $\mathfrak{m}_x$ is a maximal ideal of A. The *Zariski cotangent space* of X at the point x is defined as the quotient $\mathfrak{m}_x / \mathfrak{m}_x^2$. For each $n \geq 0$, there is an evident surjective map

$$\mathrm{Sym}^n(\mathfrak{m}_x/\mathfrak{m}_x^2) \to \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1},$$

which is an isomorphism if X is smooth at the point x. Consequently, the structure of the completed local ring $\widehat{A} = \varprojlim A/\mathfrak{m}_x^n$ is in some sense controlled by the finite-dimensional vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Now suppose that $\mathcal{C}$ is a symmetric monoidal stable ∞-category. Then one can consider commutative algebra objects $A \in \mathcal{C}$ equipped with an augmentation $\epsilon : A \to \mathbf{1}$ (here $\mathbf{1}$ denotes the unit object of $\mathcal{C}$). For every such pair (A, ϵ), one can consider an analogue of the Zariski cotangent space, which we will refer to as the *cotangent fiber* of A and denote by $\cot(A)$ (Definition 4.2.1.5). Our goal in this section is to review some elementary properties of the construction $A \mapsto \cot(A)$ which will be useful in our proof of Weil's conjecture.

Notation 4.2.0.1. Throughout this section, we let $\mathcal{C}$ denote a fixed symmetric monoidal ∞-category. We will assume that $\mathcal{C}$ is stable, admits small colimits, and that the tensor product functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

preserves colimits separately in each variable. Let $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-category of commutative algebra objects of $\mathcal{C}$ and let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, which we identify with the initial object of $\mathrm{CAlg}(\mathcal{C})$.

4.2.1 Augmented Commutative Algebras

Let A be a commutative algebra object of $\mathcal{C}$. An *augmentation on* A is a map of commutative algebra objects $\epsilon : A \to \mathbf{1}$. An *augmented commutative algebra object of* $\mathcal{C}$ is a pair (A, ϵ), where A is a commutative algebra object of $\mathcal{C}$, and ϵ is an augmentation on A. The collection of augmented commutative algebra objects of $\mathcal{C}$ can be organized into an ∞-category which we will denote by $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) = \mathrm{CAlg}(\mathcal{C})_{/\mathbf{1}}$.

If (A, ϵ) is an augmented commutative algebra object of $\mathcal{C}$, we let $\mathfrak{m}_A$ denote the fiber of the augmentation map $\epsilon : A \to \mathbf{1}$. We will refer to $\mathfrak{m}_A$ as the *augmentation ideal* of A. Note that $\mathfrak{m}_A$ inherits the structure of a nonunital commutative algebra object of $\mathcal{C}$. Moreover, the construction $A \mapsto \mathfrak{m}_A$ determines an equivalence from the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$ of augmented commutative algebra objects of $\mathcal{C}$ to the ∞-category $\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C})$ of nonunital commutative algebra objects of $\mathcal{C}$ (see Proposition [23].5.4.4.10).

Definition 4.2.1.1. Let $\mathrm{Fin}^{\mathrm{s}}$ denote the category whose objects are finite sets and whose morphisms are surjective maps. For each integer $n \geq 0$, we let $\mathrm{Fin}^{s}_{\geq n}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by those finite sets which have cardinality $\geq n$.

Suppose that A is an augmented commutative algebra object of $\mathcal{C}$. Then the construction $S \mapsto \mathfrak{m}_A^{\otimes S}$ determines a functor $\mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$. For each integer $n > 0$, we let $\mathfrak{m}_A^{(n)}$ denote the colimit $\varinjlim_{S \in \mathrm{Fin}^{s}_{\geq n}} \mathfrak{m}^{\otimes S}$. By convention, we set $\mathfrak{m}_A^{(0)} = A$.

Example 4.2.1.2. The category $\mathrm{Fin}^{s}_{\geq 1}$ has a final object, given by a 1-element set. It follows that for every augmented commutative algebra object A of $\mathcal{C}$, we have a canonical equivalence $\mathfrak{m}_A^{(1)} \simeq \mathfrak{m}_A$.

Example 4.2.1.3. Let k be a field and let A be an augmented commutative algebra over k (which we regard as a chain complex concentrated in degree zero), with augmentation ideal $\mathfrak{m}_A$. Then we can regard $\mathfrak{m}_A$ as a nonunital commutative algebra object of the symmetric monoidal ∞-category Mod_k. Then we can think of the object $\mathfrak{m}_A^{(n)} \in \mathrm{Mod}_k$ as a "derived version" of the usual nth power ideal $\mathfrak{m}_A^n \subseteq A$. Multiplication in A determines a compatible family of maps $\{\mathfrak{m}_A^{\otimes S} \to \mathfrak{m}_A^n\}_{S \in \mathrm{Fin}^s_{\geq n}}$, which can be amalgamated to give a map $\mathfrak{m}_A^{(n)} \to \mathfrak{m}_A^n$. One can show that this map is an equivalence if k is of characteristic zero and A is smooth over k (this follows from Proposition 4.2.4.2 below).

Remark 4.2.1.4. Let A be an augmented commutative algebra object of $\mathcal{C}$. Then the inclusions of categories

$$\cdots \mathrm{Fin^s}_{\geq 3} \hookrightarrow \mathrm{Fin^s}_{\geq 2} \hookrightarrow \mathrm{Fin^s}_{\geq 1}$$

determine maps

$$\cdots \to \mathfrak{m}_A^{(3)} \to \mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} \simeq \mathfrak{m}_A,$$

depending functorially on A.

Definition 4.2.1.5. Let A be an augmented commutative algebra object of $\mathcal{C}$. We let $\mathrm{cot}(A)$ denote the cofiber of the canonical map $\mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} \simeq \mathfrak{m}_A$. We will refer to $\mathrm{cot}(A)$ as the *cotangent fiber* of A.

Remark 4.2.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be stable symmetric monoidal ∞-categories which admit small colimits and for which the tensor product functors

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad \otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

preserve colimits separately in each variable, and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Then F carries augmented commutative algebra objects A of $\mathcal{C}$ to augmented commutative algebra objects $F(A)$ of $\mathcal{D}$. If F preserves colimits, then we have a canonical equivalence $\mathrm{cot}(F(A)) \simeq F(\mathrm{cot}(A))$ for each $A \in \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$.

Example 4.2.1.7. Let V be an object of $\mathcal{C}$ and let $\mathrm{Sym}^*(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$ denote the free commutative algebra object of $\mathcal{C}$ generated by V (here $\mathrm{Sym}^n(\bullet)$ denotes the derived nth symmetric power functor, which carries an object $V \in \mathcal{C}$ to the coinvariants for the action of the symmetric group Σ_n on $V^{\otimes n}$). The zero map $V \to \mathbf{1}$ determines an augmentation $\epsilon : \mathrm{Sym}^*(V) \to \mathbf{1}$, whose fiber is given by $\mathrm{Sym}^{>0}(V) \simeq \bigoplus_{n > 0} \mathrm{Sym}^n(V)$.

For any finite set S, we can identify $\mathrm{Sym}^{>0}(V)^{\otimes S}$ with the colimit $\varinjlim_{f:T\to S} V^{\otimes T}$, where the colimit is taken over all surjections $f : T \to S$. For $n > 0$, we compute

$$\begin{aligned} \mathrm{Sym}^{>0}(V)^{(n)} &\simeq \varinjlim_{|S|\geq n} \mathrm{Sym}^{>0}(V)^{\otimes S} \\ &\simeq \varinjlim_{|S|\geq n} \varinjlim_{f:T\to S} V^{\otimes T} \\ &\simeq \varinjlim_{T} \varinjlim_{f:T\to S, |S|\geq n} V^{\otimes T} \\ &\simeq \varinjlim_{T} \begin{cases} V^{\otimes T} & \text{if } |T| \geq n \\ 0 & \text{if } |T| < n. \end{cases} \\ &\simeq \bigoplus_{m\geq n} \mathrm{Sym}^m(V). \end{aligned}$$

Here in each colimit, we allow T to range over the category of finite sets and bijections and f to range over all surjections. In particular, we have a canonical equivalence $\cot(\mathrm{Sym}^*(V)) \simeq V$.

4.2.2 Cotangent Fibers and Square-Zero Extensions

Our next goal is to show that if (A, ϵ) is an augmented commutative algebra object of $\mathcal{C}$, then the cotangent fiber $\mathrm{cofib}(A)$ can be characterized by a universal property (Proposition 4.2.2.2). First, we need to establish a formal property of Definition 4.2.1.5:

Proposition 4.2.2.1. *The formation of cotangent fibers determines a functor* $\cot : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ *which preserves colimits.*

Proof. To show that the functor cot preserves all colimits, it will suffice to show that it preserves sifted colimits and finite coproducts. Since the tensor product on $\mathcal{C}$ preserves colimits separately in each variable, the functor $V \mapsto V^{\otimes S}$ preserves sifted colimits for every finite set S. It follows that the construction $A \mapsto \cot(\mathfrak{m}_A^{(n)})$ commutes with sifted colimits for each n, so that $A \mapsto \cot(A)$ commutes with sifted colimits. Since the functor cot clearly preserves initial objects, we are reduced to showing that it preserves pairwise coproducts. Let A and B be augmented commutative algebra objects of $\mathcal{C}$; we wish to show that the canonical map

$$\cot(A) \oplus \cot(B) \to \cot(A \otimes B)$$

is an equivalence. Resolving the augmentation ideal $\mathfrak{m}_A$ by free augmented commutative algebras, we can reduce to the case where $A \simeq \mathrm{Sym}^*(V)$ for some object $V \in \mathcal{C}$. Similarly, we may suppose that $B \simeq \mathrm{Sym}^*(W)$ for some $W \in \mathcal{C}$. In this case, the desired result follows from Example 4.2.1.7. □

For each object $V \in \mathcal{C}$, let $\mathbf{1} \oplus V$ denote the trivial square-zero extension of $\mathbf{1}$ by V. The construction $V \mapsto \mathbf{1} \oplus V$ determines a functor $\Omega^\infty : \mathcal{C} \to \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$.

Proposition 4.2.2.2. *The construction* $\cot : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ *is left adjoint to the formation of trivial square-zero extensions* $V \mapsto \mathbf{1} \oplus V$*. In other words, for every augmented commutative algebra* $A \in \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$ *and every object* $V \in \mathcal{C}$*, we have a canonical homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})}(A, \mathbf{1} \oplus V) \simeq \mathrm{Map}_{\mathcal{C}}(\cot(A), V).$$

Proof. Theorem [23].7.3.4.13 implies that the functor Ω^∞ exhibits $\mathcal{C}$ as a stabilization of the ∞-category $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$. In particular, we have an adjunction

$$\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathcal{C},$$

where $\Sigma^\infty : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \to \mathcal{C}$ denotes the absolute cotangent complex functor introduced in Definition [23].7.3.2.14. The functor Σ^∞ is universal among colimit-preserving functors from $\mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$ to stable ∞-categories. It follows from Proposition 4.2.2.1 that the formation of cotangent fibers factors as a composition

$$\cot : \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C}) \xrightarrow{\Sigma^\infty} \mathcal{C} \xrightarrow{\lambda} \mathcal{C},$$

where λ is some functor from $\mathcal{C}$ to itself. Using Example 4.2.1.7, we obtain equivalences of functors

$$\mathrm{id}_{\mathcal{C}} \simeq \cot \circ \mathrm{Sym}^* \simeq \lambda \circ (\Sigma^\infty \circ \mathrm{Sym}^*) \simeq \lambda,$$

so that λ is equivalent to the identity functor. □

4.2.3 Examples of Cotangent Fibers

Let k be a field and let A be an augmented commutative algebra object of Mod_k. We can identify $I = \mathrm{H}^*(\mathfrak{m}_A)$ with a maximal ideal in the graded-commutative ring $\mathrm{H}^*(A)$. We may therefore consider the (purely algebraic) Zariski cotangent space I/I^2. Note that I^2 is contained in the image of the map

$$\mathrm{H}^*(\mathfrak{m}_A^{\otimes 2}) \to \mathrm{H}^*(\mathfrak{m}_A) = I,$$

and therefore also in the kernel of the map $\mathrm{H}^*(\mathfrak{m}_A) \to \mathrm{H}^*(\cot(A))$. We therefore obtain a comparison map $I/I^2 \to \mathrm{H}^*(\cot(A))$.

Proposition 4.2.3.1. *Let* k *be a field of characteristic zero, let* A *be an augmented commutative algebra object of* CAlg_k*, and suppose that the cohomology* $\mathrm{H}^*(A)$ *is a graded polynomial ring (that is,* $\mathrm{H}^*(A)$ *is a tensor product of a polynomial ring on generators of even degree and an exterior algebra on generators of odd degree). Then the comparison map* $I/I^2 \to \mathrm{H}^*(\cot(A))$ *is an isomorphism.*

Proof. Choose homogeneous polynomial generators $\{t_i\}_{i \in I}$ of $\mathrm{H}^*(A)$ which are annihilated by the augmentation map $\epsilon : A \to k$. Let V denote the graded vector space freely generated by homogeneous elements $\{T_i\}_{i \in I}$ with $\deg(T_i) = \deg(t_i)$ and regard V as a chain complex with trivial differential. Then we can choose a map of chain complexes $\phi_0 : V \to \mathfrak{m}_A$ which carries each T_i to a cycle representing the homology class t_i. Then ϕ_0 extends to a map of augmented commutative algebras $\phi : \mathrm{Sym}^*(V) \to A$. The assumption that k has characteristic zero guarantees that the cohomology of $\mathrm{Sym}^*(V)$ is a graded polynomial ring on the generators T_i, so that ϕ is an equivalence. It follows from Example 4.2.1.7 that ϕ determines an equivalence

$$V \simeq \cot(\mathrm{Sym}^*(V)) \to \cot(A)$$

in Mod_k. □

Example 4.2.3.2. Let k be a field of characteristic zero (or, more generally, any $\mathbf{Q}$-algebra), and let A be an augmented commutative k-algebra with maximal ideal $\mathfrak{m}_A$. The *cotangent complex* $L_{A/k}$ is a chain complex of A-modules, obtained from the simplicial A-module $A \otimes_{P^\bullet} \Omega_{P^\bullet/k}$, where $P^\bullet$ is a simplicial resolution of A by free k-algebras. One can show that the cotangent fiber $\cot(A)$ is given by the (derived) tensor product $k \otimes_A L_{A/k}$. This follows from Proposition 4.2.3.1 when A is a free algebra over k, and the general case can be reduced to the case of free algebras using Proposition 4.2.2.1 below.

More generally, if k is arbitrary commutative ring and $A \in \mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Mod}_k)$, then the cotangent fiber $\cot(A)$ can be identified with the tensor product $k \otimes_A L^t_{A/k}$, where $L^t_{A/k}$ denotes the complex of *topological* André-Quillen chains of A over k (Proposition 4.2.2.2).

Example 4.2.3.3 (Rational Homotopy Theory)**.** Let X be a simply connected topological space, and assume that the cohomology ring $\mathrm{H}^*(X; \mathbf{Q})$ is finite-dimensional in each degree. Let $x \in X$ be a base point, so that x determines an augmentation $C^*(X; \mathbf{Q}) \to C^*(\{x\}; \mathbf{Q}) \simeq \mathbf{Q}$ of commutative algebra objects of the ∞-category $\mathrm{Mod}_{\mathbf{Q}}$. We will denote the augmentation ideal by $C^*_{\mathrm{red}}(X; \mathbf{Q})$. Then the cotangent fiber of $C^*_{\mathrm{red}}(X; \mathbf{Q})$ is a cochain complex M. One can show that the cohomologies of this chain complex are given by

$$\mathrm{H}^n(M) = \mathrm{Hom}(\pi_n(X, x), \mathbf{Q}).$$

Remark 4.2.3.4. Let k be a field of characteristic zero, and let A be an augmented commutative algebra object of Mod_k. One can show that the shifted dual $\Sigma^{-1} \cot(A)^\vee$ of the cotangent fiber $\cot(A)$ is quasi-isomorphic to the underlying chain complex of a differential graded Lie algebra which depends functorially on A. In other words, the construction $A \mapsto \Sigma^{-1} \cot(A)^{\vee}$ determines a contravariant functor from the ∞-category of augmented commutative algebra objects of Mod_k to the ∞-category of differential graded Lie algebras over k. This construction is adjoint to the functor $\mathfrak{g} \mapsto C^*(\mathfrak{g})$ which carries a differential graded Lie algebra $\mathfrak{g}$ to the Chevalley-Eilenberg complex which computes the Lie algebra cohomology of $\mathfrak{g}$. See §[24].IV.2 for more details.

4.2.4 The $\mathfrak{m}$-adic Filtration

Let k be a field of characteristic zero and let A be an augmented commutative algebra object of Mod_k. In good cases, one can recover the tower $\{\mathrm{cofib}(\mathfrak{m}_A^{(n)} \to A)\}_{n \geq 1}$ from the cotangent fiber $\cot(A)$ together with the Lie algebra structure on $\Sigma^{-1} \cot(A)^{\vee}$. However, for our applications in this book, it will be sufficient to describe the *successive quotients* of the filtration $\mathfrak{m}_A^{(n)}$. This does not require us to consider Lie algebras structure at all, and works without any restrictions on A or $\mathcal{C}$:

Construction 4.2.4.1. Let A be an augmented commutative algebra object of $\mathcal{C}$, so that its cotangent fiber is given by

$$\cot(A) \simeq \mathrm{cofib}(\varinjlim_{|T| \geq 2} \mathfrak{m}_A^{\otimes T} \to \varinjlim_{|T| \geq 1} \mathfrak{m}_A^{\otimes T}).$$

An easy calculation shows that for every finite set S, we can identify $\cot(\mathfrak{m}_A)^{\otimes S}$ with the cofiber

$$\mathrm{cofib}(\varinjlim_{f : T \xrightarrow{\not\sim} S} \mathfrak{m}^{\otimes T} \to \varinjlim_{f : T \to S} \mathfrak{m}_A^{\otimes T}),$$

where the colimits are taken over the category of all finite sets T equipped with a surjection $f : T \to S$ (which, on the left hand side, is required to be non-bijective).

Let $\mathcal{J}$ denote the category whose objects are finite sets T equipped with an equivalence relation E such that $|T/E| = n$, where a morphism from (T, E) to (T', E') is a surjection of finite sets $\alpha : T \to T'$ such that xEy if and only if $\alpha(x) E' \alpha(y)$. Let $\mathcal{J}_0$ denote the full subcategory of $\mathcal{J}$ spanned by those pairs (T, E) where $|T| > n$. Then the above considerations determine an equivalence

$$\mathrm{Sym}^n \cot(\mathfrak{m}_A) \simeq \mathrm{cofib}(\varinjlim_{(T,E) \in \mathcal{J}_0} \mathfrak{m}_A^{\otimes T} \to \varinjlim_{(T,E) \in \mathcal{J}} \mathfrak{m}_A^{\otimes T}).$$

We have an evident commutative diagram

$$\begin{array}{ccc} \mathcal{J}_0 & \longrightarrow & \mathcal{J} \\ \downarrow & & \downarrow \\ \mathrm{Fin}^s_{\geq n+1} & \longrightarrow & \mathrm{Fin}^s_{\geq n}, \end{array}$$

which determines a map

$$\theta : \mathrm{Sym}^n \cot(\mathfrak{m}_A) \to \mathrm{cofib}(\mathfrak{m}_A^{(n+1)} \to \mathfrak{m}_A^{(n)}).$$

Proposition 4.2.4.2. *Let A be an augmented commutative algebra object of $\mathcal{C}$. Then, for each integer $n \geq 0$, Construction 4.2.4.1 determines an equivalence* $\mathrm{Sym}^n \cot(A) \to \mathrm{cofib}(\mathfrak{m}_A^{(n+1)} \to \mathfrak{m}_A^{(n)})$. *In other words, we have a fiber sequence*

$$\mathfrak{m}_A^{(n+1)} \to \mathfrak{m}_A^{(n)} \to \mathrm{Sym}^n \cot(\mathfrak{m}_A).$$

Proof. The case $n = 0$ follows immediately from our convention $\mathfrak{m}_A^{(0)} = A$. We will therefore assume $n > 0$. Let $F : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$ denote the functor given by $F(S) = \mathfrak{m}_A^S$. For every category $\mathcal{I}$ equipped with a forgetful functor $\mathcal{I} \to \mathrm{Fin}^{\mathrm{s}}$, we let $F|_{\mathcal{I}}$ denote the restriction of F to $\mathcal{I}$, and $\varinjlim(F|_{\mathcal{I}})$ the colimit of $F|_{\mathcal{I}}$ (regarded as a diagram in $\mathcal{C}$). Unwinding the definitions, we wish to prove that the diagram σ :

$$\begin{array}{ccc} \varinjlim(F|_{\mathcal{J}_0}) & \longrightarrow & \varinjlim(F|_{\mathcal{J}}) \\ \downarrow & & \downarrow \\ \varinjlim(F|_{\mathrm{Fin}^s_{\geq n+1}}) & \longrightarrow & \varinjlim(F|_{\mathrm{Fin}^s_{\geq n}}) \end{array}$$

is a pushout diagram in the ∞-category $\mathcal{C}$. We will show that this holds for *any* functor $F : \mathrm{Fin}^{\mathrm{s}} \to \mathcal{C}$.

Let $F' : \mathrm{Fin}^s_{\geq n} \to \mathcal{C}$ be a left Kan extension of the functor $F|_{\mathrm{Fin}^s_{\geq n+1}}$ along the inclusion

$$\mathrm{Fin}^s_{\geq n+1} \hookrightarrow \mathrm{Fin}^s_{\geq n}.$$

Let $U : \mathcal{J} \to \mathrm{Fin}^s_{\geq n}$ denote the forgetful functor. Note that for every object $(T, E) \in \mathcal{J}$, the functor U induces an equivalence of categories $\mathcal{J}_{/(T,E)} \to (\mathrm{Fin}^s_{\geq n})_{/T}$. It follows that $F' \circ U$ is a left Kan extension of $F|_{\mathcal{J}_0}$ along the inclusion $\mathcal{J}_0 \hookrightarrow \mathcal{J}$. We may therefore identify σ with the commutative diagram

$$\begin{array}{ccc} \varinjlim_{(T,E)\in\mathcal{J}} F'(T) & \longrightarrow & \varinjlim_{(T,E)\in\mathcal{J}} F(T) \\ \downarrow & & \downarrow \\ \varinjlim_{T\in\mathrm{Fin}^s_{\geq n}} F'(T) & \longrightarrow & \varinjlim_{T\in\mathrm{Fin}^s_{\geq n}} F(T). \end{array}$$

For $T \in \mathrm{Fin}^s_{\geq n}$, let $F''(T)$ denote the cofiber of the canonical map $F'(T) \to F(T)$. Unwinding the definitions, we are reduced to proving that the map

$$\theta : \varinjlim_{(T,E)\in\mathcal{J}} F''(T) \to \varinjlim_{T\in\mathrm{Fin}^{\mathrm{s}}{}_{\geq n}} F''(T)$$

is an equivalence. Let $\mathrm{Fin}^s_{=n}$ denote the full subcategory of $\mathrm{Fin}^{\mathrm{s}}$ spanned by those sets having cardinality n, and let $\mathcal{J}_{=n} \subseteq \mathcal{J}$ denote the inverse image of $\mathrm{Fin}^s_{=n}$ under U. Note that $F''(T) \simeq 0$ if $|T| > n$, so that F'' is a left Kan extension of its restriction to $\mathrm{Fin}^s_{=n}$ and $F'' \circ U$ is a left Kan extension of its restriction to $\mathcal{J}_{=n}$. We may therefore identify θ with the canonical map

$$\varinjlim_{(T,E)\in\mathcal{J}_{=n}} F''(T) \to \varinjlim_{T\in\mathrm{Fin}^s_{=n}} F''(T).$$

This map is an equivalence because U induces an equivalence of categories $\mathcal{J}_{=n} \to \mathrm{Fin}^s_{=n}$. □

4.2.5 Convergence of the $\mathfrak{m}$-adic Filtration

Let A be an augmented commutative algebra object of $\mathcal{C}$. It follows from Proposition 4.2.4.2 that the successive quotients of the filtration

$$\cdots \to \mathfrak{m}_A^{(3)} \to \mathfrak{m}_A^{(2)} \to \mathfrak{m}_A^{(1)} \to \mathfrak{m}_A^{(0)} = A$$

can be functorially recovered from the cotangent fiber $\cot(A)$. We next study a condition which guarantees that this filtration is convergent, so that information about the cotangent fiber $\cot(A)$ gives information about the algebra A itself.

Proposition 4.2.5.1. *Suppose that $\mathcal{C} = \mathrm{Mod}_k$, where k is a field of characteristic zero. Let A be an augmented commutative algebra object of $\mathcal{C}$ whose augmentation ideal $\mathfrak{m}_A$ belongs to $(\mathrm{Mod}_k)_{\leq -1}$. Then, for every integer $n > 0$, the object $\mathfrak{m}_A^{(n)}$ belongs to $(\mathrm{Mod}_k)_{\leq -n}$. In particular, the inverse limit $\varprojlim \mathfrak{m}_A^{(n)}$ vanishes in $\mathcal{C}$.*

The proof of Proposition 4.2.5.1 depends on the following elementary combinatorial fact about t-structures:

Lemma 4.2.5.2. *Let $\mathcal{C}$ be a stable ∞-category which admits small colimits, equipped with a t-structure which is compatible with filtered colimits (that is, the full subcategory $\mathcal{C}_{\leq 0}$ is closed under filtered colimits). Let P be a partially ordered set, let $\lambda : P \to \mathbf{Z}_{\geq 0}$ be a strictly monotone function, and suppose we are given a functor $G : \mathrm{N}(P)^{\mathrm{op}} \to \mathcal{C}$ such that $G(x) \in \mathcal{C}_{\leq -n-\lambda(x)}$ for each $x \in P$. Then the colimit $\varinjlim G$ belongs to $\mathcal{C}_{\leq -n}$.*

Proof of Proposition 4.2.5.1. We define a category $\mathcal{I}$ as follows:

- The objects of $\mathcal{I}$ are diagrams

$$S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_d} S_d,$$

  where each S_i is a finite set of cardinality $\geq n$, and each of the maps ϕ_i is surjective but not bijective.

- Let $\vec{S} = (S_0 \to S_1 \to \cdots \to S_d)$ and $\vec{T} = (T_0 \to T_1 \to \cdots \to T_e)$ be objects of $\mathcal{I}$. A morphism from $\vec{S}$ to $\vec{T}$ in $\mathcal{I}$ consists of a map $\alpha : \{0, \ldots, e\} \to \{0, \ldots, d\}$, together with a collection of bijections $S_{\alpha(i)} \simeq T_i$ for which the diagrams

$$\begin{array}{ccc} S_{\alpha(i)} & \longrightarrow & S_{\alpha(i+1)} \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T_{i+1} \end{array}$$

commute.

We have an evident forgetful functor $\rho : \mathcal{I} \to \mathrm{Fin}^s_{\geq n}$, given by $(S_0 \to \cdots \to S_d) \mapsto S_0$. We first prove:

$(*)$ The functor ρ is left cofinal.

Fix a finite set T of cardinality $\geq n$, and define

$$\mathcal{I}_{T/} = \mathcal{I} \times_{\mathrm{Fin}^s_{\geq n}} (\mathrm{Fin}^s_{\geq n})_{T/}.$$

To prove $(*)$, we must show that each of the categories $\mathcal{I}_{T/}$ has weakly contractible nerve. Unwinding the definitions, we can identify objects of $\mathcal{I}_{T/}$ with chains of surjections

$$T \xrightarrow{\psi} S_0 \xrightarrow{\phi_1} S_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_d} S_d,$$

where the maps ϕ_i are not bijective. Let $\mathcal{I}^{\circ}_{T/}$ denote the full subcategory of $\mathcal{I}_{T/}$ spanned by those objects for which ψ is bijective. Since the inclusion $\mathcal{I}^{\circ}_{T/} \hookrightarrow \mathcal{I}_{T/}$ admits a right adjoint, it will suffice to prove that the category $\mathcal{I}^{\circ}_{T/}$ has weakly contractible nerve. This is clear, since $\mathcal{I}^{\circ}_{T/}$ contains a final object (given by the map $T \xrightarrow{\mathrm{id}} T$).

Let $F : \mathcal{I} \to \mathcal{C}$ denote the functor given by the formula

$$F(S_0 \to \cdots \to S_d) = \mathfrak{m}_A^{\otimes S_0}.$$

It follows from $(*)$ that we can identify $\mathfrak{m}^{(d)}$ with the colimit $\varinjlim(F)$. Let P denote the set of all finite subsets of $\mathbf{Z}_{\geq n}$, partially ordered by inclusion. The construction

$$(S_0 \to S_1 \to \cdots \to S_d) \mapsto \{|S_0|, |S_1|, \ldots, |S_d|\}$$

determines a functor $\rho' : \mathcal{I} \to P^{\mathrm{op}}$. Let $G : \mathrm{N}(P)^{\mathrm{op}} \to \mathcal{C}$ denote a left Kan extension of F along ρ', so that $\mathfrak{m}^{(n)} \simeq \varinjlim_{J \in P} G(J)$.

Fix a finite subset $J \subseteq \mathbf{Z}_{\geq n}$. Since ρ' is a coCartesian fibration, G is given by the formula $G(J) = \varinjlim F|_{\mathcal{J} \times_{P^{\mathrm{op}}} \{J\}}$. For every object $\vec{S} = (S_0 \to \cdots \to S_d)$ in $\mathcal{J} \times_{P^{\mathrm{op}}} \{J\}$, the set S_0 has cardinality $\geq d + |S_d| \geq d + n = |J| + n - 1$. The category $\mathcal{J} \times_{P^{\mathrm{op}}} \{J\}$ is a groupoid in which every object has a finite automorphism group. It follows that $G(J)$ can be written as a direct sum of objects of the form $(\mathfrak{m}_A^{\otimes T})_{\Gamma}$, where T is a finite set of cardinality $|J| + n - 1$ and Γ is a finite group acting on $\mathfrak{m}_A^{\otimes T}$ via permutations of T. Since k has characteristic zero, it follows that $G(J) \in (\mathrm{Mod}_k)_{\leq 1-n-|J|}$. The desired result now follows from Lemma 4.2.5.2 (take $\lambda : P \to \mathbf{Z}_{\geq 0}$ to be the function given by $\lambda(J) = |J| - 1$). □

Proof of Lemma 4.2.5.2. We will prove the following more general assertion: for every simplicial subset $K \subseteq \mathrm{N}(P)^{\mathrm{op}}$, the colimit $\varinjlim(G|_K)$ belongs to $\mathcal{C}_{\leq -n}$. Writing K as a filtered colimit of finite simplicial sets, we may reduce to the case where K is finite. We proceed by induction on the number of nondegenerate simplices of K. If K is empty, there is nothing to prove. Otherwise, we can choose a pushout diagram

$$\begin{array}{ccc} \partial \Delta^m & \longrightarrow & K_0 \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & K. \end{array}$$

Since $\varinjlim(G|_{K_0}) \in \mathcal{C}_{\leq -n}$ by the inductive hypothesis, it will suffice to prove that the cofiber of the canonical map $\theta : \varinjlim(G|_{K_0}) \to \varinjlim(G|_K)$ belongs to $\mathcal{C}_{\leq -n}$. For this, we may replace K by Δ^m and K_0 by $\partial \Delta^m$. Let $x \in P^{\mathrm{op}}$ denote the image of the final vertex $\{m\} \in \Delta^m$. We will prove that $\mathrm{cofib}(\theta) \in \mathcal{C}_{\leq -n-\lambda(x)}$. The proof proceeds by induction on m. If $m = 0$, then $\mathrm{cofib}(\theta) = G(x)$ and there is nothing to prove. If $m > 0$, then the inclusion $\Lambda^m_m \hookrightarrow \Delta^m$ is right anodyne and therefore left cofinal. It follows that the composite map

$$\varinjlim(G|_{\Lambda^m_m}) \xrightarrow{\theta'} \varinjlim(G|_{\partial \Delta^m}) \xrightarrow{\theta} \varinjlim(G_{\Delta^m})$$

is an equivalence, so that $\mathrm{cofib}(\theta) \simeq \Sigma \, \mathrm{cofib}(\theta')$. Using the pushout diagram of simplicial sets

$$\begin{array}{ccc} \partial \Delta^{m-1} & \longrightarrow & \Delta^{m-1} \\ \downarrow & & \downarrow{\scriptstyle \iota} \\ \Lambda^m_m & \longrightarrow & \partial \Delta^m, \end{array}$$

we can identify $\mathrm{cofib}(\theta')$ with the cofiber of the induced map $\theta'' : \varinjlim(G|_{\partial \Delta^{m-1}}) \to \varinjlim(G|_{\Delta^{m-1}})$. Let $y \in P$ denote the image of the final vertex of Δ^{m-1}. Then $y > x$. Since λ is monotone, we have $\lambda(y) > \lambda(x)$. Using the inductive hypothesis, we deduce that

$$\mathrm{cofib}(\theta'') \in \mathcal{C}_{\leq -n-\lambda(y)} \subseteq \mathcal{C}_{\leq -n-1-\lambda(x)},$$

so that $\mathrm{cofib}(\theta) \simeq \Sigma \, \mathrm{cofib}(\theta') \simeq \Sigma \, \mathrm{cofib}(\theta'') \in \mathcal{C}_{\leq -n-\lambda(x)}$, as desired. □

4.2.6 Application: Linearizing the Product Formula

Let us now return to the setting of algebraic geometry. Let X be an algebraic curve defined over an algebraically closed field k, and let G be a smooth affine group scheme over X. In §4.1, we constructed a canonical map

$$\rho : \int_X [\mathrm{BG}]_X \to C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell),$$

and asserted that it is an equivalence under some mild hypotheses on G (Theorem 4.1.2.1). In principle, this result gives a complete description of the ℓ-adic cohomology $\mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$. This description is somewhat unwieldy in practice, since it uses the formalism of factorization homology (Definition 4.1.1.3). Nevertheless, we will see that it yields very concrete information about the cotangent fiber of $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$.

Let BG denote the classifying stack of G. We begin by noting that the projection map $\mathrm{BG} \to X$ admits a canonical section, classifying the trivial G-bundle on X, which we can also identify with a k-valued point of the moduli stack $\mathrm{Bun}_G(X)$. By functoriality, we obtain restriction maps

$$\epsilon : [\mathrm{BG}]_X \to [X]_X = \omega_X \qquad \epsilon' : C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell) \to C^*(\mathrm{Spec}(k); \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell,$$

which equip $[\mathrm{BG}]_X$ and $C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ with the structures of augmented commutative algebra objects of the ∞-categories $\mathrm{Shv}_\ell(X)$ and $\mathrm{Mod}_{\mathbf{Z}_\ell}$, respectively.

Let $e : X \times_{\mathrm{Spec}(k)} \mathrm{Bun}_G(X) \to \mathrm{BG}$ be the evaluation map and let $\pi : X \to \mathrm{Spec}(k)$ be the projection. Then we can regard the map

$$[\mathrm{BG}]_X \xrightarrow{e^*} [X \times_{\mathrm{Spec}(k)} \mathrm{Bun}_G(X)]_X \simeq \pi^! C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$$

described in §4.1.2 as a morphism of augmented commutative algebra objects of $\mathrm{Shv}_\ell(X)$. Passing to cotangent fibers, we obtain a map

$$\mu : \cot([\mathrm{BG}]_X) \to \cot(\pi^! C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)) \simeq \pi^! \cot(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$$

(where the identification is supplied by Remark 4.2.1.6). We then have the following:

Theorem 4.2.6.1. *Let G be a smooth affine group scheme over X. Assume that the fibers of G are connected and that the generic fiber is semisimple and simply connected. Then the map $\mu : \cot([\mathrm{BG}]_X) \to \pi^! \cot(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$ described above induces an equivalence*

$$\theta : \pi_* \cot([\mathrm{BG}]_X) \to \cot(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$$

in the ∞-category $\mathrm{Shv}_\ell(\mathrm{Spec}(k)) \simeq \mathrm{Mod}_{\mathbf{Z}_\ell}$.

Proof. Fix an object $M \in \mathrm{Mod}_{\mathbf{Z}_\ell}$; we wish to show that composition with θ induces a homotopy equivalence

$$\theta_M : \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), M) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\pi_* \cot[\mathrm{BG}]_X, M).$$

Using the universal property of the cotangent fiber (Proposition 4.2.2.2), we obtain homotopy equivalences

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\cot C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), M) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), \mathbf{Z}_\ell \oplus M)$$

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}_\ell}}(\pi_* \cot[\mathrm{BG}]_X, M) \simeq \mathrm{Map}_{\mathrm{CAlg}^{\mathrm{aug}}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X \oplus \pi^! M).$$

It will therefore suffice to show that the diagram of spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), \mathbf{Z}_\ell \oplus M) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X \oplus \pi^! M) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}_\ell})}(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell), \mathbf{Z}_\ell) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Shv}_\ell(X))}([\mathrm{BG}]_X, \omega_X) \end{array}$$

is a homotopy pullback square. In fact, the horizontal maps in this diagram are homotopy equivalences by virtue of Theorem 4.1.2.1 (see Remark 4.1.2.3). □

Remark 4.2.6.2. Theorem 4.2.6.1 is strictly weaker than Theorem 4.1.2.1. In principle, Theorem 4.1.2.1 gives complete information about the cohomology of $\mathrm{Bun}_G(X)$ with coefficients in $\mathbf{Z}_\ell$ (though that information might be difficult to access). By contrast, Theorem 4.2.6.1 gives only rational information: one can show that the cotangent fibers $\cot([\mathrm{BG}]_X)$ and $\cot(C^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$ are modules over $\mathbf{Q}_\ell$.

4.3 Convergent Frob-Modules

Let k be an algebraically closed field, let X be an algebraic curve over k, and let G be a smooth affine group scheme over X. Suppose that X and G are defined over a finite field $\mathbf{F}_q \subseteq k$, so that the moduli stack $\mathrm{Bun}_G(X)$ is equipped with a geometric Frobenius map $\mathrm{Frob} : \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X)$. To prove Theorem 1.4.4.1, we need to compute the trace $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell))$. However, this requires some care: typically the cohomology groups $\mathrm{H}^n(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)$ are nonzero for infinitely many values of n. We therefore devote this section to a careful discussion of the convergence issues which arise when working with infinite sums such as

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)) = \sum_{n \geq 0} (-1)^n \,\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^n(\mathrm{Bun}_G(X); \mathbf{Z}_\ell)).$$

4.3.1 Definitions

Throughout this section, we fix a prime number ℓ and an embedding of fields $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$.

Definition 4.3.1.1. Let V^* be a graded vector space over $\mathbf{Q}_\ell$ and F an endomorphism of V^*. We will say that (V^*, F) is *convergent* if the following conditions are satisfied:

(1) The vector space V^m is finite-dimensional for every integer m.

(2) For each $\lambda \in \mathbf{C}$ and every integer m, let $d_{\lambda,m}$ denote the dimension of the generalized λ-eigenspace of F on the complex vector space $\mathbf{C} \otimes_{\mathbf{Q}_\ell} V^m$. Then the sum
$$\sum_{m,\lambda} d_{\lambda,m}|\lambda|$$
is convergent.

If (V^*, F) is convergent, we let $|V^*|_F$ denote the nonnegative real number $\sum_{m,\lambda} d_{\lambda,m}|\lambda|$; we will refer to $|V^*|_F$ as the *norm* of the pair (V^*, F). We let $\mathrm{Tr}(F|V^*)$ denote the complex number $\sum_{m,\lambda}(-1)^m d_{\lambda,m}\lambda$. Note that this sum converges absolutely, and we have $|\mathrm{Tr}(F|V^*)| \leq |V^*|_F$.

Remark 4.3.1.2. The definition of a convergent pair (V^*, F) and the trace $\mathrm{Tr}(F|V^*)$ depend on a choice of embedding $\iota : \mathbf{Q}_\ell \to \mathbf{C}$. However, for the pairs (V^*, F) of interest to us, the traces $\mathrm{Tr}(F|V^*)$ can be shown to be independent of ι.

Remark 4.3.1.3. Suppose we are given graded $\mathbf{Q}_\ell$ vector spaces V'^*, V^*, and V''^* equipped with endomorphisms F', F, and F'' respectively, together with a long exact sequence
$$\cdots \to V''^{n-1} \to V'^n \to V^n \to V''^n \to V'^{n+1} \to \cdots$$
compatible with the actions of F, F', and F''. If (V'^*, F') and (V''^*, F'') are convergent, then (V^*, F) is also convergent. Moreover, in this case we have
$$|V^*|_F \leq |V'^*|_{F'} + |V''^*|_{F''} \qquad \mathrm{Tr}(F|V^*) = \mathrm{Tr}(F'|V'^*) + \mathrm{Tr}(F''|V''^*).$$

4.3.2 The Case of an Augmented Algebra

We again fix a prime number ℓ and an embedding $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$. The following result will play a key role in our proof of Weil's conjecture (essentially, it allows us to reduce a multiplicative problem to an additive one):

Proposition 4.3.2.1. *Let A be an augmented commutative algebra object of $\mathrm{Mod}_{\mathbf{Q}_\ell}$ equipped with an automorphism F. We let $V = \cot(A)$ denote the cotangent fiber of A, so that F determines an automorphism of V (which we will also denote by F). Suppose that the following conditions are satisfied:*

(1) *The augmentation ideal $\mathfrak{m}_A$ belongs to $(\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -1}$.*

(2) *The graded $\mathbf{Q}_\ell$-vector space $\mathrm{H}^*(V)$ is finite-dimensional.*

(3) *For every integer i and every eigenvalue λ of F on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} \mathrm{H}^i(V)$, we have $|\lambda| < 1$.*

Then $(\mathrm{H}^*(A); F)$ is convergent. Moreover, we have

$$\mathrm{Tr}(F | \mathrm{H}^*(A)) = \exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(F^n | \mathrm{H}^*(V))),$$

where the sum on the right hand side is absolutely convergent.

Remark 4.3.2.2. Proposition 4.3.2.1 asserts that, under mild hypotheses, the trace of F on the cohomology of A is equal to the trace of F on the cohomology of the symmetric algebra $\mathrm{Sym}^*(V)$.

Remark 4.3.2.3. Let V^* be a finite-dimensional graded $\mathbf{Q}_\ell$-vector space equipped with an automorphism F. We define the *L-function* of the pair (V^*, F) to be the rational function of one variable t given by the formula

$$L_{V^*,F}(t) = \prod_{m \in \mathbf{Z}} \det(1 - tF | V^m)^{(-1)^{m+1}}.$$

An easy calculation yields

$$L_{V^*,F}(t) = \exp(\sum_{n>0} \frac{t^n}{n} \mathrm{Tr}(F^n | \mathrm{H}^*(V)))$$

for $|t| < C$, where $C = \sup\{\frac{1}{|\lambda|}\}$ where λ ranges over the eigenvalues of F. In particular, if all of the eigenvalues of F have complex absolute values < 1, then we have

$$L_{V^*,F}(1) = \exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(F^n | \mathrm{H}^*(V))).$$

In the situation of Proposition 4.3.2.1, we can rewrite the conclusion as

$$\mathrm{Tr}(F | \mathrm{H}^*(A)) = L_{\mathrm{H}^*(\cot A),F}(1) = \prod_{m>0} \det(1 - F | \mathrm{H}^m(V))^{(-1)^{m+1}}.$$

4.3.3 The Proof of Proposition 4.3.2.1

Fix an augmented commutative algebra $A \in \mathrm{CAlg}^{\mathrm{aug}}_{\mathbf{Q}_\ell}$ satisfying the hypotheses of Proposition 4.3.2.1, and set $V = \cot(A)$. Write the graded vector space $\mathrm{H}^*(V)$ as a direct sum $\mathrm{H}^{\mathrm{even}}(V) \oplus \mathrm{H}^{\mathrm{odd}}(V)$. Let $\{\lambda_1, \ldots, \lambda_m\}$ denote the eigenvalues of F on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{\mathrm{even}}(V)$ (counted with multiplicity), and let $\{\mu_1, \ldots, \mu_{m'}\}$ denote the eigenvalues of F on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{\mathrm{odd}}(V)$ (again counted with multiplicity). For every integer $n \geq 0$, we set

$$s_n = \sum_{n = n_1 + \cdots + n_m + |S|, S \subseteq \{1, \ldots, m'\}} \Big(\prod_{1 \leq i \leq m} |\lambda_i|^{n_i} \prod_{j \in S} |\mu_j| \Big)$$

$$\sigma_n = \sum_{n = n_1 + \cdots + n_m + |S|, S \subseteq \{1, \ldots, m'\}} \Big(\prod_{1 \leq i \leq m} \lambda_i^{n_i} \prod_{j \in S} -\mu_j \Big).$$

It follows from (3) that the sum $s_0 + s_1 + s_2 + \cdots$ converges to

$$\prod_{1 \leq i \leq m} \frac{1}{1 - |\lambda_i|} \prod_{1 \leq j \leq m'} (1 + |\mu_j|),$$

so that the sum $\sigma_0 + \sigma_1 + \sigma_2 + \cdots$ converges absolutely to

$$\prod_{1 \leq i \leq m} \frac{1}{1 - |\lambda_i|} \prod_{1 \leq j \leq m'} (1 - \mu_j).$$

For each $n \geq 0$, we can identify $\mathrm{H}^*(\mathrm{Sym}^n(V))$ with the nth symmetric power of $\mathrm{H}^*(V)$ (in the category of graded vector spaces with the usual sign convention). It follows that

$$|\,\mathrm{H}^*(\mathrm{Sym}^n(V))|_F = s_n \qquad \mathrm{Tr}(F |\, \mathrm{H}^*(\mathrm{Sym}^n(V))) = \sigma_n.$$

Let $\mathfrak{m}_A$ denote the augmentation ideal of A. For each integer $n \geq 1$, let $\mathfrak{m}_A^{(n)}$ be as in Definition 4.2.1.1, and set $\mathfrak{m}_A^{(0)} = A$. For every pair of integers $i \leq j$, let $Q_{i,j}$ denote the cofiber of the map $\mathfrak{m}_A^{(j)} \to \mathfrak{m}_A^{(i)}$. If $i < j$, then Proposition 4.2.4.2 supplies a fiber sequence

$$Q_{i+1,j} \to Q_{i,j} \to \mathrm{Sym}^i(V).$$

Applying Remark 4.3.1.3 repeatedly, we deduce that each pair $(\mathrm{H}^*(Q_{i,j}), F)$ is convergent, with

$$|\,\mathrm{H}^*(Q_{i,j})|_F \leq s_i + \cdots + s_{j-1} \qquad \mathrm{Tr}(F | H^*(Q_{i,j})) = \sigma_i + \cdots + \sigma_{j-1}.$$

We next prove the following:

$(*)$ For each integer n, the pair $(\mathrm{H}^*(\mathfrak{m}_A^{(n)}), F)$ is convergent, with

$$|\mathrm{H}^*(\mathfrak{m}_A^{(n)})|_F \leq s_n + s_{n+1} + \cdots .$$

For every integer $d \geq 0$, set $W(d)^* = \bigoplus_{i \leq d} \mathrm{H}^i(\mathfrak{m}_A^{(n)})$. To prove $(*)$, it will suffice to show that each of the pairs $(W(d)^*, F)$ is convergent with $|W(d)^*|_F \leq s_n + s_{n+1} + \cdots$. Without loss of generality we may assume that $d > n$. It follows from Proposition 4.2.5.1 that the map $\mathrm{H}^i(\mathfrak{m}_A^{(n)}) \to \mathrm{H}^i(Q_{n,d+2})$ is an isomorphism for $i \leq d$, so that

$$|W(d)^*|_F \leq |\mathrm{H}^*(Q_{n,d+2})|_F \leq s_n + \cdots + s_{d+1} \leq \sum_{n' \geq n} s_{n'} < \infty,$$

as desired.

Applying $(*)$ when $n = 0$, we deduce that $(\mathrm{H}^*(A), F)$ is convergent. Moreover, for every integer n, applying Remark 4.3.1.3 to the fiber sequence

$$\mathfrak{m}_A^{(n)} \to A \to Q_{0,n}$$

gives an inequality

$$\begin{aligned} |\mathrm{Tr}(F| \mathrm{H}^*(A)) - \sigma_0 - \cdots - \sigma_{n-1}| &= |\mathrm{Tr}(F| \mathrm{H}^*(A)) - \mathrm{Tr}(F| \mathrm{H}^*(Q_{0,n}))| \\ &= |\mathrm{Tr}(F| \mathrm{H}^*(\mathfrak{m}_A^{(n)}))| \\ &\leq |\mathrm{H}^*(\mathfrak{m}_A^{(n)})|_F \\ &\leq s_n + s_{n+1} + \cdots . \end{aligned}$$

It follows that $\mathrm{Tr}(F| \mathrm{H}^*(A))$ is given by the absolutely convergent sum

$$\sum_{n \geq 0} \sigma_n = \prod_{1 \leq i \leq m} \frac{1}{1 - \lambda_i} \prod_{1 \leq j \leq m'} (1 - \mu_j).$$

In particular, we have

$$\begin{aligned} \log \mathrm{Tr}(F| \mathrm{H}^*(A)) &= \sum_{1 \leq i \leq m} \log \frac{1}{1 - \lambda_i} - \sum_{1 \leq j \leq m'} \log \frac{1}{1 - \mu_i} \\ &= \sum_{1 \leq i \leq m} \sum_{n > 0} \frac{1}{n} \lambda_i^n - \sum_{1 \leq j \leq m'} \sum_{n > 0} \frac{1}{n} \mu_j^n \\ &= \sum_{n > 0} \frac{1}{n} (\sum_{1 \leq i \leq m} \lambda_i^n - \sum_{1 \leq j \leq m'} \mu_j^n) \\ &= \sum_{n > 0} \frac{1}{n} \mathrm{Tr}(F^n | \mathrm{H}^*(V)). \end{aligned}$$

4.4 The Trace Formula for BG

Let G be a connected linear algebraic group defined over a finite field $\mathbf{F}_q$, and let BG denote its classifying stack. A theorem of Lang asserts that every G-bundle is trivial (Theorem 1.3.2.8), and the automorphism group of the trivial G-bundle can be identified with the finite group $G(\mathbf{F}_q)$ of rational points of G. Consequently, the mass of the groupoid $\mathrm{BG}(\mathbf{F}_q)$ (in the sense of Definition 1.3.3.1) is equal to $\frac{1}{|G(\mathbf{F}_q)|}$. In this section, we will verify that the classifying stack BG satisfies the Grothendieck-Lefschetz trace formula in the form

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*(\overline{\mathrm{BG}}; \mathbf{Q}_\ell)) = \frac{|\,\mathrm{BG}(\mathbf{F}_q)|}{q^{\dim(\mathrm{BG})}} = \frac{q^{\dim(G)}}{|G(\mathbf{F}_q)|}$$

(Proposition 4.4.4.1). This is a special case of a more general result for global quotient stacks, which we will discuss in Chapter 5 (see Corollary 5.1.0.4). However, we give a direct argument in this section in order to highlight some ideas which will be useful for analyzing moduli stacks of bundles on algebraic curves in §4.5.

4.4.1 The Motive of an Algebraic Group

We begin by establishing some terminology.

Notation 4.4.1.1. Let k be a field, let ℓ be a prime number which is invertible in k, and let $\overline{k}$ be an algebraic closure of k. Let $\mathcal{Y}$ be an algebraic stack defined over k, so that $\overline{\mathcal{Y}} = \mathcal{Y} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ can be regarded as an algebraic stack over the algebraically closed field $\overline{k}$. We let $C^*_{\mathrm{gm}}(\mathcal{Y})$ denote the cochain complex $C^*(\overline{\mathcal{Y}}; \mathbf{Z}_\ell)[\ell^{-1}] \in \mathrm{Mod}_{\mathbf{Q}_\ell}$. We will refer to $C^*_{\mathrm{gm}}(\mathcal{Y})$ as the *geometric cochain complex of* $\mathcal{Y}$. We let $\mathrm{H}^*_{\mathrm{gm}}(\mathcal{Y})$ denote the cohomology of the cochain complex $C^*_{\mathrm{gm}}(\mathcal{Y})$; we will refer to $\mathrm{H}^*_{\mathrm{gm}}(\mathcal{Y})$ as the *geometric cohomology of* $\mathcal{Y}$. Note that the cochain complex $C^*_{\mathrm{gm}}(\mathcal{Y})$ and its cohomology $\mathrm{H}^*_{\mathrm{gm}}(\mathcal{Y})$ are equipped with an action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

Remark 4.4.1.2. The definition of the geometric cohomology $\mathrm{H}^*_{\mathrm{gm}}(\mathcal{Y})$ depends on the choice of an algebraic closure $\overline{k}$ of k and the choice of prime number ℓ which is invertible in k. However, to avoid making the exposition too burdensome, we will often neglect to mention these choices explicitly.

Warning 4.4.1.3. In the situation of Notation 4.4.1.1, the geometric cohomology $\mathrm{H}^*_{\mathrm{gm}}(\mathcal{Y})$ comes equipped with a canonical map

$$\theta : \mathrm{H}^*_{\mathrm{gm}}(\mathcal{Y}) \to \mathrm{H}^*(\overline{\mathcal{Y}}; \mathbf{Q}_\ell).$$

This map is an isomorphism if $\mathcal{Y}$ is of finite type over k (Proposition 3.2.5.4), but not in general. For example, if $\mathcal{Y}$ is a disjoint union of countably many copies of $\mathrm{Spec}(\mathbf{F}_q)$, then θ is given by the canonical monomorphism

$$(\prod_{i \geq 0} \mathbf{Z}_\ell)[\ell^{-1}] \to \prod_{i \geq 0} \mathbf{Q}_\ell .$$

Remark 4.4.1.4. Let Y be a smooth algebraic variety of dimension d over a finite field $\mathbf{F}_q$, so that the geometric cohomology $\mathrm{H}^*_{\mathrm{gm}}(Y)$ is equipped with a geometric Frobenius automorphism Frob. Since $\mathrm{H}^*_{\mathrm{gm}}(Y)$ is a finite-dimensional vector space over $\mathbf{Q}_\ell$, the pair $(\mathrm{H}^*_{\mathrm{gm}}(Y), \mathrm{Frob}^{-1})$ is automatically convergent (in the sense of Definition 4.3.1.1). Moreover, the Grothendieck-Lefschetz trace formula yields an equality $\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{gm}}(Y)) = q^{-d} |Y(\mathbf{F}_q)|$ (see Theorem 1.4.2.4).

Definition 4.4.1.5. Let G be a connected algebraic group defined over a field k and let $I = \mathrm{H}^{>0}_{\mathrm{gm}}(G)$ denote the (two-sided) ideal in $\mathrm{H}^*_{\mathrm{gm}}(G)$ generated by homogeneous elements of positive degree. We define the *motive of* G to be the quotient I/I^2, which we regard as a representation of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$.

Remark 4.4.1.6. For a reductive group G over a field k, the motive $M(G)$ was introduced by Gross in [14]. Definition 4.4.1.5 appears in [40]. Beware that our conventions differ from those of [14] and [40] by a Tate twist (the motive of G is defined in [14] to be the tensor product $\mathbf{Q}_\ell(1) \otimes_{\mathbf{Q}_\ell} M(G)$; see Remark 4.5.1.5 below).

Remark 4.4.1.7. In the case where $k = \mathbf{C}$ is the field of complex numbers, the motive $M(G)$ of an algebraic group G can be identified with the tensor product $\mathbf{Q}_\ell \otimes_{\mathbf{Q}} V$, where V is the rational vector space appearing in the statement of the Atiyah-Bott formula (Theorem 1.5.2.3).

Remark 4.4.1.8. Let $\phi : G \to H$ be an isogeny between connected algebraic groups over a field k. Then ϕ induces an isomorphism $\phi^* : \mathrm{H}^*_{\mathrm{gm}}(H) \to \mathrm{H}^*_{\mathrm{gm}}(G)$, which restricts to a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $M(H) \simeq M(G)$.

4.4.2 Digression: The Eilenberg-Moore Spectral Sequence

Let k be an algebraically closed field, which we regard as fixed throughout this section. If X and Y are quasi-projective k-schemes, then Theorem 3.3.1.1 supplies a quasi-isomorphism of ℓ-adic cochain complexes

$$C^*(X; \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} C^*(Y; \mathbf{Z}_\ell) \to C^*(X \times_{\mathrm{Spec}(k)} Y; \mathbf{Z}_\ell).$$

In other words, the pullback diagram of schemes

$$\begin{array}{ccc} X \times_{\mathrm{Spec}(k)} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

induces a pushout diagram

$$\begin{array}{ccc} C^*(X \times_{\mathrm{Spec}(k)} Y; \mathbf{Z}_\ell) & \longleftarrow & C^*(X; \mathbf{Z}_\ell) \\ \uparrow & & \uparrow \\ C^*(Y; \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathrm{Spec}(k); \mathbf{Z}_\ell) \end{array}$$

of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}_\ell$. We will need the following variant:

Proposition 4.4.2.1. *Let H be a connected algebraic group over k, let BH denote the classifying stack of H, let $\mathcal{Y}$ be an algebraic stack equipped with a map $\pi : \mathcal{Y} \to \mathrm{BH}$, and form a pullback square*

$$\begin{array}{ccc} \overline{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{BH}\,. \end{array}$$

Then the associated diagram of cochain complexes

$$\begin{array}{ccc} C^*(\overline{\mathcal{Y}}; \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathcal{Y}; \mathbf{Z}_\ell) \\ \uparrow & & \uparrow \\ C^*(\mathrm{Spec}(k); \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathrm{BH}; \mathbf{Z}_\ell) \end{array}$$

is a pushout diagram of $\mathbb{E}_\infty$-algebras over $\mathbf{Z}_\ell$.

Remark 4.4.2.2 (The Cohomological Eilenberg-Moore Spectral Sequence)**.** In the situation of Proposition 4.4.2.1, it follows that there exists a convergent spectral sequence

$$\mathrm{Tor}_s^{\mathrm{H}^*(\mathrm{BH}; \mathbf{Z}_\ell)}(\mathrm{H}^*(\mathcal{Y}; \mathbf{Z}_\ell), \mathbf{Z}_\ell) \Rightarrow \mathrm{H}^{*-s}(\overline{\mathcal{Y}}; \mathbf{Z}_\ell).$$

We will deduce Proposition 4.4.2.1 from the following technical result, whose proof we defer to the end of §4.4.

Lemma 4.4.2.3. *Let $A^\bullet$ be a cosimplicial object of $\mathrm{Alg}(\mathrm{Mod}_{\mathbf{Z}_\ell})$. Suppose we are given a cosimplicial right $A^\bullet$-module $M^\bullet$ and a cosimplicial left $A^\bullet$-module $N^\bullet$ satisfying the following requirements:*

(*a*) *For each integer* $n \geq 0$*, the homology groups* $\mathrm{H}_*(M^n)$*,* $\mathrm{H}_*(N^n)$*, and* $\mathrm{H}_*(A^n)$ *vanish for* $* > 0$*.*

(*b*) *For each integer* $n \geq 0$*, the unit map* $\mathbf{Z}_\ell \to \mathrm{H}_0(A^n)$ *is an isomorphism.*

(*c*) *For each integer* $n \geq 0$*, the homology group* $\mathrm{H}_{-1}(A^n)$ *is torsion-free.*

Then the canonical map

$$\theta : \mathrm{Tot}(M^\bullet) \otimes_{\mathrm{Tot}(A^\bullet)} \mathrm{Tot}(N^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_{A^\bullet} N^\bullet)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$*.*

Proof of Proposition 4.4.2.1. Let $U_0 = \mathrm{Spec}(k)$, and let $U_\bullet$ denote the simplicial scheme given by the nerve of the smooth map $U_0 \to \mathrm{BH}$ (so that $U_d \simeq H^d$). For each integer $d \geq 0$, the pullback diagram σ_d:

$$\begin{array}{ccc} \overline{\mathcal{Y}} \times_{\mathrm{BH}} U_d & \longrightarrow & \mathcal{Y} \times_{\mathrm{BH}} U_d \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) \times_{\mathrm{BH}} U_d & \longrightarrow & U_d \end{array}$$

can be rewritten as

$$\begin{array}{ccc} \overline{\mathcal{Y}} \times_{\mathrm{Spec}(k)} H^{d+1} & \longrightarrow & \overline{\mathcal{Y}} \times_{\mathrm{Spec}(k)} H^d \\ \downarrow & & \downarrow \\ H^{d+1} & \longrightarrow & H^d. \end{array}$$

Using Variant 3.4.5.5, we deduce that σ_d determines a pushout square

$$\begin{array}{ccc} C^*(\overline{\mathcal{Y}} \times_{\mathrm{BH}} U_d; \mathbf{Z}_\ell) & \longleftarrow & C^*(\mathcal{Y} \times_{\mathrm{BH}} U_d; \mathbf{Z}_\ell) \\ \uparrow & & \uparrow \\ C^*(\mathrm{Spec}(k) \times_{\mathrm{BH}} U_d; \mathbf{Z}_\ell) & \longleftarrow & C^*(U_d; \mathbf{Z}_\ell) \end{array}$$

in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. We may therefore identify $C^*(\overline{\mathcal{C}})$ with $\mathrm{Tot}(M^\bullet \otimes_{A^\bullet} N^\bullet)$, where $A^\bullet = C^*(U_\bullet; \mathbf{Z}_\ell)$, $M^\bullet = C^*(\mathcal{C} \times_{\mathrm{BH}} U_\bullet; \mathbf{Z}_\ell)$, and $N^\bullet = C^*(\mathrm{Spec}(k) \times_{\mathrm{BH}} U_\bullet; \mathbf{Z}_\ell)$. To prove Proposition 4.4.2.1, we must show that the canonical map

$$\theta : \mathrm{Tot}(M^\bullet) \otimes_{\mathrm{Tot}(A^\bullet)} \mathrm{Tot}(N^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_{A^\bullet} N^\bullet)$$

is an equivalence in $\mathrm{Mod}_{\mathbf{Z}_\ell}$. For this, it will suffice to show that $A^\bullet$, $M^\bullet$, and $N^\bullet$ satisfy the hypotheses of Lemma 4.4.2.3. Hypothesis (*a*) is obvious, and hypotheses (*b*) and (*c*) follow from our assumption that H is connected. □

4.4.3 The Motive as a Cotangent Fiber

In §4.4.1, we defined the *motive* $M(G)$ of a connected algebraic group G defined over a field k (Definition 4.4.1.5). In this section, we will provide two alternative descriptions of $M(G)$: one in terms of the cotangent fiber of the geometric cohomology of G (Proposition 4.4.3.1), and one in terms of the cotangent fiber of the geometric cohomology of BG (Remark 4.4.3.2).

Proposition 4.4.3.1. *Let G be a connected algebraic group defined over a field k and let $I = \mathrm{H}^{>0}_{\mathrm{gm}}(G)$. Then the canonical map*

$$M(G) = I/I^2 \to \mathrm{H}^*(\cot C^*_{\mathrm{gm}}(G))$$

(see §4.2.3) is an isomorphism.

Proof. The group law $m : G \times_{\mathrm{Spec}(k)} G \to G$ induces a comultiplication

$$m^* : \mathrm{H}^*_{\mathrm{gm}}(G) \to \mathrm{H}^*_{\mathrm{gm}}(G) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^*_{\mathrm{gm}}(G),$$

which endows $\mathrm{H}^*_{\mathrm{gm}}(G)$ with the structure of a finite-dimensional graded-commutative Hopf algebra over $\mathbf{Q}_\ell$. Since G is connected, it follows that $\mathrm{H}^*_{\mathrm{gm}}(G)$ is isomorphic to an exterior algebra on finitely many generators $x_1, \ldots, x_r$ of odd degrees $d_1, \ldots, d_r$ (see [26]). The desired result now follows from Proposition 4.2.3.1. □

Remark 4.4.3.2. Let G be a connected algebraic group over a field k. We have a pullback diagram of algebraic stacks

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{BG}\,. \end{array}$$

Applying Proposition 4.4.2.1, we obtain a pushout square

$$\begin{array}{ccc} C^*_{\mathrm{gm}}(G) & \longleftarrow & C^*_{\mathrm{gm}}(\mathrm{Spec}(k)) \\ \uparrow & & \uparrow \\ C^*_{\mathrm{gm}}(\mathrm{Spec}(k)) & \longleftarrow & C^*_{\mathrm{gm}}(\mathrm{BG}) \end{array}$$

of augmented commutative algebra objects of $\mathrm{Mod}_{\mathbf{Q}_\ell}$, hence a pushout diagram of cotangent fibers

$$\begin{array}{ccc} \cot C^*_{\mathrm{gm}}(G) & \longleftarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longleftarrow & \cot C^*_{\mathrm{gm}}(\mathrm{BG}); \end{array}$$

see Proposition 4.2.2.1 (here we regard BG as equipped with the base point $\mathrm{Spec}(k) \to \mathrm{BG}$ corresponding to the trivial G-bundle, and G as equipped with the base point $\mathrm{Spec}(k) \to G$ given by the identity section). In other words, we can identify $\cot C^*_{\mathrm{gm}}(G)$ with the suspension $\Sigma \cot C^*_{\mathrm{gm}}(\mathrm{BG})$. In particular, we obtain a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $M(G) \simeq \mathrm{H}^*(\cot C^*_{\mathrm{gm}}(\mathrm{BG}))$ (which shifts the grading by 1).

Remark 4.4.3.3. Let G be a connected algebraic group over a field k. One can show that the geometric cohomology ring $\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})$ is a polynomial ring on generators of even degree. It follows from Proposition 4.2.3.1 and Remark 4.4.3.2 that the motive $M(G)$ can be identified with the quotient J/J^2, where $J = \mathrm{H}^{>0}_{\mathrm{gm}}(\mathrm{BG})$ is the ideal generated by elements of positive degree.

Remark 4.4.3.4. Let G be a reductive algebraic group over a field k and let G' be a quasi-split inner form of G. Then there exists a $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism $M(G) \simeq M(G')$. To prove this, we may assume without loss of generality that G and G' are adjoint (Remark 4.4.1.8). In this case, the classifying stacks BG and BG$'$ are equivalent to one another, so the desired result follows from the characterization of $M(G)$ and $M(G')$ given in Remark 4.4.3.3.

Since G' is quasi-split, we can choose a Borel subgroup $B' \subseteq G'$. Let $T' \subseteq B'$ be a maximal torus and let Λ be the character lattice of $\mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} T'$. The Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on Λ preserving a system of positive roots, and there is a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism

$$\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BT}') \simeq \mathrm{Sym}^*(\mathbf{Q}_\ell(-1) \otimes_{\mathbf{Z}} \Lambda).$$

One can show that the restriction map $\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG}') \to \mathrm{H}^*_{\mathrm{gm}}(\mathrm{BT}')$ is injective, and its image consists of those elements of $\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BT}')$ which are invariant under the action of the Weyl group $(N(T')/T')(\overline{k})$. Combining this observation with Remark 4.4.3.3, we obtain a very explicit description of the motive $M(G) \simeq M(G')$, which agrees with the definition given in [14] (up to a Tate twist); see [40] for a more detailed explanation.

4.4.4 Proof of the Trace Formula

We now specialize to the case of algebraic groups defined over *finite* fields.

Proposition 4.4.4.1. *Let G be a connected linear algebraic group of dimension d defined over a finite field $\mathbf{F}_q$ and let* BG *denote its classifying stack. Assume that ℓ is invertible in $\mathbf{F}_q$. Then $(\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG}), \mathrm{Frob}^{-1})$ is convergent, and we have*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})) = \frac{q^d}{|G(\mathbf{F}_q)|}.$$

Moreover, both sides are equal to

$$\exp(\sum_{n>0} \frac{1}{n} \operatorname{Tr}(\operatorname{Frob}^{-n} | M(G))) = \det(1 - \operatorname{Frob}^{-1} | M(G))^{-1},$$

where $M(G)$ denotes the motive of G.

We will need the following:

Lemma 4.4.4.2. *Let G be a connected linear algebraic group over a finite field $\mathbf{F}_q$. Then each eigenvalue of the Frobenius automorphism* Frob *on the motive $M(G)$ has complex absolute value $\geq q$. If G is semisimple, then each eigenvalue has complex absolute value $\geq q^2$.*

Proof. Since $\mathbf{F}_q$ is perfect, the unipotent radical U of G is defined over $\mathbf{F}_q$. Replacing G by the quotient G/U, we may reduce to the case where G is reductive. In this case, the assertion follows immediately from the explicit description of $M(G)$ supplied by Remark 4.4.3.4. □

Proof of Proposition 4.4.4.1. Proposition 4.4.3.1 and Remark 4.4.3.2 supply Frobenius-equivariant isomorphisms

$$\mathrm{H}^*(\cot C^*_{\mathrm{gm}}(\mathrm{BG})) \simeq M(G) \simeq \mathrm{H}^*(\cot C^*_{\mathrm{gm}}(G)),$$

where the groups on the left hand side are concentrated in even degrees and the groups on the right hand side are concentrated in odd degrees. Applying Proposition 4.3.2.1 to the augmented commutative algebras $C^*_{\mathrm{gm}}(\mathrm{BG})$ and $C^*_{\mathrm{gm}}(G)$ and using Remark 4.4.1.4, we obtain

$$\begin{aligned}
\operatorname{Tr}(\operatorname{Frob}^{-1} | \mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})) &= \exp(\sum_{n>0} \frac{1}{n} \operatorname{Tr}(\operatorname{Frob}^{-n} | M(G))) \\
&= \exp(\sum_{n>0} \frac{-1}{n} \operatorname{Tr}(\operatorname{Frob}^{-n} | M(G)))^{-1} \\
&= \operatorname{Tr}(\operatorname{Frob}^{-1} | \mathrm{H}^*_{\mathrm{gm}}(G))^{-1} \\
&= (q^{-d} |G(\mathbf{F}_q)|)^{-1} \\
&= \frac{q^d}{|G(\mathbf{F}_q)|}.
\end{aligned}$$

□

Remark 4.4.4.3. We can rewrite the final assertion of Proposition 4.4.4.1 as a formula

$$|G(\mathbf{F}_q)| = q^d \det(1 - \operatorname{Frob}^{-1} | M(G)) = q^d \det(1 - \operatorname{Frob} | M(G)^{\vee})$$

for the order of the finite group $G(\mathbf{F}_q)$; this formula is due originally to Steinberg ([36]).

4.4.5 The Proof of Lemma 4.4.2.3

We conclude this section with the proof of Lemma 4.4.2.3. This will require some preliminary algebraic results.

Lemma 4.4.5.1. *Let $M^\bullet$ be a cosimplicial object of $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$ and let $N \in (\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$. Then the canonical map*

$$\mathrm{Tot}(M^\bullet) \otimes_{\mathbf{Z}_\ell} N \to \mathrm{Tot}(M^\bullet \otimes_{\mathbf{Z}_\ell} N)$$

is an equivalence.

Proof. For each integer $p \geq 0$, let $K(p)$ denote the pth partial totaliation of $M^\bullet$. Since the operation of tensoring with N is exact, we can identify $K(p) \otimes_{\mathbf{Z}_\ell} N$ with the pth partial totalization of $M^\bullet \otimes_{\mathbf{Z}_\ell} N$. It will therefore suffice to show that the canonical map

$$\theta : (\varprojlim K(p)) \otimes_{\mathbf{Z}_\ell} N \to \varprojlim (K(p) \otimes_{\mathbf{Z}_\ell} N)$$

is an equivalence. Note that for each $q \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} (\varprojlim K(p)) \otimes_{\mathbf{Z}_\ell} N & \xrightarrow{\theta} & \varprojlim (K(p) \otimes_{\mathbf{Z}_\ell} N) \\ & {\searrow}^{\phi} \quad {\swarrow}_{\psi} & \\ & K(q) \otimes_{\mathbf{Z}_\ell} N, & \end{array}$$

where the fibers of ϕ and ψ belong to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -q}$. It follows that the fiber of θ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq -q}$ for all q, so that θ is an equivalence. □

Lemma 4.4.5.2. *Let $M^\bullet$ and $N^\bullet$ be cosimplicial objects of $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 0}$. Then the canonical map*

$$\theta : \mathrm{Tot}(M^\bullet) \otimes_{\mathbf{Z}_\ell} \mathrm{Tot}(N^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_{\mathbf{Z}_\ell} N^\bullet)$$

is an equivalence.

Proof. Let $\mathbf{\Delta}$ denote the category whose objects are the nonempty linearly ordered sets $[n] = \{0, \ldots, n\}$ and whose morphisms are nondecreasing maps. Then the diagonal map $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$ is right cofinal (Lemma [25].5.5.8.4), so that we can identify θ with the natural map

$$(\varprojlim_{[m] \in \mathbf{\Delta}} M^m) \otimes_{\mathbf{Z}_\ell} (\varprojlim_{[n] \in \mathbf{\Delta}} N^n) \to \varprojlim_{[m],[n] \in \mathbf{\Delta}} (M^m \otimes_{\mathbf{Z}_\ell} N^n).$$

This follows from two applications of Lemma 4.4.5.1. □

Proof of Lemma 4.4.2.3. If A is an associative algebra object of $\mathrm{Mod}_{\mathbf{Z}_\ell}$ equipped with a right A-module M and a left A-module N, then the tensor product $M \otimes_A N$ can be computed as the geometric realization of a simplicial object $\mathrm{Bar}_A(M,N)_\bullet$ with

$$\mathrm{Bar}_A(M,N)_q \simeq M \otimes_{\mathbf{Z}_\ell} A^{\otimes q} \otimes_{\mathbf{Z}_\ell} N^\bullet.$$

For each integer d, we let $B^d_A(M,N)$ denote the realization of the d-skeleton of this simplicial object, so we have a sequence

$$M \otimes_{\mathbf{Z}_\ell} N \simeq B^0_A(M,N) \to B^1_A(M,N) \to \cdots$$

with colimit $M \otimes_A N$. Moreover, if we let $\overline{A}$ denote the cofiber of the unit map $\mathbf{Z}_\ell \to A$, then we have cofiber sequences

$$B^{d-1}_A(M,N) \to B^d_A(M,N) \to M \otimes_{\mathbf{Z}_\ell} (\Sigma\overline{A})^{\otimes d} \otimes_{\mathbf{Z}_\ell} N.$$

If $A^\bullet$, $M^\bullet$, and $N^\bullet$ are as in the statement of Lemma 4.4.2.3, then assumption (a) and Lemma 4.4.5.2 supply equivalences

$$B^d_{\mathrm{Tot}(A^\bullet)}(\mathrm{Tot}(M^\bullet), \mathrm{Tot}(N^\bullet)) \simeq \mathrm{Tot}(B^d_{A^\bullet}(M^\bullet, N^\bullet))$$

for each integer $d \geq 0$. We may therefore identify θ with the canonical map

$$\varinjlim_d \mathrm{Tot}(B^d_{A^\bullet}(M^\bullet, N^\bullet)) \to \mathrm{Tot}(\varinjlim_d B^d_{A^\bullet}(M^\bullet, N^\bullet)).$$

To prove that this map is an equivalence, it will suffice to show that there exists an integer k such that $B^d_{A^p}(M^p, N^p)$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq k}$ for all $p, d \geq 0$. We claim that this is satisfied for $k = 1$. Using the cofiber sequence above, we are reduced to proving that

$$M^p \otimes_{\mathbf{Z}_\ell} (\Sigma\overline{A}^p)^{\otimes d} \otimes_{\mathbf{Z}_\ell} N^p$$

belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 1}$ for all p and all $d \geq 0$. It follows immediately from (a) that $M^p \otimes_{\mathbf{Z}_\ell} N^p$ belongs to $(\mathrm{Mod}_{\mathbf{Z}_\ell})_{\leq 1}$. To complete the proof, it suffices to show that $\overline{A}^p$ has Tor-amplitude ≤ -1 for all $p \geq 0$, which follows from assumptions (a), (b), and (c). □

4.5 The Cohomology of $\mathrm{Bun}_G(X)$

Throughout this section, we fix a finite field $\mathbf{F}_q$, an algebraic curve X over $\mathbf{F}_q$, and a smooth affine group scheme G over X with connected fibers whose generic fiber is semisimple and simply connected. We also fix an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$, a prime number ℓ which is invertible in $\mathbf{F}_q$, and an embedding $\mathbf{Z}_\ell \hookrightarrow \mathbf{C}$. Our goal is to verify Theorem 1.4.4.1 by establishing the following:

Theorem 4.5.0.1. *The pair* $(\mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X)); \mathrm{Frob}^{-1})$ *is convergent (in the sense of Definition 4.3.1.1). Moreover, we have*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \mid \mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X))) = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G)}}{|G(\kappa(x))|},$$

where the product on the right hand side is absolutely convergent.

We will deduce Theorem 4.5.0.1 by combining the product formula (Theorem 4.1.2.1), Steinberg's formula (Proposition 4.4.4.1), and the Grothendieck-Lefschetz trace formula.

4.5.1 The Motive of a Group Scheme

In §4.4.1, we defined the motive of a connected algebraic group over a field (Definition 4.4.1.5). We now consider a variant of this definition, which makes sense for group schemes over the curve X (or over more general base schemes).

Construction 4.5.1.1. [The Relative Motive] Let $\overline{X} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} X$ and let $\overline{G} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} G$. We will regard $\mathrm{Shv}_\ell(X)$ as a symmetric monoidal ∞-category with respect to the !-tensor product of §3.3. Let $\overline{\mathrm{BG}}$ denote the classifying stack of $\overline{G}$, so that we can regard the relative cohomology sheaf $[\overline{\mathrm{BG}}]_{\overline{X}}$ as a commutative algebra object of $\mathrm{Shv}_\ell(\overline{X})$ (see §4.1.2). Note that the projection map $\overline{\mathrm{BG}} \to \overline{X}$ has a canonical section (classifying the trivial G-bundle), which induces an augmentation $\epsilon : [\overline{\mathrm{BG}}]_{\overline{X}}$. We define an ℓ-adic sheaf $\mathcal{M}(G) \in \mathrm{Shv}_\ell(\overline{X})$ by the formula

$$\mathcal{M}(G) = (\cot[\overline{\mathrm{BG}}]_{\overline{X}})[\ell^{-1}].$$

We will refer to $\mathcal{M}(G)$ as the *motive of* G *relative to* X.

Remark 4.5.1.2. In the situation of Construction 4.5.1.1, it is not necessary to invert ℓ: one can show that the cotangent fiber $\cot[\overline{\mathrm{BG}}]_{\overline{X}}$ already admits the structure of a $\mathbf{Q}_\ell$-module (see Remark 4.2.6.2).

Our interest in Construction 4.5.1.1 stems from the following consequence of Theorem 4.2.6.1:

Proposition 4.5.1.3. *There is a canonical quasi-isomorphism of chain complexes*

$$C^*(\overline{X}; \mathcal{M}(G)) \to \cot(C^*_{\mathrm{gm}}(\mathrm{Bun}_G(X))).$$

The relative motive $\mathcal{M}(G)$ is closely related to the motives defined in §4.4.

Remark 4.5.1.4. Let $x \in \overline{X}(\overline{\mathbf{F}}_q)$. Using Remark 4.2.1.6 and Proposition 3.4.3.2, we obtain equivalences

$$\begin{aligned} x^! \mathcal{M}(G) &= x^!(\cot[\overline{\mathrm{BG}}]_{\overline{X}})[\ell^{-1}] \\ &\simeq \cot(x^![\overline{\mathrm{BG}}]_{\overline{X}}[\ell^{-1}]) \\ &\simeq \cot(C^*(\overline{\mathrm{BG}}_x; \mathbf{Q}_\ell)) \end{aligned}$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Q}_\ell}$. In particular, we can identify the cohomology of the chain complex $x^! \mathcal{M}(G)$ with the motive $M(\overline{G}_x)$ (see Remark 4.4.3.2).

Remark 4.5.1.5. Let U be the largest open subset of $\overline{X}$ over which the group $\overline{G}$ is semisimple. Then we can choose a surjective étale morphism $V \to U$ and an equivalence

$$V \times_{\overline{X}} \overline{G} \simeq V \times_{\mathrm{Spec}(\overline{\mathbf{F}}_q)} H,$$

where H is a semisimple algebraic group over $\overline{\mathbf{F}}_q$. We then have

$$\begin{aligned} \mathcal{M}(G)|_V &= \cot([\overline{\mathrm{BG}}]_{\overline{X}})[\ell^{-1}]|_V \\ &\simeq \cot([V \times_{\mathrm{Spec}(\overline{\mathbf{F}}_q)} \mathrm{BH}]_V)[\ell^{-1}] \\ &\simeq \cot(C^*(\mathrm{BH}; \mathbf{Z}_\ell) \otimes \omega_V)[\ell^{-1}] \\ &\simeq \cot(C^*(\mathrm{BH}; \mathbf{Q}_\ell)) \otimes \omega_V. \end{aligned}$$

It follows that the ℓ-adic sheaf $\mathcal{M}(G)$ is lisse when restricted to U (in fact, it is even locally constant: after base change to V, it is equivalent to a direct sum of finitely many shifted copies of $\omega_V[\ell^{-1}]$). In particular, for any point $x \in U(\overline{\mathbf{F}}_q)$ we have a canonical equivalence

$$x^* \mathcal{M}(G) \simeq (x^! \mathcal{M}(G)) \otimes_{\mathbf{Q}_\ell} \Sigma^2 \mathbf{Q}_\ell(1)$$

so that the cohomology of $x^* \mathcal{M}(G)$ can be identified with the Tate-twisted motive $M(\overline{G}_x) \otimes_{\mathbf{Q}_\ell} \mathbf{Q}_\ell(1)$.

4.5.2 Estimating the Eigenvalues of Frobenius

We now address the convergence of the pair $(\mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X)), \mathrm{Frob}^{-1})$.

Proposition 4.5.2.1. *The cohomology* $\mathrm{H}^*(\overline{X}; \mathcal{M}(G))$ *is a finite-dimensional vector space over* $\mathbf{Q}_\ell$*. Moreover, each eigenvalue of the Frobenius map* Frob *on* $\mathrm{H}^*(\overline{X}; \mathcal{M}(G))$ *has complex absolute value* $\geq q$.

Proof. This can be deduced from Deligne's work on the Weil conjectures ([11]). However, we will proceed in a more elementary way. Let H denote a split form of the generic fiber of G, regarded as an algebraic group over $\mathbf{F}_q$. Choose a finite generically étale map $X' \to X$, where X' is a smooth connected curve over $\mathbf{F}_q$ (not necessarily geometrically connected) and the groups $H \times_{\mathrm{Spec}(\mathbf{F}_q)} X'$ and $G \times_X X'$ are isomorphic at the generic point of X'. Then there exists a dense open subset $U' \subseteq X'$ and an isomorphism

$$\alpha : H \times_{\mathrm{Spec}(\mathbf{F}_q)} U' \simeq G \times_X U'$$

of group schemes over U'. Shrinking U' if necessary, we may assume that U' is the inverse image of a dense open subset $U \subseteq X$, and that the map $U' \to U$ is finite étale.

Let $\{x_1, \ldots, x_n\}$ be the set of closed points of X which do not belong to U. Replacing $\mathbf{F}_q$ by a finite extension if necessary, we may assume that each x_i is defined over $\mathbf{F}_q$. Let $f_i : \mathrm{Spec}(\overline{\mathbf{F}}_q) \to \overline{X}$ denote the map determined by x_i and let $\overline{U} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} U$, so that we have an exact sequence

$$\bigoplus_{1 \leq i \leq n} \mathrm{H}^*(f_i^! \, \mathcal{M}(G)) \to \mathrm{H}^*(\overline{X}; \mathcal{M}(G)) \to \mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_U).$$

It will therefore suffice to prove the following:

- (a) For $1 \leq i \leq n$, the cohomology $\mathrm{H}^*(f_i^! \, \mathcal{M}(G))$ is finite-dimensional and each eigenvalue of Frob on $\mathrm{H}^*(f_i^! \, \mathcal{M}(G))$ has complex absolute value $\geq q$.
- (b) The cohomology $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ is finite-dimensional and each eigenvalue of Frob on $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ has complex absolute value $\geq q$.

Assertion (a) follows immediately from Lemma 4.4.4.2 and the identification

$$\mathrm{H}^*(f_i^! \, \mathcal{M}(G)) \simeq M(G_{x_i})$$

supplied by Remark 4.5.1.4. To prove (b), let $\overline{H} = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} H$, let $\overline{U}' = \mathrm{Spec}(\overline{\mathbf{F}}_q) \times_{\mathrm{Spec}(\mathbf{F}_q)} U'$ and let $\pi : \overline{U}' \to \overline{U}$ denote the projection map. Then $\mathcal{M}(G)|_{\overline{U}}$ is a direct summand of $\pi_* \pi^* \, \mathcal{M}(G)|_{\overline{U}}$, so that $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ is a direct summand of

$$\begin{aligned} \mathrm{H}^*(\overline{U}'; \mathcal{M}(G)|_{\overline{U}'}) &\simeq \mathrm{H}^*(\overline{U}'; \cot(C^*(\overline{\mathrm{BH}}; \mathbf{Q}_\ell)) \otimes \omega_{\overline{U}'}) \\ &\simeq M(H) \otimes_{\mathbf{Q}_\ell} \mathrm{H}^{*+2}(\overline{U}'; \mathbf{Q}_\ell(1)). \end{aligned}$$

The finite-dimensionality of $\mathrm{H}^*(\overline{U}; \mathcal{M}(G)|_{\overline{U}})$ follows immediately. To prove the assertion about Frobenius eigenvalues, we note that each eigenvalue of Frob on $\mathrm{H}^*(\overline{U}'; \mathbf{Q}_\ell)$ has complex absolute value ≥ 1 and therefore each eigenvalue of Frob on $\mathrm{H}^*(\overline{U}'; \mathbf{Q}_\ell(1))$ has complex absolute value $\geq q^{-1}$. We are therefore reduced to proving that each eigenvalue of Frob on $M(H)$ has complex absolute value $\geq q^2$, which follows from Lemma 4.4.4.2. □

We will also need the following assertion, whose proof will appear in the sequel to this book:

Proposition 4.5.2.2. *The moduli stack* $\mathrm{Bun}_G(X)$ *is connected.*

Using Propositions 4.5.2.1 and 4.5.1.3, we deduce that the cotangent fiber

$$\cot(C^*_{\mathrm{gm}}(\mathrm{Bun}_G(X)))$$

has finite-dimensional cohomologies, and that every eigenvalue of Frob^{-1} on the cohomology of $\cot(C^*_{\mathrm{gm}}(\mathrm{Bun}_G(X)))$ has complex absolute value < 1. Moreover, Proposition 4.5.2.2 guarantees that the cohomology group $\mathrm{H}^0_{\mathrm{gm}}(\mathrm{Bun}_G(X))$ is isomorphic to $\mathbf{Q}_\ell$. Invoking Proposition 4.3.2.1, we obtain the following preliminary version of Theorem 4.5.0.1:

Corollary 4.5.2.3. *The pair* $(\mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X)); \mathrm{Frob}^{-1})$ *is convergent. Moreover, we have*

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X))) = \exp(\sum_{n>0} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} \,|\, \mathrm{H}^*(\overline{X}; \mathcal{M}(G)))).$$

In particular, the sum on the right is absolutely convergent.

4.5.3 The Proof of Theorem 4.5.0.1

For each integer $n > 0$, the Grothendieck-Lefschetz trace formula supplies equalities

$$\begin{aligned} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} \,|\, \mathrm{H}^*(\overline{X}; \mathcal{M}(G))) &= \frac{1}{n} \sum_{\eta \in X(\mathbf{F}_{q^n})} \mathrm{Tr}(\mathrm{Frob}^{-n} \,|\, \mathrm{H}^*(\eta^! \mathcal{M}(G))) \\ &= \sum_{\eta \in X(\mathbf{F}_{q^n})} \frac{1}{n} \mathrm{Tr}(\mathrm{Frob}^{-n} \,| M(\overline{G}_\eta)) \\ &= \sum_{n = e \deg(x)} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)) \end{aligned}$$

where the latter sum is taken over all closed points $x \in X$ whose degree divides n, and Frob_x denotes the geometric Frobenius at the point x. Combining this with Corollary 4.5.2.3, we obtain an equality

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X); \mathbf{Q}_\ell)) = \exp(\sum_{n>0} \sum_{e \deg(x) = n} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)). \tag{4.5}$$

Proposition 4.5.3.1. *The double summation appearing in formula (4.5) is absolutely convergent.*

Proof. For each closed point $x \in X$, let $\lambda_{x,1}, \ldots, \lambda_{x,m_x} \in \mathbf{C}$ denote the eigenvalues of Frob_x on $\mathbf{C} \otimes_{\mathbf{Q}_\ell} M(G_x)$, so that $\mathrm{Tr}(\mathrm{Frob}_x^{-e} | M(G_x)) = \sum_{1 \leq i \leq m_x} \lambda_{x,i}^{-e}$. We will show that the triple sum

$$\sum_{n>0} \sum_{e \deg(x)=n} \frac{1}{e} \sum_{1 \leq i \leq m_x} |\lambda_{x,i}^{-e}|$$

is convergent.

For each integer d, set

$$C_d = \sum_{\deg(x)=d} \sum_{e>0} \sum_{1 \leq i \leq m_x} \frac{1}{e} |\lambda_{x,i}^{-e}|;$$

we wish to show that that each C_d is finite and that the sum $\sum_{d>0} C_d$ is convergent. Let g denote the genus of the curve X, so that we have an inequality $|X(\mathbf{F}_{q^d})| \leq q^d + 2gq^{\frac{d}{2}} + 1$. It follows that the number of closed points of X_0 having degree exactly d is bounded above by $d^{-1}(q^d + 2gq^{\frac{d}{2}} + 1)$. Let H be a split form of the generic fiber of G and let r denote the dimension of $M(H)$ as a vector space over $\mathbf{Q}_\ell$ (the number r is equal to the rank of the generic fiber of G, but we will not need to know this). For all but finitely many closed points $x \in X$, the motive $M(G_x)$ is isomorphic to $M(H)$ as a $\mathbf{Q}_\ell$-vector space (see Remark 4.5.1.5) so that $m_x = r$. In this case, each of the eigenvalues $\lambda_{x,i}$ has complex absolute value $\geq q^2$ (Lemma 4.4.4.2). For $d \gg 0$, we have

$$\begin{aligned} C_d &\leq \frac{q^d + 2gq^{\frac{d}{2}} + 1}{d} r \sum_{e>0} \frac{1}{e} q^{-2de} \\ &\leq (2g+2) q^d r \sum_{e>0} q^{-2de} \\ &\leq (2g+2) q^d r \frac{q^{-2d}}{1 - q^{-2d}} \\ &\leq \frac{(2g+2)r}{1 - q^{-2}} q^{-d}. \end{aligned}$$

It follows that the series $\sum_{d>0} C_d$ is dominated (apart from finitely many terms) by the geometric series $\sum_{d>0} \frac{(2g+2)r}{1-q^{-2}} q^{-d}$ and is therefore convergent. □

Proof of Theorem 4.5.0.1. By virtue of Proposition 4.5.3.1, we are free to rearrange the order of summation appearing in formula (4.5). We therefore obtain

$$\begin{aligned}\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathrm{Bun}_G(X))) &= \exp(\sum_{n>0} \sum_{e \deg(x)=n} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)) \\ &= \exp(\sum_{x \in X} \sum_{e>0} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)) \\ &= \prod_{x \in X} \exp(\sum_{e>0} \frac{1}{e} \mathrm{Tr}(\mathrm{Frob}_x^{-e} \,| M(G_x)) \\ &= \prod_{x \in X_0} \frac{|\kappa(x)|^{\dim(G)}}{|G_x(\kappa(x))|},\end{aligned}$$

where the last equality follows from Proposition 4.4.4.1. □

Remark 4.5.3.2. To the relative motive $\mathcal{M}(G)$ we can associate an L-function

$$L_{\mathcal{M}(G),\mathrm{Frob}^{-1}}(t) = \det(1 - t\,\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\overline{X}; \mathcal{M}(G)))^{-1},$$

which is a rational function of t. The proof of Proposition 4.5.3.1 shows that this L-function admits an Euler product expansion

$$L_{\mathcal{M}(G),\mathrm{Frob}^{-1}}(t) = \prod_{x \in X} L_{M(G_x),\mathrm{Frob}_x^{-1}}(t)$$

where the product on the right hand side converges absolutely for $|t| < q$. Combining this observation with Steinberg's formula (Proposition 4.4.4.1), we obtain

$$L_{\mathcal{M}(G),\mathrm{Frob}^{-1}}(1) = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G)}}{|G(\kappa(x))|}.$$

The right hand side of this formula is given by

$$q^{-\dim \mathrm{Bun}_G(X)} \tau(G)^{-1} \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|},$$

where $\tau(G) = \mu_{\mathrm{Tam}}(G(K_X)\backslash G(\mathbf{A}))$ denotes the Tamagawa number of G (see the discussion preceding Conjecture 1.3.3.7). We therefore obtain an equality

$$\tau(G) L_{\mathcal{M}(G),\mathrm{Frob}^{-1}}(1) = q^{-\dim \mathrm{Bun}_G(X)} \sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|},$$

which we can regard as a function field analogue of Theorem 9.9 of [14].

Chapter 5

The Trace Formula for $\mathrm{Bun}_G(X)$

Throughout this chapter, we fix a finite field $\mathbf{F}_q$ with q elements, an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$, a prime number ℓ which is invertible in $\mathbf{F}_q$, and an embedding of fields $\iota : \mathbf{Q}_\ell \hookrightarrow \mathbf{C}$.

Definition 5.0.0.1. Let $\mathcal{X}$ be a smooth algebraic stack of dimension d over $\mathbf{F}_q$. We will say that $\mathcal{X}$ *satisfies the Grothendieck-Lefschetz trace formula* if the following assertions hold:

(1) The pair $(\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}), \mathrm{Frob}^{-1})$ is convergent (in the sense of Definition 4.3.1.1), where

$$\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}) = \mathrm{H}^*(\mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q); \mathbf{Z}_\ell)[\ell^{-1}]$$

denotes the geometric cohomology of $\mathcal{X}$ (see Notation 4.4.1.1), and Frob^{-1} is the arithmetic Frobenius map.

(2) We have an equality

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})) = \frac{|\,\mathcal{X}(\mathbf{F}_q)|}{q^d},$$

where $|\,\mathcal{X}(\mathbf{F}_q)| = \sum_{\eta \in \mathcal{X}(\mathbf{F}_q)} \frac{1}{|\,\mathrm{Aut}(\eta)|}$ denotes the mass of the groupoid $\mathcal{X}(\mathbf{F}_q)$ (see Definition 1.3.3.1).

Remark 5.0.0.2. If $\mathcal{X}$ is a smooth algebraic stack over $\mathbf{F}_q$ which satisfies the Grothendieck-Lefschetz trace formula, then the mass $|\,\mathcal{X}(\mathbf{F}_q)|$ is finite.

Our goal in this chapter is to prove Theorem 1.4.4.1, which we formulate as follows:

Theorem 5.0.0.3. *Let X be an algebraic curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X. Suppose that the fibers of G are connected and that the generic fiber of G is semisimple. Then the moduli stack $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula.*

In the special case where *every* fiber of G is semisimple, Theorem 5.0.0.3 was proved by Behrend in [4]. Let us now give an outline of our proof, which will closely follow the methods used in [4].

For algebraic stacks of finite type over $\mathbf{F}_q$, the Grothendieck-Lefschetz trace formula was verified by Behrend in [4]. In §5.1, we prove a weaker version of this result: any (smooth) global quotient stack Y/H (where H is affine) satisfies the Grothendieck-Lefschetz trace formula (Corollary 5.1.0.4). This is quite relevant to the proof of Theorem 5.0.0.3, since any quasi-compact open substack $\mathrm{Bun}_G(X)$ can be presented as a global quotient stack (see Corollary 5.4.1.4).

Unfortunately, we cannot deduce Theorem 5.0.0.3 directly from the Grothendieck-Lefschetz trace formula for global quotient stacks because the moduli stack $\mathrm{Bun}_G(X)$ is usually not quasi-compact. Our strategy instead will be to decompose $\mathrm{Bun}_G(X)$ into locally closed substacks $\mathrm{Bun}_G(X)_{[P,\nu]}$ which are more directly amenable to analysis. In §5.2, we lay the foundations by reviewing the notion of a *stratification* of an algebraic stack $\mathcal{X}$. Our main result is that if $\mathcal{X}$ is a smooth algebraic stack over $\mathbf{F}_q$ which admits a stratification by locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ which satisfy the Grothendieck-Lefschetz trace formula, then $\mathcal{X}$ also satisfies the Grothendieck-Lefschetz trace formula provided that a certain convergence condition is satisfied (Proposition 5.2.2.5; see also Proposition 5.2.2.3).

To apply the results of §5.2 to our situation, we need to choose a useful stratification of $\mathrm{Bun}_G(X)$. In §5.3.2, we specialize to the case where G is a split group and review the theory of the *Harder-Narasimhan stratification*, which supplies a decomposition of $\mathrm{Bun}_G(X)$ into locally closed substacks $\mathrm{Bun}_G(X)_{P,\nu}$ where P ranges over standard parabolic subgroups of G and ν ranges over dominant regular cocharacters of the center of the Levi quotient $P/\operatorname{rad}_u P$. At the present level of generality, this theory was developed by Behrend and was the main tool used in his proof of Theorem 5.0.0.3 in the case where G is everywhere semisimple.

In order for a stratification of $\mathrm{Bun}_G(X)$ to be useful to us, we will need to know that the individual strata are more tractable than the entire moduli stack $\mathrm{Bun}_G(X)$ itself: for example, we would like to know that they are quasi-compact. In §5.4, we recall the proof that the Harder-Narasimhan strata $\mathrm{Bun}_G(X)_{P,\nu}$ are quasi-compact (Proposition 5.4.3.1) in the case of a split group G, and provide a number of tools for establishing related results (by studying the compactness properties of *morphisms* between moduli stacks of the form $\mathrm{Bun}_G(X)$ as G and X vary).

In order to prove that the moduli stack $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula, it is not enough to know that $\mathrm{Bun}_G(X)$ can be decomposed into locally closed substacks $\mathrm{Bun}_G(X)_\alpha$ which satisfy the Grothendieck-Lefschetz trace formula:

for example, we also need to know that the sum

$$|\operatorname{Bun}_G(X)(\mathbf{F}_q)| = \sum_\alpha |\operatorname{Bun}_G(X)_\alpha(\mathbf{F}_q)|$$

converges. In the case of the Harder-Narasimhan stratification, the key observation is that the infinite collection of Harder-Narasimhan strata $\{\operatorname{Bun}_G(X)_{P,\nu}\}$ can be decomposed into finitely many families whose members "look alike" (for example, members of the same family have the same ℓ-adic cohomology). In the case where G is split, one can prove this by comparing $\operatorname{Bun}_G(X)_{P,\nu}$ with the moduli stack of semistable bundles for the reductive quotient $P/\operatorname{rad}_u P$ of P. However, this maneuver does not generalize to our situation. In §5.5, we discuss a different mechanism which guarantees the same behavior: given a G-bundle $\mathcal{P}$ equipped with a reduction to a parabolic subgroup $P \subseteq G$, there is a "twisting" procedure (depending on a few auxiliary choices) for producing a *new* G-bundle $\operatorname{Tw}_{\lambda,D}(\mathcal{P})$. Roughly speaking, this twisting procedure supplies maps

$$\operatorname{Bun}_G(X)_{P,\nu} \to \operatorname{Bun}_G(X)_{P,\nu+\deg(D)\lambda}$$

which exhibit the left hand side as a fiber bundle over the right hand side, whose fibers are affine spaces (strictly speaking, this is only true if we assume that $\mathbf{F}_q$ is a field of sufficiently large characteristic; in general, the twisting construction is only defined "up to" a finite radicial map); see Proposition 5.5.6.1.

The Harder-Narasimhan stratification of §5.3 is defined only in the special case where G is split (or, more generally, an inner form of a split group). To treat the general case, we note that the generic fiber of G is a semisimple algebraic group over K_X, and therefore splits after passing to some finite Galois extension L of the fraction field K_X. The field L is then the function field of an algebraic curve $\widetilde{X}$ which is generically étale over X (though not necessarily geometrically connected as an $\mathbf{F}_q$-scheme), and the generic fiber of $G \times_X \widetilde{X}$ is split. In particular, there exists a semisimple group scheme $\widetilde{G}$ over $\widetilde{X}$ and an isomorphism β between $\widetilde{G}$ and $G \times_X \widetilde{X}$ over a dense open subset $U \subseteq \widetilde{X}$. In §5.6, we will show that the group scheme $\widetilde{G}$ can be chosen to admit an action of $\operatorname{Gal}(L/K_X)$ (compatible with the action of $\operatorname{Gal}(L/K_X)$ on $\widetilde{X}$) and that the isomorphism β can be chosen to be Γ-equivariant (Proposition 5.6.2.1). There is then a close relationship between G-bundles on X and Γ-equivariant $\widetilde{G}$-bundles on $\widetilde{X}$, which we will use to "descend" the Harder-Narasimhan stratification of $\operatorname{Bun}_{\widetilde{G}}(\widetilde{X})$ to a stratification of $\operatorname{Bun}_G(X)$ (at least after replacing G by a suitable dilitation). We will show that the latter stratification satisfies the axiomatics developed in §5.2, and thereby obtain a proof of Theorem 5.0.0.3.

Remark 5.0.0.4. It is possible to develop a theory of *compactly supported* cohomology for algebraic stacks over $\overline{\mathbf{F}}_q$. If $\mathcal{X}$ is a smooth algebraic stack of dimension d over $\mathbf{F}_q$

for which the cohomology

$$\mathrm{H}^*_c(\mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q); \mathbf{Z}_\ell)$$

is a finitely generated $\mathbf{Z}_\ell$-module in each degree, then it follows from Poincaré duality that $(\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}); \mathrm{Frob}^{-1})$ is convergent if and only if

$$(\mathrm{H}^*_c(\mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q); \mathbf{Z}_\ell)[\ell^{-1}], \mathrm{Frob})$$

is convergent, and in this event we have

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})) = \frac{\mathrm{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^*_c(\mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q); \mathbf{Z}_\ell)[\ell^{-1}])}{q^d}.$$

In this case, $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula if and only if

$$\mathrm{Tr}(\mathrm{Frob} \,|\, \mathrm{H}^*_c(\mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q); \mathbf{Z}_\ell)[\ell^{-1}]) = |\, \mathcal{X}(\mathbf{F}_q)|.$$

Note that this condition makes sense even when $\mathcal{X}$ is not smooth. However, we will confine our attention to smooth algebraic stacks in this book (since they are all that is needed for the proof of Weil's conjecture).

5.1 The Trace Formula for a Quotient Stack

Our primary goal in this section is to prove the following result:

Proposition 5.1.0.1. *Let $\mathcal{X}$ be a smooth algebraic stack over $\mathbf{F}_q$, let G be a connected linear algebraic group over $\mathbf{F}_q$, and suppose that G acts on $\mathcal{X}$. If $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula, then so does the stack-theoretic quotient $\mathcal{X}/G$.*

Remark 5.1.0.2. In the statement of Proposition 5.1.0.1, the assumption that G is affine is not needed. However, the affine case will be sufficient for our applications.

Example 5.1.0.3. Let G be a linear algebraic group over $\mathbf{F}_q$. Applying Proposition 5.1.0.1 in the case $\mathcal{X} = \mathrm{Spec}(\mathbf{F}_q)$, we deduce that the classifying stack BG satisfies the Grothendieck-Lefschetz trace formula. We proved this result in Chapter 4 as Proposition 4.4.4.1. However, the proof of Proposition 5.1.0.1 that we present in this section will *use* Proposition 4.4.4.1.

Corollary 5.1.0.4. *Let Y be a smooth $\mathbf{F}_q$-scheme of finite type and let G be a linear algebraic group over $\mathbf{F}_q$ which acts on Y. Then the stack-theoretic quotient Y/G satisfies the Lefschetz trace formula.*

Proof. Since G is affine, there exists an embedding of algebraic groups $G \hookrightarrow \mathrm{GL}_n$. Replacing Y by $(Y \times \mathrm{GL}_n)/G$ and G by GL_n, we can reduce to the case where $G = \mathrm{GL}_n$ and in particular where G is connected. In this case, the desired result follows immediately from Proposition 5.1.0.1 (together with the classical Grothendieck-Lefschetz trace formula). □

5.1.1 A Convergence Lemma

Our proof of Proposition 5.1.0.1 will use the following elementary convergence result, which will appear again in §5.2:

Lemma 5.1.1.1. *Let V be an object of the ∞-category $\mathrm{Mod}_{\mathbf{Q}_\ell}$ which is given as the inverse limit of a tower*

$$\cdots \to V(n+1) \to V(n) \to V(n-1) \to \cdots \to V(0) \to V(-1) \simeq 0.$$

Let F be an automorphism of the tower $\{V(n)\}_{n\geq 0}$, and denote also by F the induced automorphism of V. For each $n \geq 0$, let $W(n)$ denote the fiber of the map $V(n) \to V(n-1)$. Suppose that the following conditions are satisfied:

(a) Each of the pairs $(\mathrm{H}^* W(n), F)$ is convergent, in the sense of Definition 4.3.1.1.

(b) The sum $\sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_F$ converges absolutely (see Definition 4.3.1.1).

(c) For each d, there exists an integer n_0 such that $W(n) \in (\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -d}$ for $n \geq n_0$.

Then the pair $(\mathrm{H}^*(V), F)$ is convergent. Moreover, we have

$$|\,\mathrm{H}^*(V)|_F \leq \sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_F$$

$$\mathrm{Tr}(F|\,\mathrm{H}^*(V)) = \sum_{n\geq 0} \mathrm{Tr}(F|\,\mathrm{H}^*(W(n))).$$

Proof. Since each pair $(\mathrm{H}^*(W(n)), F)$ is convergent, the graded vector spaces $\mathrm{H}^*(W(n))$ are finite-dimensional in each degree. It follows by induction on n that the graded vector spaces $\mathrm{H}^*(V(n))$ are also finite-dimensional in each degree. Assumption (c) implies that for any fixed d, we have $\mathrm{H}^d(V) \simeq \mathrm{H}^d(V(n))$ for $n \gg 0$, so that the graded vector space $\mathrm{H}^*(V)$ is also finite-dimensional in each degree.

For each integer d, let $|\,\mathrm{H}^d(V)|_F$ denote the sum of the absolute values of the eigenvalues of F (counted with multiplicity) on the complex vector space $\mathrm{H}^d(V) \otimes_{\mathbf{Q}_\ell} \mathbf{C}$. Let $C = \sum_{n\geq 0} |\,\mathrm{H}^*(W(n))|_F$; we wish to show that the sum $\sum_{d\in\mathbf{Z}} |\,\mathrm{H}^d(V)|_F$ is bounded by C. To prove this, it suffices to show that for each integer d_0, the partial sum $\sum_{d\leq d_0} |\,\mathrm{H}^d(V)|_F$ is bounded above by C. Using assumption (c), we deduce that there exists an integer n such that $\mathrm{H}^d(V) \simeq \mathrm{H}^d(V(n))$ for $d \leq d_0$. It will therefore suffice to show that $\sum_{d \leq d_0} |\operatorname{H}^d(V(n))|_F \leq C$. This is clear: we have

$$\begin{aligned}\sum_{d \leq d_0} |\operatorname{H}^d(V(n))|_F &\leq \sum_{d \in \mathbf{Z}} |\operatorname{H}^d(V(n))|_F \\ &= |\operatorname{H}^*(V(n))|_F \\ &\leq \sum_{0 \leq m \leq n} |\operatorname{H}^*(W(m))|_F \\ &\leq \sum_{0 \leq m} |\operatorname{H}^*(W(m))|_F \\ &= C,\end{aligned}$$

where the second inequality follows from iterated application of Remark 4.3.1.3.

We now complete the proof by verifying the identity

$$\operatorname{Tr}(F | \operatorname{H}^*(V)) = \sum_{n \geq 0} \operatorname{Tr}(F | \operatorname{H}^*(W(n))).$$

Fix a real number $\epsilon > 0$; we will show that the difference

$$|\operatorname{Tr}(F | \operatorname{H}^*(V)) - \sum_{n \geq 0} \operatorname{Tr}(F | \operatorname{H}^*(W(n)))|$$

is bounded by ϵ. Using assumption (b), we deduce that there exists an integer $n_0 \geq 0$ for which the sum $\sum_{n > n_0} |\operatorname{H}^*(W(n))|_F$ is bounded above by $\frac{\epsilon}{2}$. Form a fiber sequence

$$U \to V \to V(n_0).$$

Applying the first part of the proof to U, we deduce that $(\operatorname{H}^*(U), F)$ is convergent with

$$|\operatorname{H}^*(U)|_F \leq \sum_{n > n_0} |\operatorname{H}^*(W(n))|_F \leq \frac{\epsilon}{2}.$$

Using Remark 4.3.1.3, we obtain

$$\begin{aligned}\operatorname{Tr}(F | \operatorname{H}^*(V)) &= \operatorname{Tr}(F | \operatorname{H}^*(U)) + \operatorname{Tr}(F | \operatorname{H}^*(V(n_0))) \\ &= \operatorname{Tr}(F | H^*(U)) + \sum_{0 \leq n \leq n_0} \operatorname{Tr}(F | \operatorname{H}^*(W(n)).\end{aligned}$$

Subtracting $\sum_{n \geq 0} \operatorname{Tr}(F | \operatorname{H}^*(W(n)))$ from both sides and taking absolute values, we obtain

$$\begin{aligned}
|\operatorname{Tr}(F|\operatorname{H}^*(V)) - \sum_{n\geq 0} \operatorname{Tr}(F|\operatorname{H}^* W(n))| &= |\operatorname{Tr}(F|\operatorname{H}^*(U)) - \sum_{n>n_0} \operatorname{Tr}(F|\operatorname{H}^* W(n))| \\
&\leq |\operatorname{Tr}(F|\operatorname{H}^*(U))| + \sum_{n>n_0} |\operatorname{Tr}(F|\operatorname{H}^* W(n)) \\
&\leq |\operatorname{H}^*(U)|_F + \sum_{n>n_0} |\operatorname{H}^*(W(n))|_F \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

as desired. □

5.1.2 The Proof of Proposition 5.1.0.1

We now turn to the proof of Proposition 5.1.0.1. Let $\mathcal{X}$ be a smooth algebraic stack over $\mathbf{F}_q$ which satisfies the Grothendieck-Lefschetz trace formula and let G be a connected linear algebraic group over $\mathbf{F}_q$ which acts on $\mathcal{X}$. We wish to show that the stack-theoretic quotient $\mathcal{X}/G$ also satisfies the Grothendieck-Lefschetz trace formula. Let BG denote the classifying stack of G, so that we have a pullback diagram

$$\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{X}/G \\
\downarrow & & \downarrow \\
\operatorname{Spec}(\mathbf{F}_q) & \longrightarrow & \mathrm{BG}.
\end{array}$$

Applying Proposition 4.4.2.1, we deduce that the induced diagram

$$\begin{array}{ccc}
C^*_{\mathrm{gm}}(\mathcal{X}) & \longleftarrow & C^*_{\mathrm{gm}}(\mathcal{X}/G) \\
\uparrow & & \uparrow \\
C^*_{\mathrm{gm}}(\operatorname{Spec}(\mathbf{F}_q)) & \longleftarrow & C^*_{\mathrm{gm}}(\mathrm{BG})
\end{array}$$

is a pushout square in $\mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Q}_\ell})$. In other words, we have a canonical equivalence

$$C^*_{\mathrm{gm}}(\mathcal{X}) \simeq C^*_{\mathrm{gm}}(\mathcal{X}/G) \otimes_{C^*_{\mathrm{gm}}(\mathrm{BG})} \mathbf{Q}_\ell .$$

Let us regard $C^*_{\mathrm{gm}}(\mathrm{BG})$ as an augmented commutative algebra object of $\mathrm{Mod}_{\mathbf{Q}_\ell}$. Let $\mathfrak{m}$ denote its augmentation ideal, and consider the filtration

$$\cdots \to \mathfrak{m}^{(3)} \to \mathfrak{m}^{(2)} \to \mathfrak{m}^{(1)} \to C^*_{\mathrm{gm}}(\mathrm{BG})$$

introduced in §4.2. We claim that this tower can be regarded as a diagram of $C^*_{\mathrm{gm}}(\mathrm{BG})$-modules, and that the induced action of $C^*_{\mathrm{gm}}(\mathrm{BG})$ on each cofiber

$$\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)} = \mathrm{cofib}(\mathfrak{m}^{(n+1)} \to \mathfrak{m}^{(n)})$$

factors through the augmentation $C^*_{\mathrm{gm}}(\mathrm{BG}) \to \mathbf{Q}_\ell$. This is a general feature of the constructions described in §4.2, but can easily be deduced in this special case from the observation that $\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})$ is a polynomial ring on generators of even degrees so that $C^*_{\mathrm{gm}}(\mathrm{BG})$ is equivalent to a symmetric algebra $\mathrm{Sym}^*(V)$ for some chain complex V concentrated in even degrees (see the proof of Proposition 4.2.3.1), together with the identifications $\mathfrak{m}^{(n)} \simeq \mathrm{Sym}^{\geq n} V$ supplied by Example 4.2.1.7.

It follows from Proposition 4.2.5.1 that each $\mathfrak{m}^{(n)}$ is belongs to $(\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -n}$ (in fact, the preceding argument even shows that $\mathfrak{m}^{(n)} \in (\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -2n}$). Let $\overline{\mathrm{BG}} = \mathrm{BG} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$. Then $C^*_{\mathrm{gm}}(\mathrm{BG}) = C^*(\overline{\mathrm{BG}}; \mathbf{Z}_\ell)[\ell^{-1}]$. We therefore have equivalences

$$\mathfrak{m}^{(n)} \otimes_{C^*_{\mathrm{gm}}(\mathrm{BG})} C^*_{\mathrm{gm}}(\mathcal{X}/G) \simeq \mathfrak{m}^{(n)} \otimes_{C^*(\overline{\mathrm{BG}}; \mathbf{Z}_\ell)} C^*_{\mathrm{gm}}(\mathcal{X}/G).$$

Applying Lemma 4.4.2.3, we conclude that each tensor product $\mathfrak{m}^{(n)} \otimes_{C^*_{\mathrm{gm}}(\mathrm{BG})} C^*_{\mathrm{gm}}(\mathcal{X}/G)$ belongs to $(\mathrm{Mod}_{\mathbf{Q}_\ell})_{\leq -n}$, so that the inverse limit

$$\varprojlim_n \mathfrak{m}^{(n)} \otimes_{C^*_{\mathrm{gm}}(\mathrm{BG})} C^*_{\mathrm{gm}}(\mathcal{X}/G)$$

vanishes. It follows that we can write $C^*_{\mathrm{gm}}(\mathcal{X}/G)$ as the limit of the tower

$$\{(C^*_{\mathrm{gm}}(\mathrm{BG})/\mathfrak{m}^{(n)}) \otimes_{C^*(\mathrm{BG})} C^*(\mathcal{X}/G)\}_{n \geq 0}$$

whose successive quotients are given by

$$\begin{aligned} W(n) &= (\mathfrak{m}^{(n)}/\mathfrak{m}^{n+1}) \otimes_{C^*_{\mathrm{gm}}(\mathrm{BG})} C^*_{\mathrm{gm}}(\mathcal{X}/G) \\ &\simeq (\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}) \otimes_{\mathbf{Q}_\ell} (\mathbf{Q}_\ell \otimes_{C^*_{\mathrm{gm}}(\mathrm{BG})} C^*_{\mathrm{gm}}(\mathcal{X}/G) \\ &\simeq (\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}) \otimes_{\mathbf{Q}_\ell} C^*_{\mathrm{gm}}(\mathcal{X}). \end{aligned}$$

Since $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula, the pair $(\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}), \mathrm{Frob}^{-1})$ is convergent. It follows that each of the pairs $(\mathrm{H}^*(W(n)), \mathrm{Frob}^{-1})$ is convergent with

$$|\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} = |\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}|_{\mathrm{Frob}^{-1}} |\,\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})|_{\mathrm{Frob}^{-1}}.$$

Since $\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})$ can be identified with the direct sum of the cohomologies of the quotients $\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}$, we get

$$\sum_{n \geq 0} |\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} = |\,\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})|_{\mathrm{Frob}^{-1}} |\,\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})|_{\mathrm{Frob}^{-1}} < \infty,$$

since $|\mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG})|_{\mathrm{Frob}^{-1}} < \infty$ by virtue of Proposition 4.4.4.1. Using Lemma 5.1.1.1, we conclude that the pair $(\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}/G), \mathrm{Frob}^{-1})$ is convergent and we obtain the identity

$$\begin{aligned}\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}/G)) &= \sum_{n \geq 0} \mathrm{Tr}(\mathrm{Frob}^{-1} \,| W(n)) \\ &= \sum_{n \geq 0} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(\mathfrak{m}^{(n)}/\mathfrak{m}^{(n+1)}))\, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})) \\ &= \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathrm{BG}))\, \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})).\end{aligned}$$

Using Proposition 4.4.4.1 and the fact that $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula, we obtain

$$\begin{aligned}\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}/G)) &= \frac{q^{\dim(G)}}{|G(\mathbf{F}_q)|} \frac{|\,\mathcal{X}(\mathbf{F}_q)|}{q^{\dim(\mathcal{X})}} \\ &= q^{-\dim(\mathcal{X}/G)} \frac{|\,\mathcal{X}(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|}.\end{aligned}$$

To complete the proof of Proposition 5.1.0.1, it will suffice to verify the identity

$$|(\mathcal{X}/G)(\mathbf{F}_q)| = \frac{|\,\mathcal{X}(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|}. \tag{5.1}$$

For each object $\eta \in (\mathcal{X}/G)(\mathbf{F}_q)$, let $\mathcal{C}_\eta$ denote the full subcategory of $\mathcal{X}(\mathbf{F}_q)$ spanned by those objects C whose image in $(\mathcal{X}/G)(\mathbf{F}_q)$ is isomorphic to η (where the isomorphism is *not* specified), so that we can write $\mathcal{X}(\mathbf{F}_q)$ as a disjoint union of the groupoids $\mathcal{C}_\eta$ where η ranges over all isomorphism classes of objects of $(\mathcal{X}/G)(\mathbf{F}_q)$. To prove (5.1), it will suffice to show that for each $\eta \in (\mathcal{X}/G)(\mathbf{F}_q)$, we have an equality

$$\frac{1}{|\,\mathrm{Aut}(\eta)|} = \frac{1}{|G(\mathbf{F}_q)|} \sum_{C \in \mathcal{C}_\eta} \frac{1}{|\,\mathrm{Aut}(C)|},$$

where the sum is taken over all isomorphism classes of objects of $\mathcal{C}_\eta$.

The object η can be regarded as a map $\mathrm{Spec}(\mathbf{F}_q) \to (\mathcal{X}/G)$, so we can consider the fiber product $Y = \mathcal{X} \times_{\mathcal{X}/G} \mathrm{Spec}(\mathbf{F}_q)$, which is a torsor for the algebraic group G. The finite group $\mathrm{Aut}(\eta)$ acts on Y, and therefore acts on the finite set $Y(\mathbf{F}_q)$. Unwinding the definitions, we can identify $\mathcal{C}_\eta$ with the groupoid-theoretic quotient of $Y(\mathbf{F}_q)$ by the action of $\mathrm{Aut}(\eta)$. We may therefore identify the set of isomorphism classes of objects of $\mathcal{C}_\eta$ with the set of orbits of $\mathrm{Aut}(\eta)$ acting on $Y(\mathbf{F}_q)$. For each $y \in Y(\mathbf{F}_q)$, the automorphism group of the corresponding object $C \in \mathcal{C}_\eta$ can be identified with the stabilizer $\operatorname{Aut}(\eta)_y = \{\phi \in \operatorname{Aut}(\eta) : \phi(y) = y\}$. We therefore have

$$\begin{aligned}\sum_{C \in \mathcal{C}_\eta} \frac{1}{|\operatorname{Aut}(C)|} &= \sum_{y \in Y(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\eta)/\operatorname{Aut}(\eta)_y|} \frac{1}{|\operatorname{Aut}(\eta)_y|} \\ &= \sum_{y \in Y(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\eta)|} \\ &= \frac{|Y(\mathbf{F}_q)|}{|\operatorname{Aut}(\eta)|}.\end{aligned}$$

To complete the proof, it will suffice to show that the finite sets $Y(\mathbf{F}_q)$ and $G(\mathbf{F}_q)$ have the same size. This follows from Lang's theorem; the $\mathbf{F}_q$-scheme Y is a G-torsor and is therefore G-equivariantly isomorphic to G (by virtue of our assumption that G is connected).

5.1.3 Application: Change of Group

Let X be an algebraic curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X with connected fibers. If the generic fiber G_0 of G is semisimple and simply connected, then Theorem 4.5.0.1 implies that the Tamagawa number $\tau(G_0)$ is given by the ratio

$$\frac{q^{-\dim(\operatorname{Bun}_G(X))} |\operatorname{Bun}_G(X)(\mathbf{F}_q)|}{\operatorname{Tr}(\operatorname{Frob}^{-1} | \operatorname{H}^*_{\mathrm{gm}}(\operatorname{Bun}_G(X)))}.$$

In particular, Weil's conjecture that $\tau(G_0) = 1$ is equivalent to the statement that $\operatorname{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula. It follows that the validity of the Grothendieck-Lefschetz trace formula for $\operatorname{Bun}_G(X)$ depends only on the generic fiber of G. Our goal in this section is to formulate a slightly weaker form of this invariance statement (Proposition 5.1.3.10) which follows directly from Proposition 5.1.0.1, without appealing to the product formula of Chapter 4 or the theory of Tamagawa measure developed in Chapter 1. This result will be used in §5.6 to reduce the proof of trace formula to a convenient special case.

We begin by reviewing a general algebro-geometric construction.

Definition 5.1.3.1. Let X be a Dedekind scheme and let $D \subseteq X$ be an effective divisor. Suppose we are given a flat morphism of schemes $\pi : Y \to X$ equipped with a section $s : X \to Y$. A *dilitation of* Y *along* D is a scheme Y' equipped with a map $\phi : Y' \to Y$ which has the following properties:

(*a*) The composite map $Y' \xrightarrow{\phi} Y \xrightarrow{\pi} X$ is flat.

(b) For every flat X-scheme Z, let $\mathrm{Hom}_X(Z, Y)$ denote the set of X-scheme morphisms from Z to Y and define $\mathrm{Hom}_X(Z, Y')$ similarly. Then composition with ϕ induces a monomorphism of sets $\mathrm{Hom}_X(Z, Y') \to \mathrm{Hom}_X(Z, Y)$, whose image consists of those maps $f : Z \to Y$ for which the diagram of schemes

$$\begin{array}{ccc} Z \times_X D & \longrightarrow & Z \\ \downarrow & & \downarrow f \\ D & \xrightarrow{s|_D} & Y \end{array}$$

is commutative.

In the situation of Definition 5.1.3.1, it is immediate that if a dilitation of Y along D exists, then it is unique up to (unique) isomorphism. For existence, we have the following:

Proposition 5.1.3.2. *Let X be a Dedekind scheme, let $D \subseteq X$ be an effective divisor, and let $\pi : Y \to X$ be a flat morphism equipped with a section s. Then there exists a map $g : Y' \to Y$ which exhibits Y' as a dilitation of Y along D. Moreover, the morphism g is affine.*

Proof. The assertion is local on X and Y. We may therefore assume that $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$ are affine, so that R is a Dedekind ring and A is a flat R-algebra equipped with an R-algebra map $\epsilon : A \to R$. Shrinking further if necessary, we may assume that the divisor $D \subseteq X$ is the vanishing locus of an element $t \in R$. Let B denote the subalgebra of $A[t^{-1}]$ generated by A together with all elements of the form $\frac{a}{t}$, where $a \in \ker(\epsilon)$. It is now easy to check that $Y' = \mathrm{Spec}(B)$ satisfies conditions (a) and (b) of Definition 5.1.3.1. □

Notation 5.1.3.3. Let X be a Dedekind scheme, let $D \subseteq X$ be an effective divisor, and let $\pi : Y \to X$ be a flat morphism equipped with a section s. We let $\mathrm{Dil}^D(Y)$ denote a dilation of Y along D (whose existence is guaranteed by Proposition 5.1.3.2). Beware that this notation is slightly abusive: the dilation $\mathrm{Dil}^D(Y)$ depends also on the choice of section s. However, we will be primarily interested in the case where Y is a group scheme over X, in which case we will take s to be the identity section.

Remark 5.1.3.4. The dilitation $\mathrm{Dil}^D(Y)$ does not depend on the entire section $s : X \to Y$, only on its restriction to the effective divisor $D \subseteq X$.

Example 5.1.3.5. In the situation of Definition 5.1.3.1, if $\pi : Y \to X$ is an isomorphism, then the identity map $\mathrm{id} : Y \to Y$ exhibits Y as a dilitation of itself along any effective divisor D.

Remark 5.1.3.6. In the situation of Definition 5.1.3.1, the dilitation $\mathrm{Dil}^D(Y)$ can be identified with an open subset of the scheme $\overline{Y}$ given by the blowup of Y along the closed subscheme $s(D) \subseteq Y$.

Remark 5.1.3.7. In the situation of Definition 5.1.3.1, suppose that the map $\pi : Y \to X$ is smooth. Then the composite map $\mathrm{Dil}^D(Y) \xrightarrow{\phi} Y \to X$ is also smooth. Moreover, the relative tangent bundle $T_{\mathrm{Dil}^D(Y)/X}$ can be identified with the pullback $\phi^* T_{Y/X}(-D)$.

Remark 5.1.3.8. In the situation of Definition 5.1.3.1, assumption (b) guarantees that the section $s : X \to Y$ lifts uniquely to a map $\widetilde{s} : X \to \mathrm{Dil}^D(Y)$, which can be regarded as a section of the composite map $\mathrm{Dil}^D(Y) \to Y \to X$.

Remark 5.1.3.9 (Functoriality)**.** Let X be a Dedekind scheme equipped with an effective divisor $D \subseteq X$. Then the construction $Y \mapsto \mathrm{Dil}^D(Y)$ determines a functor from the category of pointed flat X-schemes (that is, flat X-schemes $\pi : Y \to X$ equipped with a section $s : X \to Y$) to itself. It follows immediately from the definition that this functor preserves finite products. In particular, if G is a flat group scheme over X, then the dilitation $\mathrm{Dil}^D(G)$ inherits the structure of a group scheme over X (and the projection map $\mathrm{Dil}^D(G) \to G$ is compatible with the group structures). Moreover, if G is smooth or affine, then $\mathrm{Dil}^D(G)$ has the same property (see Proposition 5.1.3.2 and Remark 5.1.3.7).

Proposition 5.1.3.10. *Let X be a smooth complete geometrically connected curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X with connected fibers. Let G' be the smooth affine group scheme over X obtained from G by dilitation along an effective divisor $D \subseteq X$. If the algebraic stack $\mathrm{Bun}_{G'}(X)$ satisfies the Grothendieck-Lefschetz trace formula, then so does $\mathrm{Bun}_G(X)$.*

To prove Proposition 5.1.3.10, we will need the following variant of the defining property of dilitations:

Lemma 5.1.3.11. *Let X be a Dedekind scheme, let G be a smooth affine group scheme over X, and let $G' = \mathrm{Dil}^D(G)$ be the dilitation of G along an effective divisor $D \subseteq X$. For any flat X-scheme Z, the tautological map $G' \to G$ induces an equivalence of categories $\mathrm{Tors}_{G'}(Z) \to \mathrm{Tors}_G(Z, Z \times_X D)$; here $\mathrm{Tors}_G(Z, Z \times_X D)$ denotes the category of G-torsors on Z equipped with a trivialization along $Z \times_X D$.*

Proof of Proposition 5.1.3.10 from Lemma 5.1.3.11. Let H be the connected algebraic group obtained from $G \times_X D$ by Weil restriction along the finite flat map $D \to \mathrm{Spec}(\mathbf{F}_q)$. Using Lemma 5.1.3.11, we can identify $\mathrm{Bun}_{G'}(X)$ with the algebraic stack whose R-valued points are pairs $(\mathcal{P}, \gamma)$, where $\mathcal{P}$ is a G-bundle on X_R and γ is a trivialization of $\mathcal{P}$ on the divisor $D_R = D \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(R)$. The algebraic group H acts on $\mathrm{Bun}_{G'}(X)$ by changing trivializations and we can identify $\mathrm{Bun}_G(X)$ with the stack-theoretic quotient $\mathrm{Bun}_{G'}(X)/H$. The desired result now follows from Proposition 5.1.0.1. $\square$

Proof of Lemma 5.1.3.11. We first show that the functor

$$\theta : \mathrm{Tors}_{G'}(Z) \to \mathrm{Tors}_G(Z, Z \times_X D)$$

is fully faithful. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be G'-torsors on Z, and let $\mathcal{P}$ and $\mathcal{Q}$ be the associated G-torsors. We wish to show that the canonical map

$$\mathrm{Hom}_{\mathrm{Tors}_{G'}(Z)}(\mathcal{P}', \mathcal{Q}') \to \mathrm{Hom}_{\mathrm{Tors}_G(Z, Z \times_X D)}(\mathcal{P}, \mathcal{Q})$$

is bijective. This assertion can be tested locally with respect to the étale topology on Z. We may therefore assume that $\mathcal{P}'$ and $\mathcal{Q}'$ are trivial, in which case the assertion reduces to the bijectivity of the map

$$\mathrm{Hom}_X(Z, G') \to \{f : Z \to G : f|_{Z \times_X D} = \mathrm{id}\},$$

which is a defining property of the dilitation G' (Definition 5.1.3.1).

We now argue that θ is essentially surjective. Let $\mathcal{P}$ be a G-torsor on Z equipped with a trivialization s_D along $Z_D = Z \times_X D$; we wish to show that $\mathcal{P}$ belongs to the essential image of θ. By virtue of the first part of the proof, this assertion is local for the flat topology on Z. We may therefore assume without loss of generality that Z is affine and that $\mathcal{P}$ admits a trivialization $s' : \mathcal{P} \simeq Z \times_X G$. In this case, we can identify s_D with a map of X-schemes $f : Z_D \to G$. Replacing Z by either the complement of Z_D or its Henselization along Z_D (which comprise a flat covering of Z), we can reduce to the case where f extends to a map $\overline{f} : Z \to G$. Modifying the trivialization s' by f, we can reduce to the case where $s_D = s'|_{Z \times_X D}$. In this case, $\mathcal{P}$ is isomorphic to the image of the trivial G'-torsor $Z \times_X G'$ under the functor θ. □

5.2 The Trace Formula for a Stratified Stack

Let $\mathcal{X}$ be a smooth algebraic stack over $\mathbf{F}_q$. According to Corollary 5.1.0.4, if $\mathcal{X}$ can be written as the quotient of a quasi-projective variety by the action of a linear algebraic group, then $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula. Unfortunately, this observation is not sufficient for our applications: it does not apply to the moduli stack of bundles $\mathrm{Bun}_G(X)$, because $\mathrm{Bun}_G(X)$ is not quasi-compact (except in trivial cases). However, it *does* apply to any quasi-compact locally closed substack of $\mathrm{Bun}_G(X)$. We therefore encounter the following:

Question 5.2.0.1. Let $\mathcal{X}$ be a smooth algebraic stack equipped with a decomposition into locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ (see Definition 5.2.1.1). Under what conditions on the $\mathcal{X}_\alpha$ can we conclude that $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula?

In this section, we will introduce the notion of a *convergent stratification* of an algebraic stack over $\mathbf{F}_q$ (Definition 5.2.2.1) and show that it provides one answer to Question 5.2.0.1: if an algebraic stack admits a convergent stratification, then it satisfies the Grothendieck-Lefschetz trace formula (Proposition 5.2.2.3). This allows us to reduce Theorem 5.0.0.3 to the problem of finding a convergent stratification of the moduli stack $\mathrm{Bun}_G(X)$, which is the subject of the rest of this chapter.

5.2.1 Stratifications of Algebraic Stacks

We begin by introducing some terminology.

Definition 5.2.1.1. Let $\mathcal{X}$ be an algebraic stack. A *stratification* of $\mathcal{X}$ consists of the following data:

(a) A partially ordered set A.

(b) A collection of open substacks $\{\mathcal{U}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ satisfying $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ when $\alpha \leq \beta$.

This data is required to satisfy the following conditions:

- For each index $\alpha \in A$, the set $\{\beta \in A : \beta \leq \alpha\}$ is finite.
- For every field k and every map $\eta : \mathrm{Spec}(k) \to \mathcal{X}$, the set

$$\{\alpha \in A : \ \eta \text{ factors through } \mathcal{U}_\alpha\}$$

  has a least element.

Notation 5.2.1.2. Let $\mathcal{X}$ be an algebraic stack equipped with a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$. For each $\alpha \in A$, we let $\mathcal{X}_\alpha$ denote the reduced closed substack of $\mathcal{U}_\alpha$ given by the complement of $\bigcup_{\beta < \alpha} \mathcal{U}_\beta$. Each $\mathcal{X}_\alpha$ is a locally closed substack of $\mathcal{X}$; we will refer to these locally closed substacks as the *strata* of $\mathcal{X}$.

Remark 5.2.1.3. Let $\mathcal{X}$ be an algebraic stack. A stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is determined by the partially ordered set A together with the collection of locally closed substacks $\{\mathcal{X}_\alpha\}_{\alpha \in A}$: each $\mathcal{U}_\alpha$ can be characterized by the fact that it is an open substack of $\mathcal{X}$ and that, if k is a field, then a map $\eta : \mathrm{Spec}(k) \to \mathcal{X}$ factors through $\mathcal{U}_\alpha$ if and only if it factors through $\mathcal{X}_\beta$ for some $\beta \leq \alpha$. Because of this, we will generally identify stratification of $\mathcal{X}$ with the collection of locally closed substacks $\{\mathcal{X}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ (where the partial ordering of A is understood to be implicitly specified).

Remark 5.2.1.4. Let $\mathcal{X}$ be an algebraic stack equipped with a stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$. If k is a field, then for any map $\eta : \mathrm{Spec}(k) \to \mathcal{X}$ there is a *unique* index $\alpha \in A$ such that η factors through $\mathcal{X}_\alpha$. In other words, $\mathcal{X}$ is a *set-theoretic* union of the locally closed substacks $\mathcal{X}_\alpha$.

Remark 5.2.1.5 (Functoriality)**.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a map of algebraic stacks. Suppose that $\mathcal{Y}$ is equipped with a stratification $\{\mathcal{U}_\alpha \subseteq \mathcal{Y}\}_{\alpha \in A}$. Then $\{\mathcal{U}_\alpha \times_{\mathcal{Y}} \mathcal{X} \subseteq \mathcal{X}\}_{\alpha \in A}$ is a stratification of $\mathcal{X}$ (indexed by the same partially ordered set A). The corresponding strata of $\mathcal{X}$ are given by the reduced locally closed substacks

$$\mathcal{X}_\alpha = (\mathcal{Y}_\alpha \times_{\mathcal{Y}} \mathcal{X})_{\mathrm{red}}.$$

We now make some elementary observations about the behavior of stratifications with respect to the actions of finite groups, which will be useful in §5.6.

Remark 5.2.1.6 (Stratification of Fixed Point Stacks)**.** Let $\mathcal{X}$ be an algebraic stack equipped with an action of a finite group Γ. Suppose that $\mathcal{X}$ is equipped with a stratification $\{\mathcal{U}_\alpha \subseteq \mathcal{X}\}_{\alpha \in A}$ which is Γ-*equivariant* in the following sense: the group Γ acts on A (by monotone maps) and for each $\alpha \in A$, $\gamma \in \Gamma$ the open substack $\mathcal{U}_{\gamma(\alpha)}$ is the image of $\mathcal{U}_\alpha$ under the automorphism of $\mathcal{X}$ determined by γ.

Let $\mathcal{X}^\Gamma$ denote the (homotopy) fixed point stack for the action of Γ on $\mathcal{X}$, and let A^Γ denote the set of fixed points for the action of Γ on A. For each $\alpha \in A^\Gamma$, the open substack $\mathcal{U}_\alpha \subseteq \mathcal{X}$ inherits an action of Γ, and the fixed point stack $\mathcal{U}_\alpha^\Gamma$ can be regarded as an open substack of $\mathcal{X}^\Gamma$. Moreover, the collection $\{\mathcal{U}_\alpha^\Gamma \subseteq \mathcal{X}^\Gamma\}_{\alpha \in A^\Gamma}$ is a stratification of $\mathcal{X}^\Gamma$. For each $\alpha \in A^\Gamma$, the corresponding locally closed substack of $\mathcal{X}^\Gamma$ can be identified with the reduced stack $((\mathcal{X}_\alpha)^\Gamma)_{\mathrm{red}}$.

Remark 5.2.1.7. Let $\mathcal{X}$ be an algebraic stack equipped with an action of a finite group Γ, and suppose we are given a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ which is Γ-equivariant (as in Remark 5.2.1.6). Let A/Γ denote the quotient of A by the action of Γ, and for each $\alpha \in A$ let $[\alpha]$ denote its image in A/Γ. We can endow A/Γ with the structure of a partially ordered set by writing $[\alpha] \leq [\alpha']$ if there exists an element $\gamma \in \Gamma$ such that $\alpha \leq \gamma(\alpha')$. For each $[\alpha] \in A/\Gamma$, let $\mathcal{U}_{[\alpha]}$ denote the open substack of $\mathcal{X}$ given by the union $\bigcup_{\gamma \in \Gamma} \mathcal{U}_{\gamma(\alpha)}$. Then $\{\mathcal{U}_{[\alpha]}\}_{[\alpha] \in A/\Gamma}$ is a stratification of $\mathcal{X}$ indexed by the partially ordered set A/Γ. For each $[\alpha] \in A/\Gamma$, the corresponding stratum $\mathcal{X}_{[\alpha]}$ can be identified with the disjoint union $\amalg_{\alpha'} \mathcal{X}_{\alpha'}$, where α' ranges over those elements of A having the form $\gamma(\alpha)$ for some $\gamma \in \Gamma$.

Remark 5.2.1.8. Let $\mathcal{X}$ be an algebraic stack equipped with an action of a finite group Γ. Suppose we are given a stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$, where each $\mathcal{U}_\alpha$ is Γ-invariant. Then each quotient $\mathcal{U}_\alpha / \Gamma$ can be regarded as an open substack of $\mathcal{X} / \Gamma$, and the collection of open substacks $\{\mathcal{U}_\alpha / \Gamma\}_{\alpha \in A}$ determines a stratification of $\mathcal{X} / \Gamma$ whose strata can be identified with the quotients $\mathcal{X}_\alpha / \Gamma$.

More generally, suppose that the stratification $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is merely Γ-equivariant in the sense of Remark 5.2.1.6. We can then apply the preceding remark to the induced stratification $\{\mathcal{U}_{[\alpha]}\}_{\alpha \in A/\Gamma}$ by Γ-invariant open substacks. This yields a stratification of $\mathcal{X} / \Gamma$ by open substacks $\{\mathcal{U}_{[\alpha]} / \Gamma\}_{\alpha \in A/\Gamma}$, where each stratum $(\mathcal{X} / \Gamma)_{[\alpha]}$ can be identified with the quotient $\mathcal{X}_\alpha / \Gamma_\alpha$, where Γ_α denotes the subgroup of Γ which stabilizes α.

5.2.2 Convergent Stratifications

Let $\mathcal{X}$ be a smooth algebraic stack over $\mathbf{F}_q$ equipped with a stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$, where each $\mathcal{X}_\alpha$ satisfies the Grothendieck-Lefschetz trace formula. Then $\mathcal{X}$ need not satisfy the Grothendieck-Lefschetz trace formula. For example, if A is an infinite set, then the disjoint union

$$\amalg_{\alpha \in A} \operatorname{Spec}(\mathbf{F}_q)$$

does not satisfy the Grothendieck-Lefschetz trace formula. We now single out a special class of stratifications for which this problem does not arise.

Definition 5.2.2.1. Let $\mathcal{X}$ be an algebraic stack of finite type over $\operatorname{Spec}(\mathbf{F}_q)$. We will say that a stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ of $\mathcal{X}$ is *convergent* if there exists a finite collection of algebraic stacks $\mathcal{T}_1, \ldots, \mathcal{T}_n$ over $\operatorname{Spec}(\mathbf{F}_q)$ with the following properties:

(1) For each $\alpha \in A$, there exists an integer $i \in \{1, 2, \ldots, n\}$ and a diagram of algebraic stacks

$$\mathcal{T}_i \xrightarrow{f} \widetilde{\mathcal{X}}_\alpha \xrightarrow{g} \mathcal{X}_\alpha$$

where the map f is a fiber bundle (locally trivial with respect to the étale topology) whose fibers are affine spaces of some fixed dimension d_α and the map g is surjective, finite, and radicial.

(2) The nonnegative integers d_α appearing in (1) satisfy $\sum_{\alpha \in A} q^{-d_\alpha} < \infty$.

(3) For $1 \leq i \leq n$, the algebraic stack $\mathcal{T}_i$ can be written as a stack-theoretic quotient Y/G, where Y is an algebraic space of finite type over $\mathbf{F}_q$ and G is a linear algebraic group over $\mathbf{F}_q$ which acts on Y.

Remark 5.2.2.2. In the situation of Definition 5.2.2.1, hypothesis (2) guarantees that the set A is at most countable.

We can now state the main result of this section:

Proposition 5.2.2.3. *Let $\mathcal{X}$ be a smooth algebraic stack of dimension d over $\operatorname{Spec}(\mathbf{F}_q)$. If $\mathcal{X}$ admits a convergent stratification, then $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula.*

Remark 5.2.2.4. In the statement of Proposition 5.2.2.3, the hypothesis that $\mathcal{X}$ be smooth is not really important (see Remark 5.0.0.4).

We will deduce Proposition 5.2.2.3 from the following statement, whose proof we defer to §5.2.3:

Proposition 5.2.2.5. *Let $\mathcal{X}$ be a smooth algebraic stack of dimension d over $\mathrm{Spec}(\mathbf{F}_q)$. Suppose that there exists a stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ of $\mathcal{X}$ and a finite collection of algebraic stacks $\{\mathcal{T}_i\}_{1 \leq i \leq n}$ over $\mathrm{Spec}(\mathbf{F}_q)$ which satisfy conditions (1) and (2) of Definition 5.2.2.1, together with the following variant of (3):*

(3′) *Each $\mathcal{T}_i$ is smooth over $\mathrm{Spec}(\mathbf{F}_q)$ (of some fixed dimension) and satisfies the Grothendieck-Lefschetz trace formula.*

Then $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula.

Corollary 5.2.2.6. *Let X be an algebraic space which is smooth (of constant dimension) and of finite type over $\mathrm{Spec}(\mathbf{F}_q)$. Then X satisfies the Grothendieck-Lefschetz trace formula.*

Proof. Every reduced closed $Y \subseteq X$ is a quasi-compact, quasi-separated algebraic space of finite type over $\mathrm{Spec}(\mathbf{F}_q)$, and therefore contains a nonempty affine open subset $U \subseteq Y$. Since the field $\mathbf{F}_q$ is perfect, we may assume (shrinking U if necessary) that U is smooth of constant dimension over $\mathbf{F}_q$. It follows by Noetherian induction that X admits a *finite* stratification $\{X_\alpha\}_{\alpha \in A}$ where each stratum X_α is an affine scheme which is smooth over $\mathrm{Spec}(\mathbf{F}_q)$. The desired result now follows from Proposition 5.2.2.5 (taking the algebraic stacks $\mathcal{T}_i$ to be the strata X_α). □

Proof of Proposition 5.2.2.3 from Proposition 5.2.2.5. Let $\mathcal{X}$ be a smooth algebraic stack equipped with a convergent stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$. To prove that $\mathcal{X}$ satisfies the Grothendieck-Lefschetz trace formula, it will suffice (by virtue of Proposition 5.2.2.5) to show that $\mathcal{X}$ admits another stratification $\{\mathcal{Y}_\beta\}_{\beta \in B}$ which satisfies the hypotheses of Proposition 5.2.2.5.

Since the stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ is convergent, there exists a finite collection of algebraic stacks $\{\mathcal{T}_i\}_{1 \leq i \leq m}$ of finite type over $\mathrm{Spec}(\mathbf{F}_q)$ which satisfies conditions (1), (2), and (3) of Definition 5.2.2.1. In particular, condition (3) implies that we can write each $\mathcal{T}_i$ as a stack-theoretic quotient Y_i/G_i, where Y_i is an algebraic space of finite type over $\mathbf{F}_q$ and each G_i is a linear algebraic group over $\mathbf{F}_q$. Choose an integer $n \geq 0$ such that each of the algebraic spaces Y_i has dimension $\leq n$. We define a sequence of locally closed substacks

$$Z_{i,n}, Z_{i,n-1}, Z_{i,n-2}, \ldots, Z_{i,0} \subseteq Y_i$$

by descending induction as follows: for $0 \leq j \leq n$, let $Z_{i,j}$ denote the largest open subset of $(Y_i - \bigcup_{j' > j} Z_{i,j'})$ which is smooth of dimension j over $\mathrm{Spec}(\mathbf{F}_q)$ (where we regard $Y_i - \bigcup_{j' > j} Z_{i,j'}$ as a reduced closed subscheme of Y). Note that the action of G_i on Y_i preserves each $Z_{i,j}$, so that we can regard the quotient $Y_{i,j}/G_i$ as a locally closed substack $\mathcal{T}_{i,j} \subseteq \mathcal{T}_i$ which is smooth of dimension $j - \dim(G_i)$ over $\mathrm{Spec}(\mathbf{F}_q)$.

Condition (1) of Definition 5.2.2.1 implies that for each $\alpha \in A$, there exists an integer $i(\alpha) \in \{1, \ldots, m\}$ and a pair of maps

$$\mathcal{T}_{i(\alpha)} \xrightarrow{f_\alpha} \widetilde{\mathcal{X}}_\alpha \xrightarrow{g_\alpha} \mathcal{X}_\alpha,$$

where f_α is an étale fiber bundle whose fibers are affine spaces of some dimension d_α and the map g_α is surjective, finite, and radicial. In particular, the morphism f_α is smooth of constant dimension; it follows that each of the closed substacks $\mathcal{T}_{i(\alpha),j} \subseteq \mathcal{T}_{i(\alpha)}$ can be realized as a fiber product

$$\mathcal{T}_{i(\alpha)} \times_{\widetilde{\mathcal{X}}_\alpha} \widetilde{\mathcal{X}}_{\alpha,j},$$

where $\{\widetilde{\mathcal{X}}_{\alpha,j}\}_{0 \leq j \leq n}$ is the collection of locally closed substacks of $\widetilde{\mathcal{X}}_\alpha$ defined inductively by taking $\widetilde{\mathcal{X}}_{\alpha,j}$ to be the largest open substack of $(\widetilde{\mathcal{X}}_\alpha - \bigcup_{j'>j} \widetilde{\mathcal{X}}_{\alpha,j'})$ which is smooth of dimension $j - d_\alpha - \dim(G_{i(\alpha)})$ over $\mathbf{F}_q$. Since g_α is a universal homeomorphism, each of the (reduced) locally closed substacks $\widetilde{\mathcal{X}}_{\alpha,j}$ is given set-theoretically as the inverse image of a reduced locally closed substack $\mathcal{X}_{\alpha,j} \subseteq \mathcal{X}_\alpha$, and the projection map $\widetilde{\mathcal{X}}_{\alpha,j} \to \mathcal{X}_{\alpha,j}$ is surjective, finite, and radicial.

Let $B = A \times \{0, \ldots, n\}$. We will regard B as equipped with the lexicographical ordering (so that $(\alpha, j) \leq (\alpha', j')$ if either $\alpha < \alpha'$ or $\alpha = \alpha'$ and $j \leq j'$). Then $\{\mathcal{X}_{\alpha,j}\}_{(\alpha,j) \in B}$ is a stratification of $\mathcal{X}$. We claim that this stratification satisfies the hypotheses of Proposition 5.2.2.5. By construction, for each $(\alpha, j) \in B$, we have a diagram

$$\mathcal{T}_{\alpha(i),j} \to \widetilde{\mathcal{X}}_{\alpha,j} \to \mathcal{X}_{\alpha,j}$$

where the first map is an étale fiber bundle whose fibers are affine spaces of dimension d_α, and the second map is surjective, finite, and radicial. Moreover, we have

$$\sum_{(\alpha,j) \in B} q^{-d_\alpha} = (n+1) \sum_{\alpha \in A} q^{-d_\alpha} < \infty.$$

To complete the proof, it will suffice to verify condition $(3')$: each of the smooth algebraic stacks $\mathcal{T}_{i,j}$ satisfies the Grothendieck-Lefschetz trace formula. Choose an embedding $G_i \hookrightarrow \mathrm{GL}_d$, so that we can describe $\mathcal{T}_{i,j}$ as the stack-theoretic quotient of $(Y_{i,j} \times \mathrm{GL}_d)/G$ by the action of GL_d. Using Proposition 5.1.0.1, we are reduced to showing that $(Y_{i,j} \times \mathrm{GL}_d)/G$ satisfies the Grothendieck-Lefschetz trace formula, which follows from Corollary 5.2.2.6. □

5.2.3 The Proof of Proposition 5.2.2.5

The proof of Proposition 5.2.2.5 will require some preliminaries.

Lemma 5.2.3.1 (Gysin Sequence)**.** *Let X and Y be smooth quasi-projective varieties over an algebraically closed field k, let $g : Y \to X$ be a finite radicial morphism, and let $U \subseteq X$ be the complement of the image of g. Then there is a canonical fiber sequence*

$$C^{*-2d}(Y; \mathbf{Z}_\ell(-d)) \to C^*(X; \mathbf{Z}_\ell) \to C^*(U; \mathbf{Z}_\ell),$$

where d denotes the relative dimension $\dim(X) - \dim(Y)$*.*

Proof. If $f : Z' \to Z$ is a proper morphism of quasi-projective k-schemes, let $\omega_{Z'/Z} = f^! \underline{\mathbf{Z}}_{\ell Z}$ denote the relative dualizing complex of f. Note that if Z and Z' are smooth of constant dimension, we have

$$\begin{aligned}
\omega_{Z'/Z} &= f^! \underline{\mathbf{Z}}_{\ell Z} \\
&\simeq f^!(\omega_Z^{-1} \otimes \omega_Z) \\
&\simeq f^* \omega_Z^{-1} \otimes f^! \omega_Z \\
&\simeq f^* \omega_Z^{-1} \otimes \omega_{Z'} \\
&\simeq \Sigma^{-2 \dim Z} \underline{\mathbf{Z}}_{\ell Z'}(-\dim Z) \otimes \Sigma^{2 \dim(Z')} \underline{\mathbf{Z}}_{\ell Z'}(\dim Z') \\
&\simeq \Sigma^{2(\dim Z' - \dim Z)} \underline{\mathbf{Z}}_{\ell Z'}(\dim Z' - \dim Z).
\end{aligned}$$

In particular, we have $\omega_{Y/X} \simeq \Sigma^{-2d} \underline{\mathbf{Z}}_{\ell Y}(-d)$.

Let $Y_0 \subseteq X$ denote the image of g, regarded as a reduced closed subscheme of Y. Then g restricts to finite radicial surjection $g_0 : Y \to Y_0$, and we have $\omega_{Y/X} \simeq g_0^! \omega_{Y_0/X}$. Let $j : U \hookrightarrow X$ and $i : Y_0 \hookrightarrow X$ denote the inclusion maps, so that we have a fiber sequence of sheaves

$$i_* i^! \underline{\mathbf{Z}}_{\ell X} \to \underline{\mathbf{Z}}_{\ell X} \to j_* j^* \underline{\mathbf{Z}}_{\ell X}.$$

Passing to global sections, we obtain a fiber sequence

$$C^*(Y_0; \omega_{Y_0/X}) \to C^*(X; \mathbf{Z}_\ell) \to C^*(U; \mathbf{Z}_\ell).$$

The map g_0 is a finite radicial surjection, and therefore induces an equivalence between the étale sites of Y and Y_0. It follows that the counit map

$$g_{0*} g_0^! \omega_{Y_0/X} \to \omega_{Y_0/X}$$

is an equivalence, so we have equivalences

$$\begin{aligned}
C^*(Y_0; \omega_{Y_0/X}) &\simeq C^*(Y_0; g_{0*} g_0^! \omega_{Y_0/X}) \\
&\simeq C^*(Y; g_0^! \omega_{Y_0/X}) \\
&\simeq C^*(Y; \omega_{Y/X}) \\
&\simeq C^{*-2d}(Y; \mathbf{Z}_\ell(-d)).
\end{aligned}$$

□

Lemma 5.2.3.1 immediately implies a corresponding result for algebraic stacks:

Lemma 5.2.3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth algebraic stacks of constant dimension over an algebraically closed field k, let $g : \mathcal{Y} \to \mathcal{X}$ be a finite radicial morphism, and let $\mathcal{U} \subseteq \mathcal{X}$ be the open substack of $\mathcal{X}$ complementary to the image of g. Then there is a canonical fiber sequence*

$$C^{*-2d}(\mathcal{Y}; \mathbf{Z}_\ell(-d)) \to C^*(\mathcal{X}; \mathbf{Z}_\ell) \to C^*_{\mathrm{gm}}(\mathcal{Y}; \mathbf{Z}_\ell),$$

where d denotes the relative dimension $\dim(\mathcal{X}) - \dim(\mathcal{Y})$.

Proof. Let $\mathcal{C}$ denote the category whose objects are affine k-schemes X equipped with a smooth morphism $X \to \mathcal{X}$. For each object $X \in \mathcal{C}$, let $Y_X = \mathcal{Y} \times_{\mathcal{X}} X$ and let $U_X = \mathcal{U} \times_{\mathcal{X}} X$. Lemma 5.2.3.1 then supplies a fiber sequence

$$C^{*-2d}(Y_X; \mathbf{Z}_\ell(-d)) \to C^*(X; \mathbf{Z}_\ell) \to C^*(U_X; \mathbf{Z}_\ell).$$

The construction of this fiber sequence depends functorially on X. We may therefore pass to the limit to obtain a fiber sequence

$$\varprojlim_{X \in \mathcal{C}} C^{*-2d}(Y_X; \mathbf{Z}_\ell(-d)) \to \varprojlim_{X \in \mathcal{C}} C^*(X; \mathbf{Z}_\ell) \to \varprojlim_{X \in \mathcal{C}} C^*(U_X; \mathbf{Z}_\ell).$$

The desired result now follows from the identifications

$$C^{*-2d}(\mathcal{Y}; \mathbf{Z}_\ell(-d)) \simeq \varprojlim_{X \in \mathcal{C}} C^{*-2d}(Y_X; \mathbf{Z}_\ell(-d))$$

$$C^*(\mathcal{X}; \mathbf{Z}_\ell) \simeq \varprojlim_{X \in \mathcal{C}} C^*(X; \mathbf{Z}_\ell)$$

$$C^*(\mathcal{U}; \mathbf{Z}_\ell) \simeq \varprojlim_{X \in \mathcal{C}} C^*(U_X; \mathbf{Z}_\ell).$$

□

Lemma 5.2.3.3. *Let X be an affine $\mathbf{F}_q$-scheme of finite type which becomes isomorphic to an affine space $\mathbf{A}^e$ after passing to some finite extension of $\mathbf{F}_q$. Then the set $X(\mathbf{F}_q)$ has q^e elements.*

Proof. By virtue of the Grothendieck-Lefschetz trace formula, it will suffice to show that $\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(X))$ is equal to 1. Equivalently, we must show that the trace of Frob^{-1} on the reduced cohomology $\mathrm{H}^*_{\mathrm{red}}(X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q); \mathbf{Q}_\ell)$ vanishes. But this reduced cohomology itself vanishes, since $X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$ is isomorphic to an affine space over $\mathrm{Spec}(\overline{\mathbf{F}}_q)$. □

Proof of Proposition 5.2.2.5. Let $d = \dim(\mathcal{X})$ and let $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ be the given stratification of $\mathcal{X}$. The set A is at most countable (Remark 5.2.2.2). By adding additional elements to A and assigning to those additional elements the empty substack of $\mathcal{X}$, we may assume that A is infinite. Using our assumption that $\{\beta \in A : \beta \leq \alpha\}$ is finite for each $\alpha \in A$, it follows that we can choose an enumeration

$$A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$$

where each initial segment $\{\alpha_0, \ldots, \alpha_n\}$ is a downward-closed subset of A. We can then write $\mathcal{X}$ as the union of an increasing sequence of open substacks

$$\mathcal{U}_0 \hookrightarrow \mathcal{U}_1 \hookrightarrow \mathcal{U}_2 \hookrightarrow \cdots$$

where $\mathcal{U}_n$ is characterized by the requirement that if k is a field, then a map $\eta : \mathrm{Spec}(k) \to \mathcal{X}$ factors through $\mathcal{U}_n$ if and only if it factors through one of the substacks $\mathcal{X}_{\alpha_0}, \mathcal{X}_{\alpha_1}, \ldots, \mathcal{X}_{\alpha_n}$.

By hypothesis, there exists a finite collection $\{\mathcal{T}_i\}_{1 \leq i \leq m}$ of smooth algebraic stacks over $\mathrm{Spec}(\mathbf{F}_q)$, where each $\mathcal{T}_i$ has some fixed dimension d_i and satisfies the Grothendieck-Lefschetz trace formula, and for each $n \geq 0$ there exists an index $i(n) \in \{1, \ldots, m\}$ and a diagram

$$\mathcal{T}_{i(n)} \xrightarrow{f_n} \widetilde{\mathcal{X}}_{\alpha_n} \xrightarrow{g_n} \mathcal{X}_{\alpha_n},$$

where g_n is a finite radicial surjection and f_n is an étale fiber bundle whose fibers are affine spaces of some fixed dimension $e(n)$. Set

$$\overline{\mathcal{X}} = \mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$$

$$\overline{\mathcal{U}}_n = \mathcal{U}_n \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$$

$$\overline{\mathcal{T}}_i = \mathcal{T}_i \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q).$$

The map f_n induces an isomorphism on ℓ-adic cohomology. Applying Lemma 5.2.3.2 to the finite radicial map

$$g_n : \widetilde{\mathcal{X}}_{\alpha_n} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q) \to \overline{\mathcal{U}}_n,$$

we obtain fiber sequences

$$C^{*-2e'_n}(\widetilde{\mathcal{T}}_{i(n)}; \mathbf{Z}_\ell(-e'_n)) \to C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell) \to C^*(\overline{\mathcal{U}}_{n-1}; \mathbf{Z}_\ell)$$

where $e'_n = e_n + d - d_{i(n)}$ denotes the relative dimension of the map $\widetilde{\mathcal{X}}_{\alpha_n} \to \mathcal{X}$.

We have a canonical equivalence

$$\theta : C^*(\overline{\mathcal{X}}; \mathbf{Z}_\ell) \simeq \varprojlim_n C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell).$$

Our convergence assumption

$$\sum_{n \geq 0} q^{-e_n} < \infty$$

guarantees that the sequence of integers $\{e_n\}_{n \geq 0}$ tends to infinity and therefore the sequence $\{e'_n\}_{n \geq 0}$ also tends to infinity. It follows that the restriction maps

$$\mathrm{H}^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell) \to \mathrm{H}^*(\overline{\mathcal{U}}_{n-1}; \mathbf{Z}_\ell)$$

are isomorphisms for $n \gg *$, so that θ also induces an equivalence

$$C^*(\overline{\mathcal{X}}; \mathbf{Z}_\ell)[\ell^{-1}] \simeq \varprojlim_n C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell)[\ell^{-1}].$$

Set $V(n) = C^*(\overline{\mathcal{U}}_n; \mathbf{Z}_\ell)[\ell^{-1}] = C^*_{\mathrm{gm}}(\mathcal{U}_n)$ and let $W(n)$ denote the fiber of the restriction map $V(n) \to V(n-1)$ (with the convention that $W(0) = V(0)$). The above calculation gives

$$W(n) = C^{*-2e'_n}_{\mathrm{gm}}(\mathcal{T}_{i(n)})(-e'_n).$$

Since each $\mathcal{T}_i$ satisfies the Grothendieck-Lefschetz trace formula, the cohomologies of $W(n)$ are finite-dimensional in each degree and we have

$$|\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} = q^{-e'_n} |\,\mathrm{H}^*_{\mathrm{gm}}(\mathcal{T}_{i(n)}\,|_{\mathrm{Frob}^{-1}}|$$

$$\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(W(n))) = q^{-e'_n} \,\mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{T}_{i(n)})) = q^{-e_n - d} |\,\mathcal{T}_{i(n)}(\mathbf{F}_q)|.$$

In particular, we have

$$\begin{aligned} \sum_{n \geq 0} |\,\mathrm{H}^*(W(n))|_{\mathrm{Frob}^{-1}} &= \sum_{n \geq 0} q^{-e'_n} |\,\mathrm{H}^*_{\mathrm{gm}}(\mathcal{T}_{i(n)}\,|_{\mathrm{Frob}^{-1}} \\ &\leq \sum_{n \geq 0} q^{-e_n} \sum_{1 \leq i \leq m} q^{d_i - d} |\,\mathrm{H}^*_{\mathrm{gm}}(\mathcal{T}_i)|_{\mathrm{Frob}^{-1}} \\ &< \infty. \end{aligned}$$

Invoking Lemma 5.1.1.1, we conclude that $(\mathrm{H}^*_{\mathrm{gm}}(\mathcal{X}), \mathrm{Frob}^{-1})$ is convergent, with

$$\begin{aligned} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*_{\mathrm{gm}}(\mathcal{X})) &= \sum_{n \geq 0} \mathrm{Tr}(\mathrm{Frob}^{-1} \,|\, \mathrm{H}^*(W(n)) \\ &= \sum_{n \geq 0} q^{-e_n - d} |\,\mathcal{T}_{i(n)}(\mathbf{F}_q)|. \end{aligned}$$

On the other hand, the stratification $\{\mathcal{X}_{\alpha_n}\}_{n \geq 0}$ of $\mathcal{X}$ gives the identity

$$\frac{|\,\mathcal{X}(\mathbf{F}_q)|}{q^d} = \sum_{n \geq 0} q^{-d} |\,\mathcal{X}_{\alpha_n}(\mathbf{F}_q)|.$$

It will therefore suffice to prove that for each $n \geq 0$, we have

$$|\mathcal{T}_{i(n)}(\mathbf{F}_q)| = q^{e_n}|\mathcal{X}_{\alpha_n}(\mathbf{F}_q)|.$$

Since g_n is a finite radicial surjection, it induces an equivalence of categories $\widetilde{\mathcal{X}}_{\alpha_n}(\mathbf{F}_q) \simeq \mathcal{X}_{\alpha_n}(\mathbf{F}_q)$. It will therefore suffice to show that each object of the groupoid $\widetilde{\mathcal{X}}_{\alpha_n}(\mathbf{F}_q)$ can be lifted in exactly q^{e_n} ways to an object of the groupoid $\mathcal{T}_{i(n)}(\mathbf{F}_q)$ via the map f_n, which is an immediate consequence of Lemma 5.2.3.3. □

5.3 The Harder-Narasimhan Stratification

Let $\mathcal{X}$ be a smooth algebraic stack over $\mathbf{F}_q$. In §5.2, we proved that if $\mathcal{X}$ admits a convergent stratification (Definition 5.2.2.1), then it satisfies the Grothendieck-Lefschetz trace formula (Proposition 5.2.2.3). To complete the proof of Theorem 5.0.0.3, it will suffice to show that the moduli stack $\mathcal{X} = \mathrm{Bun}_G(X)$ admits a convergent stratification $\{\mathcal{X}_\alpha\}_{\alpha \in A}$, where X is an algebraic curve over $\mathbf{F}_q$ and G is a smooth affine group scheme over X with connected fibers and semisimple generic fiber. If the group scheme G is split (that is, if it is the pullback of a split semisimple algebraic group over $\mathbf{F}_q$), then this strategy can be realized by taking $\{\mathcal{X}_\alpha\}_{\alpha \in A}$ to be the *Harder-Narasimhan stratification* of $\mathrm{Bun}_G(X)$ (see Theorem 5.3.2.2). In this section, we will recall the definition of the Harder-Narasimhan stratification in the case where G is split, and describe an extension of this definition to the case of inner forms of split groups (see §5.3.5) for details) which will play an important role in our proof of Theorem 5.0.0.3.

5.3.1 Semistable G-Bundles

Throughout this section, we fix an algebraically closed field k, an algebraic curve X over k, and a reductive algebraic group G over k. Choose a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We will say that a parabolic subgroup $P \subseteq G$ is *standard* if it contains B.

Notation 5.3.1.1. For every linear algebraic group H over k, we let $\mathrm{Hom}(H, \mathbf{G}_m)$ denote the character group of H (a finitely generated abelian group). We let $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$ denote the abelian group of homomorphisms from $\mathrm{Hom}(H, \mathbf{G}_m)$ to $\mathbf{Z}$. Given elements $\mu \in \mathrm{Hom}(H, \mathbf{G}_m)$ and $\nu \in \mathrm{Hom}(H, \mathbf{G}_m)^\vee$, we let $\langle \mu, \nu \rangle \in \mathbf{Z}$ denote the integer given by evaluating ν on μ.

Definition 5.3.1.2. Let H be a linear algebraic group over k and let $\mathcal{P}$ be an H-bundle on X. For every character $\mu : H \to \mathbf{G}_m$, the H-bundle $\mathcal{P}$ determines a $\mathbf{G}_m$-bundle $\mathcal{P}_\mu$ on X, which we will identify with the corresponding line bundle. We let $\deg(\mathcal{P})$ denote the element of $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$ given by $\mu \mapsto \deg(\mathcal{P}_\mu)$. We will refer to $\deg(\mathcal{P})$ as the *degree of* $\mathcal{P}$.

Let ν be an element of $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$. We let $\mathrm{Bun}_H^\nu(X)$ denote the substack of $\mathrm{Bun}_H(X)$ whose R-valued points are H-bundles $\mathcal{P}$ on the relative curve X_R having the property that for every k-valued point $\eta : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$, the fiber $\mathcal{P}_\eta = \mathcal{P} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ has degree ν. We will refer to $\mathrm{Bun}_H^\nu(X)$ as the *moduli stack of H-bundles of degree ν on X*.

Remark 5.3.1.3. In the situation of Definition 5.3.1.2, let R be a finitely generated k-algebra and let $\mathcal{P}$ be an H-bundle on X_R. The construction $\eta \mapsto \deg(\mathcal{P}_\eta)$ determines a map from the closed points of $\mathrm{Spec}(R)$ to $\mathrm{Hom}(H, \mathbf{G}_m)^\vee$ which is locally constant for the Zariski topology. It follows that each $\mathrm{Bun}_H^\nu(X)$ is a closed and open substack of $\mathrm{Bun}_H(X)$; in particular, it is a smooth algebraic stack over k. Moreover, we can identify $\mathrm{Bun}_H(X)$ with the disjoint union

$$\coprod_{\nu \in \mathrm{Hom}(H, \mathbf{G}_m)^\vee} \mathrm{Bun}_H^\nu(X)$$

(taken in the 2-category of algebraic stacks over k).

Notation 5.3.1.4. Let H be a linear algebraic group over k and let $\mathfrak{h}$ denotes its Lie algebra. The adjoint action of H on $\mathfrak{h}$ determines a character $H \to \mathrm{GL}(\mathfrak{h}) \xrightarrow{\det} \mathbf{G}_m$, which we will denote by $2\rho_H$ and regard as an element of $\mathrm{Hom}(H, \mathbf{G}_m)$.

Remark 5.3.1.5. Specializing to the case where H is the standard Borel subgroup $B \subseteq G$, we can identify $\mathrm{Hom}(B, \mathbf{G}_m)$ with the character lattice of G. In this case, the element $2\rho_B \in \mathrm{Hom}(G, \mathbf{G}_m)$ is the sum of the positive roots of G. Beware that $2\rho_B$ is generally not divisible by 2 in $\mathrm{Hom}(B, \mathbf{G}_m)$ (however, it is divisible by 2 when G is semisimple and simply connected: in this case, ρ_B can be identified with the sum of the fundamental weights of G).

Definition 5.3.1.6. Let $\mathcal{P}$ be a G-bundle on X. We will say that $\mathcal{P}$ is *semistable* if, for every standard parabolic subgroup $P \subseteq G$ and every reduction of $\mathcal{P}$ to a P-bundle $\mathcal{Q}$, we have $\langle 2\rho_P, \deg(\mathcal{Q}) \rangle \leq 0$.

Let us identify $\mathrm{Bun}_G(X)$ with the category of pairs $(R, \mathcal{P})$ where R is a finitely generated k-algebra and $\mathcal{P}$ is a G-bundle on the relative curve X_R. We let $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ denote the full subcategory of $\mathrm{Bun}_G(X)$ spanned by those pairs $(R, \mathcal{P})$ with the following property: for every every k-valued point $\eta : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ the fiber $\mathcal{P}_\eta = \mathcal{P} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ is semistable (when viewed as a G-bundle on X). We will refer to $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ as the *semistable locus of* $\mathrm{Bun}_G(X)$.

Remark 5.3.1.7. It is not immediately obvious that the semistable locus $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ is itself an algebraic stack. However, one can show that it is an open substack of $\mathrm{Bun}_G(X)$ (and therefore algebraic): this is a special case of Theorem 5.3.2.2.

Remark 5.3.1.8. Let $P \subseteq G$ be a standard parabolic subgroup and let $\mathcal{Q}$ be a G-bundle on X. Let U denote the unipotent radical of P and let $\mathfrak{u}$ denote its Lie algebra. We then have an exact sequence

$$0 \to \mathfrak{u} \to \mathfrak{p} \to \mathfrak{p}/\mathfrak{u} \to 0$$

of representations of P. Note that the action of P on $\mathfrak{p}/\mathfrak{u}$ factors through the adjoint quotient of P/U (which is a semisimple algebraic group), and is therefore given by a map $P \to \mathrm{SL}(\mathfrak{p}/\mathfrak{u})$. It follows that the character $2\rho_P \in \mathrm{Hom}(P, \mathbf{G}_m)$ can be identified with the character

$$P \to \mathrm{GL}(\mathfrak{u}) \xrightarrow{\det} \mathbf{G}_m.$$

Remark 5.3.1.9. Let G_{ad} denote the adjoint quotient of G. For every standard parabolic subgroup $P \subseteq G$, we let P_{ad} denote the image of P in G_{ad}. If $\mathcal{Q}$ is a P-bundle, we let $\mathcal{Q}_{\mathrm{ad}}$ denote the associated P_{ad}-bundle. Note that the natural map $P \to P_{\mathrm{ad}}$ induces an isomorphism from the unipotent radical of P to the unipotent radical of P_{ad}. It follows from Remark 5.3.1.8 the induced map $\mathrm{Hom}(P_{\mathrm{ad}}, \mathbf{G}_m) \to \mathrm{Hom}(P, \mathbf{G}_m)$ carries $2\rho_{P_{\mathrm{ad}}}$ to $2\rho_P$.

Remark 5.3.1.10. Let $\mathcal{P}$ be a G-bundle on X. For every standard parabolic subgroup $P \subseteq G$, there is a canonical bijection between the set of P-reductions of $\mathcal{P}$ to the set of P_{ad}-reductions of $\mathcal{P}_{\mathrm{ad}}$, given (at the level of bundles) by the construction $\mathcal{Q} \mapsto \mathcal{Q}_{\mathrm{ad}}$. It follows from Remark 5.3.1.9 that we have $\langle 2\rho_P, \deg(\mathcal{Q})\rangle = \langle 2\rho_{P_{\mathrm{ad}}}, \deg(\mathcal{Q})_{\mathrm{ad}}\rangle$, so that $\mathcal{P}$ is semistable if and only if $\mathcal{P}_{\mathrm{ad}}$ is semistable. Consequently, we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(X)^{\mathrm{ss}} & \longrightarrow & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}}}(X)^{\mathrm{ss}} & \longrightarrow & \mathrm{Bun}_{G_{\mathrm{ad}}}(X). \end{array}$$

Variant 5.3.1.11. Let $P \subseteq G$ be a standard parabolic subgroup and let $U \subseteq P$ be its unipotent radical. Then P/U is a reductive algebraic group over X. We say that a P-bundle $\mathcal{Q}$ on X is *semistable* if the associated (P/U)-bundle on X is semistable. We let $\mathrm{Bun}_P(X)^{\mathrm{ss}}$ denote the fiber product

$$\mathrm{Bun}_P(X) \times_{\mathrm{Bun}_{P/U}(X)} \mathrm{Bun}_{P/U}(X)^{\mathrm{ss}};$$

we will refer to $\mathrm{Bun}_P(X)^{\mathrm{ss}}$ as the *moduli stack of semistable P-bundles*. For each element $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^{\vee}$, we let $\mathrm{Bun}_P^{\nu}(X)^{\mathrm{ss}}$ denote the intersection $\mathrm{Bun}_P^{\nu}(X) \cap \mathrm{Bun}_P(X)^{\mathrm{ss}}$, which we will refer to as the *moduli stack of semistable P-bundles of degree ν*.

Remark 5.3.1.12. Let $G \to G'$ be a central isogeny of reductive algebraic groups over k. We let B' and T' denote the images of B and T in G', so that B' is a Borel subgroup of G' and T' is a maximal torus in B'. For every standard parabolic subgroup $P \subseteq G$, let P' denote the image of P in G', so that P' is a standard parabolic subgroup of G'. The natural map $P \to P'$ induces an injection of finitely generated free abelian groups

$$\mathrm{Hom}(P, \mathbf{G}_m)^\vee \hookrightarrow \mathrm{Hom}(P', \mathbf{G}_m)^\vee \qquad \nu \mapsto \nu'.$$

For each $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$ we have $\mathrm{Bun}_P^\nu(X) \simeq \mathrm{Bun}_P(X) \times_{\mathrm{Bun}_{P'}(X)} \mathrm{Bun}_{P'}^{\nu'}(X)$, and Remark 5.3.1.10 gives $\mathrm{Bun}_P(X)^{\mathrm{ss}} \simeq \mathrm{Bun}_P(X) \times_{\mathrm{Bun}_{P'}(X)} \mathrm{Bun}_{P'}(X)^{\mathrm{ss}}$.

5.3.2 The Harder-Narasimhan Stratification: Split Case

Throughout this section, we fix an algebraically closed field k, a reductive algebraic group G over k, a Borel subgroup $B \subseteq G$, and a maximal torus $T \subseteq B$. To describe the Harder-Narasimhan stratification, we will need a bit more terminology:

Notation 5.3.2.1. Let $P \subseteq G$ be a standard parabolic subgroup, let $U \subseteq P$ be its unipotent radical, and let $H \subseteq P$ be the unique Levi subgroup which contains T. We have a commutative diagram

$$\begin{array}{ccc} & \mathrm{Hom}(P, \mathbf{G}_m) & \\ \nearrow & & \searrow \\ \mathrm{Hom}(P/U, \mathbf{G}_m) & \longrightarrow & \mathrm{Hom}(H, \mathbf{G}_m) \end{array}$$

where the bottom map and the left diagonal map are isomorphisms, so the right diagonal map is an isomorphism as well.

Let $\mathfrak{Z}(H)$ denote the center of H, which we regard as a subgroup of T. Since H is a reductive group, the canonical map

$$\mathrm{Hom}(H, \mathbf{G}_m) \to \mathrm{Hom}(\mathfrak{Z}(H), \mathbf{G}_m)$$

is a rational isomorphism. In particular, we have a canonical map

$$\begin{aligned} \mathrm{Hom}(T, \mathbf{G}_m) &\to \mathrm{Hom}(\mathfrak{Z}(H), \mathbf{G}_m) \\ &\to \mathrm{Hom}(\mathfrak{Z}(H), \mathbf{G}_m) \otimes \mathbf{Q} \\ &\xleftarrow{\sim} \mathrm{Hom}(H, \mathbf{G}_m) \otimes \mathbf{Q} \\ &\xleftarrow{\sim} \mathrm{Hom}(P, \mathbf{G}_m) \otimes \mathbf{Q}\,. \end{aligned}$$

In particular, every character $\alpha \in \mathrm{Hom}(T, \mathbf{G}_m)$ determines a map $\mathrm{Hom}(P, \mathbf{G}_m)^\vee \to \mathbf{Q}$, which we will denote by $\nu \mapsto \langle \alpha, \nu \rangle$.

Let $\boldsymbol{\Delta} \subseteq \mathrm{Hom}(T, \mathbf{G}_m)$ denote the set of simple roots of G and let $\boldsymbol{\Delta}_P \subseteq \boldsymbol{\Delta}$ denote the subset consisting of those roots α such that $-\alpha$ is not a root of P. We will say that an element $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$ is *dominant* if $\langle \alpha, \nu \rangle \geq 0$ for each $\alpha \in \boldsymbol{\Delta}_P$, and we will say that ν is *dominant regular* if $\langle \alpha, \nu \rangle > 0$ for each $\alpha \in \boldsymbol{\Delta}_P$. We let $\mathrm{Hom}(P, \mathbf{G}_m)^\vee_{\geq 0}$ denote the subset of $\mathrm{Hom}(P, \mathbf{G}_m)^\vee$ spanned by the dominant elements and $\mathrm{Hom}(P, \mathbf{G}_m)^\vee_{>0}$ the subset consisting of dominant regular elements.

We can now state the main result that we will need. For a proof, we refer the reader to [4] or [30].

Theorem 5.3.2.2 (The Harder-Narasimhan Stratification)**.** *Let X be an algebraic curve over k. Then:*

(a) *For each standard parabolic subgroup $P \subseteq G$, the inclusion*

$$\mathrm{Bun}_P(X)^{\mathrm{ss}} \hookrightarrow \mathrm{Bun}_P(X)$$

is an open immersion. In particular, $\mathrm{Bun}_P(X)^{\mathrm{ss}}$ *is a smooth algebraic stack over* $\mathrm{Spec}(k)$*, which can be written as a disjoint union*

$$\coprod_{\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee} \mathrm{Bun}_P^\nu(X)^{\mathrm{ss}}.$$

(b) *For each standard parabolic subgroup $P \subseteq G$ and each $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee_{>0}$, there exists a locally closed substack $\mathrm{Bun}_G(X)_{P,\nu} \subseteq \mathrm{Bun}_G(X)$ which is characterized by the following property: the natural map $\mathrm{Bun}_P(X) \to \mathrm{Bun}_G(X)$ restricts to a surjective finite radicial map*

$$\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)_{P,\nu}.$$

(c) *Let A be the collection of all pairs (P, ν), where P is a standard parabolic subgroup of G and ν is an element of* $\mathrm{Hom}(P, \mathbf{G}_m)^\vee_{>0}$*. Then, for a suitable partial ordering of A, the collection of locally closed substacks $\{\mathrm{Bun}_G(X)_{P,\nu}\}_{(P,\nu)\in A}$ determines a stratification of* $\mathrm{Bun}_G(X)$ *(see Remark 5.2.1.3).*

We will refer to the stratification of $\mathrm{Bun}_G(X)$ whose existence is guaranteed by Theorem 5.3.2.2 as the *Harder-Narasimhan stratification.*

Remark 5.3.2.3. If the field k has characteristic zero, or if $G = \mathrm{GL}_n$, or more generally if if the characteristic of k does not belong to a finite set of "bad primes" which may depend on G, then assertion (b) can be strengthened: the maps $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)_{P,\nu}$ are equivalences. However, this is not true in general; see [17] for a more thorough discussion.

Remark 5.3.2.4. Let $G \to G'$ be a central isogeny of reductive algebraic groups over k. It follows from Remark 5.3.1.12 that the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$ is the pullback of the Harder-Narasimhan stratification of $\mathrm{Bun}_{G'}(X)$.

5.3.3 Rationality Properties of the Harder-Narasimhan Stratification

We now extend the constructions of §5.3.2 to the case of a ground field k which is not assumed to be algebraically closed. Let G be a *split* reductive algebraic group over k, so that we can choose a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$ (which is split over k). If $\overline{k}$ is an algebraic closure of k and $\overline{X}$ is an algebraic curve over $\overline{k}$, then Theorem 5.3.2.2 determines a stratification of the moduli stack $\mathrm{Bun}_G(\overline{X})$.

Remark 5.3.3.1 (Functoriality in X)**.** Let ψ be an automorphism of $\overline{X}$ *as a k-scheme*, so that ψ induces an automorphism ψ_0 of the field $\overline{k} = \mathrm{H}^0(\overline{X}; \mathcal{O}_{\overline{X}})$ which we do not assume to be the identity. Then σ fits into a commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\psi} & \overline{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{k}) & \xrightarrow{\psi_0} & \mathrm{Spec}(\overline{k}). \end{array}$$

Then ψ induces an automorphism ϕ of the algebraic stack $\mathrm{Bun}_G(\overline{X})$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(\overline{X}) & \xrightarrow{\phi} & \mathrm{Bun}_G(\overline{X}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{k}) & \xrightarrow{\psi_0} & \mathrm{Spec}(\overline{k}). \end{array}$$

Unwinding the definitions, we see that the automorphism ϕ carries each Harder-Narasimhan stratum $\mathrm{Bun}_G(\overline{X})_{P,\nu}$ into itself.

Remark 5.3.3.2. Suppose that $\overline{X}$ is defined over k: that is, we have an isomorphism $\overline{X} \simeq X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$, where X is an algebraic curve over k. Let $\mathrm{Bun}_G(X)$ denote the moduli stack of G-bundles on X, which we regard as a smooth algebraic stack over k. We then have an equivalence of algebraic stacks $\mathrm{Bun}_G(\overline{X}) \simeq \mathrm{Bun}_G(X) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$. It follows that there is a bijective correspondence between open substacks of $\mathrm{Bun}_G(X)$ and $\mathrm{Gal}(\overline{k}/k)$-invariant open substacks of $\mathrm{Bun}_G(\overline{X})$. Invoking Remark 5.3.3.1, we see that the Harder-Narasimhan stratification of $\mathrm{Bun}_G(\overline{X})$ is defined over k: that is, there is a stratification of $\mathrm{Bun}_G(X)$ by reduced locally closed substacks $\{\mathrm{Bun}_G(X)_{P,\nu}\}$ satisfying

$$\mathrm{Bun}_G(\overline{X})_{P,\nu} = (\mathrm{Bun}_G(X_0)_{P,\nu} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}))_{\mathrm{red}}.$$

Remark 5.3.3.3 (Functoriality in G)**.** In the situation of Remark 5.3.3.2, the moduli stack $\mathrm{Bun}_G(X)$ depends functorially on the algebraic group G, and therefore carries an action of the automorphism group $\mathrm{Aut}(G)$. This action factors through the quotient group $\mathrm{Out}(G) = \mathrm{Aut}(G)/G^{\mathrm{ad}}$ of outer automorphisms of G. Moreover, the resulting action of $\mathrm{Out}(G)$ on $\mathrm{Bun}_G(X)$ preserves the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$, but permutes the strata. More precisely, let $A = \{(P, \nu)\}$ be as in the statement of Theorem 5.3.2.2. The automorphism group $\mathrm{Out}(G)$ acts on A by the construction

$$(\sigma \in \mathrm{Out}(G), (P, \nu) \in A) \mapsto (\sigma(P), \nu_\sigma),$$

where $\langle \mu, \nu_\sigma \rangle = \langle \mu \circ \sigma, \nu \rangle$ for $\mu \in \mathrm{Hom}(\sigma(P), \mathbf{G}_m)$. For each $\sigma \in \mathrm{Out}(G)$, the associated automorphism $\psi_\sigma : \mathrm{Bun}_G(X) \simeq \mathrm{Bun}_G(X)$ restricts to equivalences $\mathrm{Bun}_G(X)_{P,\nu} \simeq \mathrm{Bun}_G(X)_{\sigma(P),\nu_\sigma}$.

5.3.4 Digression: Inner Forms

We now introduce some terminology which will be useful for extending the theory of the Harder-Narasimhan filtration to the setting of bundles over *non-split* groups. Fix a field k, a split reductive algebraic group G_0 over k, a Borel subgroup $B_0 \subseteq G_0$, and a (k-split) maximal torus $T_0 \subseteq B_0$. We let $G_{0\,\mathrm{ad}}$ denote the adjoint quotient of G_0; for any subgroup $H_0 \subseteq G_0$, we let $H_{0\,\mathrm{ad}}$ denote the image of H_0 in G_{ad}.

If X is a k-scheme and G is a group scheme over X, we will say that G is a *form of* G_0 *over* X if there exists an étale surjection $\widetilde{X} \to X$ such that $G \times_X \widetilde{X}$ is isomorphic to $G_0 \times_{\mathrm{Spec}(k)} \widetilde{X}$ as a group scheme over X. Note that this condition implies that G is a reductive group scheme over X, and that the adjoint quotient G_{ad} of G is a form of $G_{0\,\mathrm{ad}}$ over X.

Notation 5.3.4.1. Let X be a k-scheme and let G be a form of G_0 over X. We let $\mathrm{Iso}(G, G_0)$ denote the X-scheme parametrizing isomorphisms of G with G_0 (so that the R-valued points of $\mathrm{Iso}(G, G_0)$ are isomorphisms of $G \times_X \mathrm{Spec}(R)$ with $G_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ as group schemes over R). Then $\mathrm{Iso}(G, G_0)$ is an $\mathrm{Aut}(G_0)$-torsor over X, where $\mathrm{Aut}(G_0)$ denotes the automorphism group of G_0. The automorphism group $\mathrm{Aut}(G_0)$ fits into an exact sequence

$$0 \to G_{0\,\mathrm{ad}} \to \mathrm{Aut}(G_0) \to \mathrm{Out}(G_0) \to 0,$$

where $\mathrm{Out}(G_0)$ denotes the (constant) group scheme of outer automorphisms of G_0 (if G_0 is semisimple, then $\mathrm{Out}(G_0)$ is finite). Let $\mathrm{Out}(G, G_0)$ denote the quotient $G_{0\,\mathrm{ad}} \backslash \mathrm{Iso}(G, G_0)$, which we regard as an $\mathrm{Out}(G_0)$-torsor over X. In particular, $\mathrm{Out}(G, G_0)$ is a scheme equipped with an étale surjection $\mathrm{Out}(G, G_0) \to X$ (which is finite étale in the case where G_0 is semisimple).

Definition 5.3.4.2. Let X be a k-scheme and let G be a form of G_0 over X. An *inner structure* on G is a section of the projection map $\mathrm{Out}(G, G_0) \to X$. An *inner form of*

G_0 *over* X is a pair (G, σ), where G is a form of G_0 over X and σ is an inner structure on G.

Example 5.3.4.3. Let G be a form of G_0 over a k-scheme X. Any isomorphism $\beta : G \simeq G_0 \times_{\mathrm{Spec}(k)} X$ determines an inner structure on G (in particular, the split form of G_0 over X admits a canonical inner structure), and every inner structure on G arises in this way étale locally on X. Moreover, if β' is another such isomorphism, then β' determines the same inner structure on G if and only if the isomorphism $\beta'^{-1} \circ \beta : G \to G$ is given by conjugation by an X-valued point of G_{ad}.

Example 5.3.4.4. Let X be a connected normal k-scheme with fraction field K_X and let G be a form of G_0 over X. Then every inner structure on the algebraic group $G \times_X \mathrm{Spec}(K_X)$ extends uniquely to an inner structure on G. In particular, the group scheme G admits an inner structure whenever the generic fiber of G is split.

In the special case where k is algebraically closed and X is an algebraic curve over k, the converse holds: since the fraction field K_X has dimension ≤ 1, the generic fiber G is automatically quasi-split, so that G admits an inner structure if and only if the generic fiber of G is split.

Remark 5.3.4.5. Let G be a form of G_0 over a k-scheme X. Then the group $\mathrm{Out}(G_0)$ acts on the collection of inner structures on G. If G admits an inner structure and X is connected, then this action is simply transitive.

Construction 5.3.4.6. Let (G, σ) be an inner form of G_0 over a k-scheme X. We let $\mathrm{Iso}^{\sigma}(G, G_0)$ denote the fiber product $X \times_{\mathrm{Out}(G,G_0)} \mathrm{Iso}(G, G_0)$. Then $\mathrm{Iso}(G, G_0)$ is a *bitorsor* for the groups G_{ad} and $G_{0\,\mathrm{ad}}$: that is, it is equipped with commuting actions of the X-schemes G_{ad} (on the right) and $G_{0\,\mathrm{ad}} \times_{\mathrm{Spec}(k)} X$ (on the left), each of which is simply transitive locally for the étale topology. It follows that the construction

$$\mathcal{P} \mapsto \mathrm{Iso}(G, G_0) \otimes_{G_{\mathrm{ad}}} \mathcal{P} = (\mathrm{Iso}(G, G_0) \times_X \mathcal{P})/G_{\mathrm{ad}}$$

induces an equivalence from the category $\mathrm{Tors}_{G_{\mathrm{ad}}}(X)$ of G_{ad}-torsors on X to the category $\mathrm{Tors}_{G_{0\,\mathrm{ad}}}(X)$ of $G_{0\,\mathrm{ad}}$-torsors on X.

In the case where X is an algebraic curve over k, this construction supplies an equivalence of algebraic stacks $\epsilon_\sigma : \mathrm{Bun}_{G_{\mathrm{ad}}}(X) \simeq \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)$.

Warning 5.3.4.7. In the situation of Construction 5.3.4.6, the equivalence

$$\epsilon_\sigma : \mathrm{Bun}_{G_{\mathrm{ad}}}(X) \simeq \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)$$

depends on the choice of inner structure σ. Note that the group $\mathrm{Out}(G_0)$ acts simply transitively on the set of inner structures on G; in particular, any other inner structure on G can be written as $g(\sigma)$ where $g \in \mathrm{Out}(G_0)$. In this case, we have a commutative diagram

$$\begin{array}{ccc} & \mathrm{Bun}_{G_{\mathrm{ad}}}(X) & \\ {\scriptstyle \epsilon_\sigma} \swarrow & & \searrow {\scriptstyle \epsilon_{g(\sigma)}} \\ \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X) & \longrightarrow & \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X) \end{array}$$

where the lower horizontal map is the automorphism induced by g.

5.3.5 The Harder-Narasimhan Stratification: The Inner Case

Throughout this section, we fix a field k and an algebraic curve X over k. If G is a split reductive group scheme over X, then the moduli stack $\mathrm{Bun}_G(X)$ can be equipped with the Harder-Narasimhan stratification described in Remark 5.3.3.2. In this section, we generalize the construction to the case where G is an inner form of a split reductive group.

Construction 5.3.5.1 (The Harder-Narasimhan Stratification)**.** Let G_0 be a split reductive algebraic group over k and let (G, σ) be an inner form of G_0 over X (Definition 5.3.4.2). Then σ determines a map

$$\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G_{\mathrm{ad}}}(X) \xrightarrow{\epsilon_\sigma} \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X).$$

Let A denote the set of all pairs (P_0, ν), where $P_0 \subseteq G_0$ is a standard parabolic subgroup and $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}$. For each element $(P_0, \nu) \in A$, let $\mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)_{P_{0\,\mathrm{ad}},\nu}$ denote the corresponding stratum of the Harder-Narasimhan stratification of $\mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)$ (see Remark 5.3.3.2). We let $\mathrm{Bun}_G(X)^\sigma_{P_0,\nu}$ denote the reduced locally closed substack of $\mathrm{Bun}_G(X)$ given by

$$(\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)} \mathrm{Bun}_{G_{0\,\mathrm{ad}}}(X)_{P_{0\,\mathrm{ad}},\nu})_{\mathrm{red}}.$$

Then $\{\mathrm{Bun}_G(X)^\sigma_{P_0,\nu}\}_{(P_0,\nu)\in A}$ is a stratification of $\mathrm{Bun}_G(X)$, which we will refer to as the *Harder-Narasimhan stratification.*

Warning 5.3.5.2. In the special case where the reductive group scheme G is split, the Harder-Narasimhan stratification of Construction 5.3.5.1 is not quite the same as the Harder-Narasimhan stratification of Theorem 5.3.2.2. The former stratification is indexed by the set

$$A = \{(P_0, \nu) : \ P_0 \subseteq G_0 \text{ is a standard parabolic subgroup and } \nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}\ \},$$

while the latter stratification is indexed by the set

$$B = \{P_0, \overline{\nu} : \ P_0 \subseteq G_0 \text{ is a standard parabolic subgroup and } \overline{\nu} \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee_{>0}\}.$$

For every standard parabolic subgroup $P_0 \subseteq G_0$, there is a canonical map of lattices

$$\phi_{P_0} : \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee \to \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee,$$

and for each $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee$ we have

$$\mathrm{Bun}_G(X)^\sigma_{P_0,\nu} = \amalg_{\phi_{P_0}(\overline{\nu})=\nu} \mathrm{Bun}_G(X)_{P_0,\overline{\nu}},$$

where the left hand side refers to the stratification of Construction 5.3.5.1 (where σ denotes the inner structure determined by a splitting of G) and the right hand side refers to the stratification of Theorem 5.3.2.2.

If the group G_0 is semisimple, then the map ϕ_{P_0} is injective for every standard parabolic $P_0 \subseteq G_0$. In this case, we can regard B as a subset of A, and we have

$$\mathrm{Bun}_G(X)^\sigma_{P_0,\nu} = \begin{cases} \mathrm{Bun}_G(X)_{P_0,\nu} & \text{if } (P_0,\nu) \in B \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, the only difference between the stratifications of Construction 5.3.5.1 and Theorem 5.3.2.2 is that the former includes some "superfluous" empty strata (indexed by elements of A that do not belong to B).

If the group G_0 is not semisimple, then the maps ϕ_{P_0} fail to be injective. In this case, the stratification of Theorem 5.3.2.2 is much finer than the stratification of Construction 5.3.5.1. For example, if $G_0 = \mathbf{G}_m$, then we can identify $\mathrm{Bun}_G(X) = \mathrm{Bun}_{G_0}(X)$ with the Picard stack $\mathrm{Pic}(X)$ of line bundles on X. The stratification of Construction 5.3.5.1 is trivial (there is only one stratum, consisting of the entire moduli stack $\mathrm{Pic}(X)$), but the stratification of Theorem 5.3.2.2 reproduces the decomposition of $\mathrm{Pic}(X)$ as a disjoint union $\amalg_{n\in\mathbf{Z}} \mathrm{Pic}^n(X)$, where $\mathrm{Pic}^n(X)$ denotes the moduli stack of line bundles of degree n on X.

Warning 5.3.5.3. Let (G,σ) be an inner form of G_0 over X. If the group scheme G is split, then the strata $\mathrm{Bun}_G(X)^\sigma_{P_0,\nu}$ are empty when ν does not belong to the image of the restriction map $\phi_{P_0} : \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee \to \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee$. However, this is generally not true if G is not split.

Warning 5.3.5.4. Let G be a form of G_0 over X. Suppose that G admits an inner structure σ, and let $\{\mathrm{Bun}_G(X)^\sigma_{P_0,\nu}\}_{(P_0,\nu)\in A}$ be the stratification of Construction 5.3.5.1. The collection of locally closed substacks $\{\mathrm{Bun}_G(X)^\sigma_{P_0,\nu} \subseteq \mathrm{Bun}_G(X)\}$ does not depend on the choice of σ. However, the indexing of this collection of locally closed substacks by the set A *does* depend on σ. More precisely, for each element $g \in \mathrm{Out}(G_0)$, we have

$$\mathrm{Bun}_G(X)^{g(\sigma)}_{g(P_0),\nu_g} = \mathrm{Bun}_G(X)^\sigma_{P_0,\nu}$$

(as locally closed substacks of $\mathrm{Bun}_G(X)$), where the left hand side is defined as in Remark 5.3.3.3. This equality follows immediately from Remark 5.3.3.3 together with Warning 5.3.4.7.

Remark 5.3.5.5 (Functoriality in X and G)**.** Let ψ be an automorphism of X *as an abstract scheme*, so that ψ induces an automorphism ψ_0 of the field $k = \mathrm{H}^0(X; \mathcal{O}_X)$ which we do not assume to be the identity. Let G be a form of G_0 over X and let $\overline{\psi}$ be an automorphism of G which covers the automorphism ψ of X. Then the pair $(\psi, \overline{\psi})$ determines an automorphism ϕ of the algebraic stack $\mathrm{Bun}_G(X)$, which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(X) & \xrightarrow{\phi} & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\psi_0} & \mathrm{Spec}(k). \end{array}$$

Suppose that G admits an inner structure σ. The image of σ under $\overline{\psi}$ determines another inner structure σ' on G. Using Remarks 5.3.3.1 and 5.3.3.3, we see that ϕ restricts to give equivalences of Harder-Narasimhan strata

$$\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu} \simeq \mathrm{Bun}_G(X)^{\sigma'}_{P_0,\nu} = \mathrm{Bun}^{\sigma}_G(X)_{g(P_0),\nu_g}$$

where $g \in \mathrm{Out}(G_0)$ satisfies $\sigma = g(\sigma')$. In other words, the automorphism ϕ preserves the decomposition of $\mathrm{Bun}_G(X)$ into locally closed substacks $\{\mathrm{Bun}_G(X)^{\sigma}_{P_0,\nu}\}_{(P_0,\nu)\in A}$, and permutes the strata by means of the action of the group $\mathrm{Out}(G_0)$ on A.

5.4 Quasi-Compactness Properties of Moduli Spaces of Bundles

Let X be an algebraic curve over a finite field $\mathbf{F}_q$ and let G be a smooth affine group scheme over X with connected fibers and semisimple generic fiber. In order to prove that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula (Theorem 5.0.0.3), the main obstacle that we will need to overcome is that the moduli stack $\mathrm{Bun}_G(X)$ is not quasi-compact. In this section, we collect some facts about the quasi-compactness of various loci in $\mathrm{Bun}_G(X)$ which will be needed in the proof of Theorem 5.0.0.3. These results can be summarized as follows:

- Every quasi-compact open substack $\mathcal{U} \subseteq \mathrm{Bun}_G(X)$ can be realized as the quotient of an algebraic space by the action of a linear algebraic group (Proposition 5.4.1.3).
- If the group scheme G is split reductive, then every Harder-Narasimhan stratum of $\mathrm{Bun}_G(X)$ (in the sense of §5.3.2) is quasi-compact (Proposition 5.4.3.1).
- The quasi-compactness properties of $\mathrm{Bun}_G(X)$ are largely insensitive to varying the algebraic curve X and the group scheme G (Propositions 5.4.2.1, 5.4.2.4, and 5.4.2.5).

5.4.1 Quasi-Compact Substacks of $\mathrm{Bun}_G(X)$

We begin by showing that every quasi-compact open substack of $\mathrm{Bun}_G(X)$ can be realized as a global quotient in a natural way. For this, it is convenient to introduce some notation.

Definition 5.4.1.1. Let X be an algebraic curve over a field k, let G be a smooth affine group scheme over X, and let $D \subseteq X$ be an effective divisor. We let $\mathrm{Bun}_G(X, D)$ denote the moduli stack of G-bundles on X equipped with a trivialization along D. More precisely, $\mathrm{Bun}_G(X, D)$ is the algebraic stack whose R-valued points can be identified with the category of pairs $(\mathcal{P}, \gamma)$, where $\mathcal{P}$ is a G-bundle on the relative curve X_R and γ is a trivialization of the restriction $\mathcal{P}|_{D_R}$, where $D_R = D \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$.

Remark 5.4.1.2. In the situation of Definition 5.4.1.1, Lemma 5.1.3.11 supplies an equivalence of algebraic stacks $\mathrm{Bun}_G(X, D) \simeq \mathrm{Bun}_{G'}(X)$, where $G' = \mathrm{Dil}^D(G)$ is the smooth affine group scheme over X given by the dilitation of G along D. In particular, $\mathrm{Bun}_G(X, D)$ is a smooth algebraic stack over k.

Proposition 5.4.1.3. *Let X be an algebraic curve over a field k and let G be a smooth affine group over X. Let $\mathcal{U}$ be a quasi-compact open substack of* $\mathrm{Bun}_G(X)$*. Then there exists an effective divisor $D \subseteq X$ such that the fiber product* $\mathrm{Bun}_G(X, D) \times_{\mathrm{Bun}_G(X)} \mathcal{U}$ *is an algebraic space.*

Proof. Since $\mathcal{U}$ is quasi-compact, we can choose a smooth surjection $\mathrm{Spec}(R) \to \mathcal{U}$, corresponding to a G-bundle $\mathcal{P}$ on the relative curve X_R. Since the diagonal of $\mathrm{Bun}_G(X)$ is affine, automorphisms of the G-bundle $\mathcal{P}$ are parametrized by an affine R-scheme of finite type Y. The identity automorphism of $\mathcal{P}$ determines a closed immersion of R-schemes $s : \mathrm{Spec}(R) \to Y$; let us denote the image of this map by $Y' \subseteq Y$.

Fix a closed point $x \in X$. For each $n \geq 0$, let $D_n \subseteq X$ denote the divisor given by the nth multiple of X, and let Y_n denote the closed subscheme of Y classifying automorphisms of $\mathcal{P}$ which restrict to the identity over the divisor D_n.

Let $\mathcal{O}_x$ denote the complete local ring of x at X, which we can identify with a formal power series ring $k'[[t]]$ for some finite extension k' of k. For any Noetherian R-algebra A, we can identify the formal completion of X_A along the closed subscheme $\{x\} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$ with the formal spectrum of the power series ring $A'[[t]]$, where $A' = A \otimes_k k'$. The map $\mathrm{Spec}(A'[[t]]) \to X_A$ is schematically dense, so any automorphism of $\mathcal{P} \times_{X_R} X_A$ which restricts to the identity on the $\mathrm{Spec}(A'[[t]])$ must coincide with the identity. In other words, we have $\bigcap_{n \geq 0} Y_n = Y'$ (as closed subschemes of Y). Since Y is a Noetherian scheme, we must have $Y' = Y_n$ for $n \gg 0$. It then follows that $\mathrm{Bun}_G(X, D_n) \times_{\mathrm{Bun}_G(X)} \mathcal{U}$ is an algebraic space. □

Corollary 5.4.1.4. *Let X be an algebraic curve over a field k and let G be a smooth affine group scheme over X. Suppose we are given a quasi-compact algebraic stack $\mathcal{Y}$ over k equipped with a map $f : \mathcal{Y} \to \mathrm{Bun}_G(X)$. Assume that f is representable by quasi-compact, quasi-separated algebraic spaces (in other words, for every map $\mathrm{Spec}(R) \to \mathrm{Bun}_G(X)$, the fiber product $\mathcal{Y} \times_{\mathrm{Bun}_G(X)} \mathrm{Spec}(R)$ is a quasi-compact, quasi-separated algebraic space). Then $\mathcal{Y}$ can be written as a quotient Y/H, where Y is a quasi-compact, quasi-separated algebraic space over k and H is a linear algebraic group over k.*

Proof. Since $\mathcal{Y}$ is quasi-compact, the map f factors through a quasi-compact open substack $\mathcal{U} \subseteq \mathrm{Bun}_G(X)$. Using Proposition 5.4.1.3, we can choose an effective divisor $D \subseteq X$ such that the fiber product

$$Z = \mathcal{U} \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X, D)$$

is an algebraic space. Since Z is affine over $\mathcal{U}$, it is quasi-compact and quasi-separated. Set

$$Y = \mathcal{Y} \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X, D) \simeq \mathcal{Y} \times_{\mathrm{Bun}_G(X)} Z.$$

Since f is representable by quasi-compact, quasi-separated algebraic spaces, it follows that Y is a quasi-compact, quasi-separated algebraic space. Let H denote the Weil restriction of $G \times_X D$ along the finite flat map $D \to \mathrm{Spec}(k)$. Then H is a linear algebraic group acting on $\mathrm{Bun}_G(X, D)$, and we can identify $\mathrm{Bun}_G(X)$ with the (stack-theoretic) quotient $\mathrm{Bun}_G(X, D)/H$. It follows that H acts on $Y = \mathcal{Y} \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X, D)$ (via its action on the second factor) with quotient

$$Y/H \simeq \mathcal{Y} \times_{\mathrm{Bun}\, G(X)} \mathrm{Bun}_G(X, D)/H \simeq \mathcal{Y}.$$

□

5.4.2 Varying G and X

We now collect some relative quasi-compactness results for moduli spaces of bundles.

Proposition 5.4.2.1. *Let X be an algebraic curve over a field k and let $f : G \to G'$ be a morphism of smooth affine group schemes over X. Suppose that f is an isomorphism at the generic point of X. Then the induced map $\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G'}(X)$ is quasi-compact.*

To prove Proposition 5.4.2.1, we will need a few elementary facts about the dilitation construction described in §5.1.3.

Lemma 5.4.2.2. *Let X be a Dedekind scheme, let $\pi : Y \to X$ be a flat morphism of schemes equipped with a section s, let $U \subseteq X$ be a nonempty open subscheme, and* *suppose we are given a morphism of X-schemes $\psi : Y \times_X U \to Z$, where Z is a separated scheme of finite type over X. Suppose further that the composition*

$$U \xrightarrow{s|_U} Y \times_X U \xrightarrow{\psi} Z$$

extends to a map $h : X \to Z$. Then there exists an effective divisor $D \subseteq X$ satisfying $D \cap U = \emptyset$ and a map $\overline{\psi} : \mathrm{Dil}^D(Y) \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Dil}^D(Y) \times_X U & \longrightarrow & \mathrm{Dil}^D(Y) \\ \downarrow \sim & & \downarrow \overline{\psi} \\ Y \times_X U & \xrightarrow{\psi} & Z. \end{array}$$

Proof. The assertion is local on X. We may therefore assume without loss of generality that $X = \mathrm{Spec}(R)$ is affine, that $X - U$ consists of a single point x, and that the maximal ideal $\mathfrak{m}_x \subseteq R$ defining the point x is generated by a single element $t \in R$. Choose an affine open subscheme $Y_0 \subseteq Y$ containing the point $y = s(x)$. Replacing X by $s^{-1}Y_0$, we may assume that h factors through Y_0. Note that for every effective divisor D supported at x, the dilitation $\mathrm{Dil}^D(Y)$ is covered by the open sets $\mathrm{Dil}^D(Y_0)$ and $\mathrm{Dil}^D(Y) \times_X U \simeq Y \times_X U$, which intersect in $\mathrm{Dil}^D(Y_0) \times_X U \simeq Y_0 \times_X U$. Consequently, to prove the existence of $\overline{\psi}$, we may replace Y by Y_0 and thereby reduce to the case where $Y = \mathrm{Spec}(A)$ is affine. We will abuse notation by identifying $t \in R$ with its image in A. Moreover, the section s determines an R-algebra homomorphism $\epsilon : A \to R$.

Choose an affine open subscheme $Z_0 \subseteq Z$ containing the point $z = h(x)$. Replacing X by $h^{-1}Z_0$, we may assume that h factors through Z_0. The fiber product $Z_0 \times_Z (Y_0 \times_X U)$ is a quasi-compact open subscheme of $Y_0 \times_X U$, complementary to the vanishing locus of a finitely generated ideal $I \subseteq A[t^{-1}]$. Choose a finite sequence of elements $f_1, \ldots, f_m \in A$ whose images in $A[t^{-1}]$ generate the ideal I. Since $h(X) \subseteq Z_0$, $s(U)$ is contained in $Z_0 \times_Z (Y_0 \times_X U)$, so $\epsilon(f_i) \neq 0$ for some $1 \leq i \leq m$. We can therefore write $\epsilon(f_i) = t^n u$ for some integer n, where $u \notin \mathfrak{x}$. Replacing Y by an appropriate dilitation, we can reduce to the case where $f_i - t^n u$ is divisible by t^n. In this case, we can replace f_i by $t^{-n} f_i$ and thereby reduce to the case where $\epsilon(f_i) \notin \mathfrak{m}_x$. Replacing A by $A[f_i^{-1}]$, we can further assume that f_i is invertible in A, so that ψ factors through Z_0. We may therefore replace Z by Z_0 and thereby reduce to the case where $Z \simeq \mathrm{Spec}(B)$ is affine.

Because Z is of finite type over X, we can choose a finite set of generators $b_1, \ldots, b_m$ for B as an R-algebra. Without loss of generality, we may assume that each b_j is annihilated by the R-algebra homomorphism $B \to R$ determined by h. The map ψ determines an R-algebra homomorphism $\rho : B \to A[t^{-1}]$. We can therefore write $\rho(b_j) = t^{-N} a_j$ for some $N \gg 0$ and some elements $a_j \in A$ satisfying $\epsilon(a_j) = 0$. It follows that ρ factors through the subalgebra of $A[t^{-1}]$ generated by $t^{-N} \ker(\epsilon)$, and therefore determines a map of X-schemes $\mathrm{Dil}^{Nx}(Y) \to Z$. $\square$

Applying Lemma 5.4.2.2 in the case where Y and Z are group schemes over X, we obtain the following:

Lemma 5.4.2.3. *Let X be a Dedekind scheme, let G and G' be smooth affine group schemes over X, and let $\psi : G \times_X U \to G' \times_X U$ be a morphism of group schemes, where $U \subseteq X$ is a nonempty open subset. Then there exists an effective divisor $D \subseteq X$ satisfying $D \cap U = \emptyset$ and a morphism of group schemes $\overline{\psi} : \mathrm{Dil}^D(G) \to G'$ for which the diagram*

$$\begin{array}{ccc} G \times_X U & \xrightarrow{\psi} & G' \times_X U \\ \downarrow & & \downarrow \\ \mathrm{Dil}^D(G) & \xrightarrow{\overline{\psi}} & G' \end{array}$$

commutes.

Proof of Proposition 5.4.2.1. Let X be an algebraic curve over k and let $f : G \to G'$ be a morphism of smooth affine group schemes over X which is an isomorphism over a nonempty open subset $U \subseteq X$. Then the inverse of f determines a map $g : G' \times_X U \to G \times_X U$ of group schemes over U. Using Lemma 5.4.2.3, we deduce that there is an effective divisor $D' \subseteq X$ (disjoint from U) such that, if $\overline{G}' = \mathrm{Dil}^{D'}(G')$ is the dilitation of G' along D', then g extends to a map $\overline{g} : \overline{G}' \to G$ of group schemes over X. Applying the same argument to the map $f|_U : G \times_X U \to \overline{G}' \times_X U$, we conclude that there is an effective divisor $D \subseteq X$ (again disjoint from U) such that, if $\overline{G} = \mathrm{Dil}^D(G)$ is obtained from G by dilitation along D, then $f|_U$ extends to a map $\overline{f} : \overline{G} \to \overline{G}'$. Remark 5.4.1.2 supplies equivalences

$$\mathrm{Bun}_{\overline{G}}(X) \simeq \mathrm{Bun}_G(X, D) \qquad \mathrm{Bun}_{\overline{G}'}(X) \simeq \mathrm{Bun}_{G'}(X, D'),$$

so that the maps f, $\overline{g}$, and $\overline{f}$ give a diagram of algebraic stacks

$$\mathrm{Bun}_G(X, D) \to \mathrm{Bun}_{G'}(X, D') \to \mathrm{Bun}_G(X) \to \mathrm{Bun}_{G'}(X).$$

Note that the composite map $\mathrm{Bun}_G(X, D) \to \mathrm{Bun}_G(X)$ is surjective (since any G-bundle on the relative divisor $D \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ can be trivialized étale locally on $\mathrm{Spec}(R)$), so the map $\mathrm{Bun}_{G'}(X, D') \to \mathrm{Bun}_G(X)$ is also surjective. Consequently, to prove that the map $\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G'}(X)$ is quasi-compact, it will suffice to show that the map $\phi : \mathrm{Bun}_{G'}(X, D') \to \mathrm{Bun}_{G'}(X)$ is quasi-compact. This is clear, because ϕ is an affine morphism (it is a torsor for the affine group scheme over k given by the Weil restriction of $G' \times_X D'$ along the finite flat map $D' \to \mathrm{Spec}(k)$). □

Proposition 5.4.2.4. *Let k be a field, let $f : \widetilde{X} \to X$ be a non-constant morphism of algebraic curves over k, and let G be a smooth affine group scheme over X. Then the* *canonical map of algebraic stacks* $\mathrm{Bun}_G(X) \to \mathrm{Bun}_G(\widetilde{X})$ *(given by pullback along* f*) is an affine morphism. In particular, it is quasi-compact.*

Proof. Fix a map $\mathrm{Spec}(R) \to \mathrm{Bun}_G(\widetilde{X})$, corresponding to a G-bundle $\mathcal{P}$ on the relative curve $\widetilde{X}_R$. We wish to show the fiber product $\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(\widetilde{X})} \mathrm{Spec}(R)$ is representable by an affine R-scheme. Let Y denote the fiber product $\widetilde{X}_R \times_{X_R} \widetilde{X}_R$ and let

$$\pi_1, \pi_2 : Y \to \widetilde{X}_R$$

denote the two projection maps. Let $\mathrm{Iso}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P})$ denote the affine Y-scheme whose A-valued points are G-bundle isomorphisms of $(\pi_1^* \mathcal{P}) \times_Y \mathrm{Spec}(A)$ with $(\pi_2^* \mathcal{P}) \times_Y \mathrm{Spec}(A)$. Let Z denote the affine R-scheme obtained by Weil restriction of $\mathrm{Iso}(\pi_1^* \mathcal{P}, \pi_2^* \mathcal{P})$ along the proper flat morphism $Y \to \mathrm{Spec}(R)$. It now suffices to observe that the fiber product

$$\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(\widetilde{X})} \mathrm{Spec}(R)$$

can be identified with a closed subscheme of Z: the A-valued point of Z correspond to G-bundle isomorphisms

$$\gamma : (\pi_1^* \mathcal{P} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(A)) \simeq (\pi_2^* \mathcal{P} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(A)),$$

while the A-valued points of $\mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G(\widetilde{X})} \mathrm{Spec}(R)$ correspond to such G-bundle isomorphisms which satisfy a cocycle condition (since the finite flat morphism $\widetilde{X}_A \to X_A$ is of effective descent for G-bundles). □

Proposition 5.4.2.5. *Let* X *be an algebraic curve over a field* k*, let* G *be a semisimple group scheme over* X*, and let* G_{ad} *denote the adjoint quotient of* G*. Assume that the generic fiber of* G *is split. Then the natural map* $\mathrm{Bun}_G(X) \to \mathrm{Bun}_{G_{\mathrm{ad}}}(X)$ *is quasi-compact.*

Proof. Without loss of generality, we may assume that k is algebraically closed. Let G_0 denote the generic fiber of G. Since G_0 is split, we can choose a Borel subgroup $B_0 \subseteq G_0$ and a split maximal torus $T_0 \subseteq B_0$. Since G is semisimple, the X-scheme parametrizing Borel subgroups of G is proper over X; it follows from the valuative criterion of properness that B_0 extends uniquely to a Borel subgroup $B \subseteq G$ (given by the scheme-theoretic closure of B_0 in G). Let U denote the unipotent radical of B, and let $T = B/U$. Then T is an algebraic torus over X whose generic fiber is split (since it is isomorphic to T_0); it follows that T itself is a split torus. Let B_{ad} denote the image of B in the adjoint quotient G_{ad}, and let T_{ad} denote the quotient of B_{ad} by its unipotent radical.

Let R be a finitely generated k-algebra and suppose we are given a map $f : \mathrm{Spec}(R) \to \mathrm{Bun}_{G_{\mathrm{ad}}}(X)$; we wish to prove that the fiber product $\mathrm{Spec}(R) \times_{\mathrm{Bun}_{G_{\mathrm{ad}}}(X)}$

$\mathrm{Bun}_G(X)$ is quasi-compact. This assertion can be tested locally with respect to the étale topology on $\mathrm{Spec}(R)$. We may therefore assume without loss of generality that f factors through $\mathrm{Bun}_{B_{\mathrm{ad}}}(X)$: this follows from a theorem of Drinfeld-Simpson (see [12], or the sequel to this book). Since the diagram of algebraic stacks

$$\begin{array}{ccccc} \mathrm{Bun}_G(X) & \longleftarrow & \mathrm{Bun}_B(X) & \longrightarrow & \mathrm{Bun}_T(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}}}(X) & \longleftarrow & \mathrm{Bun}_{B_{\mathrm{ad}}}(X) & \longrightarrow & \mathrm{Bun}_{T_{\mathrm{ad}}}(X) \end{array}$$

consists of pullback squares, it will suffice to show that the fiber product $\mathrm{Spec}(R)\times_{\mathrm{Bun}_{T_{\mathrm{ad}}}(X)} \mathrm{Bun}_T(X)$ is quasi-compact. We are therefore reduced to proving that the map

$$\mathrm{Bun}_T(X) \to \mathrm{Bun}_{T_{\mathrm{ad}}}(X)$$

is quasi-compact.

Let $\mathrm{Pic}(X) = \mathrm{Bun}_{\mathbf{G}_m}(X)$ denote the Picard stack of X; a choice of k-rational point $x \in X$ determines a splitting

$$\mathrm{Pic}(X) = \mathbf{Z} \times J(X) \times \mathrm{B}\,\mathbf{G}_m$$

where $J(X)$ is the Jacobian variety of X. Let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, T)$ denote the cocharacter lattice of T and let $\Lambda_{\mathrm{ad}} = \mathrm{Hom}(\mathbf{G}_m, T_{\mathrm{ad}})$ denote the cocharacter lattice of T_{ad}. We wish to show that the natural map

$$\mathrm{Bun}_T(X) \simeq \Lambda \otimes_{\mathbf{Z}} \mathrm{Pic}(X) \to \Lambda_{\mathrm{ad}} \otimes_{\mathbf{Z}} \mathrm{Pic}(X) \simeq \mathrm{Bun}_{T_{\mathrm{ad}}}(X)$$

is quasi-compact. This is clear: the preimage of each connected component of $\mathrm{Bun}_{T_{\mathrm{ad}}}(X)$ is either empty or isomorphic to a product of finitely many copies of $J(X) \times \mathrm{B}\,\mathbf{G}_m$. □

5.4.3 Quasi-Compactness of Harder-Narasimhan Strata

We now show that the Harder-Narasimhan stratification of §5.3.2 partitions $\mathrm{Bun}_G(X)$ into *quasi-compact* locally closed substacks.

Proposition 5.4.3.1. *Let X be an algebraic curve over a field k and let G be a split reductive group over k. Fix a Borel subgroup $B \subseteq G$ containing a split maximal torus $T \subseteq B$. For every standard parabolic subgroup $P \subseteq G$ and every element $\nu \in \mathrm{Hom}(P, \mathbf{G}_m)^\vee$, the algebraic stack $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}}$ is quasi-compact.*

Proof. Without loss of generality, we may assume that k is algebraically closed. We proceed in several steps.

(*a*) Suppose first that G is a torus, and let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, G)$ denote the cocharacter lattice of G. In this case, the only parabolic subgroup $P \subseteq G$ is the group G itself, and we have $\mathrm{Hom}(P, \mathbf{G}_m)^\vee \simeq \Lambda$. For each $\nu \in \Lambda$, the moduli stack $\mathrm{Bun}_P^\nu(X)^{\mathrm{ss}} = \mathrm{Bun}_P^\nu(X)$ can be identified (after choosing a k-rational point $x \in X$) with a product of finitely many copies of $\mathrm{Pic}^0(X) \simeq J(X) \times \mathrm{B}\,\mathbf{G}_m$ (as in the proof of Proposition 5.4.2.5), and is therefore quasi-compact.

(*b*) We claim that if Proposition 5.4.3.1 is valid for the quotient $G' = P/\mathrm{rad}_u(P)$ (regarded as a parabolic subgroup of itself), then it is valid for the parabolic subgroup P. To establish this, it suffices to show that the map $\mathrm{Bun}_P(X) \to \mathrm{Bun}_{G'}(X)$ is quasi-compact. Note that the unipotent radical $\mathrm{rad}_u(P)$ is equipped with a finite filtration by normal subgroups

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_m = \mathrm{rad}_u(P),$$

where each quotient U_i/U_{i-1} is isomorphic to a vector group equipped with a linear action of P (which necessarily factors through the quotient P/U_i). We claim that each of the maps $\mathrm{Bun}_{P/U_{i-1}}(X) \to \mathrm{Bun}_{P/U_i}(X)$ is quasi-compact. To prove this, fix a map $\mathrm{Spec}(R) \to \mathrm{Bun}_{P/U_i}(X)$, given by a P/U_i-torsor $\mathcal{P}$ on the relative curve X_R. Via the linear action of P/U_i on U_i/U_{i-1}, we obtain a vector bundle $\mathcal{E}_i$ on X_R. The obstruction to lifting $\mathcal{P}$ to a P/U_{i-1}-bundle is measured by a cohomology class $\eta \in \mathrm{H}^2(X_R; \mathcal{E}_i)$, which automatically vanishes since X_R is a curve over an affine scheme. Choose a lifting of $\mathcal{P}$ to a (P/U_{i-1})-torsor on X_R. Then the fiber product

$$\mathcal{Y} = \mathrm{Spec}(R) \times_{\mathrm{Bun}_{P/U_i}(X)} \mathrm{Bun}_{P/U_{i-1}}(X)$$

can be identified with the stack whose A-valued points (where A is an R-algebra) correspond to $\mathcal{E}_i$-torsors on the relative curve X_A. We wish to prove that $\mathcal{Y}$ is quasi-compact. If $D \subseteq X$ is an effective divisor, let $\mathcal{Y}_D$ denote the algebraic stack whose A-valued points are $\mathcal{E}_i$-torsors on X_A which are equipped with a trivialization along the relative divisor $D \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$. The evident forgetful functor $\mathcal{Y}_D \to \mathcal{Y}$ is surjective, so it will suffice to prove that we can choose D such that $\mathcal{Y}_D$ is quasi-compact. Note that if $\deg(D) \gg 0$, then $\mathrm{H}^0(X_R; \mathcal{E}_i(-D)) \simeq 0$ and $\mathrm{H}^1(X_R; \mathcal{E}_i(-D))$ is a projective R-module M of finite rank; in this case, we can identify $\mathcal{Y}_D$ with the affine scheme $\mathrm{Spec}(\mathrm{Sym}_R^*(M^\vee))$.

(*c*) We now prove Proposition 5.4.3.1 in general. By virtue of (*b*), it will suffice to treat the case where $P = G$. Set $\Lambda_0 = \mathrm{Hom}(G, \mathbf{G}_m)^\vee$ and $\Lambda = \mathrm{Hom}(B, \mathbf{G}_m)^\vee \simeq \mathrm{Hom}(\mathbf{G}_m, T)$. The inclusion $B \hookrightarrow G$ induces a surjective map of lattices $\chi : \Lambda \to \Lambda_0$. It follows from steps (*a*) and (*b*) that for each $\lambda \in \Lambda$, the moduli stack

$\mathrm{Bun}_B^{\lambda}(X)^{\mathrm{ss}} = \mathrm{Bun}_B^{\lambda}(X)$ is quasi-compact. It follows that any open substack of $\mathrm{Bun}_B^{\lambda}(X)$ is also quasi-compact. To complete the proof, it will suffice to show that for each $\nu \in \Lambda_0$, we can find a finite subset $S \subseteq \chi^{-1}\{\nu\}$ for which the map

$$\amalg_{\lambda \in S}(\mathrm{Bun}_B^{\lambda}(X) \times_{\mathrm{Bun}_G(X)} \mathrm{Bun}_G(X)^{\mathrm{ss}}) \to \mathrm{Bun}_G^{\nu}(X)^{\mathrm{ss}}$$

is surjective.

Let g denote the genus of the algebraic curve X, let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots of G (which we identify with elements of $\Lambda^{\vee}$), and let 2ρ denote the sum of the positive roots of G. We will show that the set

$$S = \{\lambda \in \Lambda : \chi(\lambda) = \nu, \langle 2\rho, \lambda \rangle \leq 0, \langle \alpha_i, \lambda \rangle \geq \min\{1-g, 0\}\}$$

has the desired property. Note that S can be identified with the set of lattice points belonging to the locus

$$S_{\mathbf{R}} = \{\lambda \in \Lambda \otimes \mathbf{R} : \chi(\lambda) = \nu, \langle 2\rho, \lambda \rangle \leq 0, \langle \alpha_i, \lambda \rangle \geq \min\{1-g, 0\}\}$$

which is a simplex in the real vector space $\Lambda \otimes \mathbf{R}$; this proves that S is finite. We will complete the proof by showing that if $\mathcal{P}$ is a semistable G-bundle of degree ν, then there exists $\lambda \in S$ such that $\mathcal{P}$ can be reduced to a B-bundle of degree λ. Note that in this case the conditions $\chi(\lambda) = \nu$ and $\langle 2\rho, \lambda \rangle \leq 0$ are automatic (if the second condition were violated, then $\mathcal{P}$ would not be semistable). It will therefore suffice to prove the following:

$(*)$ Let $\mathcal{P}$ be a G-bundle on X. Then $\mathcal{P}$ can be reduced to a B-bundle $\mathcal{Q}$ satisfying

$$\langle \deg(\mathcal{Q}), \alpha_i \rangle \geq \min\{1-g, 0\}$$

for $1 \leq i \leq r$.

Let $C = \{\lambda \in \Lambda : \langle \alpha_i, \lambda \rangle \geq 0 \text{ for } 1 \leq i \leq r\}$ be the dominant Weyl chamber in Λ, and let W denote the Weyl group of G. Then W acts on Λ, and every W-orbit in Λ contains an element of C. Let $\mathcal{P}$ be as in $(*)$. Then $\mathcal{P}$ admits a B-reduction $\mathcal{Q}$ (see [12], or the sequel to this book). Write $\deg(\mathcal{Q}) = w\lambda$, where $\lambda \in C$ and $w \in W$. Let us assume that $\mathcal{Q}$ and w have been chosen so that w has minimal length. We will prove $(*)$ by showing that $\langle \alpha_i, \deg(\mathcal{Q}) \rangle \geq \min\{1-g, 0\}$ for $1 \leq i \leq r$. Suppose otherwise: then there exists a simple root α such that $\langle \deg(\mathcal{Q}), \alpha_i \rangle < 0$ and $\langle \alpha_i, \deg(\mathcal{Q}) \rangle < 1-g$. Let $w_i \in W$ denote the simple reflection corresponding to the root α_i. The condition $\langle \deg(\mathcal{Q}), \alpha_i \rangle < 0$ implies that $w_i w$ has smaller length than w. We will obtain a contradiction by showing that $\mathcal{P}$ admits a reduction to a B-bundle having degree $w_i \deg(\mathcal{Q}) = (w_i w)\lambda$. For this, it suffices to establish the following:

($*'$) Let $\mathcal{Q}$ be a B-bundle on X and let α_i be a simple root of G satisfying $\langle \alpha_i, \deg(\mathcal{Q}) \rangle < 1-g$. Then there exists another B-bundle $\mathcal{Q}'$ on X such that $\deg(\mathcal{Q}') = w_i \deg(\mathcal{Q})$, and $\mathcal{Q}$ and $\mathcal{Q}'$ determine isomorphic G-bundles on X.

To prove ($*'$), let $P \subseteq G$ denote the parabolic subgroup generated by B together with the root subgroup corresponding to $-\alpha_i$, and let $\mathcal{Q}_P$ denote the P-bundle determined by $\mathcal{Q}$. We will show that $\mathcal{Q}_P$ admits a B-reduction $\mathcal{Q}'$ satisfying $\deg(\mathcal{Q}') = w_i \deg(\mathcal{Q})$. Note that there is a bijective correspondence between B-reductions of $\mathcal{Q}_P$ and $(B/\operatorname{rad}_u(P))$-reductions of the induced $(P/\operatorname{rad}_u P)$-bundle $\mathcal{Q}_{P/\operatorname{rad}_u P}$. Replacing G by $P/\operatorname{rad}_u P$, we are reduced to the problem of proving ($*'$) in the special case where G has semisimple rank 1 (that is, where α_i is the *only* root of G). In this case, we will prove that $\mathcal{Q}$ can be reduced to a T-bundle $\mathcal{Q}_0$. The element $w_i \in W$ determines an automorphism of T which becomes inner in G, and therefore induces an automorphism of the set of isomorphism classes of T-bundles with itself which does not change the isomorphism class of the associated G-bundle. This automorphism carries $\mathcal{Q}_0$ to the isomorphism class of another T-bundle $\mathcal{Q}_0^{w_i}$, and we can complete the proof of ($*'$) by taking $\mathcal{Q}'$ to be the B-bundle determined by $\mathcal{Q}_0^{w_i}$. We conclude by observing that the obstruction to choosing the reduction $\mathcal{Q}_0$ is given by an element of $\mathrm{H}^1(X; \mathcal{L})$, where $\mathcal{L}$ is the line bundle on X obtained from $\mathcal{Q}$ via the (linear) action of B on $\operatorname{rad}_u(B) \simeq \mathbf{G}_a$. An elementary calculation shows that the degree of $\mathcal{L}$ is given by $-\langle \alpha_i, \deg(\mathcal{Q}) \rangle > g-1$, so that $\mathrm{H}^1(X; \mathcal{L})$ vanishes by the Riemann-Roch theorem.

□

5.5 Comparison of Harder-Narasimhan Strata

Throughout this section, we fix a field k, an algebraic curve X over k, and a split semisimple algebraic group G_0 over k. Fix a Borel subgroup $B_0 \subseteq G_0$, a split maximal torus $T_0 \subseteq B_0$, and a parabolic subgroup $P_0 \subseteq G_0$ which contains B_0. If (G, σ) is an inner form of G_0 over X, then we can regard the moduli stack $\operatorname{Bun}_G(X)$ as equipped with the Harder-Narasimhan stratification of Construction 5.3.5.1. Using Proposition 5.4.3.1, it is not difficult to see that each of the Harder-Narasimhan strata $\operatorname{Bun}_G^{\sigma}(X)_{P_0,\nu}$ is quasi-compact. However, this observation alone is not sufficient to prove that $\operatorname{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula because there are infinitely many strata (for each standard parabolic $P_0 \subsetneq G_0$, there are infinitely many choices fo the dominant regular cocharacter ν). In order to apply Proposition 5.2.2.3, we will need to argue that the collection of Harder-Narasimhan strata can be broken into finitely many families whose members "look alike." Our goal in this section is to articulate this idea more precisely by introducing a twisting construction

(Definition 5.5.3.1) which can be used to move from one Harder-Narasimhan stratum to another.

5.5.1 Parabolic Reductions

We begin by introducing some terminology.

Definition 5.5.1.1. Let (G, σ) be an inner form of G_0 over a k-scheme Y. We will say that a parabolic subgroup $P \subseteq G$ is *of type* P_0 if there exists an étale surjection $\widetilde{Y}$ and an isomorphism

$$\beta : G \times_Y \widetilde{Y} \to G_0 \times_{\mathrm{Spec}(k)} \widetilde{Y}$$

which is compatible with σ and which restricts to an isomorphism of $P \times_Y \widetilde{Y}$ with $P_0 \times_{\mathrm{Spec}(k)} \widetilde{Y}$. Note that if Y is connected, then every parabolic subgroup $P \subseteq G$ is of type P_0 for a unique standard parabolic subgroup $P_0 \subseteq G_0$.

If $\mathcal{P}$ is a G-torsor on Y, we let $G_{\mathcal{P}}$ denote the group scheme over Y whose R-valued points are G-bundle automorphisms of $\mathcal{P} \times_Y \mathrm{Spec}(R)$. The inner structure σ on G determines an inner structure $\sigma_{\mathcal{P}}$ on $G_{\mathcal{P}}$. We will say that a subgroup $P \subseteq G_{\mathcal{P}}$ is a P_0-*structure* on $\mathcal{P}$ if it is a parabolic subgroup of type P_0.

Example 5.5.1.2. Let (G, σ) be an inner form of G_0 over a k-scheme Y, and let $P \subseteq G$ be a parabolic subgroup of type P_0. Let $\mathcal{Q}$ be a P-bundle on Y and let $\mathcal{P}$ be the associated G-bundle. Since any automorphism of $\mathcal{Q}$ (as a P-bundle) determines an automorphism of $\mathcal{P}$ (as a G-bundle), there is a canonical map of group schemes $P_{\mathcal{Q}} \to G_{\mathcal{P}}$, which exhibits $P_{\mathcal{Q}}$ as a parabolic subgroup of $G_{\mathcal{P}}$ of type P_0. This construction determines an equivalence of categories

$$\{P\text{-bundles on } Y\} \to \{G\text{-bundles } \mathcal{P} \text{ on } Y \text{ equipped with a } P_0\text{-structure}\}.$$

In particular, if $G = G_0 \times_{\mathrm{Spec}(k)} Y$ is the split form of G_0 over Y, then we obtain an equivalence

$$\{P_0\text{-bundles on } Y\} \to \{G\text{-bundles on } Y \text{ equipped with a } P_0\text{-structure}\}.$$

Remark 5.5.1.3. Suppose that G_0 is an adjoint semisimple group. If (G, σ) is an inner form of G_0 over a k-scheme Y, then Construction 5.3.4.6 determines a canonical equivalence from the category of G-bundles on Y to the category of G_0-bundles on Y, which we will denote by $\mathcal{P} \mapsto \mathcal{P}_0$. By functoriality, we can identify the automorphism group scheme $G_{\mathcal{P}}$ of $\mathcal{P}$ (as a G-bundle) with the automorphism group scheme $(G_0 \times_{\mathrm{Spec}(k)} Y)_{\mathcal{P}_0}$ of $\mathcal{P}_0$ (as a G_0-bundle). In particular, there is a canonical bijection between the set of P_0-structures on $\mathcal{P}$ and the set of P_0-structures on $\mathcal{P}_0$. Combining this observation with Example 5.5.1.2, we obtain an equivalence of categories

$$\{\ G\text{-bundles on } Y \text{ with a } P_0\text{-structure}\ \} \simeq \{\ P_0\text{-bundles on } Y\ \}.$$

Remark 5.5.1.4. Let (G, σ) be an inner form of G_0 over a k-scheme Y, and let G_{ad} denote the adjoint quotient of G. Then G_{ad} is a form of the adjoint group $G_{0\,\mathrm{ad}}$ over Y, and σ determines an inner structure σ_{ad} on G_{ad}. For any parabolic subgroup $P \subseteq G$, let P_{ad} denote the image of P in G_{ad}. The construction $P \mapsto P_{\mathrm{ad}}$ determines a bijective correspondence between parabolic subgroups of G and parabolic subgroups of G_{ad}; moreover, a parabolic subgroup $P \subseteq G$ has type P_0 if and only if $P_{\mathrm{ad}} \subseteq G_{\mathrm{ad}}$ has type $P_{0\,\mathrm{ad}} \subseteq G_{0\,\mathrm{ad}}$. It follows that if $\mathcal{P}$ is a G-bundle on Y and $\mathcal{P}_{\mathrm{ad}}$ denotes the associated G_{ad}-bundle, then there is a canonical bijection from the set of P_0-structures on $\mathcal{P}$ to the set of $P_{0\,\mathrm{ad}}$ structures on $\mathcal{P}_{\mathrm{ad}}$.

Definition 5.5.1.5. Let X be an algebraic curve over k and let (G, σ) be an inner form of G_0 over X. For every standard parabolic subgroup $P_0 \subseteq G_0$, we let $\mathrm{Bun}_{G,P_0}(X)$ denote the stack whose R-valued points are pairs $(\mathcal{P}, P)$, where $\mathcal{P}$ is a G-bundle on X_R and $P \subseteq G_{\mathcal{P}}$ is a parabolic subgroup of type P_0. We will refer to $\mathrm{Bun}_{G,P_0}(X)$ as the *moduli stack of G-bundles with a P_0-structure*.

Warning 5.5.1.6. Though it is not apparent from our notation, the moduli stack $\mathrm{Bun}_{G,P_0}(X)$ depends on the choice of inner structure σ on G. Modifying σ by an element $g \in \mathrm{Out}(G_0)$ has the effect of replacing $\mathrm{Bun}_{G,P_0}(X)$ with $\mathrm{Bun}_{G,g(P_0)}(X)$.

Example 5.5.1.7. Let X be an algebraic curve over k and let (G, σ) be an inner form of G_0 over X. If there exists a parabolic subgroup $P \subseteq G$ of type P_0, then Example 5.5.1.2 furnishes a canonical equivalence $\mathrm{Bun}_P(X) \simeq \mathrm{Bun}_{G,P_0}(X)$. Note, however, that $\mathrm{Bun}_{G,P_0}(X)$ is well-defined even if G does not have a parabolic subgroup of type P_0.

Example 5.5.1.8. Suppose that G_0 is an adjoint semisimple algebraic group over k, and let (G, σ) be an inner form of G_0 over an algebraic curve X. It follows from Remark 5.5.1.3 that the bitorsor $\mathrm{Iso}^{\sigma}(G, G_0)$ determines an equivalence $\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_{G',P_0}(X)$, where G' denotes the split form of G_0 over X. Combining this with Example 5.5.1.7, we obtain a canonical equivalence $\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_{P_0}(X)$.

Example 5.5.1.9. Let (G, σ) be an inner form of G_0 over an algebraic curve X. Then Remark 5.5.1.4 furnishes a pullback diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \longrightarrow & \mathrm{Bun}_G(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}},P_{0\,\mathrm{ad}}}(X) & \longrightarrow & \mathrm{Bun}_{G_{\mathrm{ad}}}(X). \end{array}$$

Combining this with Example 5.5.1.8, we obtain an equivalence

$$\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_G(X) \times_{\mathrm{Bun}_{G_{\mathrm{ad}}}(X)} \mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X).$$

It follows from this that $\mathrm{Bun}_{G,P_0}(X)$ is an algebraic stack which is locally of finite type over k, and that the diagonal of $\mathrm{Bun}_{G,P_0}(X)$ is affine.

Remark 5.5.1.10. If (G, σ) is an inner form of G_0 over an algebraic curve X, then the moduli stack $\mathrm{Bun}_{G,P_0}(X)$ is smooth over $\mathrm{Spec}(k)$. We will not need this fact and therefore omit the proof.

Notation 5.5.1.11. Let (G, σ) be an inner form of G_0 over an algebraic curve X. We let $\mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$ denote the fiber product

$$\mathrm{Bun}_{G,P_0}(X) \times_{\mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X)} \mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X)^{\mathrm{ss}}.$$

Then $\mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$ is an open substack of $\mathrm{Bun}_{G,P_0}(X)$ which we will refer to as the *semistable locus* of $\mathrm{Bun}_{G,P_0}(X)$.

For each element $\mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee$, we let $\mathrm{Bun}^{\nu}_{G,P_0}(X)$ denote the fiber product

$$\mathrm{Bun}_{G,P_0}(X) \times_{\mathrm{Bun}_{P_{0\,\mathrm{ad}}}(X)} \mathrm{Bun}^{\nu}_{P_{0\,\mathrm{ad}}}(X).$$

Then each $\mathrm{Bun}^{\nu}_{G,P_0}(X)$ is an open substack of $\mathrm{Bun}_{G,P_0}(X)$, and we can identify $\mathrm{Bun}_{G,P_0}(X)$ with the disjoint union $\amalg_{\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee} \mathrm{Bun}^{\nu}_{G,P_0}(X)$.

We let $\mathrm{Bun}^{\nu}_{G,P_0}(X)^{\mathrm{ss}}$ denote the intersection $\mathrm{Bun}^{\nu}_{G,P_0}(X) \cap \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}}$. It follows from Theorem 5.3.2.2 and Example 5.5.1.9 that if $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}$, then the canonical map $\mathrm{Bun}^{\nu}_{G,P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)$ restricts to a finite radicial surjection $\mathrm{Bun}^{\nu}_{G,P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_G(X)_{P_0,\nu}$.

5.5.2 Digression: Levi Decompositions

Let G be a reductive group scheme over a k-scheme Y, and let $P \subseteq G$ be a parabolic subgroup. Let $\mathrm{rad}_u(P)$ denote the unipotent radical of P, so that we have an exact sequence

$$0 \to \mathrm{rad}_u(P) \to P \xrightarrow{\pi} P/\mathrm{rad}_u(P) \to 0.$$

A *Levi decomposition of* P is a section of the map π (in the category of group schemes), which determines a semidirect product decomposition $P \simeq \mathrm{rad}_u(P) \rtimes (P/\mathrm{rad}_u(P))$.

Remark 5.5.2.1. Suppose that G is a reductive group scheme over Y and that $P \subseteq G$ is a parabolic subgroup. Then we always find a Levi decomposition ψ of P locally for the étale topology: for example, if $T \subseteq P$ is a maximal torus and $Z \subseteq T$ is the preimage in T of the center of $P/\mathrm{rad}_u(P)$, then the centralizer of Z in P is a subgroup H for which the composite map

$$H \hookrightarrow P \to P/\mathrm{rad}_u P$$

is an isomorphism, so the inverse isomorphism $P/\mathrm{rad}_u P \simeq H \hookrightarrow P$ is a Levi decomposition of P. Moreover, if P admits a Levi decomposition $\psi : P/\mathrm{rad}_u P \to P$, then ψ is unique up to conjugation by a Y-valued point of $\mathrm{rad}_u P$. More precisely, the collection of Levi decompositions of P can be regarded as a torsor for $\mathrm{rad}_u(P)$ which is locally trivial for the étale topology. Since the unipotent radical $\mathrm{rad}_u(P)$ admits a finite filtration whose successive quotients are vector groups, it follows that this torsor is trivial whenever Y is affine (in particular, it is locally trivial with respect to the Zariski topology).

Notation 5.5.2.2. Let $\mathfrak{Z}_0$ denote the center of the reductive algebraic group $P_0/\mathrm{rad}_u(P_0)$ (this is a split diagonalizable group scheme over k), and let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}_0)$ denote the cocharacter lattice of $\mathfrak{Z}_0$. There is a canonical bilinear map of abelian groups $\mathrm{Hom}(P_0, \mathbf{G}_m) \times \Lambda \to \mathbf{Z}$, which carries a pair (μ, λ) to the composite map

$$\mathbf{G}_m \xrightarrow{\lambda} \mathfrak{Z}_0 \subseteq P_0/\mathrm{rad}_u(P_0) \xrightarrow{\mu} \mathbf{G}_m,$$

regarded as an element of $\mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) \simeq \mathbf{Z}$. This bilinear map determines an injective map of lattices $\Lambda \hookrightarrow \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$. In what follows, we will generally abuse notation by identifying Λ with its image in $\mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$. We let $\Lambda_{\geq 0}$ denote the inverse image of $\mathrm{Hom}(P, \mathbf{G}_m)^\vee_{\geq 0}$ under this map (in other words, the collection of those elements $\lambda \in \Lambda$ having the property that $\langle \alpha, \lambda \rangle \geq 0$ for every simple root of G_0).

Suppose that (G, σ) is an inner form of G_0 over a k-scheme Y, and let $P \subseteq G$ be a parabolic subgroup of type P_0. Then σ determines an isomorphism $\mathfrak{Z}(P/\mathrm{rad}_u(P)) \simeq \mathfrak{Z}_0 \times_{\mathrm{Spec}(k)} Y$, where $\mathfrak{Z}(P/\mathrm{rad}_u(P))$ denotes the center of $P/\mathrm{rad}_u P$. If $\psi : P/\mathrm{rad}_u(P) \to P$ is a Levi decomposition of P, then ψ restricts to a map of group schemes $\mathfrak{Z}_0 \times_{\mathrm{Spec}(k)} Y \to P$. In particular, every element $\lambda \in \Lambda$ determines a map $\mathbf{G}_m \to P$ of group schemes over Y, which we will denote by $\psi(\lambda)$.

5.5.3 Twisting Parabolic Reductions

Let Y be a scheme. Recall that an *effective Cartier divisor* on Y is a closed subscheme $D \subseteq Y$ for which the corresponding ideal sheaf $\mathcal{I}_D \subseteq \mathcal{O}_Y$ is invertible. A *local parameter* for D is a global section of $\mathcal{I}_D$ which generates $\mathcal{I}_D$ at every point.

Definition 5.5.3.1. Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $\mathcal{P}$ be a G-bundle on Y equipped with a P_0-structure $P \subseteq G_{\mathcal{P}}$, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. A *$\lambda$-twist of* $(\mathcal{P}, P)$ *along* D is a pair $(\mathcal{P}', \gamma)$, where $\mathcal{P}'$ is a G-bundle on Y and γ is a G-bundle isomorphism $\mathcal{P} \times_Y (Y - D) \simeq \mathcal{P}' \times_Y (Y - D)$ having the following property:

(∗) Let $U \subseteq Y$ be an open subset having the property that $P \times_Y U$ admits a Levi decomposition $\psi : (P/\mathrm{rad}_u P) \times_Y U \to P \times_Y U$ and the Cartier divisor $(D \cap U) \subseteq U$ admits a local parameter t. Then the G-bundle isomorphism

$$\mathcal{P} \times_Y (U - (D \cap U)) \xrightarrow{\psi(\lambda)(t)^{-1}} \mathcal{P} \times_Y (U - (D \cap U)) \xrightarrow{\gamma} \mathcal{P}' \times_Y (U - (D \cap U))$$

extends to a G-bundle isomorphism $\overline{\gamma}_U : \mathcal{P} \times_Y U \simeq \mathcal{P}' \times_Y U$. Here the first map is given by the action of the element $\psi(\lambda)(t)^{-1} \in P(U) \subseteq G_{\mathcal{P}}(U)$, which acts on $\mathcal{P}|_U$ by G-bundle automorphisms. Note that the existence of such an extension is independent of the choice of Levi decomposition ψ.

Remark 5.5.3.2. In the situation of condition $(*)$ above, the extension $\overline{\gamma}_U$ is automatically unique (since the inclusion

$$\mathcal{P} \times_Y (U - (D \cap U)) \hookrightarrow \mathcal{P} \times_Y U$$

is complementary to a Cartier divisor, and therefore schematically dense).

Example 5.5.3.3. In the situation of Definition 5.5.3.1, suppose that $G = G_0 = \mathrm{SL}_2$, and let $P_0 \subseteq G_0$ be the standard Borel subgroup of upper-triangular matrices. Then a G-bundle with P_0-structure $(\mathcal{P}, P)$ on Y can be identified with a vector bundle $\mathcal{E}$ of rank 2, together with a line subbundle $\mathcal{L} \subseteq \mathcal{E}$ together with an isomorphism $\mathcal{E} / \mathcal{L} \simeq \mathcal{L}^{-1}$. Let us identify $\Lambda_{\geq 0}$ with the set of natural numbers. Given $(\mathcal{P}, P)$ as above, an element $\lambda \in \Lambda_{\geq 0}$, and an effective Cartier divisor $D \subseteq Y$, the pushout $\mathcal{E}' = \mathcal{L}(\lambda D) \amalg_{\mathcal{L}(-\lambda D)} \mathcal{E}(-\lambda D)$ is another rank 2 vector bundle on Y. Moreover, we have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{L}(-\lambda D) & \to & \mathcal{E}(-\lambda D) & \to & \mathcal{L}^{-1}(-\lambda D) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \to & \mathcal{L}(\lambda D) & \to & \mathcal{E}' & \to & \mathcal{L}^{-1}(-\lambda D) & \to & 0, \end{array}$$

which determines an isomorphism $\alpha : \det(\mathcal{E}') \simeq \mathcal{L}(\lambda D) \otimes \mathcal{L}^{-1}(-\lambda D) \simeq \mathcal{O}_Y$. By construction, we have a canonical vector bundle isomorphism $\gamma : \mathcal{E}|_{Y-D} \simeq \mathcal{E}'|_{Y-D}$, which is compatible with the trivializations of $\det(\mathcal{E})$ and $\det(\mathcal{E}')$ and can therefore (by slight abuse of terminology) be identified with an isomorphism of G-torsors. It is not difficult to see that the pair $(\mathcal{E}', \gamma)$ is a λ-twist of $(\mathcal{P}, P)$ along D, in the sense of Definition 5.5.3.1.

Example 5.5.3.4. In the situation of Definition 5.5.3.1, suppose that $G = G_0 = \mathrm{GL}_n$, and let $P_0 \subseteq G_0$ be the standard Borel subgroup of upper-triangular matrices. Then a G-bundle with P_0-structure $(\mathcal{P}, P)$ on Y can be identified with a flag of vector bundles

$$0 = \mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \subset \cdots \hookrightarrow \mathcal{E}_n,$$

where each $\mathcal{E}_m / \mathcal{E}_{m-1}$ is a line bundle on Y. Under the canonical isomorphism $\Lambda \simeq \mathbf{Z}^n$, we can identify $\Lambda_{\geq 0}$ with the collection of sequences $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given $(\mathcal{P}, P)$ as above, a sequence $\vec{\lambda} \in \Lambda_{\geq 0}$, and an effective

Cartier divisor $D \subseteq Y$, we let $\mathcal{E}(\vec{\lambda} D)$ denote the vector bundle on Y given by the colimit of the diagram

$$\begin{array}{ccccccccc} & \mathcal{E}_1(\lambda_2 D) & & & \cdots & & \mathcal{E}_{n-1}(\lambda_n D) & & \\ \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\ \mathcal{E}_1(\lambda_1 D) & & \mathcal{E}_2(\lambda_2 D) & \cdots & & \mathcal{E}_{m-1}(\lambda_{n-1} D) & & & \mathcal{E}_n(\lambda_n D). \end{array}$$

We have a canonical vector bundle isomorphism $\gamma : \mathcal{E}_n \,|_{Y-D} \simeq \mathcal{E}(\vec{\lambda} D)|_{Y-D}$, and the pair $(\mathcal{E}(\vec{\lambda} D), \gamma)$ is a $\vec{\lambda}$-twist of $(\mathcal{P}, P)$ along D in the sense of Definition 5.5.3.1.

5.5.4 Existence of Twists

We now show that a G-bundle with P_0 structure can be twisted along any effective divisor:

Proposition 5.5.4.1. *Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $\mathcal{P}$ be a G-bundle on X equipped with a P_0-structure $P \subseteq G_{\mathcal{P}}$, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. Then there exists a G-bundle $\mathcal{P}'$ on Y and an isomorphism*

$$\gamma : \mathcal{P} \times_Y (Y - D) \simeq \mathcal{P}' \times_Y (Y - D)$$

for which the pair $(\mathcal{P}', \gamma)$ is a λ-twist of $(\mathcal{P}, P)$ along D. Moreover, the pair $(\mathcal{P}', \gamma)$ is unique up to unique isomorphism.

Notation 5.5.4.2. Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. Suppose we are given a G-torsor $\mathcal{P}$ on Y and a P_0-structure $P \subseteq G_{\mathcal{P}}$. Proposition 5.5.4.1 implies that there exists an (essentially unique) λ-twist of $\mathcal{P}$ along D; we will denote the underlying G-bundle of this twist by $\mathrm{Tw}_{\lambda, D}(\mathcal{P}, P)$.

Example 5.5.4.3. Let $G_0 = P_0 = \mathbf{G}_m$ and let $\lambda \in \Lambda = \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m)$ be the identity map. Then G_0 has a unique inner form G over any k-scheme Y (given by the multiplicative group over Y), and we can identify G-torsors with line bundles on Y. Any such torsor admits a unique P_0-structure. If $D \subseteq Y$ is an effective Cartier divisor and $\mathcal{L}$ is a line bundle on Y, then we have

$$\mathrm{Tw}_{\lambda, D}(\mathcal{L}) = \mathcal{L}(D) = \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{I}_D^{-1}.$$

We will deduce Proposition 5.5.4.1 from the following:

Lemma 5.5.4.4. *Let (G, σ) be an inner form of G_0 over an affine k-scheme $Y = \mathrm{Spec}(R)$, let $P \subseteq G$ be a parabolic subgroup of type P_0, and let $\psi : P/\mathrm{rad}_u P \to P$ be a Levi decomposition of P. Let λ be an element of Λ, so that λ determines an action of $\mathbf{G}_m$ on P by conjugation (using the homomorphism $\mathbf{G}_m \xrightarrow{\lambda} P/\mathrm{rad}_u(P) \xrightarrow{\psi} P$). If $\lambda \in \Lambda_{\geq 0}$, then this extends to an action of $\mathbf{A}^1$ (regarded as a monoid with respect to multiplication).*

Proof. The assertion is local with respect to the étale topology on $\mathrm{Spec}(R)$. We may therefore assume without loss of generality that $G = G_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ and $P = P_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$, and that the image of ψ is $H_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$, where $H_0 \subseteq P_0$ is the unique Levi factor which contains the chosen maximal torus T_0.

Let $\{\alpha_1, \ldots, \alpha_m\}$ be an enumeration of the roots of P_0 which are not roots of H_0. For $1 \leq i \leq m$, let $f_i : \mathbf{G}_a \to P_0$ be a parametrization of the corresponding root space. It follows from the structure theory of reductive groups (and their parabolic subgroups) that the map

$$H_0 \times \mathbf{G}_a^m \to P_0 \qquad (h, y_1, \ldots, y_m) \mapsto h f_1(a_1) f_2(a_2) \ldots f_n(a_n)$$

is an isomorphism. In particular, for every R-algebra A, the preceding construction gives a bijection $H_0(A) \times A^m \to P(A)$. If $t \in A$ is invertible, then conjugation by $\psi(\lambda)(t)$ determines an automorphism of $P(A)$ which corresponds (under the preceding bijection) to the bijection of $H_0(A) \times A^m$ with itself given by

$$(h, a_1, \ldots, a_m) \mapsto (h, t^{\langle \alpha_1, \psi(\lambda) \rangle} a_1, \ldots, t^{\langle \alpha_m, \psi(\lambda) \rangle} a_m).$$

If $\lambda \in \Lambda_{\geq 0}$, then this expression is well-defined even when t is a noninvertible element of A. □

Proof of Proposition 5.5.4.1. The assertion is local on Y with respect to the Zariski topology. We may therefore assume that $Y = \mathrm{Spec}(R)$ is affine and that the Cartier divisor $D \subseteq Y$ is the vanishing locus of a regular element $t \in R$. Since Y is affine, it admits a Levi decomposition $\psi : P/\mathrm{rad}_u P \to P$. In this case, we can take $\mathcal{P}' = \mathcal{P}$ and γ to be the automorphism of $\mathcal{P} \times_Y (Y - D)$ determined by the element $\psi(\lambda)(t)$. It follows tautologically that the pair $(\mathcal{P}, \gamma)$ is characterized uniquely up to unique isomorphism by the requirement that the composite map

$$\mathcal{P} \times_Y (Y - D) \xrightarrow{\psi(\lambda)(t)^{-1}} \mathcal{P} \times_Y (Y - D) \xrightarrow{\gamma} \mathcal{P}' \times_Y (Y - D)$$

extends to an isomorphism of $\mathcal{P}$ with $\mathcal{P}'$. To complete the proof, it will suffice to show that the pair $(\mathcal{P}', \gamma)$ is a λ-twist of $\mathcal{P}$ along D: that is, after replacing Y by any open subset $U \subseteq Y$ and choosing a different local parameter t' for D and a different Levi decomposition $\psi' : P/\operatorname{rad}_u P \to P$, the composite map

$$\mathcal{P} \times_Y (Y - D) \xrightarrow{\psi'(\lambda)(t')^{-1}} \mathcal{P} \times_Y (Y - D) \xrightarrow{\gamma} \mathcal{P}' \times_Y (Y - D)$$

also extends to a G-bundle isomorphism of $\mathcal{P}$ with $\mathcal{P}'$. In other words, we wish to show that the difference $\psi'(\lambda)(t')^{-1}\psi(\lambda)(t)$ (which we regard as an element of the group $G_{\mathcal{P}}(R[t^{-1}])$) belongs to the subgroup $G_{\mathcal{P}}(R) \subseteq G_{\mathcal{P}}(R[t^{-1}])$. Note that we can write $t' = ut$, where $u \in R$ is a unit, so that

$$\psi'(\lambda)(t')^{-1}\psi(\lambda)(t) = \psi'(\lambda)(u)^{-1}\psi'(\lambda)(t)^{-1}\psi(\lambda)(t)$$

where the first factor belongs to $G_{\mathcal{P}}(R)$. It will therefore suffice to treat the case where $t' = t$.

Since Levi decompositions of P are unique up to the action of $\operatorname{rad}_u(P)$, we can choose an element $g \in \operatorname{rad}_u(P)(R) \subseteq P(R)$ such that $\psi'(\lambda)(t) = g\psi(\lambda)(t)g^{-1}$. We are therefore reduced to proving that $\psi(\lambda)(t)^{-1}g^{-1}\psi(\lambda)(t)$ belongs to $P(R)$, which follows from Lemma 5.5.4.4. □

In the situation of Notation 5.5.4.2, the twist $\operatorname{Tw}_{\lambda,D}(\mathcal{P}, P)$ comes equipped with a tautological isomorphism

$$\gamma : \operatorname{Tw}_{\lambda,D}(\mathcal{P}, P) \times_Y (Y - D) \simeq \mathcal{P} \times_Y (Y - D).$$

In particular, we obtain an isomorphism of group schemes

$$G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D) \simeq G_{\mathcal{P}} \times_Y (Y - D).$$

Under this isomorphism, the parabolic subgroup $P \subseteq \mathcal{G}_{\mathcal{P}}$ determines a parabolic subgroup $P_\gamma^\circ \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D)$.

Proposition 5.5.4.5. *Let (G, σ) be an inner form of G_0 over a k-scheme Y, let $\mathcal{P}$ be a G-bundle on X equipped with a P_0-structure $P \subseteq G_{\mathcal{P}}$, let $D \subseteq Y$ be an effective Cartier divisor, and let $\lambda \in \Lambda_{\geq 0}$. Then the subgroup $P_\gamma^\circ \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D)$ can be extended uniquely to a parabolic subgroup $P_\gamma \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)}$ of type P_0.*

Remark 5.5.4.6. The uniqueness assertion of Proposition 5.5.4.5 is immediate: if P_γ° can be extended to a parabolic subgroup $P_\gamma \subseteq G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)}$, then P_γ can be characterized as the scheme-theoretic closure of P_γ° in $G_{\operatorname{Tw}_{\lambda,D}(\mathcal{P},P)}$.

Proof of Proposition 5.5.4.5. By virtue of the uniqueness supplied by Remark 5.5.4.6, the assertion of Proposition 5.5.4.5 is local with respect to the étale topology on Y. We may therefore assume without loss of generality that the torsor $\mathcal{P}$ is trivial (so that

$G_{\mathcal{P}} \simeq G$), that $P \subseteq G$ admits a Levi decomposition ψ, that $Y = \mathrm{Spec}(R)$ is affine, and that D is the vanishing locus of a regular element $t \in R$. In this case, the proof of Proposition 5.5.4.1 shows that we can take the twist $\mathrm{Tw}_{\lambda,D}(\mathcal{P}, P)$ to be the trivial G-torsor and γ to be the map given by right multiplication by $\psi(\lambda)(t)$. It follows that the isomorphism

$$G_{\mathrm{Tw}_{\lambda,D}(\mathcal{P},P)} \times_Y (Y - D) \simeq G_{\mathcal{P}} \times_Y (Y - D)$$

corresponds to the automorphism of G given by conjugation by $\psi(\lambda)(t)$. Since $\psi(\lambda)(t)$ belongs to $P(R[t^{-1}])$, conjugation by $\psi(\lambda)(t)$ carries $P \times_Y (Y - D)$ into itself; we can therefore identify P°_{γ} with the subgroup $P \times_Y (Y - D) \subseteq G \times_Y (Y - D)$, which extends to the parabolic subgroup $P \subseteq G$. □

5.5.5 Twisting as a Morphism of Moduli Stacks

We now specialize the twisting procedure of Definition 5.5.3.1 to the case of principal bundles on algebraic curves.

Construction 5.5.5.1. Let X be an algebraic curve over k, let (G, σ) be an inner form of G_0 over X, let $D \subseteq X$ be an effective divisor, and let λ be an element of $\Lambda_{\geq 0}$. If R is a finitely generated k-algebra, $\mathcal{P}$ is a G-bundle on X_R, and $P \subseteq G_{\mathcal{P}}$ is a P_0-structure on $\mathcal{P}$, then we can regard $\mathrm{Tw}_{\lambda,D_R}(\mathcal{P}, P)$ as another G-bundle on X_R, equipped with the P_0-structure P' supplied by Proposition 5.5.4.5. The construction $(\mathcal{P}, P) \mapsto (\mathrm{Tw}_{\lambda,D_R} \mathcal{P}, P')$ depends functorially on R and therefore determines a map of algebraic stacks

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{G,P_0}(X) \to \mathrm{Bun}_{G,P_0}(X),$$

which we will refer to as *twisting by λ along D*.

Example 5.5.5.2. In the situation of Construction 5.5.5.1, suppose that $P_0 = G_0$, so that $\mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_G(X)$. In this case, the element $\lambda \in \Lambda_{\geq 0} = \Lambda$ can be regarded as a cocharacter of the center $\mathfrak{Z}(G)$, which determines an action

$$m_\lambda : \mathrm{Bun}_{\mathbf{G}_m}(X) \times_{\mathrm{Spec}(k)} \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X).$$

Unwinding the definitions, we see that if $D \subseteq X$ is an effective divisor, then the map $\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_G(X) \to \mathrm{Bun}_G(X)$ is given by $\mathcal{P} \mapsto m_\lambda(\mathcal{O}_X(D), \mathcal{P})$. In particular, $\mathrm{Tw}_{\lambda,D}$ is an automorphism of $\mathrm{Bun}_G(X)$ which preserves the semistable locus $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ and restricts to equivalences

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}^{\nu}_G(X) \simeq \mathrm{Bun}^{\nu+\deg(D)\lambda}_G(X);$$

here we identify Λ with a sublattice of $\mathrm{Hom}(G_0, \mathbf{G}_m)^{\vee}$ as in Notation 5.5.2.2.

Example 5.5.5.3. In the special case where $G = G_0 \times_{\mathrm{Spec}(k)} X$ is the split form of G_0, we can regard Construction 5.5.5.1 as giving a map $\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{P_0}(X) \to \mathrm{Bun}_{P_0}(X)$; see Example 5.5.1.7.

Remark 5.5.5.4. In the situation of Example 5.5.5.3, let us abuse notation by identifying λ with an element of $\mathrm{Hom}(P_0/\operatorname{rad}_u P_0, \mathbf{G}_m)^\vee$. Then the diagram

$$\begin{array}{ccc} \mathrm{Bun}_{P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0}(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{P_0/\operatorname{rad}_u(P_0)}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0/\operatorname{rad}_u(P_0)}(X) \end{array}$$

commutes up to canonical isomorphism. Combining this observation with Example 5.5.5.2, we deduce that $\mathrm{Tw}_{\lambda,D}$ restricts to give maps

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} \qquad \mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{P_0}^{\nu}(X) \to \mathrm{Bun}_{P_0}^{\nu+\deg(D)\lambda}(X)$$

which fit into pullback squares

$$\begin{array}{ccc} \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0}(X)^{\mathrm{ss}} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0}(X) \end{array}$$

$$\begin{array}{ccc} \mathrm{Bun}_{P_0}^{\nu}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0}^{\nu+\deg(D)\lambda}(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0}(X). \end{array}$$

Remark 5.5.5.5. In the situation of Construction 5.5.5.1, suppose that the algebraic group G_0 is semisimple and adjoint. Then the diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{P_0}(X) \end{array}$$

commutes up to canonical isomorphism, where the vertical maps are the equivalences of Example 5.5.1.8.

Remark 5.5.5.6. Let (G, σ) be an inner form of G_0 over an algebraic curve X, let $D \subseteq X$ be an effective divisor, and let $\lambda \in \Lambda_{\geq 0}$. Let λ_{ad} denote the image of λ in the lattice $\Lambda_{\mathrm{ad}} = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(P_{0\,\mathrm{ad}}/\operatorname{rad}_u P_{0\,\mathrm{ad}}))$. Then the diagram

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G_{\mathrm{ad}},P_{0\,\mathrm{ad}}}(X) & \xrightarrow{\mathrm{Tw}_{\lambda_{\mathrm{ad}},D}} & \mathrm{Bun}_{G_{\mathrm{ad}},P_{0\,\mathrm{ad}}}(X) \end{array}$$

commutes up to canonical isomorphism. Combining this observation with Remarks 5.5.5.4 and 5.5.5.5, we conclude that the $\mathrm{Tw}_{\lambda,D}$ restricts to give maps

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} \to \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} \qquad \mathrm{Tw}_{\lambda,D} : \mathrm{Bun}^{\nu}_{G,P_0}(X) \to \mathrm{Bun}^{\nu+\deg(D)\lambda}_{G,P_0}(X)$$

which fit into pullback squares

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X)^{\mathrm{ss}} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \end{array}$$

$$\begin{array}{ccc} \mathrm{Bun}^{\nu}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}^{\nu+\deg(D)\lambda}_{G,P_0}(X) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X). \end{array}$$

Remark 5.5.5.7 (Functoriality)**.** Let X be an algebraic curve over k and let ψ be an automorphism of X as an abstract scheme, so that ψ determines an automorphism ψ_0 of the field $k = \mathrm{H}^0(X; \mathcal{O}_X)$ fitting into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{\psi_0} & \mathrm{Spec}(k). \end{array}$$

Let G be a form of G_0 over X equipped with an automorphism $\overline{\psi}$ compatible with the automorphism ψ of X. Then $\overline{\psi}$ determines an automorphism of the set of inner structures on G. In particular, if σ is an inner structure on G, then we can form a new inner structure $\overline{\psi}(\sigma)$ which can be written as $g\sigma$ for some unique element $g \in \mathrm{Out}(G_0)$. The pair $(\psi, \overline{\psi})$ determines an automorphism ϕ of $\mathrm{Bun}_G(X)$

which we can lift to an equivalence $\overline{\phi} : \mathrm{Bun}_{G,P_0}(X) \simeq \mathrm{Bun}_{G,g(P_0)}(X)$ (see Warning 5.5.1.6). Each element $\lambda \in \Lambda = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(P_0/\mathrm{rad}_u(P_0))$ determines an element $g(\lambda) \in \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(g(P_0)/\mathrm{rad}_u\, g(P_0)))$, and each effective divisor $D \subseteq X$ determines a divisor $\phi(D) \subseteq X$. It follows immediately from our constructions that the diagram of algebraic stacks

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \\ \downarrow{\scriptstyle \overline{\phi}} & & \downarrow{\scriptstyle \overline{\phi}} \\ \mathrm{Bun}_{G,g(P_0)}(X) & \xrightarrow{\mathrm{Tw}_{g(\lambda),\phi(D)}} & \mathrm{Bun}_{G,g(P_0)}(X) \end{array}$$

commutes up to canonical isomorphism.

Remark 5.5.5.8 (Group Actions)**.** Let X be an algebraic curve over k and let Γ be a finite group which acts on X as an abstract scheme. Let G be a form of G_0 over X equipped with a compatible action of Γ, so that the group Γ acts on the moduli stack $\mathrm{Bun}_G(X)$.

Fix an inner structure σ on X. The collection of all inner structures on G forms a torsor Σ for the group $\mathrm{Out}(G_0)$, and the group Γ acts on Σ by $\mathrm{Out}(G_0)$-torsor automorphisms. The choice of element $\sigma \in \Sigma$ determines an isomorphism $\mathrm{Out}(G_0) \simeq \Sigma$ of $\mathrm{Out}(G_0)$-torsors, so that the action of Γ on Σ determines a group homomorphism $\rho : \Gamma \to \mathrm{Out}(G_0)$. Identifying $\mathrm{Out}(G_0)$ with the group of pinned automorphisms of G_0, the map ρ determines an action of Γ on G_0. Let P_0 be a standard parabolic subgroup of G_0 which is invariant under the action of Γ. Then the action of Γ on $\mathrm{Bun}_G(X)$ lifts canonically to an action of Γ on $\mathrm{Bun}_{G,P_0}(X)$.

The group Γ acts on the lattice $\Lambda = \mathrm{Hom}(\mathbf{G}_m, \mathfrak{Z}(P_0/\mathrm{rad}_u(P_0))$. Suppose that λ is a Γ-invariant element of $\lambda_{\geq 0}$, and let $D \subseteq X$ be an effective divisor which is Γ-invariant. Using the canonical isomorphisms of Remark 5.5.5.7, we can promote the map $\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{G,P_0}(X) \to \mathrm{Bun}_{G,P_0}(X)$ of Construction 5.5.5.1 to a Γ-equivariant morphism of algebraic stacks.

5.5.6 Classification of Untwists

Let X be an algebraic curve over k. The main property of Construction 5.5.5.1 that we will need is the following:

Proposition 5.5.6.1. *Let Γ be a finite group acting on X via k-scheme automorphisms, let G be a Γ-equivariant group scheme over X which is a form of G_0, and let σ be an inner structure on G (so that the choice of σ determines a group homomorphism $\Gamma \to \mathrm{Out}(G_0)$). Let $P_0 \subseteq G_0$ be a standard parabolic which is Γ-invariant, let $\lambda \in \Lambda_{\geq 0}$*

be Γ-invariant, and let $D \subseteq X$ be a Γ-invariant effective divisor, so that $\mathrm{Tw}_{\lambda,D}$ induces a map of (homotopy) fixed point stacks

$$\phi : \mathrm{Bun}_{G,P_0}(X)^{\Gamma} \to \mathrm{Bun}_{G,P_0}(X)^{\Gamma}.$$

If D is étale over $\mathrm{Spec}(k)$ and the action of Γ on D is free, then ϕ is a fiber bundle (locally trivial in the étale topology) whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_P, \lambda\rangle$.

Example 5.5.6.2. Let us first consider the special case of Proposition 5.5.6.1 where the group Γ is trivial, the group G_0 is SL_2, the parabolic P_0 is a Borel subgroup of SL_2, and the group scheme $G = G_0 \times_{\mathrm{Spec}(k)} X$ is constant. In this case, we can identify R-valued points of the stack $\mathrm{Bun}_{P_0}(X) = \mathrm{Bun}_{G,P_0}(X)^{\Gamma}$ with short exact sequences

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \to 0$$

of vector bundles on X_R, where $\mathcal{L}$ is a line bundle on X_R. Let us identify the weight λ of Proposition 5.5.6.1 with a nonnegative integer n. In this case, the twisting morphism $\mathrm{Tw}_{\lambda,D}$ carries a short exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \to 0$ to another short exact sequence $0 \to \mathcal{L}(nD) \to \mathcal{E}' \to \mathcal{L}^{-1}(-nD) \to 0$, where $\mathcal{E}'$ is given by the formula

$$\mathcal{E}' = \mathcal{L}(nD) \amalg_{\mathcal{L}} (\mathcal{E} \times_{\mathcal{L}^{-1}} \mathcal{L}^{-1}(-nD)).$$

Let us identify the R-valued points of $\mathrm{Bun}_{P_0}(X)$ with pairs $(\mathcal{L}, \mathcal{P})$, where $\mathcal{L}$ is a line bundle on X_R and $\mathcal{P}$ is a torsor for the line bundle $\mathcal{L}^{\otimes 2}$ (namely, the torsor parametrizing splittings of the exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}^{-1} \to 0$). Under this identification, the twisting morphism $\mathrm{Tw}_{\lambda,D}$ is given by $(\mathcal{L}, \mathcal{P}) \mapsto (\mathcal{L}(nD), \mathcal{P}')$, where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by base change along the natural map $\mathcal{L}^{\otimes 2} \to \mathcal{L}(nD)^{\otimes 2}$. From this description, it is easy to see that the map $\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{P_0}(X) \to \mathrm{Bun}_{P_0}(X)$ exhibits $\mathrm{Bun}_{P_0}(X)$ as a torsor for a vector bundle over itself: namely, the vector bundle given by the formula $(\mathcal{L}, \mathcal{P}) \mapsto \Gamma(X; \mathcal{L}^{\otimes 2} / \mathcal{L}^{\otimes 2}(-2nD)) = \Gamma(2nD; \mathcal{L}^{\otimes 2}|_{nD})$, which has rank $2n\deg(D) = \deg(D)\langle 2\rho_P, \lambda\rangle$.

To prove Proposition 5.5.6.1, we may assume without loss of generality that the field k is separably closed. It will be convenient to introduce a local variant of Construction 5.5.5.1. For each point $x \in D$, choose a local coordinate t_x for X at the point x, so that the complete local ring $\mathcal{O}_x$ can be identified with the power series ring $k[[t_x]]$.

If R is a finitely generated k-algebra, we let $X_{R,x}^{\wedge}$ denote the formal completion of X_R along the closed subscheme $\{x\} \times_X \mathrm{Spec}(R)$, which we can identify with the formal spectrum $\mathrm{Spf}(R[[t_x]])$. Let $\mathrm{Bun}_{G,P_0}(X_x^{\wedge})$ denote the (non-algebraic) moduli stack of G-bundles on $X_x^{\wedge}$ equipped with a P_0-structure. More precisely, $\mathrm{Bun}_{G,P_0}(X_x^{\wedge})$ denotes the stack whose R-valued points are pairs $(\mathcal{P}, P)$ where $\mathcal{P}$ is a G-bundle $\mathcal{P}$ on $\mathrm{Spec}(R[[t_x]])$

(or equivalently on the formal scheme $\mathrm{Spf}(R[[t_x]])$) equipped with a parabolic subgroup $P \subseteq G_{\mathcal{P}}$ of type P_0. If we let $D_{xR} \subseteq \mathrm{Spec}(R[[t_x]])$ denote the Cartier divisor given by the vanishing locus of t_x, then a variant of Construction 5.5.5.1 determines a morphism of stacks $\mathrm{Tw}_{\lambda,\{x\}} : \mathrm{Bun}_{G,P_0}(X^{\wedge}_{\{x\}}) \to \mathrm{Bun}_{G,P_0}(X^{\wedge}_{\{x\}})$. We have a commutative diagram of stacks

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X) & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X) \\ \downarrow & & \downarrow \\ \prod_{x \in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x) & \xrightarrow{\mathrm{Tw}_{\lambda,\{x\}}} & \prod_{x \in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x). \end{array}$$

which is easily seen to be a pullback square (since the operation of twisting a G-bundle $\mathcal{P}$ by λ along D does not change $\mathcal{P}$ over the open set $X - D$). Passing to Γ-invariants, we obtain another pullback square

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X)^{\Gamma} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X)^{\Gamma} \\ \downarrow & & \downarrow \\ (\prod_{x \in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x))^{\Gamma} & \xrightarrow{\mathrm{Tw}_{\lambda,\{x\}}} & (\prod_{x \in D} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x))^{\Gamma}. \end{array}$$

Let $D_0 \subseteq D$ denote a subset consisting of one element from each Γ orbit. Since the action of Γ on D is free, we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Bun}_{G,P_0}(X)^{\Gamma} & \xrightarrow{\mathrm{Tw}_{\lambda,D}} & \mathrm{Bun}_{G,P_0}(X)^{\Gamma} \\ \downarrow & & \downarrow \\ \prod_{x \in D_0} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x) & \xrightarrow{\mathrm{Tw}_{\lambda,\{x\}}} & \prod_{x \in D_0} \mathrm{Bun}_{G,P_0}(X^{\wedge}_x). \end{array}$$

Consequently, Proposition 5.5.6.1 reduces to the following local assertion (which makes no reference to the group Γ):

Proposition 5.5.6.3. *In the situation above, each of the maps*

$$\mathrm{Tw}_{\lambda,\{x\}} : \mathrm{Bun}_{G,P_0}(X^{\wedge}_x) \to \mathrm{Bun}_{G,P_0}(X^{\wedge}_x)$$

is a fiber bundle (locally trivial for the étale topology) whose fibers are affine spaces of dimension $\langle 2\rho_P, \lambda \rangle$.

Proof. Let t be a generator of the maximal ideal in the complete local ring $\mathcal{O}_x$. Since the power series ring $\mathcal{O}_x \simeq k[[t_x]]$ is strictly Henselian, the group scheme G splits over $\mathcal{O}_x$. Using Example 5.5.1.7, we obtain an identification $\mathrm{Bun}_{G,P_0}(X^{\wedge}_x) \simeq \mathrm{Bun}_{P_0}(X^{\wedge}_x)$, where $\mathrm{Bun}_{P_0}(X_x^{\wedge})$ denotes the stack whose R-valued points are P_0-bundles on $X_{R,x}^{\wedge} \simeq \mathrm{Spf}(R[[t]])$. Let $H_0 \subseteq P_0$ be the unique subgroup which contains the maximal torus T_0 and which maps isomorphically onto the reductive quotient $P_0/\mathrm{rad}_u(P_0)$. We will identify λ with a cocharacter of the center of H_0, so that we can identify $\lambda(t)$ with an element of the group $H_0(K_x)$.

Since P_0 is smooth over k, a P_0-bundle on $X_{R,x}^{\wedge}$ is trivial if and only if its restriction to the subscheme $\{x\} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \subseteq X_R$ is trivial. In particular, any P_0-bundle on $X_{R,x}^{\wedge}$ can be trivialized locally with respect to the étale topology on $\mathrm{Spec}(R)$. Moreover, the automorphism group of the trivial P_0-bundle on $X_{R,x}^{\wedge} \simeq \mathrm{Spf}(R[[t]])$ can be identified with the group $P_0(R[[t]])$. It follows that $\mathrm{Bun}_P(X_x^{\wedge})$ can be identified with the classifying stack (taken with respect to the étale topology) of the group-valued functor $R \mapsto P_0(R[[t]])$.

For every k-algebra R, let us view $P_0(R[[t]])$ as a subgroup of the larger group $P_0(R[[t]][t^{-1}])$. It follows from Lemma 5.5.4.4 that conjugation by $\lambda(t)$ determines a group homomorphism from $P_0(R[[t]])$ to itself; let us denote the image of this homomorphism by $P_0'(R[[t]])$.

Fix a map $\eta : \mathrm{Spec}(R) \to \mathrm{Bun}_{P_0}(X_x^{\wedge})$; we wish to show that the fiber product

$$Y = \mathrm{Bun}_{P_0}(X_x^{\wedge}) \times_{\mathrm{Bun}_{P_0}(X_x^{\wedge})} \mathrm{Spec}(R)$$

is representable by an affine R-scheme which is locally (with respect to the étale topology on $\mathrm{Spec}(R)$) isomorphic to $\mathbf{A}^{\langle 2\rho_{P_0}, \lambda \rangle}$. The map η classifies some P-bundle on $X_{Rx}^{\wedge}$, which we may assume to be trivial (after passing to an étale cover of $\mathrm{Spec}(R)$). Unwinding the definitions, we see that Y can be identified with the sheafification (with respect to the étale topology) of the functor

$$F : \mathrm{Ring}_R \to \mathcal{S}\mathrm{et} \qquad F(A) = P_0(A[[t]])/P_0'(A[[t]]).$$

We will compete the proof by showing that that the functor F is representable by an affine space of dimension $\langle 2\rho_{P_0}, \lambda \rangle$ over $\mathrm{Spec}(R)$ (and is therefore already a sheaf with respect to the étale topology).

Let U denote the unipotent radical of P_0, so that $P_0(A[[t]])$ factors as a semidirect product $U(A[[t]]) \rtimes H_0(A[[t]])$. This decomposition is invariant under conjugation by $\lambda(t)$ and therefore determines an analogous decomposition $P_0'(A[[t]]) \simeq U'(A[[t]]) \rtimes H_0'(A[[t]])$. Since $\lambda(t)$ is central in $H_0(A[[t]][t_x^{-1}])$, we have $H_0'(A[[t]]) = H_0(A[[t]])$. It follows that the functor $F : \mathrm{Ring}_R \to \mathcal{S}\mathrm{et}$ above can be described by the formula $F(A) = U(A[[t]])/U'(A[[t]])$.

Let $\{\alpha_1, \ldots, \alpha_m\} \subseteq \mathrm{Hom}(T_0, \mathbf{G}_m)$ be the collection of roots of P_0 which are not roots of H_0. For $1 \leq i \leq m$, let $f_i : \mathbf{G}_a \to U$ be a parametrization of the root subgroup corresponding to α_i. For $A \in \mathrm{Ring}_R$, every element of the group $U(A[[t]])$ has a unique representation as a product

$$f_1(a_1(t))f_2(a_2(t))\cdots f_m(a_m(t))$$

where $a_i(t) \in A[[t]]$. As in the proof of Lemma 5.5.4.4, we can identify $U'(A[[t]])$ with the subgroup of $U(A[[t]])$ spanned by those products where each $a_i(t)$ is divisible by $t^{\langle \alpha_i, \lambda \rangle}$.

Reordering the roots $\{\alpha_1, \ldots, \alpha_m\}$ if necessary, we may assume that for $0 \leq i \leq m$, the image of the map

$$\prod_{1 \leq i' \leq i} f_i : \mathbf{A}^i \to U_0$$

is a normal subgroup $U_i \subseteq U$. Then every $A[[t]]$-valued point of the quotient U/U_i has a unique representation as a product

$$f_{i+1}(a_{i+1}(t))f_{i+2}(a_{i+2}(t))\cdots f_m(a_m(t))$$

where $a_j(t) \in A[[t]]$. Let $V_i(A)$ denote the subgroup of $(U/U_i)(A[[t]])$ consisting of those products where each $a_j(t)$ is divisible by $t^{\langle \alpha_j, \lambda \rangle}$, and let $F_i : \mathrm{Ring}_R \to \mathcal{S}\mathrm{et}$ be the functor given by $F_i(A) = (U/U_i)(A[[t]])/V_i(A)$. We will prove the following:

- $(*)$ For $0 \leq i \leq n$, the functor F_i is representable by an affine space of dimension $\sum_{i<j\leq m} \langle \alpha_j, \lambda \rangle$ over $\mathrm{Spec}(R)$.

Note that $F_0 = F$ and that $\sum_{1 \leq j \leq m} \langle \alpha_j, \lambda \rangle = \langle 2\rho_P, \lambda \rangle$, so that when $i = 0$ assertion $(*)$ asserts that F is representable by an affine space of dimension $\langle 2\rho_P, \lambda \rangle$ over $\mathrm{Spec}(R)$. We will prove $(*)$ by descending induction on i, the case $i = m$ being trivial. To carry out the inductive step, we note that for $1 \leq i \leq m$ we have natural exact sequences

$$\begin{array}{ccccccccc} 0 & \to & t^{\langle \alpha_i, \lambda \rangle} A[[t]] & \to & V_{i-1}(A) & \to & V_i(A) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A[[u]] & \to & (U/U_{i-1})(A[[u]]) & \to & (U/U_i)(A[[u]]) & \to & 0, \end{array}$$

where the vertical maps are injective and each of the exact sequences is a central extension. It follows from a diagram chase that we can identify $F_i(A)$ with the quotient of $F_{i-1}(A)$ by a free action of the quotient $A[[t]]/t^{\langle \alpha_i, \lambda \rangle} A[[t]]$, and that this identification depends functorially on A. In other words, the functor F_{i-1} can be identified with a $\mathbf{G}_a^{\langle \alpha_i, \lambda \rangle}$-torsor over F_i. By the inductive hypothesis, the functor F_i is representable by an affine scheme, so that any $\mathbf{G}_a$-torsor over F_i is trivial. We therefore obtain

$$\begin{aligned} F_{i-1} &\simeq \mathbf{G}_a^{\langle \alpha_i, \lambda \rangle} \times F_i \\ &\simeq \mathbf{A}^{\langle \alpha_i, \lambda} \times (\mathbf{A}^{\sum_{i<j\leq m} \langle \alpha_j, \lambda \rangle} \times \mathrm{Spec}(R)) \\ &\simeq \mathbf{A}^{\sum_{i \leq j \leq m} \langle \alpha_j, \lambda \rangle} \times \mathrm{Spec}(R), \end{aligned}$$

as desired. □

5.6 Proof of the Trace Formula

Let X be an algebraic curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X, whose fibers are connected and whose generic fiber is semisimple. Our goal in this section is to prove Theorem 5.0.0.3, which asserts that the moduli stack $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula. We begin in §5.6.1 by treating the special case where the group scheme G is split: in this case, we show that the Harder-Narasimhan stratification of §5.3 is convergent (in the sense of Definition 5.2.2.1). The general case is somewhat more technical, since the group scheme G need not be split even at the generic point of X. Our strategy will be to compare $\mathrm{Bun}_G(X)$ with $\mathrm{Bun}_G(\widetilde{X})$, where $\widetilde{X}$ is the algebraic curve associated to a finite Galois extension L/K_X which splits the generic fiber of G. We will deduce Theorem 5.0.0.3 by carefully analyzing the action of the Galois group $\Gamma = \mathrm{Gal}(L/K_X)$ on the Harder-Narasimhan stratification of $\mathrm{Bun}_{G_0}(\widetilde{X})$, where G_0 is the split form of the generic fiber of G.

5.6.1 The Case of a Split Group Scheme

In the case of a split group, we can deduce Theorem 5.0.0.3 from the following:

Theorem 5.6.1.1. *Let X be an algebraic curve over $\mathbf{F}_q$ and let G be a split semisimple group scheme over X. Then the Harder-Narasimhan stratification of $\mathrm{Bun}_G(X)$ is convergent (in the sense of Definition 5.2.2.1).*

Proof of Theorem 5.6.1.1 for $G = \mathrm{SL}_2$. Assume that G is the split group SL_2, let $B \subseteq G$ denote the standard Borel subgroup of upper triangular matrices, and let us identify $\mathrm{Hom}(B, \mathbf{G}_m)^\vee$ with the group $\mathbf{Z}$ of integers. In this case, Theorem 5.3.2.2 supplies a stratification of $\mathrm{Bun}_G(X)$ by an open substack $\mathrm{Bun}_G(X)^{\mathrm{ss}}$ and locally closed substacks $\{\mathrm{Bun}_G(X)_{B,n}\}_{n>0}$. Fix a closed point $x \in X$ for which the residue field $\kappa(x)$ has degree s over $\mathbf{F}_q$. For each $m > 0$, let $\mathrm{Bun}_B^n(X)$ denote the moduli stack of B-bundles on X of degree n. We will show that the finite collection of algebraic stacks $\{\mathrm{Bun}_G(X)^{\mathrm{ss}}, \mathrm{Bun}_B^1(X), \ldots \mathrm{Bun}_B^s(X)\}$ satisfies conditions (1), (2), and (3) of Definition 5.2.2.1:

(1) Let n be a positive integer, and write $n = as + b$ for $1 \leq b \leq s$. In this case, we consider the composition

$$\mathrm{Bun}_B^b(X) \xrightarrow{\mathrm{Tw}_{a,x}} \mathrm{Bun}_B^n(X) \xrightarrow{u_n} \mathrm{Bun}_G(X)_{B,n},$$

where $\mathrm{Tw}_{a,x}$ is the twisting morphism of Construction 5.5.5.1 (which is an étale fiber bundle whose fibers are affine spaces of dimension $2as$, by virtue of Proposition 5.5.6.1) and u_n is the finite radicial surjection of Notation 5.5.1.11.

(2) For each $0 < b \leq s$, the geometric series $\sum_{a \geq 0} q^{-2as}$ converges.

(3) Each of the algebraic stacks $\{\mathrm{Bun}_G(X)^{\mathrm{ss}}, \mathrm{Bun}_B^1(X), \ldots \mathrm{Bun}_B^s(X)\}$ is quasi-compact (Proposition 5.4.3.1), and can therefore be realized as a quotient of a quasi-compact algebraic space by the action of a linear algebraic group. In fact, for any algebraic group H, a quasi-compact open substack $U \subseteq \mathrm{Bun}_H(X)$ can always be realized as the quotient $(U \times_{\mathrm{Bun}_H(X)} \mathrm{Bun}_H(X, D))/H^D$, where $\mathrm{Bun}_H(X, D)$ denotes the moduli stack of H-bundles on X equipped with a trivialization along an effective division $D \subseteq X$, and H^D denotes the algebraic group of maps from D into H; the quasi-compactness of U then guarantees that $(U \times_{\mathrm{Bun}_H(X)} \mathrm{Bun}_H(X, D))$ is an algebraic space if the divisor D is sufficiently large.

□

Proof of Theorem 5.6.1.1 in general. Once again, we fix a closed point $x \in X$ for which the residue field $\kappa(x)$ has degree s over $\mathbf{F}_q$. For each standard parabolic subgroup $P \subseteq G$, let $\Lambda_{\geq 0} \subseteq \mathrm{Hom}(P, \mathbf{G}_m)^{\vee}_{\geq 0}$ be as in Notation 5.5.2.2. Given a pair of elements $\mu, \nu \in \mathrm{Hom}(P, \mathbf{G}_m)^{\vee}_{\geq 0}$, we write $\mu \leq \nu$ if $\nu = \mu + s\lambda$ for some $\lambda \in \Lambda_{\geq 0}$. In this case, we claim that the requirements of Definition 5.2.2.1 are satisfied by the collection of algebraic stacks $\{\mathrm{Bun}_P^{\nu_0}(X)^{\mathrm{ss}}\}$, where P ranges over the (finite) collection of all standard parabolic subgroups of G and ν_0 ranges over the (finite) collection of all minimal elements of $\mathrm{Hom}(P, \mathbf{G}_m)^{\vee}_{>0}$ (with respect to the ordering described above). To verify (1), we note that each Harder-Narasimhan stratum $\mathrm{Bun}_G(X)_{P,\nu}$ admits a map

$$\mathrm{Bun}_P^{\nu_0}(X)^{\mathrm{ss}} \xrightarrow{\mathrm{Tw}_{\lambda,x}} \mathrm{Bun}_P^{\nu}(X)^{\mathrm{ss}} \xrightarrow{u} \mathrm{Bun}_G(X)_{P,\nu},$$

where ν_0 is a minimal element of $\mathrm{Hom}(P, \mathbf{G}_m)^{\vee}_{>0}$ satisfying $\nu = \nu_0 + s\lambda$, $\mathrm{Tw}_{\lambda,x}$ is the twisting morphism Construction 5.5.5.1 (which is an étale fiber bundle whose fibers are affine spaces of dimension $s\langle 2\rho_P, \lambda\rangle$, by Proposition 5.5.6.1), and u is the finite radicial surjection of Notation 5.5.1.11. Requirement (3) follows by repeating the argument given in the case $G = \mathrm{SL}_2$. To prove (2), it will suffice to show that for each standard parabolic $P \subseteq G$, the sum

$$\sum_{\lambda \in \Lambda_{\geq 0}} q^{-s\langle 2\rho_P, \lambda\rangle}$$

converges. In fact, this sum is dominated by

$$\sum_{d_1,\ldots,d_n \geq 0} q^{-s\langle 2\rho_P, d_1\lambda_1 + \cdots + d_n\lambda_n\rangle} = \prod_{1 \leq i \leq n} \sum_{d \geq 0} q^{-s\langle 2\rho_P, d\lambda_i\rangle} < \infty,$$

where $\lambda_1, \ldots, \lambda_n$ is any set of nonzero generators for the monoid $\Lambda_{\geq 0}$. □

5.6.2 Reductive Models

Let k be a field, let X be an algebraic curve over k, and let G be a reductive algebraic group over the function field K_X (which we do not assume to be split). If the field k is algebraically closed, then the function field K_X has dimension ≤ 1, so the algebraic group G is quasi-split. It follows that there is a finite Galois extension L of K_X, a split reductive group G_0 over k on which $\mathrm{Gal}(L/K_X)$ acts by pinned automorphisms, and a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism

$$G \times_{\mathrm{Spec}(K_X)} \mathrm{Spec}(K) \simeq G_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L).$$

In particular, there is a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism of $G \times_{\mathrm{Spec}(K_X)} \mathrm{Spec}(L)$ with the generic fiber of the split reductive group scheme $G_0 \times_{\mathrm{Spec}(k)} \widetilde{X}$, where $\widetilde{X}$ denotes the algebraic curve with function field L. Our goal in this section is to establish an analogous (but weaker) result which does not require the assumption that k is algebraically closed:

Proposition 5.6.2.1. *Let G be a reductive algebraic group over the fraction field K_X. Then there exists a finite Galois extension L of K_X, a reductive group scheme H over the curve $\widetilde{X}$ with function field L, an action of $\mathrm{Gal}(L/K_X)$ on H (compatible with the tautological action of $\mathrm{Gal}(L/K_X)$ on $\widetilde{X}$), and a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism*

$$G \times_{\mathrm{Spec}(K_X)} \mathrm{Spec}(L) \simeq H \times_{\widetilde{X}} \mathrm{Spec}(L).$$

Remark 5.6.2.2. In the situation of Proposition 5.6.2.1, if L is a Galois extension of K_X for which there exists a $\mathrm{Gal}(L/K_X)$-equivariant group scheme on the associated algebraic curve $\widetilde{X}$ whose generic fiber is $\mathrm{Gal}(L/K_X)$-equivariantly isomorphic to $G \times_{\mathrm{Spec}(K_X)} \mathrm{Spec}(L)$, then any larger Galois extension L' has the same property (the inclusion $L \hookrightarrow L'$ induces a map of algebraic curves $\widetilde{X}' \to \widetilde{X}$, and the pullback $H \times_{\widetilde{X}} \widetilde{X}'$ is a reductive group scheme over $\widetilde{X}'$ having the desired properties). We are therefore free to assume that the Galois extension L appearing in Proposition 5.6.2.1 is as large as we like: in particular, we may assume that the algebraic group G splits over L.

Warning 5.6.2.3. In the situation of Proposition 5.6.2.1, the algebraic curve $\widetilde{X}$ is connected, but need not be geometrically connected (when regarded as a k-scheme). If we want to guarantee that the generic fiber of H is split reductive (as in Remark 5.6.2.2), this is unavoidable: if k' is a Galois extension of k and the group scheme G is obtained by Weil restriction along the field extension $K_X \hookrightarrow K_X \otimes_k k'$, then any Galois extension L of K_X which splits G must contain k'.

Remark 5.6.2.4. There are two main differences between Proposition 5.6.2.1 (which applies over any ground field k) and the discussion which precedes it (which applies when the ground field k is algebraically closed):

- Proposition 5.6.2.1 guarantees the existence of a reductive group scheme H over $\widetilde{X}$, but does not guarantee that this reductive group scheme is constant (though we can arrange that it is split at the generic point of $\widetilde{X}$, by virtue of Remark 5.6.2.2).

- Proposition 5.6.2.1 gives no information about the action of $\mathrm{Gal}(L/K_X)$ on the group scheme H: in particular, this action need not preserve a pinning of H, even at the generic point of $\widetilde{X}$.

We will deduce Proposition 5.6.2.1 from the following local assertion:

Lemma 5.6.2.5. *Let K be the fraction field of a complete discrete valuation ring $\mathcal{O}_K$ and let G be a reductive algebraic group over K. Then there exists a finite Galois extension L of K, a reductive group scheme H over the ring of integers $\mathcal{O}_L \subseteq L$ equipped with an action of* $\mathrm{Gal}(L/K)$ *(compatible with the tautological action of* $\mathrm{Gal}(L/K)$ *over $\mathcal{O}_L$), and a* $\mathrm{Gal}(L/K)$*-equivariant isomorphism*

$$G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \simeq H \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(L).$$

Proof. Choose a maximal torus $T \subseteq G$ which is defined over K. Let L_0 be a finite Galois extension of K for which the torus T splits over L_0. We will show that if L is a Galois extension of K which contains L_0 and whose ramification degree over L_0 is divisible by $\deg(L_0/K)$, then L has the desired property.

For every Galois extension L of K which contains L_0, let $M(L)$ denote the set of isomorphism classes of pairs (H, γ), where H is a reductive group scheme over $\mathcal{O}_L$ and γ is an isomorphism

$$G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \simeq H \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(L)$$

of reductive algebraic groups over L which has the following additional property: the scheme-theoretic image of composite map

$$T \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \hookrightarrow G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \xrightarrow{\gamma} H$$

is a torus $\overline{T}$ (automatically split, since its generic fiber is split) over $\mathrm{Spec}(\mathcal{O}_L)$. The group $\mathrm{Gal}(L/K)$ acts on the set $M(L)$; to prove Lemma 5.6.2.5, it will suffice to show that there is an element of $M(L)$ which is fixed by $\mathrm{Gal}(L/K)$ (provided that L is sufficiently large).

Let G_{ad} denote the quotient of G by its center, and let T_{ad} denote the image of T in G_{ad}. For every element $g \in T_{\mathrm{ad}}(L)$, conjugation by g determines an automorphism c_g of $G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ which acts trivially on T. The construction $(H, \gamma) \mapsto (H, \gamma \circ c_g)$ determines an action of $T_{\mathrm{ad}}(L)$ on the set $M(L)$. The main fact we will need is the following:

$(*)$ The action of $T_{\mathrm{ad}}(L)$ on $M(L)$ is transitive.

To prove $(*)$, we must show that for $(H,\gamma),(H',\gamma') \in M(L)$, there exists an isomorphism $\beta : H \simeq H'$ such that the composite map

$$G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \xrightarrow{\gamma} H \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(L) \xrightarrow{\beta} H' \xrightarrow{\gamma'^{-1}} G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$$

is given by conjugation by some element $g \in T_{\mathrm{ad}}(L)$. To prove this, let $\overline{T} \subseteq H$ and $\overline{T}' \subseteq H'$ denote the scheme-theoretic images of $T \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ under γ and γ', respectively. Then $\overline{T}$ and $\overline{T}'$ are split tori over $\mathcal{O}_L$, so that the identification between their generic fibers (supplied by γ and γ') extends uniquely to an isomorphism $\beta_0 : \overline{T} \simeq \overline{T}'$. Let B be a Borel subgroup of $G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ containing $T \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. Since $\overline{T}$ is a maximal split torus in H, there is a unique Borel subgroup $\overline{B} \subseteq H$ containing $\overline{T}$ with $\gamma^{-1}\overline{B} = B$. Similarly, there is a unique Borel subgroup $\overline{B}' \subseteq H'$ which contains $\overline{T}'$ satisfying $\gamma'^{-1}\overline{B}' = B$. Since the ring of integers $\mathcal{O}_L$ is a discrete valuation ring, the pairs $(\overline{T}, \overline{B})$ and $(\overline{T}', \overline{B}')$ can be extended to pinnings of the group schemes H and H', respectively. It follows that there is a unique pinned isomorphism $\beta : H \to H'$ which restricts to the identity on the Dynkin diagram of their common generic fiber $G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. By construction, the composition

$$G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) \xrightarrow{\gamma} H \times_{\mathrm{Spec}(\mathcal{O}_L)} \mathrm{Spec}(L) \xrightarrow{\beta} H' \xrightarrow{\gamma'^{-1}} G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$$

is an automorphism of $G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ which restricts to the identity on $T \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, and is therefore given by conjugation by some element $g \in H_{\mathrm{ad}}(L)$. Since g centralizes $T \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, it belongs to the subgroup $T_{\mathrm{ad}}(L) \subseteq H_{\mathrm{ad}}(L)$. This completes the proof of $(*)$.

Let $\Lambda = \mathrm{Hom}(\mathbf{G}_m, T_{\mathrm{ad}} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))$ denote the cocharacter lattice of the split torus $T_{\mathrm{ad}} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, so that we can identify $T_{\mathrm{ad}}(L)$ with the tensor product $\Lambda \otimes L^\times$. Note that if (H,γ) is any element of $M(L)$, then (H,γ) is isomorphic to $(H, \gamma \circ c_g)$ if and only if conjugation by the element $g \in T_{\mathrm{ad}}(L)$ extends to an automorphism of the group scheme H. This condition is equivalent to the assertion that for each root α of the split group $G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$, the induced map $\Lambda \otimes L^\times \xrightarrow{\alpha} L^\times$ carries g to an element of $\mathcal{O}_L^\times$: that is, g belongs to the subgroup $\Lambda \otimes \mathcal{O}_L^\times \subseteq \Lambda \otimes L^\times \simeq T_{\mathrm{ad}}(L)$. It follows that we can regard $M(L)$ as a torsor for the quotient group

$$(\Lambda \otimes L^\times)/(\Lambda \otimes \mathcal{O}_L^\times) \simeq \Lambda \otimes \mathbf{Z}_L,$$

where $\mathbf{Z}_L = L^\times / \mathcal{O}_L^\times$ denotes the value group of L. Note that the group $\mathbf{Z}_L$ is canonically isomorphic to $\mathbf{Z}$ (so $\Lambda \otimes \mathbf{Z}_L$ is canonically isomorphic to Λ); however, in the arguments which follow, it will be convenient not to make use of this.

Let us fix an element $x_0 \in M(L_0)$, which determines an element $x_L \in M(L)$ for every finite extension L of L_0. The action of $\Lambda \otimes \mathbf{Z}_L$ on $M(L)$ determines a bijective map

$$\gamma_L : \Lambda \otimes \mathbf{Z}_L \to M(L) \qquad \gamma_L(0) = x_L.$$

The set $M(L)$ admits a unique abelian group structure for which γ_L is an isomorphism of abelian groups. Note that if L is an extension of L_0 having ramification degree d, then we can identify $\mathbf{Z}_L$ with $\frac{1}{d}\mathbf{Z}_{L_0}$. It follows that for any element $y \in M(L_0)$, the image of y in $M(L)$ is divisible by d.

The action of the Galois group $\mathrm{Gal}(L/K)$ on $M(L)$ does not preserve the group structure on $M(L)$ (because the element x_L is not necessarily $\mathrm{Gal}(L/K)$-invariant). However, the action of $\mathrm{Gal}(L/K)$ is affine-linear: that is, for each $g \in \mathrm{Gal}(L/K)$ we have the identity $g(y+z) = g(y) + g(z) - g(0)$ in $M(L)$. Suppose that L is a Galois extension of K having ramification degree divisible by $\deg(L_0/K)$. Then for each $g \in \mathrm{Gal}(L_0/K)$, the image of $g(x_0)$ in $M(L)$ is divisible by $\deg(L_0/K)$. It follows that the average $\sum_{g \in \mathrm{Gal}(L_0/K)} \frac{g(x_0)}{\deg(L_0/K)}$ is a well-defined element of $M(L)$, and this element is clearly fixed under the action of $\mathrm{Gal}(L/K)$. □

Proof of Proposition 5.6.2.1. Let G be a reductive algebraic group over K_X. Then we can choose a dense open subset $U \subseteq X$ such that G extends to a reductive group scheme G_U over U. Let S denote the finite set of closed points of X which do not belong to U. For each $x \in S$, let $\mathcal{O}_x$ denote the complete local ring of X at the point x, let K_x denote its fraction field, and let $G_x = G \times_{\mathrm{Spec}(K_X)} \mathrm{Spec}(K_x)$ be the associated reductive algebraic group over K_x. It follows from Lemma 5.6.2.5 that for each point $x \in X$, there exists a finite Galois extension L_x of K_x, a reductive algebraic group H_x over the ring of integers $\mathcal{O}_{L_x}$, an action of $\mathrm{Gal}(L_x/K_x)$ on H_x (compatible with its action on $\mathcal{O}_{L_x}$), and a $\mathrm{Gal}(L_x/K_x)$-equivariant isomorphism $G_x \times_{\mathrm{Spec}(K_x)} \mathrm{Spec}(L_x) \simeq H_x \times_{\mathrm{Spec}(\mathcal{O}_{L_x})} \mathrm{Spec}(L_x)$.

Let L be a Galois extension of K_X which is large enough that for each $x \in S$, the tensor product $L \otimes_{K_X} K_x$ contains an isomorphic copy of L_x. Enlarging the fields L_x if necessary (see Remark 5.6.2.2), we may assume that each L_x appears as a direct factor in the tensor product $L \otimes_{K_X} K_x$. Then L is the fraction field of an algebraic curve $\widetilde{X}$ (which is not necessarily geometrically connected over k). Let $\widetilde{U}$ denote the inverse image of U in $\widetilde{K}$, so that the pullback of G_U determines a $\mathrm{Gal}(L/K_X)$-equivariant reductive group scheme $G_{\widetilde{U}}$ over $\widetilde{U}$. To complete the proof, it will suffice to show that $G_{\widetilde{U}}$ admits a $\mathrm{Gal}(L/K_X)$-equivariant extension to a reductive group scheme over $\widetilde{X}$. To construct such an extension, it suffices to show that we can solve the analogous problem after replacing X by $\mathrm{Spec}(\mathcal{O}_x)$ for $x \in S$, which is precisely the content of Lemma 5.6.2.5. □

5.6.3 The Proof of Theorem 5.0.0.3

We now return to the situation of Theorem 5.0.0.3. Let X be an algebraic curve over $\mathbf{F}_q$ and let G be a smooth affine group scheme over X, whose fibers are connected and whose generic fiber is semisimple. We wish to prove that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula.

Let K_X denote the fraction field of X. According to Proposition 5.6.2.1, there is a finite Galois extension L of K_X, where L is the function field of an algebraic curve $\widetilde{X}$, a semisimple group scheme $\widetilde{G}$ over $\widetilde{X}$ equipped with a compatible action of $\mathrm{Gal}(L/K_X)$, and a $\mathrm{Gal}(L/K_X)$-equivariant isomorphism

$$G \times_X \mathrm{Spec}(L) \simeq \widetilde{G} \times_{\widetilde{X}} \mathrm{Spec}(L).$$

Moreover, we may further assume that the generic fiber of $\widetilde{G}$ is split (Remark 5.6.2.2). Note that the algebraic curve $\widetilde{X}$ is not necessarily geometrically connected when regarded as an $\mathbf{F}_q$-scheme. The algebraic closure of $\mathbf{F}_q$ in L is a finite field $\mathbf{F}_{q^d}$ with q^d elements for some $d \geq 0$; let us fix an embedding of this field into $\overline{\mathbf{F}}_q$.

Let $X' = X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$ and let $G' = G \times_X X'$; similarly we define $\widetilde{X}' = \widetilde{X} \times_{\mathrm{Spec}(\mathbf{F}_{q^d})} \mathrm{Spec}(\overline{\mathbf{F}}_q)$ and $\widetilde{G}' = \widetilde{G} \times_{\widetilde{X}} \widetilde{X}'$. For each effective divisor $Q \subseteq X'$, let $\mathrm{Dil}^Q(G')$ denote the group scheme over X' obtained by dilitation of G' along Q (see §5.1.3). Using Lemma 5.4.2.3, we see that if Q is large enough, then the equivalence $G \times_X \mathrm{Spec}(L) \simeq \widetilde{G} \times_{\widetilde{X}} \mathrm{Spec}(L)$ extends to a homomorphism $\overline{\beta} : \mathrm{Dil}^Q(G') \times_{X'} \widetilde{X}' \to \widetilde{G}'$. Enlarging Q if necessary, we may assume that Q is invariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$, so that the group scheme $\mathrm{Dil}^Q(G')$ and the map $\overline{\beta}$ are defined over $\mathbf{F}_q$: that is, we have a dilitation $\mathrm{Dil}^Q(G) \to G$ and a map of group schemes $\mathrm{Dil}^Q(G) \times_X \widetilde{X} \to \widetilde{G}$ which is an isomorphism at the generic point of $\widetilde{X}$. According to Proposition 5.1.3.10, to show that $\mathrm{Bun}_G(X)$ satisfies the Grothendieck-Lefschetz trace formula, it will suffice to show that $\mathrm{Bun}_{\mathrm{Dil}^Q(G)}(X)$ satisfies the Grothendieck-Lefschetz trace formula. We may therefore replace G by $\mathrm{Dil}^Q(G)$ and thereby reduce to the case where there exists a homomorphism of group schemes $\beta : G \times_X \widetilde{X} \to \widetilde{G}$ which is an isomorphism at the generic point of $\widetilde{X}$. Note that β is automatically $\mathrm{Gal}(L/K_X)$-equivariant (since this can be tested at the generic point of X).

Let G_0 denote the split form of $\widetilde{G}$, which we regard as a semisimple algebraic group over $\mathbf{F}_q$. Fix a Borel subgroup $B_0 \subseteq G_0$ and a split maximal torus $T_0 \subseteq B_0$. Let Σ denote the set of inner structures on the group scheme $\widetilde{G}$ (Definition 5.3.4.2). Since the generic fiber of $\widetilde{G}$ is split, the set Σ is nonempty (Example 5.3.4.4), and is therefore a torsor for the outer automorphism group $\mathrm{Out}(G_0)$ (Remark 5.3.4.5). Fix an element $\sigma \in \Sigma$, which supplies an isomorphism $\mathrm{Out}(G_0) \simeq \Sigma$ of $\mathrm{Out}(G_0)$-torsors. The group $\mathrm{Gal}(L/K_X)$ acts on the pair $(\widetilde{X}, \widetilde{G})$ and therefore acts on the set Σ by

$\mathrm{Out}(G_0)$-torsor automorphisms; let us identify this action with a group homomorphism $\rho : \mathrm{Gal}(L/K_X) \to \mathrm{Out}(G_0)$.

Let $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ denote the moduli stack of $\widetilde{G}$-bundles on $\widetilde{X}$, where we regard $\widetilde{X}$ as a geometrically connected algebraic curve over $\mathrm{Spec}(\mathbf{F}_{q^d})$ (so that $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ is a smooth algebraic stack over $\mathbf{F}_{q^d}$). Let A denote the set of all pairs (P_0, ν), where P_0 is a parabolic subgroup of G_0 which contains B_0 and $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}_{>0}$. We let $\{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}\}_{(P_0,\nu)\in A}$ denote the Harder-Narasimhan stratification of $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ determined by the choice of inner structure $\sigma \in \Sigma$ (Construction 5.3.5.1). The Galois group $\mathrm{Gal}(L/K_X)$ acts on the moduli stack $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$. According to Remark 5.3.5.5, this action permutes the Harder-Narasimhan strata (via the action of $\mathrm{Gal}(L/K_X)$ on A determined by the homomorphism ρ).

The Galois group $\mathrm{Gal}(\mathbf{F}_{q^d}/\mathbf{F}_q)$ is canonically isomorphic to the cyclic group $\mathbf{Z}/d\mathbf{Z}$ generated by the Frobenius map $t \mapsto t^q$, and the Galois group $\mathrm{Gal}(L/K_X)$ fits into a short exact sequence

$$0 \to \Gamma \to \mathrm{Gal}(L/K_X) \to \mathbf{Z}/d\mathbf{Z} \to 0$$

where Γ denotes the Galois group of L over $K_X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^d}$. Let A^{Γ} denote the set of fixed points for the action of Γ on A. According to Remark 5.2.1.6, the homotopy fixed point stack $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}$ inherits a stratification by locally closed substacks $\{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}_{P_0,\nu}\}_{(P_0,\nu)\in A^{\Gamma}}$, whose strata are defined by the formula

$$\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}_{P_0,\nu} = ((\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^{\Gamma})_{\mathrm{red}}.$$

We will regard $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}$ as an algebraic stack (not necessarily smooth) over $\mathrm{Spec}(\mathbf{F}_{q^d})$. This algebraic stack inherits a residual action of the group $\mathbf{Z}/d\mathbf{Z}$ (compatible with the action of $\mathbf{Z}/d\mathbf{Z} \simeq \mathrm{Gal}(\mathbf{F}_{q^d}/\mathbf{F}_q)$ on $\mathrm{Spec}(\mathbf{F}_{q^d})$), so we can consider the stack-theoretic quotient $\mathcal{X} = \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})$ as an algebraic stack over $\mathbf{F}_q$. Moreover, we have $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma} \simeq \mathcal{X} \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\mathbf{F}_{q^d})$.

Remark 5.6.3.1. Let $\overline{X}$ denote the stack-theoretic quotient of $\widetilde{X}$ by the action of $\mathrm{Gal}(L/K_X)$. Then $\overline{X}$ is a "stacky curve" over $\mathrm{Spec}(\mathbf{F}_q)$, and there is a natural map $\pi : \overline{X} \to X$ which exhibits X as the coarse moduli space of X'. The action of $\mathrm{Gal}(L/K_X)$ on $\widetilde{G}$ allows us to descend $\widetilde{G}$ to an affine group scheme $\overline{G} = \widetilde{G}/\mathrm{Gal}(L/K_X)$ over $\overline{X}$, and we can think of $\mathcal{X}$ as the moduli stack (defined over $\mathbf{F}_q$) of $\overline{G}$-bundles on $\overline{X}$.

Let $A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})$ be the quotient of A^{Γ} by the action of $\mathbf{Z}/d\mathbf{Z}$; for each object $(P_0, \nu) \in A^{\Gamma}$, we let $[P_0, \nu]$ denote the image of (P_0, ν) in the quotient $A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})$. It follows from Remark 5.2.1.8 that $\mathcal{X}$ inherits a stratification by locally closed substacks $\{\mathcal{X}_{[P_0,\nu]}\}_{[P_0,\nu]\in A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})}$, where each $\mathcal{X}_{[P_0,\nu]}$ can be identified with the quotient $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}_{P_0,\nu}/H$, where H denotes the subgroup of $\mathbf{Z}/d\mathbf{Z}$ which stabilizes the element $(P_0, \nu) \in A^{\Gamma}$.

Let $\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}}$ be the fiber product $\mathrm{Bun}_G(X) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\mathbf{F}_{q^d})$. The $\mathrm{Gal}(L/K_X)$-equivariant map $\beta : G \times_X \widetilde{X} \to \widetilde{G}$ induces a morphism of algebraic stacks

$$\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}$$

over $\mathbf{F}_{q^d}$, which descends to a morphism $\mathrm{Bun}_G(X) \to \mathfrak{X}$ of algebraic stacks over $\mathbf{F}_q$. Applying Remark 5.2.1.5, we obtain a stratification of $\mathrm{Bun}_G(X)$ by locally closed substacks

$$\{\mathrm{Bun}_G(X)_{[P_0,\nu]}\}_{[P_0,\nu] \in A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})},$$

where $\mathrm{Bun}_G(X)_{[P_0,\nu]} = (\mathrm{Bun}_G(X) \times_{\mathfrak{X}} \mathfrak{X}_{[P_0,\nu]})_{\mathrm{red}}$.

Proposition 5.6.3.2. *The stratification* $\{\mathrm{Bun}_G(X)_{[P_0,\nu]}\}_{[P_0,\nu] \in A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})}$ *is convergent, in the sense of Definition 5.2.2.1.*

Proof of Theorem 5.0.0.3. Combine Propositions 5.6.3.2 and 5.2.2.3. □

5.6.4 The Proof of Proposition 5.6.3.2

Throughout this section, we retain the notations of §5.6.3. We begin with a few remarks about the Harder-Narasimhan stratification of $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$. Recall that for each $(P_0, \nu) \in A$, the Harder-Narasimhan stratum $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}$ is equipped with a finite surjective radicial map $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}} \to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}$ (see Notation 5.5.1.11). If $(P_0, \nu) \in A^{\Gamma}$, then the group Γ acts on both $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}}$ and $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}$ (via its action on $\widetilde{X}$ as an algebraic curve over $\mathrm{Spec}(\mathbf{F}_{q^d})$ together with its action on the group G_0 via the homomorphism ρ), and therefore determines a map of fixed point stacks $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to (\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^{\Gamma}$.

Lemma 5.6.4.1. *For each* $(P_0, \nu) \in A^{\Gamma}$*, the map* $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to (\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^{\Gamma}$ *is a finite radicial surjection.*

Proof. Choose a map $\mathrm{Spec}(R) \to (\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^{\Gamma}$ and set $Y = \mathrm{Spec}(R) \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu}} \mathrm{Bun}^{\nu}_{G,P_0}(\widetilde{X})^{\mathrm{ss}}$. Theorem 5.3.2.2 implies that Y is a scheme and the map $Y \to \mathrm{Spec}(R)$ is surjective, finite, and radicial (see Notation 5.5.1.11). The group Γ acts on Y, and we have

$$\mathrm{Spec}(R) \times_{(\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})_{P_0,\nu})^{\Gamma}} \mathrm{Bun}^{\nu}_{G,P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \simeq Y^{\Gamma}.$$

To prove Lemma 5.6.4.1, we must show that the map $Y^{\Gamma} \to \mathrm{Spec}(R)$ is also surjective, finite, and radicial. Since Y^{Γ} can be identified with a closed subscheme of Y, the only nontrivial point is to prove surjectivity. Fix an algebraically closed field κ and a map

$\eta : \mathrm{Spec}(\kappa) \to \mathrm{Spec}(R)$. Since the map $Y \to \mathrm{Spec}(R)$ is a radicial surjection, the map η lifts *uniquely* to a map $\overline{\eta} : \mathrm{Spec}(\kappa) \to Y$. It follows from the uniqueness that $\overline{\eta}$ is invariant under the action of Γ, and therefore factors through the closed subscheme $Y^{\Gamma} \subseteq Y$. $\square$

In order to prove that $\{\mathrm{Bun}_G(X)_{[P_0,\nu]}\}_{[P_0,\nu]\in A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})}$ is a convergent stratification of the moduli stack $\mathrm{Bun}_G(X)$, we first need to isolate candidates for the algebraic stacks $\mathcal{T}_i$ which appear in Definition 5.2.2.1.

Construction 5.6.4.2. The stratification of $\{\mathrm{Bun}_G(X)_{[P_0,\nu]}\}_{[P_0,\nu]\in A^{\Gamma}/(\mathbf{Z}/d\mathbf{Z})}$ is obtained by pulling back the Harder-Narasimhan stratification of $\mathrm{Bun}_{G_0}(\widetilde{X})$ along a certain $(\mathbf{Z}/d\mathbf{Z})$-equivariant map $\mathrm{Bun}_G(X) \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\mathbf{F}_{q^d}) \to \mathrm{Bun}_{G_0}(\widetilde{X})^{\Gamma}$, and then taking quotients by the action of $\mathbf{Z}/d\mathbf{Z}$.

Let (P_0, ν) be an element of A^{Γ}. We let $\mathcal{Y}_{[P_0,\nu]}$ denote the reduced algebraic stack

$$(\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma})_{\mathrm{red}},$$

which carries an action of the subgroup $C \subseteq \mathbf{Z}/d\mathbf{Z}$ which stabilizes $(P_0, \nu) \in A^{\Gamma}$.

Note that every stratum of $\mathrm{Bun}_G(X)$ admits a surjective finite radicial morphism from a quotient stack $\mathcal{Y}_{[P_0,\nu]}/C$, for some subgroup $C \subseteq \mathbf{Z}/d\mathbf{Z}$ which stabilizes (P_0, ν) (and no two strata correspond to the same quotient). Moreover, there are only finitely many choices for the parabolic subgroup P_0 and for the subgroup C. Consequently, to prove Proposition 5.6.3.2, it will suffice to prove that for each Γ-invariant standard parabolic $P_0 \subseteq G_0$ and each subgroup $C \subseteq \mathbf{Z}/d\mathbf{Z}$ which fixes P_0, there exists a finite collection of algebraic stacks $\mathcal{T}_i$ of the form Y/H where Y is an algebraic space of finite type over $\mathbf{F}_q$ and H is a linear algebraic group over $\mathbf{F}_q$, such that for each $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}_{>0}$ which is fixed by the inverse image of C in $\mathrm{Gal}(L/K_X)$, there is a map

$$\mathcal{T}_i \to \mathcal{Y}_{[P_0,\nu]}/C$$

which exhibits $\mathcal{T}_i$ as a fiber bundle over $\mathcal{Y}_{[P_0,\nu]}/C$ whose fibers are affine spaces of dimension d_{ν}, and the sum $\sum_{\nu} q^{-d_{\nu}}$ converges. We will prove this in two steps:

(*i*) We first show that each of the algebraic stacks $\mathcal{Y}_{[P_0,\nu]}/C$ can be individually written as the quotient of a quasi-compact, quasi-separated algebraic space by the action of a linear algebraic group.

(*ii*) Applying the twisting procedure of Construction 5.5.5.1 with respect to a suitably chosen Galois-invariant divisor $\widetilde{D} \subseteq \widetilde{X}$, we can construct an abundant supply of Galois-equivariant maps $\mathcal{Y}_{[P_0,\nu]} \to \mathcal{Y}_{[P_0,\nu+\lambda]}$. Using these maps, we show that the requirements of Definition 5.2.2.1 can be satisfied by choosing $\{\mathcal{T}_i\}$ to be a finite collection of stacks of the form $\mathcal{Y}_{[P_0,\nu]}/C$, where $C \subseteq \mathbf{Z}/d\mathbf{Z}$ is a subgroup stabilizing (P_0, ν) (here it is convenient *not* to require that C is the entire stabilizer of (P_0, ν)).

Steps (i) and (ii) of our strategy can be articulated precisely in the following pair of results, which immediately imply Proposition 5.6.3.2:

Proposition 5.6.4.3. *Let $(P_0, \nu) \in A^\Gamma$ and let $C \subseteq \mathbf{Z}/d\mathbf{Z}$ be a subgroup which stabilizes (P_0, ν). Then, as an algebraic stack over $\mathbf{F}_q$, the quotient $\mathcal{Y}_{[P_0,\nu]}/C$ can be written as a stack-theoretic quotient Y/H, where Y is a quasi-compact quasi-separated algebraic space over $\mathbf{F}_q$ and H is a linear algebraic group over $\mathbf{F}_q$.*

Proposition 5.6.4.4. *Let d' be a divisor of d, let $P_0 \subseteq G_0$ be a standard parabolic subgroup which is fixed under the action of the subgroup*

$$\Gamma' = \mathrm{Gal}(L/K_X) \times_{\mathbf{Z}/d\mathbf{Z}} (d'\mathbf{Z}/d\mathbf{Z}) \subseteq \mathrm{Gal}(L/K_X),$$

and let $S \subseteq \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}$ be the subset consisting of those elements ν which are fixed by Γ'. Then there exists a finite subset $S_0 \subseteq S$ with the following properties:

(1) *For each $\nu \in S$, there exists a $\nu_0 \in S_0$ and a $(d'\mathbf{Z}/d\mathbf{Z})$-equivariant map of algebraic stacks*

$$\mathcal{Y}_{[P_0,\nu_0]} \to \mathcal{Y}_{[P_0,\nu]}$$

which exhibits $\mathcal{Y}_{[P_0,\nu_0]}$ as a fiber bundle (locally trivial with respect to the étale topology) of some rank e_ν over $\mathcal{Y}_{[P_0,\nu]}$.

(2) *For every real number $r > 1$, the infinite sum $\sum_{\nu \in S} r^{-e_\nu}$ converges.*

Proof of Proposition 5.6.4.3. Let $C' \subseteq \mathbf{Z}/d\mathbf{Z}$ be the stabilizer of (P_0, ν) in A^Γ. It follows from Lemma 5.6.4.1 that $\mathcal{Y}_{[P_0,\nu]}/C'$ admits a surjective finite radicial map $\mathcal{Y}_{[P_0,\nu]}/C' \to \mathrm{Bun}_G(X)_{P_0,\nu}$. The projection map $\mathcal{Y}_{[P_0,\nu]}/C \to \mathcal{Y}_{[P_0,\nu]}/C'$ is finite étale and the inclusion $\mathrm{Bun}_G(X)_{P_0,\nu} \hookrightarrow \mathrm{Bun}_G(X)$ is a locally closed immersion. It follows that the composite map

$$\mathcal{Y}_{[P_0,\nu]}/C \to \mathcal{Y}_{[P_0,\nu]}/C' \to \mathrm{Bun}_G(X)_{P_0,\nu} \hookrightarrow \mathrm{Bun}_G(X)$$

is quasi-finite. By virtue of Corollary 5.4.1.4, Proposition 5.6.4.3 is equivalent to the statement that the algebraic stack $\mathcal{Y}_{[P_0,\nu]}/C$ is quasi-compact. Since the quotient map $\mathcal{Y}_{[P_0,\nu]} \to \mathcal{Y}_{[P_0,\nu]}/C$ is surjective, it will suffice to show that $\mathcal{Y}_{[P_0,\nu]}$ is quasi-compact.

Using Propositions 5.4.2.4 and 5.4.2.1, we deduce that the composite map

$$\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \xrightarrow{g} \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma \to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$$

is quasi-compact. Since the algebraic stack $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ has affine diagonal, the map $\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma} \to \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})$ is affine and, in particular, quasi-separated. It follows that g is quasi-compact. Consequently, to prove that

$$\mathcal{Y}_{[P_0,\nu]} = (\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^{\Gamma}} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma})_{\mathrm{red}}$$

is quasi-compact, it will suffice to show that $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}$ is quasi-compact. Using the affine morphism $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}}$, we are reduced to proving that $\mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}}$ is quasi-compact. It now suffices to observe that we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}} & \longrightarrow & \mathrm{Bun}^{\nu}_{P_{0\,\mathrm{ad}}}(\widetilde{X})^{\mathrm{ss}} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\widetilde{G}}(X) & \longrightarrow & \mathrm{Bun}_{\widetilde{G}_{\mathrm{ad}}}(X), \end{array}$$

where the lower horizontal map is quasi-compact (Proposition 5.4.2.5) and the upper right hand corner is quasi-compact (Proposition 5.4.3.1). □

To prove Proposition 5.6.4.4, we are free to replace X by $X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\mathbf{F}_{q^{d'}})$ and G by $G \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\mathbf{F}_{q^{d'}})$ (keeping $\widetilde{X}$ and $\widetilde{G}$ the same), and thereby reduce to the case where $d' = 1$. For the remainder of this section, we will fix a standard parabolic subgroup $P_0 \subseteq G_0$ which is invariant under the action of the Galois group $\mathrm{Gal}(L/K_X)$; we will prove that Proposition 5.6.4.4 is valid for P_0 (in the case $d' = 1$). To simplify our notation, for $\nu \in S$ we will denote the algebraic stack $\mathcal{Y}_{[P_0,\nu]}$ simply by $\mathcal{Y}_{\nu}$.

Let $\Lambda \subseteq \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}$ be as in Notation 5.5.2.2. Let $\mathbf{\Delta}_{P_0} = \{\alpha_1, \ldots, \alpha_m\}$ be the collection of simple roots α of G_0 such that $-\alpha$ is not a root of P_0. The construction

$$\lambda \mapsto \{\langle \alpha_i, \lambda\rangle\}_{1 \leq i \leq m}$$

determines an injective map $\Lambda \hookrightarrow \mathbf{Z}^m$ between finitely generated abelian groups of the same rank. It follows that we can choose an integer $N > 0$ such that the image of Λ contains $N\mathbf{Z}^m$: in other words, we can find elements $\lambda_1, \ldots, \lambda_m \in \Lambda$ satisfying

$$\langle \alpha_j, \lambda_i \rangle = \begin{cases} N & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Every element $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^{\vee}$ can be written uniquely in the form $\sum c_i \lambda_i$, where the elements c_i are rational numbers. We observe that ν belongs to $\mathrm{Hom}(P_0, \mathbf{G}_m)^{\vee}_{>0}$ if and only if each of the rational numbers c_i is positive, and that ν is fixed by the action of $\mathrm{Gal}(L/K_0)$ if and only if we have $c_i = c_j$ whenever the roots α_i and α_j are conjugate by the action of $\mathrm{Gal}(L/K_X)$ (which acts on the set $\mathbf{\Delta}_{P_0}$ via the group homomorphism $\rho : \mathrm{Gal}(L/K_X) \to \mathrm{Out}(G_0)$).

Since L is a Galois extension of K_X, the map $\pi : \widetilde{X} \to X$ is generically étale. Choose a closed point $x \in X$ such that π is étale over the point x and the map $\beta : G \times_X \widetilde{X} \to \widetilde{G}$ is an isomorphism when restricted to the inverse image of x. Let $D \subseteq \widetilde{X}$ be the effective divisor given by the inverse image of x, and let $\deg(D)$ denote the degree of D over $\mathbf{F}_{q^d}$. Let us say that an element $\nu = \sum c_i \lambda_i \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$ is *minimal* if each of the coefficients c_i satisfies the inequality $0 < c_i \leq \deg(D)$. Note that every element $\nu \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$ can be written uniquely in the form $\nu_0 + \sum c_i \deg(D) \lambda_i$, where ν_0 is minimal and each c_i is an integer. Moreover, ν belongs to $\mathrm{Hom}(P_0, \mathbf{G}_m)^\vee_{>0}$ if and only if each of the integers c_i is nonnegative, and ν is fixed by $\mathrm{Gal}(L/K_X)$ if and only if ν_0 and $\sum c_i \lambda_i$ are both fixed by $\mathrm{Gal}(L/K_X)$. We will deduce Proposition 5.6.4.4 from the following more precise result:

Proposition 5.6.4.5. *Let $\nu \in \mathrm{Hom}(P_0, \mathbf{G}_m)^\vee$ be an element which is minimal and fixed by the action of $\mathrm{Gal}(L/K_X)$. For every element $\lambda \in \Lambda_{\geq 0}$ which is fixed by the action of $\mathrm{Gal}(L/K_X)$, there exists a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map $\mathcal{Y}_\nu \to \mathcal{Y}_{\nu + \deg(D)\lambda}$ which exhibits $\mathcal{Y}_\nu$ as a fiber bundle (locally trivial with respect to the étale topology) whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda \rangle$.*

Proof. For each $\nu \in \mathrm{Hom}(P_{0\,\mathrm{ad}}, \mathbf{G}_m)^\vee_{>0}$, let $\mathcal{Z}_\nu$ denote the fiber product

$$\mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma} \mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}.$$

Let $U = X - \{x\}$, which we regard as an open subset of X, and let $\widetilde{U}$ denote the inverse image of U in $\widetilde{X}$. Let $\mathrm{Bun}_G(U)$ denote the moduli stack of G-bundles on U: that is, the (non-algebraic) stack over $\mathbf{F}_q$ whose R-valued points are given by G-torsors on the open curve $U_R = U \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(R)$, and define $\mathrm{Bun}_{\widetilde{G}}(\widetilde{U})$ similarly. Since the map $\widetilde{X} \to X$ is étale over the point x and the map $G \times_X \widetilde{X} \to \widetilde{G}$ is an isomorphism over $\{x\}$, the diagram

$$\begin{array}{ccc} \mathrm{Bun}_G(X)_{\mathbf{F}_{q^d}} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(\widetilde{X})^\Gamma \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G(U)_{\mathbf{F}_{q^d}} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(\widetilde{U})^\Gamma \end{array}$$

is a pullback square. It follows that we can identify $\mathcal{Z}_\nu$ with the fiber product

$$\mathrm{Bun}_G(U)_{\mathbf{F}_{q^d}} \times_{\mathrm{Bun}_{\widetilde{G}}(\widetilde{U})^\Gamma} \mathrm{Bun}^\nu_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}.$$

For each $\lambda \in \Lambda_{\geq 0}$, the twisting map

$$\mathrm{Tw}_{\lambda,D} : \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X}) \to \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X})$$

is $\mathrm{Gal}(L/K_X)$-equivariant (Remark 5.5.5.8) and therefore induces a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map

$$u : \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X})^{\Gamma} \to \mathrm{Bun}_{\widetilde{G},P_0}(\widetilde{X})^{\Gamma}.$$

It follows from Remark 5.5.5.6 that u restricts to a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map

$$u_0 : \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} \to \mathrm{Bun}^{\nu+\deg(D)\lambda}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma}.$$

The map u_0 is a pullback of u, and therefore (by virtue of Proposition 5.5.6.1) is an étale fiber bundle whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle$. Note that the twisting construction does not modify bundles over the open set $\widetilde{U}$: in other words, the diagram

$$\begin{array}{ccc} \mathrm{Bun}^{\nu}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(\widetilde{U})^{\Gamma} \\ \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Bun}^{\nu+\deg(D)\lambda}_{\widetilde{G},P_0}(\widetilde{X})^{\mathrm{ss}\,\Gamma} & \longrightarrow & \mathrm{Bun}_{\widetilde{G}}(\widetilde{U})^{\Gamma} \end{array}$$

commutes up to canonical isomorphism. This isomorphism allows us to lift u_0 to a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map $\overline{u}_0 : \mathcal{Z}_\nu \to \mathcal{Z}_{\nu+\deg(D)\lambda}$ which is a pullback of u_0 and therefore also an étale fiber bundle whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle$.

By definition, we have $\mathcal{Y}_\nu = (\mathcal{Z}_\nu)_{\mathrm{red}}$ and $\mathcal{Y}_{\nu+\deg(D)\lambda} = (\mathcal{Z}_{\nu+\deg(D)})_{\mathrm{red}}$. Consequently, $\overline{u}_0$ induces a $(\mathbf{Z}/d\mathbf{Z})$-equivariant map $v : \mathcal{Y}_\nu \to \mathcal{Y}_{\nu+\deg(D)\lambda}$ which factors as a composition

$$\mathcal{Y}_\nu \xrightarrow{v'} \mathcal{Y}_{\nu+\deg(D)\lambda} \times_{\mathcal{Z}_{\nu+\deg(D)\lambda}} \mathcal{Z}_\nu \xrightarrow{v''} \mathcal{Y}_{\nu+\deg(D)\lambda}.$$

The map v'' is a pullback of $\overline{u}_0$ and is therefore an étale fiber bundle whose fibers are affine spaces of dimension $\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle$. To complete the proof, it will suffice to show that v' is an equivalence. It is clear that v' induces an equivalence of the underlying reduced substacks. Since $\mathcal{Y}_\nu$ is reduced, we only need to show that the fiber product $\mathcal{Y}_{\nu+\deg(D)\lambda} \times_{\mathcal{Z}_{\nu+\deg(D)\lambda}} \mathcal{Z}_\nu$ is also reduced. This follows from the fact that $\mathcal{Y}_{\nu+\deg(D)\lambda}$ is reduced, since the morphism v'' is smooth. □

Proof of Proposition 5.6.4.4. We will show that the subset

$$S_0 = \{\nu \in S : \ \nu \text{ is minimal}\} \subseteq S$$

satisfies the requirements of Proposition 5.6.4.4. The only nontrivial point is to prove the convergence of the infinite sum $\sum_{\nu \in S} r^{-e_\nu}$ for $r > 1$. By virtue of Proposition 5.6.4.5, we can write this sum as

$$|S_0| \sum_{\lambda \in Z} r^{-\frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda\rangle},$$

where the sum is taken over the set of all $\mathrm{Gal}(L/K_X)$-invariant elements $\lambda \in \Lambda$ which can be written in the form $\sum_{1 \leq i \leq m} c_i \lambda_i$ where each c_i is a nonnegative integer. Up to a constant factor of $|S_0|$, this sum is dominated by the larger infinite sum

$$\begin{aligned}
\sum_{c_1,\ldots,c_m \geq 0} r^{-\frac{N}{|\Gamma|}\langle 2\rho_{P_0}, \sum c_i \lambda_i\rangle} &= \sum_{c_1,\ldots,c_m \in \mathbf{Z}_{\geq 0}} (r^{-\frac{\deg(D)}{|\Gamma|}})^{\sum c_i \langle 2\rho_{P_0}, \lambda_i\rangle} \\
&= \prod_{1 \leq i \leq m} (\frac{r^{a_i}}{r^{a_i} - 1}) \\
&< \infty,
\end{aligned}$$

where $a_i = \frac{\deg(D)}{|\Gamma|}\langle 2\rho_{P_0}, \lambda_i\rangle$. □

Bibliography

[1] M. Artin, A. Grothendieck, and J.-L. Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer, 1971.

[2] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[3] C. Barwick, S. Glasman, and D. Nardin. Dualizing cartesian and cocartesian fibrations. *Theory Appl. Categ.*, 33:Paper No. 4, 67–94, 2018.

[4] K. A. Behrend. *The Lefschetz trace formula for the moduli stack of principal bundles*. ProQuest LLC, Ann Arbor, MI, 1991. Thesis (Ph.D.)–University of California, Berkeley.

[5] A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[6] A. Borel. Some finiteness properties of adele groups over number fields. *Inst. Hautes Études Sci. Publ. Math.*, (16):5–30, 1963.

[7] S. Bosch. Néron models. In *Proceedings of the conference on p-adic analysis (Houthalen, 1987)*, pages 31–39. Vrije Univ. Brussel, Brussels, 1986.

[8] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984.

[9] B. Conrad. Finiteness theorems for algebraic groups over function fields. *Compos. Math.*, 148(2):555–639, 2012.

[10] P. Deligne. *Cohomologie étale*, volume 569 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1977. Séminaire de géométrie algébrique du Bois-Marie SGA $4\frac{1}{2}$.

[11] P. Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, (52):137–252, 1980.

[12] V. G. Drinfeld and C. Simpson. B-structures on G-bundles and local triviality. *Math. Res. Lett.*, 2(6):823–829, 1995.

[13] E. Freitag and R. Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[14] B. H. Gross. On the motive of a reductive group. *Invent. Math.*, 130(2):287–313, 1997.

[15] A. Grothendieck. *Séminaire de géométrie algébrique du Bois-Marie 1965-66, Cohomologie l-adique et fonctions L, SGA5*, volume 589 of *Springer Lecture Notes*. Springer-Verlag, 1977.

[16] G. Harder. über die Galoiskohomologie halbeinfacher algebraischer Gruppen. III. *J. Reine Angew. Math.*, 274/275:125–138, 1975. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III.

[17] J. Heinloth. Bounds for Behrend's conjecture on the canonical reduction. *Int. Math. Res. Not. IMRN*, (14):Art. ID rnn045, 17, 2008.

[18] R. E. Kottwitz. Tamagawa numbers. *Ann. of Math. (2)*, 127(3):629–646, 1988.

[19] K. F. Lai. Tamagawa number of reductive algebraic groups. *Compositio Math.*, 41(2):153–188, 1980.

[20] S. Lang. Algebraic groups over finite fields. *Amer. J. Math.*, 78:555–563, 1956.

[21] R. P. Langlands. The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 143–148. Amer. Math. Soc., Providence, R.I., 1966.

[22] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[23] J. Lurie. *Higher algebra*. Unpublished manuscript.

[24] J. Lurie. *Spectral Algebraic Geometry.* Unfinished manuscript.

[25] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2009.

[26] J. W. Milnor and J. C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[27] H. Minkowski. Untersuchungen über quadratische Formen. *Acta Math.*, 7(1):201–258, 1885. I. Bestimmung der Anzahl verschiedener Formen, welche ein gegebenes Genus enthält.

[28] J. Oesterlé. Nombres de Tamagawa et groupes unipotents en caractéristique p. *Invent. Math.*, 78(1):13–88, 1984.

[29] T. Ono. On the relative theory of Tamagawa numbers. *Bull. Amer. Math. Soc.*, 70:325–326, 1964.

[30] S. Schieder. The Harder-Narasimhan stratification of the moduli stack of G-bundles via Drinfeld's compactifications. *Selecta Math. (N.S.)*, 21(3):763–831, 2015.

[31] J.-P. Serre. *A course in arithmetic.* Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.

[32] J.-P. Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, fifth edition, 1994.

[33] C. L. Siegel. über die analytische Theorie der quadratischen Formen. *Ann. of Math. (2)*, 36(3):527–606, 1935.

[34] H. T. S. Smith. On the orders and genera of quadratic forms containing more than three indeterminates. *Proceedings of the Royal Society of London*, 13(0):199–203, 1863.

[35] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[36] R. Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

[37] J.-L. Verdier. A duality theorem in the etale cohomology of schemes. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 184–198. Springer, Berlin, 1967.

[38] A. Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, 55:497–508, 1949.

[39] A. Weil. *Adeles and algebraic groups*, volume 23 of *Progress in Mathematics*. Birkhäuser, Boston, Mass., 1982. With appendices by M. Demazure and Takashi Ono.

[40] J.-K. Yu. On the motive and cohomology of a reductive group. Unpublished manuscript.