THE C_{2^n} BOREL DUAL STEENROD ALGEBRA

NICK GEORGAKOPOULOS

Abstract. In this very short note, we expand the Hu-Kriz computation of the G-equivariant Borel dual Steenrod algebra in characteristic 2, from the group $G = C_2$ to all power-2 cyclic groups $G = C_{2^n}$.

Contents

1. Introduction

In this companion piece to [Geo21a], we show that the C_2-equivariant Borel dual Steenrod algebra computation in [HK96] generalizes to all groups $G = C_{2^n}$. More precisely, we give an explicit description of the $RO(C_{2^n})$-graded ring of the homotopy fixed points $(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{hC_{2^n}}_\bigstar$ as a Hopf algebroid over $(H\mathbb{F}_2)^{hC_{2^n}}_\bigstar$, where $\mathbb{F}_2$ stands for the constant C_{2^n}-Green functor associated to the field of two elements. We also compare our description to the dual description of the Borel Steenrod algebra of [Gre88].

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2. Conventions and notations

We will use the letter k to denote the field $\mathbb{F}_2$ with trivial $G = C_{2^n}$ action, the constant G-Mackey functor $k = \underline{\mathbb{F}_2}$ and the corresponding equivariant Eilenberg-MacLane spectrum Hk. The meaning should always be clear from the context.

Henceforth all our co/homology will be in k coefficients. We use $k_\bigstar(X)$ to denote the $RO(G)$-graded Mackey functor of G-equivariant homology in k-coefficients. The value of $k_\bigstar(X)$ on the G/H orbit is denoted by $k^H_\bigstar(X)$.

The real representation ring $RO(C_{2^n})$ is spanned by the irreducible representations $1, \sigma, \lambda_{s,k}$ where σ is the 1-dimensional sign representation and $\lambda_{s,m}$ is the 2-dimensional representation given by rotation by $2\pi s(m/2^n)$ degrees for $1 \le m$ dividing 2^{n-2} and odd $1 \le s < 2^n/m$. Note that 2-locally, $S^{\lambda_{s,m}} \simeq S^{\lambda_{1,m}}$ as

C_{2^n}-equivariant spaces, by the s-power map. Therefore, to compute $k_\bigstar(X)$ for $\bigstar \in RO(C_{2^n})$ it suffices to only consider $\bigstar$ in the span of $1, \sigma, \lambda_k := \lambda_{1,2^k}$ for $0 \le k \le n-2$ ($\lambda_{n-1} = 2\sigma$ and $\lambda_n = 2$).

For $V = \sigma$ or $V = \lambda_m$, denote by $a_V \in k^{C_{2^n}}_{-V}$ the Euler class induced by the inclusion of north and south poles $S^0 \hookrightarrow S^V$; also denote by $u_V \in k^{C_{2^n}}_{|V|-V}$ the orientation class generating the Mackey functor $k_{|V|-V} = k$ ([HHR16]).

3. Borel cohomology

Let EG be a contractible free G-space and $\tilde{E}G$ be the cofiber of the collapse map $EG_+ \to S^0$. For a spectrum X we use the notation $X_h = EG_+ \wedge X$, $X^h = F(EG_+, X)$ and $X^t = \tilde{E}G \wedge X^h$; there is a cofiber sequence

$$X_h \to X^h \to X^t$$

The G-fixed points of X_h, X^h, X^t are the nonequivariant spectra of homotopy orbits X_{hG}, homotopy fixed points X^{hG} and Tate fixed points X^{tG} respectively.

The orientation classes $u_V : k \wedge S^{|V|} \to k \wedge S^V$ are nonequivariant equivalences, hence induce G-equivalences in X_h, X^h, X^t for a k-module X, so they act invertibly on $X_{h\bigstar}, X^h_\bigstar$ and $X^t_\bigstar$. This implies that

$$X_{h\bigstar} \approx X_{h|\bigstar|}\ ,\ X^h_\bigstar = X^h_{|\bigstar|}\ ,\ X^t_\bigstar = X^t_{|\bigstar|}$$

and the $RO(G)$ graded part is determined by the integer graded part.

Proposition 3.1. *For $G = C_{2^n}$ and $n > 1$:*

$$k^{hG}_\bigstar = k[a_\sigma, a_{\lambda_0}, u^\pm_\sigma, u^\pm_{\lambda_0}, ..., u^\pm_{\lambda_{n-2}}]/a_\sigma^2$$
$$k^{tG}_\bigstar = k[a_\sigma, a^\pm_{\lambda_0}, u^\pm_\sigma, u^\pm_{\lambda_0}, ..., u^\pm_{\lambda_{n-2}}]/a_\sigma^2$$

and $k_{hG\bigstar} = \Sigma^{-1} k^{tG}_\bigstar / k^{hG}_\bigstar$ (forgetting the ring structure). The map $k_{hG\bigstar} \to k^{hG}_\bigstar$ is trivial.

Proof. The homotopy fixed point spectral sequence becomes:

$$H^*(G;k)[u^\pm_\sigma, u^\pm_{\lambda_0}, ..., u^\pm_{\lambda_{n-2}}] \implies k^{hG}_\bigstar$$

We have $H^*(G;k) = k^*BG = k[a]/a^2 \otimes k[b]$ where $|a| = 1$ and $|b| = 2$. The spectral sequence collapses with no extensions and we can identify $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$. Finally, $\tilde{E}G = S^{\infty\lambda_0} = \operatorname{colim}(S^{\lambda_0} \xrightarrow{a_{\lambda_0}} S^{\lambda_0} \xrightarrow{a_{\lambda_0}} \cdots)$ so to get $k^{tG}_\bigstar$ we are additionally inverting a_{λ_0}.

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For $G = C_2$ we have

$$k^{hC_2}_\bigstar = k[a_\sigma, u^\pm_\sigma]$$
$$k^{tC_2}_\bigstar = k[a^\pm_\sigma, u^\pm_\sigma]$$

and $k_{hC_2\bigstar} = \Sigma^{-1} k^{tC_2}_\bigstar / k^{hC_2}_\bigstar$ (forgetting the ring structure). The map $k_{hC_2\bigstar} \to k^{hC_2}_\bigstar$ is trivial.

4. The Borel dual Steenrod algebra

The G-Borel dual Steenrod algebra is

$$(k \wedge k)^{hG}_{\bigstar}$$

This is a Hopf algebroid over $k^{hG}_{\bigstar}$.

We will implicitly be completing it at the ideal generated by a_σ for $G = C_2$, and at the ideal generated by a_{λ_0} for $G = C_{2^n}$, $n > 1$ (see [HK96] pg. 373 for more details in the case of $G = C_2$). With this convention, Hu-Kriz computed the C_2-Borel dual Steenrod algebra to be

$$(k \wedge k)^{hC_2}_{\bigstar} = k^{hC_2}_{\bigstar}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ ($\xi_0 = 1$). The generators ξ_i restrict to the Milnor generators in the nonequivariant dual Steenrod algebra and

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k$$
$$\epsilon(\xi_i) = 0 \ , \ i \geq 1$$
$$\eta_R(a_\sigma) = a_\sigma$$
$$\eta_R(u_\sigma)^{-1} = \sum_{i=0}^{\infty} a_\sigma^{2^i - 1} u_\sigma^{-2^i} \xi_i$$

Proposition 4.1. *For $G = C_{2^n}$, $n > 1$,*

$$(k \wedge k)^{hG}_{\bigstar} = k^{hG}_{\bigstar}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ restricting to the $C_{2^{n-1}}$ generators ξ_i, with

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k$$
$$\epsilon(\xi_i) = 0 \ , \ i \geq 1$$
$$\eta_R(a_\sigma) = a_\sigma \ , \ \eta_R(a_{\lambda_0}) = a_{\lambda_0}$$
$$\eta_R(u_\sigma) = u_\sigma + a_\sigma \xi_1$$
$$\eta_R(u_{\lambda_m}) = u_{\lambda_m} \ , \ m > 0$$
$$\eta_R(u_{\lambda_0})^{-1} = \sum_i a_{\lambda_0}^{2^i - 1} u_{\lambda_0}^{-2^i} \xi_i^2$$

Proof. The computation of $(k \wedge k)^{hG}_* = (k \wedge k)^*(BG)$ follows from the computation of $k^{hG}_* = k^*(BG) = k[a]/a^2 \otimes k[b]$ and the fact that nonequivariantly, $k \wedge k$ is a free k-module. To see that the homotopy fixed point spectral sequence for $k \wedge k$ converges strongly, let $F^i BG$ be the skeletal filtration on the Lens space $BG = S^\infty / C_{2^n}$; we can then compute directly that $\lim^1_i (k \wedge k)^*(F^i BG) = \lim^1_i (k[a]/a^2 \otimes k[b]/b^i) = 0$.

Thus we get $(k \wedge k)^{hG}_{\bigstar} = k^{hG}_{\bigstar}[\xi_i]$ and the diagonal Δ and augmentation ϵ are the same as in the nonequivariant case. The Euler classes a_σ, a_{λ_0} are maps of spheres so they are preserved under η_R. The action of η_R on u_σ, u_{λ_0} can be computed through the right coaction on $k^{hG}_\bigstar$: The (completed) coaction of the nonequivariant dual Steenrod algebra on $k^*(BG) = k[a]/a^2 \otimes k[b]$ is

$$a \mapsto a \otimes 1$$
$$b \mapsto \sum_i b^{2^i} \otimes \xi_i^2$$

To verify the formula for the coaction on b we need to check that $Sq^1(b) = 0$ (the alternative is $Sq^1(b) = ab$). From the long exact sequence associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, we can see that the vanishing of the Bockstein on b follows from $H^2(C_{2^n}; \mathbb{Z}/4) = \mathbb{Z}/4$ ($n > 1$).

After identifying $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$ we get the formula for $\eta_R(u_{\lambda_0})$ and also that

$$\eta_R(u_\sigma) = u_\sigma + \epsilon a_\sigma \xi_1$$

where ϵ is either 0 or 1. This is equivalent to

$$\eta_R(u_\sigma^{-1}) = u_\sigma^{-1} + \epsilon a_\sigma u_\sigma^{-2} \xi_1$$

and to see that $\epsilon = 1$ we use the map $k^{hC_2} = k^{h(C_{2^n}/C_{2^{n-1}})} \to k^{hC_{2^n}}$ that sends a_σ, u_σ to a_σ, u_σ respectively. Finally, to compute $\eta_R(u_{\lambda_m})$ for $m > 0$ note that

$$k^{hC_{2^{n-m}}} = k^{hC_{2^n}/C_{2^m}} \to k^{hC_{2^n}}$$

sends $a_{\lambda_0}, u_{\lambda_0}$ to $a_{\lambda_m} = 0, u_{\lambda_m}$ respectively. □

5. Comparison with Greenlees's description

We now compare our result with the description of the Borel Steenrod algebra given in [Gre88], which is dual to our calculation.

In our notation, the G-spectrum b of [Gre88] is $b = k^h$ and $b^V(X)$ corresponds to $(k^h)^{|V|}_G(X)$; to get $(k^h)^V_G(X)$ we need to multiply with the invertible element $u_V \in k^{hG}_{|V|-V}$. The Borel Steenrod algebra is $b^\bigstar_G b = (k^h)^\bigstar_G(k^h)$ and the Borel dual Steenrod algebra is $b^G_\bigstar b = (k^h)^G_\bigstar(k^h) = (k \wedge k)^{hG}_\bigstar$.

Greenlees proves that the Borel Steenrod algebra is given by the Massey-Peterson twisted tensor product ([MP65]) of the nonequivariant Steenrod algebra k^*k and the Borel cohomology of a point $(k^h)^\bigstar_G = k^{hG}_{-\bigstar}$. The twisting has to do with the fact that the action of the Borel Steenrod algebra on $x \in (k^h)^\bigstar_G(X)$ is given by:

$$(\theta \otimes a)(x) = \theta(ax)$$

where $\theta \in k^*k$ and $a \in k^{hG}_\bigstar$. The product of elements $\theta \otimes a$ and $\theta' \otimes a'$ in the Borel Steenrod algebra is not $\theta\theta' \otimes aa'$, since θ does not commute with cup-products, but rather satisfies the Cartan formula:

$$\theta(ab) = \sum_i \theta_i'(a)\theta_i''(b)\ ,\ \Delta\theta = \sum_i \theta_i' \otimes \theta_i''$$

Therefore:

$$(\theta \otimes a)(\theta' \otimes a')(x) = \theta(a\theta'(a'x)) = \sum_i \theta_i'(a)(\theta_i''\theta')(a'x)$$

so

$$(\theta \otimes a)(\theta' \otimes a') = \sum_i \theta_i'(a)(\theta_i''\theta' \otimes a') \tag{1}$$

(we have ignored signs as we are working in characteristic 2).
So the Borel Steenrod algebra is $k^*k \otimes k^{hG}_\bigstar$ with twisted algebra structured defined by (1).

Moreover, Greenlees expresses the action of k^*k on $(k^h)^\bigstar_G(X)$ in terms of the action of k^*k on the orientation classes u_V and the usual (nonequivariant) action of k^*k on $(k^h)^*_G(X) = k^*(X \wedge_G EG_+)$. This is done through the Cartan formula: If $x \in (k^h)^V_G(X)$ then $u_V^{-1}x \in (k^h)^{|V|}_G(X)$ and

$$\theta(x) = \theta(u_V u_V^{-1}x) = \sum_i \theta_i'(u_V)\theta_i''(u_V^{-1}x)$$

What remains to compute is $\theta_i'(u_V)$, namely the action of k^*k on orientation classes.

In our case, for $G = C_{2^n}$, we can see that:

Proposition 5.1. *The action of k^*k on orientation classes is determined by:*

$$Sq^i(u_\sigma) = \begin{cases} u_\sigma & i = 0 \\ a_\sigma & i = 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$Sq^i(u_{\lambda_m}) = \begin{cases} u_{\lambda_m} & i = 0 \\ a_{\lambda_0} & i = 2, m = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Compare with the proof of Proposition 4.1. □

The twisting in the case of the Borel dual Steenrod algebra corresponds to the fact that $(k \wedge k)^{hG}_\bigstar$ is a Hopf algebroid and not a Hopf algebra; computationally this amounts to the formula for η_R of Proposition 4.1.

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Department of Mathematics, University of Chicago
E-mail: nickg@math.uchicago.edu
Website: math.uchicago.edu/∼nickg