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Enriched ∞-categories via non-symmetric ∞-operads

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ABSTRACT

We set up a general theory of weak or homotopy-coherent enrichment in an arbitrary monoidal ∞-category. Our theory of enriched ∞-categories has many desirable properties; for instance, if the enriching ∞-category $\mathcal{V}$ is presentably symmetric monoidal then $\mathrm{Cat}_\infty^{\mathcal{V}}$ is as well. These features render the theory useful even when an ∞-category of enriched ∞-categories comes from a model category (as is often the case in examples of interest, e.g. dg-categories, spectral categories, and (∞, n)-categories). This is analogous to the advantages of ∞-categories over more rigid models such as simplicial categories — for example, the resulting ∞-categories of functors between enriched ∞-categories automatically have the correct homotopy type.

We construct the homotopy theory of $\mathcal{V}$-enriched ∞-categories as a certain full subcategory of the ∞-category of "many-object associative algebras" in $\mathcal{V}$. The latter are defined using a non-symmetric version of Lurie's ∞-operads, and we develop the basics of this theory, closely following Lurie's treatment of symmetric ∞-operads. While we may regard these "many-object" algebras as enriched ∞-categories, we show that it is precisely the full subcategory of "complete" objects (in the sense of Rezk, i.e. those whose spaces of objects are equivalent to their spaces of equivalences) that are local with respect to the class of fully faithful and essentially surjective functors. We also consider an alternative model of enriched

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∞-categories as certain presheaves of spaces satisfying analogues of the "Segal condition" for Rezk's Segal spaces. Lastly, we present some applications of our theory, most notably the identification of associative algebras in $\mathcal{V}$ as a coreflective subcategory of pointed $\mathcal{V}$-enriched ∞-categories as well as a proof of a strong version of the Baez–Dolan stabilization hypothesis.

Contents

1. Introduction

Over the past decade, taking the higher-categorical nature of various mathematical structures seriously has turned out to be a very fruitful idea in several areas of mathematics. In particular, the theory of ∞-*categories* (or more precisely $(\infty, 1)$-*categories*) has found many applications in algebraic topology and in other fields. However, despite the large amount of work that has been carried out on the foundations of ∞-category theory, above all by Joyal and Lurie, the theory is in many ways still in its infancy, and the analogues of many concepts from ordinary category theory remain to be explored. In this paper we begin to study the natural analogue in the ∞-categorical context of one such concept, namely that of *enriched categories*.

Enriched categories in the usual sense are ubiquitous in modern mathematics: the morphisms between objects in naturally occurring categories often have more structure than just that of a set. However, there are a number of important situations where the classical theory of enriched categories has turned out to be insufficient in ways that lead us towards considering the higher-categorical version of enrichment. In algebraic topology, for example, the categories that arise typically have a *space* of morphisms between any two objects, but it is usually only the (weak) homotopy types of these spaces that matter. Naïvely, we might guess that this means we should consider these categories as enriched in the *homotopy category* of spaces, but this turns out to lose information that is important for most applications. We are therefore forced to consider the homotopy theory of categories enriched in topological spaces (or any other model for the homotopy theory of spaces, such as simplicial sets), with respect to the appropriate notion of weak equivalences, which takes us outside the usual theory of enriched categories. It is possible to consider this homotopy theory in the context of Quillen's model categories (as was originally done by Bergner [8] for simplicial categories), but the resulting model structures are in some ways not very well-behaved, essentially because these "strictly enriched" categories are in a sense too rigid. This makes it hard to understand the correct homotopy types of the spaces of functors between them, and also makes homotopy-invariant constructions (such as homotopy limits and colimits) problematic to set up.

An additional problem is that many naturally occurring composition laws between spaces are not strictly associative, but only associative up to coherent homotopy. This makes them difficult to model as simplicial or topological categories. It is therefore often more convenient to work with a notion of "category enriched in spaces" where composition of morphisms is only associative up to coherent homotopy. This is the idea behind the theory of ∞-*categories*. Roughly speaking, the notion of ∞-category is a generalization of the notion of category where in addition to objects and morphisms we also have homotopies between morphisms, homotopies between homotopies, and so on, and composition is only associative up to a coherent choice of higher homotopies. There are several ways to make this idea precise, such as *Segal categories*, *complete Segal spaces*, and *quasicategories*. It turns out that working with ∞-categories also avoids the other problems with simplicial or topological categories mentioned above, such as the difficulty of constructing functor categories.

A similar situation arises in other areas of mathematics, such as algebraic geometry or representation theory, where there are many examples of *derived* categories. These have traditionally been thought of as *additive* categories, which is to say categories enriched in abelian groups, equipped with the additional data of a *triangulation*. Recently, however, it has been understood that derived categories, or more generally triangulated categories, are not rich enough for many applications — the extra structure of the triangulation must be replaced by the more refined and intrinsic notion of a *differential graded* structure, i.e. an enrichment in chain complexes. The correct notion of an equivalence between these *dg-categories* does not require a dg-functor to be given by isomorphisms on chain complexes of maps, however — instead, the functor need only induce *quasi-isomorphisms*. On the other hand, it is again not enough to consider these categories as simply enriched in the homotopy category of chain complexes (i.e. the derived category of abelian groups): just as a differential graded algebra (or more generally an A_∞-algebra) is a much richer and more subtle object than a homotopy-associative multiplication on a chain complex, the composition in a dg-category contains far more information than an enrichment in the homotopy category of chain complexes.

Homotopy-coherent compositions also occur in this context — a key example here is the *Fukaya categories* of symplectic geometry. These can often be described using *A_∞-categories*, but unfortunately the theory of A_∞-categories is not as well-behaved as a replacement for that of dg-categories as ∞-categories are as a replacement for simplicial categories.

A third example of this type is *spectral* categories (or categories enriched in spectra), of which there are many interesting examples in algebraic topology. These are much more general than dg-categories, and tend to arise in examples where the mapping spectra can only be extracted up to homotopy. To emphasize the subtleties of the situation, the very existence of a symmetric monoidal model for the homotopy theory of spectra (under the smash product) was only fairly recently resolved, after being an open question for several decades. Moreover, in this context no notion of homotopy-coherent enrichment has so far been proposed; this is a problem, for example because many important functors that are known to preserve A_∞-structures, such as algebraic K-theory or topological Hochschild homology, cannot be realized as lax monoidal functors to a model category of spectra.

Now, just as spaces are the higher-categorical analogue of sets, spectra are the higher categorical analogue of abelian groups or chain complexes, and the sophisticated nature of these objects means that we require a more conceptual and less ad hoc approach to the homotopy theory of spectral categories than what is often sufficient in the theory of dg-categories. One way to do this is to set up model category structures on enriched categories — it is possible to treat the homotopy theory of dg-categories [40], spectral categories [41], or even categories enriched in other sufficiently nice monoidal model categories [25,7,39,32] in this way. However, the resulting model categories suffer from the same problems as that of simplicial categories. In the case of dg-categories, for example, the correct spaces of dg-functors have only recently been explicitly described by Toën [43], using a fairly complex construction; there are earlier constructions of functor categories between A_∞-categories [29], but these are also problematic.

In this paper we propose a different approach, namely to set up a general theory of weak or homotopy-coherent enrichment. Specifically, we will define and study ∞-categories enriched in *monoidal ∞-categories*, which are ∞-categories equipped with a tensor product that is associative and unital up to coherent homotopy. This theory encompasses, for example, analogues of spectral categories and dg-categories where composition is only associative up to coherent homotopy. For the former we consider ∞-categories enriched in the ∞-category of spectra, while for the latter we enrich in the *derived ∞-category* of abelian groups, in the sense of [28, §1.3.2], i.e. the ∞-category obtained by inverting the quasi-isomorphisms between chain complexes of abelian groups. The resulting homotopy theories of enriched ∞-categories are much better behaved than those of strictly enriched categories — for example, we have naturally defined enriched ∞-categories of functors between enriched ∞-categories. Moreover, the resulting homotopy theories are *equivalent* to those of ordinary enriched categories, as is proved in [19]. Thus, our theory gives a more flexible approach to the homotopy theory of dg-categories and spectral categories, which we expect will make many construction in these settings easier to carry out.

The idea of "weak" enrichment is also implicit in the concept of higher category theory itself: an n-category should have k-morphisms between $(k-1)$-morphisms for $k = 1, \dots, n$, so there is an $(n-1)$-category of maps between any two objects. As is well known, however, to obtain a good notion of n-category for $n > 2$ it is not sufficient to just consider n-categories as *strictly* enriched in $(n-1)$-categories, as in most naturally occurring examples composition is only associative up to invertible higher morphisms. We can avoid this issue by instead applying our ∞-categorical theory of enrichment: iterating the enrichment procedure starting with the category of sets gives an inductively defined notion of fully weak n-category. Starting instead with spaces we obtain a theory of (∞, n)-categories, and we can also similarly define (weak) (n, k)-categories, which are n-categories where the i-morphisms are all invertible for $i > k$. Moreover, the resulting homotopy theories are equivalent to those of existing models for n-categories and (∞, n)-categories (as is also proved in [19]).

Thanks to the foundational work of Lurie, we are able to set up our theory of enrichment entirely within the context of ∞-categories (rather than working with model categories, say). Apart from greater generality, working in this setting gives a theory with many good properties, including the following:

(a) Weak (or homotopy-coherent) enrichment is the only natural notion of enrichment which is possible in this language, which allows us to define our enriched ∞-categories in the obvious way as "many-object associative algebras" in a given monoidal ∞-category. (In other words, the ∞-categorical analogue of "strictly enriched" categories automatically results in the appropriate "weakly enriched" theory.)
(b) It is both easy and natural to consider enriched categories with *spaces* of objects rather than just sets of objects, which turns out to make the resulting homotopy theory both nicer and simpler to set up, analogously to the way in which (complete) Segal spaces are better behaved than Segal categories.
(c) We automatically get very good naturality properties, some of which would have been difficult even to formulate in a model-categorical framework — for example, our ∞-categories are natural with respect to functors between monoidal ∞-categories that are lax monoidal in the appropriate ∞-categorical sense. This means that we can easily apply functors such as group completion, algebraic K-theory, and topological Hochschild homology (which are lax monoidal as functors of ∞-categories, but do not arise from lax monoidal functors between model categories) to construct spectral ∞-categories.
(d) We obtain the correct ∞-categories of enriched functors between enriched ∞-categories simply as the internal Hom objects right adjoint to the natural tensor product of enriched ∞-categories. From the point of view of the model-categorical approach to enrichment this is in some sense the most subtle and useful feature — subtle because the homotopically correct internal Hom must be invariant under enriched equivalences (the primary defect of simplicial categories as a model for ∞-categories) and useful because the existence of these functor ∞-categories makes constructions in, and the further development of, enriched higher category theory possible.
(e) Beyond just constructing a homotopy theory, our theory gives a good setting in which to develop ∞-categorical analogues of many concepts from enriched category theory, as we hope to demonstrate in future work.

In addition to setting up the homotopy theory of enriched ∞-categories, we also construct several non-trivial examples: We show that Lurie's *stable ∞-categories* from [28, §1.1] are all enriched in the ∞-category of spectra, and that the *R-linear ∞-categories* of [27, §6] are enriched in the ∞-category of R-modules, where R is an $\mathbb{E}_2$-ring spectrum. Moreover, we prove that every closed monoidal ∞-category is enriched in itself. This gives us, for example, the natural n-category of functors between any two n-categories, generalizing the familiar fact that the category of categories is enriched over itself.

We also discuss a number of simple applications of the theory. As mentioned above, we provide a reasonable definition of the ∞-category of weak (n, m)-categories for any n and m, which has the advantage of not relying on families of diagrams parametrizing coherence conditions and which agrees with those of Barwick, Bergner, Rezk, Joyal, and others. In this context we give a proof of "Baez–Dolan stabilization" for (weak) n-categories (generalizing that of Lurie for $(n, 1)$-categories). This is the idea that, for

$m \geq n+2$, an m-tuply monoidal weak n-category is precisely an $(n+2)$-tuply monoidal weak n-category (for example, putting two compatible monoidal structures on a category makes it a *braided* monoidal category, while three or more monoidal structures makes it *symmetric* monoidal). We also show that (for $m \leq \infty$ and $m \geq k \neq \infty$) an $\mathbb{E}_n$-monoidal (m, k)-category is the same thing as an $(m+n, k+n)$-category with a single (distinguished) object and a single j-morphism for $j = 1, \ldots, n-1$.

The theory we set up in this article is the first completely general theory of weak enrichment. Weak enrichment in *Cartesian* monoidal model categories has previously been defined as *Segal enriched categories* as studied by Pellissier [33], Lurie [26], and Simpson [38] (generalizing Bergner's model structure on Segal categories [9]). It is important to note that many of the interesting examples of enriched categories are cases (such as abelian groups, chain complexes, and spectra) in which the monoidal structure is not Cartesian; so, while more complicated to describe, allowing for non-Cartesian enrichment is necessary to support the standard examples of interest.

In the non-Cartesian case, there is a theory of A_∞-categories, which gives a notion of weak enrichment in chain complexes, and more recently Bacard [1,2] has set up a model-categorical theory of weak enrichment in a class of symmetric monoidal model categories that can be applied to many interesting examples. A definition of enriched ∞-categories different from ours has also been given by Lurie [28, Definition 4.2.1.28], but he does not develop this theory beyond defining the objects. We will see in §7 that in many cases we can extract an enriched ∞-category in our sense from one of Lurie's, and we hope to be able to extend this construction to a comparison between our theory and Lurie's in the future.

1.1. Overview

In §2 we introduce our definition of enriched ∞-categories in terms of (generalized) non-symmetric ∞-operads, and motivate it by explaining how it relates to ordinary enriched categories.

In §3 we briefly describe the non-symmetric version of Lurie's theory of (generalized) ∞-operads, and prove some (straightforward, for the most part) extensions of Lurie's results. The most technical results, particularly those building towards the construction of colimits of algebras, have been relegated to Appendix A.

The theory of ∞-operads lets us define, for a monoidal ∞-category $\mathcal{V}$, an ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ of $\mathcal{V}$-enriched ∞-categories; this is our object of study in §4. The main result is that if the ∞-category $\mathcal{V}$ is presentable and its tensor product preserves colimits in each variable, then this ∞-category is also presentable. We also compare this model of enriched ∞-categories to a certain ∞-category of presheaves that satisfy analogues of the Segal condition for Segal spaces.

In §5 we construct the correct ∞-category of enriched ∞-categories by inverting the fully faithful and essentially surjective functors in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. Here we prove the main theorem of this article: we can obtain this localization as the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$

spanned by the *complete* $\mathcal{V}$–∞-categories — those $\mathcal{V}$–∞-categories $\mathcal{C}$ such that the underlying space of objects in $\mathcal{C}$ is equivalent to the classifying space of equivalences in $\mathcal{C}$. We also prove that the resulting ∞-category has the expected naturality properties.

In §6 we describe some simple applications of our construction: First we set up a theory of (n, k)-categories and prove the "homotopy hypothesis" in this setting. We then prove that enriching in an $(n, 1)$-category gives an $(n+1, 1)$-category of enriched ∞-categories; from this the Baez–Dolan stabilization hypothesis for k-tuply monoidal n-categories follows easily if we define n-categories to be (∞, n)-categories enriched in sets. We also show that $\mathbb{E}_n$-algebras in an $\mathbb{E}_n$-monoidal ∞-category $\mathcal{V}$ embed fully faithfully into *pointed* $\mathcal{V}$-enriched (∞, n)-categories. This last result has a number of interesting corollaries, such as a description of $\mathbb{E}_n$-monoidal ∞-categories as $(\infty, n+1)$-categories with a single object and a single j-morphism for $j < n$, and a simple construction of endomorphism algebras.

In §7 we construct an important class of examples of enriched ∞-categories: If an ∞-category $\mathcal{C}$ is right-tensored over a monoidal ∞-category $\mathcal{V}$ in such a way that the tensor product $C \otimes (-)$ has a right adjoint $F(C, -) \in \mathcal{V}$ for all $C \in \mathcal{C}$, we show that $\mathcal{C}$ is enriched in $\mathcal{V}$ with the maps from C to D given by $F(C, D)$. There are several interesting special cases: a closed monoidal ∞-category is enriched in itself, and all stable ∞-categories are enriched in the ∞-category of spectra. We prove this result by considering Lurie's definition of enriched ∞-categories and observing that we can extract an enriched ∞-category in our sense by means of Lurie's construction of an ∞-category of "enriched strings".

Finally, in Appendix A we prove some more technical results about non-symmetric ∞-operads.

1.2. Notation and terminology

In this article we will work throughout in the setting of $(\infty, 1)$-categories, by which we mean (heuristically) higher categories in which the n-morphisms are invertible for $n > 1$. Specifically, we will make use of the theory of quasicategories, as, due to the work of Joyal and Lurie, it is currently by far the most highly developed theory of $(\infty, 1)$-categories. Following Lurie we will refer to these objects as ∞-categories, however. We generally recycle the notation and terminology used by Lurie in [25,28]; here are some exceptions and reminders:

- $\mathbf{\Delta}$ is the simplicial indexing category, with objects the non-empty finite totally ordered sets $[n] := \{0, 1, \dots, n\}$ and morphisms order-preserving functions between them.
- $\mathbf{\Gamma}^{\mathrm{op}}$ is the category of pointed finite sets (so, by our convention, $\mathbf{\Gamma}$ is the *opposite* of the category of pointed finite sets).
- Generic categories are generally denoted by single capital bold-face letters $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and generic ∞-categories by single caligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Specific categories

and ∞-categories both get names in the normal text font: thus the category of small **V**-categories is denoted $\mathrm{Cat}^{\mathbf{V}}$ and the ∞-category of small $\mathcal{V}$–∞-categories is denoted $\mathrm{Cat}_\infty^{\mathcal{V}}$.

- Set_Δ is the category of simplicial sets, i.e. the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ of set-valued presheaves on $\mathbf{\Delta}$.
- $\mathcal{S}$ is the ∞-category of spaces; this can be defined as the coherent nerve $\mathrm{N}\,\mathrm{Set}_\Delta^\circ$ of the full simplicial subcategory $\mathrm{Set}_\Delta^\circ$ of Set_Δ spanned by the Kan complexes.
- We say a class of morphisms in an ∞-category *satisfies the 2-out-of-3 property* if given morphisms $f: x \to y$ and $g: y \to z$, if any two out of f, g, $g \circ f$ are in the class, so is the third.
- If $\mathcal{C}$ is an ∞-category and A and B are objects of $\mathcal{C}$, then we write $\mathrm{Map}_{\mathcal{C}}(A, B)$ (or just $\mathrm{Map}(A, B)$ if the ∞-category $\mathcal{C}$ is obvious from the context) for the space of maps from A to B in $\mathcal{C}$. In the context of quasicategories there are a number of explicit models for these mapping spaces as simplicial sets (cf. [25, §1.2.2], [12]), but for our purposes it suffices to think of $\mathrm{Map}_{\mathcal{C}}(A, B)$ as an object of the ∞-category of spaces. Constructions of such a "mapping space functor" $\mathrm{Map}_{\mathcal{C}}: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ can be found in [25, §5.1.3] and [28, §5.2.1].
- To distinguish the ∞-categories of non-symmetric ∞-operads and their algebras from their symmetric counterparts we use a superscript "ns" for the non-symmetric versions and a superscript "Σ" for the symmetric versions. Thus the ∞-category of non-symmetric ∞-operads is denoted $\mathrm{Opd}_\infty^{\mathrm{ns}}$ and the ∞-category of symmetric ∞-operads $\mathrm{Opd}_\infty^{\Sigma}$. However, we take the non-symmetric versions to be the default ones in this paper and thus often do not include the superscript — for example, if $\mathcal{O}$ and $\mathcal{P}$ are non-symmetric ∞-operads we will generally denote the ∞-category of $\mathcal{O}$-algebras in $\mathcal{P}$ by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$.
- We make use of the elegant theory of *Grothendieck universes* to allow us to define (∞-)categories without being limited by set-theoretical size issues; specifically, we fix three nested universes, and refer to sets contained in them as *small*, *large*, and *very large*. When $\mathcal{C}$ is an ∞-category of small objects of a certain type, we generally refer to the corresponding ∞-category of large objects as $\widehat{\mathcal{C}}$, without explicitly defining this object. For example, Cat_∞ is the (large) ∞-category of small ∞-categories, and $\widehat{\mathrm{Cat}}_\infty$ is the (very large) ∞-category of large ∞-categories.
- If $\mathcal{C}$ is an ∞-category, we write $\iota\mathcal{C}$ for the *interior* or *underlying space* of $\mathcal{C}$, i.e. the largest subspace of $\mathcal{C}$ that is a Kan complex.
- We write $\mathrm{LFib}(\mathcal{C})$ for the ∞-category of left fibrations over $\mathcal{C}$ (for example obtained from the covariant model structure on $(\mathrm{Set}_\Delta)_{/\mathcal{C}}$). Similarly, we write $\mathrm{Cart}(\mathcal{C})$ and $\mathrm{CoCart}(\mathcal{C})$ for the ∞-categories of Cartesian and coCartesian fibrations to $\mathcal{C}$, respectively, i.e. the ∞-categories associated to the Cartesian and coCartesan model structures on $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}}$.
- We denote by Pres_∞ the ∞-category of presentable ∞-categories and colimit-preserving functors.

- If $f: \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $g: \mathcal{D} \to \mathcal{C}$, we will refer to the adjunction as $f \dashv g$.
- If K is a simplicial set we write $K^{\triangleleft} := \Delta^0 \star K$ and $K^{\triangleright} := K \star \Delta^0$, where $\star$ is the *join* operation. If $\mathcal{C}$ is an ∞-category, we can interpret $\mathcal{C}^{\triangleleft}$ and $\mathcal{C}^{\triangleright}$ as the ∞-categories obtained by freely adjoining an initial object and a final object to $\mathcal{C}$, respectively. We denote the "cone points" coming from Δ^0 in $K^{\triangleleft}$ and $K^{\triangleright}$ by $-\infty$ and ∞, respectively.
- A simplicial set K is *sifted* if it is non-empty and the diagonal map $K \to K \times K$ is cofinal; see [25, §5.5.8] for alternative characterizations. The key point is that sifted colimits are generated by filtered colimits and colimits of simplicial objects, and small colimits are generated by sifted colimits and finite coproducts.

Warning 1.2.1. As far as possible we argue using the "high-level" language of ∞-categories, without referring to their specific implementation as quasicategories. Following this philosophy we have generally not distinguished notationally between categories and their nerves, since categories are a special kind of ∞-category. However, we do indicate the nerve (using N) when we think of the nerve of a category as being a specific simplicial set; by the same principle we always indicate the nerves of simplicial categories. This should hopefully not cause any confusion.

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2. From enriched categories to enriched ∞-categories

The goal of this section is to introduce our definition of enriched ∞-categories, and to motivate it by explaining how it relates to ordinary enriched categories. In the process, we also give an expository introduction to (non-symmetric) ∞-operads.

2.1. Multicategories and enrichment

Recall the usual definition of an enriched category: if $\mathbf{V}$ is a monoidal category, a $\mathbf{V}$-*enriched category* (or $\mathbf{V}$-*category*) $\mathbf{C}$ consists of:

- a set $\operatorname{ob}\mathbf{C}$ of objects,
- for all pairs $X, Y \in \operatorname{ob}\mathbf{C}$ an object $\mathbf{C}(X,Y)$ in $\mathbf{V}$,
- composition maps $\mathbf{C}(X,Y) \otimes \mathbf{C}(Y,Z) \to \mathbf{C}(X,Z)$,
- units $\mathrm{id}_X \colon I \to \mathbf{C}(X,X)$.

The composition must be associative (this involves the associator isomorphism for $\mathbf{V}$) and unital. When formulated in this way, it is not obvious how this notion ought to be generalized in the setting of ∞-categories. We should therefore look for an alternative, more conceptual, way of defining enriched categories — this is provided by the theory of *multicategories*.

A multicategory is, roughly speaking, a category where a morphism has a *list* of objects as its source. More precisely, a *multicategory* (or *non-symmetric coloured operad*) $\mathbf{M}$ consists of

- a set $\operatorname{ob}\mathbf{M}$ of objects,
- for objects $X_1, \ldots, X_n, Y$ (where n can be 0) a set $\mathbf{M}(X_1, \ldots, X_n; Y)$ of "multimorphisms" from $(X_1, \ldots, X_n)$ to Y,
- an identity multimorphism $\mathrm{id}_X \colon (X) \to X$ for all objects X,
- an associative and unital composition law, in the sense that we can compose multimorphisms

  $$(Z_1, \ldots, Z_{i_1}) \to Y_1, \quad \ldots, \quad (Z_{i_{n-1}+1}, \ldots, Z_{i_n}) \to Y_n$$

  with a multimorphism $(Y_1, \ldots, Y_n) \to X$ to get a composite multimorphism $(Z_1, \ldots, Z_{i_n}) \to X$.

A multicategory with a single object is precisely a non-symmetric operad.[1]

If $\mathbf{M}$ and $\mathbf{N}$ are multicategories, a *multifunctor* $F \colon \mathbf{M} \to \mathbf{N}$ assigns an object $F(X)$ in $\mathbf{N}$ to each object X of $\mathbf{M}$, and to each multimorphism $(X_1, \ldots, X_n) \to Y$ in $\mathbf{M}$ a multimorphism

$$\big(F(X_1), \ldots, F(X_n)\big) \to F(Y)$$

in $\mathbf{N}$ such that this assignment is compatible with units and composition. We can view a monoidal category $\mathbf{V}$ as a multicategory by defining

[1] Note that later we will refer to the ∞-categorical version of (non-symmetric) coloured operads as just (non-symmetric) ∞-operads, for consistency with the terminology used by Lurie [28] and Barwick [5].

$$\mathbf{V}(X_1, \ldots, X_n; Y) := \mathbf{V}(X_1 \otimes \cdots \otimes X_n, Y).$$

An *algebra* for a multicategory $\mathbf{M}$ in a monoidal category $\mathbf{V}$ is then just a multifunctor from $\mathbf{M}$ to $\mathbf{V}$ viewed as a multicategory.

Given a set S, there is a simple multicategory $\mathbf{O}_S$ such that $\mathbf{O}_S$-algebras in a monoidal category $\mathbf{V}$ are precisely $\mathbf{V}$-categories with set of objects S: the set of objects of $\mathbf{O}_S$ is $S \times S$, and the multimorphism sets are defined by

$$\mathbf{O}_S\big((X_0, Y_1), (X_1, Y_2), \ldots, (X_{n-1}, Y_n); (Y_0, X_n)\big) := \begin{cases} *, & \text{if } Y_i = X_i, \ i = 0, \ldots, n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus an $\mathbf{O}_S$-algebra $\mathbf{C}$ in $\mathbf{V}$ assigns an object $\mathbf{C}(X, Y)$ to each pair (X, Y) of elements of S, with a unit $I \to \mathbf{C}(X, X)$ from the unique map $() \to (X, X)$, and a composition map $\mathbf{C}(X, Y) \otimes \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$ from the unique multimorphism $((X, Y), (Y, Z)) \to (X, Z)$. Looking at triples of pairs we see that this composition is associative, and it is also clearly unital, so $\mathbf{C}$ is a $\mathbf{V}$-category. If $\mathbf{C}$ and $\mathbf{D}$ are $\mathbf{V}$-categories, with sets of objects S and T, respectively, then from this perspective a $\mathbf{V}$-functor $\mathbf{C} \to \mathbf{D}$ consists of a function $f \colon S \to T$ and a multicategorical natural transformation from $\mathbf{C}$ to $f^*\mathbf{D}$ of multifunctors $\mathbf{O}_S \to \mathbf{V}$, where $f^*\mathbf{D}$ denotes the composite of $\mathbf{D}$ with the obvious multifunctor $\mathbf{O}_S \to \mathbf{O}_T$ induced by f: this natural transformation precisely assigns to each pair $X, Y \in S$ a morphism $\mathbf{C}(X, Y) \to \mathbf{D}(f(X), f(Y))$ compatible with units and composition.

Remark 2.1.1. This definition of enriched categories via multicategories is certainly classical, and it is not clear to us who first introduced it. In more recent work it can be seen, for example, as a starting point for Leinster's theory of enrichment in **fc**-multicategories and more general classes of multicategories associated to Cartesian monads [23].

This construction suggests that we can use an ∞-categorical version of multicategories to define enriched ∞-categories. In the next subsection we will describe such an ∞-categorical theory of multicategories, namely a non-symmetric version of Lurie's ∞-*operads*; this includes as a special case a notion of monoidal ∞-category, and if $\mathcal{V}$ is a monoidal ∞-category we will see that we can define a $\mathcal{V}$-enriched ∞-category with set of objects S as an $\mathbf{O}_S$-algebra in $\mathcal{V}$.

2.2. ∞-Operads

To generalize multicategories to the ∞-categorical setting it is possible to use *simplicial multicategories*, i.e. multicategories enriched in simplicial sets. However, these suffer from the same technical problems as simplicial categories considered as a model for ∞-categories (most notably, it is difficult to compute the correct space of simplicial multifunctors between simplicial multicategories in this rigid setting). Just as for ∞-categories, it is better to use a model where composition is only associative up to coherent homotopy. We will now introduce one such definition, namely a non-symmetric variant of Lurie's ∞-operads.[2]

Before we state the definition, it is helpful to consider an alternative definition of ordinary multicategories:

Definition 2.2.1. If $\mathbf{M}$ is a multicategory, then the *category of operators* $\mathbf{M}^{\otimes}$ of $\mathbf{M}$ has objects lists $(X_1, \ldots, X_n)$ of objects $X_i \in \mathbf{M}$, $n = 0, 1, \ldots$, and a morphism

$$(X_1, \ldots, X_n) \to (Y_1, \ldots, Y_m)$$

is given by a morphism $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ and for each $j = 1, \ldots, m$ a multimorphism

$$(X_{\phi(j-1)+1}, X_{\phi(j-1)+2}, \ldots, X_{\phi(j)}) \to Y_j$$

in $\mathbf{M}$. There is an obvious projection $\mathbf{M}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$, sending $(X_1, \ldots, X_n)$ to $[n]$.

Remark 2.2.2. This is the non-symmetric version of the category of operators of a symmetric operad introduced by May and Thomason [30].

We can characterize those categories over $\mathbf{\Delta}^{\mathrm{op}}$ that are equivalent to categories of operators of multicategories; to state this characterization it is convenient to first introduce some notation:

Definition 2.2.3. We say that a morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}$ is *inert* if it is the inclusion of a sub-interval of $[m]$, i.e. if $\phi(i) = \phi(0) + i$ for $i = 0, \ldots, n$. We denote the inert morphism $[1] \to [n]$ given by the inclusions $\{i-1, i\} \hookrightarrow [n]$ by ρ_i for $i = 1, \ldots, n$.

Definition 2.2.4. Let $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$ denote the subcategory of $\mathrm{Cat}_{/\mathbf{\Delta}^{\mathrm{op}}}$ defined as follows: The objects of $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$ are functors $\pi\colon \mathbf{C} \to \mathbf{\Delta}^{\mathrm{op}}$ such that the following conditions hold:

(i) For every inert morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X \in \mathbf{C}_{[n]}$ there exists a π-coCartesian morphism $X \to \phi_! X$ over ϕ.

(ii) For every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ the functor

$$\mathbf{C}_{[n]} \to \mathbf{C}_{[1]}^{\times n}$$

induced by the coCartesian arrows over the inert maps ρ_i $(i = 1, \ldots, n)$ is an equivalence of categories.

(iii) For every morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and $Y \in \mathbf{C}_{[m]}$, composition with coCartesian morphisms $Y \to Y_i$ over the inert morphisms ρ_i gives an isomorphism

[2] An alternative approach to ∞-operads is the theory of *dendroidal sets* introduced by Moerdijk and Weiss [31], which we will not discuss here.

$$\mathrm{Hom}^{\phi}_{\mathbf{C}}(X,Y) \xrightarrow{\sim} \prod_i \mathrm{Hom}^{\rho_i\circ\phi}_{\mathbf{C}}(X,Y_i),$$

where $\mathrm{Hom}^{\phi}_{\mathbf{C}}(X,Y)$ denotes the subset of $\mathrm{Hom}_{\mathbf{C}}(X,Y)$ of morphisms that map to ϕ in $\mathbf{\Delta}^{\mathrm{op}}$.

The morphisms of $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$ from $\mathbf{C} \to \mathbf{\Delta}^{\mathrm{op}}$ to $\mathbf{D} \to \mathbf{\Delta}^{\mathrm{op}}$ are the functors $\mathbf{C} \to \mathbf{D}$ over $\mathbf{\Delta}^{\mathrm{op}}$ that preserve the coCartesian morphisms over inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$.

Proposition 2.2.5. *The functor $(-)^{\otimes}$ from multicategories to categories over $\mathbf{\Delta}^{\mathrm{op}}$ gives an equivalence between the category of multicategories and $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$.*

Proof. It is easy to see that the category of operators of a multicategory $\mathbf{M}$ satisfies conditions (i)–(iii):

(i) The coCartesian map from $(X_1,\dots,X_n)$ over an inert map $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$ is the projection $(X_1,\dots,X_n) \to (X_{\phi(1)},\dots,X_{\phi(n)})$ determined by the identity maps of the X_i's.

(ii) Clearly $\mathbf{M}^{\otimes}_{[n]}$ is equivalent to $(\mathbf{M}^{\otimes}_{[1]})^{\times n}$ via these projections.

(iii) This is immediate from the definition of the morphisms in $\mathbf{M}^{\otimes}$.

Moreover, any functor of multicategories $F\colon \mathbf{M} \to \mathbf{N}$ induces a functor $\mathbf{M}^{\otimes} \to \mathbf{N}^{\otimes}$ that preserves coCartesian arrows over inert maps: this simply says that $(X_1,\dots,X_n)$ is sent to $(F(X_1),\dots,F(X_n))$. Thus the functor $(-)^{\otimes}$ does indeed factor through $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$.

Conversely, if $\phi\colon \mathbf{M}^{\otimes} \to \mathbf{N}^{\otimes}$ is a functor over $\mathbf{\Delta}^{\mathrm{op}}$ that preserves these coCartesian morphisms, then condition (iii) implies that ϕ is completely determined by the maps $\mathbf{M}(X_1,\dots,X_n;Y) \to \mathbf{N}(\phi(X_1),\dots,\phi(X_n);\phi(Y))$, and so comes from a functor of multicategories. This shows that $(-)^{\otimes}$ is fully faithful.

It remains to show that the functor is essentially surjective. Suppose $\pi\colon \mathbf{C} \to \mathbf{\Delta}^{\mathrm{op}}$ is an object of $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$. Then we can define a multicategory $\mathbf{M}_\pi$ as follows:

- The objects of $\mathbf{M}_\pi$ are the objects of $\mathbf{C}_{[1]}$.
- By condition (ii) we can think of the objects of $\mathbf{C}_{[n]}$ as lists $(X_1,\dots,X_n)$ where the X_i's are objects of $\mathbf{C}_{[1]}$. We define the multimorphism set $\mathbf{M}_\pi(X_1,\dots,X_n;Y)$ to be $\mathrm{Hom}^{\alpha_n}_{\mathbf{C}}((X_1,\dots,X_n),Y)$ where α_n denotes the map $[1] \to [n]$ that sends 0 to 0 and 1 to n.
- The identity $\mathrm{id}_X \in \mathbf{M}_\pi(X;X)$ is just the identity map of X in $\mathbf{C}_{[1]}$.
- To define the composition

$$\begin{aligned}\mathbf{M}_\pi(X_1,\dots,X_{n_1};Y_1) \times \cdots \times \mathbf{M}_\pi(X_{n_{k-1}+1},\dots,X_{n_k};Y_k) \times \mathbf{M}_\pi(Y_1,\dots Y_k;Z)\\ \to \mathbf{M}_\pi(X_1,\dots,X_{n_k};Z)\end{aligned}$$

observe that by (iii) we can describe the source as

$$\mathrm{Hom}^{\beta}_{\mathbf{C}}\big((X_1,\ldots,X_{n_k}),(Y_1,\ldots,Y_k)\big)\times \mathrm{Hom}^{\alpha_k}_{\mathbf{C}}\big((Y_1,\ldots,Y_k),Z\big),$$

where $\beta\colon [k]\to[n_k]$ sends 0 to 0 and i to n_i for $i>0$. Thus composition in $\mathbf{C}$ gives the desired composition in $\mathbf{M}_\pi$.

- To see that the composition is associative and unital, we apply the equivalences from (iii) similarly, and use the associativity and unitality of composition in $\mathbf{C}$.

It is then easy to check that the category of operators $\mathbf{M}_\pi^{\otimes}$ is equivalent to $\mathbf{C}$ over $\mathbf{\Delta}^{\mathrm{op}}$. Thus the functor $(-)^{\otimes}$ is essentially surjective, which completes the proof. □

We can thus equivalently *define* a multicategory to be a functor $\mathbf{C}\to\mathbf{\Delta}^{\mathrm{op}}$ satisfying (i)–(iii). Using the theory developed in [25], these conditions moreover have obvious ∞-categorical analogues, which leads us to the following definition:

Definition 2.2.6. A *non-symmetric* ∞*-operad* is an inner fibration $\pi\colon \mathcal{O}\to\mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For every inert morphism $\phi\colon[n]\to[m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X\in\mathcal{O}_{[n]}$ there exists a π-coCartesian morphism $X\to\phi_!X$ over ϕ.

(ii) For every $[n]\in\mathbf{\Delta}^{\mathrm{op}}$ the functor

$$\mathcal{O}_{[n]}\to(\mathcal{O}_{[1]})^{\times n}$$

induced by the coCartesian arrows over the inert maps ρ_i $(i=1,\ldots,n)$ is an equivalence of ∞-categories.

(iii) For every morphism $\phi\colon[n]\to[m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and $Y\in\mathcal{O}_{[m]}$, composition with coCartesian morphisms $Y\to Y_i$ over the inert morphisms ρ_i gives an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(X,Y)\xrightarrow{\sim}\prod_i \mathrm{Map}^{\rho_i\circ\phi}_{\mathcal{O}}(X,Y_i),$$

where $\mathrm{Map}^{\phi}_{\mathcal{O}}(X,Y)$ denotes the subspace of $\mathrm{Map}_{\mathcal{O}}(X,Y)$ of morphisms that map to ϕ in $\mathbf{\Delta}^{\mathrm{op}}$.

Remark 2.2.7. This is a special case of Barwick's notion of an ∞-operad over an operator category [5], namely the case where the operator category is the category of finite ordered sets.

Remark 2.2.8. Being an inner fibration is a technical condition that does not have an analogue for ordinary categories; among other things it implies that the simplicial set $\mathcal{O}$ must be an ∞-category. Every functor of ∞-categories can be replaced by an equivalent one that is an inner fibration.

Remark 2.2.9. The proof of Proposition 2.2.5 indicates how to interpret a non-symmetric ∞-operad $\mathcal{O}\to\mathbf{\Delta}^{\mathrm{op}}$ as a multicategory "weakly enriched in spaces":

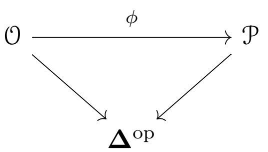
- By condition (ii), the objects of $\mathcal{O}$ can be identified with lists $(X_1, \ldots, X_n)$ where the X_i's are objects of $\mathcal{O}_{[1]}$ (which we think of as the underlying ∞-category of the multicategory)
- By condition (iii), the spaces of maps in $\mathcal{O}$ are determined by the mapping spaces of the form

$$\mathrm{Map}_{\mathcal{O}}^{\alpha_n}\big((X_1, \ldots, X_n), Y\big),$$

  which we think of as the space of multimorphisms in $\mathcal{O}$ from $(X_1, \ldots, X_n)$ to Y.
- The composition of these multimorphisms is determined using condition (iii) by ordinary composition of morphisms in $\mathcal{O}$, as in the proof of Proposition 2.2.5.

Since our definition takes place in the context of ∞-categories, which already encode the notion of coherently homotopy-associative composition of morphisms, this means that the composition of multimorphisms in $\mathcal{O}$ is also coherently homotopy-associative, as expected.

Definition 2.2.10. If $\mathcal{O}$ and $\mathcal{P}$ are non-symmetric ∞-operads, a *morphism of non-symmetric ∞-operads* from $\mathcal{O}$ to $\mathcal{P}$ is a commutative diagram



such that ϕ carries coCartesian morphisms in $\mathcal{O}$ that map to inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to coCartesian morphisms in $\mathcal{P}$. We will also refer to a morphism of non-symmetric ∞-operads $\mathcal{O} \to \mathcal{P}$ as an $\mathcal{O}$-*algebra* in $\mathcal{P}$.

Remark 2.2.11. One advantage of working with ∞-operads over simplicial or topological multicategories is that they can be described as the fibrant objects in a model category where every object is cofibrant. This means that we can work with simple objects like the associative operad rather than having to use a cofibrant replacement, i.e. an A_∞-operad: the ∞-category of algebras for the associative operad in a non-symmetric ∞-operad is always equivalent to the ∞-category of A_∞-algebras.

We now want to define monoidal ∞-categories as a special class of non-symmetric ∞-operads. The appropriate definition is suggested by the following observation:

Lemma 2.2.12.

(i) An object $\pi \colon \mathbf{C} \to \mathbf{\Delta}^{\mathrm{op}}$ in $\mathrm{Cat}^{\mathrm{mult}}_{/\mathbf{\Delta}^{\mathrm{op}}}$ is equivalent to the category of operators of the multicategory associated to a monoidal category if and only if π is a Grothendieck opfibration.

(ii) A morphism $\phi \colon \mathbf{C} \to \mathbf{D}$ between two such objects corresponds to a lax monoidal functor between the associated monoidal categories.

(iii) Under this correspondence the (strong) monoidal functors give precisely the morphisms that preserve all *coCartesian morphisms.*

Proof. Let $\mathbf{M}$ be the multicategory corresponding to $\pi \colon \mathbf{C} \to \mathbf{\Delta}^{\mathrm{op}}$, and write $\mathbf{M}_0 \cong \mathbf{C}_{[1]}$ for its underlying category. The existence of coCartesian morphisms for $\alpha_n \colon [1] \to [n]$ implies that there is a functor $\otimes_n \colon \mathbf{C}_{[1]}^{\times n} \simeq \mathbf{C}_{[n]} \to \mathbf{C}_{[1]}$ such that $\mathbf{M}(X_1, \ldots, X_n; Y) \cong \mathbf{M}_0(\otimes_n(X_1, \ldots, X_n), Y)$. But writing α_n as a composite of elementary face maps in $\mathbf{\Delta}$ in various ways, we get canonical equivalences between $\otimes_n$ and the various ways of successively applying $\otimes_2$ to adjacent elements. Moreover, the coCartesian morphism over the degeneracy $[1] \to [0]$ in $\mathbf{\Delta}$ gives a map $* \simeq \mathbf{C}_{[0]} \to \mathbf{C}_{[1]}$, which amounts to a unit $I \in \mathbf{M}_0$. This implies that if we define $X \otimes Y := \otimes_2(X, Y)$ then $\otimes$ is a monoidal structure on $\mathbf{M}_0$ such that $\mathbf{M}$ is the multicategory associated to this monoidal category. This proves (i). (ii) is then clear, since lax monoidal functors clearly correspond to functors between the associated multicategories, and (iii) follows since a functor preserves all coCartesian arrows precisely if we have natural isomorphisms $F(X) \otimes F(Y) \cong F(X \otimes Y)$ and $F(I) \cong I$. □

In the ∞-categorical case we therefore make the following definitions:

Definition 2.2.13. A *monoidal ∞-category* is a non-symmetric ∞-operad $\mathcal{V}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ that is also a coCartesian fibration. We will generally denote the fibre $\mathcal{V}^{\otimes}_{[1]}$ by $\mathcal{V}$; by abuse of notation we will allow ourselves to say "let $\mathcal{V}$ be a monoidal ∞-category" as shorthand for "let $\mathcal{V}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ be a monoidal ∞-category".

Definition 2.2.14. If $\mathcal{V}^{\otimes}$ and $\mathcal{W}^{\otimes}$ are monoidal ∞-categories, we will refer to a morphism of non-symmetric ∞-operads from $\mathcal{V}^{\otimes}$ to $\mathcal{W}^{\otimes}$ as a *lax monoidal functor*. A *monoidal functor* from $\mathcal{V}^{\otimes}$ to $\mathcal{W}^{\otimes}$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{V}^{\otimes} & \xrightarrow{\phi} & \mathcal{W}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{array}$$

such that ϕ preserves *all* coCartesian morphisms.

Remark 2.2.15. For a coCartesian fibration $\pi\colon \mathcal{C} \to \mathbf{\Delta}^{\mathrm{op}}$, condition (iii) in the definition of non-symmetric ∞-operads follows from condition (ii), since the coCartesian morphisms in $\mathcal{C}$ allow us to identify the space of maps over $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ with a space of maps in $\mathcal{C}_{[n]}$, which decomposes as a product due to condition (ii). This means that, under the equivalence between coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ and functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, monoidal ∞-categories precisely correspond to simplicial ∞-categories $\mathcal{C}_\bullet$ that satisfy the *Segal condition*: the map $\mathcal{C}_n \to \mathcal{C}_1^{\times n}$ induced by the maps $\rho_i\colon [n] \to [1]$ in $\mathbf{\Delta}^{\mathrm{op}}$ are equivalences. The idea that simplicial objects satisfying this condition give a model for A_∞-algebras goes back to Segal (as an unpublished variant of the definition of E_∞-algebras using $\mathbf{\Gamma}$-spaces in [36]) — thus we can interpret monoidal ∞-categories as A_∞-algebras (or just associative algebras, since we are working in the "fully weak" context of ∞-categories) in Cat_∞.

Remark 2.2.16. A monoidal ∞-category $\mathcal{V}^\otimes$ corresponds to the data of a homotopy-coherently associative tensor product on $\mathcal{V}$. To see this, let us unpack the data we get from a monoidal ∞-category, interpreted as a simplical ∞-category $\mathcal{V}_\bullet$ satisfying the Segal condition:

- The map $d_1\colon [2] \to [1]$ gives a tensor product $\otimes\colon \mathcal{V}^{\times 2} \simeq \mathcal{V}_2 \to \mathcal{V}$.
- The map $s_0\colon [0] \to [1]$ gives a unit $* \simeq \mathcal{V}_0 \to \mathcal{V}$.
- The map $\alpha_3\colon [3] \to [1]$ gives a map $\otimes_3\colon \mathcal{V}^{\times 3} \simeq \mathcal{V}_3 \to \mathcal{V}$. The two factorizations $\alpha_3 = d_1 \circ d_1 = d_1 \circ d_2$ give 2-simplices in Cat_∞ that can be interpreted as natural equivalences between $\otimes_3(A, B, C)$ and the composites $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, respectively. Composing these gives the expected natural associator equivalence $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$.
- Similarly, the different ways of decomposing $\alpha_4\colon [4] \to [1]$ as a composite of 3 face maps gives 3-simplices in Cat_∞ that determine homotopies between the different ways of using the associator to pass between different 4-fold tensor products.
- In general, the different ways of decomposing α_n as a composite of $n-1$ face maps gives $(n-1)$-simplices in Cat_∞ that determine the coherence data for n-fold tensor products.

If $\mathbf{M}$ is an ordinary multicategory, then it is clear that (the nerve of) its category of operators $\mathbf{M}^\otimes$ is a non-symmetric ∞-operad — by abuse of notation we will also refer to this non-symmetric ∞-operad as $\mathbf{M}$ in contexts where this does not cause confusion. We can then define enriched ∞-categories as follows:

Definition 2.2.17. If S is a set and $\mathcal{V}^\otimes$ is a monoidal ∞-category, a $\mathcal{V}$*-enriched* ∞*-category* (or $\mathcal{V}$*–*∞*-category*) with set of objects S is an $\mathbf{O}_S$-algebra in $\mathcal{V}$, i.e. a morphism of non-symmetric ∞-operads $\mathbf{O}_S^\otimes \to \mathcal{V}^\otimes$. If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$–∞-categories with sets of objects S and T, respectively, then a $\mathcal{V}$*-functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of a function $f\colon S \to T$

and a natural transformation $\eta\colon \mathcal{C} \to f^*\mathcal{D}$ of functors $\mathbf{O}_S^{\otimes} \to \mathcal{V}^{\otimes}$, where $f^*\mathcal{D}$ denotes the composite of $\mathcal{D}$ with the functor $\mathbf{O}_S^{\otimes} \to \mathbf{O}_T^{\otimes}$ induced by f.

Example 2.2.18. For a one-element set, $\mathbf{O}_*$ is just the associative operad, and $\mathbf{O}_*^{\otimes}$ is $\mathbf{\Delta}^{\mathrm{op}}$. Thus one-object $\mathcal{V}$–∞-categories are precisely ∞-categorical associative algebras, i.e. A_∞-algebras, just as we would expect.

Remark 2.2.19. We saw at the end of §2.1 that $\mathbf{O}_S$-algebras in a monoidal category $\mathbf{V}$ correspond to $\mathbf{V}$-enriched categories with S as their set of objects. Similarly, an $\mathbf{O}_S$-algebra $\mathcal{C}$ in a monoidal ∞-category $\mathcal{V}$ corresponds to the data we would expect to have in an enriched ∞-category. Speaking somewhat informally, to make the underlying ideas clearer, we have for example the following data:

- The object (X, Y) in $\mathbf{O}_S^{\otimes}$ is sent to an object $\mathcal{C}(X, Y) \in \mathcal{V}$.
- The morphism $((X, Y), (Y, Z)) \to (X, Z)$ in $\mathbf{O}_S^{\otimes}$ is sent to a morphism $\mu_{X,Y,Z}\colon \mathcal{C}((X, Y), (Y, Z)) \to \mathcal{C}(X, Z)$ in $\mathcal{V}^{\otimes}$. Since $\mathcal{C}$ preserves coCartesian morphisms over inert maps in $\mathbf{\Delta}^{\mathrm{op}}$, under the equivalence $\mathcal{V}^{\otimes}_{[2]} \simeq \mathcal{V}^{\times 2}$ the object $\mathcal{C}((X, Y), (Y, Z))$ is equivalent to $(\mathcal{C}(X, Y), \mathcal{C}(Y, Z))$, and so using the coCartesian morphism over the map $d_1\colon [2] \to [1]$, we can interpret this as a composition morphism $\mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z)$.
- Similarly, the morphism $() \to (X, X)$ is sent to a morphism we may interpret as a map $I \to \mathcal{C}(X, X)$ where I is the unit of the tensor product on $\mathcal{V}$.
- The morphism $((X, Y), (Y, Z), (Z, W)) \to (X, W)$ in $\mathbf{O}_S^{\otimes}$ factors as $((X, Y), (Y, Z), (Z, W)) \to ((X, Z), (Z, W)) \to (X, W)$ and also as $((X, Y), (Y, Z), (Z, W)) \to ((X, Y), (Y, W)) \to (X, W)$. Pushing the associated data in $\mathcal{V}^{\otimes}$ into $\mathcal{V}$ using the coCartesian morphisms, this gives:
  - an object $\otimes_3(\mathcal{C}(X, Y), \mathcal{C}(Y, Z), \mathcal{C}(Z, W))$ with equivalences α to $\mathcal{C}(X, Y) \otimes (\mathcal{C}(Y, Z) \otimes \mathcal{C}(Z, W))$ and β to $(\mathcal{C}(X, Y) \otimes (\mathcal{C}(Y, Z)) \otimes \mathcal{C}(Z, W)$
  - a morphism $\mu_{X,Y,Z,W}\colon \otimes_3(\mathcal{C}(X, Y), \mathcal{C}(Y, Z), \mathcal{C}(Z, W)) \to \mathcal{C}(X, W)$
  - homotopies between $\mu_{X,Y,Z,W} \circ \alpha^{-1}$ and $\mu_{X,Y,W} \circ (\mathrm{id} \otimes \mu_{Y,Z,W})$ and between $\mu_{X,Y,Z,W} \circ \beta^{-1}$ and $\mu_{X,Z,W} \circ (\mu_{X,Y,Z} \otimes \mathrm{id})$.

  The latter two homotopies can then be composed to get a homotopy between $\mu_{X,Y,W} \circ (\mathrm{id} \otimes \mu_{Y,Z,W})$ and $\mu_{X,Z,W} \circ (\mu_{X,Y,Z} \otimes \mathrm{id})$, which is the first homotopy-coherence data for the associativity of the composition operation.
- Similarly, the data derived from the different decompositions of $((X, Y), (Y, Z), (Z, W), (W, V)) \to (X, V)$ as composites of "face maps" gives the coherence data for 3-fold compositions, and so forth.

If $\mathcal{O}$ and $\mathcal{P}$ are non-symmetric ∞-operads, we get an ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ of $\mathcal{O}$-algebras in $\mathcal{P}$ by taking the full subcategory spanned by the morphisms of non-symmetric ∞-operads in the ∞-category $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{O}, \mathcal{P})$ of functors over $\mathbf{\Delta}^{\mathrm{op}}$. By abuse of notation, if $\mathcal{O}$ is a non-symmetric ∞-operad and $\mathcal{V}^{\otimes}$ is a monoidal ∞-category we will usually write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ instead of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}^{\otimes})$.

In §3.2 we will construct an ∞-category $\mathrm{Opd}_{\infty}^{\mathrm{ns}}$ of non-symmetric ∞-operads and see that the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is functorial in $\mathcal{O}$ and $\mathcal{P}$. This allows us to construct a Cartesian fibration

$$\mathrm{Alg}(\mathcal{P}) \to \mathrm{Opd}_{\infty}^{\mathrm{ns}}$$

whose fibre at $\mathcal{O}$ is $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$. Pulling this back along the functor $\mathrm{Set} \to \mathrm{Opd}_{\infty}^{\mathrm{ns}}$ that sends a set S to $\mathbf{O}_S^{\otimes}$ we get an ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{P})$ with a projection to Set. If $\mathcal{V}$ is a monoidal ∞-category, the objects of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ are clearly $\mathcal{V}$-enriched ∞-categories and the morphisms are precisely $\mathcal{V}$-functors.

A $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ is given by a function $f: S \to T$ of sets of objects and a morphism $\eta: \mathcal{C} \to f^*\mathcal{D}$ of $\mathbf{O}_S$-algebras. This morphism is an equivalence in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ if and only if f is a *bijection* of sets and η is an equivalence of $\mathbf{O}_S$-algebras (i.e. the morphism is *fully faithful*). This is obviously not the correct notion of equivalence for $\mathcal{V}$–∞-categories — we want the equivalences to be the morphisms that are fully faithful and *essentially surjective* (in the usual sense that every object of $\mathcal{D}$ is *equivalent* to an object in the image of f; we will define this precisely below in §5.3 after discussing equivalences in enriched ∞-categories in §5.2). We therefore want to invert these morphisms. In the ∞-categorical setting it is always possible to formally invert any collection of morphisms, but to understand the resulting localized ∞-category we need it to be an *accessible* localization. This is the ∞-categorical analogue of left Bousfield localization of model categories, and means that we can find the localized ∞-category as the full subcategory of *local* objects inside the original ∞-category. However, this is easily seen to be impossible using our current definition of enriched ∞-categories: For example, if we enrich in the monoidal category of sets with the Cartesian product, then $\mathrm{Alg}_{\mathrm{cat}}(\mathrm{Set})$ is just the ordinary category of small categories and functors. But if we invert the fully faithful and essentially surjective functors we get the $(2, 1)$-category of categories, functors, and natural equivalences, which obviously cannot be a full subcategory of an ordinary category.

To avoid this problem we need another definition of enriched ∞-categories for which this localization is well-behaved. It will turn out that we get a much nicer ∞-category of enriched ∞-categories if we allow them to have *spaces* of objects rather than just sets — this is also aligned with the philosophy of higher category theory, whereby spaces should be thought of as the ∞-categorical analogue of sets in ordinary category theory. One way to do this would be to define simplicial multicategories $\mathbf{O}_S$ where S is now a simplicial groupoid, and then work with the associated ∞-operads. We will, in fact, define such simplicial multicategories and briefly make use of them below in §4.2, but it turns out that there is an easier and more natural way to carry out this generalization: We will base our theory of enriched ∞-categories on the ∞-categorical version of a slightly different approach to enriched categories, using *virtual double categories* instead of multicategories, which we describe in the next subsection.

2.3. Virtual double categories and enrichment

Virtual double categories[3] are a common generalization of double categories and multicategories. Roughly speaking, a virtual double category has objects and vertical and horizontal morphisms between them, but in addition to a collection of "squares" there are cells with a list of vertical arrows as source; we refer the reader to [11] or [24] for an explicit definition along this point of view.

Here, we will instead consider virtual double categories from the category of operators perspective: they are exactly what we get if we allow the fibre $\mathbf{O}_{[0]}$ at $[0]$ in a category of operators to be non-trivial, and require $\mathbf{O}_{[n]}$ to be the n-fold iterated fibre product

$$\mathbf{O}_{[1]} \times_{\mathbf{O}_{[0]}} \cdots \times_{\mathbf{O}_{[0]}} \mathbf{O}_{[1]}.$$

To state the precise definition we first introduce some notation:

Definition 2.3.1. Let $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$ denote the subcategory of $\mathbf{\Delta}^{\mathrm{op}}$ where the morphisms are the inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$. We write $\mathcal{G}^{\mathbf{\Delta}}$ for the full subcategory of $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$ spanned by the objects $[0]$ and $[1]$, and $\mathcal{G}^{\mathbf{\Delta}}_{[n]/}$ for the category $(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}})_{[n]/} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}^{\mathbf{\Delta}}$ of inert morphisms from $[n]$ to $[1]$ and $[0]$.

Definition 2.3.2. A *virtual double category* is a functor $\pi\colon \mathbf{M} \to \mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For every inert morphism $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X \in \mathbf{M}_{[n]}$ there exists a π-coCartesian morphism $X \to \phi_! X$ over ϕ.

(ii) For every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ the functor

$$\mathbf{M}_{[n]} \to \lim_{[n]\to[i]\in \mathcal{G}^{\mathbf{\Delta}}_{[n]/}} \mathbf{M}_{[i]} \simeq \mathbf{M}_{[1]} \times_{\mathbf{M}_{[0]}} \cdots \times_{\mathbf{M}_{[0]}} \mathbf{M}_{[1]}$$

induced by the coCartesian arrows over the inert maps in $\mathcal{G}^{\mathbf{\Delta}}_{[n]/}$ is an equivalence of categories.

(iii) For every morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and $Y \in \mathbf{M}_{[m]}$, composition with coCartesian morphisms $Y \to Y_\alpha$ over the inert morphisms $\alpha\colon [m] \to [i]$ in $\mathcal{G}^{\mathbf{\Delta}}_{[m]/}$ gives an isomorphism

$$\mathrm{Hom}^{\phi}_{\mathbf{M}}(X,Y) \xrightarrow{\sim} \lim_{\alpha\in \mathcal{G}^{\mathbf{\Delta}}_{[m]/}} \mathrm{Hom}^{\alpha\circ\phi}_{\mathbf{M}}(X,Y_\alpha),$$

where $\mathrm{Hom}^{\phi}_{\mathbf{M}}(X,Y)$ denotes the subset of $\mathrm{Hom}_{\mathbf{M}}(X,Y)$ of morphisms that map to ϕ in $\mathbf{\Delta}^{\mathrm{op}}$.

[3] Also known as **fc**-multicategories; note that, for consistency with Lurie's terminology, we will refer to their ∞-categorical generalization as *generalized non-symmetric ∞-operads*.

Remark 2.3.3. A virtual double category $\mathbf{M} \to \mathbf{\Delta}^{\mathrm{op}}$ corresponds to a double category precisely when this functor is a Grothendieck opfibration.

Definition 2.3.4. If $\mathbf{M} \to \mathbf{\Delta}^{\mathrm{op}}$ and $\mathbf{N} \to \mathbf{\Delta}^{\mathrm{op}}$ are virtual double categories, a *functor of virtual double categories* from $\mathbf{M}$ to $\mathbf{N}$ is a functor $F: \mathbf{M} \to \mathbf{N}$ over $\mathbf{\Delta}^{\mathrm{op}}$ that preserves coCartesian morphisms over inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$.

Given a set S, we can define a double category with set of objects S where the vertical morphisms are trivial, and there is a unique horizontal morphism between any two elements of S. In terms of categories of operators, this corresponds to the category $\mathbf{\Delta}^{\mathrm{op}}_S$ whose objects are non-empty sequences $(X_0, \ldots, X_n)$ of elements $X_i \in S$, with a unique morphism

$$(X_0, \ldots, X_n) \to (X_{\phi(0)}, \ldots, X_{\phi(m)})$$

for each $\phi: [m] \to [n]$ in $\mathbf{\Delta}$. If $\mathbf{V}$ is a monoidal category and $\mathbf{V}^{\otimes}$ is its category of operators, a functor of virtual double categories $\mathbf{C}: \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{V}^{\otimes}$ is a functor over $\mathbf{\Delta}^{\mathrm{op}}$ such that $\mathbf{C}(X_0, \ldots, X_n) = (\mathbf{C}(X_0, X_1), \ldots, \mathbf{C}(X_{n-1}, X_n))$. This is precisely a $\mathbf{V}$-category with set of objects S: for each $X \in S$ the unique map $X \to (X, X)$ gives an identity $I \to \mathbf{C}(X, X)$, and for objects $X, Y, Z \in S$ the map $(X, Y, Z) \to (X, Z)$ over $d_1: [2] \to [1]$ gives a composition map $\mathbf{C}(X, Y) \otimes \mathbf{C}(Y, Z) \to \mathbf{C}(X, Z)$, which is associative because the two composite maps $(X, Y, Z, W) \to (X, Y, W) \to (X, W)$ and $(X, Y, Z, W) \to (X, Z, W) \to (X, W)$ are equal.

A functor between $\mathbf{V}$-categories $\mathbf{C}$ and $\mathbf{D}$ with sets of objects S and T, respectively, can then be described as a function $f: S \to T$ together with a natural transformation $\mathbf{C} \to f^*\mathbf{D}$ of functors $\mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{V}^{\otimes}$, where $f^*\mathbf{D}$ denotes the composite of $\mathbf{D}$ with the functor $\mathbf{\Delta}^{\mathrm{op}}_f: \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}_T$ induced by f: this natural transformation precisely gives maps $\mathbf{C}(X, Y) \to \mathbf{D}(f(X), f(Y))$ compatible with units and composition.

Remark 2.3.5. Using the virtual double categories $\mathbf{\Delta}^{\mathrm{op}}_S$ to define enrichment gives the right notion also when considering enrichment in more general settings, such as enrichment in double categories or in general virtual double categories (cf. [23]).

2.4. Generalized ∞-operads

It is now clear how to generalize the notion of virtual double category to the ∞-categorical setting, analogously to our definition of non-symmetric ∞-operads above:

Definition 2.4.1. A *generalized non-symmetric ∞-operad* is an inner fibration $\pi: \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For every inert morphism $\phi: [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X \in \mathcal{M}_{[n]}$ there exists a π-coCartesian morphism $X \to \phi_! X$ over ϕ.

(ii) For every $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ the functor

$$\mathcal{M}_{[n]} \to \lim_{[n]\to[i]\in\mathcal{G}^{\mathbf{\Delta}}_{[n]/}} \mathcal{M}_{[i]} \simeq \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \cdots \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]}$$

induced by the coCartesian arrows over the inert maps in $\mathcal{G}^{\mathbf{\Delta}}_{[n]/}$ is an equivalence of ∞-categories.

(iii) For every morphism $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and $Y \in \mathcal{M}_{[m]}$, composition with coCartesian morphisms $Y \to Y_\alpha$ over the inert morphisms $\alpha\colon [m] \to [i]$ in $\mathcal{G}^{\mathbf{\Delta}}_{[m]/}$ gives an equivalence

$$\mathrm{Map}^{\phi}_{\mathcal{M}}(X, Y) \xrightarrow{\sim} \lim_{\alpha\in\mathcal{G}^{\mathbf{\Delta}}_{[m]/}} \mathrm{Map}^{\alpha\circ\phi}_{\mathcal{M}}(X, Y_\alpha),$$

where $\mathrm{Map}^{\phi}_{\mathcal{M}}(X, Y)$ denotes the subspace of $\mathrm{Map}_{\mathcal{M}}(X, Y)$ of morphisms that map to ϕ in $\mathbf{\Delta}^{\mathrm{op}}$.

Definition 2.4.2. If $\mathcal{M}$ and $\mathcal{N}$ are generalized non-symmetric ∞-operads, a *morphism of generalized non-symmetric ∞-operads* from $\mathcal{M}$ to $\mathcal{N}$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\ & \searrow \quad \swarrow & \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{array}$$

such that ϕ carries coCartesian morphisms in $\mathcal{M}$ that map to inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to coCartesian morphisms in $\mathcal{N}$. We will also refer to a morphism of generalized non-symmetric ∞-operads $\mathcal{M} \to \mathcal{N}$ as an $\mathcal{M}$-*algebra* in $\mathcal{N}$.

Definition 2.4.3. A *double ∞-category* is a generalized non-symmetric ∞-operad $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ that is also a coCartesian fibration.

Remark 2.4.4. Again, as in Remark 2.2.15, for a coCartesian fibration condition (iii) in the definition of a generalized non-symmetric ∞-operad is implied by condition (ii). Thus, under the equivalence between coCartesian fibrations to $\mathbf{\Delta}^{\mathrm{op}}$ and functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, double ∞-categories correspond to simplicial ∞-categories $\mathcal{C}_\bullet$ satisfying the "Rezk–Segal condition":

$$\mathcal{C}_n \to \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1$$

is an equivalence. In general simplicial objects in an ∞-category $\mathcal{X}$ with finite limits satisfying this condition can be thought of as *internal categories* in $\mathcal{X}$ — in particular, taking $\mathcal{X}$ to be the ∞-category of spaces these are precisely the *Segal spaces* introduced by Rezk [34] as a model for ∞-categories. This justifies the term double ∞-category, since double categories are precisely internal categories in Cat.

We can now introduce a generalization of the virtual double categories $\mathbf{\Delta}^{\mathrm{op}}_S$: If $S \in \mathcal{S}$ is a space, there is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}$ that sends $[n]$ to $S^{\times n+1}$, face maps to projections to the corresponding factors, and degeneracies to the corresponding diagonal maps; a more precise definition will be given in §4.1. It is easy to see that this simplicial space satisfies the Rezk–Segal condition, so if we let $\mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$ be a left fibration corresponding to this functor then this is a double ∞-category by Remark 2.4.4. When S is a set this obviously agrees with the previous definition.

Using this we can state our improved definition of enriched ∞-categories:

Definition 2.4.5. Let $S \in \mathcal{S}$ be a space and let $\mathcal{V}$ be a monoidal ∞-category. A $\mathcal{V}$-*enriched* ∞-*category* (or $\mathcal{V}$–∞-*category*) with space of objects S is a $\mathbf{\Delta}^{\mathrm{op}}_S$-algebra in $\mathcal{V}$.

Example 2.4.6. Any associative algebra object in $\mathcal{V}$ can be regarded as a $\mathcal{V}$–∞-category with a contractible space of objects. In particular, the unit I of the tensor product in $\mathcal{V}$ has a unique associative algebra structure (by Proposition 3.1.18) so we can regard I as a $\mathcal{V}$–∞-category with a single object whose endomorphisms are given by I.

Remark 2.4.7. We will define the generalized non-symmetric ∞-operads $\mathbf{\Delta}^{\mathrm{op}}_S$ more carefully below in §4.1. It will sometimes be useful, for example to distinguish our definition from other possible definitions of enriched ∞-categories, to refer to a $\mathbf{\Delta}^{\mathrm{op}}_S$-algebra in $\mathcal{V}$ as a *categorical algebra* in $\mathcal{V}$ with space of objects S.

Remark 2.4.8. This definition clearly does not require $\mathcal{V}$ to be a monoidal ∞-category — we can define ∞-categories enriched in any generalized non-symmetric ∞-operad as $\mathbf{\Delta}^{\mathrm{op}}_S$-algebras. This gives an ∞-categorical version of Leinster's notion of enrichment in an **fc**-multicategory [23]. However, as there are technical obstacles in the theory of ∞-operads to extending most of our results beyond the case of monoidal ∞-categories, we will not consider this generalization here.

Definition 2.4.9. Suppose $\mathcal{V}$ is a monoidal ∞-category, and $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{V}$–∞-categories with spaces of objects S and T, respectively. A $\mathcal{V}$-*functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of a morphism of spaces $f \colon S \to T$ and a natural transformation $\mathcal{C} \to f^*\mathcal{D}$, where $f^*\mathcal{D}$ denotes the composite of $\mathcal{D}$ with the morphism $\mathbf{\Delta}^{\mathrm{op}}_f \colon \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}_T$ induced by f.

If $\mathcal{M}$ and $\mathcal{N}$ are generalized non-symmetric ∞-operads we get an ∞-category $\mathrm{Alg}_{\mathcal{M}}(\mathcal{N})$ of $\mathcal{M}$-algebras in $\mathcal{N}$ by taking the full subcategory of the ∞-category $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{N})$ of functors over $\mathbf{\Delta}^{\mathrm{op}}$ that is spanned by the morphisms of generalized non-symmetric ∞-operads. Just as for ∞-operads, we will construct (in §3.2) an ∞-category $\mathrm{Opd}^{\mathrm{ns,gen}}_\infty$ of generalized non-symmetric ∞-operads, and the ∞-categories $\mathrm{Alg}_{\mathcal{M}}(\mathcal{N})$ are functorial in $\mathcal{M}$ and $\mathcal{N}$. As before, we then get a Cartesian fibration $\mathrm{Alg}(\mathcal{N}) \to \mathrm{Opd}^{\mathrm{ns,gen}}_\infty$ whose fibre at $\mathcal{M}$ is $\mathrm{Alg}_{\mathcal{M}}(\mathcal{N})$. We can pull this back along the functor $\mathcal{S} \to \mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ that sends $S \in \mathcal{S}$ to $\mathbf{\Delta}_S^{\mathrm{op}}$ to get an ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{N})$. If $\mathcal{V}$ is a monoidal ∞-category, the objects of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ are $\mathcal{V}$–∞-categories and the morphisms are $\mathcal{V}$-functors.

Remark 2.4.10. We refer to the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ (which we will construct more carefully below in §4.3) as the *∞-category of categorical algebras* in $\mathcal{V}$, reserving the name *∞-category of $\mathcal{V}$–∞-categories* for the localization of this at the fully faithful and essentially surjective functors.

We will prove in §5.3 that inverting the fully faithful and essentially surjective functors in the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ as we have just defined it gives the same ∞-category as inverting them in the version considered above where we only allowed sets of objects. Now, however, we can find the localized ∞-category as a full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. The local objects turn out to be the *complete* $\mathcal{V}$–∞-categories, which are those whose space of objects is equivalent to their classifying space of equivalences, in a sense we will make precise below in §5.2. If we write $\mathrm{Cat}_\infty^{\mathcal{V}}$ for the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ spanned by these complete $\mathcal{V}$–∞-categories, the main result of this article is the following:

Theorem 2.4.11. *Let $\mathcal{V}$ be a monoidal ∞-category. The inclusion*

$$\mathrm{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$$

has a left adjoint, and this exhibits $\mathrm{Cat}_\infty^{\mathcal{V}}$ as the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to the fully faithful and essentially surjective functors.

2.5. Enriched categories as presheaves

As discussed above, our main construction of the ∞-category $\mathrm{Cat}_\infty^{\mathcal{V}}$ of ∞-categories enriched in $\mathcal{V}$ will be as a localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, an ∞-category of algebras for a family of (generalized) ∞-operads. Although useful for many purposes — for example, it is easy to relate $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ to model categories of strictly enriched categories (cf. [19]) — when working with a presentable ∞-category it can also often be useful to have a construction of it as an explicit localization of an ∞-category of presheaves on a small ∞-category of generators. In much the same way as Cat_∞ itself embeds into $\mathcal{P}(\mathbf{\Delta})$ as the full subcategory of complete Segal spaces, one might imagine that $\mathrm{Cat}_\infty^{\mathcal{V}}$ embeds into presheaves on a $\mathcal{V}$-enriched version of $\mathbf{\Delta}$ whose objects classify "composable strings of morphisms" in a $\mathcal{V}$-enriched ∞-category $\mathcal{C}$.

In fact, this $\mathcal{V}$-enriched version of $\mathbf{\Delta}$ nearly comes to us for free from our monoidal ∞-category $p: \mathcal{V}^\otimes \to \mathbf{\Delta}^{\mathrm{op}}$. The functor p is a coCartesian fibration, and so arises as the unstraightening of a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which satisfies the usual Segal condition. But we may also unstraighten p to a Cartesian fibration $q: \mathcal{V}_\otimes^\vee \to \mathbf{\Delta}$ — this is our desired $\mathcal{V}$-enriched version of $\mathbf{\Delta}$. Roughly speaking, the objects of $\mathcal{V}_\otimes^\vee$ are ordered tuples

$(V_1, \ldots, V_n)$ of objects of $\mathcal{V}$, which we can interpret as the free $\mathcal{V}$-enriched ∞-category on the $\mathcal{V}$-enriched graph

$$0 \xrightarrow{V_1} 1 \xrightarrow{V_2} 2 \to \cdots \to n-1 \xrightarrow{V_n} n,$$

which we denote $\Delta^{(V_1,\ldots,V_n)}$. The free $\mathcal{V}$–∞-category on this graph has composition determined by the monoidal structure on $\mathcal{V}$, so for example the maps from $i-1$ to j in $\Delta^{(V_1,\ldots,V_n)}$ are given by $V_i \otimes V_{i+1} \otimes \cdots \otimes V_j$.

A $\mathcal{V}$-enriched ∞-category $\mathcal{C}$ then determines a presheaf

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(-, \mathcal{C})\colon (\mathcal{V}_\otimes^\vee)^{\mathrm{op}} \longrightarrow \mathcal{S}$$

by sending $\Delta^{(V_0,\ldots,V_n)}$ to the space of $\mathcal{V}$-enriched functors from $\Delta^{(V_0,\ldots,V_n)}$ to $\mathcal{C}$. This construction induces a functor

$$\mathrm{Cat}_\infty^{\mathcal{V}} \longrightarrow \mathcal{P}(\mathcal{V}_\otimes^\vee).$$

We will rigorously construct this in §4.5 and show that it is fully faithful, from which it follows almost immediately that $\mathrm{Cat}_\infty^{\mathcal{V}}$ is an accessible localization of $\mathcal{P}(\mathcal{V}_\otimes^\vee)$. Moreover, the essential image of this embedding can be identified with the "complete Segal spaces" in a sense entirely analogous to that of Rezk [34], and the categorical algebras $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ embed in $\mathcal{P}(\mathcal{V}_\otimes^\vee)$ as analogues of the Segal spaces. We use this ambient presheaf ∞-category in §5.6 to prove a crucial technical result about the "completion" functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Cat}_\infty^{\mathcal{V}}$.

3. Non-symmetric ∞-operads

In this section we give the definitions and results we need about (generalized) non-symmetric ∞-operads. These are a special case of Barwick's ∞-operads over an operator category [5], and are also studied by Lurie in [28, §4.7.1] (though in a somewhat *ad hoc* manner).

For the most part the theory of non-symmetric ∞-operads is completely analogous to Lurie's theory of (symmetric) ∞-operads developed in [28], with the category $\mathbf{\Gamma}^{\mathrm{op}}$ of pointed finite sets replaced by the category $\mathbf{\Delta}^{\mathrm{op}}$. In order to keep this article to a reasonable length we only give references to the corresponding results in [28] when the proofs are essentially the same.

3.1. Basic definitions revisited

In this subsection we restate, in a slightly more technical form, the basic definitions of (generalized) non-symmetric ∞-operads — see §2 for some motivation for these definitions. We begin by describing a factorization system on the category $\mathbf{\Delta}^{\mathrm{op}}$.

Definition 3.1.1. Let $\mathbf{\Delta}$ be the usual simplicial indexing category. A morphism $f\colon [n] \to [m]$ in $\mathbf{\Delta}$ is *inert* if it is the inclusion of a sub-interval of $[m]$, i.e. $f(i) = f(0) + i$ for all i, and *active* if it preserves the extremal elements, i.e. $f(0) = 0$ and $f(n) = m$. We say a morphism in $\mathbf{\Delta}^{\mathrm{op}}$ is *active* or *inert* if it is so when considered as a morphism in $\mathbf{\Delta}$, and write $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{act}}$ and $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$ for the subcategories of $\mathbf{\Delta}^{\mathrm{op}}$ with active and inert morphisms, respectively. We write $\rho_i\colon [n] \to [1]$ for the inert map in $\mathbf{\Delta}^{\mathrm{op}}$ corresponding to the inclusion $\{i-1, i\} \hookrightarrow [n]$.

Lemma 3.1.2. *The active and inert morphisms form a factorization system on $\mathbf{\Delta}^{\mathrm{op}}$.*

Proof. This is a special case of [5, Lemma 8.3]; it is also easy to check by hand. □

Definition 3.1.3. A *non-symmetric ∞-operad* is an inner fibration $\pi\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For each inert map $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X \in \mathcal{O}$ such that $\pi(X) = [n]$, there exists a π-coCartesian edge $X \to \phi_! X$ over ϕ.

(ii) For every $[n]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the functor

$$\mathcal{O}_{[n]} \to \prod_{i=1}^{n} \mathcal{O}_{[1]}$$

induced by the inert maps $\rho_i\colon [n] \to [1]$ in $\mathbf{\Delta}^{\mathrm{op}}$ is an equivalence.

(iii) Given $C \in \mathcal{O}_{[n]}$ and a coCartesian map $C \to C_i$ over each inert map $\rho_i\colon [n] \to [1]$, the object C is a π-limit of the C_i's.

Remark 3.1.4. It is immediate from the definition of relative limits in [25, §4.3.1] that Definition 3.1.3 is equivalent to Definition 2.2.6: Recall that a diagram $\bar{p}\colon K^{\triangleleft} \to \mathcal{O}$ is a π-limit if and only if the natural map

$$\lambda\colon \mathcal{O}_{/\bar{p}} \to \mathcal{O}_{/p} \times_{\mathbf{\Delta}^{\mathrm{op}}_{/\pi p}} \mathbf{\Delta}^{\mathrm{op}}_{/\pi \bar{p}}$$

is a categorical equivalence, where $p := \bar{p}|_K$. But the projections $\mathcal{O}_{/\bar{p}} \to \mathcal{O}$ and $\mathcal{O}_{/p} \times_{\mathbf{\Delta}^{\mathrm{op}}_{/\pi p}} \mathbf{\Delta}^{\mathrm{op}}_{/\pi \bar{p}} \to \mathcal{O}$ are both right fibrations, so the map λ is an equivalence if and only if the induced map on fibres over any $o \in \mathcal{O}$ is an equivalence. Since $K^{\triangleleft}$ has an initial object, we may identify $\mathcal{O}_{/\bar{p}}$ with $\mathcal{O}_{/x}$ where $x = \bar{p}(-\infty)$ and $\mathbf{\Delta}^{\mathrm{op}}_{/\pi \bar{p}}$ with $\mathbf{\Delta}^{\mathrm{op}}_{/[n]}$ where $[n] = \pi(x)$. If $[m] = \pi(o)$ then the induced map on fibres is therefore

$$\mathrm{Map}_{\mathcal{O}}(o, x) \to \mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}([m], [n]) \times_{\lim_{k \in K} \mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}([m], \pi p(k))} \lim_{k \in K} \mathrm{Map}_{\mathcal{O}}(o, p(k)).$$

This is an equivalence if and only if the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{O}}(o,x) & \longrightarrow & \lim_{k\in K} \mathrm{Map}_{\mathcal{O}}(o,p(k)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}([m],[n]) & \longrightarrow & \lim_{k\in K} \mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}([m],\pi p(k)) \end{array}$$

is Cartesian, i.e. if and only if for every map $\phi\colon [m] \to [n]$ the map on fibres over ϕ

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(o,x) \to \lim_{k\in K} \mathrm{Map}^{\bar{p}(\psi_k)\circ\phi}_{\mathcal{O}}(o,p(k))$$

is an equivalence, where ψ_k is the unique map $-\infty \to k$ in $K^{\triangleleft}$. Applying this to the coCartesian projections $c \to c_i$ for some $c \in \mathcal{O}_{[n]}$, we get that c is a π-limit of the c_i's if and only if for every $o \in \mathcal{O}_{[m]}$ and every map $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the map

$$\mathrm{Map}^{\phi}_{\mathcal{O}}(o,c) \to \prod_{i=1}^{n} \mathrm{Map}^{\rho_i\phi}_{\mathcal{O}}(o,c_i)$$

is an equivalence, which was the condition used in Definition 2.2.6. Similarly, Definition 3.1.13 below is equivalent to Definition 2.4.1.

Remark 3.1.5. We will see below in §3.7 that there is a natural map $c\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ such that if $\mathcal{O} \to \mathbf{\Gamma}^{\mathrm{op}}$ is a (generalized) symmetric ∞-operad, in the sense of [28], then the pullback $c^*\mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ along c is a (generalized) non-symmetric ∞-operad. Moreover, if $\mathcal{O}$ is a symmetric monoidal ∞-category then $c^*\mathcal{O}$ is a monoidal ∞-category. We will occasionally refer to the pullback $c^*\mathcal{O}$ also as $\mathcal{O}$. For example, if $\mathcal{C}$ is an ∞-category with finite products we will denote the monoidal ∞-category pulled back from the Cartesian symmetric monoidal structure $\mathcal{C}^{\times} \to \mathbf{\Gamma}^{\mathrm{op}}$ by $\mathcal{C}^{\times}$ too.

A useful way of constructing non-symmetric ∞-operads is taking the nerve of the category of operators associated to a simplicial multicategory:

Definition 3.1.6. A *simplicial multicategory* $\mathbf{O}$ consists of a set $\mathrm{ob}\,\mathbf{O}$ of objects and simplicial sets $\mathbf{O}(X_1,\ldots,X_n;Y)$ of multimorphisms for all $X_1,\ldots,X_n,Y \in \mathrm{ob}\,\mathbf{O}$, together with composition maps

$$\begin{aligned} &\mathbf{O}\big(X^1_1,\ldots,X^1_{n_1};Y_1\big) \times \cdots \times \mathbf{O}\big(X^k_1,\ldots,X^k_{n_k};Y_k\big) \times \mathbf{O}(Y_1,\ldots,Y_k;Z) \\ &\quad \to \mathbf{O}\big(X^1_1,\ldots,X^k_{n_k};Z\big), \end{aligned}$$

as well as identity maps, satisfying the usual associativity law for multicategories. A simplicial multicategory $\mathbf{O}$ is *fibrant* if all the simplicial sets $\mathbf{O}((X_1,\ldots,X_n),Y)$ are Kan complexes.

Definition 3.1.7. Let $\mathbf{O}$ be a simplicial multicategory. Define $\mathbf{O}^{\otimes}$ to be the simplicial category with objects finite lists $(X_1, \ldots, X_n)$ $(n = 0, 1, \ldots)$ of objects of $\mathbf{O}$ and morphisms given by

$$\mathbf{O}^{\otimes}\big((X_1, \ldots, X_n), (Y_1, \ldots, Y_m)\big) = \coprod_{\phi : [m] \to [n]} \prod_{i=1}^{m} \mathbf{O}(X_{\phi(i-1)+1}, \ldots, X_{\phi(i)}; Y_i),$$

with composition defined using composition in $\mathbf{O}$. The simplicial category $\mathbf{O}^{\otimes}$ has an obvious projection to $\mathbf{\Delta}^{\mathrm{op}}$.

Lemma 3.1.8. *Suppose* $\mathbf{O}$ *is a fibrant simplicial multicategory. Then the projection* $\mathrm{N}\mathbf{O}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ *is a non-symmetric* ∞*-operad.*

Proof. As [28, Proposition 2.1.1.27]. □

Remark 3.1.9. A non-symmetric variant of the work of Cisinski and Moerdijk [10] should give a model category structure on simplicial multicategories whose fibrant objects are the fibrant simplicial multicategories. The resulting homotopy theory of simplicial multicategories is (partially) known to be equivalent to that of ∞-operads, at least in the symmetric case, but currently the only known relation is via the homotopy theory of dendroidal sets: Cisinski and Moerdijk [10] construct a Quillen equivalence between simplicial symmetric multicategories and dendroidal sets, and Heuts, Hinich, and Moerdijk [20] construct a zig-zag of Quillen equivalences between dendroidal sets and symmetric ∞-operads (but unfortunately their comparison is currently restricted to the special case of ∞-operads without nullary operations). No doubt a version of dendroidal sets defined using planar trees would lead to a similar comparison between simplicial multicategories and non-symmetric ∞-operads.

Definition 3.1.10. A *monoidal* ∞*-category* is a non-symmetric ∞-operad $\mathcal{V}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ that is also a coCartesian fibration.

Remark 3.1.11. We will see below in §3.7 that this is equivalent to Lurie's definition of monoidal ∞-categories in [28].

Example 3.1.12. Suppose $\mathcal{V}^{\otimes}$ is a monoidal ∞-category. Then $d_1 : [2] \to [1]$ induces a functor $d_{1,!} : \mathcal{V} \times \mathcal{V} \simeq \mathcal{V}^{\otimes}_{[2]} \to \mathcal{V}$ — a tensor product on $\mathcal{V}$. Similarly $s_0 : [0] \to [1]$ gives a functor $s_{0,!} : * \simeq \mathcal{V}^{\otimes}_{[0]} \to \mathcal{V}$ which picks out a unit object $I_{\mathcal{V}} := s_{0,!}*$ in $\mathcal{V}$.

Definition 3.1.13. A *generalized non-symmetric* ∞*-operad* is an inner fibration $\pi : \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ such that:

(i) For each inert map $\phi : [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and every $X \in \mathcal{M}$ such that $\pi(X) = [n]$, there exists a π-coCartesian edge $X \to \phi_! X$ over ϕ.

(ii) For every $[n]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the map

$$\mathcal{M}_{[n]} \to \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \cdots \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]}$$

induced by the inert maps $[n] \to [1], [0]$ is an equivalence.

(iii) Given $C \in \mathcal{M}_{[n]}$ and a coCartesian map $C \to C_\alpha$ over each inert map α in $\mathcal{G}^{\mathbf{\Delta}}_{[n]/}$ (i.e. each inert map from $[n]$ to $[1]$ and $[0]$), the object C is a π-limit of the C_α's.

Definition 3.1.14. A *double* ∞*-category* is a generalized non-symmetric ∞-operad that is also a coCartesian fibration.

Definition 3.1.15. Let $\pi\colon \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ be a (generalized) non-symmetric ∞-operad. We say that a morphism f in $\mathcal{M}$ is *inert* if it is coCartesian and $\pi(f)$ is an inert morphism in $\mathbf{\Delta}^{\mathrm{op}}$. We say that f is *active* if $\pi(f)$ is an active morphism in $\mathbf{\Delta}^{\mathrm{op}}$.

Lemma 3.1.16. *The active and inert morphisms form a factorization system on any generalized non-symmetric* ∞*-operad.*

Proof. This is a special case of [28, Proposition 2.1.2.5]. □

Definition 3.1.17. A morphism of (generalized) non-symmetric ∞-operads is a commutative diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\phi} & \mathcal{N} \\ & \searrow \quad \swarrow & \\ & \mathbf{\Delta}^{\mathrm{op}} & \end{array}$$

such that ϕ carries inert morphisms in $\mathcal{M}$ to inert morphisms in $\mathcal{N}$. We will also refer to a morphism of (generalized) non-symmetric ∞-operads $\mathcal{M} \to \mathcal{N}$ as an $\mathcal{M}$*-algebra* in $\mathcal{N}$; we write $\mathrm{Alg}_{\mathcal{M}}(\mathcal{N})$ for the full subcategory of the ∞-category $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{N})$ of functors over $\mathbf{\Delta}^{\mathrm{op}}$ spanned by the morphisms of (generalized) non-symmetric ∞-operads.

Proposition 3.1.18. *Suppose* $\mathcal{V}$ *is a monoidal* ∞*-category. Then* $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})$ *has an initial object* $I_{\mathcal{V}}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{V}^{\otimes}$*, which is the unique associative algebra structure on the unit object* $I_{\mathcal{V}}$ *of* $\mathcal{V}$*.*

Proof. As [28, Corollary 3.2.1.9]. □

Definition 3.1.19. A map of (generalized) non-symmetric ∞-operads is a *fibration of (generalized) non-symmetric* ∞*-operads* if it is also a categorical fibration and a *coCartesian fibration of (generalized) non-symmetric* ∞*-operads* if it is also a coCartesian fibration.

Definition 3.1.20. We will also refer to a map of non-symmetric ∞-operads between monoidal ∞-categories as a *lax monoidal functor*. A *monoidal functor* is a lax monoidal functor that preserves *all* coCartesian arrows. If $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories, we denote the full subcategory of $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V}^{\otimes}, \mathcal{W}^{\otimes})$ spanned by the monoidal functors by $\mathrm{Fun}^{\otimes}(\mathcal{V}^{\otimes}, \mathcal{W}^{\otimes})$. We also use the same notation for the analogous ∞-category of functors between double ∞-categories that preserve all coCartesian morphisms.

It will be useful to know that monoidal ∞-categories are well-behaved with respect to certain localizations:

Definition 3.1.21. Let $\mathcal{V}$ be a monoidal ∞-category and suppose $\mathcal{W}$ is a full subcategory of $\mathcal{V}$ such that the inclusion $i: \mathcal{W} \hookrightarrow \mathcal{V}$ has a left adjoint $L: \mathcal{V} \to \mathcal{W}$. We say that the localization L is *monoidal* if the tensor product of two L-equivalences is again an L-equivalence.

Proposition 3.1.22. *Let $\mathcal{V}$ be a monoidal ∞-category and suppose $L: \mathcal{V} \to \mathcal{W}$ is a monoidal localization with fully faithful right adjoint $i: \mathcal{W} \hookrightarrow \mathcal{V}$. Write $\mathcal{W}^{\otimes}$ for the full subcategory of objects X of $\mathcal{V}^{\otimes}$ such that $\rho_{i,!}X \in \mathcal{W}$ for $i = 1, \ldots, n$ (if $X \in \mathcal{V}^{\otimes}_{[n]}$). Then*

(i) The inclusion $i^{\otimes}: \mathcal{W}^{\otimes} \hookrightarrow \mathcal{V}^{\otimes}$ has a left adjoint $L^{\otimes}: \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ over $\mathbf{\Delta}^{\mathrm{op}}$.
(ii) The projection $\mathcal{W}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ exhibits $\mathcal{W}^{\otimes}$ as a monoidal ∞-category.
(iii) The inclusion $i^{\otimes}$ is a lax monoidal functor and $L^{\otimes}$ is a monoidal functor.

Proof. As [28, Proposition 2.2.1.9]. □

Definition 3.1.23. Suppose $\mathcal{V}$ is a monoidal ∞-category. If K is a simplicial set, we say that $\mathcal{V}$ is *compatible with K-indexed colimits* if

(1) the ∞-category $\mathcal{V}$ has K-indexed colimits (hence so does $\mathcal{V}^{\otimes}_{[n]} \simeq \prod \mathcal{V}$ and $\phi_!$ preserves them for any inert map ϕ),
(2) for all (active) maps $\phi: [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the map

$$\phi_!: \prod_{i=1}^{n} \mathcal{V} \simeq \mathcal{V}^{\otimes}_{[n]} \to \mathcal{V}^{\otimes}_{[m]}$$

preserves K-indexed colimits separately in each variable.

Recall that the ∞-category Pres_{∞} of presentable ∞-categories and colimit-preserving functors has a symmetric monoidal structure, constructed by Lurie in [28, §4.8.1]. The tensor product has the universal property that a colimit-preserving functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ corresponds to a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that preserves colimits separately in each variable. The unit for this tensor product is the ∞-category $\mathcal{S}$ of spaces.

Definition 3.1.24. Let $\mathrm{Mon}_\infty^{\mathrm{Pr}}$ be the ∞-category $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Pres}_\infty)$ of associative algebra objects in Pres_∞ equipped with the tensor product of presentable ∞-categories. Thus $\mathrm{Mon}_\infty^{\mathrm{Pr}}$ is the ∞-category of monoidal ∞-categories $\mathcal{C}^\otimes$ compatible with small colimits such that $\mathcal{C}$ is presentable, with 1-morphisms monoidal functors that preserve colimits. We will refer to the objects of $\mathrm{Mon}_\infty^{\mathrm{Pr}}$ as *presentably monoidal ∞-categories*.

Remark 3.1.25. By Proposition 3.1.18 the ∞-category $\mathrm{Mon}_\infty^{\mathrm{Pr}}$ has an initial object given by the unique presentably monoidal structure on the unit $\mathcal{S}$, which is clearly the Cartesian monoidal structure.

3.2. The ∞-category of ∞-operads

Our goal in this subsection is to construct ∞-categories and $(\infty, 2)$-categories of (generalized) non-symmetric ∞-operads. For this we make use of Lurie's theory of *categorical patterns* from [28, §B].

A number of important objects in higher category theory can be regarded as forming (non-full) subcategories of slice categories of the ∞-category Cat_∞ of ∞-categories — in particular, we have seen above that this is the case for (non-symmetric) ∞-operads and monoidal ∞-categories, which form subcategories of $(\mathrm{Cat}_\infty)_{/\mathbf{\Delta}^{\mathrm{op}}}$. The theory of categorical patterns provides a machine for generating model structures describing ∞-categories of this kind. Specifically, these are model structures on the slice category of marked simplicial sets over some fixed marked simplicial set — the marking, which is a collection of 1-simplices in a simplicial set, allows us to easily consider subcategories of slice categories where some type of map must be preserved (the *inert* maps in the case of ∞-operads, and the *coCartesian* maps in the case of monoidal ∞-categories). Although we could construct the desired ∞-categories of ∞-operads or monoidal ∞-categories directly as subcategories of $(\mathrm{Cat}_\infty)_{/\mathbf{\Delta}^{\mathrm{op}}}$, having the model structure around makes it easy to see that these ∞-categories have all colimits, and indeed are presentable, and also allows us to construct certain functors as Quillen adjunctions.

Definition 3.2.1. A categorical pattern $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\})$ consists of

- an ∞-category $\mathcal{C}$,
- a marking of $\mathcal{C}$, i.e. a collection S of 1-simplices in $\mathcal{C}$ that includes all the degenerate ones,
- a collection of diagrams of ∞-categories $p_\alpha\colon K_\alpha^\triangleleft \to \mathcal{C}$ such that p_α takes every edge in $K_\alpha^\triangleleft$ to a marked edge of $\mathcal{C}$.

Remark 3.2.2. Lurie's definition of a categorical pattern in [28, §B] is more general than this: in particular, he includes the data of a *scaling* of the simplicial set $\mathcal{C}$, i.e. a collection T of 2-simplices in $\mathcal{C}$ that includes all the degenerate ones. In all the examples we consider, however, the scaling consists of *all* 2-simplices of the simplicial set $\mathcal{C}$. We restrict ourselves to this special case as it gives a clearer description of the $\mathfrak{P}$-fibrant objects, and also simplifies the notation.

From a categorical pattern, Lurie constructs a model category that encodes the ∞-category of $\mathfrak{P}$-fibrant objects, in the following sense:

Definition 3.2.3. Suppose $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\})$ is a categorical pattern. A map of simplicial sets $Y \to \mathcal{C}$ is $\mathfrak{P}$-*fibrant* if the following criteria are satisfied:

(1) The underlying map $\pi: Y \to \mathcal{C}$ is an inner fibration. (In particular, Y is an ∞-category.)
(2) Y has all π-coCartesian edges over the morphisms in S.
(3) For every α, the coCartesian fibration $\pi_\alpha: Y \times_{\mathcal{C}} K_\alpha^{\triangleleft} \to K_\alpha^{\triangleleft}$, obtained by pulling back π along p_α, is classified by a limit diagram $K_\alpha^{\triangleleft} \to \mathrm{Cat}_\infty$.
(4) For every α, the composite of any coCartesian section $s: K_\alpha^{\triangleleft} \to Y \times_{\mathcal{C}} K_\alpha^{\triangleleft}$ of π_α with the projection $Y \times_{\mathcal{C}} K_\alpha^{\triangleleft} \to Y$ is a π-limit diagram.

Examples 3.2.4.

(i) Let $\mathfrak{O}_{\mathrm{ns}}$ be the categorical pattern

$$(\mathbf{\Delta}^{\mathrm{op}}, I_{\mathrm{ns}}, \{p_{[n]}: K_{[n]}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}\}),$$

where I_{ns} is the set of inert morphisms and $K_{[n]}$ is the set of inert morphisms $[n] \to [1]$ in $\mathbf{\Delta}^{\mathrm{op}}$. It is immediate from Definition 3.1.3 that a map $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is $\mathfrak{O}_{\mathrm{ns}}$-fibrant precisely if it is a non-symmetric ∞-operad.

(ii) Let $\mathfrak{M}$ denote the categorical pattern

$$(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{N}\mathbf{\Delta}_1^{\mathrm{op}}, \{p_{[n]}: K_{[n]}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}\}).$$

Then a map $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is $\mathfrak{M}$-fibrant precisely if $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is a monoidal ∞-category.

(iii) Let $\mathfrak{O}_{\mathrm{ns}}^{\mathrm{gen}}$ be the categorical pattern

$$(\mathbf{\Delta}^{\mathrm{op}}, I_{\mathrm{ns}}, \{(\mathcal{G}^{\mathbf{\Delta}})_{[n]/}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}\}).$$

It is immediate from Definition 3.1.13 that a map $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is $\mathfrak{O}_{\mathrm{ns}}^{\mathrm{gen}}$-fibrant if and only if $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is a generalized non-symmetric ∞-operad.

(iv) Let $\mathfrak{D}$ denote the categorical pattern

$$(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{N}\mathbf{\Delta}_1^{\mathrm{op}}, \{(\mathcal{G}^{\mathbf{\Delta}})_{[n]/}^{\triangleleft} \to \mathbf{\Delta}^{\mathrm{op}}\}).$$

Then a map $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is $\mathfrak{D}$-fibrant if and only if $Y \to \mathbf{\Delta}^{\mathrm{op}}$ is a double ∞-category.

Theorem 3.2.5. *(Lurie, [28, Theorem B.0.20].) Let $\mathfrak{P} = (\mathcal{C}, S, \{p_\alpha\})$ be a categorical pattern, and let $\overline{\mathcal{C}}$ denote the marked simplicial set $(\mathcal{C}, S)$. There is a left proper combinatorial simplicial model structure on the category $(\mathrm{Set}_\Delta^+)_{/\overline{\mathcal{C}}}$ such that:*

(1) The cofibrations are the morphisms whose underlying maps of simplicial sets are monomorphisms. In particular, all objects are cofibrant.
(2) An object $(X, T) \to \overline{\mathcal{C}}$ is fibrant if and only if $X \to \mathcal{C}$ is $\mathfrak{P}$-fibrant and T is precisely the collection of coCartesian morphisms over the morphisms in S.

We denote the category $(\mathrm{Set}_\Delta^+)_{/\overline{\mathcal{C}}}$ equipped with this model structure by $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}}$.

Applying this in the case $\mathfrak{P} = \mathfrak{O}_{\mathrm{ns}}$, we get:

Corollary 3.2.6. *There is a left proper combinatorial simplicial model structure on $(\mathrm{Set}_\Delta^+)_{/(\mathbf{\Delta}^{\mathrm{op}}, I_{\mathrm{ns}})}$ such that*

(1) The cofibrations are the morphisms whose underlying maps of simplicial sets are monomorphisms. In particular, all objects are cofibrant.
(2) An object $(X, T) \to \mathbf{\Delta}^{\mathrm{op}}$ is fibrant if and only if $X \to \mathbf{\Delta}^{\mathrm{op}}$ is a non-symmetric ∞-operad and T is precisely the collection of inert morphisms of X.

We call this the non-symmetric ∞-operad model structure*.*

Definition 3.2.7. The ∞-category $\mathrm{Opd}_\infty^{\mathrm{ns}}$ of non-symmetric ∞-operads is the ∞-category associated to the simplicial model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$, i.e. the coherent nerve of the simplicial category of fibrant objects. Thus the objects of $\mathrm{Opd}_\infty^{\mathrm{ns}}$ can be identified with non-symmetric ∞-operads. Moreover, since the maps between these in $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$ are precisely the maps that preserve inert morphisms, it is also easy to see that the space of maps from $\mathcal{O}$ to $\mathcal{P}$ in $\mathrm{Opd}_\infty^{\mathrm{ns}}$ is equivalent the subspace of $\mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{O}, \mathcal{P})$ given by the components corresponding to inert-morphism-preserving maps, as expected. This justifies calling $\mathrm{Opd}_\infty^{\mathrm{ns}}$ the *∞-category of non-symmetric ∞-operads*.

Remark 3.2.8. This ∞-category of non-symmetric ∞-operads is a special case of the ∞-categories of ∞-operads over an operator category constructed by Barwick in [5, Theorem 8.15]. By [5, Proposition 8.17] a morphism $\mathcal{O} \to \mathcal{P}$ in $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$ between non-symmetric ∞-operads marked by their inert morphisms is a weak equivalence if and only if the underlying morphism $\mathcal{O} \to \mathcal{P}$ is an equivalence of ∞-categories, as we would expect.

Definition 3.2.9. Similarly, applying Theorem 3.2.5 to the categorical patterns $\mathfrak{M}$, $\mathfrak{O}_{\mathrm{ns}}^{\mathrm{gen}}$, and $\mathfrak{D}$ gives simplicial model categories $(\mathrm{Set}_\Delta^+)_{\mathfrak{M}}$, $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}^{\mathrm{gen}}}$, and $(\mathrm{Set}_\Delta^+)_{\mathfrak{D}}$ whose fibrant objects are, respectively, monoidal ∞-categories, generalized non-symmetric ∞-operads, and double ∞-categories. We write Mon_∞, $\mathrm{Opd}_\infty^{\mathrm{ns,gen}}$, and Dbl_∞ for the ∞-categories associated to these simplicial model categories, and refer to them as the *∞-categories of monoidal ∞-categories, generalized non-symmetric ∞-operads, and double ∞-categories.*

Definition 3.2.10. The morphisms in Mon_∞ are the (strong) monoidal functors between monoidal ∞-categories. We write $\mathrm{Mon}_\infty^{\mathrm{lax}}$ for the ∞-category of monoidal ∞-categories and lax monoidal functors, i.e. the full subcategory of $\mathrm{Opd}_\infty^{\mathrm{ns}}$ spanned by the monoidal ∞-categories.

Examples 3.2.11. Several other ∞-categories we will encounter can be constructed using model categories coming from categorical patterns:

- If $\mathcal{C}$ is an ∞-category, let $\mathfrak{P}_{\mathcal{C}}^{\mathrm{coCart}}$ be the categorical pattern $(\mathcal{C}, \mathcal{C}_1, \emptyset)$. Then $(\mathcal{E}, T) \to \mathcal{C}^\sharp$ is $\mathfrak{P}_{\mathcal{C}}^{\mathrm{coCart}}$-fibrant if and only if $\pi\colon \mathcal{E} \to \mathcal{C}$ is a coCartesian fibration, and T is the set of π-coCartesian edges in $\mathcal{E}$. The model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}_{\mathcal{C}}^{\mathrm{coCart}}}$ is the coCartesian model structure on $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}^\sharp}$. Thus the associated ∞-category is the ∞-category $\mathrm{CoCart}(\mathcal{C})$ of coCartesian fibrations over $\mathcal{C}$, which is equivalent to $\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty)$.
- If $\mathcal{C}$ is an ∞-category, let $\mathfrak{P}_{\mathcal{C}}^{\mathrm{eq}}$ be the categorical pattern $(\mathcal{C}, \iota\mathcal{C}_1, \emptyset)$. Then $(\mathcal{E}, T) \to \mathcal{C}^\natural$ is $\mathfrak{P}_{\mathcal{C}}^{\mathrm{eq}}$-fibrant if and only if $\mathcal{E}$ is an ∞-category, the map $\pi\colon \mathcal{E} \to \mathcal{C}$ is a categorical fibration, and T is the set of equivalences in $\mathcal{E}$. (This follows from the description of categorical fibrations to ∞-categories in [25, Corollary 2.4.6.5].) The model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}_{\mathcal{C}}^{\mathrm{eq}}}$ is the over-category model structure on $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}^\natural}$ from the model structure on Set_Δ^+. The associated ∞-category is thus the over-category $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$.
- If $\mathcal{C}$ is an ∞-category and $\mathcal{D}$ is a subcategory of $\mathcal{C}$, let $\mathfrak{P}_{\mathcal{C},\mathcal{D}}^{\mathrm{coCart}}$ be the categorical pattern $(\mathcal{C}, \mathcal{D}_1, \emptyset)$. Then $(\mathcal{E}, T) \to (\mathcal{C}, \mathcal{D}_1)$ is $\mathfrak{P}_{\mathcal{C},\mathcal{D}}^{\mathrm{coCart}}$-fibrant if and only if $\mathcal{E}$ is an ∞-category, the map $\pi\colon \mathcal{E} \to \mathcal{C}$ is an inner fibration, $\mathcal{E}$ has all π-coCartesian edges over morphisms in $\mathcal{D}$, and T consists precisely of these coCartesian edges. The model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}_{\mathcal{C},\mathcal{D}}^{\mathrm{coCart}}}$ gives an ∞-category of functors $\mathcal{E} \to \mathcal{C}$ that have coCartesian morphisms over the morphisms in $\mathcal{D}$; we write $\mathrm{CoCart}(\mathcal{C}, \mathcal{D})$ for this ∞-category.

Remark 3.2.12. For any categorical pattern $\mathfrak{P}$, the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}}$ is enriched in the model category of marked simplicial sets — this follows from [28, Remark B.2.5] (taking $\mathfrak{P}'$ to be the trivial categorical pattern on Δ^0). Passing to the subcategories of fibrant objects we therefore get fibrant marked simplicial categories of (generalized) non-symmetric ∞-operads. Marked simplicial categories are one model for the theory of $(\infty, 2)$-categories, so we get $(\infty, 2)$-categories $\mathrm{OPD}_\infty^{\mathrm{ns}}$ and $\mathrm{OPD}_\infty^{\mathrm{ns,gen}}$ with underlying ∞-categories $\mathrm{Opd}_\infty^{\mathrm{ns}}$ and $\mathrm{Opd}_\infty^{\mathrm{ns,gen}}$. If $\mathcal{M}$ and $\mathcal{N}$ are (generalized) non-symmetric ∞-operads, we can identify the ∞-category $\mathrm{Alg}_{\mathcal{M}}(\mathcal{N})$ with the ∞-category of maps from $\mathcal{M}$ to $\mathcal{N}$ in the fibrant marked simplicial category $\mathrm{OPD}_\infty^{\mathrm{ns,gen}}$.

Proposition 3.2.13. *The identity is a left (marked simplicially enriched) Quillen functor* $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}^{\mathrm{gen}}} \to (\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$.

Proof. As [28, Corollary 2.3.2.6]. □

Corollary 3.2.14. *The inclusion* $\mathrm{Opd}_\infty^{\mathrm{ns}} \to \mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ *has a left adjoint* $L_{\mathrm{gen}}\colon \mathrm{Opd}_\infty^{\mathrm{ns,gen}} \to \mathrm{Opd}_\infty^{\mathrm{ns}}$.

3.3. Filtered colimits of ∞*-operads*

Colimits of (generalized) non-symmetric ∞-operads are in general difficult to describe explicitly. However, we will now show that filtered colimits can be computed in Cat_∞:

Theorem 3.3.1. *The forgetful functors* $\mathrm{Opd}_\infty^{\mathrm{ns}}$, $\mathrm{Opd}_\infty^{\mathrm{ns,gen}} \to \mathrm{Cat}_\infty$ *detect filtered colimits.*

For this we need some preliminary technical results:

Proposition 3.3.2. *Let* $p\colon \mathcal{I} \to (\mathrm{Cat}_\infty)_{/\mathcal{B}}$ *be a filtered diagram, and let* $f\colon B \to B'$ *be a morphism in* $\mathcal{B}$ *such that for each* $\alpha \in \mathcal{I}$ *the functor* $p(\alpha)\colon \mathcal{C}_\alpha \to \mathcal{B}$ *has* $p(\alpha)$*-coCartesian morphisms* $C \to f_!C$ *over* f *for each* $C \in (\mathcal{C}_\alpha)_B$*, and the functors* $p(\phi)$ *preserve these for all morphisms* $\phi\colon \alpha \to \beta$ *in* $\mathcal{I}$*. Then:*

(i) The colimit $\mathcal{C} \to \mathcal{B}$ *of* p *also has coCartesian morphisms over* f*.*
(ii) The functors $\mathcal{C}_\alpha \to \mathcal{C}$ *preserve these coCartesian morphisms for all* $\alpha \in \mathcal{I}$*.*
(iii) A functor $\mathcal{C} \to \mathcal{D}$ *over* $\mathcal{B}$ *preserves coCartesian morphisms over* f *if and only if all the composites* $\mathcal{C}_\alpha \to \mathcal{C} \to \mathcal{D}$ *do so.*

Proof. For $\alpha \in \mathcal{I}$, let $r_{\alpha,!}$ denote the canonical functor $\mathcal{C}_\alpha \to \mathcal{C}$. Suppose $X \in \mathcal{C}_B$; then there exists $\alpha \in \mathcal{I}$ and $X' \in (\mathcal{C}_\alpha)_B$ such that $X \simeq r_{\alpha,!}X'$. Let $\bar{f}\colon X' \to f_!X'$ be a coCartesian morphism over f; we wish to prove that $r_{\alpha,!}\bar{f}$ is coCartesian in $\mathcal{C}$. To see this we must show that for all $Y \in \mathcal{C}_A$ the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(r_{\alpha,!}f_!X', Y) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{B}}(B', A) & \longrightarrow & \mathrm{Map}_{\mathcal{B}}(B, A) \end{array}$$

is a pullback diagram. Changing α if necessary, we may without loss of generality assume there is a $Y' \in \mathcal{C}_\alpha$ such that $r_{\alpha,!}Y' \simeq Y$. Since filtered colimits commute with finite limits in spaces, and the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is the fibre of the projection

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \simeq \operatorname*{colim}_\alpha \mathrm{Fun}(\Delta^1, \mathcal{C}_\alpha) \to \operatorname*{colim}_\alpha \mathcal{C}_\alpha \times \mathcal{C}_\alpha \simeq \mathcal{C} \times \mathcal{C}$$

at (X, Y), it is easy to see that we can describe $\mathrm{Map}_{\mathcal{C}}(X, Y)$ as the filtered colimit

$$\operatorname*{colim}_{\phi\colon \alpha \to \beta \in \mathcal{I}_{\alpha/}} \mathrm{Map}_{\mathcal{C}_\beta}(\phi_! X', \phi_! Y'),$$

and the commutative square as the colimit square

$$\begin{array}{ccc} \operatorname*{colim}_{\phi\colon \alpha\to\beta\in\mathcal{I}_{\alpha/}} \mathrm{Map}_{\mathcal{C}_\beta}(\phi_! f_! X', \phi_! Y') & \longrightarrow & \operatorname*{colim}_{\phi\colon \alpha\to\beta\in\mathcal{I}_{\alpha/}} \mathrm{Map}_{\mathcal{C}_\beta}(\phi_! X', \phi_! Y') \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{\phi\colon \alpha\to\beta\in\mathcal{I}_{\alpha/}} \mathrm{Map}_{\mathcal{B}}(B', A) & \longrightarrow & \operatorname*{colim}_{\phi\colon \alpha\to\beta\in\mathcal{I}_{\alpha/}} \mathrm{Map}_{\mathcal{B}}(B, A). \end{array}$$

Each of the squares in this colimit are pullback squares since by assumption $\phi_!\bar{f}$ is coCartesian in $\mathcal{C}_\beta$ for all $\phi\colon \alpha \to \beta$. Hence, since filtered colimits in $\mathcal{S}$ commute with finite limits, it follows that the colimit square is also a pullback. Thus $r_{\alpha,!}\bar{f}$ is coCartesian in $\mathcal{C}$, as required. This proves claims (i) and (ii), and (iii) is then clear from this description of the coCartesian morphisms in $\mathcal{C}$. □

Corollary 3.3.3. *The forgetful functor* $\mathrm{CoCart}(\mathcal{C}) \to (\mathrm{Cat}_\infty)_{/\mathcal{C}}$ *detects filtered colimits.*

Proof. We can describe $\mathrm{CoCart}(\mathcal{C})$ as the subcategory of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ whose objects are the coCartesian fibrations and whose morphisms are the functors that preserve coCartesian morphisms. This is clear if we consider the functor of fibrant simplicial categories induced by the functor from the coCartesian model structure on $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}}$ to the over-category model structure on $(\mathrm{Set}_\Delta^+)_{/\mathcal{C}^\natural}$ that forgets the markings that do not map to equivalences in $\mathcal{C}$. The result then follows from Proposition 3.3.2. □

Corollary 3.3.4. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D}$ a subcategory of $\mathcal{C}$. The forgetful functor* $\mathrm{CoCart}(\mathcal{C}, \mathcal{D}) \to (\mathrm{Cat}_\infty)_{/\mathcal{C}}$ *detects filtered colimits.*

Proof. The ∞-category $\mathrm{CoCart}(\mathcal{C}, \mathcal{D})$ can be identified with the full subcategory of the pullback $\mathrm{CoCart}(\mathcal{D}) \times_{(\mathrm{Cat}_\infty)_{/\mathcal{D}}} (\mathrm{Cat}_\infty)_{/\mathcal{C}}$ spanned by those maps $\mathcal{E} \to \mathcal{C}$ that have coCartesian arrows over the morphisms in $\mathcal{D}$ — this is clear from the definition of the mapping spaces in the fibrant simplicial categories associated to the corresponding model categories. The result therefore follows from Proposition 3.3.2. □

Lemma 3.3.5. *Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is an adjunction. Then:*

(i) If the right adjoint U preserves κ-filtered colimits, then F preserves κ-compact objects.
(ii) If in addition $\mathcal{C}$ is κ-accessible, then U preserves κ-filtered colimits if and only if F preserves κ-compact objects.

Proof. For the first claim, suppose $X \in \mathcal{C}$ is a κ-compact object and $p\colon K \to \mathcal{D}$ is a κ-filtered diagram. Then we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{D}}\big(F(X), \operatorname{colim} p\big) &\simeq \mathrm{Map}_{\mathcal{C}}\big(X, G(\operatorname{colim} p)\big) \simeq \mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} G \circ p) \\ &\simeq \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, G \circ p) \simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}\big(F(X), p\big).\end{aligned}$$

Thus $\mathrm{Map}_{\mathcal{D}}(F(X), –)$ preserves κ-filtered colimits, i.e. $F(X)$ is κ-compact. For the second claim, suppose F preserves κ-compact objects, and $p: K \to \mathcal{D}$ is a κ-filtered diagram; we wish to prove that the natural map $\operatorname{colim} G \circ p \to G(\operatorname{colim} p)$ is an equivalence. Since $\mathcal{C}$ is κ-accessible, to prove this it suffices to show that the induced map

$$\mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} G \circ p) \to \mathrm{Map}_{\mathcal{C}}\big(X, G(\operatorname{colim} p)\big)$$

is an equivalence for all κ-compact objects $X \in \mathcal{C}$. But when X is κ-compact, we have equivalences

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}\big(X, G(\operatorname{colim} p)\big) &\simeq \mathrm{Map}_{\mathcal{D}}\big(F(X), \operatorname{colim} p\big) \simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}\big(F(X), p\big) \\ &\simeq \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, G \circ p) \simeq \mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} G \circ p),\end{aligned}$$

so this is true. □

Lemma 3.3.6. *Let $\mathcal{C}$ be an ∞-category and let C be an object of $\mathcal{C}$. Then the forgetful functor $F: \mathcal{C}_{/C} \to \mathcal{C}$ reflects colimits, i.e. a diagram $\bar{p}: K^{\triangleright} \to \mathcal{C}_{/C}$ is a colimit diagram if the composite $F \circ \bar{p}: K^{\triangleright} \to \mathcal{C}$ is a colimit diagram. Moreover, if $\mathcal{C}$ has finite products, then F creates colimits, i.e. $\bar{p}$ is a colimit diagram if and only if $F \circ \bar{p}$ is a colimit diagram.*

Proof. Write C' for $\bar{p}(\infty)$ and p for $\bar{p}|_K$. For any map $f: D \to C$ we have a commutative square

$$\begin{array}{ccc}\lim_{x \in K} \mathrm{Map}_{\mathcal{C}}(p(x), D) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C', D) \\ \downarrow & & \downarrow \\ \lim_{x \in K} \mathrm{Map}_{\mathcal{C}}(p(x), C) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(C', C).\end{array}$$

If $F \circ \bar{p}$ is a colimit diagram in $\mathcal{C}$ then the horizontal morphisms in this square are both equivalences, hence so are all induced maps on fibres. But for any object $g: X \to C$ in $\mathcal{C}_{/C}$ the space $\mathrm{Map}_{\mathcal{C}_{/C}}(X, D)$ is the pullback

$$\begin{array}{ccc}\mathrm{Map}_{\mathcal{C}_{/C}}(X, D) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, D) \\ \downarrow & & \downarrow \\ \{g\} & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, C),\end{array}$$

and so since limits commute one map on fibres is

$$\lim_{x\in K} \mathrm{Map}_{\mathcal{C}_{/C}}\big(p(x), D\big) \to \mathrm{Map}_{\mathcal{C}_{/C}}(C', D).$$

Thus this is an equivalence for all $D \to C$ if $F \circ \bar{p}$ is a colimit diagram in $\mathcal{C}$, which shows that $\bar{p}$ is a colimit diagram in $\mathcal{C}_{/C}$ if $F \circ \bar{p}$ is a colimit diagram.

Conversely, suppose $\bar{p}$ is a colimit diagram, so that

$$\lim_{x\in K} \mathrm{Map}_{\mathcal{C}_{/C}}\big(p(x), D\big) \to \mathrm{Map}_{\mathcal{C}_{/C}}(C', D)$$

is an equivalence for all $D \to C$. If $\mathcal{C}$ has finite products, then for any $Y \to C$ in $\mathcal{C}_{/C}$ and any $X \in \mathcal{C}$ we have a natural equivalence

$$\mathrm{Map}_{\mathcal{C}_{/C}}(Y, X \times C) \simeq \mathrm{Map}_{\mathcal{C}}(Y, X)$$

where $X \times C \to C$ is the product projection. Thus, taking D to be $X \times C$ we get by naturality an equivalence

$$\lim_{x\in K} \mathrm{Map}_{\mathcal{C}}\big(p(x), X\big) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(C', X),$$

and thus $F \circ \bar{p}$ is a colimit diagram in $\mathcal{C}$. □

Proposition 3.3.7. *Suppose $\mathcal{C}$ is a κ-accessible ∞-category with finite products such that the Cartesian product preserves κ-filtered colimits separately in each variable. Then an object $X \to C$ is κ-compact in $\mathcal{C}_{/C}$ if and only if X is a κ-compact object of $\mathcal{C}$.*

Proof. The forgetful functor $r_!\colon \mathcal{C}_{/C} \to \mathcal{C}$ creates colimits by Lemma 3.3.6 and admits a right adjoint $r^*\colon \mathcal{C} \to \mathcal{C}_{/C}$ given by sending $X \in \mathcal{C}$ to the projection $X \times C \to C$. By assumption the composite $r_!r^*$, which sends X to $X \times C$, preserves κ-filtered colimits, hence so does r^*. By Lemma 3.3.5 the left adjoint $r_!$ preserves κ-compact objects. Thus if $X \to C$ is κ-compact in $\mathcal{C}_{/C}$, then X is κ-compact in $\mathcal{C}$.

Conversely, suppose $X \to C$ is an object of $\mathcal{C}_{/C}$ such that X is κ-compact in $\mathcal{C}$, and $p\colon K \to \mathcal{C}_{/C}$ is a κ-filtered diagram in $\mathcal{C}_{/C}$. We then have a diagram

$$\begin{array}{ccc}
\operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, r_! \circ p) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, \operatorname{colim} r_! \circ p) \\
\downarrow & & \downarrow \\
\operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X, C) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, C)
\end{array}$$

where the horizontal maps are equivalences. Since κ-filtered colimits commute with κ-small limits in $\mathcal{S}$, hence in particular finite limits, we have a pullback diagram

$$\begin{array}{ccc} \operatorname{colim} \mathrm{Map}_{\mathcal{C}_{/C}}(X,p) & \longrightarrow & \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X,p) \\ \downarrow & & \downarrow \\ \operatorname{colim} * & \longrightarrow & \operatorname{colim} \mathrm{Map}_{\mathcal{C}}(X,C) \end{array}$$

where the obvious map $\operatorname{colim} * \to *$ is an equivalence. Thus the canonical map

$$\operatorname{colim} \mathrm{Map}_{\mathcal{C}_{/C}}(X,p) \to \mathrm{Map}_{\mathcal{C}_{/C}}(X, \operatorname{colim} p)$$

can be identified with the pullback along the inclusion $\{X \to C\} \to \mathrm{Map}_{\mathcal{C}}(X,C)$ of an equivalence and so is itself an equivalence. Hence $X \to C$ is indeed κ-compact in $\mathcal{C}_{/C}$. □

Corollary 3.3.8. *Suppose $\mathcal{C}$ is a κ-accessible ∞-category with finite limits, such that the Cartesian product preserves κ-filtered colimits separately in each variable. Then for every morphism $f: C \to D$ in $\mathcal{C}$ the pullback functor $f^*: \mathcal{C}_{/D} \to \mathcal{C}_{/C}$ preserves κ-filtered colimits.*

Proof. The functor f^* is right adjoint to the functor $f_!: \mathcal{C}_{/C} \to \mathcal{C}_{/D}$ given by composition with f. By Proposition 3.3.7 the functor $f_!$ preserves κ-compact objects, and so by Lemma 3.3.5 the right adjoint f^* preserves κ-filtered colimits. □

Proof of Theorem 3.3.1. We consider first the case of the forgetful functor $\mathrm{Opd}_\infty^{\mathrm{ns}} \to \mathrm{Cat}_\infty$. For any categorical pattern $\mathfrak{P} = (X, S, \{p_\alpha\})$, it follows from the proof of [28, Theorem B.0.20] that the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}}$ is a left Bousfield localization of the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{P}^-}$, where $\mathfrak{P}^-$ be the categorical pattern $(X, S, \emptyset)$. Thus the ∞-category $\mathrm{Opd}_\infty^{\mathrm{ns}}$ is a localization of $\mathrm{CoCart}(\boldsymbol{\Delta}^{\mathrm{op}}, \boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{int}})$, and by Corollary 3.3.4 the forgetful functor $\mathrm{CoCart}(\boldsymbol{\Delta}^{\mathrm{op}}, \boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{int}}) \to (\mathrm{Cat}_\infty)_{/\boldsymbol{\Delta}^{\mathrm{op}}}$ detects filtered colimits. It follows that the colimit of a filtered diagram of ∞-operads is the localization of the colimit of the corresponding diagram in $\mathrm{CoCart}(\boldsymbol{\Delta}^{\mathrm{op}}, \boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{int}})$, and this colimit can be computed in $(\mathrm{Cat}_\infty)_{/\boldsymbol{\Delta}^{\mathrm{op}}}$ or equivalently in Cat_∞, by Lemma 3.3.6. Thus, to show that the forgetful functor from $\mathrm{Opd}_\infty^{\mathrm{ns}}$ to Cat_∞ preserves filtered colimits it suffices to show that the colimit in $(\mathrm{Cat}_\infty)_{/\boldsymbol{\Delta}^{\mathrm{op}}}$ of such a diagram is also an ∞-operad.

Let $p: \mathcal{I} \to \mathrm{Opd}_\infty^{\mathrm{ns}}, \alpha \mapsto \mathcal{O}_\alpha$ be a filtered diagram, and let $\mathcal{O}$ be the colimit in Cat_∞ of the diagram obtained by composing with the forgetful functor. By Proposition 3.3.2 the induced map $\mathcal{O} \to \boldsymbol{\Delta}^{\mathrm{op}}$ has coCartesian arrows over inert morphisms in $\boldsymbol{\Delta}^{\mathrm{op}}$, so it suffices to prove that the two other conditions for being an ∞-operad are satisfied.

Since pullbacks in Cat_∞ preserve filtered colimits by Corollary 3.3.8, and these commute with finite limits in Cat_∞, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{[n]} & \longrightarrow & \operatorname{colim}_\alpha \mathcal{O}_{\alpha,[n]} \\ \downarrow & & \downarrow \\ (\mathcal{O}_{[1]})^{\times n} & \longrightarrow & \operatorname{colim}_\alpha (\mathcal{O}_{\alpha,[1]})^{\times n} \end{array}$$

where all but the left vertical map are known to be equivalences, hence this is also an equivalence.

Now suppose Y is an object of $\mathcal{O}_{[n]}$ and $\eta_i \colon Y \to Y_i$ are coCartesian arrows over the inert maps $\rho_i \colon [n] \to [1]$ in $\mathbf{\Delta}^{\mathrm{op}}$. We must show that for every $X \in \mathcal{O}_{[m]}$ and every map $\phi \colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the morphism

$$\operatorname{Map}^{\phi}_{\mathcal{O}}(X, Y) \to \prod_i \operatorname{Map}^{\rho_i \phi}_{\mathcal{O}}(X, Y_i)$$

is an equivalence. We can choose $\alpha \in \mathcal{I}$ and objects X_α and Y_α in $\mathcal{O}_\alpha$ that map to X and Y; coCartesian morphisms $Y_\alpha \to \rho_{i,!} Y_\alpha$ over ρ_i will then map to η_i. As in the proof of Proposition 3.3.2, since $\mathcal{O}$ is a filtered colimit in Cat_∞ we get a diagram

$$\begin{array}{ccc} \operatorname{Map}^{\phi}_{\mathcal{O}}(X, Y) & \longrightarrow & \prod_i \operatorname{Map}^{\rho_i \phi}_{\mathcal{O}}(X, Y_i) \\ \downarrow & & \downarrow \\ \operatorname*{colim}_{\psi \colon \alpha \to \beta \in \mathcal{I}_{\alpha/}} \operatorname{Map}^{\phi}_{\mathcal{O}_\beta}(\psi_! X_\alpha, \psi_! Y_\alpha) & \longrightarrow & \prod_i \operatorname*{colim}_{\psi \colon \alpha \to \beta \in \mathcal{I}_{\alpha/}} \operatorname{Map}^{\rho_i \phi}_{\mathcal{O}_\beta}(\psi_! X_\alpha, \psi_! \rho_{i,!} Y_\alpha) \end{array}$$

where the vertical maps are equivalences. But since filtered colimits commute with finite limits in $\mathcal{S}$, the bottom horizontal map is also an equivalence, as $\mathcal{O}_\beta$ is an ∞-operad for all β. It follows that the top horizontal map is also an equivalence, which completes the proof that $\mathcal{O}$ is an ∞-operad.

The proof for $\mathrm{Opd}_\infty^{\mathrm{ns,gen}}$ is similar — the only difference is the we replace the finite products with limits over the categories $\mathcal{G}^{\mathbf{\Delta}}_{[n]/}$, which are also finite. □

3.4. *Trivial ∞-operads*

In this subsection we will associate to any non-symmetric ∞-operad $\mathcal{O}$ a *trivial* ∞-operad $\mathcal{O}_{\mathrm{triv}}$ with a map $\mathcal{O}_{\mathrm{triv}} \to \mathcal{O}$, such that for any ∞-operad $\mathcal{P}$ the ∞-category $\mathrm{Alg}_{\mathcal{O}_{\mathrm{triv}}}(\mathcal{P})$ of $\mathcal{O}_{\mathrm{triv}}$-algebras in $\mathcal{P}$ is equivalent to the functor ∞-category $\mathrm{Fun}(\mathcal{O}_{[1]}, \mathcal{P}_{[1]})$; an analogous result also holds for generalized non-symmetric ∞-operads.

Definition 3.4.1. Let $\mathcal{M}$ be a generalized non-symmetric ∞-operad. Define the generalized non-symmetric ∞-operad $\mathcal{M}_{\mathrm{triv}}$ by the pullback diagram

$$\begin{array}{ccc} \mathcal{M}_{\mathrm{triv}} & \xrightarrow{\tau_{\mathcal{M}}} & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}} \end{array}$$

This is the *trivial generalized non-symmetric* ∞*-operad over* $\mathcal{M}$.

Definition 3.4.2. Let $\mathfrak{O}^{\mathrm{triv}}_{\mathrm{ns}}$ denote the categorical pattern

$$\big(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}, \mathrm{N}\big(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}\big)_1, \{(\mathcal{G}^{\mathbf{\Delta}})^{\triangleleft}_{[n]/} \to \mathbf{\Delta}^{\mathrm{op}}\}\big).$$

Remark 3.4.3. An object (X, S) of $(\mathrm{Set}^+_\Delta)_{/(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}, \mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}})_1)}$ is $\mathfrak{O}^{\mathrm{triv}}_{\mathrm{ns}}$-fibrant if $X \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$ is a coCartesian fibration, S is the set of coCartesian edges, and the Segal morphisms $X_{[n]} \to X_{[1]} \times_{X_{[0]}} \cdots \times_{X_{[0]}} X_{[1]}$ are equivalences.

Under the equivalence between coCartesian fibrations and functors the ∞-category associated to the model category $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{triv}}_{\mathrm{ns}}}$ corresponds to the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{triv}}, \mathrm{Cat}_\infty)$ spanned by the functors that are right Kan extensions along the inclusion $\gamma\colon \mathcal{G}^{\mathbf{\Delta}} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$. Thus we have proved the following:

Lemma 3.4.4. *The* ∞*-category associated to the model category* $(\mathrm{Set}^+_\Delta)_{\mathfrak{O}^{\mathrm{triv}}_{\mathrm{ns}}}$ *is equivalent to* $\mathrm{Fun}(\mathcal{G}^{\mathbf{\Delta}}, \mathrm{Cat}_\infty)$*.*

The obvious map of categorical patterns $\mathfrak{O}^{\mathrm{triv}}_{\mathrm{ns}} \to \mathfrak{O}^{\mathrm{gen}}_{\mathrm{ns}}$ then induces an adjoint pair of functors

$$\gamma_!\colon\ \mathrm{Fun}(\mathcal{G}^{\mathbf{\Delta}}, \mathrm{Cat}_\infty) \rightleftarrows \mathrm{Opd}^{\mathrm{ns,gen}}_\infty\colon\ \gamma^*.$$

Since composition with the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} \to \mathbf{\Delta}^{\mathrm{op}}$ takes $\mathfrak{O}^{\mathrm{triv}}_{\mathrm{ns}}$-fibrant objects to $\mathfrak{O}^{\mathrm{gen}}_{\mathrm{ns}}$-fibrant objects, the left adjoint $\gamma_!$ sends a functor $\mathcal{G}^{\mathbf{\Delta}} \to \mathrm{Cat}_\infty$ to its right Kan extension to $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} \to \mathrm{Cat}_\infty$, then to the composite $\mathcal{E} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}} \to \mathbf{\Delta}^{\mathrm{op}}$, where $\mathcal{E} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$ is the associated coCartesian fibration. In particular, if $\mathcal{M}$ is a generalized non-symmetric ∞-operad, then $\mathcal{M}_{\mathrm{triv}}$ is $\gamma_!\gamma^*\mathcal{M}$, and the natural map $\mathcal{M}_{\mathrm{triv}} \to \mathcal{M}$ is the adjunction morphism.

Taking the $(\infty, 2)$-categories associated to the categorical patterns into account, we get the following:

Proposition 3.4.5. *Let* $F\colon \mathcal{G}^{\mathbf{\Delta}} \to \mathrm{Cat}_\infty$ *be a functor, and* $\mathcal{F} \to \mathcal{G}^{\mathbf{\Delta}}$ *the associated coCartesian fibration. If* $\mathcal{M}$ *is a generalized non-symmetric* ∞*-operad let* $\mathcal{M}_{\mathrm{glob}}$ *denote the pullback of* $\mathcal{M}$ *along* $\mathcal{G}^{\mathbf{\Delta}} \to \mathbf{\Delta}^{\mathrm{op}}$*. Then there is a natural equivalence between* $\mathrm{Alg}_{\gamma_! F}(\mathcal{M})$ *and the full subcategory* $\mathrm{Fun}^{\mathrm{coCart}}_{\mathcal{G}^{\mathbf{\Delta}}}(\mathcal{F}, \mathcal{M}_{\mathrm{glob}})$ *of* $\mathrm{Fun}_{\mathcal{G}^{\mathbf{\Delta}}}(\mathcal{F}, \mathcal{M}_{\mathrm{glob}})$ *spanned by functors that preserve coCartesian arrows. In particular, if* $\mathcal{O}$ *is a non-symmetric* ∞*-operad, then* $\mathrm{Alg}_{\gamma_! F}(\mathcal{O}) \simeq \mathrm{Fun}(F([1]), \mathcal{O}_{[1]})$*.*

3.5. Monoid and category objects

We will now observe that if $\mathcal{V}$ is an ∞-category with finite products and $\mathcal{M}$ is a (generalized) non-symmetric ∞-operad, then the $\mathcal{M}$-algebras in the Cartesian monoidal ∞-category $\mathcal{V}^{\times}$ are equivalent to a certain class of functors $\mathcal{M} \to \mathcal{V}$, namely the $\mathcal{M}$-*monoids*.

Definition 3.5.1. Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad and $\mathcal{V}$ an ∞-category with finite products. An $\mathcal{M}$-*monoid object* in $\mathcal{V}$ is a functor $F\colon \mathcal{M} \to \mathcal{V}$ such that its restriction $F|_{\mathcal{M}_{\mathrm{triv}}}$ is a right Kan extension of $F|_{\mathcal{M}_{[1]}}$ along the inclusion $\mathcal{M}_{[1]} \hookrightarrow \mathcal{M}_{\mathrm{triv}}$. Write $\mathrm{Mon}_{\mathcal{M}}(\mathcal{V})$ for the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{V})$ spanned by the $\mathcal{M}$-monoid objects.

Definition 3.5.2. Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad and $\mathcal{V}$ is an ∞-category with finite limits. An $\mathcal{M}$-*category object* in $\mathcal{V}$ is a functor $F\colon \mathcal{M} \to \mathcal{V}$ such that its restriction $F|_{\mathcal{M}_{\mathrm{triv}}}$ is a right Kan extension of $F|_{\mathcal{M}_{\mathrm{glob}}}$ along the inclusion $\mathcal{M}_{\mathrm{glob}} \hookrightarrow \mathcal{M}_{\mathrm{triv}}$. Write $\mathrm{Cat}_{\mathcal{M}}(\mathcal{V})$ for the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{V})$ spanned by the $\mathcal{M}$-category objects. When $\mathcal{M}$ is $\mathbf{\Delta}^{\mathrm{op}}$ we refer to $\mathbf{\Delta}^{\mathrm{op}}$-category objects as just *category objects*.

Proposition 3.5.3. *Suppose $\mathcal{V}$ is an ∞-category with finite products, and consider $\mathcal{V}$ as a monoidal ∞-category via the pullback of the Cartesian symmetric monoidal structure. Then for any generalized non-symmetric ∞-operad $\mathcal{M}$ we have $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V}) \simeq \mathrm{Mon}_{\mathcal{M}}(\mathcal{V})$.*

Proof. As [28, Proposition 2.4.2.5]. □

Proposition 3.5.4. *We have equivalences $\mathrm{Mon}_{\infty} \simeq \mathrm{Mon}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_{\infty})$ and $\mathrm{Dbl}_{\infty} \simeq \mathrm{Cat}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_{\infty})$.*

Proof. We can identify Mon_{∞} with the full subcategory of the ∞-category of coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ spanned by the monoidal ∞-categories. Under the equivalence between coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ and functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$ these correspond precisely to those functors satisfying the condition for a monoid object. Similarly, the double ∞-categories correspond to the category objects. □

3.6. The algebra fibration

In this subsection we define, given a non-symmetric ∞-operad $\mathcal{O}$, a Cartesian fibration $\mathrm{Alg}(\mathcal{O}) \to \mathrm{Opd}_{\infty}^{\mathrm{ns}}$ with fibre $\mathrm{Alg}_{\mathcal{P}}(\mathcal{O})$ at $\mathcal{P} \in \mathrm{Opd}_{\infty}^{\mathrm{ns}}$ — the objects of $\mathrm{Alg}(\mathcal{O})$ are thus pairs $(\mathcal{P}, A)$ where $\mathcal{P}$ is a non-symmetric ∞-operad and A is a $\mathcal{P}$-algebra in $\mathcal{O}$. We then study the ∞-category $\mathrm{Alg}(\mathcal{V})$ in the special case when $\mathcal{V}$ is a monoidal ∞-category and consider its behaviour as we vary the monoidal ∞-category $\mathcal{V}$.

Definition 3.6.1. Let $\mathcal{O}$ be a non-symmetric ∞-operad. Recall that $(\mathrm{Set}_\Delta^+)^{\mathrm{op}}_{\mathfrak{O}_{\mathrm{ns}}}$ is a marked simplicial model category, so we have a functor

$$(\mathrm{Set}_\Delta^+)^{\mathrm{op}}_{\mathfrak{O}_{\mathrm{ns}}} \to \mathrm{Set}_\Delta^+$$

represented by $\mathcal{O}$. This restricts to a functor between the fibrant objects in these marked simplicial model categories; forgetting from the marked simplicial enrichment down to enrichment in simplicial sets (by forgetting the unmarked 1-simplices) and taking nerves we get a functor

$$(\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}} \to \mathrm{Cat}_\infty;$$

this sends a non-symmetric ∞-operad $\mathcal{P}$ to $\mathrm{Alg}_{\mathcal{P}}(\mathcal{O})$. We define

$$\mathrm{Alg}(\mathcal{O}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$$

to be a Cartesian fibration corresponding to this functor.

Remark 3.6.2. We could also construct $\mathrm{Alg}(\mathcal{O})$ as a full subcategory of the source of a Cartesian fibration associated to the functor $(\mathrm{Cat}_\infty)_{/\mathbf{\Delta}^{\mathrm{op}}} \to \mathrm{Cat}_\infty$ that sends $\mathcal{C} \to \mathbf{\Delta}^{\mathrm{op}}$ to $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{C}, \mathcal{O})$.

Remark 3.6.3. Let $\mathcal{V}$ be an ∞-category with finite products. Then we can similarly define a fibration $\mathrm{Mon}(\mathcal{V}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$ with fibre $\mathrm{Mon}_{\mathcal{O}}(\mathcal{V})$ at $\mathcal{O}$. The proof of [28, Proposition 2.4.1.7] implies that the equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathcal{V})$ is natural in $\mathcal{O}$, which gives an equivalence $\mathrm{Alg}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Mon}(\mathcal{V})$ when $\mathcal{V}$ is considered as a monoidal ∞-category via the Cartesian product.

Definition 3.6.4. For $\mathcal{O}$ a non-symmetric ∞-operad, let

$$\mathrm{Alg}_{\mathrm{triv}}(\mathcal{O}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$$

be the pullback of $\mathrm{Alg}(\mathcal{O})$ along the functor $\gamma_!\gamma^*$ from $\mathrm{Opd}_\infty^{\mathrm{ns}}$ to itself that sends $\mathcal{P}$ to $\mathcal{P}_{\mathrm{triv}}$. The natural maps $\tau_{\mathcal{P}}^*\colon \mathcal{P}_{\mathrm{triv}} \to \mathcal{P}$ then induce a functor

$$\tau^*\colon \mathrm{Alg}(\mathcal{O}) \to \mathrm{Alg}_{\mathrm{triv}}(\mathcal{O}).$$

Remark 3.6.5. The natural equivalence $\mathrm{Alg}_{\mathcal{P}_{\mathrm{triv}}}(\mathcal{V}) \simeq \mathrm{Fun}(\mathcal{P}_{[1]}, \mathcal{V})$ of Proposition 3.4.5 implies that there is a pullback diagram

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{triv}}(\mathcal{V}) & \longrightarrow & \mathcal{F}_{\mathcal{V}} \\ \downarrow & & \downarrow \\ \mathrm{Opd}_\infty^{\mathrm{ns}} & \longrightarrow & \mathrm{Cat}_\infty, \end{array}$$

where the lower horizontal map sends an ∞-operad $\mathcal{O}$ to $\mathcal{O}_{[1]}$, and the right vertical map is a Cartesian fibration associated to the functor $\mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{C}, \mathcal{V})$.

Lemma 3.6.6. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits. Then the projection $\mathrm{Alg}(\mathcal{V}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$ is both Cartesian and coCartesian.*

Proof. By [25, Corollary 5.2.2.5] it suffices to prove that for each map $f : \mathcal{O} \to \mathcal{P}$ in $\mathrm{Opd}_\infty^{\mathrm{ns}}$ the map $f^* : \mathrm{Alg}_{\mathcal{P}}^{\mathrm{ns}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}^{\mathrm{ns}}(\mathcal{V})$ has a left adjoint. This is precisely the content of Theorem A.4.6. □

Lemma 3.6.7. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits. Then the functor τ^* has a left adjoint*

$$\tau_! : \mathrm{Alg}_{\mathrm{triv}}(\mathcal{V}) \to \mathrm{Alg}(\mathcal{V})$$

relative to $\mathrm{Opd}_\infty^{\mathrm{ns}}$.

Proof. By [28, Proposition 7.3.2.6] it suffices to prove that τ^* admits fibrewise left adjoints, which we showed in Theorem A.4.6, and that τ^* preserves Cartesian arrows, which is clear since it is the functor associated to a natural transformation between the corresponding functors to Cat_∞. □

Lemma 3.6.8. *The functor $\mathrm{Alg}_{(-)}(\mathcal{V}) : (\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}} \to \mathrm{Cat}_\infty$ takes colimits in $\mathrm{Opd}_\infty^{\mathrm{ns}}$ to limits.*

Proof. For any categorical pattern $\mathfrak{P}$, the product

$$\mathrm{Set}_\Delta^+ \times (\mathrm{Set}_\Delta^+)_{\mathfrak{P}} \to (\mathrm{Set}_\Delta^+)_{\mathfrak{P}}$$

is a left Quillen bifunctor by [28, Remark B.2.5]. Thus the induced functor of ∞-categories preserves colimits in each variable. In particular, the product

$$\mathrm{Cat}_\infty \times \mathrm{Opd}_\infty^{\mathrm{ns}} \to \mathrm{Opd}_\infty^{\mathrm{ns}}$$

preserves colimits in each variable. Now $\mathrm{Alg}_{(-)}(-)$ is defined as a right adjoint to this, so for any ∞-category $\mathcal{C}$ we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{Cat}_\infty}\big(\mathcal{C}, \mathrm{Alg}_{\mathrm{colim}_\alpha \mathcal{O}_\alpha}(\mathcal{P})\big) &\simeq \mathrm{Map}_{\mathrm{Opd}_\infty^{\mathrm{ns}}}(\mathcal{C} \times \operatorname*{colim}_\alpha \mathcal{O}_\alpha, \mathcal{P}) \\ &\simeq \mathrm{Map}_{\mathrm{Opd}_\infty^{\mathrm{ns}}}\big(\operatorname*{colim}_\alpha(\mathcal{C} \times \mathcal{O}_\alpha), \mathcal{P}\big) \\ &\simeq \lim_\alpha \mathrm{Map}_{\mathrm{Opd}_\infty^{\mathrm{ns}}}(\mathcal{C} \times \mathcal{O}_\alpha, \mathcal{P}) \end{aligned}$$

$$\simeq \lim_{\alpha} \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathrm{Alg}_{\mathcal{O}_\alpha}(\mathcal{P}))$$
$$\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \lim_{\alpha} \mathrm{Alg}_{\mathcal{O}_\alpha}(\mathcal{P})).$$

Thus $\mathrm{Alg}_{\mathrm{colim}\,\mathcal{O}_\alpha}(\mathcal{P}) \simeq \lim_\alpha \mathrm{Alg}_{\mathcal{O}_\alpha}(\mathcal{P})$. □

Proposition 3.6.9. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits. Then $\mathrm{Alg}(\mathcal{V})$ admits small colimits.*

Proof. By Lemma 3.6.6, the fibration $\pi\colon \mathrm{Alg}(\mathcal{V}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$ is coCartesian. Moreover, its fibres have all colimits by Corollary A.5.7 and the functors $f_!$ induced by morphisms f in $\mathrm{Opd}_\infty^{\mathrm{ns}}$ preserve colimits, being left adjoints. Thus π satisfies the conditions of [15, Lemma 9.8]. □

Proposition 3.6.10. *Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal ∞-categories compatible with small colimits. Suppose $F\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ is a monoidal functor such that $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Then $F_*\colon \mathrm{Alg}(\mathcal{V}) \to \mathrm{Alg}(\mathcal{W})$ preserves colimits.*

Proof. Since $\mathcal{V}$ and $\mathcal{W}$ are compatible with small colimits, the projections

$$\mathrm{Alg}(\mathcal{V}), \quad \mathrm{Alg}(\mathcal{W}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$$

are coCartesian fibrations. Thus a diagram in $\mathrm{Alg}(\mathcal{W})$ is a colimit diagram if and only if it is a relative colimit diagram whose projection to $\mathrm{Opd}_\infty^{\mathrm{ns}}$ is a colimit diagram.

It therefore suffices to prove that F_* preserves coCartesian arrows and preserves colimits fibrewise. The former follows from Lemma A.4.7, and the latter from Proposition A.5.10. □

Proposition 3.6.11. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category. Then the ∞-category $\mathrm{Alg}(\mathcal{V})$ is presentable and the projection $\mathrm{Alg}(\mathcal{V}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$ is an accessible functor.*

Proof. This follows from [15, Theorem 9.3] together with Theorem A.4.6, Corollary A.5.7, and Lemma 3.6.8. □

Next we observe that the ∞-category $\mathrm{Alg}(\mathcal{O})$ is functorial in $\mathcal{O}$:

Definition 3.6.12. Since the model category $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$ is enriched in marked simplicial sets, the enriched Yoneda functor

$$\mathfrak{H}\colon (\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}^{\mathrm{op}} \times (\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}} \to \mathrm{Set}_\Delta^+$$

induces a functor of ∞-categories $(\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}} \times \mathrm{Opd}_\infty^{\mathrm{ns}} \to \mathrm{Cat}_\infty$ sending $(\mathcal{O}, \mathcal{P})$ to $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$. Let $\mathrm{Alg}_{\mathrm{co}} \to \mathrm{Opd}_\infty^{\mathrm{ns}} \times (\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}}$ be a Cartesian fibration corresponding to this functor.

The fibre of $\mathrm{Alg}_{\mathrm{co}}$ at $\mathcal{O}$ in the second component is $\mathrm{Alg}(\mathcal{O})$. The composite $\mathrm{Alg}_{\mathrm{co}} \to (\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}}$ with projection to the second factor is then a Cartesian fibration corresponding to a functor $\mathrm{Opd}_\infty^{\mathrm{ns}} \to \mathrm{Cat}_\infty$ that sends $\mathcal{O}$ to $\mathrm{Alg}(\mathcal{O})$. Thus we see that $\mathrm{Alg}(\mathcal{O})$ is functorial in $\mathcal{O}$.

Definition 3.6.13. Let $\mathrm{Alg} \to \mathrm{Opd}_\infty^{\mathrm{ns}}$ be a coCartesian fibration corresponding to the functor $\mathcal{O} \mapsto \mathrm{Alg}(\mathcal{O})$.

Next we show that the algebra fibration is compatible with products of non-symmetric ∞-operads:

Proposition 3.6.14. $\mathrm{Alg}(-)$ *is lax monoidal with respect to the Cartesian product of non-symmetric ∞-operads.*

Proof. The Cartesian product on $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$ gives a symmetric monoidal structure on $(\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}^{\mathrm{op}} \times (\mathrm{Set}_\Delta^+)_{\mathfrak{O}_{\mathrm{ns}}}$ by taking products in both variables. The functor $\mathfrak{H}$ is lax monoidal with respect to this, and so induces an $((\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}} \times \mathrm{Opd}_\infty^{\mathrm{ns}})^{\times}$-monoid in Cat_∞. From this we get a Cartesian fibration $\mathrm{Alg}_{\mathrm{co}}^{\times} \to (((\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}} \times \mathrm{Opd}_\infty^{\mathrm{ns}})^{\times})^{\mathrm{op}}$. Projecting to the second factor gives a Cartesian fibration that corresponds to a monoid $(\mathrm{Opd}_\infty^{\mathrm{ns}})^{\times} \to \mathrm{Cat}_\infty$, and so a lax monoidal functor $(\mathrm{Opd}_\infty^{\mathrm{ns}})^{\times} \to \mathrm{Cat}_\infty^{\times}$. This shows that $\mathrm{Alg}(-)$ is a lax monoidal functor. □

This construction gives an "external product"

$$\boxtimes \colon \mathrm{Alg}(\mathcal{O}) \times \mathrm{Alg}(\mathcal{P}) \to \mathrm{Alg}(\mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{P}).$$

Our next result is that for algebras in monoidal ∞-categories compatible with colimits this preserves colimits in each variable; this requires a preliminary observation:

Lemma 3.6.15. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories compatible with small colimits. Then the external product $\boxtimes$ preserves free algebras, i.e. given non-symmetric ∞-operads $\mathcal{O}$ and $\mathcal{P}$, algebras $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ and $B \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{W})$, and morphisms of non-symmetric ∞-operads $f \colon \mathcal{O} \to \mathcal{Q}$ and $g \colon \mathcal{P} \to \mathcal{R}$, we have $f_! A \boxtimes g_! B \simeq (f \times g)_!(A \boxtimes B)$ in $\mathrm{Alg}_{\mathcal{Q} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{R}}(\mathcal{V} \times \mathcal{W})$.*

Proof. This follows from Lemma A.2.3. □

Proposition 3.6.16. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories compatible with small colimits, and let $\mathcal{O}$ and $\mathcal{P}$ be non-symmetric ∞-operads and $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ an $\mathcal{O}$-algebra. Then*

$$A \boxtimes (-) \colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{W}) \to \mathrm{Alg}_{\mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{P}}(\mathcal{V} \times \mathcal{W})$$

preserves colimits.

Proof. First we consider the case of trivial non-symmetric ∞-operads. Suppose A' is an $\mathcal{O}_{\mathrm{triv}}$-algebra. Then

$$A' \boxtimes -\colon \mathrm{Alg}_{\mathcal{P}_{\mathrm{triv}}}(\mathcal{W}) \to \mathrm{Alg}_{\mathcal{O}_{\mathrm{triv}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{P}_{\mathrm{triv}}}(\mathcal{V} \times \mathcal{W})$$

clearly preserves colimits, since it is equivalent to the functor

$$A'|_{\mathcal{O}_{[1]}} \times -\colon \mathrm{Fun}(\mathcal{P}, \mathcal{W}) \to \mathrm{Fun}(\mathcal{O}_{[1]} \times \mathcal{P}_{[1]}, \mathcal{V} \times \mathcal{W}).$$

Since we have $\tau^*_{\mathcal{V}\times\mathcal{W}}(A \boxtimes B) \simeq \tau^*_{\mathcal{V}} A \boxtimes \tau^*_{\mathcal{W}} B$ and $\tau^*_{\mathcal{V}\times\mathcal{W}}$ detects sifted colimits by Corollary A.5.4, it follows that $A \boxtimes -$ preserves sifted colimits for any A.

Next we consider the case where A is a free algebra $\tau_{\mathcal{V},!}A'$ for some $\mathcal{O}_{\mathrm{triv}}$-algebra A' in $\mathcal{V}$. By Lemma 3.6.15 we have

$$\tau_{\mathcal{V},!}A' \boxtimes \tau_{\mathcal{W},!}B' \simeq \tau_{\mathcal{V}\times\mathcal{W},!}(A' \boxtimes B'),$$

so the functor $\tau_{\mathcal{V},!}A \boxtimes -$ preserves colimits of free algebras. Thus it must preserve all colimits, by monadicity (Corollary A.5.6).

Finally, suppose $A_\bullet$ is a free resolution of A, and $\alpha \mapsto B_\alpha$ is any diagram. Then since $\boxtimes$ preserves sifted colimits we have

$$A \boxtimes \operatorname{colim} B_\alpha \simeq |A_\bullet| \boxtimes \operatorname{colim} B_\alpha \simeq |A_\bullet \boxtimes \operatorname{colim} B_\alpha|.$$

From the case of free algebras we then get that this is equivalent to

$$|\operatorname{colim}(A_\bullet \boxtimes B_\alpha)| \simeq \operatorname{colim} |A_\bullet \boxtimes B_\alpha|.$$

But since $\boxtimes$ preserves sifted colimits in each variable, this is $\operatorname{colim}(|A_\bullet| \boxtimes B_\alpha) \simeq \operatorname{colim}(A \boxtimes B_\alpha)$. □

Remark 3.6.17. The Cartesian product of non-symmetric ∞-operads does not in general preserve colimits, so it is not possible for the external product, considered as a functor $A \boxtimes (-)\colon \mathrm{Alg}(\mathcal{W}) \to \mathrm{Alg}(\mathcal{V} \times \mathcal{W})$ to preserve colimits.

Finally, we observe that the algebra fibration is well-behaved with respect to adjunctions and monoidal localizations:

Proposition 3.6.18. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are presentably monoidal ∞-categories and $F\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ is a monoidal functor such that the underlying functor $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Let $g\colon \mathcal{W} \to \mathcal{V}$ be a right adjoint of $F_{[0]}$. Then there exists a lax monoidal functor $G\colon \mathcal{W}^\otimes \to \mathcal{V}^\otimes$ extending g such that we have an adjunction*

$$F_* : \mathrm{Alg}(\mathcal{V}) \rightleftarrows \mathrm{Alg}(\mathcal{W}) : G_*$$

over $\mathrm{Opd}^{\mathrm{ns}}_\infty$.

Proof. This is immediate from (the dual of) [28, Proposition 7.3.2.6] as its hypotheses are satisfied by Lemma A.4.7 and Proposition A.5.11. □

Corollary 3.6.19. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category and $L\colon \mathcal{V} \to \mathcal{W}$ is an accessible monoidal localization with fully faithful right adjoint $i\colon \mathcal{W} \hookrightarrow \mathcal{V}$. Then we have an adjunction*

$$L_*^{\otimes} : \mathrm{Alg}(\mathcal{V}) \rightleftarrows \mathrm{Alg}(\mathcal{W}) : i_*^{\otimes}$$

over $\mathrm{Opd}_\infty^{\mathrm{ns}}$. *Moreover, $i_*^{\otimes}$ is fully faithful.*

Proof. This follows from combining Proposition 3.6.18 and Lemma A.5.12. □

3.7. Non-symmetric and symmetric ∞-operads

In this subsection we briefly discuss the relation between non-symmetric and symmetric ∞-operads and their algebras. We will use the terminology and notation of [28] for (symmetric) ∞-operads, except that we use superscript Σ's to distinguish the symmetric case from the non-symmetric case discussed so far.

Definition 3.7.1. Let $c\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ be the functor defined as in [28, Construction 4.1.2.5] (this is the same as the functor introduced by Segal in [36]). This takes inert morphisms in $\mathbf{\Delta}^{\mathrm{op}}$ to inert morphisms in $\mathbf{\Gamma}^{\mathrm{op}}$, and moreover induces a morphism of categorical patterns from $\mathfrak{O}_{\mathrm{ns}}$ to the analogous categorical pattern $\mathfrak{O}_{\Sigma}$ for symmetric ∞-operads. Thus c induces adjoint functors

$$c_! : \mathrm{Opd}_\infty^{\mathrm{ns}} \rightleftarrows \mathrm{Opd}_\infty^{\Sigma} : c^*.$$

Moreover, since the induced Quillen functors are enriched in marked simplicial sets, we get equivalences

$$\mathrm{Alg}_{\mathcal{O}}(c^*\mathcal{P}) \simeq \mathrm{Alg}^{\Sigma}_{c_!\mathcal{O}}(\mathcal{P}),$$

where $\mathcal{O}$ is a non-symmetric ∞-operad and $\mathcal{P}$ is a symmetric ∞-operad.

Remark 3.7.2. This Quillen adjunction is a special case of the Quillen adjunction induced by a morphism of operator categories defined in [5, Proposition 8.18].

Proposition 3.7.3.

(i) *The symmetric ∞-operad $c_!\mathbf{\Delta}^{\mathrm{op}}$ is equivalent to the symmetric ∞-operad $\mathbb{E}_1 \simeq \mathrm{Ass}$ of [28, Definition 4.1.1.3].*

(ii) The ∞*-category* Mon_∞ *of monoidal* ∞*-categories is equivalent to the* ∞*-category* $\mathrm{Mon}_\infty^{\Sigma,\mathbb{E}_1}$ *of* $\mathbb{E}_1$*-monoidal* ∞*-categories.*

(iii) The ∞*-category* $\mathrm{Mon}_\infty^{\Sigma,\mathbb{E}_n}$ *of* $\mathbb{E}_n$*-monoidal (or* n*-tuply monoidal)* ∞*-categories is equivalent to the* ∞*-category* $\mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}(\mathrm{Mon}_\infty)$ *of* $\mathbb{E}_{n-1}$*-algebras in monoidal* ∞*-categories.*

Proof.

(i) This follows from [28, Proposition 4.1.2.15].

(ii) We have an equivalence

$$\begin{aligned}\mathrm{Mon}_\infty &\simeq \mathrm{Mon}_{\boldsymbol{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty) \simeq \mathrm{Alg}_{\boldsymbol{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty) \simeq \mathrm{Alg}^{\Sigma}_{c_!\boldsymbol{\Delta}^{\mathrm{op}}}(\mathrm{Cat}_\infty)\\ &\simeq \mathrm{Mon}^{\Sigma}_{\mathbb{E}_1}(\mathrm{Cat}_\infty) \simeq \mathrm{Mon}_\infty^{\Sigma,\mathbb{E}_1}.\end{aligned}$$

(iii) Since $\mathbb{E}_n \simeq \mathbb{E}_{n-1} \otimes \mathbb{E}_1$, using the equivalences from (ii) we get an equivalence

$$\mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}(\mathrm{Mon}_\infty) \simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\left(\mathrm{Alg}^{\Sigma}_{\mathbb{E}_1}(\mathrm{Cat}_\infty)\right) \simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathrm{Cat}_\infty) \simeq \mathrm{Mon}_\infty^{\Sigma,\mathbb{E}_n}. \qquad \square$$

Remark 3.7.4. In fact, though we do not need it here, the functor $c_!$ induces an equivalence $\mathrm{Opd}_\infty^{\mathrm{ns}} \simeq (\mathrm{Opd}_\infty^{\Sigma})_{/\mathbb{E}_1}$ — this is [28, Proposition 4.7.1.1].

Remark 3.7.5. By Proposition 3.7.3, the ∞-category $\mathrm{Mon}_\infty^{\mathrm{Pr}}$ of presentably monoidal ∞-categories is equivalent to the ∞-category $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Pres}_\infty)$ of $\mathbb{E}_1$-algebras in Pres_∞. Using [28, Proposition 3.2.4.3] we therefore see that the tensor product on Pres_∞ induces a symmetric monoidal structure on $\mathrm{Mon}_\infty^{\mathrm{Pr}}$. The unit for this tensor product is given by the unique presentably monoidal structure on the unit $\mathcal{S}$, namely the Cartesian monoidal structure.

On the ∞-operads corresponding to ordinary multicategories, the functor $c_!$ corresponds to the usual symmetrization, i.e. it adds free actions by the symmetric groups:

Definition 3.7.6. Let $\mathbf{M}$ be a multicategory. The *symmetrization* $\mathrm{Sym}(\mathbf{M})$ is the symmetric multicategory with objects those of $\mathbf{M}$, and multimorphism sets

$$\mathrm{Sym}(\mathbf{M})(X_1,\ldots,X_n;Y) = \coprod_{\sigma\in\Sigma_n} \mathbf{M}(X_{\sigma(1)},\ldots,X_{\sigma(n)};Y);$$

composition in $\mathrm{Sym}(\mathbf{M})$ is defined using the usual maps $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. The units in Σ_n give an obvious map $\mu\colon \mathbf{M}^{\otimes} \to \mathrm{Sym}(\mathbf{M})^{\otimes}$.

Proposition 3.7.7. *Let* $\mathbf{M}$ *be a multicategory. The map* $\mu\colon \mathbf{M}^{\otimes} \to \mathrm{Sym}(\mathbf{M})^{\otimes}$ *over* $\boldsymbol{\Gamma}^{\mathrm{op}}$ *is an approximation of symmetric* ∞*-operads (cf. [28, Definition 2.3.3.6]).*

Proof. This follows by a variant of the argument in the proof of [28, Proposition 4.1.2.10]. □

Corollary 3.7.8. *The map* $\mathbf{M}^{\otimes} \to \mathrm{Sym}(\mathbf{M})^{\otimes}$ *induces an equivalence of symmetric* ∞*-operads*

$$c_!\mathbf{M}^{\otimes} \xrightarrow{\sim} \mathrm{Sym}(\mathbf{M})^{\otimes}.$$

In particular, if $\mathcal{O}$ *is any symmetric* ∞*-operad we have a natural equivalence*

$$\mathrm{Alg}_{\mathbf{M}}(c^*\mathcal{O}) \simeq \mathrm{Alg}^{\Sigma}_{\mathrm{Sym}(\mathbf{M})}(\mathcal{O}).$$

4. Categorical algebras

Our main goal in this section is to define the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ of categorical algebras in a monoidal ∞-category $\mathcal{V}$ and prove that this has various good properties. First, in §4.1, we carefully define the double ∞-categories $\mathbf{\Delta}^{\mathrm{op}}_S$ for S a space, and make some observations about the functor $S \mapsto \mathbf{\Delta}^{\mathrm{op}}_S$. Next, in §4.2, we identify the non-symmetric ∞-operad associated to $\mathbf{\Delta}^{\mathrm{op}}_S$ as one arising from a certain simplicial multicategory; this allows us to prove a crucial property of the double ∞-categories $\mathbf{\Delta}^{\mathrm{op}}_S$. We are then ready, in §4.3, to use the algebra fibration from §3.6 to construct the ∞-categories $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ and study these; in particular, we will prove that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is a lax monoidal functor of $\mathcal{V}$, and that it is presentable if $\mathcal{V}$ is presentable and equipped with a colimit-preserving monoidal product. In §4.4 we then prove that categorical algebras in spaces are equivalent to Segal spaces, which will prove useful in the next section as it allows us to reduce several proofs to the known case of Segal spaces. Finally, in §4.5 we show that categorical algebras are equivalent to an alternative model for enriched ∞-categories as certain presheaves.

4.1. The double ∞*-categories* $\mathbf{\Delta}^{\mathrm{op}}_S$

We begin with an abstract definition of double ∞-categories $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}$, where $\mathcal{C}$ is any ∞-category:

Definition 4.1.1. Let i denote the inclusion $\{[0]\} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$. Taking right Kan extensions along i gives a functor $i_*\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty)$. If $\mathcal{C}$ is an ∞-category, we write $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$ for a coCartesian fibration corresponding to the functor $i_*\mathcal{C}$.

Remark 4.1.2. If $\mathcal{C}$ is an ∞-category, then $i_*\mathcal{C}$ is the simplicial ∞-category with nth space $\mathcal{C}^{\times n+1}$, face maps given by the appropriate projections, and degeneracies by the appropriate diagonal maps.

Lemma 4.1.3. *Let $\mathcal{C}$ be an ∞-category. The coCartesian fibration $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$ is a double ∞-category.*

Proof. It is clear that $i_*\mathcal{C}$ is a category object, hence $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}$ is a double ∞-category by Proposition 3.5.4. □

Remark 4.1.4. We can also give a more explicit description of the simplicial sets $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}$, as follows: Consider the forgetful functor $\mathbf{\Delta} \to \mathrm{Set}$ that sends $[n]$ to the set $\{0, \ldots, n\}$, and let $\mathbf{P} \to \mathbf{\Delta}^{\mathrm{op}}$ be an associated Grothendieck fibration. Then define $E_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$ to be the simplicial set satisfying the universal property

$$\mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}(K, E_{\mathcal{C}}) \cong \mathrm{Hom}(\mathbf{P} \times_{\mathbf{\Delta}^{\mathrm{op}}} K, \mathcal{C})$$

The map $E_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$ is a coCartesian fibration by [25, Proposition 3.2.2.13], and the corresponding functor is that sending $[n]$ to $\mathrm{Fun}(\mathbf{P}_{[n]}, \mathcal{C}) \simeq \mathcal{C}^{\times(n+1)}$ by [15, Proposition 7.3]. Thus the fibration $E_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$ is the same as the coCartesian fibration $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}} \to \mathbf{\Delta}^{\mathrm{op}}$.

Remark 4.1.5. The functor

$$\mathbf{\Delta}^{\mathrm{op}}_{(-)} \colon \mathrm{Cat}_\infty \to \mathrm{Opd}^{\mathrm{ns,gen}}_\infty$$

is a right adjoint to the functor $\mathrm{Opd}^{\mathrm{ns,gen}}_\infty \to \mathrm{Cat}_\infty$ that sends a generalized non-symmetric ∞-operad $\mathcal{M}$ to its fibre $\mathcal{M}_{[0]}$ at $[0]$: it is a composite of the right Kan extension functor $i_* \colon \mathrm{Cat}_\infty \to \mathrm{Dbl}_\infty$, which is right adjoint to the fibre-at-$[0]$ functor, and the inclusion $\mathrm{Dbl}_\infty \hookrightarrow \mathrm{Opd}^{\mathrm{ns,gen}}_\infty$, right adjoint to the monoidal envelope functor, which preserves fibres at $[0]$ (cf. §A.1).

Remark 4.1.6. It follows from Remark 4.1.5 that the functor $\mathbf{\Delta}^{\mathrm{op}}_{(-)} \colon \mathrm{Cat}_\infty \to \mathrm{Opd}^{\mathrm{ns,gen}}_\infty$ is fully faithful, since using the adjunction we have

$$\mathrm{Map}\big(\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}, \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{D}}\big) \simeq \mathrm{Map}\big(\big(\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}\big)_{[0]}, \mathcal{D}\big) \simeq \mathrm{Map}(\mathcal{C}, \mathcal{D}).$$

Proposition 4.1.7. *The functor $\mathbf{\Delta}^{\mathrm{op}}_{(-)} \colon \mathrm{Cat}_\infty \to \mathrm{Opd}^{\mathrm{ns,gen}}_\infty$ preserves filtered colimits.*

Proof. Suppose we have a filtered diagram of ∞-categories $p \colon \mathcal{I} \to \mathrm{Cat}_\infty$ with colimit $\mathcal{C}$. Since $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}$ is a generalized non-symmetric ∞-operad, by Theorem 3.3.1 it suffices to show that $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{C}}$ is the colimit of $\mathbf{\Delta}^{\mathrm{op}}_{p(-)}$ in Cat_∞. Now this composite functor

$$\mathrm{Cat}_\infty \xrightarrow{\mathbf{\Delta}^{\mathrm{op}}_{(-)}} \mathrm{Opd}^{\mathrm{ns,gen}}_\infty \to \mathrm{Cat}_\infty$$

factors as

$$\mathrm{Cat}_\infty \xrightarrow{i_*} \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{CoCart}(\mathbf{\Delta}^{\mathrm{op}}) \xrightarrow{q} \mathrm{Cat}_\infty,$$

where $\mathrm{CoCart}(\mathbf{\Delta}^{\mathrm{op}})$ is the ∞-category of coCartesian fibrations over $\mathbf{\Delta}^{\mathrm{op}}$ and the rightmost functor q is the forgetful functor that sends a fibration $\mathcal{E} \to \mathbf{\Delta}^{\mathrm{op}}$ to the ∞-category $\mathcal{E}$. The functor q preserves filtered colimits by Corollary 3.3.4, so it suffices to prove that i_* preserves them. Colimits in functor categories are computed pointwise, so to see this it suffices to show that for each $[n]$ the composite functor $\mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ induced by composing with evaluation at $[n]$ preserves filtered colimits. This functor sends $\mathcal{D}$ to the product $\mathcal{D}^{\times(n+1)}$, and so preserves filtered (and even sifted) colimits by [25, Proposition 5.5.8.6], since the Cartesian product of ∞-categories preserves colimits separately in each variable. $\square$

4.2. The ∞-operad associated to $\mathbf{\Delta}^{\mathrm{op}}_S$

By Corollary 3.2.14 there is a universal non-symmetric ∞-operad $L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_S$ receiving a map from the double ∞-category $\mathbf{\Delta}^{\mathrm{op}}_S$. In this subsection we describe a concrete model for $L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_S$ as the ∞-operad associated to a simplicial multicategory. We will use this below in §5.3 to see that our theory of enriched ∞-categories is equivalent to the definition sketched in §2.2, and it will also allow us to conclude that the functor that sends S to $L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_S$ preserves products.

Remark 4.2.1. Although it is obvious that the functor $\mathbf{\Delta}^{\mathrm{op}}_{(-)}$ preserves products, since it's a right adjoint by Remark 4.1.5, it is not clear that the localization functor

$$L_{\mathrm{gen}}\colon \mathrm{Opd}_\infty^{\mathrm{ns,gen}} \to \mathrm{Opd}_\infty^{\mathrm{ns}}$$

preserves products — in fact, this may well be false in general.

First we define simplicial categories $\mathcal{D}(\mathcal{C})$ that model $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ when $\mathcal{C}$ is a simplicial category:

Definition 4.2.2. Given a simplicial category $\mathcal{C}$, the simplicial category $\mathcal{D}(\mathcal{C})$ has objects finite sequences $(c_0, \ldots, c_n)$ of objects of $\mathcal{C}$; morphisms are given by

$$\mathcal{D}(\mathcal{C})\big((c_0, \ldots, c_n), (d_0, \ldots, d_m)\big) := \coprod_{\phi\colon [m]\to[n]} \prod_{i=0}^{m} \mathcal{C}(c_{\phi(i)}, d_i),$$

with the obvious composition maps induced by those in $\mathcal{C}$.

Proposition 4.2.3. *Suppose $\mathcal{C}$ is a fibrant simplicial category. Then:*

- *(i) The projection $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathrm{N}\mathbf{\Delta}^{\mathrm{op}}$ is a coCartesian fibration.*
- *(ii) The fibre $\mathrm{N}\mathcal{D}(\mathcal{C})_{[0]}$ is equivalent to $\mathrm{N}\mathcal{C}$.*
- *(iii) There is a natural map $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$, and this preserves coCartesian edges.*
- *(iv) This map is an equivalence of ∞-categories.*

Proof.

(i) It is clear that $\mathcal{D}(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ is a fibration in the model structure on simplicial categories; since N is a right Quillen functor, it follows that $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathrm{N}\mathbf{\Delta}^{\mathrm{op}}$ is a categorical fibration. It therefore suffices to check that $\mathrm{N}\mathcal{D}(\mathcal{C})$ has coCartesian morphisms. Given an object $C = (c_0, \ldots, c_n)$ in $\mathcal{D}(\mathcal{C})$ and a map $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}$, let $\bar{\phi}$ denote the obvious map $C \to C' = (c_{\phi(0)}, \ldots, c_{\phi(m)})$ in $\mathcal{D}(\mathcal{C})$. We apply the criterion of [25, Proposition 2.4.1.10] to see that $\bar{\phi}$ is coCartesian in $\mathrm{N}\mathcal{D}(\mathcal{C})$; thus we need to show that for every $X \in \mathcal{D}(\mathcal{C})$ over $[k] \in \mathbf{\Delta}^{\mathrm{op}}$ the commutative diagram

$$\begin{array}{ccc} \mathcal{D}(\mathcal{C})(C', X) & \longrightarrow & \mathcal{D}(\mathcal{C})(C, X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}([m], [k]) & \longrightarrow & \mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}([n], [k]) \end{array}$$

is a homotopy Cartesian square of simplicial sets. Since the simplicial category $\mathcal{C}$ is fibrant, so is $\mathcal{D}(\mathcal{C})$, hence the vertical maps are Kan fibrations. It therefore suffices to show that the induced maps on fibres are weak equivalences, which is clear from the definition of $\mathcal{D}(\mathcal{C})$.

(ii) We have a pullback diagram of simplicial categories

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \{[0]\} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}}. \end{array}$$

Since the simplicial nerve is a right adjoint, it follows that $\mathrm{N}\mathcal{C}$ is the fibre of the map of simplicial sets $\mathrm{N}\mathcal{D}(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}$ at $[0]$. Since this map is a coCartesian fibration, by [25, Corollary 3.3.1.4] $\mathrm{N}\mathcal{C}$ is also the homotopy fibre in the Joyal model structure.

(iii) By definition $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ corresponds to the right Kan extension $i_*\mathrm{N}\mathcal{C}$ of $\mathrm{N}\mathcal{C}$ along the inclusion $i\colon \{[0]\} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$. The functor i_* is right adjoint to the fibre-at-$[0]$ functor i^*, and from (ii) we know that $i^*\mathrm{N}\mathcal{D}(\mathcal{C}) \simeq \mathrm{N}\mathcal{C}$. The adjunction $i^* \dashv i_*$ then gives the required map $\mathcal{D}(\mathcal{C}) \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ (which preserves coCartesian edges since by definition i_* lands in the ∞-category of coCartesian fibrations and coCartesian-morphism-preserving functors).

(iv) By [25, Corollary 2.4.4.4] it suffices to show that for each $[n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ the induced map on fibres

$$\big(\mathrm{N}\mathcal{D}(\mathcal{C})\big)_{[n]} \to \big(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}\big)_{[n]}$$

is a categorical equivalence. As in (ii) we can identify the fibre $(\mathrm{N}\mathcal{D}(\mathcal{C}))_{[n]}$ with $\mathrm{N}\mathcal{C}^{\times n}$, via the Segal maps, so by naturality we have a commutative diagram

$$\begin{array}{ccc} (\mathrm{N}\mathcal{D}(\mathcal{C}))_{[n]} & \longrightarrow & (\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}})_{[n]} \\ \downarrow & & \downarrow \\ \mathrm{N}\mathcal{C}^{\times n} & \longrightarrow & \mathrm{N}\mathcal{C}^{\times n}, \end{array}$$

where all but the top horizontal map are known to be categorical equivalences. Hence this must also be a categorical equivalence, by the 2-out-of-3 property. □

Definition 4.2.4. Let $\mathcal{C}$ be a simplicial category. The simplicial multicategory $\mathcal{O}_{\mathcal{C}}$ has objects $\mathrm{ob}\,\mathcal{C} \times \mathrm{ob}\,\mathcal{C}$ and multimorphism spaces defined by

$$\begin{aligned} &\mathcal{O}_{\mathcal{C}}\big((x_0, y_1), \ldots, (x_{n-1}, y_n); (y_0, x_n)\big) \\ &\quad := \mathcal{C}(y_0, x_0) \times \mathcal{C}(y_1, x_1) \times \cdots \times \mathcal{C}(y_{n-1}, x_{n-1}) \times \mathcal{C}(y_n, x_n). \end{aligned}$$

Composition is defined in the obvious way, using composition in $\mathcal{C}$. Write $\mathcal{O}_{\mathcal{C}}^{\otimes}$ for the associated simplicial category of operators over $\mathbf{\Delta}^{\mathrm{op}}$.

If $\mathcal{C}$ is a fibrant simplicial category then $\mathcal{O}_{\mathcal{C}}$ is a fibrant simplicial multicategory in the sense of Definition 3.1.6, and so $\mathrm{N}\mathcal{O}_{\mathcal{C}}^{\otimes}$ is a non-symmetric ∞-operad by Lemma 3.1.8.

Remark 4.2.5. If S is a set (regarded as a category with no non-identity morphisms), then the multicategory $\mathcal{O}_S$ is clearly the same as $\mathbf{O}_S$ as defined in §2.1.

The simplicial multicategory $\mathcal{O}_{\mathcal{C}}$ is only a model for $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{N}\mathcal{C}}$ when $\mathrm{N}\mathcal{C}$ is a space, but is easier to define than the version that works more generally. Indeed there is not even a natural map from $\mathcal{D}(\mathcal{C})$ to $\mathcal{O}_{\mathcal{C}}^{\otimes}$ in general; however, we can construct one if we restrict ourselves to simplicial *groupoids*.

A simplicial category can be viewed as a simplicial object in categories whose simplicial set of objects is constant, so by analogy we take a *simplicial groupoid* to be a simplicial object in groupoids with constant set of objects. There is a model structure on simplicial groupoids, due to Dwyer and Kan [14, Theorem 2.5], where the weak equivalences are the usual Dwyer–Kan equivalences of simplicial categories, restricted to groupoids. The simplicial nerve functor restricts to a right Quillen equivalence from this to the usual model structure on simplicial sets by [14, Theorem 3.3]. In particular, it follows that every space is modelled by a fibrant object in simplicial groupoids, which is a simplicial groupoid whose mapping spaces are Kan complexes.

Since a simplicial category can be viewed as a simplicial object in categories with constant set of objects, a simplicial groupoid $\mathcal{G}$ can also be regarded as a simplicial category with an involution $i\colon \mathcal{G} \to \mathcal{G}^{\mathrm{op}}$ such that $i^{\mathrm{op}} \circ i = \mathrm{id}_{\mathcal{G}}$, which sends a morphism to its inverse. Using this we can construct a functor $\mathcal{D}(\mathcal{G}) \to \mathcal{O}_{\mathcal{G}}^{\otimes}$:

Definition 4.2.6. Suppose $\mathcal{G}$ is a simplicial groupoid. Let $\Phi\colon \mathcal{D}_{\mathcal{G}} \to \mathcal{O}_{\mathcal{G}}^{\otimes}$ be the functor that sends an object $(c_0, \ldots, c_n)$ of $\mathcal{D}(\mathcal{G})$ to $((c_0, c_1), (c_1, c_2), \ldots, (c_{n-1}, c_n))$ and is given on morphisms by applying i on the first factor and inserting identities into the factors that are missing in $\mathcal{D}(\mathcal{G})$ in the obvious way.

Theorem 4.2.7. *Let $\mathcal{G}$ be a fibrant simplicial groupoid. Then the map*

$$\mathrm{N}\Phi\colon \mathrm{N}\mathcal{D}(\mathcal{G}) \to \mathrm{N}\mathcal{O}_{\mathcal{G}}^{\otimes}$$

exhibits $\mathrm{N}\mathcal{O}_{\mathcal{G}}^{\otimes}$ as the operadic localization $L_{\mathrm{gen}}\mathrm{N}\mathcal{D}(\mathcal{G})$ of $\mathrm{N}\mathcal{D}(\mathcal{G})$.

Proof. By Corollary A.6.9 it suffices to show that for all $(x, y) \in \mathcal{G} \times \mathcal{G}$ the induced map

$$g\colon \left(\mathrm{N}\mathcal{D}(\mathcal{G})_{\mathrm{act}}\right)_{/(x,y)} \to \left(\mathrm{N}\left(\mathcal{O}_{\mathcal{G}}^{\otimes}\right)_{\mathrm{act}}\right)_{/(x,y)}$$

is cofinal. We will prove that g is a categorical equivalence; to see this we show that g is essentially surjective and induces equivalences on mapping spaces.

We first observe that g is essentially surjective: an active morphism to (x, y) in $\mathcal{O}_{\mathcal{G}}^{\otimes}$ is determined by an object $T = ((t_0, s_1), (t_1, s_2), \ldots, (t_{n-1}, s_n))$ and morphisms $\alpha\colon x \to t_0$, $\beta_1\colon s_1 \to t_1, \ldots, \beta_{n-1}\colon s_{n-1} \to t_{n-1}$, $\gamma\colon s_n \to y$ in $\mathcal{G}$. Such a morphism is in the image of g if and only if the β_i's are all identities. Since $\mathcal{G}$ is by assumption a simplicial groupoid all morphisms in $\mathcal{G}$ are equivalences, and so the morphism

$$\big((t_0, s_1), (s_1, s_2), \ldots, (s_{n-1}, s_n)\big) \to \big((t_0, s_1), (t_1, s_2), \ldots, (t_{n-1}, s_n)\big)$$

given by $(\mathrm{id}, \mathrm{id}, \beta_1, \mathrm{id}, \beta_2, \ldots, \mathrm{id})$ is an equivalence from an object in the image of g to T.

It remains to show that g is fully faithful. Given objects $Z = (z_0, \ldots, z_n)$ and $Z' = (z'_0, \ldots, z'_m)$ in $\mathcal{D}(\mathcal{G})$ we must show that for each active map $\phi\colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ the map

$$\mathrm{Map}^{\phi}_{\mathrm{N}\mathcal{D}(\mathcal{G})_{/(x,y)}}(Z, Z') \to \mathrm{Map}^{\phi}_{(\mathrm{N}\mathcal{O}_{\mathcal{G}}^{\otimes})_{/(x,y)}}\big(g(Z), g(Z')\big)$$

is an equivalence, where the superscripts denote the fibres over ϕ in $\mathbf{\Delta}^{\mathrm{op}}$. Let α be the unique active map $[1] \to [n]$ in $\mathbf{\Delta}$; then we can identify this as a map of homotopy fibres from the commutative square

$$\begin{array}{ccc} \mathcal{D}(\mathcal{G})^{\phi}(Z, Z') & \longrightarrow & \mathcal{D}(\mathcal{G})^{\alpha}(Z, (x, y)) \\ \downarrow & & \downarrow \\ (\mathcal{O}_{\mathcal{G}}^{\otimes})^{\phi}(g(Z), g(Z')) & \longrightarrow & (\mathcal{O}_{\mathcal{G}}^{\otimes})^{\alpha}(g(Z), (x, y)), \end{array}$$

where the superscripts again denote the fibres of these spaces over maps in $\mathbf{\Delta}^{\mathrm{op}}$. To see that our map of homotopy fibres is an equivalence it suffices to show that this diagram is homotopy Cartesian.

We have equivalences

$$\begin{aligned}
\mathcal{D}(\mathcal{G})^{\phi}(Z, Z') &\simeq \prod_{i=0}^{m} \mathcal{G}(z_{\phi(i)}, z'_i), \\
\mathcal{D}(\mathcal{G})^{\alpha}\big(Z, (x, y)\big) &\simeq \mathcal{G}(z_0, x) \times \mathcal{G}(z_n, y), \\
(\mathcal{O}_{\mathcal{G}}^{\otimes})^{\phi}\big(g(Z), g(Z')\big) &\simeq \mathcal{G}(z'_0, z_{\phi(0)}) \times \mathcal{G}(z_{\phi(0)+1}, z_{\phi(0)+1}) \times \cdots \times \mathcal{G}(z_{\phi(1)-1}, z_{\phi(1)-1}) \\
&\quad \times \mathcal{G}(z_{\phi(1)}, z'_1) \times \mathcal{G}(z'_1, z_{\phi(1)}) \times \cdots \times \mathcal{G}(z_{\phi(m)}, z'_m), \\
(\mathcal{O}_{\mathcal{G}}^{\otimes})^{\alpha}\big(g(Z), (x, y)\big) &\simeq \mathcal{G}(x, z_0) \times \mathcal{G}(z_1, z_1) \times \cdots \times \mathcal{G}(z_{n-1}, z_{n-1}) \times \mathcal{G}(z_n, y).
\end{aligned}$$

Under these equivalences our commutative square is the product of the squares

$$\begin{array}{ccc}
* & \longrightarrow & * \\
\downarrow & & \downarrow \\
\mathcal{G}(z_j, z_j) & \longrightarrow & \mathcal{G}(z_j, z_j)
\end{array}$$

for j not in the image of ϕ,

$$\begin{array}{ccc}
\mathcal{G}(z_0, z'_0) \times \mathcal{G}(z_n, z'_m) & \longrightarrow & \mathcal{G}(z_0, x) \times \mathcal{G}(z_n, y) \\
{\scriptstyle (i, \mathrm{id})}\downarrow & & \downarrow{\scriptstyle (i, \mathrm{id})} \\
\mathcal{G}(z'_0, z_0) \times \mathcal{G}(z_n, z'_m) & \longrightarrow & \mathcal{G}(x, z_0) \times \mathcal{G}(z_n, y),
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{G}(z_{\phi(i)}, z'_i) & \longrightarrow & * \\
{\scriptstyle (\mathrm{id}, i)}\downarrow & & \downarrow \\
\mathcal{G}(z_{\phi(i)}, z'_i) \times \mathcal{G}(z'_i, z_{\phi(i)}) & \longrightarrow & \mathcal{G}(z_{\phi(i)}, z_{\phi(i)})
\end{array}$$

for $i = 1, \dots, m-1$. The squares of the first kind are clearly homotopy Cartesian, the second square is homotopy Cartesian since the maps induced by the involution i are equivalences, and the squares of the third kind are homotopy Cartesian since $\mathcal{G}$ is a simplicial groupoid. □

Corollary 4.2.8. *Let X be a space and $\mathcal{X}$ a fibrant simplicial groupoid such that the Kan complex* $\mathrm{N}\mathcal{X}$ *is equivalent to X. Then the composite map*

$$\mathbf{\Delta}_X^{\mathrm{op}} \simeq \mathrm{N}\mathcal{D}(\mathcal{X}) \to \mathrm{N}\mathcal{O}_{\mathcal{X}}^{\otimes}$$

induces an equivalence of non-symmetric ∞-operads $L_{\mathrm{gen}}\mathbf{\Delta}_X^{\mathrm{op}} \xrightarrow{\sim} \mathrm{N}\mathcal{O}_{\mathcal{X}}^{\otimes}$.

Corollary 4.2.9. *The functor* $L_{\mathrm{gen}}(\mathbf{\Delta}^{\mathrm{op}}_{(-)})\colon \mathcal{S} \to \mathrm{Opd}^{\mathrm{ns}}_{\infty}$ *preserves products.*

Proof. Given spaces X and Y, there exist fibrant simplicial groupoids $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathrm{N}\mathcal{X} \simeq X$ and $\mathrm{N}\mathcal{Y} \simeq Y$. Then by Corollary 4.2.8 we have a commutative diagram

$$\begin{array}{ccc} L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{X\times Y} & \longrightarrow & L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{X} \times_{\mathbf{\Delta}^{\mathrm{op}}} L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{Y} \\ \downarrow & & \downarrow \\ \mathrm{N}\mathcal{O}^{\otimes}_{\mathcal{X}\times\mathcal{Y}} & \longrightarrow & \mathrm{N}(\mathcal{O}^{\otimes}_{\mathcal{X}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{O}^{\otimes}_{\mathcal{Y}}) \end{array}$$

where the vertical maps are equivalences. It is clear from the definition that $\mathcal{O}_{\mathcal{X}\times\mathcal{Y}} \simeq \mathcal{O}_{\mathcal{X}} \times \mathcal{O}_{\mathcal{Y}}$, so the natural map $\mathcal{O}^{\otimes}_{\mathcal{X}\times\mathcal{Y}} \to \mathcal{O}^{\otimes}_{\mathcal{X}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{O}^{\otimes}_{\mathcal{Y}}$ is a weak equivalence of fibrant simplicial categories. By the 2-out-of-3 property the top horizontal map in the commutative square is therefore an equivalence of ∞-categories. □

4.3. The ∞-category of categorical algebras

We are now ready to define and study the ∞-categories $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ of categorical algebras:

Definition 4.3.1. Suppose $\mathcal{V}$ is a monoidal ∞-category. The ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is defined by the pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) & \longrightarrow & \mathrm{Alg}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{(-)}]{} & \mathrm{Opd}^{\mathrm{ns}}_{\infty}, \end{array}$$

where the right vertical map is the algebra fibration from §3.6 and the lower horizontal map sends a space S to the non-symmetric ∞-operad $L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{S}$ associated to the generalized non-symmetric ∞-operad $\mathbf{\Delta}^{\mathrm{op}}_{S}$. The objects of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ are thus categorical algebras in $\mathcal{V}$ and its 1-morphisms are $\mathcal{V}$-functors as defined in §2.4. We will refer to $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ as the *∞-category of categorical algebras.*

Remark 4.3.2. Since $\mathcal{V}$ is a monoidal ∞-category, and so in particular a non-symmetric ∞-operad, we could equivalently have defined $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ using the analogue of the algebra fibration over the base $\mathrm{Opd}^{\mathrm{ns,gen}}_{\infty}$, since there is natural equivalence $\mathrm{Alg}_{L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{S}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{S}}(\mathcal{V})$ for every space S.

Pulling back the fibration of trivial algebras in the same way, we get the functor that forms the free $\mathcal{V}$–∞-category on a *graph*:

Definition 4.3.3. Let $\mathcal{V}$ be a monoidal ∞-category. The ∞-category $\mathrm{Graph}_\infty(\mathcal{V})$ of $\mathcal{V}$-*graphs* is defined by the pullback

$$\begin{array}{ccc} \mathrm{Graph}_\infty(\mathcal{V}) & \longrightarrow & \mathrm{Alg}_{\mathrm{triv}}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[L\boldsymbol{\Delta}^{\mathrm{op}}_{(-)}]{} & \mathrm{Opd}_\infty^{\mathrm{ns}}. \end{array}$$

Thus the fibre of $\mathrm{Graph}_\infty(\mathcal{V})$ at $X \in \mathcal{S}$ is $\mathrm{Fun}(X \times X, \mathcal{V})$. By Remark 3.6.5 we also get a pullback square

$$\begin{array}{ccc} \mathrm{Graph}_\infty(\mathcal{V}) & \longrightarrow & \mathcal{F}_{\mathcal{V}} \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[\Delta]{} & \mathcal{S}, \end{array}$$

where Δ is the diagonal functor that sends S to $S \times S$, and $\mathcal{F}_{\mathcal{V}} \to \mathcal{S}$ is the Cartesian fibration associated to the functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ sending S to $\mathrm{Fun}(S, \mathcal{V})$.

Remark 4.3.4. The pullback of the left adjoint $\tau_!$ of τ^* gives a functor

$$F \colon \mathrm{Graph}_\infty(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$$

left adjoint to the forgetful functor $U \colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Graph}_\infty(\mathcal{V})$.

Proposition 4.3.5. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category, i.e. the ∞-category $\mathcal{V}$ is presentable and the tensor product on $\mathcal{V}$ preserves small colimits separately in each variable. Then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is a presentable ∞-category.*

Remark 4.3.6. Proposition 4.3.5 can be seen as an ∞-categorical version of a theorem of Kelly and Lack [22, Theorem 4.5]. The fact that this 1-categorical result is comparatively recent, whereas the ∞-categorical variant is one of the first steps in our setup, underscores the importance of presentability in the ∞-categorical context.

We first observe that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has colimits:

Lemma 4.3.7. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits (i.e. the tensor product on $\mathcal{V}$ preserves colimits separately in each variable). Then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has all small colimits.*

Proof. By Lemma 3.6.6, the fibration $\pi \colon \mathrm{Alg}(\mathcal{V}) \to \mathrm{Opd}_\infty^{\mathrm{ns}}$ is both Cartesian and co-Cartesian, hence the same is true of its pullback $p \colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$. Moreover, the fibres

$\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_X}(\mathcal{V})$ have all colimits by Corollary A.5.7 and the functors $f_!$ induced by morphisms f in $\mathcal{S}$ preserve colimits, being left adjoints. Thus p satisfies the conditions of [15, Lemma 9.8], which implies that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has small colimits. □

Proof of Proposition 4.3.5. The ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has colimits by Lemma 4.3.7, so it remains to prove that it is accessible. But in the pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) & \longrightarrow & \mathrm{Alg}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{(-)}]{} & \mathrm{Opd}^{\mathrm{ns}}_\infty. \end{array}$$

the right vertical morphism is an accessible functor between accessible ∞-categories by Proposition 3.6.11. Moreover, by Proposition 4.1.7 the bottom horizontal morphism preserves filtered colimits (since L_{gen} is a left adjoint and so preserves all colimits), and thus is in particular also an accessible functor. It then follows from [25, Proposition 5.4.6.6] that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is also an accessible ∞-category. □

Our next goal is to prove that the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is a lax monoidal functor in $\mathcal{V}$, with respect to the Cartesian product of monoidal ∞-categories and the Cartesian product of ∞-categories. Knowing this will allow us to conclude, for example, that if $\mathcal{V}$ is a symmetric monoidal ∞-category then there is an induced symmetric monoidal structure on $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. We first observe that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is indeed functorial in $\mathcal{V}$:

Definition 4.3.8. As in §3.6, let $\mathrm{Alg}_{\mathrm{co}} \to \mathrm{Opd}^{\mathrm{ns}}_\infty \times (\widehat{\mathrm{Opd}}^{\mathrm{ns}}_\infty)^{\mathrm{op}}$ be a Cartesian fibration classifying the functor $\mathrm{Alg}_{(-)}(-)$. Then we define $\mathrm{Alg}_{\mathrm{cat,co}}$ by the pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{cat,co}} & \longrightarrow & \mathrm{Alg}_{\mathrm{co}} \\ \downarrow & & \downarrow \\ \mathcal{S} \times (\widehat{\mathrm{Mon}}^{\mathrm{lax}}_\infty)^{\mathrm{op}} & \longrightarrow & \mathrm{Opd}^{\mathrm{ns}}_\infty \times (\widehat{\mathrm{Opd}}^{\mathrm{ns}}_\infty)^{\mathrm{op}}, \end{array}$$

where the bottom horizontal functor is the product of the functor $\mathbf{\Delta}^{\mathrm{op}}_{(-)}$ and the opposite of the inclusion of the full subcategory of large monoidal ∞-categories into $\widehat{\mathrm{Opd}}^{\mathrm{ns}}_\infty$.

Lemma 4.3.9. $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ *is functorial in* $\mathcal{V}$ *with respect to lax monoidal functors.*

Proof. The composite $\mathrm{Alg}_{\mathrm{cat,co}} \to (\widehat{\mathrm{Mon}}^{\mathrm{lax}}_\infty)^{\mathrm{op}}$ is a Cartesian fibration classifying a functor $\mathcal{V}^{\otimes} \mapsto \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. □

Remark 4.3.10. If $\mathbf{V}$ is an ordinary monoidal category, we can identify the usual category of $\mathbf{V}$-enriched categories with the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathbf{V})$ spanned by the $\mathbf{V}$-enriched ∞-categories with sets of objects. In particular, taking $\mathbf{V}$ to be the category Set of sets, with the Cartesian product as monoidal structure, we can identify the usual category Cat of categories with a full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathrm{Set})$. Since the inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$ preserves products, this allows us to consider ordinary categories as $\mathcal{S}$–∞-categories.

Proposition 4.3.11. $\mathrm{Alg}_{\mathrm{cat}}(-)$ *is lax monoidal with respect to the Cartesian product of monoidal ∞-categories.*

Proof. The functor $L_{\mathrm{gen}}\mathbf{\Delta}^{\mathrm{op}}_{(-)}$ is monoidal with respect to the Cartesian products of spaces and non-symmetric ∞-operads, by Corollary 4.2.9. The result therefore follows by the same proof as that of Proposition 3.6.14. □

Corollary 4.3.12. *Let $\mathcal{O}$ be a symmetric ∞-operad and suppose $\mathcal{V}$ is an $\mathcal{O} \otimes \mathbb{E}_1$-monoidal ∞-category. Then* $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ *is an $\mathcal{O}$-monoidal ∞-category.*

Proof. It follows from Proposition 4.3.11 that for any symmetric ∞-operad $\mathcal{O}$, the functor $\mathrm{Alg}_{\mathrm{cat}}(-)$ takes an $\mathcal{O}$-algebra in $\widehat{\mathrm{Mon}}_\infty$ to an $\mathcal{O}$-algebra in $\widehat{\mathrm{Cat}}_\infty$. The inclusion $\widehat{\mathrm{Mon}}_\infty \to \widehat{\mathrm{Mon}}^{\mathrm{lax}}_\infty$ clearly preserves Cartesian products, and by Proposition 3.7.3 we can identify $\widehat{\mathrm{Mon}}_\infty$ with the ∞-category $\mathrm{Alg}^{\Sigma}_{\mathbb{E}_1}(\widehat{\mathrm{Cat}}_\infty)$ of $\mathbb{E}_1$-monoidal ∞-categories. By [28, Remark 2.4.2.6] a large $\mathcal{O} \otimes \mathbb{E}_1$-monoidal ∞-category is the same thing as an $\mathcal{O}$-algebra in $\mathrm{Alg}^{\Sigma}_{\mathbb{E}_1}(\widehat{\mathrm{Cat}}_\infty)$, and so the functor $\mathrm{Alg}_{\mathrm{cat}}(-)$ indeed takes $\mathcal{O}\otimes\mathbb{E}_1$-monoidal ∞-categories to $\mathcal{O}$-monoidal ∞-categories. □

Corollary 4.3.13.

(i) Suppose $\mathcal{V}$ is an $\mathbb{E}_n$-monoidal ∞-category. Then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ *is an $\mathbb{E}_{n-1}$-monoidal ∞-category.*
(ii) Suppose $\mathcal{V}$ is a symmetric monoidal ∞-category. Then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ *is a symmetric monoidal ∞-category.*

Proof. This follows from combining Corollary 4.3.12 with [28, Theorem 5.1.2.2]. □

Our next goal is to show that the functor $\mathrm{Alg}_{\mathrm{cat}}(-)$, when restricted to presentably monoidal ∞-categories, is lax monoidal with respect to the tensor product of presentable ∞-categories. We first observe that restricting in this way does indeed give a functor to presentable ∞-categories:

Proposition 4.3.14. *The restriction of* $\mathrm{Alg}_{\mathrm{cat}}(-)$ *to the ∞-category* $\mathrm{Mon}^{\mathrm{Pr}}_\infty$ *of presentably monoidal ∞-categories factors through the subcategory* Pres_∞ *of* $\widehat{\mathrm{Cat}}_\infty$ *of presentable ∞-categories and colimit-preserving functors.*

Proof. If $\mathcal{V}$ is presentably monoidal, then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is presentable by Proposition 4.3.5. Moreover, it follows by the same proof as that of Proposition 3.6.10 that a monoidal functor $F\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ such that $F_{[1]}$ preserves colimits induces a colimit-preserving functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})$. □

Next we see that when restricted to categorical algebras, the external product $\boxtimes$ of §3.6 preserves colimits in each variable:

Proposition 4.3.15. *Let $\mathcal{V}$ be a monoidal ∞-category, and suppose that $\mathcal{C}$ is a categorical algebra in $\mathcal{V}$. Then $\mathcal{C} \boxtimes -\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V} \times \mathcal{W})$ preserves colimits.*

Proof. Since the Cartesian product of spaces preserves colimits in each variable, it suffices to prove that $\mathcal{C} \boxtimes (-)$ preserves colimits fibrewise and preserves coCartesian arrows. This follows from Lemma 3.6.15 and Proposition 3.6.16. □

Corollary 4.3.16. *The functor $\mathrm{Alg}_{\mathrm{cat}}(-)\colon \mathrm{Mon}_{\infty}^{\mathrm{Pr}} \to \mathrm{Pres}_{\infty}$ is lax monoidal with respect to the tensor product of presentable ∞-categories.*

Proof. We have constructed a lax monoidal functor

$$\mathrm{Alg}_{\mathrm{cat}}(-)\colon (\widehat{\mathrm{Mon}}_{\infty}^{\mathrm{lax}})^{\times} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}.$$

By Proposition 4.3.15 and Proposition 4.3.14, the composite

$$(\mathrm{Mon}_{\infty}^{\mathrm{Pr}})^{\otimes} \to (\widehat{\mathrm{Mon}}_{\infty}^{\mathrm{lax}})^{\times} \to \widehat{\mathrm{Cat}}_{\infty}^{\times}$$

factors through $\mathrm{Pres}_{\infty}^{\otimes}$ as defined in [28, Notation 4.8.1.2]. □

Corollary 4.3.17. *If $\mathcal{V}$ is presentably monoidal, then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$, and the tensoring operation*

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \times \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$$

preserves colimits separately in each variable.

Proof. By Remark 3.7.5, the unit of the tensor product of presentably monoidal ∞-categories is $\mathcal{S}^{\times}$, and so this is a commutative algebra object in the ∞-category $\mathrm{Mon}_{\infty}^{\mathrm{Pr}}$ by [28, Corollary 3.2.1.9]. Any presentably monoidal ∞-category $\mathcal{V}^{\otimes}$ is moreover canonically a module over this commutative algebra object. Since $\mathrm{Alg}_{\mathrm{cat}}(-)$ is lax monoidal with respect to $\otimes$, it follows that the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is a module over the presentably symmetric monoidal ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$ in Pres_{∞}. In other words, the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$ and the tensoring operation preserves colimits separately in each variable. □

Definition 4.3.18. If $\mathcal{V}$ is a presentably monoidal ∞-category, $\mathcal{C}$ is a $\mathcal{V}$–∞-category, and $\mathcal{X}$ is an $\mathcal{S}$–∞-category, then we denote their tensor by $\mathcal{C} \otimes \mathcal{X}$. For fixed $\mathcal{X}$ the functor $\mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{X}$ preserves colimits, and hence has a right adjoint — i.e. $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is also cotensored over $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$; we denote the cotensor of $\mathcal{C}$ and $\mathcal{X}$ by $\mathcal{C}^{\mathcal{X}}$. If $\mathcal{D}$ is another $\mathcal{V}$–∞-category we thus have a canonical equivalence

$$\mathrm{Map}\big(\mathcal{D}, \mathcal{C}^{\mathcal{X}}\big) \simeq \mathrm{Map}(\mathcal{D} \otimes \mathcal{X}, \mathcal{C}).$$

The ∞-category of categorical algebras is well-behaved with respect to adjunctions:

Lemma 4.3.19. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are presentably monoidal ∞-categories and $F\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ is a monoidal functor such that the underlying functor $f\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Let $g\colon \mathcal{W} \to \mathcal{V}$ be a right adjoint of f, and let $G\colon \mathcal{W}^{\otimes} \to \mathcal{V}^{\otimes}$ be the lax monoidal structure on g given by Proposition A.5.11. Then the functors*

$$F_* : \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W}) : G_*$$

are adjoint.

Proof. Let $\mathcal{C}$ be a $\mathcal{V}$–∞-category with space of objects S, and let $\mathcal{D}$ be a $\mathcal{W}$–∞-category with space of objects T. We must show that the natural map $\mathrm{Map}(\mathcal{C}, G_*\mathcal{D}) \to \mathrm{Map}(F_*\mathcal{C}, \mathcal{D})$ is an equivalence. We have a commutative triangle of spaces

$$\begin{array}{ccc} \mathrm{Map}(\mathcal{C}, G_*\mathcal{D}) & \longrightarrow & \mathrm{Map}(F_*\mathcal{C}, \mathcal{D}) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}(X, Y) & \end{array}$$

so it suffices to show that we have an equivalence on the fibres over each $\phi\colon X \to Y$. But we can identify the map on this fibre with

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_S}(\mathcal{V})}(\mathcal{C}, G_*\phi^*\mathcal{D}) \to \mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_S}(\mathcal{W})}(F_*\mathcal{C}, \phi^*\mathcal{D}),$$

which is an equivalence since F_* and G_* are adjoint functors on $\mathbf{\Delta}^{\mathrm{op}}_S$-algebras by Proposition A.5.11. □

Example 4.3.20. Suppose $\mathcal{V}$ is a presentably monoidal ∞-category. Then there is a unique colimit-preserving functor $t\colon \mathcal{S} \to \mathcal{V}$ that sends $*$ to $I_{\mathcal{V}}$. This has right adjoint $u\colon \mathcal{V} \to \mathcal{S}$ given by $\mathrm{Map}_{\mathcal{V}}(I_{\mathcal{V}}, -)$. Using the monoidal structure on $\mathrm{Mon}_{\infty}^{\mathrm{Pr}}$ we get a monoidal functor

$$T\colon \mathcal{S}^{\times} \simeq \mathcal{S}^{\times} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times_{\mathbf{\Delta}^{\mathrm{op}}} I_{\mathcal{V}}} \mathcal{S}^{\times} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{V}^{\otimes} \to \mathcal{S}^{\times} \otimes \mathcal{V}^{\otimes} \xrightarrow{\sim} \mathcal{V}^{\otimes}$$

extending t. By Proposition A.5.11 there is a lax monoidal functor $U\colon \mathcal{V}^{\otimes} \to \mathcal{S}^{\times}$ extending u such that for any non-symmetric ∞-operad $\mathcal{O}$ we have an adjunction

$$T_* : \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) : U_*.$$

Then by Lemma 4.3.19 we have an adjunction

$$T_* : \mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) : U_*.$$

Unravelling the definitions, it is clear that we can identify the functor T_* with the operation of tensoring with the unit $I_{\mathcal{V}} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$.

Our final goal in this subsection is to show that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ behaves very nicely with respect to monoidal localizations of $\mathcal{V}$. First we must introduce some notation:

Definition 4.3.21. Let $\mathcal{V}$ be a presentably monoidal ∞-category. The functor

$$\mathrm{Fun}(\{0,1\} \times \{0,1\}, \mathcal{V}) \to \mathcal{V}$$

given by evaluation at $(0,1)$ clearly has a left adjoint given by sending $V \in \mathcal{V}$ to the functor $\{0,1\} \times \{0,1\} \to \mathcal{V}$ that takes $(0,1)$ to V and the other elements to $\emptyset$. Let $\Sigma\colon \mathcal{V} \to \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\{0,1\}}}(\mathcal{V})$ denote the composite of this with the free algebra functor $\tau_!\colon \mathrm{Fun}(\{0,1\} \times \{0,1\}, \mathcal{V}) \to \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\{0,1\}}}(\mathcal{V})$. Thus for any categorical algebra $\mathcal{C}$ in $\mathcal{V}$ with space of objects $\{0,1\}$ we have

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\{0,1\}}}(\mathcal{V})}(\Sigma V, \mathcal{C}) \simeq \mathrm{Map}_{\mathcal{V}}\big(V, \mathcal{C}(0,1)\big).$$

We also write Σ for the functor $\mathcal{V} \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ obtained by composing this with the inclusion of the fibre at $\{0,1\}$. Thus for any $\mathcal{V}$–∞-category $\mathcal{C}$ with space of objects S the fibre of

$$\mathrm{Map}(\Sigma V, \mathcal{C}) \to \mathrm{Map}(\{0,1\}, S) \simeq S \times S$$

at (X,Y) is $\mathrm{Map}_{\mathcal{V}}(V, \mathcal{C}(X,Y))$.

Proposition 4.3.22. *Let $\mathcal{V}$ be a presentably monoidal ∞-category and suppose $L\colon \mathcal{V} \to \mathcal{W}$ is a monoidal accessible localization with fully faithful right adjoint $i\colon \mathcal{W} \hookrightarrow \mathcal{V}$. Let $i^{\otimes}\colon \mathcal{W}^{\otimes} \hookrightarrow \mathcal{V}^{\otimes}$ and $L^{\otimes}\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be as in Proposition 3.1.22. Suppose L exhibits $\mathcal{W}$ as the localization of $\mathcal{V}$ with respect to a set of morphisms S. Then there is an adjunction*

$$L^{\otimes}_* : \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W}) : i^{\otimes}_*$$

which exhibits $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})$ as the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to $\Sigma(S)$. Moreover, if $\mathcal{V}$ is at least $\mathbb{E}_2$-monoidal then this localization is again monoidal.

Proof. It follows from Lemma A.5.12 that the lax monoidal structure on i provided by Proposition A.5.11 is given by $i^{\otimes}$, so by Lemma 4.3.19 we indeed have an adjunction $L_*^{\otimes} \dashv i_*^{\otimes}$.

To see that this is a localization we must show that $i_*^{\otimes}$ is fully faithful. To prove this it suffices to show that for every categorical algebra $\mathcal{C} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with space of objects X the counit $L_*^{\otimes} i_*^{\otimes} \mathcal{C} \to \mathcal{C}$ is an equivalence in $\mathrm{Alg}_{\mathbf{\Delta}_X^{\mathrm{op}}}(\mathcal{V})$. By Lemma A.5.5 this is equivalent to the induced morphism of underlying graphs being an equivalence, i.e. to $Li\mathcal{C}(C, D) \to \mathcal{C}(C, D)$ being an equivalence in $\mathcal{V}$ for all $C, D \in \mathcal{C}$. But this is true since i is fully faithful.

Next we must show that $\mathcal{C} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ lies in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})$ if and only if it is local with respect to the morphisms in $\Sigma(S)$. Consider a map $f: A \to B$ in $\mathcal{V}$. Then the induced map

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\Sigma B, \mathcal{C}) \to \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\Sigma A, \mathcal{C})$$

is an equivalence in $\mathcal{S}$ if and only if it induces an equivalence on the fibres over all points of $\mathrm{Map}_{\mathcal{S}}(S^0, X)$. Using the universal property of Σ we conclude that it is an equivalence if and only if for all objects $C, D \in \mathcal{C}$ the induced map

$$\mathrm{Map}_{\mathcal{V}}\big(B, \mathcal{C}(C, D)\big) \to \mathrm{Map}_{\mathcal{V}}\big(A, \mathcal{C}(C, D)\big)$$

is an equivalence. Thus $\mathcal{C}$ is local with respect to the maps in $\Sigma(S)$ if and only if all the mapping objects $\mathcal{C}(C, D)$ are local with respect to the maps in S, i.e. if and only if these all lie in $\mathcal{W}$. Thus $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})$ is indeed the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to $\Sigma(S)$.

Finally, it is clear from the construction of the monoidal structure on $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ that the localization will again be monoidal when this exists. □

4.4. Categorical algebras in spaces

In this subsection we will prove that the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$ of categorical algebras in spaces is equivalent to the ∞-category Seg_∞ of Segal spaces. These are an alternative definition of $(\infty, 1)$-categories introduced by Rezk [34]. We begin by briefly reviewing the definition in the ∞-categorical context:

Definition 4.4.1. Suppose $\mathcal{C}$ is an ∞-category with finite limits. A *category object* in $\mathcal{C}$ is a simplicial object $F: \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that for each n the map

$$F_n \to F_1 \times_{F_0} \cdots \times_{F_0} F_1$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$ and $\{i\} \hookrightarrow [n]$ is an equivalence. A *Segal space* is a category object in the ∞-category $\mathcal{S}$ of spaces.

Let δ_n denote the simplicial space obtained from the simplicial set Δ^n by composing with the inclusion $\mathrm{Set} \hookrightarrow \mathcal{S}$. A simplicial space is then a Segal space if and only if it is local with respect to the map

$$\mathrm{seg}_n \colon \delta_n \to \delta_1 \amalg_{\delta_0} \cdots \amalg_{\delta_0} \delta_1.$$

Definition 4.4.2. Let $\mathrm{Seg}(\mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ spanned by the Segal spaces; this is the localization of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ with respect to the maps seg_*.

The key result for the comparison is the following:

Proposition 4.4.3. *Let S be a space, and let $\pi \colon \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$ be the usual projection. Let $\pi_! \colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_S, \mathcal{S}) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ be the functor given by left Kan extension along π. Then a functor $F \colon \mathbf{\Delta}^{\mathrm{op}}_S \to \mathcal{S}$ is a $\mathbf{\Delta}^{\mathrm{op}}_S$-monoid if and only if $\pi_! F$ is a Segal space.*

Proof. It is clear that $\pi_! F([0])$ is equivalent to S. We must thus show that the Segal morphism

$$\pi_! F([n]) \to \pi_! F([1]) \times_S \cdots \times_S \pi_! F([1]) =: (\pi_! F)^{\mathrm{Seg}}_{[n]}$$

is an equivalence if and only if F is a $\mathbf{\Delta}^{\mathrm{op}}_S$-monoid. Since π is a coCartesian fibration, we have an equivalence $\pi_! F([n]) \simeq \mathrm{colim}_{\xi \in S^{\times(n+1)}} F(\xi)$. It thus suffices to show that $(\pi_! F)^{\mathrm{Seg}}_{[n]}$ is also a colimit of this diagram if and only if F is a $\mathbf{\Delta}^{\mathrm{op}}_S$-monoid. There is a natural transformation $(S^{\times(n+1)})^{\triangleright} \to \mathrm{Fun}(\Delta^1, \mathcal{S})$ that sends $\xi \in S^{\times(n+1)}$ to $F(\xi) \to \xi$ and ∞ to $(\pi_! F)^{\mathrm{Seg}}_{[n]} \to S^{\times(n+1)}$; since $\mathcal{S}$ is an ∞-topos, by [25, Theorem 6.1.3.9] the colimit is $(\pi_! F)^{\mathrm{Seg}}_{[n]}$ if and only if this natural transformation is Cartesian. Since $S^{\times(n+1)}$ is a space, this is equivalent to the square

$$\begin{array}{ccc} F(\xi) & \longrightarrow & (\pi_! F)^{\mathrm{Seg}}_{[n]} \\ \big\downarrow & & \big\downarrow \\ \xi & \longrightarrow & S^{\times(n+1)} \end{array}$$

being a pullback for all ξ, so we are reduced to showing that the fibre of $(\pi_! F)^{\mathrm{Seg}}_{[n]} \to S^{\times(n+1)}$ at ξ is $F(\xi)$ if and only if F is a $\mathbf{\Delta}^{\mathrm{op}}_S$-monoid. Since limits commute, if ξ is $(s_0, \ldots, s_n)$ this fibre is the iterated fibre product

$$(\pi_! F[1])_{(s_0,s_1)} \times_{(\pi_! F[0])_{(s_1)}} \cdots \times_{(\pi_! F[0])_{(s_{n-1})}} (\pi_! F[1])_{(s_{n-1},s_n)}.$$

But using [25, Theorem 6.1.3.9] again it is clear that the natural maps $F(x,y) \to (\pi_! F[1])_{(x,y)}$ and $* \simeq F(x) \to (\pi_! F)_{(x)}$ are equivalences for all $x, y \in S$. Thus the map $F(\xi) \to (\pi_! F)^{\mathrm{Seg}}_{[n],\xi}$ is equivalent to the natural map

$$F(\xi) \to F(s_0, s_1) \times \cdots \times F(s_{n-1}, s_n).$$

By definition this is an equivalence for all $\xi \in \mathbf{\Delta}^{\mathrm{op}}_S$ if and only if F is a $\mathbf{\Delta}^{\mathrm{op}}_S$-monoid, which completes the proof. □

Corollary 4.4.4. *Let S be a space, and let $\pi \colon \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$ denote the canonical projection. By [15, Corollary 8.6] the functor*

$$\pi_! \colon \mathrm{Fun}\big(\mathbf{\Delta}^{\mathrm{op}}_S, \mathcal{S}\big) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})_{/i_*S}$$

given by left Kan extension is an equivalence. Under this equivalence, the full subcategory $\mathrm{Mon}_{\mathbf{\Delta}^{\mathrm{op}}_S}(\mathcal{S})$ of $\mathbf{\Delta}^{\mathrm{op}}_S$-monoids corresponds to the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})_{/i_*S}$ spanned by the Segal spaces $Y_\bullet$ such that $Y_0 \simeq S$ and the map $Y_\bullet \to i_*S$ is given by the adjunction unit $Y_\bullet \to i_*i^*Y_\bullet \simeq i_*S$.

Proof. It is clear that $\pi_!$ takes $\mathrm{Mon}_{\mathbf{\Delta}^{\mathrm{op}}_S}(\mathcal{S})$ into the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})_{/i_*S}$ spanned by simplicial spaces $Y_\bullet$ with $Y_0 \simeq S$ and the map $Y_\bullet \to i_*S$ given by the adjunction unit $Y_\bullet \to i_*i^*Y \simeq i_*S$. The result therefore follows by Proposition 4.4.3. □

Corollary 4.4.5. *Let S be a space, and let $\pi \colon \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$ denote the canonical projection. The functor $\pi_! \colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_S, \mathcal{S}) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S})$ given by left Kan extension along π gives an equivalence of the full subcategory $\mathrm{Mon}_{\mathbf{\Delta}^{\mathrm{op}}_S}(\mathcal{S})$ of $\mathbf{\Delta}^{\mathrm{op}}_S$-monoids with the subcategory $(\mathrm{Seg}_\infty)_S$ of Segal spaces with 0th space S and morphisms that are the identity on the 0th space.*

Lemma 4.4.6. *Let $\mathcal{E}$ and $\mathcal{B}$ be ∞-categories and $p \colon \mathcal{E} \to \mathcal{B}$ an inner fibration. Suppose*

(1) $\mathcal{E}$ has finite limits and p preserves these,
(2) p has a right adjoint $r \colon \mathcal{B} \to \mathcal{E}$ such that $p \circ r \simeq \mathrm{id}_{\mathcal{B}}$.

Then p is a Cartesian fibration.

Proof. Given $x \in \mathcal{E}$ and a morphism $f \colon b \to p(x)$, we must show there exists a Cartesian arrow in $\mathcal{E}$ lying over f with target x. Define $\overline{f} \colon y \to x$ by the pullback diagram

$$\begin{array}{ccc} y & \xrightarrow{\overline{f}} & x \\ \downarrow & & \downarrow \\ r(b) & \xrightarrow[r(f)]{} & rp(x). \end{array}$$

Since p preserves pullbacks, the morphism $p(\overline{f})$ is equivalent to f. Moreover, for any $z \in \mathcal{E}$ we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(z,y) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(z,x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{E}}(z,r(b)) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(z,rp(x)). \end{array}$$

Under the adjunction this corresponds to the commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{E}}(z,y) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(z,x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{B}}(p(z),b) & \longrightarrow & \mathrm{Map}_{\mathcal{E}}(p(z),p(x)) \end{array}$$

induced by the functor p. But then $\bar{f}$ is Cartesian by [25, Proposition 2.4.4.3]. $\square$

Theorem 4.4.7. *There is an equivalence* $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Seg}_{\infty}$, *given by sending a* $\mathbf{\Delta}^{\mathrm{op}}_S$*-algebra* $\mathcal{C}$ *to the left Kan extension* $\pi_!\mathcal{C}'$ *of the composite*

$$\mathcal{C}' \colon \mathbf{\Delta}^{\mathrm{op}}_S \xrightarrow{\mathcal{C}} \mathcal{S}^{\times} \to \mathcal{S}$$

along $\pi \colon \mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$, *where the second map (which sends* $(S_1, \ldots, S_n) \in \mathcal{S}^{\times}_{[n]}$ *to* $S_1 \times \cdots \times S_n$*) comes from a Cartesian structure in the sense of [28, Definition 2.4.1.1].*

Proof. If $\mathcal{V}$ is an ∞-category with finite products, pulling back the monoid fibration $\mathrm{Mon}(\mathcal{V}) \to \mathrm{Opd}^{\mathrm{ns}}_{\infty}$ of Remark 3.6.3 along $\mathbf{\Delta}^{\mathrm{op}}_{(-)}$ gives a Cartesian fibration $\mathrm{Mon}_{\mathrm{cat}}(\mathcal{V})$ with an equivalence

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Mon}_{\mathrm{cat}}(\mathcal{V})$$

over $\mathcal{S}$. Taking left Kan extensions along the projections $\mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}}$ for all $S \in \mathcal{S}$ we get (using Proposition 4.4.3) a commutative square

$$\begin{array}{ccc} \mathrm{Mon}_{\mathrm{cat}}(\mathcal{S}) & \xrightarrow{\Phi} & \mathrm{Seg}_{\infty} \\ & \searrow \quad \swarrow_{\mathrm{ev}_{[0]}} & \\ & \mathcal{S}. & \end{array}$$

By Lemma 4.4.6 it is clear that $\mathrm{ev}_{[0]} \colon \mathrm{Seg}_{\infty} \to \mathcal{S}$ is a Cartesian fibration, and the functor Φ preserves Cartesian morphisms by [25, Theorem 6.1.3.9]. It thus suffices to prove that for each $S \in \mathcal{S}$ the functor on fibres $\mathrm{Mon}_{\mathbf{\Delta}^{\mathrm{op}}_S}(\mathcal{S}) \to (\mathrm{Seg}_{\infty})_S$ is an equivalence, which is the content of Corollary 4.4.5. $\square$

4.5. A presheaf model for categorical algebras

In this subsection we will give an alternative characterization of the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ (for $\mathcal{V}$ a presentably monoidal ∞-category) as a localization of an ∞-category of presheaves. We thank Jeremy Hahn for suggesting this model; similar models have also been considered in unpublished work of Charles Rezk in the setting of model categories. Throughout this subsection we assume that $\mathcal{V}$ is a presentably monoidal ∞-category.

Definition 4.5.1. Let $\mathcal{V}^{\vee}_{\otimes} \to \mathbf{\Delta}$ be a Cartesian fibration corresponding to the same functor as the coCartesian fibration $\mathcal{V}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$. A presheaf $\phi\colon (\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}} \to \mathcal{S}$ is a *Segal presheaf* if it satisfies the following conditions:

(1) The functor $\mathcal{V}^{\mathrm{op}} \simeq (\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}}_{[1]} \to \mathcal{S}_{/\phi()^{\times 2}}$, induced by the Cartesian morphisms over the face maps $[0] \to [1]$ in $\mathbf{\Delta}$, takes colimit diagrams in $\mathcal{V}$ to limit diagrams in $\mathcal{S}_{/\phi()^{\times 2}}$.
(2) For every object $X \in \mathcal{V}^{\vee}_{\otimes}$, lying over $[n] \in \mathbf{\Delta}$, the diagram

$$\begin{array}{ccc} \phi(X) & \longrightarrow & \phi(d_n^* X) \\ \downarrow & & \downarrow \\ \phi(\alpha^*(X)) & \longrightarrow & \phi(), \end{array}$$

where $\alpha\colon [1] \to [n]$ is the map sending 0 to $n-1$ and 1 to n, is a pullback square.

Write $\mathcal{P}(\mathcal{V}^{\vee}_{\otimes})^{\mathrm{Seg}}$ for the full subcategory of $\mathcal{P}(\mathcal{V}^{\vee}_{\otimes})$ spanned by the Segal presheaves.

Remark 4.5.2. If $\phi\colon (\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}} \to \mathcal{S}$ is a Segal presheaf, then for every n the functor

$$(\mathcal{V}^{\times n})^{\mathrm{op}} \simeq (\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}}_{[n]} \to \mathcal{S}_{/\phi()^{\times (n+1)}},$$

induced by the Cartesian morphisms over the inclusions $[0] \hookrightarrow [n]$, takes colimits in $\mathcal{V}^{\times n}$ to limits in $\mathcal{S}_{/\phi()^{\times (n+1)}}$.

Since filtered ∞-categories are contractible, it is easy to see that a filtered diagram in $\mathcal{S}_{/\phi()^{\times (n+1)}}$ is a limit diagram if and only if the diagram in $\mathcal{S}$ obtained by composing with the forgetful functor $\mathcal{S}_{/\phi()^{\times (n+1)}} \to \mathcal{S}$ is a limit diagram. Thus if ϕ is a Segal presheaf the functors $(\mathcal{V}^{\times n})^{\mathrm{op}} \simeq (\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}}_{[n]} \to \mathcal{S}$ all take filtered colimits in $\mathcal{V}^{\times n}$ to limits in $\mathcal{S}$. If $\mathcal{V}$ is a κ-presentable ∞-category we may therefore regard a Segal presheaf on $\mathcal{V}$ as a presheaf on the full subcategory $(\mathcal{V}^{\vee}_{\otimes})^{\kappa}$ spanned by the objects that lie in the image of $(\mathcal{V}^{\kappa})^{\times n}$ in $(\mathcal{V}^{\vee}_{\otimes})_{[n]} \simeq \mathcal{V}^{\times n}$ for all n. Moreover, a presheaf $\phi\colon (\mathcal{V}^{\vee}_{\otimes})^{\kappa,\mathrm{op}} \to \mathcal{S}$ corresponds to a Segal presheaf if and only if it is local with respect to a set of maps in $\mathcal{P}((\mathcal{V}^{\vee}_{\otimes})^{\kappa})$, hence $\mathcal{P}(\mathcal{V}^{\vee}_{\otimes})^{\mathrm{Seg}}$ is an accessible localization of $\mathcal{P}((\mathcal{V}^{\vee}_{\otimes})^{\kappa})$.

We now prove that Segal presheaves give an alternative model for categorical algebras:

Theorem 4.5.3. *There is an equivalence between* $\mathcal{P}(\mathcal{V}^{\vee}_{\otimes})^{\mathrm{Seg}}$ *and* $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$.

Proof. Given a Cartesian fibration of ∞-categories $p\colon \mathcal{E} \to \mathcal{B}$, let $\mathcal{E}^{\triangleright}_{\mathcal{B}}$ be the pushout $\mathcal{B} \amalg_{\mathcal{E}\times\{0\}} \mathcal{E} \times \Delta^1$ and let $j\colon \mathcal{B} \to \mathcal{E}^{\triangleright}_{\mathcal{B}}$ be the obvious inclusion. By [15, §8], the functor $j^*\colon \mathcal{P}(\mathcal{E}^{\triangleright}_{\mathcal{B}}) \to \mathcal{P}(\mathcal{B})$ is a Cartesian fibration corresponding to the functor $\mathcal{P}(\mathcal{B}) \simeq \mathrm{RFib}(\mathcal{B}) \to \mathrm{Cat}_\infty$ that sends a right fibration $\mathcal{Y} \to \mathcal{B}$ to $\mathrm{Fun}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{Y}^{\mathrm{op}}, \mathcal{P}_{\mathcal{B}}(\mathcal{E}))$, where $\mathcal{P}_{\mathcal{B}}(\mathcal{E}) \to \mathcal{B}^{\mathrm{op}}$ is the coCartesian fibration corresponding to the functor $\mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends $b \in \mathcal{B}^{\mathrm{op}}$ to $\mathcal{P}(\mathcal{E}_b)$. Let $\delta\colon \mathcal{S} \to \mathrm{LFib}(\mathbf{\Delta}^{\mathrm{op}}) \simeq \mathcal{P}(\mathbf{\Delta})$ denote the functor that sends X to $\mathbf{\Delta}^{\mathrm{op}}_X \to \mathbf{\Delta}^{\mathrm{op}}$. Write $\mathcal{Q}$ for the pullback

$$\begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathcal{P}((\mathcal{V}^{\vee}_{\otimes})^{\triangleleft}_{\mathbf{\Delta}}) \\ {\scriptstyle q}\downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[\delta]{} & \mathcal{P}(\mathbf{\Delta}). \end{array}$$

Then by [15, Corollary 8.7] the functor q is the Cartesian fibration corresponding to the functor that sends $X \in \mathcal{S}$ to $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathbf{\Delta}^{\mathrm{op}}_X, \mathcal{P}_{\mathbf{\Delta}}(\mathcal{V}^{\vee}_{\otimes}))$.

Let $\mathcal{Q}_1$ be the full subcategory of $\mathcal{Q}$ spanned by presheaves $\phi\colon ((\mathcal{V}^{\vee}_{\otimes})^{\triangleleft}_{\mathbf{\Delta}})^{\mathrm{op}} \to \mathcal{S}$ whose restrictions to $(\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}}$ are Segal presheaves and for which the restriction $\phi|_{\{()\}\times\Delta^1}\colon \Delta^1 \to \mathcal{S}$ is an equivalence. Then the restriction functor $\mathcal{P}((\mathcal{V}^{\vee}_{\otimes})^{\triangleleft}_{\mathbf{\Delta}}) \to \mathcal{P}(\mathcal{V}^{\vee}_{\otimes})$ gives an equivalence between $\mathcal{Q}_1$ and $\mathcal{P}(\mathcal{V}^{\vee}_{\otimes})^{\mathrm{Seg}}$ — this is clear, since for every space X the composite

$$(\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\delta(X)} \mathcal{S}$$

is the final Segal presheaf that sends $()$ to X.

We can identify $\mathcal{V}^{\otimes}$ with the full subcategory of $\mathcal{P}_{\mathbf{\Delta}}(\mathcal{V}^{\vee}_{\otimes})$ spanned fibrewise by the representable presheaves. Let $\mathcal{Q}_2$ denote the full subcategory of $\mathcal{Q}$ spanned by the presheaves that correspond to categorical algebras in $\mathcal{V}$, i.e. that under the identification above correspond to functors $\mathbf{\Delta}^{\mathrm{op}}_X \to \mathcal{P}_{\mathbf{\Delta}}(\mathcal{V}^{\vee}_{\otimes})$ that land in the full subcategory $\mathcal{V}^{\otimes}$ and preserve inert morphisms. Then we can identify the ∞-category $\mathcal{Q}_2$ with $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$.

It remains to observe that the full subcategories $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have the same objects. It is clear that a presheaf $\phi\colon (\mathcal{V}^{\vee}_{\otimes})^{\triangleleft\,\mathrm{op}}_{\mathbf{\Delta}} \to \mathcal{S}$ whose restriction to $\{()\} \times \Delta^1$ is an equivalence corresponds to a functor $F\colon \mathbf{\Delta}^{\mathrm{op}}_X \to \mathcal{V}^{\otimes}$ if and only if for every $[n]$ the functor $(\mathcal{V}^{\times n})^{\mathrm{op}} \simeq (\mathcal{V}^{\vee}_{\otimes})^{\mathrm{op}}_{[n]} \to \mathcal{S}_{/\phi()^{\times(n+1)}}$ takes colimits in $\mathcal{V}^{\times n}$ to limits in $\mathcal{S}_{/\phi()^{\times(n+1)}}$. Moreover, the functor F preserves inert morphisms if and only if for every object $T \in \mathbf{\Delta}^{\mathrm{op}}_X$, the morphism $F(T) \to F(\alpha_! T)$ is coCartesian, where $\alpha\colon [1] \to [n]$ is the morphism in $\mathbf{\Delta}$ that sends 0 to $n-1$ and 1 to n, or equivalently, under the identification $\mathcal{V}^{\otimes}_{[n]} \simeq \mathcal{V}^{\times n}$, the objects $F(T)$ and $(F(d_{n,!}T), F(\alpha_! T))$ are equivalent. In terms of ϕ, this condition, for all $T \in \phi()^{\times(n+1)}$ is precisely the condition that the diagram

$$\begin{array}{ccc} \phi(X) & \longrightarrow & \phi(d_n^* X) \\ \downarrow & & \downarrow \\ \phi(\alpha^*(X)) & \longrightarrow & \phi() \end{array}$$

is a pullback square for all $X \in (\mathcal{V}^\vee_\otimes)_{[n]}$. Thus ϕ is a Segal presheaf if and only if F is a categorical algebra. □

5. The ∞-category of enriched ∞-categories

Our goal in this section is to prove our main result: we can always localize the ∞-category of categorical algebras at the fully faithful and essentially surjective functors by restricting to the full subcategory of *complete* objects. Along the way, we will introduce analogues of a number of familiar concepts from ordinary enriched category theory in our setting.

In §5.1 we define objects, morphisms, and equivalences in enriched ∞-categories. Then in §5.2 we study the classifying space of equivalences in an enriched ∞-category; the *complete* enriched ∞-categories are those whose classifying spaces of equivalences are equivalent to their underlying spaces of objects. Next we study three types of equivalences of $\mathcal{V}$–∞-categories: in §5.3 we define *fully faithful* and *essentially surjective* functors, in §5.4 *local equivalences* (those in the saturated class of a certain map) and finally in §5.5 *categorical equivalences* (those with an inverse up to natural equivalence). In §5.6 we prove that for ∞-categories enriched in a presentably monoidal ∞-category the fully faithful and essentially surjective functors are the same as the local equivalences, hence the full subcategory of complete objects gives the localization; we can extend this result to ∞-categories enriched in a general large monoidal ∞-category by embedding this in a presentable ∞-category in a larger universe. Finally in §5.7 we prove that the localized ∞-category inherits the functoriality properties of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$.

5.1. Some basic concepts

In this subsection we define the basic notions of objects, morphisms, and equivalences in an enriched ∞-category, an observe that these have the expected properties. We first consider objects:

Definition 5.1.1. Suppose $\mathcal{V}$ is a monoidal ∞-category. The unit of $\mathcal{V}$ defines an (essentially unique) associative algebra object $I_{\mathcal{V}} \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{V}^\otimes$ by Proposition 3.1.18. We write $[0]_{\mathcal{V}}$ (or sometimes $I_{\mathcal{V}}$ or $E^0_{\mathcal{V}}$ depending on context) for this associative algebra regarded as an enriched ∞-category. We view this as the trivial $\mathcal{V}$–∞-category with one object, and so we refer to a map $[0]_{\mathcal{V}} \to \mathcal{C}$ as an *object* of the $\mathcal{V}$–∞-category $\mathcal{C}$.

This definition justifies calling the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}([0]_{\mathcal{V}}, \mathcal{C})$ the *space of objects* in $\mathcal{C}$. However, if $\mathcal{C}$ is a $\mathbf{\Delta}^{\mathrm{op}}_X$-algebra in $\mathcal{V}$ then we also think of X as being the space of objects of $\mathcal{C}$. Luckily, it is easy to see that the two concepts agree:

Lemma 5.1.2. *Let $\mathcal{C}\colon \mathbf{\Delta}^{\mathrm{op}}_X \to \mathcal{V}^{\otimes}$ be a $\mathcal{V}$–∞-category. Then the map*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}([0]_{\mathcal{V}}, \mathcal{C}) \to \mathrm{Map}_{\mathcal{S}}(*, X) \simeq X$$

induced by the Cartesian fibration $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$ is an equivalence.

Proof. It suffices to check that the fibres of this map are contractible. By [25, Proposition 2.4.4.2] the fibre at a point $p\colon * \to X$ is

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})}(I_{\mathcal{V}}, p^*\mathcal{C}),$$

which is contractible since the unit $I_{\mathcal{V}}$ is the initial associative algebra object of $\mathcal{V}$. □

Next, we consider morphisms in an enriched ∞-category:

Definition 5.1.3. Write $[1]$ for the category corresponding to the ordered set $\{0, 1\}$, regarded as an $\mathcal{S}$–∞-category by Remark 4.3.10. Suppose $\mathcal{V}$ is a presentably monoidal ∞-category; then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$ by Corollary 4.3.17. We write $[1]_{\mathcal{V}}$ for the $\mathcal{V}$–∞-category $[1] \otimes I_{\mathcal{V}}$. A *morphism* in a $\mathcal{V}$–∞-category $\mathcal{C}$ is a map $[1]_{\mathcal{V}} \to \mathcal{C}$.

Lemma 5.1.4. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category and $\mathcal{C}$ is a $\mathcal{V}$–∞-category. The two objects 0 and 1 of $[1]_{\mathcal{V}}$ induce two maps $i_0, i_1\colon [0]_{\mathcal{V}} \to [1]_{\mathcal{V}}$; composing with these gives for any $\mathcal{V}$–∞-category $\mathcal{C}$ a map of spaces*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}([1]_{\mathcal{V}}, \mathcal{C}) \to \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}([0]_{\mathcal{V}}, \mathcal{C})^{\times 2}.$$

The fibre $\mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})_{X,Y}$ of this map $\mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})$ at points $X, Y \in \mathrm{Map}([0]_{\mathcal{V}}, \mathcal{C})$ is equivalent to $\mathrm{Map}(I, \mathcal{C}(X, Y))$.

Proof. Let $U\colon \mathcal{V}^{\otimes} \to \mathcal{S}^{\times}$ be the lax monoidal functor defined in Example 4.3.20. We then have

$$\mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})_{X,Y} \simeq \mathrm{Map}([1], U_*\mathcal{C})_{X,Y}.$$

Since $[1]_{\mathcal{S}}$ is the free $\mathcal{S}$–∞-category on the $\mathcal{S}$-graph having a single edge from 0 to 1, using the adjunction between $\mathcal{S}$–∞-categories and $\mathcal{S}$-graphs from Remark 4.3.4 we see that this is given by $U_*\mathcal{C}(X, Y) \simeq \mathrm{Map}(I_{\mathcal{V}}, \mathcal{C}(X, Y))$. □

Remark 5.1.5. This means that a morphism in $\mathcal{C}$ from X to Y is the same thing as a map $I \to \mathcal{C}(X, Y)$. This definition, of course, makes sense for any monoidal ∞-category $\mathcal{V}$.

We now define *equivalences* in enriched ∞-categories, and prove that these satisfy some of the expected properties. We will define an equivalence in a $\mathcal{V}$–∞-category $\mathcal{C}$ to be a functor $E^1 \to \mathcal{C}$ where E^1 is the generic $\mathcal{V}$–∞-category with two objects and an equivalence between them. More precisely, E^1 is a special case of a more general notion of a *trivial* enriched ∞-category, which we now define:

Definition 5.1.6. For any space S, the *trivial* $\mathcal{V}$*–*∞*-category* $E_S^{\mathcal{V}}$ with objects S is given by the composite

$$\mathbf{\Delta}^{\mathrm{op}}_S \to \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{I_{\mathcal{V}}} \mathcal{V}^{\otimes}.$$

We will generally drop the $\mathcal{V}$ from the notation and just write E_S when the monoidal ∞-category in question is obvious from the context. The $\mathcal{V}$–∞-categories E_S are functorial in S. We abbreviate $E^n := E_{\{0,\ldots,n\}}$; restricting to order-preserving maps between the sets $\{0, \ldots, n\}$ $(n = 0, 1, \ldots)$ we then have a cosimplicial $\mathcal{V}$–∞-category $E^{\bullet}$.

Remark 5.1.7. When S is a set, E_S is the enriched ∞-category associated to the trivial category with objects S and a unique morphism $A \to B$ for any pair of objects $A, B \in S$. This is also known as the *coarse* category with objects S, to distinguish it from the "discrete" trivial category with objects S (which has only identity morphisms).

We think of E^n as the generic $\mathcal{V}$–∞-category with $n + 1$ equivalent objects, so a map $E^n \to \mathcal{C}$ for some $\mathcal{V}$–∞-category $\mathcal{C}$ is a choice of $n + 1$ equivalent objects of $\mathcal{C}$. In particular, we have:

Definition 5.1.8. Suppose $\mathcal{C}$ is a $\mathcal{V}$–∞-category. An *equivalence* in $\mathcal{C}$ is a $\mathcal{V}$-functor $E^1 \to \mathcal{C}$.

Remark 5.1.9. We will see below, in Proposition 5.1.15, that this is equivalent to other reasonable definitions of an equivalence in a $\mathcal{V}$–∞-category.

Definition 5.1.10. Let

$$T : \mathcal{S}^{\times} \rightleftarrows \mathcal{V}^{\otimes} : U$$

be the adjoint functors described in Example 4.3.20, which induce an adjunction

$$T_* : \mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \rightleftarrows \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) : U_*,$$

by Lemma 4.3.19. If $\mathcal{C}$ is a $\mathcal{V}$–∞-category, we refer to $U_*\mathcal{C}$ as the *underlying* $\mathcal{S}$*–*∞*-category* of $\mathcal{C}$. By Theorem 4.4.7 we can identify $U_*\mathcal{C}$ with a Segal space.

We now make the very useful observation that the equivalences in a $\mathcal{V}$–∞-category $\mathcal{C}$ only depend on the underlying Segal space $U_*\mathcal{C}$:

Proposition 5.1.11. *Let $\mathcal{V}$ be a presentably monoidal ∞-category. Then for any space S there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(E_S^{\mathcal{V}}, \mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})}(E_S^{\mathcal{S}}, U_*\mathcal{C}).$$

This follows from the following lemma:

Lemma 5.1.12.

(i) Let $\mathcal{V}$ be a monoidal ∞-category. By Proposition 4.3.11, the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}(*)$, since the unique monoidal structure on the trivial one-object ∞-category $*$ is the unit for the Cartesian product of monoidal ∞-categories. There is a natural equivalence between the $\mathcal{V}$–∞-category $E_S^{\mathcal{V}}$ and the tensor $E_S^* \otimes I_{\mathcal{V}}$.

(ii) Let $\mathcal{V}$ be a presentably monoidal ∞-category; then the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is tensored over $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$ by Corollary 4.3.17. In this case there is a natural equivalence between $E_S^{\mathcal{V}}$ and the tensor $E_S^{\mathcal{S}} \otimes I_{\mathcal{V}}$.

Proof. We first prove (i). Considering the construction of the external product in Alg, we see that $E_S^* \otimes I_{\mathcal{V}}$ is given by

$$E_S^* \times_{\mathbf{\Delta}^{\mathrm{op}}} I_{\mathcal{V}} \colon \mathbf{\Delta}_S^{\mathrm{op}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{V}^{\otimes} \simeq \mathcal{V}^{\otimes}.$$

We can factor this as

$$\mathbf{\Delta}_S^{\mathrm{op}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{E_S^* \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{id}} \mathbf{\Delta}^{\mathrm{op}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times_{\mathbf{\Delta}^{\mathrm{op}}} I_{\mathcal{V}}} \mathbf{\Delta}^{\mathrm{op}} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{V}^{\otimes},$$

which is clearly the same as $E_S^{\mathcal{V}}$.

Now in the situation of (ii), part (i) then gives an equivalence

$$E_S^{\mathcal{S}} \otimes I_{\mathcal{V}} \simeq (E_S^* \otimes I_{\mathcal{S}}) \otimes I_{\mathcal{V}} \simeq E_S^* \otimes (I_{\mathcal{S}} \otimes I_{\mathcal{V}}) \simeq E_S^* \otimes I_{\mathcal{V}} \simeq E_S^{\mathcal{V}},$$

since it is easy to see that the tensorings with $\mathrm{Alg}_{\mathrm{cat}}(*)$ and $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S})$ are compatible. □

Proof of Proposition 5.1.11. By Lemma 5.1.12, the $\mathcal{V}$–∞-category $E_S^{\mathcal{V}}$ is naturally equivalent to $T_* E_S^{\mathcal{S}}$. We now complete the proof by recalling that the functor T_* is left adjoint to U_*. □

Definition 5.1.13. We write $\iota_1 \mathcal{C} := \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(E^1, \mathcal{C})$ for the *space of equivalences* in a $\mathcal{V}$–∞-category $\mathcal{C}$. More generally, we write $\iota_n \mathcal{C}$ for the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(E^n, \mathcal{C})$ — we can think of this as the space of n composable equivalences in $\mathcal{C}$, together with all the coherence data for the compositions. These spaces form a simplicial space $\iota_\bullet \mathcal{C}$ — here the face maps can be thought of as composing equivalences, and the degeneracy maps as inserting identity maps.

Remark 5.1.14. By Proposition 5.1.11 there is a natural equivalence $\iota_\bullet \mathcal{C} \simeq \iota_\bullet U_* \mathcal{C}$. This will allow us to reduce many of our arguments below to the case of spaces, where we can make use of results of Rezk from [34].

In particular, we will now use this to prove that our definition of equivalence agrees with a number of other reasonable definitions:

Proposition 5.1.15. *Suppose $\mathcal{V}$ is a monoidal ∞-category and $\mathcal{C}$ is a $\mathcal{V}$–∞-category. Let X, Y be objects of $\mathcal{C}$ and $\alpha \colon I_{\mathcal{V}} \to \mathcal{C}(X,Y)$ a morphism in $\mathcal{C}$. Then the following are equivalent:*

(i) α is an equivalence, i.e. it extends to a functor $E^1 \to \mathcal{C}$.
(ii) For all $Z \in \iota_0\mathcal{C}$, the composite map in $\mathcal{V}^\otimes$

$$\mathcal{C}(Y,Z) \to \big(I_{\mathcal{V}}, \mathcal{C}(Y,Z)\big) \to \big(\mathcal{C}(X,Y), \mathcal{C}(Y,Z)\big) \to \mathcal{C}(X,Z)$$

given by composing with α is an equivalence.
(iii) For all $Z \in \iota_0\mathcal{C}$, the composite map in $\mathcal{V}^\otimes$

$$\mathcal{C}(Z,X) \to \big(\mathcal{C}(Z,X), I_{\mathcal{V}}\big) \to \big(\mathcal{C}(Z,X), \mathcal{C}(X,Y)\big) \to \mathcal{C}(X,Y)$$

given by composing with α is an equivalence.
(iv) α has an inverse, i.e. a map $I \to \mathcal{C}(Y,X)$ such that the composites

$$I \to (I,I) \xrightarrow{(\beta,\alpha)} \big(\mathcal{C}(X,Y), \mathcal{C}(Y,X)\big) \to \mathcal{C}(X,X),$$
$$I \to (I,I) \xrightarrow{(\alpha,\beta)} \big(\mathcal{C}(Y,X), \mathcal{C}(X,Y)\big) \to \mathcal{C}(Y,Y)$$

are homotopic to the identity maps of X and Y, respectively.

Proof. We first show that (i) is equivalent to (ii). Suppose (i) holds, and let $\hat{\alpha} \colon E^1 \to \mathcal{C}$ be an equivalence extending α. Composing with the inverse equivalence from Y to X gives an inverse to composition with α, since the composite map is composing with the composite $X \to Y \to X$, which is the identity.

Now suppose (ii) holds. Without loss of generality, we may assume that $\mathcal{V}$ is presentably monoidal (by embedding in a presentably monoidal ∞-category of presheaves in a larger universe, if necessary). Then a map $E^1_{\mathcal{V}} \to \mathcal{C}$ is adjoint to a map $E^1_{\mathcal{S}} \to U_*\mathcal{C}$ where $U \colon \mathcal{V} \to \mathcal{S}$ is again as in Example 4.3.20. If (ii) holds for α then the analogous condition also holds for α considered as a morphism in $U_*\mathcal{C}$. It thus suffices to show that (ii) implies (i) in the case where $\mathcal{V}$ is $\mathcal{S}$. We again use the equivalence between $\mathcal{S}$–∞-categories and Segal spaces of Theorem 4.4.7; the map α is clearly a "homotopy equivalence" in the sense of [34, §5.5], and so extends to a map from E^1 by [34, Theorem 6.2].

The proof that (i) is equivalent to (iii) is similar, so it remains to prove that (i) is equivalent to (iv). Since equivalences are detected in $U_*\mathcal{C}$, this is immediate from [34, Theorem 6.2]. □

The inclusion $[1]_{\mathcal{V}} \to E^1$ of the map from 0 to 1 induces a map $\iota_1\mathcal{C} \to \mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C})$. The two inclusions of $E^0 \simeq [0]_{\mathcal{V}}$ into $[1]_{\mathcal{V}}$ and E^1 then give a commutative triangle

$$\begin{array}{ccc} \iota_1\mathcal{C} & \longrightarrow & \mathrm{Map}([1]_{\mathcal{V}}, \mathcal{C}) \\ & \searrow \quad \swarrow & \\ & \iota_0\mathcal{C} \times \iota_0\mathcal{C}. & \end{array}$$

We end this section by showing that on fibres, this map is an inclusion of components:

Definition 5.1.16. Suppose $\mathcal{C}$ is a $\mathcal{V}$–∞-category and X, Y are objects of $\mathcal{C}$. We let $\mathrm{Map}(I_{\mathcal{V}}, \mathcal{C}(X,Y))_{\mathrm{eq}}$ be the subspace of $\mathrm{Map}(I_{\mathcal{V}}, \mathcal{C}(X,Y))$ consisting of the components in the image of $\iota_1\mathcal{C}_{X,Y}$ under the induced map on fibres in the diagram above.

Proposition 5.1.17. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category, $\mathcal{C}$ is a $\mathcal{V}$–∞-category, and X, Y are objects of $\mathcal{C}$. Then the map $(\iota_1\mathcal{C})_{X,Y} \to \mathrm{Map}(I_{\mathcal{V}}, \mathcal{C}(X,Y))_{\mathrm{eq}}$ is an equivalence.*

Proof. By Proposition 5.1.11 it again suffices to prove this for $\mathcal{S}$–∞-categories. Using the identification of $\mathcal{S}$–∞-categories with Segal spaces of Theorem 4.4.7, this therefore follows from the corresponding statement in that setting. The latter is a consequence of [34, Theorem 6.2], since if $\mathcal{C}$ is a Segal space with objects X, Y, a point of $\mathcal{C}(X,Y)$ is a "homotopy equivalence" in the sense of [34, §5.5] if and only if it extends to a map from $E^1_{\mathcal{S}}$, by [34, Proposition 11.1]. □

5.2. *The classifying space of equivalences*

In this section we define the classifying space of equivalences in an enriched ∞-category, and use this to define *complete* enriched ∞-categories. We then prove that the simplicial space of equivalences is always a groupoid object, which allows us to give a simpler description of the completeness condition.

Definition 5.2.1. Let $\mathcal{C}$ be a $\mathcal{V}$–∞-category. The *classifying space of equivalences* $\iota\mathcal{C}$ of $\mathcal{C}$ is the geometric realization $|\iota_\bullet\mathcal{C}|$ of the simplicial space $\iota_\bullet\mathcal{C} := \mathrm{Map}(E^\bullet, \mathcal{C})$.

We regard $\iota\mathcal{C}$ as the "correct" space of objects of $\mathcal{C}$, and by analogy with Rezk's notion of complete Segal space we say that an enriched ∞-category is *complete* if its underlying space is the correct one:

Definition 5.2.2. A $\mathcal{V}$–∞-category $\mathcal{C}$ is *complete* if the natural map $\iota_0\mathcal{C} \to \iota\mathcal{C}$ is an equivalence.

Our next goal is to prove that the simplicial space $\iota\mathcal{C}$ is always a groupoid object; we prove this by showing that the cosimplicial object $E^\bullet$ satisfies the dual condition of being a *cogroupoid* object:

Definition 5.2.3. A cosimplicial object $X\colon \mathbf{\Delta} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is a *cogroupoid object* if for every partition $[n] = S \cup S'$ such that $S \cap S'$ consists of a single element, the diagram

$$\begin{array}{ccc} X(S\cap S') & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ X(S') & \longrightarrow & X([n]) \end{array}$$

is a pushout square.

Lemma 5.2.4. *If $X\colon \mathbf{\Delta} \to \mathcal{C}$ is a cogroupoid object in an ∞-category $\mathcal{C}$, then for every object $Y \in \mathcal{C}$ the simplicial space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a groupoid object in spaces.*

Theorem 5.2.5. *If $\mathcal{V}$ is a presentably monoidal ∞-category then the cosimplicial object $E^\bullet$ is a cogroupoid object.*

Proof. We will show that $E^N \amalg_{E_{\{N\}}} E_{\{N,N+1\}} \to E^{N+1}$ is an equivalence; by induction this will imply that $E^\bullet$ is a cogroupoid object, as the ordering of the objects is arbitrary. Since $\mathcal{V}$ is presentably monoidal, by Proposition 5.1.11 it suffices to prove this when $\mathcal{V}$ is $\mathcal{S}$.

Under the equivalence $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Seg}_\infty$ of Theorem 4.4.7 the $\mathcal{S}$–∞-category E_S clearly corresponds to the Segal space i_*S. If S is a set it follows that in the model category structure on bisimplicial sets modelling Segal spaces, E_S corresponds to $\pi^* N\mathcal{I}_S$ where $\mathcal{I}_S$ is the ordinary category with objects S and a unique morphism between any pair of objects, and $\pi\colon \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ is the projection onto the first factor.

Define $G_N := \mathrm{N}\mathcal{I}_{\{0,\ldots,N\}}$. By [34, Remark 10.2], for $0 < i < n$ the map $\pi^*\Lambda^n_k \to \pi^*\Delta^n$ is a Segal equivalence, so (since π^* is a left adjoint and thus preserves colimits) it suffices to prove that $G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} \hookrightarrow G_{N+1}$ is an inner anodyne morphism of simplicial sets. To prove this we consider a series of nested filtrations of the simplices of G_{N+1}. First we must introduce some notation:

An n-simplex σ of G_{N+1} can be described by a list $a_0 \cdots a_n$ of elements $a_i \in \{0, \ldots, N+1\}$; it is non-degenerate if $a_i \neq a_{i+1}$ for all i. If σ is a non-degenerate simplex, let $\beta(\sigma)$ be the number of times the sequence jumps between $\{0, \ldots, N\}$ and $\{N, N+1\}$.

Also let $\tau(\sigma)$ be the position of the first $N+1$ where the sequence jumps from $\{N, N+1\}$ to $\{0, \ldots, N\}$; if there is no such jump let $\tau(\sigma) = \infty$ and let $\tau'(\sigma)$ denote the position of the first jump from $\{0, \ldots, N\}$ to $\{N, N+1\}$. Then we make the following definitions:

- If $t \neq \infty$, let $S_n^{b,t}$ be the set of non-degenerate n-simplices σ in G_{N+1} such that $\beta(\sigma) = b$, $\tau(\sigma) = t$, and $a_{t+1} \neq N$. Let $S_n^{1,\infty,t}$ be the set of non-degenerate n-simplices in G_{N+1} such that $\beta(\sigma) = 1$, $\tau(\sigma) = \infty$, $\tau'(\sigma) = t$, and $a_{t-1} \neq N$.
- If $t \neq \infty$, let $T_n^{b,t}$ be the set of non-degenerate $(n+1)$-simplices σ in G_{N+1} such that $\beta(\sigma) = b$, $\tau(\sigma) = t$ and $a_{t+1} = N$. Let $T_n^{1,\infty,t}$ be the set of non-degenerate $(n+1)$-simplices σ in G_{N+1} such that $\beta(\sigma) = 1$, $\tau(\sigma) = \infty$, $\tau'(\sigma) = t+1$, and $a_t = N$.

Define a filtration

$$G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} =: \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq G_{N+1}$$

by letting $\mathcal{F}_n$ be the subspace of G_{N+1} whose non-degenerate simplices are those of $\mathcal{F}_0$ together with all the non-degenerate i-simplices for $i \leq n$ and the $(n+1)$-simplices in $T_n^{b,t}$ and $T_n^{1,\infty,t}$ for all b, t. Then $G_{N+1} = \bigcup_n \mathcal{F}_n$, so to prove that $G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} \hookrightarrow G_{N+1}$ is inner anodyne it suffices to prove that the inclusions $\mathcal{F}_{n-1} \hookrightarrow \mathcal{F}_n$ are inner anodyne.

Next define a filtration

$$\mathcal{F}_{n-1} =: \mathcal{F}_n^0 \subseteq \mathcal{F}_n^1 \subseteq \cdots \subseteq \mathcal{F}_n^{n-1} := \mathcal{F}_n$$

by setting $\mathcal{F}_n^b$ to be the subspace of $\mathcal{F}_n$ containing $\mathcal{F}_{n-1}$ together with the simplices in $S_n^{i,t}$ and $T_n^{i,t}$ for all $i \leq b$ together with $S_n^{1,\infty,t}$ and $T_n^{1,\infty,t}$ for all t. To prove that the inclusions $\mathcal{F}_{n-1} \hookrightarrow \mathcal{F}_n$ are inner anodyne it suffices to prove that the inclusions $\mathcal{F}_n^{b-1} \hookrightarrow \mathcal{F}_n^b$ are all inner anodyne.

Finally define a filtration

$$\mathcal{F}_n^{b-1} =: \mathcal{F}_n^{b,n+1} \subseteq \mathcal{F}_n^{b,n} \subseteq \cdots \subseteq \mathcal{F}_n^{b,0} := \mathcal{F}_n^b,$$

by setting $\mathcal{F}_n^{b,t}$ to be the subspace of $\mathcal{F}_n^b$ containing $\mathcal{F}_n^{b-1}$ together with the simplices in $S_n^{b,j}$ and $T_n^{b,j}$ (as well as $S_n^{1,\infty,j}$ and $T_n^{1,\infty,j}$ if $b = 1$) for all $j \geq t$. To prove that the inclusions $\mathcal{F}_n^{b-1} \hookrightarrow \mathcal{F}_n^b$ are inner anodyne it suffices to show that the inclusions $\mathcal{F}_n^{b,t-1} \hookrightarrow \mathcal{F}_n^{b,t}$ are all inner anodyne.

Now observe that (for $b > 1$) if $\sigma \in T_n^{b,t}$ then $d_t\sigma \in S_n^{b,t}$ and $d_i\sigma \in \mathcal{F}_n^{b,t-1}$ for $i \neq t$, and σ is uniquely determined by $d_t\sigma$. Thus we get a pushout diagram

$$\begin{array}{ccc} \coprod_{\sigma\in T_n^{b,t}} \Lambda^{n+1}_t & \longrightarrow & \coprod_{\sigma\in T_n^{b,t}} \Delta^{n+1} \\ \downarrow & & \downarrow \\ \mathcal{F}_n^{b,t-1} & \longrightarrow & \mathcal{F}_n^{b,t} \end{array}$$

where we always have $0 < t < n+1$. Thus the bottom horizontal map is inner anodyne. The proof is similar when $b = 1$, expect that we must also consider the simplices in $S_n^{1,\infty,t}$, so we conclude that $G_N \amalg_{G_{\{N\}}} G_{\{N,N+1\}} \to G_{N+1}$ is indeed inner anodyne. □

Remark 5.2.6. We can generalize this to the case of an arbitrary large monoidal ∞-category $\mathcal{V}$ as follows: by [28, Remark 4.8.1.8] there exists a presentably monoidal structure on the (very large) presentable ∞-category $\widehat{\mathcal{P}}(\mathcal{V})$ of presheaves of large spaces on $\mathcal{V}$, such that the Yoneda embedding $\mathcal{V} \to \widehat{\mathcal{P}}(\mathcal{V})$ is a monoidal functor. This induces a fully faithful embedding

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \widehat{\mathrm{Alg}}_{\mathrm{cat}}\big(\widehat{\mathcal{P}}(\mathcal{V})\big);$$

moreover, if X a small space then $E_X^{\widehat{\mathcal{P}}(\mathcal{V})}$ is clearly the image of $E_X^{\mathcal{V}}$. Thus if a diagram of $E_X^{\mathcal{V}}$'s is a colimit diagram in $\widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V}))$ it must also be a colimit diagram in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ — in particular $E_{\mathcal{V}}^{\bullet}$ is a cogroupoid object in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$.

Corollary 5.2.7. *The simplicial space $\iota_\bullet\mathcal{C}$ is a groupoid object in spaces for all $\mathcal{V}$–∞-categories $\mathcal{C}$.*

Lemma 5.2.8. *Suppose $X_\bullet$ is a category object in an ∞-category $\mathcal{C}$. Then the following are equivalent:*

(i) The functor $X_\bullet$ is constant.
(ii) The map $s_0 \colon X_0 \to X_1$ is an equivalence.

Proof. It is obvious that (i) implies (ii). To show that (ii) implies (i) first observe that if $s_0 \colon X_0 \to X_1$ is an equivalence, then by the 2-out-of-3 property $d_0, d_1 \colon X_1 \to X_0$ are also equivalences. Since $X_\bullet$ is a category object we have pullback diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{d_0} & X_{n-1} \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0, \end{array}$$

and so the face maps $d_0 \colon X_n \to X_{n-1}$ are equivalences for all i and n. Combining this with the simplicial identities we see inductively that all face maps and degeneracies are equivalences. □

Lemma 5.2.9. *Suppose $U_\bullet$ is a groupoid object in $\mathcal{S}$. The following are equivalent:*

(i) *The map $U_0 \to |U_\bullet|$ is an equivalence.*
(ii) *The map $s_0\colon U_0 \to U_1$ is an equivalence.*
(iii) *The simplicial object $U_\bullet$ is constant, i.e. for every map $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ the induced map $\iota_n\mathcal{C} \to \iota_m\mathcal{C}$ is an equivalence.*

Proof. We first show that (i) implies (ii): Since $\mathcal{S}$ is an ∞-topos, the groupoid object $U_\bullet$ is effective, i.e. it is equivalent to the Čech nerve of the map $U_0 \to |U_\bullet|$. Thus we have a pullback diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{d_0} & U_0 \\ {\scriptstyle d_1}\downarrow & & \downarrow \\ U_0 & \longrightarrow & |U_\bullet|, \end{array}$$

so the maps d_0, d_1 are equivalences. From the 2-out-of-3 property it follows that s_0 is also an equivalence. It follows from Lemma 5.2.8 that (ii) implies (iii). Finally (iii) implies (i) since the simplicial set $\mathbf{\Delta}^{\mathrm{op}}$ is weakly contractible. □

We can now give a simpler characterization of the completeness condition for $\mathcal{V}$–∞-categories:

Corollary 5.2.10. *Let $\mathcal{C}$ be a $\mathcal{V}$–∞-category. The following are equivalent:*

(i) *$\mathcal{C}$ is complete.*
(ii) *The natural map $s_0\colon \iota_0\mathcal{C} \to \iota_1\mathcal{C}$ is an equivalence.*
(iii) *The simplicial space $\iota_\bullet\mathcal{C}$ is constant (i.e. for every map $\phi\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ the induced map $\iota_n\mathcal{C} \to \iota_m\mathcal{C}$ is an equivalence).*

Proof. Apply Lemma 5.2.9 to the groupoid object $\iota_\bullet\mathcal{C}$. □

5.3. Fully faithful and essentially surjective functors

In this subsection we introduce analogues of *fully faithful* and *essentially surjective* functors in the context of enriched ∞-categories, and show that these have the expected properties.

Definition 5.3.1. Let $\mathcal{V}$ be a monoidal ∞-category. A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is *fully faithful* if the maps $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ are equivalences in $\mathcal{V}$ for all X, Y in $\iota_0\mathcal{C}$.

Lemma 5.3.2. *A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if it is a Cartesian morphism in* $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ *with respect to the projection* $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$.

Proof. If $f: S \to \iota_0\mathcal{D}$ is a map of spaces, then a Cartesian morphism over f with target $\mathcal{D}$ has source $f^*\mathcal{D} = \mathcal{D} \circ \mathbf{\Delta}_f^{\mathrm{op}}$; in particular a Cartesian morphism induces equivalences $f^*\mathcal{D}(x,y) \to \mathcal{D}(f(x), f(y))$ for all $x, y \in X$.

Conversely, suppose $F: \mathcal{C} \to \mathcal{D}$ gives an equivalence on all mapping spaces. The functor F factors as

$$\mathcal{C} \xrightarrow{F'} (\iota_0 F)^*\mathcal{D} \xrightarrow{F''} \mathcal{D},$$

where F'' is Cartesian. The morphism F' induces an equivalence on underlying spaces and is given by equivalences $\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ for all $X, Y \in \iota_0\mathcal{C}$. By Lemma A.5.5 it follows that F' is an equivalence in $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}}}(\mathcal{V})$ and so in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. In particular F' is a Cartesian morphism and hence so is the composite $F \simeq F'' \circ F'$. □

Definition 5.3.3. A functor $F: \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for every point $X \in \iota_0\mathcal{D}$ there exists an equivalence $E^1 \to \mathcal{D}$ from X to a point in the image of $\iota_0 F$.

Lemma 5.3.4. *A functor $F: \mathcal{C} \to \mathcal{D}$ is essentially surjective if and only if the induced map $\pi_0\iota F: \pi_0\iota\mathcal{C} \to \pi_0\iota\mathcal{D}$ is surjective.*

Proof. Since $\iota_\bullet\mathcal{D}$ is a groupoid object, the set $\pi_0\iota\mathcal{D}$ is the quotient of $\pi_0\iota_0\mathcal{D}$ where we identify two components of $\iota_0\mathcal{D}$ if there exists a point of $\iota_1\mathcal{D}$, i.e. an equivalence $E^1 \to \mathcal{D}$, connecting them. Thus $F: \mathcal{C} \to \mathcal{D}$ is essentially surjective if and only if $\pi_0\iota F$ is surjective. □

Lemma 5.3.5. *Suppose $F: \mathcal{C} \to \mathcal{D}$ is a fully faithful functor of $\mathcal{V}$–∞-categories. Then for every $X, Y \in \mathcal{C}$ the induced map $(\iota_1\mathcal{C})_{X,Y} \to (\iota_1\mathcal{D})_{FX,FY}$ is an equivalence.*

Proof. By Proposition 5.1.17, we can identify the map $(\iota_1\mathcal{C})_{X,Y} \to (\iota_1\mathcal{D})_{FX,FY}$ with the map

$$\mathrm{Map}\big(I, \mathcal{C}(X,Y)\big)_{\mathrm{eq}} \to \mathrm{Map}\big(I, \mathcal{D}(FX,FY)\big)_{\mathrm{eq}}$$

induced by F. Since F is fully faithful the map $\mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ is an equivalence in $\mathcal{V}$, hence $\mathrm{Map}(I, \mathcal{C}(X,Y)) \to \mathrm{Map}(I, \mathcal{D}(FX,FY))$ is an equivalence in $\mathcal{S}$. To complete the proof it therefore suffices to show that $\mathrm{Map}(I, \mathcal{C}(X,Y))_{\mathrm{eq}} \to \mathrm{Map}(I, \mathcal{D}(FX,FY))_{\mathrm{eq}}$ is surjective on components — i.e. if $\alpha: I \to \mathcal{D}(FX,FY)$ is an equivalence then it is the image of an equivalence $\beta: I \to \mathcal{C}(X,Y)$. We know that α is the image of some map β, so it suffices to show that such a β must be an equivalence. By Proposition 5.1.15 the map β is an equivalence if and only if for every $Z \in \iota_0\mathcal{C}$ the map $\mathcal{C}(Z,X) \to \mathcal{C}(Z,Y)$ induced by composition with β is an equivalence. Consider the diagram

$$\begin{array}{ccc} \mathcal{C}(Z,X) & \longrightarrow & \mathcal{D}(FZ,FX) \\ \downarrow & & \downarrow \\ \mathcal{C}(Z,Y) & \longrightarrow & \mathcal{D}(FZ,FY), \end{array}$$

where the vertical maps are given by composition with β and α, respectively. Since F is fully faithful and α is an equivalence, all morphisms in this diagram except the left vertical map are known to be equivalences. By the 2-out-of-3 property this must also be an equivalence for all Z, so β is indeed an equivalence. □

Proposition 5.3.6. *If a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective, then the induced map $\iota F\colon \iota\mathcal{C} \to \iota\mathcal{D}$ is an equivalence.*

Proof. The simplicial spaces $\iota_\bullet\mathcal{C}$ and $\iota_\bullet\mathcal{D}$ are groupoid objects by Corollary 5.2.7, and since F is essentially surjective the map ιF is surjective on π_0 by Lemma 5.3.4. By [26, Remark 1.2.17] it therefore suffices to show that the diagram

$$\begin{array}{ccc} \iota_1\mathcal{C} & \longrightarrow & \iota_1\mathcal{D} \\ \downarrow & & \downarrow \\ \iota_0\mathcal{C} \times \iota_0\mathcal{C} & \longrightarrow & \iota_0\mathcal{D} \times \iota_0\mathcal{D} \end{array}$$

is a pullback square. To prove this we must show that for all $X, Y \in \mathcal{C}$ the map on fibres is an equivalence, which we proved in Lemma 5.3.5. □

Corollary 5.3.7. *A fully faithful $\mathcal{V}$-functor F is essentially surjective if and only if ιF is an equivalence.*

Corollary 5.3.8. *A fully faithful and essentially surjective functor between complete $\mathcal{V}$-∞-categories is an equivalence in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$.*

Proof. This follows by combining Proposition 5.3.6 and Lemma A.5.5. □

Proposition 5.3.9. *Fully faithful and essentially surjective $\mathcal{V}$-functors satisfy the 2-out-of-3 property.*

Proof. Suppose we have $\mathcal{V}$-functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. There are three cases to consider:

(1) Suppose F and G are fully faithful and essentially surjective. It is obvious that $G \circ F$ is fully faithful. Since $\pi_0\iota F$ and $\pi_0\iota G$ are surjective, so is their composite $\pi_0\iota(G \circ F)$, thus $G \circ F$ is also essentially surjective.

(2) Suppose G and $G \circ F$ are fully faithful and essentially surjective. Then F is also Cartesian, i.e. fully faithful, by [25, Proposition 2.4.1.7]. By Proposition 5.3.6 the maps ιG and $\iota(G \circ F)$ are equivalences, hence so is ιF, thus F is also essentially surjective.

(3) Suppose F and $G \circ F$ are fully faithful and essentially surjective. By Proposition 5.3.6 the maps ιF and $\iota(G \circ F)$ are equivalences, hence so is ιG, and thus G is essentially surjective. To see that G is fully faithful, we must show that for any X, Y in $\iota_0 G$ the map $\mathcal{D}(X, Y) \to \mathcal{E}(GX, GY)$ is an equivalence. But since F is essentially surjective there exist objects X', Y' in $\iota_0 \mathcal{C}$ and equivalences $FX' \simeq X$, $FY' \simeq Y$ in $\mathcal{D}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(FX', FY') & \longrightarrow & \mathcal{E}(GFX', GFY') \\ \downarrow & & \downarrow \\ \mathcal{D}(X, Y) & \longrightarrow & \mathcal{E}(GX, GY), \end{array}$$

where the vertical maps are given by composition with the chosen equivalences and so are equivalences in $\mathcal{V}$ by Proposition 5.1.15. The top horizontal map is also an equivalence, since in the commutative triangle

$$\begin{array}{ccc} \mathcal{C}(X', Y') & \longrightarrow & \mathcal{E}(GFX', GFY') \\ & \searrow \quad \nearrow & \\ & \mathcal{D}(FX', FY') & \end{array}$$

the other two maps are equivalences. Thus by the 2-out-of-3 property the bottom horizontal map $\mathcal{D}(X, Y) \to \mathcal{E}(GX, GY)$ is also an equivalence, and so G is also fully faithful. $\square$

Remark 5.3.10. Under the equivalence $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Seg}_\infty$ of Theorem 4.4.7, the fully faithful and essentially surjective functors correspond to the Dwyer–Kan equivalences in the sense of [34, §7.4].

The "correct" ∞-category of $\mathcal{V}$–∞-categories is obtained by inverting the fully faithful and essentially surjective morphisms in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. We will now show that doing this produces the same ∞-category as inverting the fully faithful and essentially surjective functors in the subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ where we only have sets of objects. First, we will briefly review the general notion of localization of ∞-categories and prove a basic fact about these (generalizing [13, Corollary 3.6]):

Definition 5.3.11. The inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ has left and right adjoints. The right adjoint, $\iota\colon \mathrm{Cat}_\infty \to \mathcal{S}$, sends an ∞-category $\mathcal{C}$ to its maximal Kan complex, i.e. its subcategory of equivalences. The left adjoint $\kappa\colon \mathrm{Cat}_\infty \to \mathcal{S}$ sends an ∞-category $\mathcal{C}$ to a Kan complex $\kappa\mathcal{C}$ such that $\mathcal{C} \to \kappa\mathcal{C}$ is a weak equivalence in the usual model structure on simplicial sets.

Definition 5.3.12. Suppose $\mathcal{C}$ is an ∞-category and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ that contains all the equivalences. The *localization* $\mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ with respect to $\mathcal{W}$ is the ∞-category with the universal property that for any ∞-category $\mathcal{E}$, a functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{E}$ is the same thing as a functor $\mathcal{C} \to \mathcal{E}$ that sends morphisms in $\mathcal{W}$ to equivalences in $\mathcal{E}$. More precisely, we have for every $\mathcal{E}$ a pullback square

$$\begin{array}{ccc} \mathrm{Map}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{E}) & \longrightarrow & \mathrm{Map}(\mathcal{W}, \iota\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathcal{C}, \mathcal{E}) & \longrightarrow & \mathrm{Map}(\mathcal{W}, \mathcal{E}). \end{array}$$

Remark 5.3.13. It follows that, in the situation above, the ∞-category $\mathcal{C}[\mathcal{W}^{-1}]$ is given by the pushout square in Cat_∞

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \kappa\mathcal{W} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}[\mathcal{W}^{-1}]. \end{array}$$

Lemma 5.3.14. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories and $\mathcal{W}_\mathcal{C} \subseteq \mathcal{C}$ and $\mathcal{W}_\mathcal{D} \subseteq \mathcal{D}$ are subcategories containing all the equivalences. Let $\mathcal{C}[\mathcal{W}_\mathcal{C}^{-1}]$ and $\mathcal{D}[\mathcal{W}_\mathcal{D}^{-1}]$ be localizations with respect to $\mathcal{W}_\mathcal{C}$ and $\mathcal{W}_\mathcal{D}$. Suppose*

$$F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$$

is an adjunction such that

(1) $F(\mathcal{W}_\mathcal{C}) \subseteq \mathcal{W}_\mathcal{D}$,
(2) $G(\mathcal{W}_\mathcal{D}) \subseteq \mathcal{W}_\mathcal{C}$,
(3) the unit morphism $\eta_C\colon C \to GFC$ is in $\mathcal{W}_\mathcal{C}$ for all $C \in \mathcal{C}$,
(4) the counit morphism $\gamma_D\colon FGD \to D$ is in $\mathcal{W}_\mathcal{D}$ for all $D \in \mathcal{D}$.

Then F and G induce an equivalence $\mathcal{C}[\mathcal{W}_\mathcal{C}^{-1}] \simeq \mathcal{D}[\mathcal{W}_\mathcal{D}^{-1}]$.

Proof. Let $\kappa\mathcal{W}_\mathcal{C}$ and $\kappa\mathcal{W}_\mathcal{D}$ be Kan complexes that are fibrant replacements for $\mathcal{W}_\mathcal{C}$ and $\mathcal{W}_\mathcal{D}$ in the usual model structure on simplicial sets. Then the ∞-categories $\mathcal{C}[\mathcal{W}_\mathcal{C}^{-1}]$ and $\mathcal{D}[\mathcal{W}_\mathcal{D}^{-1}]$ can be described as the homotopy pushouts

$$\begin{array}{ccccccc} \mathcal{W}_{\mathcal{C}} & \longrightarrow & \kappa\mathcal{W}_{\mathcal{C}} & \qquad & \mathcal{W}_{\mathcal{D}} & \longrightarrow & \kappa\mathcal{W}_{\mathcal{D}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}], & & \mathcal{D} & \longrightarrow & \mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}] \end{array}$$

in the Joyal model structure. Then from (1) and (2) it is clear that the functors F and G induce functors $F'\colon \mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}] \to \mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}]$ and $G'\colon \mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}] \to \mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}]$, and the natural transformations η and γ induce natural transformations $\eta'\colon \mathrm{id} \to G'F'$ and $\gamma'\colon F'G' \to \mathrm{id}$. The objects of $\mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}]$ and $\mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}]$ can be taken to be the same as those of $\mathcal{C}$ and $\mathcal{D}$, so by (3) and (4) the morphisms η'_c and γ'_d are equivalences for all $c \in \mathcal{C}[\mathcal{W}_{\mathcal{C}}^{-1}]$ and $d \in \mathcal{D}[\mathcal{W}_{\mathcal{D}}^{-1}]$. Thus η' and γ' are natural equivalences and F' and G' are hence equivalences of ∞-categories. □

Lemma 5.3.15. *Suppose $\mathcal{W}$ is an ∞-category and $\pi\colon \mathcal{E} \to \kappa\mathcal{W}$ is a Cartesian fibration. Let $\pi'\colon \mathcal{E}' \to \mathcal{W}$ denote the pullback of π along the canonical map $\eta\colon \mathcal{W} \to \kappa\mathcal{W}$. Then $\mathcal{E}$ is the localization of $\mathcal{E}'$ with respect to $\mathcal{W} \times_{\kappa\mathcal{W}} \iota\mathcal{E}$, i.e. the morphisms in $\mathcal{E}'$ that map to equivalences in $\mathcal{E}$.*

Proof. Let $F\colon \kappa\mathcal{W}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a functor classified by π. Then π' is classified by the composite functor $\eta^{\mathrm{op}} \circ F\colon \mathcal{W}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. By [25, Corollary 4.1.2.6], the functor η^{op} is cofinal, hence by [25, Proposition 4.1.1.8] the functors F and $\eta^{\mathrm{op}} \circ F$ have the same colimit. But by [25, Corollary 3.3.4.3], the colimit of F is the localization of $\mathcal{E}$ with respect to the π-Cartesian morphisms, and the colimit of $\eta^{\mathrm{op}} \circ F$ is the localization of $\mathcal{E}'$ with respect to the π'-Cartesian morphisms. But since $\kappa\mathcal{W}$ is a Kan complex, the π-Cartesian morphisms in $\mathcal{E}$ are precisely the equivalences, hence it follows that $\mathcal{E}$ is the localization of $\mathcal{E}'$ with respect to the π'-Cartesian morphisms. But the π'-Cartesian morphisms in $\mathcal{E}'$ are precisely the morphisms that map to equivalences in $\mathcal{E}$, by [25, Remark 2.4.1.12]. □

Proposition 5.3.16. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{W}$ a subcategory of $\mathcal{C}$ containing the equivalences. Suppose we have a pushout square in* Cat_∞

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \kappa\mathcal{W} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C}[\mathcal{W}^{-1}], \end{array}$$

and a Cartesian fibration $\pi\colon \mathcal{E} \to \mathcal{C}[\mathcal{W}^{-1}]$. Write $\pi'\colon \mathcal{E}' \to \mathcal{C}$ for the pullback of π along $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$. Then the map $\mathcal{E}' \to \mathcal{E}$ exhibits $\mathcal{E}$ as the localization of $\mathcal{E}'$ with respect to $\mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \iota\mathcal{E}$, i.e. the morphisms in $\mathcal{E}'$ that map to equivalences in $\mathcal{E}$ and to $\mathcal{W}$ under the projection to $\mathcal{C}$.

Proof. Since π is a Cartesian fibration it follows from [25, Corollary 2.4.4.5] that the given pushout square pulls back along π to a pushout square

$$\begin{array}{ccc} \mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \mathcal{E} & \longrightarrow & \kappa\mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & \mathcal{E}. \end{array}$$

It therefore suffices to show that we have a pushout square

$$\begin{array}{ccc} \mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \iota\mathcal{E} & \longrightarrow & \kappa\mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \iota\mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \mathcal{E} & \longrightarrow & \kappa\mathcal{W} \times_{\mathcal{C}[\mathcal{W}^{-1}]} \mathcal{E}, \end{array}$$

which follows from Lemma 5.3.15. □

Theorem 5.3.17. *Suppose $\mathcal{V}$ is a monoidal ∞-category. Define $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}$ by the pullback*

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}} & \xrightarrow{i} & \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathrm{Set} & \longrightarrow & \mathcal{S} \end{array}$$

where the bottom horizontal map is the obvious inclusion. Then the functor i induces an equivalence

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}[\mathrm{FFES}^{-1}] \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})[\mathrm{FFES}^{-1}]$$

after inverting the fully faithful and essentially surjective functors.

Proof. Considering $\mathcal{S}$ as the ∞-category associated to the usual model structure on simplicial sets, we get a functor $j \colon \mathrm{Set}_\Delta \to \mathcal{S}$ that exhibits $\mathcal{S}$ as the localization of Set_Δ with respect to the weak equivalences. Let $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_\Delta$ be the ∞-category defined by the pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_\Delta & \xrightarrow{j'} & \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathrm{Set}_\Delta & \xrightarrow[j]{} & \mathcal{S}. \end{array}$$

Then $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}$ is the pullback of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\Delta}$ along the inclusion $\mathrm{Set} \to \mathrm{Set}_{\Delta}$ of the constant simplicial sets. This has a right adjoint $(-)_0\colon \mathrm{Set}_{\Delta} \to \mathrm{Set}$ that sends a simplicial set to its set of 0-simplices. The inclusion

$$i'\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\Delta}$$

therefore has a right adjoint

$$s\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\Delta} \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}$$

that sends an object $(X \in \mathrm{Set}_{\Delta}, \mathcal{C} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}))$ to the pullback of $\mathcal{C}$ along the morphism $X_0 \to X \to \iota_0\mathcal{C}$. It is clear that i' preserves fully faithful and essentially surjective functors, as does s by the 2-out-of-3 property. Moreover, $si \simeq \mathrm{id}$ and the counit $is(\mathcal{C}) \to \mathcal{C}$ is fully faithful and essentially surjective for all $\mathcal{C}$. It then follows from Lemma 5.3.14 that i' induces an equivalence

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}\big[\mathrm{FFES}^{-1}\big] \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\Delta}\big[\mathrm{FFES}^{-1}\big]$$

after inverting the fully faithful and essentially surjective functors. Moreover, by Proposition 5.3.16 the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\Delta}$ with respect to the morphisms that induce weak equivalences in Set_{Δ} and project to equivalences in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. These are obviously among the fully faithful and essentially surjective functors, and so j' induces an equivalence

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\Delta}\big[\mathrm{FFES}^{-1}\big] \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})\big[\mathrm{FFES}^{-1}\big].$$

Composing these two equivalences gives the result. □

Remark 5.3.18. Combining this result with Corollary 4.2.8 it follows that the localized ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})[\mathrm{FFES}^{-1}]$ is equivalent to the preliminary definition of an ∞-category of $\mathcal{V}$–∞-categories we discussed in §2.2, using the ∞-operads associated to the multicategories $\mathbf{O}_S$ with S a set.

5.4. Local equivalences

In this subsection we consider the strongly saturated class of maps generated by $s^0\colon E^1 \to E^0$; we call these the *local equivalences*. We assume throughout that $\mathcal{V}$ is a presentably monoidal ∞-category, so that $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is a presentable ∞-category by Proposition 4.3.5.

Definition 5.4.1. The *local equivalences* in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ are the elements of the strongly saturated class of morphisms generated by the map $s^0\colon E^1 \to E^0$.

Proposition 5.4.2. *The following are equivalent, for a $\mathcal{V}$–∞-category $\mathcal{C}$:*

(i) *$\mathcal{C}$ is complete.*
(ii) *$\mathcal{C}$ is local with respect to $E^1 \to E^0$, i.e. the map $\mathrm{Map}(E^0, \mathcal{C}) \to \mathrm{Map}(E^1, \mathcal{C})$ is an equivalence.*
(iii) *For every local equivalence $\mathcal{A} \to \mathcal{B}$, the induced map*

$$\mathrm{Map}(\mathcal{B}, \mathcal{C}) \to \mathrm{Map}(\mathcal{A}, \mathcal{C})$$

is an equivalence.

Proof. (i) is equivalent to (ii) by Corollary 5.2.10, and (ii) is equivalent to (iii) by [25, Proposition 5.5.4.15(4)]. □

Definition 5.4.3. Write $\mathrm{Cat}_\infty^{\mathcal{V}}$ for the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ spanned by the complete $\mathcal{V}$–∞-categories.

Proposition 5.4.4. *The inclusion $\mathrm{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has a left adjoint, which exhibits $\mathrm{Cat}_\infty^{\mathcal{V}}$ as the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to the local equivalences.*

Proof. The ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is presentable by Proposition 4.3.5, and the local equivalences are generated by a set of maps. The existence of the left adjoint therefore follows from [25, Proposition 5.5.4.15(4)] and Proposition 5.4.2. □

Corollary 5.4.5. *The ∞-category $\mathrm{Cat}_\infty^{\mathcal{V}}$ is presentable.*

Proof. This follows from [25, Proposition 5.5.4.15(3)]. □

Theorem 5.4.6. *$\mathrm{Cat}_\infty^{\mathcal{S}}$ is equivalent to Cat_∞.*

Proof. Under the equivalence $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{Seg}_\infty$ of Theorem 4.4.7, the subcategory $\mathrm{Cat}_\infty^{\mathcal{S}}$ corresponds to the subcategory of *complete* Segal spaces. It is proved in [21] that this is equivalent to Cat_∞. □

Lemma 5.4.7. *The map $\mathrm{id} \otimes s^0 \colon E^1 \otimes E^1 \to E^1 \otimes E^0 \simeq E^1$ is a local equivalence.*

Proof. Using Proposition 5.1.11 it suffices to prove this when $\mathcal{V}$ is $\mathcal{S}$. We can then identify $E^1 \otimes E^1$ with $E^{\{0,1\} \times \{0,1\}} \simeq E^3$; under this identification the map $E^1 \otimes E^1 \to E^1$ is induced by the map from $\{0,1,2,3\}$ to $\{0,1\}$ that sends $0, 1$ to 0 and $2, 3$ to 1. Under the equivalence $E^3 \simeq E^{\{0,1\}} \amalg_{E^{\{1\}}} E^{\{1,2\}} \amalg_{E^{\{2\}}} E^{\{2,3\}}$ implied by Theorem 5.2.5 this corresponds to

$$s^0 \cup \mathrm{id} \cup s^0 \colon E^1 \amalg_{E_0} E^1 \amalg_{E^0} E^1 \to E^0 \amalg_{E^0} E^1 \amalg_{E^0} E^0,$$

which is clearly in the strongly saturated class generated by s^0. □

Lemma 5.4.8. *If $\mathcal{C}$ is a complete $\mathcal{V}$–∞-category, then the $\mathcal{V}$–∞-category $\mathcal{C}^{E^1}$ is also complete.*

Proof. We need to show that the natural map $\iota_0\mathcal{C}^{E^1} \to \iota_1\mathcal{C}^{E^1}$ is an equivalence. Using the adjunction between cotensoring and tensoring we can identify this with the map $\mathrm{Map}(E^1, \mathcal{C}) \to \mathrm{Map}(E^1 \otimes E^1, \mathcal{C})$ induced by composition with $\mathrm{id} \otimes s^0$. This map is an equivalence since $\mathcal{C}$ is complete and $\mathrm{id} \otimes s^0$ is a local equivalence by Lemma 5.4.7. □

5.5. Categorical equivalences

In this subsection we study *categorical equivalences* between enriched ∞-categories, which are functors with an inverse up to natural equivalence. Our main result is that categorical equivalences are always local equivalences as well as fully faithful and essentially surjective. We begin by defining natural equivalences between $\mathcal{V}$-functors:

Definition 5.5.1. Suppose $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{V}$–∞-categories and $F, G : \mathcal{A} \to \mathcal{B}$ are $\mathcal{V}$-functors. A *natural equivalence* from F to G is a functor $H : \mathcal{A} \otimes E^1 \to \mathcal{B}$ such that $H \circ (\mathrm{id} \otimes d^1) \simeq F$ and $H \circ (\mathrm{id} \otimes d^0) \simeq G$. We say that F and G are *naturally equivalent* if there exists a natural equivalence from F to G.

Definition 5.5.2. A functor $F : \mathcal{A} \to \mathcal{B}$ is a *categorical equivalence* if there exists a functor $G : \mathcal{B} \to \mathcal{A}$ and natural equivalences ϕ from $F \circ G$ to $\mathrm{id}_{\mathcal{B}}$ and ψ from $G \circ F$ to $\mathrm{id}_{\mathcal{A}}$. Such a functor G is called a *pseudo-inverse* of F; we refer to (F, G, ϕ, ψ) as a *categorical equivalence datum*.

Proposition 5.5.3. *Categorical equivalences are fully faithful and essentially surjective.*

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a categorical equivalence, and let (F, G, ϕ, ψ) be a categorical equivalence datum. For each object X in $\iota_0\mathcal{D}$ the natural equivalence ψ supplies an equivalence between X and $FG(X)$, which is in the image of F, so F is essentially surjective.

To prove that F is fully faithful, we must show that, given X, Y in $\mathcal{C}$, the map $\alpha : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ induced by F is an equivalence in $\mathcal{V}$.

The natural equivalence ϕ supplies an equivalence

$$\beta : \mathcal{C}(GFX, GFY) \to \mathcal{C}(X, Y)$$

and a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(X, Y) & \xrightarrow{\qquad\qquad} & \mathcal{C}(GFX, GFY) \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \beta} & \\ & \mathcal{C}(X, Y). & \end{array}$$

The top map is the composite

$$\mathcal{C}(X,Y) \xrightarrow{\alpha} \mathcal{D}(FX,FY) \xrightarrow{\gamma} \mathcal{C}(GFX,GFY),$$

where γ is the map induced by G, and so we see that $\beta \circ \gamma \circ \alpha \simeq \mathrm{id}$.

From $F \circ \phi$ we likewise get an equivalence

$$\epsilon: \mathcal{D}(FGFX, FGFY) \to \mathcal{D}(FX, FY)$$

and a commutative diagram

$$\begin{array}{ccc} \mathcal{D}(FX,FY) & \longrightarrow & \mathcal{D}(FGFX,FGFY) \\ & {\scriptstyle \mathrm{id}}\searrow \quad \swarrow{\scriptstyle \epsilon} & \\ & \mathcal{D}(FX,FY), & \end{array}$$

where the top map is the composite

$$\mathcal{D}(FX,FY) \xrightarrow{\gamma} \mathcal{C}(GFX,GFY) \xrightarrow{\delta} \mathcal{D}(FGFX,FGFY),$$

and so $\epsilon \circ \delta \circ \gamma \simeq \mathrm{id}$. Moreover, we have a commutative square

$$\begin{array}{ccc} \mathcal{C}(GFX,GFY) & \xrightarrow{\delta} & \mathcal{D}(FGFX,FGFY) \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ \mathcal{C}(X,Y) & \xrightarrow[\alpha]{} & \mathcal{D}(FX,FY), \end{array}$$

thus we get $\alpha \circ \beta \circ \gamma \simeq \epsilon \circ \delta \circ \gamma \simeq \mathrm{id}$. This shows that $\beta \circ \gamma$ is an inverse of α, and so α is an equivalence in $\mathcal{V}$. Thus F is fully faithful. □

Corollary 5.5.4. *A categorical equivalence between complete $\mathcal{V}$-∞-categories is an equivalence.*

Proof. Combine Proposition 5.5.3 and Corollary 5.3.8. □

Our next goal is to prove that categorical equivalences are local equivalences; this will require some preliminary results:

Proposition 5.5.5. *Categorical equivalences satisfy the 2-out-of-3 property.*

Proof. Suppose we have functors $F: \mathcal{C} \to \mathcal{D}$ and $F': \mathcal{D} \to \mathcal{E}$. There are three cases to consider:

(1) Suppose (F, G, ϕ, ψ) and (F', G', ϕ', ψ') are categorical equivalence data. Then $G \circ \phi' \circ (F \otimes \mathrm{id})$ is a natural equivalence from $GG'F'F$ to GF. Combining this with ϕ gives a map

$$\left(\mathcal{C} \otimes E^1\right) \amalg_{\mathcal{C} \otimes E^0} \left(\mathcal{C} \otimes E^1\right) \to \mathcal{C}.$$

But tensoring with $\mathcal{C}$ preserves colimits, and $E^1 \amalg_{E^0} E^1 \simeq E^2$ by Theorem 5.2.5, so we get a map $\mathcal{C} \otimes E^2 \to \mathcal{C}$. Composing with $\mathrm{id} \otimes d^1 \colon \mathcal{C} \otimes E^1 \to \mathcal{C} \otimes E^2$ we get a natural equivalence from $GG'F'F$ to the identity. Using the same argument we can also combine $F' \circ \psi \circ (G' \otimes \mathrm{id})$ and ψ' to get a natural equivalence from $F'FGG'$ to the identity. Thus $F'F$ is a categorical equivalence with pseudo-inverse GG'.

(2) Suppose (F, G, ϕ, ψ) and $(F'F, H, \alpha, \beta)$ are categorical equivalence data. We will show that FH is a pseudo-inverse of F'. Since β is a natural equivalence from $F'(FH)$ to id it remains to construct a natural equivalence from FHF' to id. Let $\overline{\psi}$ denote $\psi \circ (\mathrm{id} \otimes E_\sigma)$, where $\sigma \colon \{0,1\} \to \{0,1\}$ is the map that interchanges 0 and 1 (thus $\overline{\psi}$ is ψ considered as a natural equivalence from id to FG). Combining $FHF' \circ \overline{\psi}$, $F \circ \alpha \circ g$ and ψ we get a map

$$\mathcal{D} \otimes E^3 \simeq \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \to \mathcal{D}$$

and composing with $\mathcal{D} \otimes E_{\{0,3\}} \to \mathcal{D} \otimes E^3$ we get the required natural equivalence.

(3) Suppose (F', G', ϕ', ψ') and $(F'F, H, \alpha, \beta)$ are categorical equivalence data. We will show that HF' is a pseudo-inverse of F. Since α is a natural equivalence from $HF'F$ to id it remains to construct a natural equivalence from FHF' to id. Let $\overline{\phi}'$ denote $\phi \circ (\mathrm{id} \otimes E_\sigma)$; combining $\overline{\phi}' \circ FHF'$, $G' \circ \beta \circ F'$ and ϕ' we get a map

$$\mathcal{D} \otimes E^3 \simeq \mathcal{D}e \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \amalg_{\mathcal{D}} \mathcal{D} \otimes E^1 \to \mathcal{D},$$

and composing with $\mathcal{D} \otimes E_{\{0,3\}} \to \mathcal{D} \otimes E^3$ we get the required natural equivalence. □

For the rest of this subsection we will for convenience assume that $\mathcal{V}$ is a presentably monoidal ∞-category.

Corollary 5.5.6. *Suppose $f \colon S \to T$ is a map of sets. Then $E_f \colon E_S \to E_T$ is a categorical equivalence.*

Proof. By Proposition 5.1.11 it suffices to prove this in $\mathcal{S}$. First suppose f is surjective; let $g \colon T \hookrightarrow S$ be a section of f. We claim that E_g is a pseudo-inverse to E_f. We have $E_f \circ E_g \simeq E_{f \circ g} \simeq \mathrm{id}$, so it suffices to construct a natural equivalence $E_S \times E^1 \simeq E_{S \times \{0,1\}} \to E_S$ from $E_{g \circ f}$ to the identity. This is given by E_h where $h \colon S \times \{0,1\} \to S$ sends $(s, 0)$ to $gf(s)$ and $(s, 1)$ to s.

By the dual argument the result holds if f is injective. By Proposition 5.5.5 we can therefore conclude that it holds for a general f. □

Lemma 5.5.7. *Suppose $F: \mathcal{A} \to \mathcal{B}$ is a categorical equivalence of $\mathcal{S}$–∞-categories. Then for any $\mathcal{V}$–∞-category $\mathcal{C}$ the induced map $\mathcal{C}^F: \mathcal{C}^{\mathcal{B}} \to \mathcal{C}^{\mathcal{A}}$ is a categorical equivalence.*

Proof. A natural equivalence $\mathcal{A} \otimes E^1 \to \mathcal{A}$ induces a natural equivalence

$$\mathcal{C}^{\mathcal{A}} \otimes E^1 \to \mathcal{C}^{\mathcal{A}}$$

by taking the adjoint of the induced map $\mathcal{C}^{\mathcal{A}} \to \mathcal{C}^{\mathcal{A} \otimes E^1} \simeq (\mathcal{C}^{\mathcal{A}})^{E^1}$. □

Lemma 5.5.8. *If $\mathcal{C}$ is a complete $\mathcal{V}$–∞-category, then the natural map*

$$\mathcal{C}^{s^0}: \mathcal{C} \simeq \mathcal{C}^{E^0} \to \mathcal{C}^{E^1}$$

is an equivalence.

Proof. The map $s^0: E^1 \to E^0$ is a categorical equivalence by Corollary 5.5.6, so it follows by Lemma 5.5.7 that $\mathcal{C} \to \mathcal{C}^{E^1}$ is also a categorical equivalence. But $\mathcal{C}^{E^1}$ is complete by Lemma 5.4.8, and a categorical equivalence between complete objects is an equivalence by Corollary 5.5.4. □

Proposition 5.5.9. *For any $\mathcal{V}$–∞-category $\mathcal{C}$, the map $\mathrm{id} \otimes s^0: \mathcal{C} \otimes E^1 \to \mathcal{C} \otimes E^0 \simeq \mathcal{C}$ is a local equivalence.*

Proof. We must show that for any complete $\mathcal{V}$–∞-category $\mathcal{D}$ the map

$$\mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}\big(\mathcal{C} \otimes E^1, \mathcal{D}\big)$$

is an equivalence. Using the adjunction between tensoring and cotensoring with E^1, we see that this map is equivalent to the map

$$\mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}\big(\mathcal{C}, \mathcal{D}^{E^1}\big)$$

given by composing with $\mathcal{D}^{s^0}: \mathcal{D} \to \mathcal{D}^{E^1}$. This is an equivalence by Lemma 5.5.8. □

Corollary 5.5.10. *Suppose $\mathcal{D}$ is a complete $\mathcal{V}$–∞-category; then for any $\mathcal{V}$–∞-category $\mathcal{C}$ we have*

$$|\,\mathrm{Map}(\mathcal{C} \otimes E^{\bullet}, \mathcal{D})| \simeq \mathrm{Map}(\mathcal{C}, \mathcal{D}).$$

Proof. The simplicial space $\mathrm{Map}(\mathcal{C} \otimes E^{\bullet}, \mathcal{D})$ is a groupoid object in spaces, since $E^{\bullet}$ is a cogroupoid object by Theorem 5.2.5 and tensoring preserves colimits. By Lemma 5.2.9 it therefore suffices to show that $\mathrm{Map}(\mathcal{C} \otimes E^0, \mathcal{D}) \to \mathrm{Map}(\mathcal{C} \otimes E^1, \mathcal{D})$ is an equivalence, which holds by Proposition 5.5.9. □

Remark 5.5.11. The left-hand side here is what we would expect the mapping space to be in the ∞-category underlying an $(\infty, 2)$-category of $\mathcal{V}$–∞-categories, functors, and natural transformations. This shows that the mapping spaces between complete $\mathcal{V}$–∞-categories are the correct ones.

Lemma 5.5.12. *Suppose $\mathcal{D}$ is a complete $\mathcal{V}$–∞-category. Then for any $\mathcal{V}$–∞-category $\mathcal{C}$ the two maps*

$$\big(\mathrm{id} \otimes d^0\big)^*, \big(\mathrm{id} \otimes d^1\big)^*\colon \mathrm{Map}\big(\mathcal{C} \otimes E^1, \mathcal{D}\big) \to \mathrm{Map}(\mathcal{C}, \mathcal{D})$$

are homotopic.

Proof. Clearly $(\mathrm{id}\otimes s^0)^* \circ (\mathrm{id}\otimes d^i)^*\colon \mathrm{Map}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}(\mathcal{C}, \mathcal{D})$ is homotopic to the identity for $i = 0, 1$. But by Proposition 5.5.9, the map $(\mathrm{id} \otimes s^0)$ is a local equivalence, hence $(\mathrm{id} \otimes s^0)^*$ is an equivalence since $\mathcal{D}$ is complete. Composing with its inverse we get that

$$(\mathrm{id} \otimes d^0)^* \simeq (\mathrm{id} \otimes d^1)^*,$$

as required. □

Theorem 5.5.13. *Categorical equivalences are local equivalences.*

Proof. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a categorical equivalence and (F, G, ϕ, ψ) is a categorical equivalence datum. If $\mathcal{E}$ is a complete $\mathcal{V}$–∞-category we must show that the map

$$F^*\colon \mathrm{Map}(\mathcal{C}, \mathcal{E}) \to \mathrm{Map}(\mathcal{D}, \mathcal{E})$$

given by composition with F is an equivalence of spaces. By Lemma 5.5.12 we have equivalences

$$G^*F^* \simeq \phi^* \circ \big(\mathrm{id} \otimes d^1\big)^* \simeq \phi^* \circ \big(\mathrm{id} \otimes d^0\big)^* \simeq \mathrm{id},$$
$$F^*G^* \simeq \psi^* \circ \big(\mathrm{id} \otimes d^1\big)^* \simeq \psi^* \circ \big(\mathrm{id} \otimes d^0\big)^* \simeq \mathrm{id}.$$

Thus G^* is an inverse of F^*, and so F^* is indeed an equivalence. □

5.6. Completion

We will now construct an explicit completion functor, analogous to Rezk's completion functor for Segal spaces in [34, §14], when $\mathcal{V}$ is a presentably monoidal ∞-category. Using this we can then show that the local equivalences are precisely the fully faithful and essentially surjective functors.

Definition 5.6.1. If $\mathcal{C}$ is a $\mathcal{V}$–∞-category, let $\widehat{\mathcal{C}}$ denote the geometric realization $|\mathcal{C}^{E^\bullet}|$.

Theorem 5.6.2. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category and $\mathcal{C}$ is a $\mathcal{V}$–∞-category. The natural map $\mathcal{C} \to \widehat{\mathcal{C}}$ is both a local equivalence and fully faithful and essentially surjective. Moreover, the $\mathcal{V}$–∞-category $\widehat{\mathcal{C}}$ is complete.*

Proof. The functors $E^n \to E^m$ induced by the maps $[n] \to [m]$ in $\mathbf{\Delta}$ are categorical equivalences by Corollary 5.5.6, so the induced functors $\mathcal{C}^{E^m} \to \mathcal{C}^{E^n}$ are also categorical equivalences by Lemma 5.5.7. These functors are therefore all fully faithful and essentially surjective by Proposition 5.5.3, and local equivalences by Theorem 5.5.13. Local equivalences are by definition closed under colimits, so it follows that the map $\mathcal{C} \to \widehat{\mathcal{C}}$ is a local equivalence.

Since ι_0 preserves colimits, the map $\iota_0\mathcal{C} \to \iota_0\widehat{\mathcal{C}} \simeq \iota\mathcal{C}$ is surjective on π_0, and so the functor $\mathcal{C} \to \widehat{\mathcal{C}}$ is essentially surjective. To see that this functor is also fully faithful, we consider the model for categorical algebras as Segal presheaves from §4.5. If $\mathcal{V}$ is κ-presentable, then the colimit $\widehat{\mathcal{C}}$ in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \simeq \mathcal{P}(\mathcal{V}_{\otimes}^{\vee})^{\mathrm{Seg}}$ can be described as a localization of the colimit $\widehat{F}$ of the diagram $F_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{P}((\mathcal{V}_{\otimes}^{\vee})^{\kappa})$ corresponding to $\mathcal{C}^{E^\bullet}$. The colimit $\widehat{F}$ can be computed objectwise, and in fact is already local: Given $X \in (\mathcal{V}_{\otimes}^{\vee})_{[k]}$, we know that for every $\phi \colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ the diagram

$$\begin{array}{ccc} F_m(X) & \longrightarrow & F_n(X) \\ \downarrow & & \downarrow \\ F_m()^{\times(k+1)} & \longrightarrow & F_n()^{\times(k+1)} \end{array}$$

is a pullback square. Since $\mathcal{S}$ is an ∞-topos, by [25, Theorem 6.1.3.9] it follows that the square

$$\begin{array}{ccc} F_0(X) & \longrightarrow & \widehat{F}(X) \\ \downarrow & & \downarrow \\ F_0()^{\times(k+1)} & \longrightarrow & \widehat{F}()^{\times(k+1)} \end{array}$$

is also a pullback square. From this we conclude that $\widehat{F}$ is also a Segal presheaf, since the map $\widehat{F}(X) \to \widehat{F}()^{\times(k+1)} \simeq |F_\bullet()|^{\times(k+1)}$ has the same fibres as $F_0(X) \to F_0()^{\times(k+1)}$ and $F_0() \to \widehat{F}()$ is surjective on π_0. Using the equivalence between Segal presheaves and categorical algebras of Theorem 4.5.3 we conclude that $\mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful, as the object $\widehat{\mathcal{C}}(x, y)$ is determined by the fibre of $\widehat{F}(A) \to \widehat{F}()^{\times 2} \simeq (\iota\mathcal{C})^{\times 2}$ at (x, y) for all $A \in \mathcal{V}$.

It remains to prove that $\widehat{\mathcal{C}}$ is complete, i.e. that the map $\iota_0\widehat{\mathcal{C}} \to \iota_1\widehat{\mathcal{C}}$ is an equivalence. We have a commutative diagram

$$\begin{array}{ccc} |\iota_0 \mathcal{C}^{E^\bullet}| & \longrightarrow & \iota_0 \widehat{\mathcal{C}} \\ \downarrow & & \downarrow \\ |\iota_1 \mathcal{C}^{E^\bullet}| & \longrightarrow & \iota_1 \widehat{\mathcal{C}}, \end{array}$$

where the top horizontal morphism is an equivalence since ι_0 preserves colimits. The left vertical map is also an equivalence: We have equivalences $\iota_1 \mathcal{C}^{E^n} \simeq \mathrm{Map}(E^1 \otimes E^n, \mathcal{C}) \simeq \iota_n \mathcal{C}^{E^1}$, so $|\iota_1 \mathcal{C}^{E^\bullet}| \simeq \iota \mathcal{C}^{E^1}$, and under this equivalence the left vertical map corresponds to that induced by the natural map $\mathcal{C} \to \mathcal{C}^{E^1}$; we know that this is fully faithful and essentially surjective, and so induces an equivalence on ι by Proposition 5.3.6. In order to show that $\widehat{\mathcal{C}}$ is complete, it thus suffices to show that the bottom horizontal map $|\iota_1 \mathcal{C}^{E^\bullet}| \to \iota_1 \widehat{\mathcal{C}}$ is an equivalence.

Consider the commutative diagram

$$\begin{array}{ccc} |\iota_1 \mathcal{C}^{E^\bullet}| & \longrightarrow & \iota_1 \widehat{\mathcal{C}} \\ \downarrow & & \downarrow \\ |\iota_0 \mathcal{C}^{E^\bullet}|^{\times 2} & \longrightarrow & \iota_0 \widehat{\mathcal{C}}^{\times 2}, \end{array}$$

with the vertical maps coming from the maps $d^0, d^1 \colon E^0 \to E^1$. Here the bottom horizontal map is an equivalence, so to prove that the top horizontal map is an equivalence it suffices to prove that this is a pullback square. Since $\mathcal{C} \to \widehat{\mathcal{C}}$ is essentially surjective, to see this we need only show that for all $(X, Y) \in \iota_0 \mathcal{C}^{\times 2}$ the induced map on fibres $|\iota_1 \mathcal{C}^{E^\bullet}|_{(X,Y)} \to \iota_1 \widehat{\mathcal{C}}_{(X,Y)}$ is an equivalence.

Since $\mathcal{C}^{E^m} \to \mathcal{C}^{E^n}$ is fully faithful and essentially surjective for all $[n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$, the map $\iota \mathcal{C}^{E^m} \to \iota \mathcal{C}^{E^n}$ is an equivalence by Proposition 5.3.6. Therefore, as the groupoid objects $\iota_\bullet \mathcal{C}^{E^m}$ and $\iota_\bullet \mathcal{C}^{E^n}$ are effective, the diagram

$$\begin{array}{ccc} \iota_1 \mathcal{C}^{E^m} & \longrightarrow & \iota_1 \mathcal{C}^{E^n} \\ \downarrow & & \downarrow \\ (\iota_0 \mathcal{C}^{E^m})^{\times 2} & \longrightarrow & (\iota_0 \mathcal{C}^{E^n})^{\times 2} \end{array}$$

is a pullback square. In other words, the natural transformation $\iota_1 \mathcal{C}^{E^\bullet} \to (\iota_0 \mathcal{C}^{E^\bullet})^{\times 2}$ is Cartesian. Applying [25, Theorem 6.1.3.9] again, we see that the extended natural transformation of functors $(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathcal{S}$ that includes the colimits is also Cartesian. Thus we have a pullback square

$$\begin{array}{ccc} \iota_1\mathcal{C} & \longrightarrow & |\iota_1\mathcal{C}^{E^\bullet}| \\ \big\downarrow & & \big\downarrow \\ \iota_0\mathcal{C}^{\times 2} & \longrightarrow & |\iota_0\mathcal{C}^{E^\bullet}|^{\times 2}. \end{array}$$

In particular, for $(X, Y) \in \iota_0\mathcal{C}^{\times 2}$ the induced map on fibres $\iota_1\mathcal{C}_{(X,Y)} \to |\iota_1\mathcal{C}^{E^\bullet}|_{(X,Y)}$ is an equivalence. Since $\mathcal{C} \to \widehat{\mathcal{C}}$ is fully faithful and essentially surjective, the map $\iota_1\mathcal{C}_{(X,Y)} \to \iota_1\widehat{\mathcal{C}}_{(X,Y)}$ is also an equivalence by Lemma 5.3.5. By the 2-out-of-3 property it then follows that $|\iota_1\mathcal{C}^{E^\bullet}|_{(X,Y)} \to \iota_1\widehat{\mathcal{C}}_{(X,Y)}$ is an equivalence too. This completes the proof that $\widehat{\mathcal{C}}$ is complete. □

Corollary 5.6.3. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category. The following are equivalent, for a functor $F: \mathcal{C} \to \mathcal{D}$ of $\mathcal{V}$–∞-categories:*

(i) F is a local equivalence.
(ii) F is fully faithful and essentially surjective.

Proof. By Theorem 5.6.2 we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{C}} & \xrightarrow[\widehat{F}]{} & \widehat{\mathcal{D}}, \end{array}$$

where the vertical maps are both local equivalences and fully faithful and essentially surjective, and $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are complete.

Since local equivalences form a strongly saturated class of morphisms, it follows from the 2-out-of-3 property that F is a local equivalence if and only if $\widehat{F}$ is a local equivalence, i.e. if and only if $\widehat{F}$ is an equivalence, since $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are complete.

Fully faithful and essentially surjective functors also satisfy the 2-out-of-3 property, by Proposition 5.3.9, so F is fully faithful and essentially surjective if and only if $\widehat{F}$ is. But by Corollary 5.3.8 the functor $\widehat{F}$ is fully faithful and essentially surjective if and only if it is an equivalence, since $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are complete. Thus F is a local equivalence if and only if it is fully faithful and essentially surjective. □

Corollary 5.6.4. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category. The ∞-category $\mathrm{Cat}_\infty^{\mathcal{V}}$ is the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to the fully faithful and essentially surjective functors.*

Remark 5.6.5. We might expect that the fully faithful and essentially surjective functors also coincide with the categorical equivalences, but this turns out *not* to be the case when we allow spaces of objects. To see this, first observe that if $F\colon \mathcal{A} \to \mathcal{B}$ is a categorical equivalence, then for every $\mathcal{V}$–∞-category $\mathcal{C}$ the map

$$F_*\colon |\operatorname{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{A})| \to |\operatorname{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{B})|$$

is surjective on π_0: suppose $G\colon \mathcal{B} \to \mathcal{A}$ is a pseudo-inverse of F, then given a functor $\phi\colon \mathcal{C} \to \mathcal{B}$ the natural equivalence from $F \circ G$ to id gives a natural equivalence from $F \circ G \circ \phi$ to ϕ, so up to natural equivalence ϕ is in the image of F_*. Now if $\mathcal{B} \to \widehat{\mathcal{B}}$ is a categorical equivalence where $\widehat{\mathcal{B}}$ is complete, then by Corollary 5.5.10 we have $|\operatorname{Map}(\mathcal{C} \otimes E^\bullet, \widehat{\mathcal{B}})| \simeq \operatorname{Map}(\mathcal{C}, \widehat{\mathcal{B}})$, and since $\operatorname{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{B})$ is a groupoid object the map $\operatorname{Map}(\mathcal{C}, \mathcal{B}) \to |\operatorname{Map}(\mathcal{C} \otimes E^\bullet, \mathcal{B})|$ is surjective on π_0. Thus $\operatorname{Map}(\mathcal{C}, \mathcal{B}) \to \operatorname{Map}(\mathcal{C}, \widehat{\mathcal{B}})$ is surjective on π_0.

Now suppose $\iota_0\mathcal{B}$ is discrete and $\iota\mathcal{B}$ is not; then there clearly exists for some $n > 0$ a map from the n-sphere $S^n \to \iota\mathcal{B}$ that does not factor through $\iota_0\mathcal{B}$. But we have a $\mathcal{V}$–∞-category $S^n \otimes E^0$ such that $\operatorname{Map}(S^n \otimes E^0, \mathcal{B}) \simeq \operatorname{Map}(S^n, \iota_0\mathcal{B})$ — so if $\mathcal{B} \to \widehat{\mathcal{B}}$ were a categorical equivalence then $\operatorname{Map}(S^n, \iota_0\mathcal{B}) \to \operatorname{Map}(S^n, \iota\mathcal{B})$ would have to be surjective on π_0, a contradiction. Completion maps $\mathcal{B} \to \widehat{\mathcal{B}}$ therefore cannot be categorical equivalences in general.

We now deduce our main result for a general large monoidal ∞-category $\mathcal{V}$ from the presentable case, by embedding in a larger universe:

Theorem 5.6.6. *Let $\mathcal{V}$ be a large monoidal ∞-category. The inclusion of the full subcategory of complete $\mathcal{V}$–∞-categories $\mathrm{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ has a left adjoint that exhibits $\mathrm{Cat}_\infty^{\mathcal{V}}$ as the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ with respect to the fully faithful and essentially surjective functors.*

Proof. Let $\widehat{\mathcal{P}}(\mathcal{V})$ be the ∞-category of presheaves of large spaces on $\mathcal{V}$. By [28, Proposition 4.8.1.10] there exists a monoidal structure on $\widehat{\mathcal{P}}(\mathcal{V})$ such that the Yoneda embedding $j\colon \mathcal{V} \to \widehat{\mathcal{P}}(\mathcal{V})$ is a monoidal functor. Let $\widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V}))$ be the (very large) ∞-category of large categorical algebras in $\widehat{\mathcal{P}}(\mathcal{V})$; this is a presentable ∞-category, and writing $\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})}$ for its subcategory of complete $\widehat{\mathcal{P}}(\mathcal{V})$–$\infty$-categories we know from Corollary 5.6.4 that the inclusion

$$\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})} \hookrightarrow \widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V}))$$

has a left adjoint $\widehat{L}$ that exhibits $\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})}$ as the localization with respect to the fully faithful and essentially surjective functors.

If $\mathcal{C}$ is in the essential image of the fully faithful inclusion

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \hookrightarrow \widehat{\mathrm{Alg}}_{\mathrm{cat}}(\widehat{\mathcal{P}}(\mathcal{V})),$$

then the natural map $\mathcal{C} \to \widehat{L}\mathcal{C}$ is fully faithful and essentially surjective. But then $\iota_0\widehat{L}\mathcal{C} \simeq \iota\mathcal{C}$, so $\iota_0\widehat{L}\mathcal{C}$ is an (essentially) small space, and the mapping objects in $\widehat{L}\mathcal{C}$ are in the essential image of $\mathcal{V}$ in $\widehat{\mathcal{P}}(\mathcal{V})$. Thus $\widehat{L}\mathcal{C}$ is in the essential image of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, and so the functor $\widehat{L}$ restricts to a functor $L\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Cat}_\infty^{\mathcal{V}}$, since $\mathrm{Cat}_\infty^{\mathcal{V}}$ is equivalent to the full subcategory of $\widehat{\mathrm{Cat}}_\infty^{\widehat{\mathcal{P}}(\mathcal{V})}$ spanned by objects in the essential image of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. □

5.7. Properties of the localized ∞-category

In this subsection we observe that the localized ∞-category $\mathrm{Cat}_\infty^{\mathcal{V}}$ inherits the naturality properties of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. We first show that $\mathrm{Cat}_\infty^{\mathcal{V}}$ is functorial in $\mathcal{V}$:

Proposition 5.7.1. *Let*

$$\mathrm{Alg}_{\mathrm{cat}} \to \widehat{\mathrm{Mon}}_\infty^{\mathrm{lax}}$$

be a coCartesian fibration corresponding to the functor $\mathrm{Alg}_{\mathrm{cat}}(-)$. *Define* Enr_∞ *to be the full subcategory of* $\mathrm{Alg}_{\mathrm{cat}}$ *whose objects are the complete enriched ∞-categories. Then the restricted projection*

$$\mathrm{Enr}_\infty \to \widehat{\mathrm{Mon}}_\infty^{\mathrm{lax}}$$

is a coCartesian fibration, and the inclusion $\mathrm{Enr}_\infty \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}$ *admits a left adjoint over* $\widehat{\mathrm{Mon}}_\infty^{\mathrm{lax}}$.

This follows from a general result about fibrewise localizations of coCartesian fibrations that we prove first:

Lemma 5.7.2. *Suppose* $\mathcal{E} \to \Delta^1$ *is a coCartesian fibration, and* $\mathcal{E}'$ *is a full subcategory of* $\mathcal{E}$ *such that the inclusion* $\mathcal{E}'_1 \hookrightarrow \mathcal{E}_1$ *admits a left adjoint* $L\colon \mathcal{E}_1 \to \mathcal{E}'_1$. *Then the restriction* $\mathcal{E}' \to \Delta^1$ *is also a coCartesian fibration.*

Proof. We must show that for each $x \in \mathcal{E}'_0$ there exists a coCartesian arrow with source x over $0 \to 1$ in Δ^1. Suppose $\phi\colon x \to y$ is such a coCartesian arrow in $\mathcal{E}$, and let $y \to Ly$ be the unit of the adjunction. Then the composite $x \xrightarrow{\phi} y \to Ly$ is a coCartesian arrow in $\mathcal{E}'$: by [25, Proposition 2.4.4.3] it suffices to show that for all $z \in \mathcal{E}'_1$ the map $\mathrm{Map}_{\mathcal{E}'}(Ly, z) \to \mathrm{Map}_{\mathcal{E}'}(x, z)$ is an equivalence, which is clear since $\mathrm{Map}_{\mathcal{E}'}(Ly, z) \simeq \mathrm{Map}_{\mathcal{E}}(y, z)$ as $z \in \mathcal{E}'_1$, $\mathrm{Map}_{\mathcal{E}'}(x, z) \simeq \mathrm{Map}_{\mathcal{E}}(x, z)$ as $\mathcal{E}'$ is a full subcategory of $\mathcal{E}$, and $x \to y$ is a coCartesian morphism in $\mathcal{E}$. □

Lemma 5.7.3. *Let* $\mathcal{E} \to \mathcal{B}$ *be a locally coCartesian fibration and* $\mathcal{E}^0$ *a full subcategory of* $\mathcal{E}$ *such that for each* $b \in \mathcal{B}$ *the induced map on fibres* $\mathcal{E}^0_b \hookrightarrow \mathcal{E}_b$ *admits a left adjoint* $L_b\colon \mathcal{E}_b \to \mathcal{E}^0_b$. *Assume these localization functors are compatible in the sense that the following condition is satisfied:*

(∗) *Suppose $f: b \to b'$ is a morphism in $\mathcal{B}$ and e is an object of $\mathcal{E}_b$. Let $e \to e'$ and $L_b e \to e''$ be locally coCartesian arrows lying over f, and let $L_{b'}e' \to L_{b'}e''$ be the unique morphism such that the diagram*

$$\begin{array}{ccccc} e & \longrightarrow & e' & \longrightarrow & L_{b'}e' \\ \downarrow & & \downarrow & & \downarrow \\ L_b e & \longrightarrow & e'' & \longrightarrow & L_{b'}e'' \end{array}$$

commutes. Then the morphism $L_{b'}e' \to L_{b'}e''$ is an equivalence.

Then

(i) the composite map $\mathcal{E}^0 \to \mathcal{B}$ is also a locally coCartesian fibration,
(ii) the inclusion $\mathcal{E}^0 \hookrightarrow \mathcal{E}$ admits a left adjoint $L: \mathcal{E} \to \mathcal{E}^0$ relative to $\mathcal{B}$.

Proof. (i) is immediate from the previous lemma, and then (ii) follows from [28, Proposition 7.3.2.11] — condition (2) of this result is satisfied since, in the notation of condition (∗), a locally coCartesian arrow in $\mathcal{E}^0$ over f with source $L_b e$ is given by the composite $L_b e \to e'' \to L_{b'}e''$. □

Proposition 5.7.4. *Let $\mathcal{E} \to \mathcal{B}$ be a coCartesian fibration and $\mathcal{E}^0$ a full subcategory of $\mathcal{E}$. Suppose that for each $b \in \mathcal{B}$ the induced map on fibres $\mathcal{E}^0_b \hookrightarrow \mathcal{E}_b$ admits a left adjoint $L_b: \mathcal{E}_b \to \mathcal{E}^0_b$ and that the functors $\phi_!: \mathcal{E}_b \to \mathcal{E}_{b'}$ corresponding to morphisms $\phi: b \to b'$ in $\mathcal{B}$ preserve the fibrewise local equivalences. Then*

(i) the composite map $\mathcal{E}^0 \to \mathcal{B}$ is a coCartesian fibration,
(ii) the inclusion $\mathcal{E}^0 \hookrightarrow \mathcal{E}$ admits a left adjoint $L: \mathcal{E} \to \mathcal{E}^0$ over $\mathcal{B}$, and L preserves coCartesian arrows.

Proof. Lemma 5.7.3 implies (ii) and also that $\mathcal{E}^0 \to \mathcal{E} \to \mathcal{B}$ is a locally coCartesian fibration, since for a coCartesian fibration condition (∗) says precisely that fibrewise local equivalences are preserved by the functors $\phi_!$. By [25, Proposition 2.4.2.8] it remains to show that locally coCartesian morphisms are closed under composition. Suppose $f: b \to b'$ and $g: b' \to b''$ are morphisms in $\mathcal{B}$, and that $e \in \mathcal{E}^0_b$. Let $e \to e'$ be a coCartesian arrow in $\mathcal{E}$ over f, and let $e' \to e''_1$ and $L_{b'}e' \to e''_2$ be coCartesian arrows in $\mathcal{E}$ over g. Then a locally coCartesian arrow over f in $\mathcal{E}^0$ is given by $e \to e' \to L_{b'}e'$ and a locally coCartesian arrow over g is given by $L_{b'}e' \to e''_2 \to L_{b''}e''_2$. We have a commutative diagram

$$\begin{array}{ccccccc} e & \longrightarrow & e' & \longrightarrow & e_1'' & \longrightarrow & L_{b''}e_1'' \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & L_{b'}e' & \longrightarrow & e_2'' & \longrightarrow & L_{b''}e_2'' \end{array}$$

Here the composite along the top row is a locally coCartesian arrow for gf, and the composite along the bottom is the composite of locally coCartesian arrows for g and f. By condition $(*)$ of Lemma 5.7.3, the rightmost vertical morphism is an equivalence, hence the composite map $e \to L_{b''}e_2''$ is locally coCartesian. □

Lemma 5.7.5. *Suppose $\phi\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ is a lax monoidal functor. Then the induced functor*

$$\phi_*\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})$$

preserves fully faithful and essentially surjective morphisms.

Proof. It is obvious from the definitions that ϕ_* preserves fully faithful functors. To see that it preserves essentially surjective ones we note that if two points of $\iota_0\mathcal{C}$ are equivalent as objects of $\mathcal{C}$ then they are also equivalent as objects of $\phi_*\mathcal{C}$, since the map $I_{\mathcal{W}} \to \phi(I_{\mathcal{V}})$ induces a functor $E^1_{\mathcal{W}} \to \phi_* E^1_{\mathcal{V}}$. □

Proof of Proposition 5.7.1. The result follows by combining Proposition 5.7.4 and Lemma 5.7.5. □

Corollary 5.7.6. $\mathrm{Cat}_\infty^{\mathcal{V}}$ *is functorial in $\mathcal{V}$ with respect to lax monoidal functors of monoidal ∞-categories.*

Proof. The coCartesian fibration $\mathrm{Enr}_\infty \to \widehat{\mathrm{Mon}}_\infty^{\mathrm{lax}}$ of Proposition 5.7.1 classifies a functor $\widehat{\mathrm{Mon}}_\infty^{\mathrm{lax}} \to \mathrm{Cat}_\infty$ that sends a monoidal ∞-category $\mathcal{V}$ to $\mathrm{Cat}_\infty^{\mathcal{V}}$. □

Lemma 5.7.7. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories compatible with small colimits, and $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a monoidal functor such that $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Then the induced functor $F_*\colon \mathrm{Cat}_\infty^{\mathcal{V}} \to \mathrm{Cat}_\infty^{\mathcal{W}}$ preserves colimits.*

Proof. This functor F_* is the composite

$$\mathrm{Cat}_\infty^{\mathcal{V}} \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \xrightarrow{F_*^{\mathrm{Alg}}} \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W}) \xrightarrow{L_{\mathcal{W}}} \mathrm{Cat}_\infty^{\mathcal{W}},$$

where $L_{\mathcal{W}}$ is the completion functor for $\mathcal{W}$ and we write F_*^{Alg} for the functor on $\mathrm{Alg}_{\mathrm{cat}}$ induced by composition with F for clarity. By Lemma 5.7.5 the functor F_*^{Alg} preserves local equivalences, so $F_*^{\mathrm{Alg}} L_{\mathcal{V}}\mathcal{C}$ and $F_*^{\mathrm{Alg}}\mathcal{C}$ are locally equivalent for all $\mathcal{C}$; it follows that

$L_{\mathcal{W}} \circ F_*^{\mathrm{Alg}} \circ L_{\mathcal{V}} \simeq L_{\mathcal{W}} \circ F_*^{\mathrm{Alg}}$. If $\alpha \mapsto \mathcal{C}_\alpha$ is a diagram in $\mathrm{Cat}_\infty^{\mathcal{V}}$ then its colimit is $L_{\mathcal{V}}(\mathrm{colim}\, \mathcal{C}_\alpha)$ where this colimit is computed in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$. Thus we have

$$F_*(\mathrm{colim}\, \mathcal{C}_\alpha) \simeq L_{\mathcal{W}} F_*^{\mathrm{Alg}} L_{\mathcal{V}}(\mathrm{colim}\, \mathcal{C}_\alpha) \simeq L_{\mathcal{W}} F_*^{\mathrm{Alg}}(\mathrm{colim}\, \mathcal{C}_\alpha) \simeq \mathrm{colim}\, L_{\mathcal{W}} F_*^{\mathrm{Alg}}(\mathcal{C}_\alpha) \simeq \mathrm{colim}\, F_* \mathcal{C}_\alpha. \qquad \square$$

Proposition 5.7.8. *The restriction of the functor* $\mathrm{Cat}_\infty^{(-)}$ *to* $\mathrm{Mon}_\infty^{\mathrm{Pr}}$ *factors through* Pres_∞.

Proof. This follows from Lemma 5.7.7 and Corollary 5.4.5. $\square$

Proposition 5.7.9. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories and let $\mathcal{A}$ be a complete $\mathcal{V}$–∞-category and $\mathcal{B}$ a complete $\mathcal{W}$–∞-category. Then $\mathcal{A} \boxtimes \mathcal{B}$ is a complete $\mathcal{V} \times \mathcal{W}$–$\infty$-category.*

This follows from the following observation:

Lemma 5.7.10. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories and let $\mathcal{A}$ be a $\mathcal{V}$–∞-category and $\mathcal{B}$ a $\mathcal{W}$–∞-category. Then $\iota_\bullet(\mathcal{A} \boxtimes \mathcal{B})$ is naturally equivalent to $\iota_\bullet \mathcal{A} \times \iota_\bullet \mathcal{B}$, and $\iota(\mathcal{A} \boxtimes \mathcal{B})$ is naturally equivalent to $\iota \mathcal{A} \times \iota \mathcal{B}$,*

Proof. The "external product" $\boxtimes$ is clearly the Cartesian product in the ∞-category $\mathrm{Alg}_{\mathrm{cat}}$, and so it is easy to see that for any $\mathcal{V} \times \mathcal{W}$–$\infty$-category $\mathcal{C}$ we have

$$\mathrm{Map}(\mathcal{C}, \mathcal{A} \boxtimes \mathcal{B}) \simeq \mathrm{Map}(\pi_{1,*}\mathcal{C}, \mathcal{A}) \times \mathrm{Map}(\pi_{2,*}\mathcal{C}, \mathcal{B}),$$

where π_1 and π_2 denote the projections from $\mathcal{V} \times \mathcal{W}$ to $\mathcal{V}$ and $\mathcal{W}$, respectively. Moreover, $\pi_{i,*}E^S \simeq E^S$ for all S (since π_i obviously preserves the unit of the monoidal structure). Thus

$$\iota_\bullet(\mathcal{A} \boxtimes \mathcal{B}) \simeq \iota_\bullet \mathcal{A} \times \iota_\bullet \mathcal{B}.$$

Since colimits of simplicial objects commute with products it follows that $\iota(\mathcal{A} \boxtimes \mathcal{B}) \simeq \iota \mathcal{A} \times \iota \mathcal{B}$. $\square$

Proof of Proposition 5.7.9. By Lemma 5.7.10 we have a natural map

$$\iota_0(\mathcal{A} \boxtimes \mathcal{B}) \simeq \iota_0 \mathcal{A} \times \iota_0 \mathcal{B} \to \iota \mathcal{A} \times \iota \mathcal{B} \simeq \iota(\mathcal{A} \boxtimes \mathcal{B}).$$

This is an equivalence if $\mathcal{A}$ and $\mathcal{B}$ are complete, i.e. $\mathcal{A} \boxtimes \mathcal{B}$ is indeed also complete. $\square$

Corollary 5.7.11. $\mathrm{Cat}_\infty^{(-)}$ *is a lax monoidal functor with respect to the Cartesian product of monoidal ∞-categories.*

Proof. By Proposition 5.7.9 the complete enriched ∞-categories are closed under the exterior product in $\mathrm{Alg}_{\mathrm{cat}}$, and so the definition of the lax monoidal structure on the functor $\mathrm{Alg}_{\mathrm{cat}}(-)$ implies that the restriction to $\mathrm{Cat}_\infty^{(-)}$ is also lax monoidal. □

Corollary 5.7.12. *Let $\mathcal{O}$ be a symmetric ∞-operad, and suppose $\mathcal{V}$ is an $\mathcal{O} \otimes \mathbb{E}_1$-monoidal ∞-category. Then $\mathrm{Cat}_\infty^{\mathcal{V}}$ is an $\mathcal{O}$-monoidal ∞-category. In particular, if $\mathcal{V}$ is an $\mathbb{E}_n$-monoidal ∞-category then $\mathrm{Cat}_\infty^{\mathcal{V}}$ is $\mathbb{E}_{n-1}$-monoidal, and if $\mathcal{V}$ is symmetric monoidal then so is $\mathrm{Cat}_\infty^{\mathcal{V}}$.*

Proof. This follows by the same proof as that of Corollaries 4.3.12 and 4.3.13. □

Remark 5.7.13. If $\mathcal{V}$ is an $\mathbb{E}_n$-monoidal ∞-category, we can therefore iterate the enrichment functor k times for $k \leq n$ to obtain ∞-categories $\mathrm{Cat}_{(\infty,k)}^{\mathcal{V}}$ of (∞, k)-*categories enriched in* $\mathcal{V}$.

Proposition 5.7.14. *Suppose $\mathcal{V}$ is an $\mathbb{E}_2$-monoidal ∞-category. Then the localization $L\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Cat}_\infty^{\mathcal{V}}$ is monoidal.*

Proof. We must show that if $f\colon \mathcal{C} \to \mathcal{C}'$ and $g\colon \mathcal{D} \to \mathcal{D}'$ are fully faithful and essentially surjective functors in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, then their tensor product $f \otimes g\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D}'$ is also fully faithful and essentially surjective. By definition, the tensor product $\mathcal{C} \otimes \mathcal{C}'$ is given by $\mu_*(\mathcal{C} \boxtimes \mathcal{C}')$, where μ is the tensor product functor $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, which is monoidal since $\mathcal{V}$ is $\mathbb{E}_2$-monoidal.

By Lemma 5.7.5 it therefore suffices to check that the external product $f \boxtimes g$ is fully faithful and essentially surjective in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V} \times \mathcal{V})$. It is obvious that $f \boxtimes g$ is fully faithful, and it is essentially surjective since $\iota(f \boxtimes g)$ is naturally equivalent to $\iota f \times \iota g$ by Lemma 5.7.10. □

Combining this with Proposition 3.1.22, we get:

Corollary 5.7.15. *Suppose $\mathcal{V}$ is an $\mathbb{E}_2$-monoidal ∞-category. Then the localization $L\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathrm{Cat}_\infty^{\mathcal{V}}$ is a monoidal functor.*

Proposition 5.7.16. *When restricted to $\mathrm{Mon}_\infty^{\mathrm{Pr}}$, the functor $\mathrm{Cat}_\infty^{(-)}$ is lax monoidal with respect to the tensor product of presentable ∞-categories.*

Proof. This follows from Corollary 4.3.16, since by Proposition 5.7.9 the complete enriched ∞-categories are closed under the exterior product. □

Proposition 5.7.17. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are presentably monoidal ∞-categories and $F\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ is a monoidal functor such that the underlying functor $f\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Let $g\colon \mathcal{W} \to \mathcal{V}$ be a right adjoint of f, and let $G\colon \mathcal{W}^\otimes \to \mathcal{V}^\otimes$ be the lax monoidal structure on g given by Proposition A.5.11. Then:*

(i) The functor $G_*\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{W}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ *preserves complete objects.*
(ii) The functors

$$L_{\mathcal{W}}F_* : \mathrm{Cat}_\infty^{\mathcal{V}} \rightleftarrows \mathrm{Cat}_\infty^{\mathcal{W}} : G_*$$

are adjoint.

Proof. Since F is monoidal and f preserves colimits, it is clear that for any $\mathcal{S}$–∞-category $\mathcal{C}$ we have $F_*(I_{\mathcal{V}} \otimes \mathcal{C}) \simeq I_{\mathcal{W}} \otimes \mathcal{C}$. Hence for any $\mathcal{W}$–∞-category $\mathcal{D}$ we have natural equivalences

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})}(E^n, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})}(F_*E^n, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(E^n, G_*\mathcal{D}),$$

and so in particular $\iota G_*\mathcal{D} \simeq \iota\mathcal{D}$ and $G_*\mathcal{D}$ is complete if $\mathcal{D}$ is. This proves (i).

To prove (ii), observe that using Lemma 4.3.19 we have natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{W}}}(L_{\mathcal{W}}F_*\mathcal{C}, \mathcal{D}) &\simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{W})}(F_*\mathcal{C}, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{C}, G_*\mathcal{D}) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(\mathcal{C}, L_{\mathcal{V}}G_*\mathcal{D}). \qquad \square\end{aligned}$$

Proposition 5.7.18. *Let* $\mathcal{V}$ *be a presentably monoidal* ∞*-category and suppose* $L\colon \mathcal{V} \to \mathcal{W}$ *is a monoidal accessible localization with fully faithful right adjoint* $i\colon \mathcal{W} \hookrightarrow \mathcal{V}$*. Let* $i^\otimes\colon \mathcal{W}^\otimes \hookrightarrow \mathcal{V}^\otimes$ *and* $L^\otimes\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ *be as in Proposition 3.1.22. Suppose* L *exhibits* $\mathcal{W}$ *as the localization of* $\mathcal{V}$ *with respect to a set of morphisms* S*. Then the resulting adjunction*

$$L_*^\otimes : \mathrm{Cat}_\infty^{\mathcal{W}} \rightleftarrows \mathrm{Cat}_\infty^{\mathcal{V}} : i_*^\otimes$$

exhibits $\mathrm{Cat}_\infty^{\mathcal{W}}$ *as the localization of* $\mathrm{Cat}_\infty^{\mathcal{V}}$ *with respect to* $\Sigma(S)$*. Moreover, if* $\mathcal{V}$ *is at least* $\mathbb{E}_2$*-monoidal then this localization is again monoidal.*

Proof. The adjunction exists by combining Lemma A.5.12 and Proposition 5.7.17. The functor $i_*^\otimes$ is fully faithful since the functor on categorical algebras induced by $i^\otimes$ is fully faithful by Proposition 4.3.22 and preserves complete objects by Proposition 5.7.17(i). Thus this adjunction is a localization. The remaining statements follow by the same argument as in the proof of Proposition 4.3.22. $\square$

6. Some applications

In this section we describe some simple applications of our machinery: In §6.1 we use iterated enrichment to define ∞-categories of n-groupoids and (n, k)-categories for all n and $0 \leq k \leq n$ and prove the "homotopy hypothesis" in this context. Then in §6.2 we show that enriching in a monoidal $(n, 1)$-category gives an $(n + 1, 1)$-category, and use this to prove the Baez–Dolan stabilization hypothesis for k-tuply monoidal n-categories, and finally in §6.3 we prove that for any monoidal ∞-category $\mathcal{V}$ there is a fully faithful embedding of associative algebras in $\mathcal{V}$ into pointed $\mathcal{V}$–∞-categories.

6.1. (n,k)-categories as enriched ∞-categories

In this subsection we explain how to define (n,k)-categories in the context of enriched ∞-categories, and deduce some simple results that describe the resulting homotopy theories as localizations, including a version of the "homotopy hypothesis".

We begin by inductively defining n-groupoids and (n,k)-categories:

Definition 6.1.1. Assuming we have already defined $\mathrm{Cat}_{(n,1)}$, let $\mathrm{Gpd}_n \hookrightarrow \mathrm{Cat}_{(n,1)}$ be the full subcategory of objects local with respect to the obvious map $[1] \to E^0$; we refer to the objects of Gpd_n as *n-groupoids*. Then we define the ∞-category $\mathrm{Cat}_{(n+k,k)}$ of *$(n+k,k)$-categories* to be the ∞-category $\mathrm{Cat}^{\mathrm{Gpd}_n}_{(\infty,k)}$ of (∞,k)-categories enriched in Gpd_n. To start off the induction we define 0-groupoids to be sets, i.e. we define $\mathrm{Gpd}_0 := \mathrm{Set}$. We also extend the notation by setting $\mathrm{Cat}_{(n,0)} := \mathrm{Gpd}_n$.

Remark 6.1.2. Since the objects of $\mathrm{Cat}_{(n,1)}$ are already local with respect to $E^1 \to E^0$ we can equivalently define Gpd_n as the full subcategory of objects local with respect to either of the inclusions $[1] \to E^1$. Thus an $(n,1)$-category is an n-groupoid precisely if all of its 1-morphisms are equivalences.

Remark 6.1.3. Observe that the ∞-category $\mathrm{Cat}_{(n,n)}$ is defined by iterated enrichment starting with sets: $\mathrm{Cat}_{(n,n)} := \mathrm{Cat}^{\mathrm{Set}}_{(\infty,n)}$. For $n < \infty$ we will refer to (n,n)-categories as *n-categories* and write $\mathrm{Cat}_n := \mathrm{Cat}_{(n,n)}$. The comparison results in [19] imply that this ∞-category of n-categories is equivalent to Tamsamani's homotopy theory of n-categories [42].

Remark 6.1.4. As observed by Bartels and Dolan (cf. [4]), the definition can also be extended to allow $n = -2$ and $n = -1$: We can take $\mathrm{Cat}_{(-2,1)} = \mathrm{Gpd}_{-2} := *$; then $\mathrm{Cat}_{(-1,1)}$ consists of the empty category and E^0. These are both -1-groupoids, so $\mathrm{Gpd}_{-1} \simeq \mathrm{Cat}_{(-1,1)}$. Next it is easy to identify $\mathrm{Cat}_{(0,1)}$ with partially ordered sets, so Gpd_0 consists of partially ordered sets where all morphisms are isomorphisms. These are equivalent to partially ordered sets with only identity morphisms, i.e. just sets, so $\mathrm{Gpd}_0 \simeq \mathrm{Set}$ as before.

For $n = \infty$ we define (∞,k)-categories by starting with spaces instead:

Definition 6.1.5. Let $\mathrm{Cat}_{(\infty,0)} := \mathcal{S}$, and $\mathrm{Cat}_{(\infty,k)} := \mathrm{Cat}^{\mathcal{S}}_{(\infty,k)}$.

We now wish to identify $\mathrm{Cat}_{(n,k)}$ as a localization of $\mathrm{Cat}_{(\infty,k)}$, starting from the following trivial observation:

Lemma 6.1.6.

(i) $\mathrm{Set} \hookrightarrow \mathcal{S}$ *is the full subcategory of objects local with respect to the maps* $S^n \to *$ *for* $n > 0$.
(ii) $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ *is the full subcategory of objects local with respect to the map* $[1] \to E^0$.

Proof. (i) is obvious, and (ii) is easy to prove if we take complete Segal spaces as our model for ∞-categories — a Segal space is local with respect to $[1] \to E^0$ if and only if it is constant. □

Combining this with Proposition 5.7.18 we immediately get the following:

Proposition 6.1.7.

(i) *The inclusion* $\mathrm{Cat}_{(n,k)} \hookrightarrow \mathrm{Cat}_{(n,k+1)}$ *induced by the inclusion* $\mathrm{Gpd}_{n-k} \hookrightarrow \mathrm{Cat}_{(n-k,1)}$ *exhibits* $\mathrm{Cat}_{(n,k)}$ *as the localization with respect to* $\Sigma^k[1] \to \Sigma^k E^0$.
(ii) *The inclusion* $\mathrm{Cat}_{(n,k)} \hookrightarrow \mathrm{Cat}_{(n,l)}$, $k < l \leq n$, *exhibits* $\mathrm{Cat}_{(n,k)}$ *as the localization with respect to* $\Sigma^i[1] \to \Sigma^i E^0$, $i = k, k+1, \ldots, l-1$.
(iii) *The inclusion* $\mathrm{Cat}_{(\infty,k)} \hookrightarrow \mathrm{Cat}_{(\infty,k+1)}$ *induced by the inclusion* $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ *exhibits* $\mathrm{Cat}_{(\infty,n)}$ *as the localization with respect to* $\Sigma^k[1] \to \Sigma^k E^0$.
(iv) *The inclusion* $\mathrm{Cat}_{(\infty,k)} \hookrightarrow \mathrm{Cat}_{(\infty,n)}$ *for* $k < n$ *exhibits* $\mathrm{Cat}_{(\infty,k)}$ *as the localization with respect to* $\Sigma^i[1] \to \Sigma^i E^0$, $i = k, k+1, \ldots, n-1$.
(v) *The inclusion* $\mathrm{Cat}_n \hookrightarrow \mathrm{Cat}_{(\infty,n)}$ *induced by the inclusion* $\mathrm{Set} \hookrightarrow \mathcal{S}$ *exhibits* Cat_n *as the localization of* $\mathrm{Cat}_{(\infty,n)}$ *with respect to* $\Sigma^n S^k \to \Sigma^n *$ *for* $k > 0$.

Theorem 6.1.8. *The composite functor* $\mathrm{Cat}_{(n,k)} \hookrightarrow \mathrm{Cat}_n \hookrightarrow \mathrm{Cat}_{(\infty,n)}$ *factors through* $\mathrm{Cat}_{(\infty,k)}$, *and the resulting inclusion* $\mathrm{Cat}_{(n,k)} \hookrightarrow \mathrm{Cat}_{(\infty,k)}$ *exhibits* $\mathrm{Cat}_{(n,k)}$ *as the localization with respect to* $\Sigma^k S^j \to \Sigma^k *$ *for* $j > n-k$.

For the proof we need the following observation:

Lemma 6.1.9. *Let* $\kappa\colon \mathrm{Cat}_\infty \to \mathcal{S}$ *denote the left adjoint to the inclusion* $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$. *Then if* X *is a space, the space* $\kappa\Sigma X$ *is the (unreduced) suspension of* X.

Proof. We take complete Segal spaces as our model for ∞-categories; then the inclusion of $\mathcal{S}$ corresponds to the inclusion of *constant* simplicial spaces and κ corresponds to geometric realization. Let $\mathbf{\Delta}_s$ denote the subcategory of $\mathbf{\Delta}$ where the morphisms are the *surjective* morphisms of simplicial sets. Let $S(X)\colon \mathbf{\Delta}_s^{\mathrm{op}} \to \mathcal{S}$ be the semisimplicial space with $S(X)_0 = \{0,1\}$, $S(X)_1 = X$ with $d_1(X) = 0$ and $d_0(X) = 1$, and $S(X)_n = \emptyset$ for $n > 1$. If j denotes the inclusion $\mathbf{\Delta}_s^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ then it is easy to see that the left Kan extension $j_!S(X)$ is a (complete) Segal space. Moreover, using the adjunction $j_! \dashv j^*$ it is clear that $j_!S(X)$ satisfies the universal property of ΣX. Thus $\kappa\Sigma X$ is the colimit of the functor $j_!S(X)$, i.e. the left Kan extension $q_!j_!S(X)$ along $q\colon \mathbf{\Delta}^{\mathrm{op}} \to *$. But this is equivalent to $(qj)_!S(X)$, which is the colimit of the semisimplicial space $S(X)$. Using the standard model-categorical approach to homotopy colimits we can describe this as the quotient of $\Delta^1 \times X$ where we identify $\{0\} \times X$ and $\{1\} \times X$ with points, which is precisely the unreduced suspension of the space X. □

Proof of Theorem 6.1.8. From Proposition 6.1.7 we see that $\mathrm{Cat}_{(n,k)}$ is the localization of $\mathrm{Cat}_{(\infty,n)}$ with respect to $\Sigma^i[1] \to \Sigma^i E^0$, $i = k, k+1, \dots, n-1$ and $\Sigma^n S^j \to \Sigma^n *$ for $j > 0$. On the other hand, $\mathrm{Cat}_{(\infty,k)}$ is the localization of $\mathrm{Cat}_{(\infty,n)}$ with respect to just the first class of maps, so the inclusion $\mathrm{Cat}_{(n,k)} \hookrightarrow \mathrm{Cat}_{(\infty,n)}$ certainly factors through $\mathrm{Cat}_{(\infty,k)}$. To prove the result it therefore suffices to show that the image of $\Sigma^n S^j \to \Sigma^n *$ under the localization $\mathrm{Cat}_{(\infty,n)} \to \mathrm{Cat}_{(\infty,k)}$ is $\Sigma^k S^{j+n-k} \to \Sigma^k *$. This follows by induction from the case $k = 0$, which is a special case of Lemma 6.1.9. □

In the case $k = 0$, this gives a version of the "homotopy hypothesis" in our setting:

Corollary 6.1.10 *(Homotopy hypothesis). There is an inclusion* $\mathrm{Gpd}_n \hookrightarrow \mathcal{S}$ *that exhibits* Gpd_n *as the localization of* $\mathcal{S}$ *with respect to the maps* $S^j \to *$, $j > n$. *In other words, the* ∞*-category* Gpd_n *of* n*-groupoids is equivalent to the* ∞*-category* $\mathcal{S}^{\leq n}$ *of* n-types*, i.e. spaces whose homotopy groups vanish in degrees* $> n$.

6.2. Enriching in $(n,1)$*-categories and Baez–Dolan stabilization*

In this subsection we prove that enriching in an $(n,1)$-category $\mathcal{V}$ gives an $(n+1,1)$-category of $\mathcal{V}$–∞-categories. We begin by recalling the appropriate definition of an $(n,1)$-category in the context of ∞-categories:

Definition 6.2.1. An ∞-category $\mathcal{C}$ is an $(n,1)$-*category* if the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X,Y)$ are $(n-1)$-types for all $X, Y \in \mathcal{C}$, i.e. $\pi_k \mathrm{Map}_{\mathcal{C}}(X,Y) = 0$ for $k \geq n$. In other words, there are no non-trivial k-morphisms in $\mathcal{C}$ for $k > n$.

Remark 6.2.2. Using the equivalence $\mathrm{Cat}_\infty^{\mathcal{S}} \simeq \mathrm{Cat}_\infty$ of Theorem 5.4.6 and the case $k = 1$ of Theorem 6.1.8 we can identify $(n,1)$-categories in this sense with those defined in the previous subsection.

Remark 6.2.3. Suppose $\mathcal{V}$ is a monoidal ∞-category such that $\mathcal{V}$ is an $(n,1)$-category. Then clearly $\mathcal{V}^\otimes$ is also an $(n,1)$-category. The phrase *monoidal* $(n,1)$-*category* is thus unambiguous.

Proposition 6.2.4. *Suppose* $\mathcal{V}$ *is a monoidal* $(n,1)$*-category and* $\mathcal{C}$ *is a* $\mathcal{V}$*–*∞*-category. Then the space* $\iota\mathcal{C}$ *is an* n*-type.*

Proof. Let $s\colon \pi_0(\iota_0\mathcal{C}) \to \iota_0\mathcal{C}$ be a section of the projection $\iota_0\mathcal{C} \to \pi_0\iota_0\mathcal{C}$. Then the Cartesian morphism $s^*\mathcal{C} \to \mathcal{C}$ is fully faithful and essentially surjective, and so induces an equivalence $\iota(s^*\mathcal{C}) \to \iota\mathcal{C}$ by Proposition 5.3.6. Without loss of generality we may therefore assume that the space $\iota_0\mathcal{C}$ is discrete.

The simplicial space $\iota_\bullet\mathcal{C}$ is a groupoid object by Corollary 5.2.7. By [25, Corollary 6.1.3.20] this groupoid object is effective, and so we have a pullback diagram

$$\begin{array}{ccc} \iota_1\mathcal{C} & \longrightarrow & \iota_0\mathcal{C} \\ \downarrow & & \downarrow \\ \iota_0\mathcal{C} & \longrightarrow & \iota\mathcal{C}. \end{array}$$

If X is a point of $\iota_0\mathcal{C}$, we get a pullback diagram

$$\begin{array}{ccc} \iota_1\mathcal{C}_{\{X\}} & \longrightarrow & \iota_0\mathcal{C} \\ \downarrow & & \downarrow \\ \{X\} & \longrightarrow & \iota\mathcal{C}, \end{array}$$

where $\iota_1\mathcal{C}_{\{X\}}$ is the fibre of $\iota_1\mathcal{C} \to \iota_0\mathcal{C}$ at X. Since the map $\iota_0\mathcal{C} \to \iota\mathcal{C}$ is surjective on components, by considering the long exact sequence of homotopy groups associated to this fibre sequence we see that $\iota\mathcal{C}$ is an n-type provided the spaces $\iota_1\mathcal{C}_{\{X\}}$ are $(n-1)$-types for all $X \in \iota_0\mathcal{C}$.

The space $\iota_1\mathcal{C}_{\{X\}}$ is a union of components of $\iota_1\mathcal{C}$, so it suffices to show that $\iota_1\mathcal{C}$ is an $(n-1)$-type. Since $\iota_0\mathcal{C}$ is discrete, i.e. a 0-type, by [25, Lemma 5.5.6.14] this is equivalent to proving that the fibres of the map $\iota_1\mathcal{C} \to \iota_0\mathcal{C} \times \iota_0\mathcal{C}$ are $(n-1)$-types. But by Proposition 5.1.17 we can identify the fibre $\iota_1\mathcal{C}_{X,Y}$ at $(X,Y) \in \iota_0\mathcal{C}^{\times 2}$ with the space $\mathrm{Map}(I, \mathcal{C}(X,Y))_{\mathrm{eq}}$ that is the union of the components of $\mathrm{Map}(I, \mathcal{C}(X,Y))$ corresponding to equivalences. Since $\mathcal{V}$ is by assumption an n-category, the space $\mathrm{Map}(I, \mathcal{C}(X,Y))$ is necessarily an $(n-1)$-type, hence so is $\mathrm{Map}(I, \mathcal{C}(X,Y))_{\mathrm{eq}}$. □

Theorem 6.2.5. *Suppose $\mathcal{V}$ is a monoidal $(n,1)$-category. Then $\mathrm{Cat}_\infty^{\mathcal{V}}$ is an $(n+1,1)$-category.*

Proof. We need to show that if $\mathcal{C}$ and $\mathcal{D}$ are complete $\mathcal{V}$–∞-categories then the space

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{C}, \mathcal{D})$$

is an n-type. By Proposition 6.2.4, the space $\iota_0\mathcal{D} \simeq \iota\mathcal{D}$ is an n-type, hence the space $\mathrm{Map}_{\mathcal{S}}(\iota_0\mathcal{C}, \iota_0\mathcal{D})$ is as well. It follows from [25, Lemma 5.5.6.14] that, in order to prove that $\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{C}, \mathcal{D})$ is an n-type, it suffices to show that the fibres of the map

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathcal{S}}(\iota_0\mathcal{C}, \iota_0\mathcal{D})$$

induced by the projection $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$ are n-types.

Since the projection $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$ is a Cartesian fibration, by [25, Proposition 2.4.4.2] we can identify the fibre of this map at $f\colon \iota_0\mathcal{C} \to \iota_0\mathcal{D}$ with

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}}}(\mathcal{V})}(\mathcal{C}, f^*\mathcal{D}).$$

This space is the fibre of

$$\mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}\left(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}} \times \Delta^1, \mathcal{V}^{\otimes}\right) \to \mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}\left(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^{\otimes}\right) \times \mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}\left(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^{\otimes}\right)$$

at $(\mathcal{C}, f^*\mathcal{D})$. Since n-types are closed under all limits by [25, Proposition 5.5.6.5], it suffices to show that the spaces $\mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^{\otimes})$ and $\mathrm{Map}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}} \times \Delta^1, \mathcal{V}^{\otimes})$ are n-types. Now these spaces are fibres of $\mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}}, \mathcal{V}^{\otimes}) \to \mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{V}^{\otimes})$ and $\mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}_{\iota_0\mathcal{C}} \times \Delta^1, \mathcal{V}^{\otimes}) \to \mathrm{Map}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{V}^{\otimes})$, so by the same argument it's enough to show that these mapping spaces are n-types. But $\mathcal{V}^{\otimes}$ is by assumption an $(n, 1)$-category, so this holds by [25, Proposition 2.3.4.18]. □

Corollary 6.2.6. *The ∞-category Cat_n of n-categories is an $(n+1, 1)$-category. More generally, the ∞-category $\mathrm{Cat}_{(n,k)}$ of (n, k)-categories is an $(n+1, 1)$-category.*

Proof. Since Set is obviously a monoidal $(1, 1)$-category, applying Theorem 6.2.5 inductively we see that Cat_n is an $(n+1, 1)$-category. Similarly, if we know that Gpd_n is an $(n+1, 1)$-category it follows by induction that $\mathrm{Cat}_{(n+k,k)}$ is an $(n+k+1, 1)$-category. In particular $\mathrm{Cat}_{(n+1,1)}$ is an $(n+2, 1)$-category, and so its full subcategory Gpd_{n+1} of $(n+1)$-groupoids is also an $(n+2, 1)$-category. Since $\mathrm{Gpd}_0 = \mathrm{Set}$ is a $(1, 1)$-category we see by induction that $\mathrm{Cat}_{(n,k)}$ is an $(n+1, 1)$-category for all (n, k). □

It follows that if $\mathcal{V}$ is a symmetric monoidal $(n, 1)$-category, then $\mathbb{E}_k$-algebras in $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ are equivalent to $\mathbb{E}_\infty$-algebras for k sufficiently large:

Corollary 6.2.7. *Let $\mathcal{V}$ be a symmetric monoidal $(n, 1)$-category. Then*

(i) the map $\mathbb{E}_k \to \mathbf{\Gamma}^{\mathrm{op}}$ induces an equivalence

$$\mathrm{Alg}^{\Sigma}_{\mathbb{E}_k}\left(\mathrm{Cat}^{\mathcal{V}}_{\infty}\right) \xrightarrow{\sim} \mathrm{Alg}^{\Sigma}_{\mathbf{\Gamma}^{\mathrm{op}}}\left(\mathrm{Cat}^{\mathcal{V}}_{\infty}\right)$$

for $k \geq n+1$,

(ii) the stabilization map $i\colon \mathbb{E}_k \to \mathbb{E}_{k+1}$ (defined in [28, §5.1.1]) induces an equivalence

$$i^*\colon \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{k+1}}\left(\mathrm{Cat}^{\mathcal{V}}_{\infty}\right) \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_k}\left(\mathrm{Cat}^{\mathcal{V}}_{\infty}\right)$$

for $k \geq n+1$.

Proof. (i) is immediate from [28, Corollary 5.1.1.7], and (ii) follows by the 2-out-of-3 property. □

We end this subsection by observing that when $\mathcal{V}$ is the monoidal ∞-category of n-categories, this yields the Baez–Dolan stabilization hypothesis, by the same proof as Lurie's version for $(n, 1)$-categories [25, Example 5.1.2.3]:

Definition 6.2.8. A *k-tuply monoidal n-category* is an $\mathbb{E}_k$-algebra in Cat_n, i.e. an $\mathbb{E}_k$-monoidal n-category.

Corollary 6.2.9 *(Baez–Dolan stabilization hypothesis). The stabilization map $i\colon \mathbb{E}_k \to \mathbb{E}_{k+1}$ induces an equivalence*

$$i^*\colon \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{k+1}}(\mathrm{Cat}_n) \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_k}(\mathrm{Cat}_n)$$

for $k \geq n+2$, i.e. k-tuply monoidal n-categories stabilize at $k = n+2$.

Proof. Apply Corollary 6.2.7 to Cat_n. □

Remark 6.2.10. The Baez–Dolan stabilization hypothesis was originally stated by Baez and Dolan in [3]. A version of it was proved by Simpson [37], who showed that for $k \geq n+2$ a k-tuply monoidal n-category can be "delooped" to a $(k+1)$-tuply monoidal n-category; the ∞-categorical version above extends this by showing that this construction gives an equivalence of ∞-categories.

6.3. $\mathbb{E}_n$-algebras as enriched (∞, n)-categories

In ordinary enriched category theory, it is obvious that associative algebra objects in a monoidal category $\mathbf{V}$ are equivalent to $\mathbf{V}$-categories with a single object. Similarly, if $\mathcal{V}$ is a monoidal ∞-category, we can identify the ∞-category $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})$ of associative algebra objects with the full subcategory of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ spanned by $\mathcal{V}$–∞-categories whose space of objects is a point. In this subsection we will prove that after localizing with respect to the fully faithful and essentially surjective $\mathcal{V}$-functors we still get a fully faithful functor from $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})$ provided we consider *pointed* $\mathcal{V}$–∞-categories. It then follows by induction that, if $\mathcal{V}$ is at least $\mathbb{E}_n$-monoidal, the same is true for the natural map from $\mathbb{E}_n$-algebras to pointed enriched (∞, n)-categories.

Definition 6.3.1. Let $\mathcal{V}$ be a monoidal ∞-category. By Proposition 3.1.18, the unit object of $\mathcal{V}$ is the initial object in the ∞-category $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})$ of associative algebra objects. The inclusion $j\colon \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V}) \hookrightarrow \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ therefore factors through $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/}$. Composing this with the localization functor we get a functor $B\colon \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V}) \to (\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}$.

Theorem 6.3.2. *Let $\mathcal{V}$ be a monoidal ∞-category. Then:*

(i) The functor $B\colon \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V}) \to (\mathrm{Cat}_\infty^{\mathcal{V}})_{E^0/}$ is fully faithful.
(ii) If $\mathcal{V}$ is $\mathbb{E}_2$-monoidal, then B is a monoidal functor.
(iii) B admits a right adjoint $\Omega\colon (\mathrm{Cat}_\infty^{\mathcal{V}})_{E^0/} \to \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})$.
(iv) If $\mathcal{V}$ is presentably $\mathbb{E}_2$-monoidal, then Ω is a lax monoidal functor.

For the proof of (iii) we first make some simple observations:

Lemma 6.3.3. *Let $\pi\colon \mathcal{E} \to \mathcal{B}$ be a Cartesian fibration. For any $B \in \mathcal{B}$, the functor $\mathcal{E}_B \to \mathcal{E}_{B/} := \mathcal{E} \times_{\mathcal{B}} \mathcal{B}_{B/}$ admits a right adjoint.*

Proof. First suppose B is an initial object of $\mathcal{B}$. Then there is an obvious map $\mathcal{B}^{\triangleleft} \to \mathcal{B}$ that sends $-\infty$ to B and is the identity when restricted to $\mathcal{B}$. Let $\pi'\colon \mathcal{E}' \to \mathcal{B}^{\triangleleft}$ be the pullback of π along this map; then π' is a Cartesian fibration. Since the obvious projection $\mathcal{B}^{\triangleleft} \to (\Delta^0)^{\triangleleft} = \Delta^1$ is clearly a Cartesian fibration, the composite functor $\mathcal{E}' \to \Delta^1$ is also a Cartesian fibration. But this is clearly also the coCartesian fibration associated to the inclusion $\mathcal{E}_B \hookrightarrow \mathcal{E}$, hence this functor does indeed have a right adjoint.

For the general case we reduce to the case already proved by pulling back along the forgetful functor $\mathcal{B}_{B/} \to \mathcal{B}$. □

Lemma 6.3.4. *Suppose given an adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

and suppose $D \in \mathcal{D}$ is an object such that the counit map $FGD \to D$ is an equivalence. Then the induced functor $\mathcal{D}_{D/} \to \mathcal{C}_{GD/}$ admits a left adjoint, given by $\mathcal{C}_{GD/} \to \mathcal{D}_{FGD/} \simeq \mathcal{D}_{D/}$.

Proof. The (dual of) the argument in the proof of [25, Lemma 5.2.5.2] applies under our assumptions without assuming any colimits exist in $\mathcal{D}$. □

Proof of Theorem 6.3.2. To prove (i), let R and S be two $\mathbf{\Delta}^{\mathrm{op}}$-algebras in $\mathcal{V}$. We have a fibre sequence

$$\mathrm{Map}_{(\mathrm{Cat}_\infty^{\mathcal{V}})_{E^0/}}(BR, BS) \to \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(BR, BS) \to \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}\big(E^0, BS\big).$$

Since BS is the completion $L_{\mathcal{V}}j(S)$ of S regarded as a $\mathcal{V}$–∞-category, we have equivalences

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(BR, BS) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}\big(j(R), BS\big)$$

and

$$\mathrm{Map}_{\mathrm{Cat}_\infty^{\mathcal{V}}}(E^0, BS) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(E^0, BS).$$

The projection $\iota_0\colon \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \to \mathcal{S}$ gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(j(R), BS) & \longrightarrow & \mathrm{Map}_{\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})}(E^0, BS) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}}(*, \iota_0 BS) & \longrightarrow & \mathrm{Map}_{\mathcal{S}}(*, \iota_0 BS) \end{array}$$

where the right vertical map is an equivalence by Lemma 5.1.2 and the bottom horizontal map is the identity, since $E^0 \to j(R)$ is the identity on ι_0. Thus we can identify the fibre of the top horizontal map at the functor $E^0 \to BS$ corresponding to a point $p\colon * \to \iota_0 BS$ with the corresponding fibre of the left vertical map, which is $\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})}(R, p^*BS)$ by [25, Proposition 2.4.4.2].

Take p to be the underlying map of spaces of the completion functor $j(S) \to BS$; since this is fully faithful the induced map $j(S) \to p^*BS$ is an equivalence, and in particular

$$\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})}(R, S) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})}(R, p^*BS).$$

Thus the map $\mathrm{Map}_{\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})}(R, S) \to \mathrm{Map}_{(\mathrm{Cat}_\infty^{\mathcal{V}})_{E^0/}}(BR, BS)$ is also an equivalence, which completes the proof of (i).

We now prove (ii). It is clear from the definition of the monoidal structures that the functor $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/}$ is monoidal. Since it follows from Corollary 5.7.15 that the localization $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/} \to (\mathrm{Cat}_\infty^{\mathcal{V}})_{E^0/}$ is monoidal (by regarding the overcategories as ∞-categories of $\mathbb{E}_0$-algebras, for example), it follows that B is monoidal.

To prove (iii), we first observe that the adjunction $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \rightleftarrows \mathrm{Cat}_\infty^{\mathcal{V}}$ descends to an adjunction $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/} \rightleftarrows (\mathrm{Cat}_\infty^{\mathcal{V}})_{E^0/}$ by Lemma 6.3.4. It therefore suffices to show that the functor $j\colon \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V}) \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/}$ admits a right adjoint. To see this we first show that the obvious functor $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/} \to \mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \times_{\mathcal{S}} \mathcal{S}_*$ is an equivalence. It is clear that this functor is essentially surjective, so it suffices to show that for $\mathcal{C}, \mathcal{D}$ in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{E^0/}$ the induced map

$$\mathrm{Map}_{E^0/}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Map}(\iota_0\mathcal{C}, \iota_0\mathcal{D})} \mathrm{Map}_{*/}(\iota_0\mathcal{C}, \iota_0\mathcal{D})$$

is an equivalence. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Map}_{E^0/}(\mathcal{C},\mathcal{D}) & \longrightarrow & \mathrm{Map}(\mathcal{C},\mathcal{D}) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{*/}(\iota_0\mathcal{C},\iota_0\mathcal{D}) & \longrightarrow & \mathrm{Map}(\iota_0\mathcal{C},\iota_0\mathcal{D}) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathrm{Map}(*,\iota_0\mathcal{D}).
\end{array}$$

Here the bottom square is clearly a pullback square, and the outer rectangle is a pullback square because of the natural equivalence $\mathrm{Map}(E^0,\mathcal{D}) \xrightarrow{\sim} \mathrm{Map}(*,\iota_0\mathcal{D})$. Thus the top square is also a pullback square. (iii) therefore follows by applying Lemma 6.3.3.

Finally, (iv) now follows from Proposition A.5.11. □

Remark 6.3.5. A pointed $\mathcal{V}$–∞-category $\mathcal{C}$ is in the essential image of the functor B if and only if $\iota\mathcal{C}$ is connected, since then the functor $p^*\mathcal{C} \to \mathcal{C}$ induced by the chosen point $p : * \to \iota_0\mathcal{C}$ is fully faithful and essentially surjective, and $p^*\mathcal{C}$ is a $\mathbf{\Delta}^{\mathrm{op}}$-algebra. In other words, $\mathbf{\Delta}^{\mathrm{op}}$-algebras in $\mathcal{V}$ are equivalent to pointed $\mathcal{V}$–∞-categories with a single object up to homotopy.

Definition 6.3.6. Let $\mathcal{V}$ be an $\mathbb{E}_2$-monoidal ∞-category. A *monoidal $\mathcal{V}$–∞-category* is a $\mathbf{\Delta}^{\mathrm{op}}$-algebra in $\mathrm{Cat}_\infty^{\mathcal{V}}$.

Corollary 6.3.7. *Let $\mathcal{V}$ be an $\mathbb{E}_2$-monoidal ∞-category. Then monoidal $\mathcal{V}$–∞-categories are equivalent to pointed $\mathcal{V}$–$(\infty,2)$-categories with a single object (up to homotopy).*

Proof. By definition $\mathcal{V}$–$(\infty,2)$-categories are ∞-categories enriched in $\mathrm{Cat}_\infty^{\mathcal{V}}$, so this follows from Remark 6.3.5. □

Remark 6.3.8. In particular, taking $\mathcal{V}$ to be Gpd_n, we see that monoidal (n,k)-categories are equivalent to pointed $(n+1,k+1)$-categories with a single object. Taking $\mathcal{V}$ to be $\mathcal{S}$, this remains true for $n=\infty$.

Definition 6.3.9. If $\mathcal{C}$ is a $\mathcal{V}$–∞-category and X is an object of $\mathcal{C}$, we write $\Omega_X\mathcal{C} \in \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{V})$ for the value of the functor Ω on the corresponding map $E^0 \to \mathcal{C}$. This is the *endomorphism algebra* of X.

By applying Theorem 6.3.2 inductively we can generalize it to the $\mathbb{E}_n$-monoidal setting:

Definition 6.3.10. By Proposition 3.7.3 monoidal ∞-categories are equivalent to $\mathbb{E}_1$-monoidal ∞-categories, and $\mathbf{\Delta}^{\mathrm{op}}$-algebras in a monoidal ∞-category are equivalent to $\mathbb{E}_1$-algebras in the associated $\mathbb{E}_1$-monoidal ∞-category. Since $\mathbb{E}_n \otimes \mathbb{E}_m \simeq \mathbb{E}_{n+m}$ for all n, m, by Theorem 6.3.2(ii) we get maps

$$\mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathcal{V}) \simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\big(\mathrm{Alg}^{\Sigma}_{\mathbb{E}_1}(\mathcal{V})\big) \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\big((\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}\big).$$

We can identify $(\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}$ with $\mathrm{Alg}^{\Sigma}_{\mathbb{E}_0}(\mathrm{Cat}^{\mathcal{V}}_{\infty})$, so

$$\begin{aligned}\mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\big((\mathrm{Cat}^{\mathcal{V}}_{\infty})_{E^0/}\big) &\simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\big(\mathrm{Alg}^{\Sigma}_{\mathbb{E}_0}(\mathrm{Cat}^{\mathcal{V}}_{\infty})\big) \\ &\simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}\otimes\mathbb{E}_0}\big(\mathrm{Cat}^{\mathcal{V}}_{\infty}\big) \\ &\simeq \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\big(\mathrm{Cat}^{\mathcal{V}}_{\infty}\big).\end{aligned}$$

Thus we have maps

$$\mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathcal{V}) \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_{n-1}}\big(\mathrm{Cat}^{\mathcal{V}}_{\infty}\big) \to \cdots \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_1}\big(\mathrm{Cat}^{\mathcal{V}}_{(\infty,n-1)}\big) \to \big(\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)}\big)_{E^0/}.$$

Applying Theorem 6.3.2 (and the symmetric counterparts of some of the results we used in its proof) inductively, we get the following:

Corollary 6.3.11. *Suppose $\mathcal{V}$ is an $\mathbb{E}_n$-monoidal ∞-category.*

(i) The functor

$$B^n \colon \mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathcal{V}) \to \big(\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)}\big)_{E^0/}$$

is fully faithful.

(ii) If $\mathcal{V}$ is $\mathbb{E}_{n+1}$-monoidal, then B^n is a monoidal functor.

(iii) If $\mathcal{V}$ is presentably $\mathbb{E}_n$-monoidal, then B^n admits a right adjoint $\Omega^n \colon (\mathrm{Cat}^{\mathcal{V}}_{(\infty,n)})_{E^0/} \to \mathrm{Alg}^{\Sigma}_{\mathbb{E}_n}(\mathcal{V})$.

(iv) *If $\mathcal{V}$ is presentably $\mathbb{E}_{n+1}$-monoidal, then Ω^n is a lax monoidal functor.*

Definition 6.3.12. Let $\mathcal{V}$ be an $\mathbb{E}_{n+1}$-monoidal ∞-category; then $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ is $\mathbb{E}_n$-monoidal by Corollary 5.7.12. An *$\mathbb{E}_n$-monoidal $\mathcal{V}$-∞-category* is an $\mathbb{E}_n$-algebra in $\mathrm{Cat}^{\mathcal{V}}_{\infty}$.

Corollary 6.3.13. *Let $\mathcal{V}$ be an $\mathbb{E}_{n+1}$-monoidal ∞-category. Then $\mathbb{E}_n$-monoidal $\mathcal{V}$-∞-categories are equivalent to pointed $\mathcal{V}$-$(\infty, n+1)$-categories with a single object and only identity j-morphisms for $j = 1, \ldots, n-1$.*

Remark 6.3.14. In particular, taking $\mathcal{V}$ to be Gpd_n, we see that $\mathbb{E}_m$-monoidal (n,k)-categories are equivalent to pointed $(n+m, k+m)$-categories with a single object and only identity j-morphisms for $j = 1, \ldots, m-1$. Taking $\mathcal{V}$ to be $\mathcal{S}$, this remains true for $n = \infty$.

Definition 6.3.15. If $\mathcal{C}$ is a $\mathcal{V}$–(∞, n)-category and X is an object of $\mathcal{C}$, we write $\Omega^n_X\mathcal{C}$ for Ω^n applied to the corresponding map $E^0 \to \mathcal{C}$. This is the *endomorphism* $\mathbb{E}_n$*-algebra* of X.

Remark 6.3.16. If $\mathcal{C}$ is a $\mathcal{V}$–(∞, n)-category and X is an object of $\mathcal{C}$, the underlying object in $\mathcal{V}$ of the $\mathbb{E}_n$-algebra $\Omega^n_X\mathcal{C}$ is the endomorphisms of the $(n-1)$-fold identity map of the identity map of ... of the identity map of X.

7. Enriching ∞-categories tensored over a monoidal ∞-category

Suppose $\mathbf{V}$ is a monoidal category and $\mathbf{C}$ is an ordinary category that is right-tensored over $\mathbf{V}$, i.e. there is a functor

$$(-) \otimes (-) \colon \mathbf{C} \times \mathbf{V} \to \mathbf{C},$$

compatible with the tensor product of $\mathbf{V}$. If for every $C \in \mathbf{C}$ the functor $C \otimes (-)$ has a right adjoint $F(C, -) \colon \mathbf{C} \to \mathbf{V}$, then it is easy to see that we can enrich $\mathbf{C}$ in $\mathbf{V}$, with the morphism object from C to D given by $F(C, D) \in \mathbf{V}$. In particular, if the monoidal category $\mathbf{V}$ is left-closed, then it is enriched in itself. Our goal in this section is to prove the analogous statement in the context of enriched ∞-categories, which will allow us to construct a number of interesting examples of these.

To prove this we will consider a variant of Lurie's definition of enriched ∞-categories from [28, §4.2.1]. After introducing the natural generalized non-symmetric ∞-operads that parametrize modules in §7.1 (and proving that the resulting ∞-categories of modules are equivalent to those of [28]), we review Lurie's definition in §7.2. It is easy to see that an ∞-category right-tensored over a monoidal ∞-category with adjoints as above defines an enriched ∞-category in this sense; by applying Lurie's construction of enriched strings from [28, §4.7.2], which we review in §7.3, we can quite easily extract a categorical algebra from this in §7.4.

7.1. Modules

Definition 7.1.1. Write BM for the category of simplices $\mathrm{Simp}(\Delta^1)$ of the simplicial set Δ^1. The objects of BM can be described as sequences of integers $(i_0, \ldots, i_k)$ where $0 \leq i_0 \leq \cdots \leq i_k \leq 1$, and there is a unique morphism $(i_0, \ldots, i_k) \to (i_{\phi(0)}, \ldots, i_{\phi(m)})$ for every map $\phi \colon [m] \to [k]$ in $\boldsymbol{\Delta}$. The obvious projection $\mathrm{BM} \to \boldsymbol{\Delta}^{\mathrm{op}}$ exhibits BM as a double ∞-category. If $\mathcal{M}$ is a generalized non-symmetric ∞-operad, a *bimodule* in $\mathcal{M}$ is a BM-algebra in $\mathcal{M}$. We write $\mathrm{Bimod}(\mathcal{M})$ for the ∞-category $\mathrm{Alg}_{\mathrm{BM}}(\mathcal{M})$ of bimodules in $\mathcal{M}$.

Definition 7.1.2. The obvious inclusions $i, j \colon \boldsymbol{\Delta}^{\mathrm{op}}_{\{0\}}, \boldsymbol{\Delta}^{\mathrm{op}}_{\{1\}} \to \mathrm{BM}$ are maps of generalized ∞-operads. We say a bimodule M in a generalized non-symmetric ∞-operad $\mathcal{M}$ is an

i^*M–j^*M-*bimodule*. If A and B are associative algebras in a generalized non-symmetric ∞-operad $\mathcal{M}$, we write $\mathrm{Bimod}_{A,B}(\mathcal{M})$ for the fibre of the projection $(i^*, j^*)\colon \mathrm{Bimod}(\mathcal{M}) \to \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}) \times \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})$ at (A, B), i.e. the ∞-category of A–B-bimodules.

Definition 7.1.3. Let LM denote the full subcategory of BM spanned by the objects of the form $(0, \ldots, 0, 1)$ and $(0, \ldots, 0)$. The restricted projection $\mathrm{LM} \to \mathbf{\Delta}^{\mathrm{op}}$ exhibits LM as a double ∞-category. A *left module* in a generalized non-symmetric ∞-operad $\mathcal{M}$ is an LM-algebra in $\mathcal{M}$. We write $\mathrm{LMod}(\mathcal{M})$ for the ∞-category $\mathrm{Alg}_{\mathrm{LM}}(\mathcal{M})$ of left modules in $\mathcal{M}$.

Definition 7.1.4. Let RM denote the full subcategory of BM spanned by the objects of the form $(0, 1 \ldots, 1)$ and $(1, \ldots, 1)$. The restricted projection $\mathrm{RM} \to \mathbf{\Delta}^{\mathrm{op}}$ exhibits RM as a double ∞-category. A *right module* in a generalized non-symmetric ∞-operad $\mathcal{M}$ is an RM-algebra in $\mathcal{M}$. We write $\mathrm{RMod}(\mathcal{M})$ for the ∞-category $\mathrm{Alg}_{\mathrm{RM}}(\mathcal{M})$ of right modules in $\mathcal{M}$.

Definition 7.1.5. The obvious inclusions $i\colon \mathbf{\Delta}^{\mathrm{op}}_{\{0\}} \hookrightarrow \mathrm{LM}$ and $j\colon \mathbf{\Delta}^{\mathrm{op}}_{\{1\}} \hookrightarrow \mathrm{RM}$ are maps of generalized non-symmetric ∞-operads. If $M\colon \mathrm{LM} \to \mathcal{M}$ is a left module in a generalized non-symmetric ∞-operad $\mathcal{M}$, we say M is a *left* i^*M*-module*. Similarly, if M is a right module in $\mathcal{M}$ we say that it is a *right* j^*M*-module*. If A is an associative algebra in $\mathcal{M}$, we write $\mathrm{LMod}_A(\mathcal{M})$ and $\mathrm{RMod}_A(\mathcal{M})$ for the fibres of the projections $i^*\colon \mathrm{LMod}(\mathcal{M}) \to \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})$ and $j^*\colon \mathrm{RMod}(\mathcal{M}) \to \mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M})$ at A, respectively.

It is easy to describe the ∞-operad localizations of the generalized non-symmetric ∞-operads BM, LM, and RM in terms of multicategories:

Definition 7.1.6. Let **BM** be the multicategory with objects $\mathfrak{l}$, $\mathfrak{m}$ and $\mathfrak{r}$ and multimorphisms

$$\mathbf{BM}(\mathfrak{l}, \ldots, \mathfrak{l}, \mathfrak{m}, \mathfrak{r}, \ldots, \mathfrak{r}; \mathfrak{m}) = *$$
$$\mathbf{BM}(\mathfrak{l}, \ldots, \mathfrak{l}; \mathfrak{l}) = *$$
$$\mathbf{BM}(\mathfrak{r}, \ldots, \mathfrak{r}; \mathfrak{r}) = *$$

(where there can be zero $\mathfrak{l}$'s and $\mathfrak{r}$'s in the lists), with all other multimorphism sets empty. We then define **LM** to be the full submulticategory of **BM** with objects $\mathfrak{l}$ and $\mathfrak{m}$ and **RM** to be the full submulticategory with objects $\mathfrak{r}$ and $\mathfrak{m}$.

Proposition 7.1.7. *There are obvious maps* $\mathrm{BM} \to \mathbf{BM}^{\otimes}$*,* $\mathrm{LM} \to \mathbf{LM}^{\otimes}$*, and* $\mathrm{RM} \to \mathbf{RM}^{\otimes}$*. These induce equivalences of non-symmetric ∞-operads* $L_{\mathrm{gen}}\,\mathrm{BM} \xrightarrow{\sim} \mathbf{BM}^{\otimes}$*,* $L_{\mathrm{gen}}\,\mathrm{LM} \xrightarrow{\sim} \mathbf{LM}^{\otimes}$ *and* $L_{\mathrm{gen}}\,\mathrm{RM} \xrightarrow{\sim} \mathbf{RM}^{\otimes}$*.*

Proof. It is easy to see that these maps satisfy the criterion of Corollary A.6.9. □

Corollary 7.1.8. *Let $\mathcal{O}$ be a non-symmetric ∞-operad. Then there are natural equivalences* $\mathrm{Bimod}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathbf{BM}}(\mathcal{O})$, $\mathrm{LMod}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathbf{LM}}(\mathcal{O})$ *and* $\mathrm{RMod}(\mathcal{O}) \simeq \mathrm{Alg}_{\mathbf{RM}}(\mathcal{O})$.

Remark 7.1.9. The symmetric ∞-operads used in [28] to define bimodules, left modules, and right modules clearly arise from the symmetrizations of the multicategories **BM**, **LM** and **RM**, respectively. By Proposition 3.7.7 it therefore follows that the ∞-categories of modules defined here are equivalent to those defined in [28].

7.2. Lurie's enriched ∞-categories

In this section we describe a variant of Lurie's definition of enriched ∞-categories in [28, §4.2.1].

Definition 7.2.1. A *weakly enriched ∞-category* is a fibration of generalized non-symmetric ∞-operads $q: \mathcal{M} \to \mathrm{RM}$ such that the fibres $\mathcal{M}_{(0)}$ and $\mathcal{M}_{(1)}$ are contractible. We write $\mathcal{M}_{\mathfrak{r}}^{\otimes}$ for the non-symmetric ∞-operad $j^*\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ and $\mathcal{M}_{\mathfrak{m}}$ for the fibre $\mathcal{M}_{(0,1)}$ and say that q *exhibits* $\mathcal{M}_{\mathfrak{m}}$ *as weakly enriched in* $\mathcal{M}_{\mathfrak{r}}^{\otimes}$.

Example 7.2.2. Let $\mathcal{O}$ be any non-symmetric ∞-operad. The pullback $\pi^*\mathcal{O} \to \mathrm{RM}$ along the projection $\pi: \mathrm{RM} \to \mathbf{\Delta}^{\mathrm{op}}$ exhibits $\mathcal{O}_{[1]}$ as weakly enriched in $\mathcal{O}$.

Example 7.2.3. Let $q: \mathcal{M} \to \mathrm{RM}$ be a weakly enriched ∞-category such that q is also a coCartesian fibration. Then we say that q exhibits $\mathcal{M}_{\mathfrak{m}}$ as *right-tensored* over $\mathcal{M}_{\mathfrak{r}}$, which is a monoidal ∞-category. Clearly, an ∞-category $\mathcal{C}$ is right-tensored over a monoidal ∞-category $\mathcal{V}$ if and only if there exists an RM-algebra $F: \mathrm{RM} \to \mathrm{Cat}_{\infty}^{\times}$ such that $F(0,1) \simeq \mathcal{C}$ and j^*F is an associative algebra corresponding to $\mathcal{V}^{\otimes}$.

Definition 7.2.4. Let $q: \mathcal{M} \to \mathrm{RM}$ be a weakly enriched ∞-category. Given $C_1, \ldots, C_n \in \mathcal{M}_{\mathfrak{r}}$ and $M, N \in \mathcal{M}_{\mathfrak{m}}$, we write

$$\mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}\big(M \boxtimes (C_1, \ldots, C_n), N\big)$$

for $\mathrm{Map}_{\mathcal{M}}^{\phi}((M, C_1, \ldots, C_n), N)$, where $\phi: [n+1] \to [1]$ is the unique active map.

Definition 7.2.5. A *pseudo-enriched ∞-category* is a weakly enriched ∞-category $q: \mathcal{M} \to \mathrm{RM}$ such that $\mathcal{M}_{\mathfrak{r}}^{\otimes}$ is a monoidal ∞-category, and for all $C_1, \ldots, C_n \in \mathcal{M}_{\mathfrak{r}}$ $(n = 0, 1, \ldots)$ and $M, N \in \mathcal{M}_{\mathfrak{m}}$, the canonical map

$$\mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}\big(M \boxtimes (C_1 \otimes \cdots \otimes C_n), N\big) \to \mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}\big(M \boxtimes (C_1, \ldots, C_n), N\big)$$

is an equivalence.

Remark 7.2.6. Taking $n = 0$ in this definition, we see that in a pseudo-enriched ∞-category $\mathcal{M}$ we have

$$\mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}(M \boxtimes I, N) \simeq \mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}(M, N).$$

Example 7.2.7. The pullback $\pi^*\mathcal{O} \to \mathrm{RM}$ exhibits $\mathcal{O}_{[1]}$ as pseudo-enriched in $\mathcal{O}$ if and only if $\mathcal{O}$ is a monoidal ∞-category.

Example 7.2.8. If a weakly enriched ∞-category $q: \mathcal{M} \to \mathrm{RM}$ is a coCartesian fibration, then it is clearly pseudo-enriched.

Definition 7.2.9. Let $\mathcal{M} \to \mathrm{RM}$ be a pseudo-enriched ∞-category. Suppose M and N are objects of $\mathcal{M}_{\mathfrak{m}}$; a *morphism object* for M, N is an object $F(M, N) \in \mathcal{M}_{\mathfrak{r}}$ together with a map $\alpha \in \mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}(M \boxtimes F(M, N), N)$ such that for every $C \in \mathcal{M}_{\mathfrak{r}}$ composition with α induces an equivalence

$$\mathrm{Map}_{\mathcal{M}_{\mathfrak{r}}}\big(C, F(M, N)\big) \to \mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}(M \boxtimes C, N).$$

We say that $\mathcal{M} \to \mathrm{RM}$ is a *Lurie-enriched ∞-category* if there exists a morphism object in $\mathcal{M}_{\mathfrak{r}}$ for all $M, N \in \mathcal{M}_{\mathfrak{m}}$.

Remark 7.2.10. From Remark 7.2.6 we see that in a Lurie-enriched ∞-category $\mathcal{M}$ there is a natural equivalence

$$\mathrm{Map}_{\mathcal{M}_{\mathfrak{r}}}\big(I, F(M, N)\big) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}_{\mathfrak{m}}}(M, N).$$

Example 7.2.11. A monoidal ∞-category $\mathcal{V}$ is *left-closed* if and only if for every $C \in \mathcal{V}$ the functor $C \otimes (-): \mathcal{V} \to \mathcal{V}$ has a right adjoint. If $\mathcal{V}$ is a monoidal ∞-category, the pullback $\pi^*\mathcal{V}^{\otimes}$ exhibits $\mathcal{V}$ as Lurie-enriched in $\mathcal{V}$ if and only if $\mathcal{V}$ is left-closed.

Example 7.2.12. More generally, suppose the ∞-category $\mathcal{C}$ is right-tensored over the monoidal ∞-category $\mathcal{V}$. The associated coCartesian weakly enriched ∞-category $q: \mathcal{M} \to \mathrm{RM}$ is Lurie-enriched if and only if for every $M \in \mathcal{C}$, the right-tensoring functor $M \otimes (-): \mathcal{V} \to \mathcal{C}$ has a right adjoint $F(M, -)$ (so that $\mathrm{Map}_{\mathcal{V}}(V, F(M, N)) \simeq \mathrm{Map}_{\mathcal{C}}(M \otimes V, N)$).

Remark 7.2.13. We use *right* modules rather than the left modules used in [28, §4.2.1] so that the composition maps of morphism objects are compatible with those for categorical algebras: If $\mathcal{M}$ is a Lurie-enriched ∞-category in our sense, then for a triple A, B, C of objects in $\mathcal{M}_{\mathfrak{m}}$ we get a composition map $F(A, B) \otimes F(B, C) \to F(A, C)$, whereas [28, Definition 4.2.1.28] gives composition maps $F(B, C) \otimes F(A, B) \to F(A, C)$. This is why we get Lurie-enriched ∞-categories from *left*-closed monoidal ∞-categories rather than right-closed ones as in [28, Example 4.2.1.32].

Definition 7.2.14. Let $\mathrm{Enr}_{\mathrm{Lur}}$ be the full subcategory of $(\mathrm{Opd}_\infty^{\mathrm{ns,gen}})_{/\mathrm{RM}}$ spanned by the Lurie-enriched ∞-categories. Pullback along the inclusion $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{RM}$ induces a projection $\mathrm{Enr}_{\mathrm{Lur}} \to \mathrm{Mon}_\infty$; we write $\mathrm{Enr}_{\mathrm{Lur}}^{\mathcal{V}}$ for the fibre at $\mathcal{V}^\otimes \in \mathrm{Mon}_\infty$. This is the ∞-category of *Lurie-$\mathcal{V}$-enriched ∞-categories*.

We expect that the ∞-category $\mathrm{Enr}_{\mathrm{Lur}}^{\mathcal{V}}$ is equivalent to the ∞-category $\mathrm{Cat}_\infty^{\mathcal{V}}$ of complete categorical algebras in $\mathcal{V}$ defined above, but we will not attempt to prove this here.

7.3. Enriched strings

We now describe the analogue for our variant definition of Lurie's construction of an ∞-category of *enriched strings* in [28, §4.7.2].

Definition 7.3.1. Let Po denote the full subcategory of $\mathrm{Fun}([1], \mathbf{\Delta})$ spanned by the inert morphisms. In other words, an object of Po is an inert morphism $\alpha\colon [i] \to [n]$, or equivalently an object $[n] \in \mathbf{\Delta}$ together with a subinterval $\{j, j+1, \ldots, j+i\} \subseteq [n]$. A morphism from α to $\beta\colon [j] \to [m]$ is a commutative diagram

$$\begin{array}{ccc} [i] & \xrightarrow{\alpha} & [n] \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\phi} \\ [j] & \xrightarrow[\beta]{} & [m]. \end{array}$$

Note that, since α and β are inert, a morphism ψ factoring $\phi \circ \alpha$ through β is uniquely determined, if it exists. The inclusions $i_0, i_1\colon [0] \hookrightarrow [1]$ taking the unique object of $[0]$ to $0, 1$, respectively, induce functors $\Phi, \Theta\colon \mathrm{Po} \to \mathbf{\Delta}$. We write Po' for the full subcategory of Po spanned by the (necessarily inert) morphisms $[0] \to [n]$.

Definition 7.3.2. We define a map $\chi\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{RM}$ by sending $[n]$ to the object $(0, 1, \ldots, 1)$ over $[n+1]$ and $\phi\colon [m] \to [n]$ to the coCartesian map over $[0] \star \phi\colon [m+1] \to [n+1]$. Thus the composite $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{RM} \to \mathbf{\Delta}^{\mathrm{op}}$ is given by $[0] \star -$.

Definition 7.3.3. Suppose $\mathcal{M} \to \mathrm{RM}$ is a weakly enriched ∞-category. Define simplicial sets $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}$ and $\mathrm{Str}\,\mathcal{M}$ over $\mathbf{\Delta}^{\mathrm{op}}$ by the universal properties

$$\mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}(K, \overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}) \simeq \mathrm{Hom}_{\mathrm{RM}}(K \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Po}^{\mathrm{op}}, \mathcal{M}),$$
$$\mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}(K, \mathrm{Str}\,\mathcal{M}) \simeq \mathrm{Hom}_{\mathrm{RM}}\big(K \times_{\mathbf{\Delta}^{\mathrm{op}}} (\mathrm{Po}')^{\mathrm{op}}, \mathcal{M}\big),$$

where the map $\mathrm{Po}^{\mathrm{op}} \to \mathrm{RM}$ is given by the composite

$$\mathrm{Po}^{\mathrm{op}} \xrightarrow{\Phi} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\chi} \mathrm{RM}.$$

Lemma 7.3.4. *The* ∞*-category* $\operatorname{Str}\mathcal{M}$ *is equivalent to* $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}_{\mathfrak{m}}}$.

Proof. This is immediate from Remark 4.1.4. $\square$

Definition 7.3.5. Let $\mathrm{Po}_{[n]}$ denote the fibre of $\Theta\colon \mathrm{Po} \to \mathbf{\Delta}$ at $[n]$, i.e. the full subcategory of $\mathbf{\Delta}_{/[n]}$ spanned by inert morphisms. A morphism in $\mathrm{Po}_{[n]}$ is thus a commutative diagram

$$\begin{array}{ccc} [i] & \xrightarrow{\phi} & [j] \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & [n] & \end{array}$$

where α and β are inert — if such a morphism exists then the morphism ϕ is clearly uniquely determined by α and β, and must also be inert. We can thus equivalently describe $\mathrm{Po}_{[n]}$ as the category associated to the partially ordered set of subintervals of $[n]$. We write $\Phi_{[n]}$ for $\Phi|_{\mathrm{Po}_{[n]}}$.

Definition 7.3.6. The unique map $[0] \to [-1]$ in $\mathbf{\Delta}^{\mathrm{op}}_{+}$ induces a natural transformation $[0]\star(-) \to [-1]\star(-) = \mathrm{id}$ of functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$; this is given by $d_0\colon [n] \to [n-1]$ for all $n = 1, \ldots$. Since d_0 is inert, we can define a natural transformation $\bar{\chi}\colon \Delta^1 \times \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{RM}$ by taking the coCartesian lift of this starting at χ. Thus $\bar{\chi}_{[n]}$ is given by $d_0\colon (0, 1, \ldots, 1) \to (1, \ldots, 1)$.

Definition 7.3.7. Let $q\colon \mathcal{M} \to \mathrm{RM}$ be a weakly enriched ∞-category. An *enriched* n*-string* in $\mathcal{M}$ is a functor $\sigma\colon \mathrm{Po}^{\mathrm{op}}_{[n]} \to \mathcal{M}$ such that:

(1) The composite $q \circ \sigma$ is

$$\mathrm{Po}^{\mathrm{op}}_{[n]} \xrightarrow{\Phi^{\mathrm{op}}_{[n]}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\chi} \mathrm{RM}.$$

(2) If

$$\begin{array}{ccc} [i] & \xrightarrow{\phi} & [j] \\ & \searrow^{\alpha} \quad {}^{\beta}\swarrow & \\ & [n] & \end{array}$$

is a morphism in $\mathrm{Po}_{[n]}$ such that $\alpha(0) = \beta(0)$ (or equivalently $\phi(0) = 0$), then $\sigma(\phi)$ is inert. (Notice that if $\phi\colon [n] \to [m]$ is an inert map in $\mathbf{\Delta}^{\mathrm{op}}$, then $[0] \star \phi$ is inert if and only if $\phi(0) = 0$, so these are precisely the maps ϕ so that $\sigma(\phi)$ lies over an inert map in $\mathbf{\Delta}^{\mathrm{op}}$.)

(3) Let $\sigma \to \sigma'$ be a coCartesian lift of $\bar{\chi}|_{\Delta^1 \times \mathrm{Po}_{[n]}^{\mathrm{op}}}$. Then for any morphism ϕ in $\mathrm{Po}_{[n]}$, the morphism $\sigma'(\phi)$ is inert in $\mathcal{M}_{\mathfrak{r}}^{\otimes}$.

Remark 7.3.8. An enriched 0-string is a map $* \simeq \mathrm{Po}_{[0]}^{\mathrm{op}} \to \mathcal{M}$ over the map $* \to \mathrm{RM}$ sending $*$ to $(0,1)$, i.e. just an object of $\mathcal{M}_{\mathfrak{m}}$. An enriched 1-string corresponds to a map $(M, C) \to N$ over $d_1 \colon (0,1,1) \to (0,1)$; if $\mathcal{M}$ is a Lurie-enriched ∞-category then this is equivalent to a map $C \to F(M,N)$. In general, an enriched n-string corresponds to a sequence of maps

$$(M_0, C_1, \ldots, C_n) \to (M_1, C_2, \ldots, C_n) \to \cdots \to (M_{n-1}, C_n) \to M_n,$$

together with coherence data, where each map is the identity on the components after the first two. If $\mathcal{M}$ is a Lurie-enriched ∞-category, then this is equivalent to a sequence of maps

$$C_1 \to F(M_0, M_1), \quad C_2 \to F(M_1, M_2), \quad \ldots \quad C_n \to F(M_{n-1}, M_n).$$

Definition 7.3.9. The fibre of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}$ at $[n]$ is clearly $\mathrm{Fun}_{\mathrm{RM}}(\mathrm{Po}_{[n]}^{\mathrm{op}}, \mathcal{M})$. We write $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ for the full subcategory of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en}}$ spanned by the enriched n-strings for all n.

Proposition 7.3.10. *Let $q \colon \mathcal{M} \to \mathrm{RM}$ be a weakly enriched ∞-category. Then:*

- *(i) The projection $p \colon \mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}$ is a categorical fibration.*
- *(ii) For every $X \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ and every inert morphism $\alpha \colon p(X) \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ there exists a p-coCartesian morphism $X \to \alpha_! X$ over α.*
- *(iii) Let $\bar{\alpha} \colon X \to Y$ be a morphism in $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ such that $p(\bar{\alpha}) \colon [m] \to [n]$ is an inert morphism. Then $\bar{\alpha}$ is p-coCartesian if and only if for all $\phi \colon [k] \to [n]$ in $\mathrm{Po}_{[n]}^{\mathrm{op}}$ the induced map $X(\alpha \circ \phi) \to Y(\phi)$ is an equivalence.*
- *(iv) Suppose q is a coCartesian fibration. Then so is p, and a morphism $\bar{\alpha} \colon X \to Y$ in $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ over $\alpha \colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ is p-coCartesian if and only if for every $\phi \colon [k] \to [n]$ in $\mathrm{Po}_{[n]}^{\mathrm{op}}$ the induced map $X(\alpha \circ \phi) \to Y(\phi)$ is q-coCartesian.*

Proof. As [28, Proposition 4.7.2.23]. □

Proposition 7.3.11. *Let $q \colon \mathcal{M} \to \mathrm{RM}$ be a weakly enriched ∞-category. Then the projection $p \colon \mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}$ satisfies the Segal condition, i.e. for each $[n]$, the map*

$$\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[n]} \to \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]} \times_{\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[0]}} \cdots \times_{\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[0]}} \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{[1]}$$

is an equivalence.

Proof. As [28, Proposition 4.7.2.13]. □

Proposition 7.3.12. *Let $q\colon \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$ be a weakly enriched ∞-category. Then:*

(i) The projection $r\colon \mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}}$ is a categorical fibration.
(ii) Given $X \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ and an inert morphism $\alpha\colon r(X) \to Y$ in $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}}$, there exists an r-coCartesian morphism $X \to \alpha_! X$ over α.
(iii) Suppose $X \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$, $\alpha\colon r(X) \to Y$ is a morphism in $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}}$, and $\alpha_0\colon [m] \to [n]$ is the underlying morphism in $\mathbf{\Delta}^{\mathrm{op}}$. Then a morphism $\bar{\alpha}\colon X \to \overline{Y}$ over α is r-coCartesian if and only if $\bar{\alpha}$ induces an equivalence $X(\alpha \circ \phi) \to \overline{Y}(\phi)$ for all $\phi\colon [k] \to [n]$ in $\mathrm{Po}^{\mathrm{op}}_{[n]}$.
(iv) Suppose q is a coCartesian fibration, and let r_0 denote the projection $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}} \to \mathbf{\Delta}^{\mathrm{op}}$. Given $X \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ and an r_0-coCartesian morphism $\alpha\colon r(X) \to Y$ in $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}}$, there exists an r-coCartesian morphism $\bar{\alpha}\colon X \to \alpha_! X$ in $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ over α. Moreover, if $X \in \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ and $\alpha\colon r(X) \to Y$ in $\mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}}$ is r_0-coCartesian over $\alpha_0\colon [m] \to [n]$ in $\mathbf{\Delta}^{\mathrm{op}}$, then a morphism $X \to \overline{Y}$ in $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ over α is r-coCartesian if and only if the induced map $X(\alpha \circ \phi) \to \overline{Y}(\phi)$ in $\mathcal{M}$ is q-coCartesian for all $\phi \in \mathrm{Po}^{\mathrm{op}}_{[n]}$.

Proof. As [28, Lemma 4.7.2.27]. □

Definition 7.3.13. Define $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+} \to \mathbf{\Delta}^{\mathrm{op}}$ by

$$\mathrm{Hom}_{\mathbf{\Delta}^{\mathrm{op}}}(K, \overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+}) \cong \mathrm{Hom}_{\mathrm{RM}}\big(\Delta^1 \times K \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Po}^{\mathrm{op}}, \mathcal{M}\big),$$

where the map $\Delta^1 \times \mathrm{Po}^{\mathrm{op}} \to \mathrm{RM}$ is the composite of $\mathrm{id} \times \Phi^{\mathrm{op}}\colon \Delta^1 \times \mathrm{Po}^{\mathrm{op}} \to \Delta^1 \times \mathbf{\Delta}^{\mathrm{op}}$ with the natural transformation $\bar{\chi}$.

Definition 7.3.14. Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a weakly enriched ∞-category. Let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}$ denote the full subcategory of $\overline{\mathrm{Str}\,\mathcal{M}}^{\mathrm{en},+} \to \mathbf{\Delta}^{\mathrm{op}}$ spanned by objects $F\colon \Delta^1 \times \mathrm{Po}^{\mathrm{op}}_{[n]} \to \mathcal{M}$ such that $F|_{\{0\}\times \mathrm{Po}^{\mathrm{op}}_{[n]}}$ is an enriched n-string and F is a q-left Kan extension of $F|_{\{0\}\times \mathrm{Po}^{\mathrm{op}}_{[n]}}$.

Lemma 7.3.15. *The projection $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \to \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}$ is a trivial fibration.*

Proof. Immediate from [25, Proposition 4.3.2.15]. □

Definition 7.3.16. Let $i\colon \mathbf{\Delta}^{\mathrm{op}} \to \Delta^1 \times \mathrm{Po}^{\mathrm{op}}$ be the functor that sends $[n]$ to $(1, \mathrm{id}\colon [n] \to [n])$. Then composition with i induces a functor $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \to \mathcal{M}^{\otimes}_{\mathfrak{r}}$ over $\mathbf{\Delta}^{\mathrm{op}}$.

Lemma 7.3.17. *Let $q\colon \mathcal{M} \to \mathrm{RM}$ be a weakly enriched ∞-category. The functor $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \to \mathcal{M}^{\otimes}_{\mathfrak{r}}$ preserves inert morphisms.*

Proof. This is obvious from the definitions and Proposition 7.3.10. □

7.4. Extracting a categorical algebra

In this subsection we will extract a categorical algebra from a coCartesian Lurie-enriched ∞-category, and consider some examples of enriched ∞-categories that arise in this way.

Definition 7.4.1. Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a weakly enriched ∞-category. Let $\mathrm{Str}\,\mathcal{M}_\iota^{\mathrm{en}}$ be defined by the pullback

$$\begin{array}{ccc} \mathrm{Str}\,\mathcal{M}_\iota^{\mathrm{en}} & \longrightarrow & \mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \\ \downarrow & & \downarrow \\ \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}_{\mathfrak{m}}} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}_{\mathfrak{m}}}. \end{array}$$

Lemma 7.4.2. *Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a coCartesian weakly enriched ∞-category. Then the projection $\mathrm{Str}\,\mathcal{M}_\iota^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}}$ is a coCartesian fibration.*

Proof. This follows immediately from Proposition 7.3.12 since the projection $\pi\colon \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}} \to \mathbf{\Delta}^{\mathrm{op}}$ is a left fibration, so all morphisms in $\mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}}$ are π-coCartesian. □

Remark 7.4.3. We expect that Lemma 7.4.2 is also true for pseudo-enriched ∞-categories that are not coCartesian fibrations, but since this is not needed for the examples we are interested in we will not consider this generalization here.

Definition 7.4.4. Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a Lurie-enriched ∞-category. Let $\mathrm{Str}\,\mathcal{M}_{\mathrm{eq}}^{\mathrm{en}}$ be the full subcategory of $\mathrm{Str}\,\mathcal{M}_\iota^{\mathrm{en}}$ spanned by enriched n-strings $\sigma\colon \mathrm{Po}^{\mathrm{op}}_{[n]} \to \mathcal{M}$ such that for $i = 1, \ldots, n$, the map

$$\sigma(\{i, i+1\} \hookrightarrow [n]) \simeq (M_i, C_i) \to M_{i+1} \simeq \sigma(\{i+1\} \hookrightarrow [n])$$

exhibits C_i as the morphism object $F(M_i, M_{i+1})$.

Proposition 7.4.5. *Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a coCartesian Lurie-enriched ∞-category. Then the projection $\mathrm{Str}\,\mathcal{M}_{\mathrm{eq}}^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}}$ is a trivial fibration.*

Proof. The universal property of the morphism object $F(M, N)$ implies that the universal map $(M, F(M, N)) \to N$ is the final object in the fibre of $\mathrm{Str}\,\mathcal{M}^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathcal{M}}$ over (M, N). The Segal condition (Proposition 7.3.11) implies that $\mathrm{Str}\,\mathcal{M}_{\mathrm{eq}}^{\mathrm{en}}$ is precisely the full subcategory of $\mathrm{Str}\,\mathcal{M}_\iota^{\mathrm{en}}$ spanned by the objects that are final in their fibre. It therefore follows by [25, Proposition 2.4.4.9(1)] that the projection $\mathrm{Str}\,\mathcal{M}_{\mathrm{eq}}^{\mathrm{en}} \to \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}}$ is a trivial fibration. □

Definition 7.4.6. Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a Lurie-enriched ∞-category. Let $\mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}_{\mathrm{eq}}$ be defined by the pullback square

$$\begin{array}{ccc} \mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}_{\mathrm{eq}} & \longrightarrow & \mathrm{Str}\,\mathcal{M}^{\mathrm{en},+} \\ \downarrow & & \downarrow \\ \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{\mathrm{eq}} & \longrightarrow & \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}. \end{array}$$

Theorem 7.4.7. *Suppose $q\colon \mathcal{M} \to \mathrm{RM}$ is a coCartesian Lurie-enriched ∞-category. The composite*

$$\overline{\mathcal{M}}_{\mathfrak{m}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{M}} \xleftarrow{\sim} \mathrm{Str}\,\mathcal{M}^{\mathrm{en}}_{\mathrm{eq}} \xleftarrow{\sim} \mathrm{Str}\,\mathcal{M}^{\mathrm{en},+}_{\mathrm{eq}} \to \mathcal{M}^{\otimes}_{\mathfrak{r}}$$

is a categorical algebra in $\mathcal{M}_{\mathfrak{r}}$.

Proof. It follows from Lemma 7.3.17 that this map preserves inert morphisms, so it is a categorical algebra. □

Remark 7.4.8. We expect that, as suggested by Remark 7.2.10, in the situation above the underlying ∞-category of $\overline{\mathcal{M}}_{\mathfrak{m}}$ is equivalent to $\mathcal{M}_{\mathfrak{m}}$. This would imply that the categorical algebra $\overline{\mathcal{M}}_{\mathfrak{m}}$ is in fact complete. We will not prove this here, however, as this requires developing more of the theory of Lurie-enriched ∞-categories than is appropriate here.

Using Example 7.2.12 we can restate this as:

Corollary 7.4.9. *Suppose $\mathcal{V}$ is a monoidal ∞-category and $\mathcal{C}$ is an ∞-category that is right-tensored over $\mathcal{V}$ so that the tensor product $C \otimes (\text{–})$ has a right adjoint $F(C, \text{–})\colon \mathcal{C} \to \mathcal{V}$ for all $C \in \mathcal{C}$. Then $\mathcal{C}$ is enriched in $\mathcal{V}$; more precisely, there is a categorical algebra $\overline{\mathcal{C}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{C}} \to \mathcal{V}^{\otimes}$ such that $\overline{\mathcal{C}}(C, D) \simeq F(C, D)$.*

This construction allows us to construct several interesting examples of enriched ∞-categories:

Corollary 7.4.10. *Suppose $\mathcal{V}$ is a left-closed monoidal ∞-category. Then $\mathcal{V}$ is enriched in itself; more precisely, there exists a categorical algebra $\overline{\mathcal{V}}\colon \mathbf{\Delta}^{\mathrm{op}}_{\iota\mathcal{V}} \to \mathcal{V}^{\otimes}$ such that $\overline{\mathcal{V}}(V, W)$ in $\mathcal{V}$ is the internal hom from V to W.*

Example 7.4.11. Suppose $\mathcal{V}$ is a presentably $\mathbb{E}_2$-monoidal ∞-category; then $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ is a presentably monoidal ∞-category, and so is in particular right-closed. Thus there exists a $\mathcal{V}$–$(\infty, 2)$-category $\overline{\mathrm{Cat}}^{\mathcal{V}}_{\infty}$ of $\mathcal{V}$–∞-categories. More generally, if $\mathcal{V}$ is presentably $\mathbb{E}_k$-monoidal (or presentably symmetric monoidal), there exists a $\mathcal{V}$–$(\infty, n+1)$-category $\overline{\mathrm{Cat}}^{\mathcal{V}}_{(\infty,n)}$ of $\mathcal{V}$–(∞, n)-categories for all $n < k$. For example, taking $\mathcal{V}$ to be $\mathcal{S}$ there is an $(\infty, n+1)$-category $\overline{\mathrm{Cat}}^{\mathcal{S}}_{(\infty,n)}$ of (∞, n)-categories, and taking $\mathcal{V}$ to be Set there is an $(n+1)$-category $\overline{\mathrm{Cat}}_n$ of n-categories.

Remark 7.4.12. Several homotopy theories that can easily be constructed as spectral presheaves $\mathrm{Fun}^{\mathrm{Sp}}(\mathcal{A}^{\mathrm{op}}, \overline{\mathrm{Sp}})$, where $\mathcal{A}$ is a small spectral category, can (conjecturally) be identified with more familiar homotopy theories:

(i) Suppose G is a finite group, and let $\mathcal{B}^G$ denote the *Burnside (2,1)-category* of G; this has objects finite G-sets, 1-morphisms spans of finite G-sets, and 2-morphisms isomorphisms of spans. We can regard this as a category enriched in symmetric monoidal groupoids, via the coproduct, and hence as an ∞-category enriched in E_∞-spaces. Group completion of E_∞-spaces is a lax monoidal functor from E_∞-spaces to (connective) spectra, so applying this to the mapping spaces in $\mathcal{B}^G$ gives a spectral ∞-category $\mathcal{B}^G_+$. The presheaf spectral ∞-category $\mathrm{Fun}^{\mathrm{Sp}}(\mathcal{B}^{G,\mathrm{op}}_+, \overline{\mathrm{Sp}})$ is the spectral ∞-category of *genuine G-spectra* — a version of this comparison has recently been proved by Guillou and May [17,16,18] using enriched model categories. (It has also been observed by Barwick that (as group-completion is a left adjoint) it is not necessary to group-complete the mapping spaces in $\mathcal{B}^G$ to describe G-spectra; this is the basis for the ∞-categorical description of G-spectra in [6].)

(ii) Let $\mathcal{B}$ denote the *global Burnside (2,1)-category of finite groups*. This has objects finite groups, 1-morphisms from G to H are finite free H-sets equipped with a compatible G-action, and 2-morphisms are isomorphisms of these. This can also be regarded as enriched in symmetric monoidal groupoids via coproducts, and by group-completing we obtain a spectral ∞-category $\mathcal{B}_+$. The presheaf spectral ∞-category $\mathrm{Fun}^{\mathrm{Sp}}(\mathcal{B}^{\mathrm{op}}_+, \overline{\mathrm{Sp}})$ is the spectral ∞-category of *global equivariant spectra* for finite groups, as studied by Schwede [35].

Corollary 7.4.13. *Suppose $\mathcal{V}$ is a presentably monoidal ∞-category, and $\mathcal{C}$ is a right $\mathcal{V}$-module in* Pres_∞ *(with respect to the tensor product of presentable ∞-categories). Then $\mathcal{C}$ is enriched in $\mathcal{V}$.*

Example 7.4.14. By [28, Proposition 4.8.2.18], presentable stable ∞-categories are precisely Sp-modules in Pres_∞, hence presentable stable ∞-categories are enriched in spectra. But any stable ∞-category is a full subcategory of its Ind-completion, hence it follows that all stable ∞-categories are enriched in spectra.

Example 7.4.15. In [27, §6], Lurie defines *R-linear ∞-categories* for an $\mathbb{E}_2$-ring spectrum R to be left LMod_R-modules in Pres_∞. If we instead consider right LMod_R-modules we get ∞-categories enriched in left R-modules from R-linear ∞-categories. Moreover, if R is at least $\mathbb{E}_3$-monoidal (so that LMod_R is at least $\mathbb{E}_2$-monoidal), then these two notions coincide.

Appendix A. Technicalities on ∞-operads

In this appendix we collect the more technical results we need about non-symmetric ∞-operads.

A.1. Monoidal envelopes

In this subsection we describe the non-symmetric version of Lurie's *monoidal envelope* of an ∞-operad $\mathcal{O}$, which gives a monoidal structure on the ∞-category $\mathcal{O}_{\mathrm{act}}$ of active morphisms in $\mathcal{O}$ that we will make use of below to define operadic colimits.

Definition A.1.1. Let $\mathrm{Act}(\boldsymbol{\Delta}^{\mathrm{op}})$ be the full subcategory of $\mathrm{Fun}(\Delta^1, \boldsymbol{\Delta}^{\mathrm{op}})$ spanned by the active morphisms. If $\mathcal{M}$ is a generalized non-symmetric ∞-operad, we define $\mathrm{Env}(\mathcal{M})$ to be the fibre product

$$\mathcal{M} \times_{\mathrm{Fun}(\{0\}, \boldsymbol{\Delta}^{\mathrm{op}})} \mathrm{Act}(\boldsymbol{\Delta}^{\mathrm{op}}).$$

Proposition A.1.2. *The map* $\mathrm{Env}(\mathcal{M}) \to \boldsymbol{\Delta}^{\mathrm{op}}$ *induced by evaluation at* 1 *in* Δ^1 *is a double* ∞*-category.*

Proof. As [28, Proposition 2.2.4.4]. □

Proposition A.1.3. *Suppose* $\mathcal{N}$ *is a double* ∞*-category and* $\mathcal{M}$ *is a generalized non-symmetric* ∞*-operad. The inclusion* $\mathcal{M} \to \mathrm{Env}(\mathcal{M})$ *induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{M}), \mathcal{N}) \to \mathrm{Alg}_{\mathcal{M}}(\mathcal{N}).$$

Proof. As [28, Proposition 2.2.4.9]. □

Corollary A.1.4. *Suppose* $\mathcal{O}$ *is a non-symmetric* ∞*-operad. Then* $\mathrm{Env}(\mathcal{O})$ *is a monoidal* ∞*-category, and if* $\mathcal{C}^{\otimes}$ *is a monoidal* ∞*-category then*

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathcal{C}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}).$$

Proof. The only object of $\boldsymbol{\Delta}$ that admits an active map from $[0]$ is $[0]$, hence for any generalized non-symmetric ∞-operad $\mathcal{M}$ we have $\mathrm{Env}(\mathcal{M})_{[0]} \simeq \mathcal{M}_{[0]}$. In particular $\mathrm{Env}(\mathcal{O})_{[0]} \simeq *$ for a non-symmetric ∞-operad $\mathcal{O}$, so the result follows from Proposition A.1.2 and Proposition A.1.3. □

Definition A.1.5. If $\mathcal{O}$ is a non-symmetric ∞-operad, the monoidal ∞-category $\mathrm{Env}(\mathcal{O})$ is called the *monoidal envelope* of $\mathcal{O}$. This gives a monoidal structure on the subcategory $\mathcal{O}_{\mathrm{act}}$ of $\mathcal{O}$ determined by the active morphisms. We denote this tensor product on $\mathcal{O}_{\mathrm{act}}$ by $\oplus$.

A.2. Operadic colimits

We wish to prove that, under reasonable hypotheses, if $\mathcal{V}$ is a monoidal ∞-category and $f\colon \mathcal{O} \to \mathcal{P}$ is a morphism of non-symmetric ∞-operads then the functor

$$f^*\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$$

given by composition with f has a left adjoint. This depends on an existence theorem for *operadic left Kan extensions*, which makes use of the concept of *operadic colimits* that we introduce in this subsection.

Definition A.2.1. Suppose $q\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ is a non-symmetric ∞-operad. Given a diagram $p\colon K \to \mathcal{O}_{\mathrm{act}}$ we write $\mathcal{O}^{\mathrm{act}}_{[1],p/} := \mathcal{O}_{[1]} \times_{\mathcal{O}} (\mathcal{O}_{\mathrm{act}})_{p/}$. A diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{O}_{\mathrm{act}}$ is a *weak operadic colimit diagram* if the induced map $\psi\colon \mathcal{O}^{\mathrm{act}}_{[1],\bar{p}/} \to \mathcal{O}^{\mathrm{act}}_{[1],p/}$ is a categorical equivalence.

A diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{O}_{\mathrm{act}}$ is an *operadic colimit diagram* if the composite functors

$$K^{\triangleright} \to \mathcal{O}_{\mathrm{act}} \xrightarrow{-\oplus X} \mathcal{O}_{\mathrm{act}}$$
$$K^{\triangleright} \to \mathcal{O}_{\mathrm{act}} \xrightarrow{X\oplus -} \mathcal{O}_{\mathrm{act}}$$

are weak operadic colimit diagrams for all $X \in \mathcal{O}$.

Remark A.2.2. By [25, Proposition 2.1.2.1], the map ψ in the definition of weak operadic colimits is always a left fibration, hence it is a categorical equivalence if and only if it is a trivial Kan fibration.

Lemma A.2.3. *Suppose $\mathcal{O}$ and $\mathcal{P}$ are non-symmetric ∞-operads, and $\bar{p}\colon K^{\triangleright} \to \mathcal{O}_{\mathrm{act}}$ and $\bar{q}\colon L^{\triangleright} \to \mathcal{P}_{\mathrm{act}}$ are weak operadic colimit diagrams. Then the composite*

$$\bar{r}\colon (K \times_{\mathbf{\Delta}^{\mathrm{op}}} L)^{\triangleright} \to K^{\triangleright} \times_{\mathbf{\Delta}^{\mathrm{op}}} L^{\triangleright} \to \mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{P}$$

is also a weak operadic colimit diagram. Moreover, if $\bar{p}$ and $\bar{q}$ are operadic colimit diagrams, so is $\bar{r}$.

Proof. Let $r := \bar{r}|_{K \times_{\mathbf{\Delta}^{\mathrm{op}}} L}$. Then we must show that the map $(\mathcal{O}_{[1]} \times \mathcal{P}_{[1]})^{\mathrm{act}}_{\bar{r}/} \to (\mathcal{O}_{[1]} \times \mathcal{P}_{[1]})^{\mathrm{act}}_{r/}$ is a categorical equivalence. We have a commutative diagram

$$\begin{array}{ccc} (\mathcal{O}_{[1]} \times \mathcal{P}_{[1]})^{\mathrm{act}}_{(\bar{p},\bar{q})/} & \longrightarrow & (\mathcal{O}_{[1]} \times \mathcal{P}_{[1]})^{\mathrm{act}}_{r/} \\ & \searrow \quad \nearrow & \\ & (\mathcal{O}_{[1]} \times \mathcal{P}_{[1]})^{\mathrm{act}}_{\bar{r}/}. & \end{array}$$

We clearly have an equivalence $(\mathcal{O}_{[1]} \times \mathcal{P}_{[1]})^{\mathrm{act}}_{(\bar{p},\bar{q})/} \simeq \mathcal{O}^{\mathrm{act}}_{[1],\bar{p}/} \times \mathcal{P}^{\mathrm{act}}_{[1],\bar{q}/}$, and so the top horizontal map is the product of the equivalences $\mathcal{O}^{\mathrm{act}}_{[1],\bar{p}/} \to \mathcal{O}^{\mathrm{act}}_{[1],p/}$ and $\mathcal{P}^{\mathrm{act}}_{[1],\bar{q}/} \to \mathcal{P}^{\mathrm{act}}_{[1],q/}$ and hence is an equivalence. By the 2-out-of-3 property it therefore suffices to show that the left diagonal map in the diagram is an equivalence. But this is true because the inclusion $(K \times L)^{\triangleright} \hookrightarrow K^{\triangleright} \times L^{\triangleright}$ is right anodyne. (By [25, Proposition 4.1.2.1] it suffices to prove this inclusion is cofinal, and the criterion of "Theorem A", [25, Theorem 4.1.3.1], clearly holds in this case.) It is then clear from the definition of the monoidal structure on $(\mathcal{O} \times_{\mathbf{\Delta}^{\mathrm{op}}} \mathcal{P})_{\mathrm{act}}$ that if $\bar{p}$ and $\bar{q}$ are operadic colimits, then so is $\bar{r}$. □

Proposition A.2.4. *Let $\mathcal{O}$ be a non-symmetric ∞-operad, and suppose given finitely many operadic colimit diagrams $\bar{p}_i \colon K_i^{\triangleright} \to \mathcal{O}_{\mathrm{act}}$, $i = 0, \dots, n$. Let $K := \prod_i K_i$, and let $\bar{p}$ be the composite*

$$K^{\triangleright} \to \prod_i K_i^{\triangleright} \to \prod_i \mathcal{O}_{\mathrm{act}} \simeq \mathrm{Env}(\mathcal{O})_{[n]} \xrightarrow{\oplus} \mathcal{O}_{\mathrm{act}}.$$

Then $\bar{p}$ is an operadic colimit diagram.

Proof. As [28, Proposition 3.1.1.8]. □

Lemma A.2.5. *Suppose K is a sifted simplicial set, and $\mathcal{V}$ is a monoidal ∞-category that is compatible with K-indexed colimits. Then $\phi_! \colon \mathcal{V}^{\otimes}_{[n]} \to \mathcal{V}^{\otimes}_{[m]}$ preserves K-indexed colimits for all ϕ in $\mathbf{\Delta}^{\mathrm{op}}$.*

Proof. As [28, Lemma 3.2.3.7]. □

Proposition A.2.6. *Let $\mathcal{V}$ be a monoidal ∞-category, and let $\bar{p} \colon K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]}$ be a diagram. Then $\bar{p}$ is a weak operadic colimit diagram if and only if the composite*

$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \xrightarrow{r_!} \mathcal{V}$$

is a colimit diagram, where r is the unique active map $[m] \to [1]$.

Proof. This follows as in the proof of [28, Proposition 3.1.1.7]. □

Corollary A.2.7. *Let $\mathcal{V}$ be a monoidal ∞-category, and let $\bar{p} \colon K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]}$ be a diagram. Then $\bar{p}$ is an operadic colimit diagram if and only if for every object $Y \in \mathcal{V}^{\otimes}$ the composites*

$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \xrightarrow{-\oplus Y} \mathcal{V}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{C}$$
$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \xrightarrow{Y \oplus -} \mathcal{V}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{V}$$

are colimit diagrams in $\mathcal{V}$, Y lies over $[n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ and r is the unique active map $[n+m] \to [1]$.

Proposition A.2.8. *Let $q\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}}$ be a non-symmetric ∞-operad, and suppose given a map $\bar{h}\colon \Delta^1 \times K^{\triangleright} \to \mathcal{O}_{\mathrm{act}}$; write $\bar{h}_i := \bar{h}|_{\{i\}\times K^{\triangleright}}$, $i = 0, 1$. Suppose that*

(a) For every vertex x of $K^{\triangleright}$, the restriction $\bar{h}|_{\Delta^1\times\{x\}}$ is a q-coCartesian edge of $\mathcal{O}$.
(b) The composite map

$$\Delta^1 \times \{\infty\} \hookrightarrow \Delta^1 \times K^{\triangleright} \xrightarrow{\bar{h}} \mathcal{O} \xrightarrow{q} \mathbf{\Delta}^{\mathrm{op}}$$

is an equivalence in $\mathbf{\Delta}^{\mathrm{op}}$.

Then $\bar{h}_0$ is a weak operadic colimit diagram if and only if $\bar{h}_1$ is a weak operadic colimit diagram. Moreover, if $\mathcal{O}$ is a monoidal ∞-category, then $\bar{h}_0$ is an operadic colimit diagram if and only if $\bar{h}_1$ is an operadic colimit diagram.

Proof. As [28, Proposition 3.1.1.15]. □

Corollary A.2.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be monoidal ∞-categories compatible with small colimits, and suppose $F\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ is a monoidal functor such that $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Then composition with F preserves operadic colimit diagrams.*

Proof. Suppose $\bar{p}\colon K^{\triangleright} \to \mathcal{V}^{\otimes}$ is an operadic colimit diagram. We wish to show that the composite map $K^{\triangleright} \to \mathcal{W}^{\otimes}$ is also an operadic colimit diagram. By Proposition A.2.8 we may assume that $\bar{p}$ lands in a fibre $\mathcal{V}^{\otimes}_{[m]}$. We now apply Corollary A.2.7 to conclude that it suffices to show that the composites

$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \to \mathcal{W}^{\otimes}_{[m]} \xrightarrow{-\oplus Y} \mathcal{W}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{W}$$
$$K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \to \mathcal{W}^{\otimes}_{[m]} \xrightarrow{Y\oplus -} \mathcal{W}^{\otimes}_{[n+m]} \xrightarrow{r_!} \mathcal{W},$$

where r is the unique active map $[n+m] \to [1]$, are colimit diagrams, for all $[n]$ and all $Y \in \mathcal{V}^{\otimes}_{[n]}$. Observe that the functors $r_!(-\oplus Y)$ and $r_!(Y \oplus -)$ are equivalently given by $\mu_!(r'_!(-) \oplus r''_!(Y))$ and $\mu_!(r''_!(Y) \oplus r'_!(-))$, where $r'\colon [m] \to [1]$, $r''\colon [n] \to [1]$ and $\mu\colon [2] \to [1]$ are the unique active maps between these objects. Since $\mu_!$ preserves colimits in each variable in both $\mathcal{V}^{\otimes}$ and $\mathcal{W}^{\otimes}$, it suffices to show that

$$K^{\triangleright} \to \mathcal{W}^{\otimes}_{[m]} \xrightarrow{r'_!} \mathcal{W}$$

is a colimit diagram. But we have a commutative diagram

$$\begin{array}{ccc} \mathcal{V}^{\otimes}_{[m]} & \xrightarrow{F^{\otimes}_{[m]}} & \mathcal{W}^{\otimes}_{[m]} \\ \downarrow{\scriptstyle r'_!} & & \downarrow{\scriptstyle r'_!} \\ \mathcal{V} & \xrightarrow[F]{} & \mathcal{W} \end{array}$$

so this is true since $K^{\triangleright} \to \mathcal{V}^{\otimes}_{[m]} \to \mathcal{V}$ is a colimit diagram and $F_{[1]}$ preserves colimits. □

Proposition A.2.10. *Let $q\colon \mathcal{V}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ be a monoidal ∞-category compatible with K-indexed colimits for some simplicial set K. Suppose given a diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{V}^{\otimes}_{\mathrm{act}}$ that sends the cone point ∞ to an object in $\mathcal{V}$. Let $\bar{q}\colon K^{\triangleright} \to \mathcal{V}^{\otimes}$ be a coCartesian lift of $\bar{p}$ along the active maps to $[1]$. Then $\bar{p}$ is an operadic colimit diagram if and only if $\bar{q}$ is a colimit diagram. In particular, given a diagram $p\colon K \to \mathcal{V}^{\otimes}_{\mathrm{act}}$ there exists an operadic colimit diagram $\bar{p}\colon K^{\triangleright} \to \mathcal{V}^{\otimes}_{\mathrm{act}}$ extending p that sends ∞ to an object of $\mathcal{V}$.*

Proof. As [28, Proposition 3.1.1.20]. □

A.3. Operadic Kan extensions

We now discuss operadic Kan extensions in the non-symmetric case. Here we work in slightly more generality than for the corresponding results in [28] — the proof of Lurie's existence theorem can also be used to construct operadic Kan extensions along a restricted class of morphisms of generalized non-symmetric ∞-operads.

Definition A.3.1. Let $\mathcal{C}$ be an ∞-category. A *$\mathcal{C}$-family of (generalized) non-symmetric ∞-operads* is a categorical fibration $\pi\colon \mathcal{O} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathcal{C}$ such that:

(i) For $C \in \mathcal{C}$, $X \in \mathcal{O}_C$, and α an inert morphism in $\mathbf{\Delta}^{\mathrm{op}}$ from the image of X, there exists a coCartesian morphism $X \to Y$ over α in $\mathcal{O}_C$.
(ii) For $X \in \mathcal{O}_C$ with image $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ let $p_X\colon K^{\triangleleft}_{[n]} \to \mathcal{O}$ be a coCartesian lift of $p_{[n]}\colon K_{[n]} \to \mathbf{\Delta}^{\mathrm{op}}$ (or consider a lift of $\mathcal{G}^{\mathrm{ns}}_{[n]/} \to \mathbf{\Delta}^{\mathrm{op}}$ for a generalized non-symmetric ∞-operad). Then p_X is a π-limit diagram.
(iii) For each $C \in \mathcal{C}$, the induced map $\mathcal{O}_C \to \mathbf{\Delta}^{\mathrm{op}}$ is a (generalized) non-symmetric ∞-operad.

A Δ^1-family will also be referred to as a *correspondence of (generalized) non-symmetric ∞-operads.*

Definition A.3.2. We say a correspondence $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}} \times \Delta^1$ of generalized non-symmetric ∞-operads is *constant over* $[0]$ if the restriction $\mathcal{M}_{[0]} \to \Delta^1$ is a coCartesian fibration whose associated functor $\Delta^1 \to \mathrm{Cat}_\infty$ is an equivalence.

Definition A.3.3. Let $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}} \times \Delta^1$ be a correspondence from a generalized non-symmetric ∞-operad $\mathcal{A}$ to a generalized non-symmetric ∞-operad $\mathcal{B}$ that is constant over $[0]$ and such that $\mathcal{A}_{[0]}$ and $\mathcal{B}_{[0]}$ are Kan complexes, let $\mathcal{O}$ be a non-symmetric ∞-operad, and let $\overline{F}\colon \mathcal{M} \to \mathcal{O}$ be a map of generalized non-symmetric ∞-operads. The map $\overline{F}$ is an *operadic left Kan extension* of $F = \overline{F}|_{\mathcal{A}}$ if for every $B \in \mathcal{B}_{[1]}$ the composite map

$$\big((\mathcal{M}_{\mathrm{act}})_{/B} \times_{\mathcal{M}} \mathcal{A}\big)^{\triangleright} \to (\mathcal{M}_{/B})^{\triangleright} \to \mathcal{M} \xrightarrow{\overline{F}} \mathcal{O}$$

is an operadic colimit diagram.

Theorem A.3.4.

(i) Suppose given a Δ^1-family of generalized non-symmetric ∞-operads $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}} \times \Delta^1$ constant over $[0]$ and such that $\mathcal{M}_{[0],i}$ is a Kan complex for $i = 0, 1$, a non-symmetric ∞-operad $\mathcal{O}$ and a commutative diagram of generalized non-symmetric ∞-operad family maps

$$\begin{array}{ccc} \mathcal{M} \times_{\Delta^1} \{0\} & \xrightarrow{f} & \mathcal{O} \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}}. \end{array}$$

Then there exists an operadic left Kan extension $\overline{f}$ of f if and only if for every B in $\mathcal{M} \times_{\Delta^1} \{1\}$, the diagram

$$(\mathcal{M}_{\mathrm{act}})_{/B} \times_{\Delta^1} \{0\} \to \mathcal{M} \times_{\Delta^1} \{0\} \xrightarrow{f} \mathcal{O}$$

can be extended to an operadic colimit diagram lifting

$$\big((\mathcal{M}_{\mathrm{act}})_{/B} \times_{\Delta^1} \{0\}\big)^{\triangleright} \to \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}.$$

(ii) Suppose given a Δ^n-family of generalized non-symmetric ∞-operads $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}} \times \Delta^n$ with $n \geq 1$ such that all sub-Δ^1-families are constant over $[0]$ and the fibres $\mathcal{M}_{[0],i}$ are all Kan complexes, a non-symmetric ∞-operad $\mathcal{O}$, and a commutative diagram of generalized non-symmetric ∞-operad family maps

$$\begin{array}{ccc} \mathcal{M} \times_{\Delta^n} \Lambda^n_0 & \xrightarrow{f} & \mathcal{O} \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathbf{\Delta}^{\mathrm{op}} \end{array}$$

such that the restriction of f to $\mathcal{M} \times_{\Delta^n} \Delta^{\{0,1\}}$ is an operadic left Kan extension of $f|_{\mathcal{M} \times_{\Delta^n} \{0\}}$. Then there exists a morphism $\overline{f} : \mathcal{M} \to \mathcal{O}$ extending f.

Proof. As [28, Theorem 3.1.2.3]. □

A.4. Free algebras

Let $\mathcal{V}$ be a monoidal ∞-category compatible with small colimits and let $f : \mathcal{A} \to \mathcal{B}$ be a functor of generalized non-symmetric ∞-operads that is an equivalence over $[0]$ and such that $\mathcal{A}_{[0]}$ and $\mathcal{B}_{[0]}$ are Kan complexes. Using the existence theorem for operadic left Kan extensions, we can now construct an adjoint to the functor

$$f^*\colon \mathrm{Alg}_{\mathcal{B}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{A}}(\mathcal{V})$$

given by composition with f. This is given by forming *free* algebras:

Definition A.4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be generalized non-symmetric ∞-operads, let $\mathcal{O}$ be a non-symmetric ∞-operad, and let $f\colon \mathcal{A} \to \mathcal{B}$ be a map of generalized non-symmetric ∞-operads that is an equivalence over $[0]$ and such that $\mathcal{A}_{[0]}$ and $\mathcal{B}_{[0]}$ are Kan complexes. Suppose $A \in \mathrm{Alg}_{\mathcal{A}}(\mathcal{O})$, $B \in \mathrm{Alg}_{\mathcal{B}}(\mathcal{O})$, and $\phi\colon A \to f^*B$ is a map of $\mathcal{A}$-algebras in $\mathcal{O}$. For $b \in \mathcal{B}_{[1]}$, let $(\mathcal{A}_{\mathrm{act}})_{/b} := \mathcal{A} \times_{\mathcal{B}} (\mathcal{B}_{\mathrm{act}})_{/b}$. Then A and B induce maps $\alpha, \beta\colon (\mathcal{A}_{\mathrm{act}})_{/b} \to \mathcal{O}_{\mathrm{act}}$ and ϕ determines a natural transformation $\eta\colon \alpha \to \beta$. The map β clearly extends to $\overline{\beta}\colon (\mathcal{A}_{\mathrm{act}})_{/b} \to (\mathcal{O}_{\mathrm{act}})_{/B(b)}$. Since the projection

$$(\mathcal{O}_{\mathrm{act}})_{/B(b)} \to \mathcal{O}_{\mathrm{act}} \times_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{act}}} \left(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{act}}\right)_{/[n]}$$

(where b lies over $[n] \in \mathbf{\Delta}^{\mathrm{op}}$) is a right fibration, we can lift η to an essentially unique map $\overline{\eta}\colon \overline{\alpha} \to \overline{\beta}$ (over $\mathbf{\Delta}^{\mathrm{op}}$). We say that ϕ *exhibits B as a free $\mathcal{B}$-algebra generated by A* if for every $b \in \mathcal{B}_{[1]}$ the map $\overline{\alpha}$ determines an operadic q-colimit diagram $(\mathcal{A}_{\mathrm{act}})^{\triangleright}_{/b} \to \mathcal{O}$.

Remark A.4.2. The map $\phi\colon A \to f^*B$ above determines a map

$$H\colon (\mathcal{A} \times \Delta^1) \amalg_{\mathcal{A} \times \{1\}} \mathcal{B} \to \mathcal{O} \times \Delta^1.$$

Choose a factorization of H as

$$H\colon \left(\mathcal{A} \times \Delta^1\right) \amalg_{\mathcal{A} \times \{1\}} \mathcal{B} \xrightarrow{H'} \mathcal{M} \xrightarrow{H''} \mathcal{O} \times \Delta^1,$$

where H' is a categorical equivalence and $\mathcal{M}$ is an ∞-category. The composite map $\mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}} \times \Delta^1$ exhibits $\mathcal{M}$ as a correspondence of non-symmetric ∞-operads. Then the map ϕ exhibits B as a free $\mathcal{B}$-algebra generated by A if and only if H'' is an operadic left Kan extension.

Proposition A.4.3. *Suppose $\phi\colon A \to f^*B$ exhibits B as a free $\mathcal{B}$-algebra in $\mathcal{O}$ generated by A. Then for every $B' \in \mathrm{Alg}_{\mathcal{B}}(\mathcal{O})$ composition with ϕ induces a homotopy equivalence*

$$\mathrm{Map}(B, B') \to \mathrm{Map}(A, f^*B').$$

Proof. As [28, Proposition 3.1.3.2]. □

Proposition A.4.4. *Suppose $A \in \mathrm{Alg}_{\mathcal{A}}(\mathcal{O})$. Then there exists a free $\mathcal{B}$-algebra B generated by A if and only if for every $b \in \mathcal{B}_{[1]}$ the induced map*

$$(\mathcal{A}_{\mathrm{act}})_{/b} \to \mathcal{A}_{\mathrm{act}} \xrightarrow{A} \mathcal{O}$$

can be extended to an operadic colimit diagram lying over

$$(\mathcal{A}_{\mathrm{act}})^{\triangleright}_{/b} \to \mathcal{B}_{\mathrm{act}} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{act}}.$$

Proof. As [28, Proposition 3.1.3.3]. □

Corollary A.4.5. *Let $\mathcal{O}$ be a non-symmetric ∞-operad, and suppose $f\colon \mathcal{A} \to \mathcal{B}$ is a map of generalized non-symmetric ∞-operads that is an equivalence over $[0]$ and such that $\mathcal{A}_{[0]}$ and $\mathcal{B}_{[0]}$ are Kan complexes. The functor $f^*\colon \mathrm{Alg}_{\mathcal{B}}(\mathcal{O}) \to \mathrm{Alg}_{\mathcal{A}}(\mathcal{O})$ admits a left adjoint $f_!$, provided that for every $\mathcal{A}$-algebra A in $\mathcal{O}$ and every $b \in \mathcal{B}_*$, the diagram*

$$(\mathcal{A}_{\mathrm{act}})_{/b} \to \mathcal{A}_{\mathrm{act}} \xrightarrow{A} \mathcal{O}$$

can be extended to an operadic colimit diagram lying over

$$(\mathcal{A}_{\mathrm{act}})^{\triangleright}_{/b} \to \mathcal{B}_{\mathrm{act}} \to \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{act}}.$$

Proof. As [28, Corollary 3.1.3.4]. □

Combining this with Proposition A.2.10, we get:

Theorem A.4.6. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with κ-small colimits for some uncountable regular cardinal κ, and $f\colon \mathcal{A} \to \mathcal{B}$ is a map of generalized non-symmetric ∞-operads that is an equivalence over $[0]$ and such that $\mathcal{A}_{[0]}$ and $\mathcal{B}_{[0]}$ are Kan complexes, with $\mathcal{A}$ and $\mathcal{B}$ essentially κ-small. Then the functor $f^*\colon \mathrm{Alg}_{\mathcal{B}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{A}}(\mathcal{V})$ admits a left adjoint $f_!$.*

Lemma A.4.7. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are monoidal ∞-categories which are compatible with small colimits, and let $F\colon \mathcal{V}^{\otimes} \to \mathcal{W}^{\otimes}$ be a monoidal functor such that $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Then for every generalized non-symmetric ∞-operad $\mathcal{M}$ the induced functor*

$$F_*\colon \mathrm{Alg}_{\mathcal{M}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{M}}(\mathcal{W})$$

preserves free algebras, i.e. for all maps of generalized non-symmetric ∞-operads $f\colon \mathcal{N} \to \mathcal{M}$ that are equivalences over $[0]$ and such that $\mathcal{M}_{[0]}$ and $\mathcal{N}_{[0]}$ are Kan complexes, the natural map $f_!F_* \to F_*f_!$ (adjoint to $F_* \to F_*f^*f_! \simeq f^*F_*f_!$) is an equivalence.

Proof. This follows immediately from Corollary A.2.9. □

We can also give a more explicit description of the left adjoint $\tau_{\mathcal{M},!}$, where $\mathcal{M}$ is a generalized non-symmetric ∞-operad such that $\mathcal{M}_{[0]}$ is a Kan complex. Recall that by Proposition 3.4.5 if $\mathcal{O}$ is a non-symmetric ∞-operad then we have $\mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{O}) \simeq$

$\mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{O}_{[1]})$. We can therefore regard $\tau_{\mathcal{M},!}$ as a functor

$$\mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{O}_{[1]}) \to \mathrm{Alg}_{\mathcal{M}}(\mathcal{O}).$$

Definition A.4.8. For $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ and $X \in \mathcal{M}_{[1]}$, let $\mathcal{P}^{\mathcal{M}}_{X,n}$ be the full subcategory of $\mathcal{M}_{\mathrm{triv}} \times_{\mathcal{M}} \mathcal{M}_{/X}$ of morphisms $Y \to X$ over the active map $[n] \to [1]$.

Suppose $\mathcal{V}$ is a monoidal ∞-category and $F: \mathcal{M}_{[1]} \to \mathcal{V}$ is a functor. Let $\overline{F}$ be the associated $\mathcal{M}_{\mathrm{triv}}$-algebra in $\mathcal{V}$. We have a canonical map $h: \mathcal{P}^{\mathcal{M}}_{X,n} \times \Delta^1 \to \mathcal{M}$, a natural transformation from $\mathcal{P}^{\mathcal{M}}_{X,n} \to \mathcal{M}_{\mathrm{triv}} \hookrightarrow \mathcal{M}$ to the constant functor at X. Since $\mathcal{V}^{\otimes} \to \mathbf{\Delta}^{\mathrm{op}}$ is coCartesian, from $\overline{F} \circ h$ we get a coCartesian natural transformation $\bar{h}$ from a functor $g: \mathcal{P}^{\mathcal{M}}_{X,n} \to \mathcal{V}$ to the constant functor at $F(X)$. We let $\mathrm{P}^n_{\mathcal{M},X}(F)$ denote a colimit of g, if it exists.

Proposition A.4.9. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with κ-small colimits, and $\mathcal{M}$ is a κ-small generalized non-symmetric ∞-operad such that $\mathcal{M}_{[0]}$ is a Kan complex. Suppose moreover that A is an $\mathcal{M}$-algebra in $\mathcal{V}$ and $F: \mathcal{M}_{[1]} \to \mathcal{V}$ is a functor. Then a map $F \to (\tau_{\mathcal{M}})^* A$ is adjoint to an equivalence $\tau_{\mathcal{M},!}F \xrightarrow{\sim} A$ if and only if for every $X \in \mathcal{M}_{[1]}$ the maps $\mathrm{P}^n_{\mathcal{M},X}(F) \to A(X)$ exhibit $A(X)$ as a coproduct*

$$\coprod_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{P}^n_{\mathcal{M},X}(F) \to A(X)$$

Proof. As [28, Proposition 3.1.3.13]. □

A.5. Colimits of algebras in monoidal ∞-categories

In this subsection we show that colimits exist in the ∞-categories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ for all small non-symmetric ∞-operads $\mathcal{O}$ when $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits. We first consider the case of sifted colimits:

Lemma A.5.1. *Suppose K is a sifted simplicial set and $\mathcal{V}$ is a monoidal ∞-category that is compatible with K-indexed colimits. Then for every $\phi: [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ the associated functor $\phi_!: \mathcal{V}^{\otimes}_{[n]} \to \mathcal{V}^{\otimes}_{[m]}$ preserves K-indexed colimits.*

Proof. As [28, Lemma 3.2.3.7]. □

Lemma A.5.2. *Suppose $p: X \to S$ is a coCartesian fibration, and let $\bar{r}: K^{\triangleright} \to \mathrm{Fun}(\Delta^1, X)$ be a colimit diagram such that for every $i \in K$ the edge $\bar{r}(i,0) \to \bar{r}(i,1)$ is coCartesian. Then the edge $\bar{r}(\infty,0) \to \bar{r}(\infty,1)$ is also coCartesian.*

Proof. Since colimits in functor categories are pointwise, we must show that for all $x \in X$ the diagram

$$\begin{array}{ccc} \mathrm{Map}_X(\mathrm{colim}_i\, \bar{r}(i,1), x) & \longrightarrow & \mathrm{Map}_X(\mathrm{colim}_i\, \bar{r}(i,0), x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_S(\mathrm{colim}_i\, p\bar{r}(i,1), p(x)) & \longrightarrow & \mathrm{Map}_S(\mathrm{colim}_i\, p\bar{r}(i,0), p(x)) \end{array}$$

is Cartesian, which is clear since limits commute. □

To describe sifted colimits of algebras, we need the following result, which is due to Jacob Lurie — we thank him for explaining the proof to us.

Theorem A.5.3. *Let K be a weakly contractible simplicial set. Suppose $p: X \to S$ is a coCartesian fibration such that for all $s \in S$ the fibre X_s admits K-indexed colimits, and for all edges $f: s \to t$ in S the functor $f_!: X_s \to X_t$ preserves K-indexed colimits. Then for any map $g: T \to S$,*

(i) the ∞-category $\mathrm{Fun}_S(T, X)$ admits K-indexed colimits,
(ii) a map $K^{\triangleright} \to \mathrm{Fun}_S(T, X)$ is a colimit diagram if and only if for all $t \in T$ the composite

$$K^{\triangleright} \to \mathrm{Fun}_S(T, X) \to X_{g(t)}$$

is a colimit diagram,
(iii) if E is a set of edges of T, the full subcategory of $\mathrm{Fun}_S(T, X)$ spanned by functors that take the edges in E to coCartesian edges of X is closed under K-indexed colimits in $\mathrm{Fun}_S(T, X)$.

Proof. The ∞-category $\mathrm{Fun}_S(T, X)$ is a fibre of the functor $p_*: \mathrm{Fun}(T, X) \to \mathrm{Fun}(T, S)$ induced by composition with p. The functor p_* is a coCartesian fibration by [25, Proposition 3.1.2.1]. Since the functors $f_!$ preserve K-indexed colimits, by [25, Proposition 4.3.1.10] a diagram $\bar{q}: K^{\triangleright} \to \mathrm{Fun}_S(T, X)$ is a colimit diagram if and only if the composite $\bar{q}': K^{\triangleright} \to \mathrm{Fun}_S(T, X) \to \mathrm{Fun}(T, X)$ is a p_*-colimit diagram. By [25, Corollary 4.3.1.11], K-indexed p_*-colimits exist in $\mathrm{Fun}(T, X)$, which proves (i).

Moreover, a diagram in $\mathrm{Fun}(T, X)$ is a colimit diagram if and only if it is a p_*-colimit diagram and its image in $\mathrm{Fun}(T, S)$ is a colimit diagram. Since $\bar{q}'$ lands in one of the fibres of p_*, the projection to $\mathrm{Fun}(T, S)$ is constant, which means it is a colimit as K is weakly contractible. Thus $\bar{q}'$ is a p_*-colimit diagram if and only if it is a colimit diagram in $\mathrm{Fun}(T, X)$. By [25, Corollary 5.1.2.3] this means that $\bar{q}'$ is a colimit diagram if and only if for all $t \in T$ the induced maps $K^{\triangleright} \to X$ are colimit diagrams. A diagram in X is a colimit if and only if it is a p-colimit and the projection to S is a colimit. Since K is weakly contractible, applying [25, Proposition 4.3.1.10] we see that this is true if and only if the induced map $K^{\triangleright} \to X_{g(t)}$ is a colimit diagram in $X_{g(t)}$. This proves (ii).

Suppose $e: t \to t'$ is an edge of T and $q: K \to \mathrm{Fun}_S(T, X)$ is a diagram such that for all vertices $k \in K$ the functor $q(k): T \to X$ takes e to a p-coCartesian edge of X.

Let $\bar{q}\colon K^{\triangleright} \to \mathrm{Fun}_S(T, X)$ be a colimit diagram extending q. To prove (iii) we must show that the functor $\bar{q}(\infty)$ also takes e to a coCartesian edge of X. From our description of colimits in $\mathrm{Fun}_S(T, X)$ it follows that this is equivalent to showing that coCartesian edges of X are closed under colimits, which is true by Lemma A.5.2. □

Corollary A.5.4. *Suppose K is a sifted simplicial set and $\mathcal{V}$ is a monoidal ∞-category that is compatible with K-indexed colimits. Then for any generalized non-symmetric ∞-operad $p\colon \mathcal{M} \to \mathbf{\Delta}^{\mathrm{op}}$, we have:*

- *(i) The ∞-category $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$ admits K-indexed colimits.*
- *(ii) A map $K^{\triangleright} \to \mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$ is a colimit diagram if and only if for every $X \in \mathcal{M}$ the induced diagram $K^{\triangleright} \to \mathcal{V}^{\otimes}_{p(X)}$ is a colimit diagram.*
- *(iii) The full subcategory $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ of $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$ is stable under K-indexed colimits.*
- *(iv) A map $K^{\triangleright} \to \mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$ is a colimit diagram if and only if, for every $X \in \mathcal{M}_{[1]}$, the induced diagram $K^{\triangleright} \to \mathcal{V}$ is a colimit diagram.*
- *(v) The restriction functor $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V}) \to \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{V})$ detects K-indexed colimits.*

Proof. Sifted simplicial sets are weakly contractible by [25, Proposition 5.5.8.7] so (i)–(iii) follow from Theorem A.5.3 (which is implicit in the proof of [28, Proposition 3.2.3.1]). Then (iv) and (v) follow as in the proof of [28, Proposition 3.2.3.1]. □

We now use this to show that the adjunction $\tau_{\mathcal{M},!} \dashv \tau_{\mathcal{M}}^{*}$ is monadic; we first check that $\tau_{\mathcal{M}}^{*}$ is conservative:

Lemma A.5.5. *Suppose $\mathcal{V}$ is a monoidal ∞-category and $\mathcal{M}$ is a generalized non-symmetric ∞-operad. Then the forgetful functor*

$$\tau_{\mathcal{M}}^{*}\colon \mathrm{Alg}_{\mathcal{M}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{V}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{V})$$

is conservative.

Proof. The ∞-category $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ is a full subcategory of $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$. Therefore a map of algebras $f\colon A \to B$ is an equivalence in $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ if and only if it is an equivalence in $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$. Applying Theorem A.5.3 to Δ^0-indexed colimits, we see that a morphism $f\colon A \to B$ is an equivalence in $\mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{M}, \mathcal{V}^{\otimes})$ if and only if $f_X\colon A(X) \to B(X)$ is an equivalence in $\mathcal{V}^{\otimes}$ for all $X \in \mathcal{M}$. Thus equivalences are detected after restricting to $\mathcal{M}_{\mathrm{triv}}$. □

Corollary A.5.6. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits, and $\mathcal{M}$ is a generalized non-symmetric ∞-operad such that $\mathcal{M}_{[0]}$ is a Kan complex. Then the adjunction*

$$(\tau_{\mathcal{M}})_!\colon \mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathcal{M}}(\mathcal{V})\colon (\tau_{\mathcal{M}})^{*}$$

is monadic.

Proof. We showed that the functor $\tau_{\mathcal{M}}^*$ is conservative in Lemma A.5.5, and that it preserves sifted colimits in Corollary A.5.4. The adjunction $(\tau_{\mathcal{M}})_! \dashv \tau_{\mathcal{M}}^*$ is therefore monadic by [25, Corollary 5.5.2.9]. □

Corollary A.5.7. *Suppose $\mathcal{V}$ is a monoidal ∞-category compatible with small colimits and $\mathcal{M}$ is a generalized non-symmetric ∞-operad such that $\mathcal{M}_{[0]}$ is a Kan complex. Then $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ has all small colimits. Moreover, if $\mathcal{V}$ is presentable, so is $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$.*

This is an immediate consequence of the following general facts about monadic adjunctions:

Lemma A.5.8. *Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a monadic adjunction such that $\mathcal{C}$ has all small colimits, $\mathcal{D}$ has sifted colimits, and U preserves sifted colimits. Then $\mathcal{D}$ has all small colimits.*

Proof. Since $\mathcal{D}$ by assumption has all sifted colimits, it suffices to prove that $\mathcal{D}$ has finite coproducts. Since $\mathcal{C}$ has coproducts and F preserves colimits, the ∞-category $\mathcal{D}$ has coproducts for objects in the essential image of F.

Let $A^1, \dots, A^n$ be a finite collection of objects in $\mathcal{D}$. By [28, Proposition 4.7.4.14], there exist simplicial objects $A^i_\bullet$ in $\mathcal{D}$ such that each A^i_k is in the essential image of F and $|A^i_\bullet| \simeq A^i$. Since coproducts of elements in the essential image of F exist, we can form a simplicial diagram $\coprod_i A^i_\bullet$. By [25, Lemma 5.5.2.3], the geometric realization $|\coprod_i A^i_\bullet|$ is a coproduct of the A^i's. □

Proposition A.5.9. *Suppose $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a monadic adjunction such that $\mathcal{C}$ is κ-presentable, $\mathcal{D}$ has small colimits, and the right adjoint U preserves κ-filtered colimits. Then $\mathcal{D}$ is κ-presentable.*

Proof. Since $\mathcal{C}$ is κ-presentable, every object of $\mathcal{C}$ is a colimit of κ-compact objects. Since U preserves κ-filtered colimits, F preserves κ-compact objects by Lemma 3.3.5. Therefore every object in the essential image of F is a colimit of κ-compact objects. But by [28, Proposition 4.7.4.14], every object of $\mathcal{D}$ is a colimit of objects in the essential image of F, so every object of $\mathcal{D}$ is a colimit of κ-compact objects. Since by assumption $\mathcal{D}$ has all small colimits, this implies that $\mathcal{D}$ is κ-presentable. □

Proof of Corollary A.5.7. Apply Lemma A.5.8 and Proposition A.5.9 to the monadic adjunction $\tau_{\mathcal{M},!} \dashv \tau_{\mathcal{M}}^*$. □

Proposition A.5.10. *Let $\mathcal{M}$ be a generalized non-symmetric ∞-operad such that $\mathcal{M}_{[0]}$ is a Kan complex, and let $\mathcal{V}$ and $\mathcal{W}$ be monoidal ∞-categories compatible with small colimits. Suppose $F\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ is a monoidal functor such that $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Then the induced functor*

$$F_*\colon \mathrm{Alg}_{\mathcal{M}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{M}}(\mathcal{W})$$

preserves colimits.

Proof. Write F_*^{triv} for the induced functor $\mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{W})$. Under the equivalences $\mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{V}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{V})$ and $\mathrm{Alg}_{\mathcal{M}_{\mathrm{triv}}}(\mathcal{W}) \simeq \mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{W})$ this corresponds to composition with $F_{[1]}$, and so preserves colimits. Clearly $\tau_{\mathcal{M}}^* F_* \simeq F_*^{\mathrm{triv}} \tau_{\mathcal{M}}^*$. Since $\tau_{\mathcal{M}}^*$ detects sifted colimits, it follows that F_* preserves sifted colimits. To prove that it preserves all colimits, it thus remains to prove it preserves finite coproducts.

Since F is a monoidal functor, by Lemma A.4.7 the functor F_* preserves free algebras, i.e. $F_* \tau_{\mathcal{M},!} \simeq \tau_{\mathcal{M},!} F_*^{\mathrm{triv}}$. Therefore F_* preserves colimits of free algebras. Let A and B be objects of $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ and let $A_\bullet$ and $B_\bullet$ be free resolutions of A and B. Then we have natural equivalences

$$\begin{aligned} F_*(A \amalg B) &\simeq F_*(|A_\bullet \amalg B_\bullet|) \simeq |F_*(A_\bullet \amalg B_\bullet)| \simeq |F_*(A_\bullet) \amalg F_*(B_\bullet)| \\ &\simeq |F_*(A_\bullet)| \amalg |F_*(B_\bullet)| \simeq F_*(|A_\bullet|) \amalg F_*(|B_\bullet|) \simeq F_*(A) \amalg F_*(B), \end{aligned}$$

so F_* does indeed preserve coproducts. □

Proposition A.5.11. *Suppose $\mathcal{V}$ and $\mathcal{W}$ are presentably monoidal ∞-categories and $F\colon \mathcal{V}^\otimes \to \mathcal{W}^\otimes$ is a monoidal functor such that the underlying functor $F_{[1]}\colon \mathcal{V} \to \mathcal{W}$ preserves colimits. Let $g\colon \mathcal{W} \to \mathcal{V}$ be a right adjoint of $F_{[0]}$. Then there exists a lax monoidal functor $G\colon \mathcal{W}^\otimes \to \mathcal{V}^\otimes$ extending g such that for any small non-symmetric ∞-operad $\mathcal{O}$ we have an adjunction*

$$F_*\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{W}) : G_*.$$

Proof. By Proposition A.5.10 the functor $F_*\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{W})$ is colimit-preserving, and by Corollary A.5.7 these ∞-categories of $\mathcal{O}$-algebras are presentable. It follows by [25, Corollary 5.5.2.9] that F_* has a right adjoint

$$R_{\mathcal{O}}\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{W}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}).$$

Moreover, since F_* is natural in $\mathcal{O}$ so is $R_{\mathcal{O}}$, by [25, Corollary 5.2.2.5]. Taking the underlying spaces of the ∞-categories of algebras, we see that $R_{(-)}$ induces a natural transformation $\rho\colon \mathrm{Map}(-, \mathcal{W}^\otimes) \to \mathrm{Map}(-, \mathcal{V}^\otimes)$ of functors $(\mathrm{Opd}_\infty^{\mathrm{ns}})^{\mathrm{op}} \to \mathcal{S}$. The full subcategory $\mathcal{W}_\kappa^\otimes$ of $\mathcal{W}^\otimes$ spanned by objects coming from the full subcategory $\mathcal{W}^\kappa \subseteq \mathcal{W}$ spanned by κ-compact objects is a small non-symmetric ∞-operad. Applying $R_{\mathcal{W}_\kappa^\otimes}$ to the inclusion $\mathcal{W}_\kappa^\otimes \to \mathcal{W}^\otimes$ gives compatible maps $G^\kappa\colon \mathcal{W}_\kappa^\otimes \to \mathcal{V}^\otimes$. Combining these gives $G\colon \mathcal{W}^\otimes \to \mathcal{V}^\otimes$. Since every map $\mathcal{O} \to \mathcal{W}^\otimes$ where $\mathcal{O}$ is a small non-symmetric ∞-operad factors through $\mathcal{W}_\kappa^\otimes$ for some κ, we see that ρ is given by composition with G. Moreover, the functor $R_{(-)}$ must also be given by composition with G, since $\mathrm{Alg}_{\mathcal{O}}(\mathcal{W})$ is the ∞-category associated to the simplicial space $\mathrm{Map}(\mathcal{O} \otimes \Delta^\bullet, \mathcal{W}^\otimes)$.

It remains to show that G is indeed a lax monoidal extension of g. This follows from taking $\mathcal{O}$ to be the trivial non-symmetric ∞-operad $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}$: then $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}}(\mathcal{V}) \simeq \mathcal{V}$ and $\mathrm{Alg}_{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{int}}}(\mathcal{W}) \simeq \mathcal{W}$, and under these identifications F_* corresponds to $F_{[1]}$ and G_* to the functor $G_{[1]}$. Thus g and $G_{[1]}$ are both right adjoint to F and so must be equivalent. □

In the case of monoidal localizations we can explicitly identify this lax monoidal structure on the right adjoint:

Lemma A.5.12. *Suppose $\mathcal{O}$ is a small non-symmetric ∞-operad, $\mathcal{V}$ is a monoidal ∞-category and $L: \mathcal{V} \to \mathcal{W}$ is a monoidal localization with fully faithful right adjoint $i: \mathcal{W} \hookrightarrow \mathcal{V}$. Then the monoidal functor $L^{\otimes}$ and the lax monoidal inclusion $i^{\otimes}: \mathcal{W}^{\otimes} \hookrightarrow \mathcal{V}^{\otimes}$ of Proposition 3.1.22 induce an adjunction*

$$L^{\otimes}_* : \mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{W}) : i^{\otimes}_*.$$

Moreover, $i^{\otimes}_*$ is fully faithful.

Proof. Since $L^{\otimes}$ is left adjoint to $i^{\otimes}$, it is easy to see that we get an adjunction

$$L^{\otimes}_* : \mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{O}, \mathcal{V}^{\otimes}) \rightleftarrows \mathrm{Fun}_{\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{O}, \mathcal{W}^{\otimes}) : i^{\otimes}_*.$$

But this clearly restricts to an adjunction between the full subcategories $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ and $\mathrm{Alg}_{\mathcal{O}}(\mathcal{W})$, as required.

To prove that $i^{\otimes}_*$ is fully faithful, it suffices to show that for every $\mathcal{O}$-algebra A in $\mathcal{W}$ the counit $L^{\otimes}_* i^{\otimes}_* A \to A$ is an equivalence. By Lemma A.5.5 we need only show that the induced natural transformation of functors $\mathcal{O}_{[1]} \to \mathcal{W}$ is an equivalence, i.e. that for every $X \in \mathcal{O}_{[1]}$ the map $LiA(X) \to A(X)$ is an equivalence in $\mathcal{W}$, which is true since i is fully faithful. □

A.6. Approximations of ∞-operads

In this subsection we use Lurie's theory of *approximations* to give a criterion for a map $\mathcal{M} \to \mathcal{O}$ to exhibit a non-symmetric ∞-operad $\mathcal{O}$ as the operadic localization $L_{\mathrm{gen}}\mathcal{M}$ of a generalized non-symmetric ∞-operad $\mathcal{M}$.

Definition A.6.1. Suppose $\mathcal{M}$ is a generalized non-symmetric ∞-operad, $\mathcal{O}$ is a non-symmetric ∞-operad, and $f: \mathcal{M} \to \mathcal{O}$ is a fibration of generalized non-symmetric ∞-operads. Then f is an *approximation* if for all $C \in \mathcal{M}$ and $\alpha: X \to f(C)$ active in $\mathcal{O}$ there exists an f-Cartesian morphism $\bar{\alpha}: \overline{X} \to C$ lifting α, and a *weak approximation* if given $C \in \mathcal{M}$ and $\alpha: X \to f(C)$ an arbitrary morphism in $\mathcal{O}$, the full subcategory of

$$\mathcal{M}_{/C} \times_{\mathcal{O}_{/f(C)}} \mathcal{O}_{X//f(C)}$$

corresponding to pairs $(\beta: C' \to C, \gamma: X \to f(C'))$ with γ inert is weakly contractible. More generally, a map $f: \mathcal{M} \to \mathcal{O}$ is a (*weak*) *approximation* if it factors as a composition

$$\mathcal{M} \xrightarrow{f'} \mathcal{M}' \xrightarrow{f''} \mathcal{O}$$

where f' is an equivalence of generalized non-symmetric ∞-operads and f'' is a categorical fibration that is a (weak) approximation.

Proposition A.6.2. *An approximation is a weak approximation.*

Proof. As [28, Lemma 2.3.3.10]. □

Proposition A.6.3. *A fibration of generalized non-symmetric ∞-operads $f: \mathcal{M} \to \mathcal{O}$, where $\mathcal{O}$ is a non-symmetric ∞-operad, is a weak approximation if and only if for every object $C \in \mathcal{M}$ and every active morphism $\alpha: X \to f(C)$ in $\mathcal{O}$, the ∞-category $\mathcal{M}_{/C} \times_{\mathcal{O}_{/f(C)}} \{X\}$ is weakly contractible.*

Proof. As [28, Proposition 2.3.3.11]. □

Proposition A.6.4. *Let $f: \mathcal{M} \to \mathcal{O}$ be a fibration of generalized non-symmetric ∞-operads, where $\mathcal{O}$ is a non-symmetric ∞-operad. If $\mathcal{O}_{[1]}$ is a Kan complex, then f is a weak approximation if and only if f is an approximation.*

Proof. As [28, Corollary 2.3.3.17]. □

Theorem A.6.5. *Suppose $f: \mathcal{M} \to \mathcal{O}$ is a weak approximation such that $f_{[1]}: \mathcal{M}_{[1]} \to \mathcal{O}_{[1]}$ is a categorical equivalence. Then for any non-symmetric ∞-operad $\mathcal{P}$, the induced map*

$$f^*: \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \to \mathrm{Alg}_{\mathcal{M}}(\mathcal{P})$$

is an equivalence.

Proof. As [28, Theorem 2.3.3.23]. □

Corollary A.6.6. *Suppose $f: \mathcal{M} \to \mathcal{O}$ is a weak approximation such that $f_{[1]}$ is a categorical equivalence. Then the induced map of non-symmetric ∞-operads $L_{\mathrm{gen}}\mathcal{M} \to \mathcal{O}$ is an equivalence.*

Proposition A.6.7. *Suppose $f: \mathcal{O} \to \mathcal{P}$ is a map of non-symmetric ∞-operads, and $\mathcal{P}_{[1]}$ is a Kan complex. The commutative diagram*

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}) & \xrightarrow{f^*} & \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}) \\ \downarrow{\tau^*_{\mathcal{P}}} & & \downarrow{\tau^*_{\mathcal{O}}} \\ \mathrm{Fun}(\mathcal{P},\mathcal{S}) & \xrightarrow[f^*_{[1]}]{} & \mathrm{Fun}(\mathcal{O},\mathcal{S}) \end{array}$$

induces a natural transformation $\alpha\colon \tau_{\mathcal{O},!} \circ f^*_{[1]} \to f^* \circ \tau_{\mathcal{P},!}$*. If* α *induces an equivalence* $\tau_{\mathcal{O},!} f^*_{[1]} A \xrightarrow{\sim} f^* \tau_{\mathcal{P},!} A$ *where* A *is the constant functor* $\mathcal{P} \to \mathcal{S}$ *with value* $*$*, then* f *is an approximation.*

Proof. As [28, Proposition 2.3.4.8]. □

Corollary A.6.8. *Let* $\mathcal{O}$ *be a non-symmetric* ∞*-operad such that* $\mathcal{O}_{[1]}$ *is a Kan complex, and let* $f\colon \mathcal{M} \to \mathcal{O}$ *be a map of generalized non-symmetric* ∞*-operads such that* $f_{[1]}\colon \mathcal{M}_{[1]} \to \mathcal{O}_{[1]}$ *is an equivalence. Write* A *for the constant functor* $\mathcal{M}_{[1]} \simeq \mathcal{O}_{[1]} \to \mathcal{S}$ *with value* $*$*. If the natural map* $\tau_{\mathcal{M},!} A \to f^* \tau_{\mathcal{O},!} A$ *is an equivalence, then* f *exhibits* $\mathcal{O}$ *as the operadic localization of* $\mathcal{M}$*.*

Proof. Applying Proposition A.6.7 to the induced map $f'\colon L_{\mathrm{gen}}\mathcal{M} \to \mathcal{O}$, we see that this map is an approximation and induces an equivalence $L_{\mathrm{gen}}\mathcal{M}_{[1]} \to \mathcal{O}_{[1]}$. By Theorem A.6.5, it follows that f' is an equivalence. □

Corollary A.6.9. *Let* $\mathcal{O}$ *be a non-symmetric* ∞*-operad such that* $\mathcal{O}_{[1]}$ *is a Kan complex, and* $f\colon \mathcal{M} \to \mathcal{O}$ *be a map of generalized non-symmetric* ∞*-operads such that* $f_{[1]}\colon \mathcal{M}_{[1]} \to \mathcal{O}_{[1]}$ *is an equivalence and* $\mathcal{M}_{[0]}$ *is a Kan complex. If the induced map* $(\mathcal{M}_{\mathrm{act}})_{/x} \to (\mathcal{O}_{\mathrm{act}})_{/x}$ *is cofinal for all* $x \in \mathcal{M}_{[1]} \simeq \mathcal{O}_{[1]}$*, then* f *exhibits* $\mathcal{O}$ *as the operadic localization of* $\mathcal{M}$*.*

Proof. By Corollary A.6.8 it suffices to show that the natural map of $\mathcal{M}$-algebras $\tau_{\mathcal{M},!} A \to f^* \tau_{\mathcal{O},!} A$ is an equivalence. Since $\tau^*_{\mathcal{M}}$ detects equivalences by Lemma A.5.5, to see this it suffices to show that for all $x \in \mathcal{M}_{[1]}$ the map of spaces $(\tau_{\mathcal{M},!} A)(x) \to (\tau_{\mathcal{O},!} A)(x)$ is an equivalence. Since $\mathcal{M}_{[0]}$ is a Kan complex, we can describe $\tau_{\mathcal{M},!} A$ using the results of §A.3. We thus see that this map can be identified with the map

$$\operatorname*{colim}_{(\mathcal{M}_{\mathrm{act}})_{/x}} * \to \operatorname*{colim}_{(\mathcal{O}_{\mathrm{act}})_{/x}} *$$

of colimits induced by $(\mathcal{M}_{\mathrm{act}})_{/x} \to (\mathcal{O}_{\mathrm{act}})_{/x}$. If this map is cofinal, then the induced map on colimits is an equivalence. □

Remark A.6.10. The same argument shows that for any presentably monoidal ∞-category $\mathcal{V}$ the natural map $\tau_{\mathcal{M},!} F \to f^* \tau_{\mathcal{O},!} F$ is an equivalence for any functor $F\colon \mathcal{M}_{[1]} \to \mathcal{V}$. It follows that $\tau_{\mathcal{M},!}$ and $\tau_{\mathcal{O},!}$ are given by the same monad on $\mathrm{Fun}(\mathcal{M}_{[1]}, \mathcal{V})$, hence the ∞-categories of algebras $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ and $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V})$ must be equivalent, since they are both

∞-categories of algebras for this monad. An alternative proof of Corollary A.6.9 (not using the notion of approximation) should be possible by embedding any small non-symmetric ∞-operad $\mathcal{P}$ in a presentably monoidal ∞-category $\widehat{\mathcal{P}}$ and showing that $\mathrm{Alg}_{\mathcal{M}}(\mathcal{P})$ and $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ are the same subcategory of $\mathrm{Alg}_{\mathcal{M}}(\widehat{\mathcal{P}}) \simeq \mathrm{Alg}_{\mathcal{O}}(\widehat{\mathcal{P}})$.

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