

ON HOPKINS' PICARD GROUPS FOR THE PRIME 3 AND CHROMATIC LEVEL 2

PAUL GOERSS, HANS-WERNER HENN, MARK MAHOWALD AND CHARLES REZK

ABSTRACT. We give a calculation of Picard groups Pic_2 of $K(2)$ -local invertible spectra and $Pic(\mathcal{L}_2)$ of $E(2)$ -local invertible spectra, both at the prime 3. The main contribution of this paper is to calculation the subgroup κ_2 of invertible spectra X with $(E_2)_*X \cong (E_2)_*S^0$ as twisted modules over the Morava stabilizer group \mathbb{G}_2 .

1. INTRODUCTION

Let \mathcal{C} be a symmetric monoidal category with product \wedge and unit object I . An object $X \in \mathcal{C}$ is *invertible* if there exists an object Y and an isomorphism $X \wedge Y \cong I$. If the collection of isomorphism classes of invertible objects is a set, then \wedge defines a group structure on this set. This group is called the *Picard group* of \mathcal{C} and is denoted by $Pic(\mathcal{C})$. It is a basic invariant of the category \mathcal{C} .

An immediate example from stable homotopy is the category \mathcal{S} of spectra and homotopy classes of maps. However, the answer turns out to be unsurprising: the only invertible spectra are the sphere spectra S^n , $n \in \mathbb{Z}$ and the map $\mathbb{Z} \rightarrow Pic(\mathcal{S})$ sending n to S^n is an isomorphism. See [22].

An insight due to Mike Hopkins is that the problem becomes considerably more interesting if we pass to the localized stable homotopy categories which arise in chromatic stable homotopy theory; that is, if we localize the stable category with respect to complex orientable cohomology theories. We fix a prime p and work with p -local theories – for example, the Morava K -theories $K(n)$, $n \geq 1$, or the closely related Johnson-Wilson theories $E(n)$. We will write L_E for localization with respect to E ; however it is customary to abbreviate $L_{E(n)}$ as L_n . We will also write \mathcal{K}_n for the homotopy category of $K(n)$ -local spectra and \mathcal{L}_n for the category of $E(n)$ -local spectra. See [15] for a great deal of information about the basic structure of these categories.

If E is any homology theory, then the homotopy category of E -local spectra has a symmetric monoidal structure given by smash product with unit $L_E S^0$. The main project of this paper is to study the Picard group of the categories \mathcal{K}_2 and \mathcal{L}_2 at the prime 3. The Picard groups of \mathcal{K}_1 and \mathcal{L}_1 were investigated in [13]. A feature of that paper is the observation that $Pic(\mathcal{L}_1)$ at $p = 2$ had an exotic element not immediately detected by algebraic methods; we will investigate similar phenomena.

There are two steps to the calculation of the Picard group of these localized stable homotopy categories. The first is algebraic. If E_* is a homology theory with E_*E flat over E_* , we denote by $Pic(E)_{\text{alg}}$ the Picard group of invertible E_*E -comodules. For \mathcal{L}_2 and \mathcal{K}_2 this

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group is known and recorded in Section 2.2 below. In both of these cases, there is then a homomorphism from the Picard group of the E -local stable homotopy category

$$\mathrm{Pic}(\mathcal{L}_E) \longrightarrow \mathrm{Pic}(E)_{\mathrm{alg}}$$

sending X to E_*X . In our examples this will be onto and the interest then turns to the kernel. This is the group of *exotic* elements: those E -local invertible spectra so that $E_*X \cong E_*$ as E_*E comodules. For \mathcal{K}_2 and \mathcal{L}_2 for $p > 3$, there are no non-trivial exotic elements. Write κ_2 and $\kappa(\mathcal{L}_2)$ for the group of exotic elements in \mathcal{K}_2 and \mathcal{L}_2 respectively at $p = 3$. Kamiya and Shimomura [17] have shown that $\kappa(\mathcal{L}_2)$ is either of order 3 or order 9; in particular, it is known to be non-trivial. Our main result shows that it is of order 9.

Theorem 1.1. *Let κ_2 be the group of exotic elements in the $K(2)$ -local Picard group $\mathrm{Pic}(\mathcal{K}_2)$ at $p = 3$. Then there is an isomorphism*

$$\kappa_2 \cong \mathbb{Z}/3 \times \mathbb{Z}/3.$$

Furthermore the localization map

$$\kappa(\mathcal{L}_2) \longrightarrow \kappa_2$$

is an isomorphism.

The first part of this result is proved in section 5; the second in section 6. Along the way we will take the opportunity to revisit some of the standard results of $K(2)$ -local homotopy theory; for example, in section 4, we will show that the $K(2)$ -local Adams-Novikov Spectral Sequence for any exotic element of the Picard group has a global vanishing line at the E_∞ -page, although it has no such line at E_2 .

Combining the algebraic calculation with Theorem 1.1 now completes the calculation for $p = 3$. Let \mathbb{Z}_p denote the p -adic integers.

Theorem 1.2. *There are isomorphisms*

$$\mathrm{Pic}(\mathcal{K}_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}/16 \times \mathbb{Z}/3 \times \mathbb{Z}/3.$$

and

$$\mathrm{Pic}(\mathcal{L}_2) \cong \mathbb{Z} \times \mathbb{Z}/3 \times \mathbb{Z}/3.$$

For more precise statements and details see Theorem 2.11 and Proposition 6.2.

It is worth emphasizing that the two summands of κ_2 do not have equal dignity. At $p = 2$ and $n = 1$, the exotic element of κ_1 can be detected by real K -theory; that is, there is an element of the Picard group so that $K_*X \cong K_*S^0$, but $KO_*X \not\cong KO_*S^0$. In the same way, at $p = 3$ and $n = 2$, there are elements of the Picard group so that $(E_2)_*X \cong (E_2)_*S^0$, but for which $(E_2^{hG})_*X \not\cong (E_2^{hG})_*S^0$ for some finite group G acting on E_2 . However, there are also non-trivial elements of κ_2 so that

$$(E_2^{hG})_*X \cong (E_2^{hG})_*S^0$$

for all finite groups acting on E_2 . Detecting and constructing these truly exotic elements is the novel part of our argument.

2. BACKGROUND

In this section we work on the basic technology and known results behind our work. Recall we have a fixed prime p , which will eventually be 3.

2.1. Morava K -theory and related homology theories. We will work with p -local 2-periodic complex oriented cohomology theories E equipped with a p -typical formal group law F . Thus $E_* \cong E_0[u^{\pm 1}]$ where u is a unit in degree -2 and the associated formal group law F is determined by the p -series

$$[p]_F(x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots$$

where v_i is in degree $2(p^i - 1)$.

Basic examples of such theories are the Morava K -theories $K(n)_*(-)$ with

$$K(n)_* = \mathbb{F}_p[u^{\pm 1}]$$

where u is in degree 2; the formal group law is the Honda formal group law Γ_n of height n with p -series $[p]_{\Gamma_n}(x) = u^{-(p^n-1)}x^{p^n}$.

There is an associated Landweber exact homology theory $(E_n)_*$ – the Morava E -theory, also known as the Lubin-Tate theory. It is customary to choose the p -typical formal group law over $(E_n)_*$ with

$$(2.1) \quad v_i = \begin{cases} u^{-(p^i-1)}u_i, & 1 \leq i < n; \\ u^{-(p^n-1)}, & i = n; \\ 0, & i > n. \end{cases}$$

This equation defines elements u_i in $(E_n)_0$ and an isomorphism

$$W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] \cong (E_n)_0$$

where $W(-)$ is the Witt vector functor. The associated formal group law is the universal deformation of the Honda formal group law over $\mathbb{F}_{p^n} \otimes K(n)_*$.

Define the (big) *Morava stabilizer group* \mathbb{G}_n to be the automorphisms of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$. Since Γ_n is defined over \mathbb{F}_p , we have

$$\mathbb{G}_n = \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

where \mathbb{S}_n is the automorphisms of Γ_n over \mathbb{F}_{p^n} .

The group \mathbb{G}_n acts on the deformations of $(\mathbb{F}_{p^n}, \mathbb{G}_n)$ and hence on $(E_n)_*$. By the Hopkins-Miller Theorem [2] this action can be lifted uniquely to E_∞ -ring spectra; that is, the group \mathbb{G}_n acts on the spectrum E_n through E_∞ -ring maps. In [1], Devinatz and Hopkins showed that for all closed subgroups $K \subseteq \mathbb{G}_n$ there is a homotopy fixed point spectrum E_n^{hK} and a spectral sequence

$$(2.2) \quad H^s(K, (E_n)_t) \implies \pi_{t-s} E_n^{hK}.$$

Here the group cohomology is continuous cohomology. If K is finite, this is the usual fixed point spectral sequence. By contrast, if $K = \mathbb{G}_n$ itself, then $E_n^{hK} \simeq L_{K(n)} S^0$ and this is the Adams-Novikov Spectral Sequence. Compare (2.5) below.

The 2-periodic Johnson-Wilson theories $E(n)$ have

$$E(n)_0 \cong \mathbb{Z}_{(p)}[u_1, \dots, u_{n-1}, u_n^{\pm 1}]$$

and p -typical formal group law defined by the Equation (2.1). There is a morphism of homotopy commutative ring spectra $E(n) \rightarrow E_n$ which, on coefficients, is the map indicated by the notation. The associated homology theories have the same acyclics and induce the same localization functor, which we will call L_n , but note that the notation $(E_n)_*(-)$ does **not** mean the homology theory associated to E_n .

Remark 2.1 (Local smash products). If X and Y are E -local, then $X \wedge Y$ need not be E -local; hence the natural smash product for E -local spectra is

$$X \wedge_E Y = L_E(X \wedge Y).$$

The notation \wedge_E is very cumbersome, however, so we will drop it when it is understood. In particular, throughout most of this paper, X and Y are $K(n)$ -local spectra, and $X \wedge Y$ means $X \wedge_{K(n)} Y$.

From this point of view, the natural definition of $(E_n)_*(-)$ is

$$(E_n)_*X \stackrel{\text{def}}{=} \pi_*L_{K(n)}(E_n \wedge X).$$

Under appropriate circumstances (see [15]) $(E_n)_*X$ is a completion of $\pi_*(E_n \wedge X)$; however, $(E_n)_*(-)$ is not a homology theory because it does not take wedges to sums. Since \mathbb{G}_n acts on E_n , \mathbb{G}_n acts on $(E_n)_*X$. The $(E_n)_*$ -module $(E_n)_*X$ is equipped with the \mathfrak{m} -adic topology where \mathfrak{m} is the maximal ideal in $(E_n)_0$. This topology is always topologically complete, but need not be separated. With respect to this topology, the group \mathbb{G}_n acts through continuous maps and the action is twisted because it is compatible with the action of \mathbb{G}_n on the coefficient ring E_* . See [5] §2 for some precise assumptions which guarantee that $(E_n)_*X$ is complete and separated. We will call these modules twisted \mathbb{G}_n -modules.

Let $\text{map}(\mathbb{G}_n, (E_n)_*)$ be the $(E_n)_*$ -module of continuous maps out of \mathbb{G}_n . We give this the diagonal \mathbb{G}_n -action; that is,

$$(2.3) \quad (g\phi)(x) = g\phi(g^{-1}x).$$

Then (by [23] among other sources) there is an isomorphism of twisted \mathbb{G}_n -modules

$$(2.4) \quad (E_n)_*E_n \cong \text{map}(\mathbb{G}_n, (E_n)_*).$$

and we have an Adams-Novikov Spectral Sequence

$$(2.5) \quad H^s(\mathbb{G}_n, (E_n)_*X) \implies \pi_{t-s}L_{K(n)}X.$$

The E_2 -term is continuous group homology.

Remark 2.2. We give some more detail about the Morava stabilizer group \mathbb{G}_n . Write \mathbb{W} for the Witt vectors $W(\mathbb{F}_{p^n})$ and let σ be the lift of Frobenius ϕ on \mathbb{F}_{p^n} to \mathbb{W} . Define $\mathbb{W}\langle S \rangle$ to be the non-commutative polynomial ring over \mathbb{W} with $wS = S\sigma(w)$ for $w \in \mathbb{W}$. Then define

$$\mathcal{O}_n = \mathbb{W}\langle S \rangle / (S^n = p).$$

There is an isomorphism $\mathcal{O}_n \cong \text{End}(\Gamma_n)$. From this it follows that $\mathbb{S}_n \cong \mathcal{O}_n^\times$.

Note the canonical right action of \mathbb{S}_n on \mathcal{O}_n determines a homomorphism

$$\mathbb{S}_n \longrightarrow \text{Aut}(\mathcal{O}_n) \cong \text{GL}_n(\mathbb{W}).$$

It is easy to check that the image of the composition

$$\mathbb{S}_n \longrightarrow \text{GL}_n(\mathbb{W}) \xrightarrow{\det} \mathbb{W}^\times$$

is exactly $\mathbb{Z}_p^\times \subseteq \mathbb{W}^\times$, so that we get an induced homomorphism

$$\mathbb{G}_n \longrightarrow \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \times \mathbb{Z}_p^\times.$$

The element $1+p \in \mathbb{Z}_p^\times$ defines an isomorphism $\mathbb{Z}_p \cong \mathbb{Z}_p^\times/\mu$, where μ is the maximal finite subgroup. Using this identification we get the *reduced norm*.

$$(2.6) \quad N : \mathbb{G}_n \longrightarrow \mathbb{Z}_p.$$

Define \mathbb{G}_n^1 to be the kernel of this homomorphism.

The inclusion $a \mapsto a + 0S \in \mathbb{S}_n$ defines an injection $\mathbb{Z}_p^\times \rightarrow \mathbb{G}_n$ onto the center of \mathbb{G}_n . We use $1+p$ to identify \mathbb{Z}_p with the subgroup of \mathbb{Z}_p^\times of elements congruent to 1 modulo p . Then the composition

$$\mathbb{Z}_p \rightarrow \mathbb{G}_n \rightarrow \mathbb{Z}_p$$

of the inclusion followed by the reduced norm of Equation (2.6) is multiplication by n . Hence if n is prime to p , we have a splitting isomorphism

$$(2.7) \quad \mathbb{G}_n^1 \times \mathbb{Z}_p \xrightarrow{\cong} \mathbb{G}_n.$$

Remark 2.3. The action of the center on $(E_n)_*$ is quite simple. The inclusion $\mathbb{Z}_p \rightarrow \text{End}(\mathbb{F}_{p^n}, \Gamma_n)$ is the completion of the morphism $\mathbb{Z} \rightarrow \text{End}(\mathbb{F}_{p^n}, \Gamma_n)$ sending q to the group homomorphism $q : \Gamma_n \rightarrow \Gamma_n$. This has a canonical lift to an action of the universal deformation and, hence, $q \in \mathbb{Z}_p^\times$ acts trivially on $(E_n)_0$ and by

$$q_* u = qu$$

in degree -2 .

2.2. Picard groups. The completed tensor product over $(E_n)_*$ endows the category of twisted \mathbb{G}_n -modules with a symmetric monoidal structure whose unit is $(E_n)_* S^0$. We begin our analysis of the Picard group Pic_n of \mathcal{K}_n with the following result from [13].

Theorem 2.4. *Let X be a $K(n)$ -local spectrum. Then the following conditions are equivalent:*

- a) X is invertible in \mathcal{K}_n .
- b) $(E_n)_* X$ is a free $(E_n)_*$ -module of rank 1.
- c) $(E_n)_* X$ is invertible with respect to the tensor product in the category of twisted \mathbb{G}_n -modules.

Write $(\text{Pic}_n)_{\text{alg}}$ for the Picard group of the category of twisted \mathbb{G}_n -modules. Then Theorem 2.4 implies that the assignment $X \mapsto (E_n)_* X$ defines a group homomorphism

$$\varepsilon_n : \text{Pic}_n \rightarrow (\text{Pic}_n)_{\text{alg}}.$$

The group $(\text{Pic}_n)_{\text{alg}}$ contains the index 2-subgroup $(\text{Pic}_n)_{\text{alg}}^0$ of invertible twisted \mathbb{G}_n -modules which are concentrated in even degrees. Likewise Pic_n contains a subgroup Pic_n^0 of index 2 and ε restricts to a homomorphism

$$\varepsilon_n^0 : \text{Pic}_n^0 \rightarrow (\text{Pic}_n)_{\text{alg}}^0$$

If $M_* \in (\text{Pic}_n)_{\text{alg}}^0$, then M_0 is an invertible $(E_n)_0$ -module with an action of \mathbb{G}_n . Standard considerations now imply the following result.

Proposition 2.5. *There is a canonical isomorphism*

$$(\text{Pic}_n)_{\text{alg}}^0 \cong H^1(\mathbb{G}_n, (E_n)_0^\times).$$

Remark 2.6. There are two basic elements of $H^1(\mathbb{G}_n, (E_n)_0^\times)$ which play a particularly important role in all calculations; for example, in case $n = 2$ they will be topological generators of $H^1(\mathbb{G}_n, (E_n)_0^\times)$. Recall that 1-cocycles in group cohomology can be represented by crossed homomorphisms; that is, functions

$$\phi : \mathbb{G}_n \longrightarrow (E_n)_0^\times$$

so that $\phi(gh) = [g\phi(h)]\phi(h)$. This formula is multiplicative because the group operation on $(E_n)_0^\times$ is multiplication.

1.) The image of the the $K(n)$ -local 2 sphere $L_{K(n)}S^2$ under the homomorphism ε_n determines a crossed homomorphism

$$\eta : \mathbb{G}_n \longrightarrow (E_n)_0^\times$$

given by the formula

$$g_*(u) = \eta(g)u$$

where $u \in (E_n)_{-2} = (E_n)_0 S^2$ is the canonical generator.

2.) The second element, which we denote as $(E_n)_0\langle \det \rangle$, is the (genuine) homomorphism given as the composition of the determinant map and the canonical inclusion

$$\mathbb{G}_n \xrightarrow{\det} \mathbb{Z}_p^\times \xrightarrow{\subseteq} (E_n)_0^\times .$$

Remark 2.7. The element $(E_n)_0\langle \det \rangle$ of $(\text{Pic}_n)_0^{\text{alg}}$ has a canonical topological realization, which we now define. We assume $p > 2$.

Let SG_n be the kernel of the determinant $\det : \mathbb{G}_n \rightarrow \mathbb{Z}_p^\times$ and let $\mu \subseteq \mathbb{Z}_p^\times$ be the subgroup of $(p-1)$ st roots of unity. Then μ acts on $(E_n)^{h\text{SG}_n}$. Let $(E_n)_\chi^{h\text{SG}_n}$ be the wedge summand of $(E_n)^{h\text{SG}_n}$ defined by the character $\chi : \mu \rightarrow \mathbb{Z}_p^\times$ with $\chi(g) = g^{-1}$. Let $\psi^{p+1} = (p+1) + 0S \in \mathbb{G}_n$ be our chosen generator of the subgroup of the center given by elements congruent to 1 modulo p ; see Remark 2.2. Then ψ^{p+1} maps to a topological generator of \mathbb{G}_n/SG_n . Define $S^0\langle \det \rangle$ by the fiber sequence using the difference

$$(2.8) \quad S^0\langle \det \rangle \longrightarrow (E_n)_\chi^{h\text{SG}_n} \xrightarrow{\psi^{p+1} - \det(\psi^{p+1})} (E_n)_\chi^{h\text{SG}_n} .$$

Then we assert that $(E_n)_0 S^0\langle \det \rangle = (E_n)_0\langle \det \rangle$ as twisted \mathbb{G}_n -module.

From [5] §2, we have an isomorphism of Morava modules

$$(2.9) \quad (E_n)_0 E_n^{h\text{SG}_n} \cong \text{map}(\mathbb{G}_n/\text{SG}_n, (E_n)_0) \cong \text{map}(\mathbb{Z}_p^\times, (E_n)_0) .$$

where the twisted \mathbb{G}_n -action on the set of continuous maps is given by $(g\phi)(x) = g\phi(g^{-1}x)$. It follows that

$$(E_n)_0 (E_n)_\chi^{h\text{SG}_n} \cong \text{map}(\mathbb{Z}_p, (E_n)_0 \otimes \chi) .$$

The calculation of $(E_n)_0 S^0\langle \det \rangle$ now follows from the long exact sequence in $(E_n)_*$ of the fibration (2.8).

Definition 2.8. Define the group κ_n of **exotic elements** of Pic_n to be the kernel of

$$\varepsilon_n^0 : \text{Pic}_n^0 \rightarrow (\text{Pic}_n)_0^{\text{alg}} .$$

The group κ_n measures the difference between the homotopy theoretic and algebraic definition of the Picard group in the $K(n)$ -local category. The first result is negative: the following Proposition says that $\kappa_n = 0$ if n is large with respect to p . See [22].

Proposition 2.9. *Suppose n is not divisible by $p-1$ and $n^2 \leq 2p-2$. Then*

$$\kappa_n = 0 .$$

Remark 2.10. The case $n=1$ was studied for all primes in [13]. Here E_1 can be chosen to be the p -complete K -theory and the group $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$ acts through Adams operations. This group acts trivially on $(E_1)_0 = K_0 = \mathbb{Z}_p$. If $p > 2$, then

$$(\text{Pic}_1)_0^{\text{alg}} \cong H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1) \cong \mathbb{Z}_p^\times .$$

and $\text{Pic}_1 \cong (\text{Pic}_1)_{\text{alg}} \cong \mathbb{Z}_p \times \mathbb{Z}/2(p-1)$. Explicit topological generators and other elements are given in [13].

If $p=2$, then $\mathbb{G}_1 = \mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times \mathbb{Z}/2$ contains 2-torsion. As a result,

$$(\text{Pic}_1)_0^{\text{alg}} \cong H^1(\mathbb{Z}_2^\times, \mathbb{Z}_2^\times) \cong \mathbb{Z}_2^\times \times \mathbb{Z}/2 .$$

The evaluation map

$$(2.10) \quad \text{Pic}_1^0 \rightarrow (\text{Pic}_1)_0^{\text{alg}} \cong \mathbb{Z}_2^\times \times \mathbb{Z}/2$$

sends X to (a_3, a_{-1}) where $\psi^3(x) = a_3x$ and $\psi^{-1}(x) = a_{-1}x$ for some generator $x \in K_0X$. For example, S^2 maps to $(3, -1)$. Notice that $(\text{Pic}_1)_0^{\text{alg}}$ has two generators of order 2, with invariants $(1, -1)$ and $(-1, 1)$ respectively. We have

$$(\text{Pic}_1)_{\text{alg}} \cong \mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$$

generated by these elements and K_*S^1 .

The map $\text{Pic}_1 \rightarrow (\text{Pic}_1)_{\text{alg}}$ is a split surjection, but now $\kappa_1 \cong \mathbb{Z}/2$. An explicit generator for κ_1 can be given by $L_{K(1)}DQ$ where DQ is the dual of the “question mark complex”. If we extend $\eta : S^1 \rightarrow S^0$ to a map from the Moore spectrum

$$\bar{\eta} : \Sigma M_2 \rightarrow S^0$$

then DQ is the cone of $\bar{\eta}$. A classical calculation shows $K_*DQ \cong K_*S^0$ as graded modules over the Adams operations; however, $KO \wedge DQ \simeq \Sigma^4 KO$.

We now begin to consider the case $n = 2$ and $p > 2$. The case $p = 2$ is considerably harder and not discussed here at all. The following algebraic results are due to Hopkins [10] if $p > 3$ and to Karamanov if $p = 3$ [18, 19].

Theorem 2.11. *Let $n = 2$ and $p > 2$. Then*

- (1) $(\text{Pic}_n)_{\text{alg}}^0 \cong H^1(\mathbb{G}_n, (E_n)_0^\times) \cong \mathbb{Z}_p^2 \times \mathbb{Z}/(p^2 - 1)$ topologically generated by $(E_2)_0 S^2$ and $(E_2)_0 \langle \det \rangle$;
- (2) $(\text{Pic}_n)_{\text{alg}} \cong \mathbb{Z}_p^2 \times \mathbb{Z}/2(p^2 - 1)$;
- (3) Both ε_n and ε_n^0 are surjective, and if $p > 3$ they are both isomorphisms.

The difficult part is the group cohomology calculation. Part (2) follows from the observation, also in [13], that the extension

$$0 \rightarrow (\text{Pic}_n)_{\text{alg}}^0 \rightarrow (\text{Pic}_n)_{\text{alg}} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

cannot be split. Part (3) follows from Proposition 2.9 and by noting, as in [19], that the elements of Remark 2.6 generate the cohomology group.

2.3. Resolutions of \mathbb{G}_2 -modules. The group \mathbb{G}_n is a p -adic analytic group in the sense of Lazard [20] and such groups are of finite mod p cohomological dimension unless they contain elements of order p . We saw above in Remark 2.10 that the element of order 2 in \mathbb{G}_1 at 2 created extra elements in Pic_1 . One reason that this paper is interesting is that if $p = 3$, then \mathbb{G}_2 contains elements of order 3.

From now on we will fix $p = 3$ and work at $n = 2$.

An explicit element of order 3 in \mathbb{G}_2 is given by

$$a = -\frac{1}{2}(1 + \omega S)$$

where ω is a fixed chosen primitive 8-th root of unity in $\mathbb{W} = W(\mathbb{F}_p)$. This element defines an inclusion $C_3 \rightarrow \mathbb{G}_2$ of the cyclic group of order 3 and, by [8] Theorem 1.9, the induced map

$$H^*(\mathbb{G}_2, \mathbb{F}_3) \longrightarrow H^*(C_3, \mathbb{F}_3)$$

is surjective. Thus \mathbb{G}_2 (at $p = 3$) does not have finite cohomological dimension. The further study of the cohomology of \mathbb{G}_2 (see Theorem 2.13 below) uses two subgroups. Write $\langle - \rangle$ for the subgroup generated by a list of elements.

Definition 2.12. Define two finite subgroups of \mathbb{G}_2 as follows. Let $\phi \in \text{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ be the Frobenius.

- (1) $G_{24} = \langle a, \omega^2, \omega\phi \rangle$. Note that ω^2 acts non-trivially on $C_3 = \langle a \rangle$ and $\omega\phi$ acts trivially on $C_3 = \langle a \rangle$; hence, $G_{24} \cong C_3 \rtimes Q_8$ where Q_8 is the quaternion group of order 8.
- (2) $SD_{16} = \langle \omega, \phi \rangle$. This group is isomorphic to the semidihedral group of order 16.

Recall the splitting

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_3$$

of Equation 2.7. Here \mathbb{G}_2^1 is the kernel of the reduced norm. The finite subgroups of \mathbb{G}_2 are automatically finite subgroups of \mathbb{G}_2^1 .

We now give a resolution of the trivial \mathbb{G}_2 -module \mathbb{Z}_3 . If $X = \lim_{\alpha} X_{\alpha}$ is a profinite set, let

$$\mathbb{Z}_p[[X]] = \lim_{i,\alpha} \mathbb{Z}/p^i[X_{\alpha}].$$

Let χ be the character of SD_{16} with $\chi(\omega) = \chi(\phi) = -1$.

Theorem 2.13. [5] *There is an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$ -modules of the following form*

$$0 \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z}_3$$

with $C_0 = C_3 \cong \mathbb{Z}_3[[\mathbb{G}_2^1/G_{24}]]$ and $C_1 = C_2 \cong \mathbb{Z}_3[[\mathbb{G}_2^1]] \otimes_{\mathbb{Z}_3[SD_{16}]} \mathbb{Z}_3(\chi)$.

Remark 2.14. We can use this resolution to obtain a resolution of the trivial \mathbb{G}_2 -module \mathbb{Z}_3 . If C_s is the \mathbb{G}_2^1 -module of Theorem 2.13, define D_s to be the \mathbb{G}_2 -module obtained from C_s by inducing up from \mathbb{G}_2^1 . Thus, for example,

$$D_0 \cong \mathbb{Z}_3[[\mathbb{G}_2/G_{24}]].$$

Using the isomorphism $\mathbb{G}_2^1 \times \mathbb{Z}_3 \cong \mathbb{G}_2$ of (2.7), D_s can be obtained by taking the completed tensor product of C_s with $\mathbb{Z}_3[[\mathbb{Z}_3]]$. There is a larger resolutions by \mathbb{G}_2 -modules

$$(2.11) \quad 0 \rightarrow D_3 \rightarrow D_3 \oplus D_2 \rightarrow D_2 \oplus D_1 \rightarrow D_1 \oplus D_0 \rightarrow D_0 \rightarrow \mathbb{Z}_3$$

To see this, let $\psi^{p+1} = \psi^4$ be the chosen topological generator for \mathbb{Z}_3 . Then we get a very short resolution

$$(2.12) \quad 0 \longrightarrow \mathbb{Z}_3[[\mathbb{Z}_3]] \xrightarrow{\psi^4 - 1} \mathbb{Z}_3[[\mathbb{Z}_3]] \longrightarrow \mathbb{Z}_3 \longrightarrow 0.$$

If we write P_{\bullet} for the complex

$$P_{\bullet} \stackrel{\text{def}}{=} \{ \mathbb{Z}_3[[\mathbb{Z}_3]] \xrightarrow{\psi^4 - 1} \mathbb{Z}_3[[\mathbb{Z}_3]] \}$$

then the resolution D_{\bullet} is the completion of the tensor product $C_{\bullet} \otimes P_{\bullet}$, where C_{\bullet} is the resolution of Theorem 2.13.

We recall that a continuous $\mathbb{Z}_3[[\mathbb{G}_2]]$ -module M is *profinite* if there is an isomorphism $M \cong \lim_{\alpha} M_{\alpha}$ where each M_{α} is a finite $\mathbb{Z}_3[[\mathbb{G}_2]]$ module.

Corollary 2.15. *Let M be a profinite $\mathbb{Z}_3[[\mathbb{G}_2^1]]$ -module. Then there is a first quadrant cohomology spectral sequence*

$$E_1^{s,t}(M) \cong \text{Ext}_{\mathbb{Z}_3[[\mathbb{G}_2^1]]}^s(C_t, M) \implies H^{s+t}(\mathbb{G}_2^1, M)$$

with

$$E_1^{s,0}(M) = E_1^{s,3}(M) \cong H^s(G_{24}, M)$$

and

$$E_1^{s,1}(M) = E_1^{s,2}(M) \cong \begin{cases} \text{Hom}_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3(\chi), M) & s = 0 \\ 0 & s > 0. \end{cases}$$

Remark 2.16. If M is a profinite $\mathbb{Z}_3[[\mathbb{G}_2]]$ -module, there is a similar spectral sequence for computing $H^*(\mathbb{G}_2, M)$ using the second resolution (2.11), although the E_1 term is slightly more complicated. We do record that for $s > 0$, $E_1^{s,t} = 0$ if $t = 2$, and if $t = 0, 1, 3$ or 4 , then

$$E_1^{s,t}(M) = H^s(G_{24}, M).$$

Furthermore, using Remark 2.14, we have

$$d_1 = \psi^4 - 1 : E_1^{s,t}(M) \longrightarrow E_1^{s,t+1}(M)$$

for $s > 0$ and $t = 0, 3$.

Remark 2.17. We have considerable input for these spectral sequences. For example, for the spectral sequence of Corollary 2.15 the terms $E_1^{*,1}$ and $E_1^{*,2}$ contribute only to $H^1(\mathbb{G}_2^1, M)$ and $H^2(\mathbb{G}_2^1, M)$. These terms also can be rewritten if $M = (E_2)_n X$ for some spectrum X . The representation $\mathbb{Z}_3(\chi)$ is self-dual and $\mathbb{Z}_3(\chi) \otimes (E_2)_0 \cong (E_2)_0 S^8 \cong (E_2)_{-8}$; hence,

$$\mathrm{Hom}_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3(\chi), (E_2)_* X) \cong H^0(SD_{16}, (E_2)_* \Sigma^8 X).$$

Since SD_{16} is a 2-group, there are no higher cohomology groups. In particular, if $X = S^0$, we have

$$H^0(SD_{16}, (E_2)_*) \cong \pi_* \Sigma^8 E_2^{hSD_{16}}$$

and we know from [5] §3 that there is an isomorphism

$$\mathbb{Z}_3[[y]][v_1, v_2^{\pm 1}]/(v_2 y = v_1^4) \cong \pi_* \Sigma^8 E_2^{hSD_{16}}.$$

For the terms where the group G_{24} appears, we will use Theorem 2.18 below. There are invariant elements c_4, c_6 and Δ in $H^0(G_{24}, (E_2)_*)$ of internal degrees 8, 12 and 24 respectively. The element Δ is invertible and there is a relation ¹

$$c_4^3 - c_6^2 = (12)^3 \Delta.$$

Define $j = c_4^3/\Delta$ and let M_* be the graded ring

$$M_* = \mathbb{Z}_3[[j]][c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = (12)^3 \Delta, \Delta j = c_4^3).$$

Furthermore, there are also elements $\alpha \in H^1(G_{24}, (E_2)_4)$ and $\beta \in H^2(\mathbb{G}_{24}, (E_2)_{12})$ and there are relations

$$(2.13) \quad \begin{aligned} 3\alpha &= 3\beta = \alpha^2 &= 0 \\ c_4\alpha &= c_4\beta &= 0 \\ c_6\alpha &= c_6\beta &= 0 \end{aligned}$$

The fixed point spectral sequence is also computed in §3 in [5]; the results will be useful in section 4 and 5.

Theorem 2.18. *Let $R \subseteq M_*[\alpha, \beta]$ be the ideal generated by the relations of (2.13). Then the induced map*

$$M_*[\alpha, \beta]/R \longrightarrow H^*(G_{24}, (E_2)_*)$$

is an isomorphism.

Theorem 2.19. 1.) *In the fixed-point spectral sequence*

$$H^s(G_{24}, (E_2)_t) \Longrightarrow \pi_{t-s} E_2^{hG_{24}}$$

all differentials are determined by

$$d_5(\Delta) = \pm \alpha \beta^2$$

and

$$d_9(\alpha \Delta^2) = \pm \beta^5.$$

2.) *The class Δ^3 is a permanent cycle and extends to an equivalence*

$$\Sigma^{72} E_2^{hG_{24}} \simeq E_2^{hG_{24}}.$$

3.) *The kernel of the Hurewicz map*

$$\pi_n E_2^{hG_{24}} \rightarrow (E_2)_n E_2^{hG_{24}}, \quad 0 \leq n \leq 72$$

¹In [5] we wrote this relation as $c_4^3 - c_6^2 = 3^3 \Delta$. However, we can replace c_4 and c_6 by $c_4/2^2$ and $c_6/2^3$ respectively to get the indicated relation, which has the aesthetic value of coinciding with the standard relation among modular forms. The connection can be made using the formal group of a supersingular elliptic curve. See [7].

is a $\mathbb{Z}/3$ module generated by the classes

$$\beta^i, 1 \leq i \leq 4; \quad \alpha\beta^i, i = 0, 1$$

and classes x and βx where $x \in \pi_{27}E_2^{hG_{24}}$ is the Toda bracket $\langle \alpha, \alpha, \beta^2 \rangle$ detected by $\pm\alpha\Delta$. Furthermore $\alpha x = \pm\beta^3$.

3. EXOTIC Pic AT $p = 3$

The main theorem of the paper is that κ_2 is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. We will state a refined version of this result below in Theorem 3.5 once we have assembled the necessary preliminaries. This refined version will be proved in the section 5.

Recall that we have an Adams-Novikov Spectral Sequence

$$E_2^{s,t} = H^s(\mathbb{G}_2, (E_2)_t) \Rightarrow \pi_{t-s}(L_{K(2)}S^0).$$

A key algebraic result is the following.

Proposition 3.1. *If t is not divisible by $4 = 2(p-1)$, then*

$$H^s(\mathbb{G}_2, (E_2)_t) = 0.$$

Furthermore, there is a splittable short exact sequence

$$(3.1) \quad 0 \rightarrow H^1(G_{24}, (E_2)_4) \rightarrow H^5(\mathbb{G}_2, (E_2)_4) \rightarrow H^5(G_{24}, (E_2)_4) \rightarrow 0.$$

A choice of splitting yields an isomorphism

$$H^5(\mathbb{G}_2, (E_2)_4) \cong \mathbb{Z}/3 \times \mathbb{Z}/3.$$

Proof. The first statement is simply the standard sparseness result for the Adams-Novikov Spectral Sequence. For a proof in this context, see [4], Proposition 4.1.

For the second statement we calculate using the spectral sequence of Remark 2.16, which is a variant of the spectral sequence of Corollary 2.15. From this spectral sequence and Theorem 2.18 we see that there is a short exact sequence

$$0 \rightarrow E_2^{1,4} \rightarrow H^5(\mathbb{G}_2, (E_2)_4) \rightarrow E_2^{5,0} \rightarrow 0$$

where $E_2^{1,4}$ is cokernel of

$$d_1 : E_1^{1,3} = H^1(G_{24}, (E_2)_4) \rightarrow H^1(G_{24}, (E_2)_4) = E_1^{1,4}$$

and $E_2^{5,0}$ is the kernel of

$$d_1 : E_1^{5,0} = H^5(G_{24}, (E_2)_4) \rightarrow H^5(G_{24}, (E_2)_4) = E_1^{5,1}.$$

This differential is completely determined by the action of the center of \mathbb{G}_2 ; indeed, in both cases

$$d_1 = (\psi^4 - 1)_*$$

where ψ^4 is a generator for the central \mathbb{Z}_3 . It then follows from Remark 2.3 that $d_1 = 0$.

We now use Theorem 2.18 to note that

$$H^5(G_{24}, (E_2)_4) \cong \mathbb{Z}/3$$

generated by $\alpha\beta^2\Delta^{-1}$ and

$$H^1(G_{24}, (E_2)_4) \cong \mathbb{Z}/3$$

generated by α .

It remains to show that the exact sequence (3.1) is split. For this we compare it with the spectral sequence for the module $M = (E_2/(3, u_1))_4 = (\mathbb{F}_9[u^{\pm 1}])_4$. If the sequence of (3.1) does not split, then the class α would map to zero under the induced map

$$H^5(\mathbb{G}_2, (E_2)_4) \rightarrow H^5(\mathbb{G}_2, M).$$

We will show that, in fact, α does not map to zero. The short exact sequence of (3.1) maps to the analogous short exact sequence

$$(3.2) \quad 0 \rightarrow H^1(G_{24}, M) \rightarrow H^5(\mathbb{G}_2, M) \rightarrow H^5(G_{24}, M) \rightarrow 0 .$$

Thus it is sufficient to argue that the reduction induces an isomorphism

$$H^1(G_{24}, (E_2)_4) \cong H^1(G_{24}, M)$$

and this follows immediately from Theorem 2.18. \square

Remark 3.2. We can choose one generator of $H^5(\mathbb{G}_2, (E_2)_4)$ which restricts to the generator $\alpha\beta^2\Delta^{-1}$ of $H^5(G_{24}, (E_2)_4)$. Despite the fact that it is not unique, we still call this generator $\alpha\beta^2\Delta^{-1}$. The other generates the kernel of

$$H^5(\mathbb{G}_2, (E_2)_4) \rightarrow H^5(\mathbb{G}_2^1, (E_2)_4).$$

In the notation of [3], the image of this element in $H^5(\mathbb{G}_2, (E_2)/(p, u_1)_4)$ is $\zeta\alpha\beta\alpha_{35}\Delta^{-2}$. We won't use this notation later in this paper.

Construction 3.3. We next construct a homomorphism

$$(3.3) \quad \tau : \kappa_2 \rightarrow H^5(\mathbb{G}_2, (E_2)_4)$$

from the group of exotic elements in Pic_2 . The ideas here can be found in [14] and [17].

Let $Z \in \kappa_2$ and consider the Adams Novikov Spectral Sequence

$$H^s(\mathbb{G}_2, (E_2)_t Z) \cong H^2(\mathbb{G}_2, (E_2)_t Z) \implies \pi_{t-s} Z .$$

A choice of isomorphism of twisted \mathbb{G}_2 -modules

$$f : (E_2)_* \xrightarrow{\cong} (E_2)_* Z$$

defines a commutative diagram

$$\begin{array}{ccc} H^0(\mathbb{G}_2, (E_2)_0) & \xrightarrow{\phi} & H^5(\mathbb{G}_2, (E_2)_4) \\ f_* \downarrow \cong & & \cong \downarrow f_* \\ H^0(\mathbb{G}_2, (E_2)_0 Z) & \xrightarrow{d_5} & H^5(\mathbb{G}_2, (E_2)_4 Z) . \end{array}$$

We now set

$$\tau(Z) = \phi(\iota) = f_*^{-1} d_5 f_*(\iota)$$

where $\iota \in H^0(\mathbb{G}_2, (E_2)_0) \cong \mathbb{Z}_3$ is the unit. To check that this definition is independent of the chosen isomorphism f , note that if g is any other isomorphism, there is a unit $a \in \mathbb{Z}_3^\times$ so that $g = af$. Finally, to check that τ is a homomorphism, we use the Künneth isomorphism

$$(E_2)_* Z_1 \otimes_{(E_2)_*} (E_2)_* Z_2 \cong (E_2)_*(Z_1 \wedge Z_2)$$

and the multiplicative structure of the Adams Novikov Spectral Sequence, which gives a Leibniz rule for differentials.

Remark 3.4. In the case of Pic_1 and $p = 2$, there is an analogous homomorphism

$$\tau : \kappa_1 \rightarrow H^3(\mathbb{Z}_2^\times, (E_1)_2) \cong \mathbb{Z}/2$$

which is an isomorphism. Compare Remark 2.10.

Our main technical theorem is the following result, proved in the next section.

Theorem 3.5. *The homomorphism*

$$\tau : \kappa_2 \rightarrow H^5(\mathbb{G}_2, (E_2)_4)$$

is an isomorphism.

The following result will be needed later when calculating with our topological resolutions. Recall from Construction 3.3 that a choice of isomorphism $f : (E_2)_* \cong (E_2)_* Z$ defines a unit element

$$\iota_Z = f_*(\iota) \in H^0(\mathbb{G}_2, E_0 Z) \cong \mathbb{Z}_3$$

unique up to unit in \mathbb{Z}_3 .

Lemma 3.6. *Let $Z \in \kappa_2$ be an exotic element in the Picard group.*

1.) *There is an integer k , $0 \leq k \leq 2$, so that there is an equivalence of $E_2^{hG_{24}}$ -modules*

$$\Sigma^{24k} E_2^{hG_{24}} \simeq E_2^{hG_{24}} \wedge Z.$$

2.) *There is an equivalence of $E_2^{hSD_{16}}$ -modules*

$$E_2^{hSD_{16}} \simeq E_2^{hSD_{16}} \wedge Z.$$

Proof. For Part (1) we examine the fixed point spectral sequence reads

$$(3.4) \quad H^s(G_{24}, (E_2)_t Z) \implies \pi_{t-s}(E_2 \wedge Z)^{hG_{24}}.$$

We note $(E_2 \wedge Z)^{hG_{24}} \simeq E_2^{hG_{24}} \wedge Z$ because Z is a dualizable object in the $K(2)$ -local category. This spectral sequence (3.4) is a module over the spectral sequence

$$H^s(G_{24}, (E_2)_t) \implies \pi_{t-s} E_2^{hG_{24}}.$$

and the E_2 term of (3.4) is free of rank 1 over $H^s(G_{24}, (E_2)_t)$ on the class $\iota_Z \in H^0(G_{24}, (E_2)_0)$. There is an element $b \in \mathbb{F}_3$ so that

$$d_5(\iota_Z) = b\alpha\beta^2 \Delta^{-1} \iota_Z.$$

Then the differential $d_5(\Delta) = \pm\alpha\beta^2$ implies there is a k , $0 \leq k \leq 2$, so that

$$d_5(\Delta^k \iota_Z) = 0.$$

For degree reasons, $\Delta^k \iota_Z$ is a permanent cycle. Comparing the E_∞ -terms of the spectral sequence show that after extending the resulting map $S^{24k} \rightarrow E_2^{hG_{24}} \wedge Z$ to a module map $\Sigma^{24k} E_2^{hG_{24}} \rightarrow E_2^{hG_{24}} \wedge Z$ gives the needed equivalence.

Part (2) is similar, but easier, as $H^s(SD_{16}, (E_2)_t Z) = H^s(SD_{16}, (E_2)_t) = 0$ for $s > 0$. \square

4. TOPOLOGICAL RESOLUTIONS AND VANISHING LINES

In order to prove Theorem 3.5 and complete the calculation of κ_2 we must use that the algebraic resolution of Theorem 2.13 has a topological refinement. This is the main theorem of [5], and we begin the section by recalling those results. This has other uses as well, and we will give a proof of the existence of a horizontal vanishing line for the Adams-Novikov Spectral Sequence for any exotic element in the Picard group.

4.1. Topological resolutions. The isomorphism $(E_2)_*E_2 \cong \text{map}(\mathbb{G}_2, (E_2)_*)$ of Equation (2.4) has the following refinement for any closed subgroup of K of \mathbb{G}_2 . As in §2 of [5], there is an isomorphism, natural in K ,

$$(4.1) \quad (E_2)_*E_2^{hK} \cong \text{map}(\mathbb{G}_2/K, (E_2)_*).$$

From this it follows that if we apply the functor

$$\text{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2]]}(-, (E_2)_*E_2) = \text{Hom}_{\mathbb{Z}_3[[\mathbb{G}_2]]}(-, \text{map}(\mathbb{G}_2, (E_2)_*))$$

to the first resolution of Theorem 2.13 induced up from \mathbb{G}_2^1 to \mathbb{G}_2 we get a resolution of $(E_2)_*E_2^{h\mathbb{G}_2^1}$ by twisted \mathbb{G}_2 -modules

$$(4.2) \quad \begin{aligned} (E_2)_*E_2^{h\mathbb{G}_2^1} &\rightarrow (E_2)_*E_2^{hG_{24}} \rightarrow (E_2)_*\Sigma^8 E_2^{hSD_{16}} \\ &\rightarrow (E_2)_*\Sigma^{40} E_2^{hSD_{16}} \rightarrow (E_2)_*\Sigma^{48} E_2^{hG_{24}} \rightarrow 0. \end{aligned}$$

We have $\Sigma^8 E_2^{hSD_{16}}$ because C_1 is twisted by a character; also, $\Sigma^8 E_2^{hSD_{16}} \simeq \Sigma^{40} E_2^{hSD_{16}}$. See Remark 2.17. However

$$\Sigma^{48} E_2^{hG_{24}} \not\cong E_2^{hG_{24}}$$

even though

$$(E_2)_*\Sigma^{48} E_2^{hG_{24}} \cong (E_2)_*E_2^{hG_{24}}.$$

This suspension is crucial to realization of the resolution of (4.3) of Theorem 4.1. See also Remark 5.3 for more on the role of suspensions.

For the sphere itself we use the second resolution of Theorem 2.13 to get a resolution of $(E_2)_*$ as a twisted \mathbb{G}_2 -module:

$$(4.3) \quad \begin{aligned} (E_2)_* &\rightarrow (E_2)_*E_2^{hG_{24}} \rightarrow (E_2)_*\Sigma^8 E_2^{hSD_{16}} \times (E_2)_*E_2^{hG_{24}} \\ &\rightarrow (E_2)_*\Sigma^{40} E_2^{hSD_{16}} \times (E_2)_*\Sigma^8 E_2^{hSD_{16}} \\ &\rightarrow (E_2)_*\Sigma^{48} E_2^{hG_{24}} \times (E_2)_*\Sigma^{40} E_2^{hSD_{16}} \rightarrow (E_2)_*\Sigma^{48} E_2^{hG_{24}} \rightarrow 0. \end{aligned}$$

The following theorem is the main result of §5 of [5].

Theorem 4.1. *There is a sequence of maps between spectra*

$$\begin{aligned} L_{K(2)}S^0 &\rightarrow E_2^{hG_{24}} \rightarrow \Sigma^8 E_2^{hSD_{16}} \times E_2^{hG_{24}} \rightarrow \Sigma^{40} E_2^{hSD_{16}} \times \Sigma^8 E_2^{hSD_{16}} \\ &\rightarrow \Sigma^{48} E_2^{hG_{24}} \times \Sigma^{40} E_2^{hSD_{16}} \rightarrow \Sigma^{48} E_2^{hG_{24}} \end{aligned}$$

realizing the resolution (4.3) and with the property that any two successive maps are null-homotopic and all possible Toda brackets are zero modulo indeterminacy.

There is an analogous topological resolution of $E_2^{h\mathbb{G}_2^1}$ realizing the algebraic resolution of (4.2).

Let us write F_s for the successive terms in the topological resolution of Theorem 4.1. Thus $F_0 = E_2^{hG_{24}}$,

$$F_1 = \Sigma^8 E_2^{hSD_{16}} \times E_2^{hG_{24}},$$

on so on through F_4 . Then Theorem 4.1 implies that there is a tower of fibrations

$$(4.4) \quad \begin{array}{ccccccc} L_{K(2)}S^0 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & E^{hG_{24}} = F_0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \Sigma^{-4}F_4 & & \Sigma^{-3}F_3 & & \Sigma^{-2}F_2 & & \Sigma^{-1}F_1 & & \end{array}$$

This tower yields a spectral sequence

$$E_1^{s,t} = \pi_t F_s \implies \pi_{t-s} L_{K(2)} S^0.$$

4.2. The vanishing line. The E_∞ -term of the Adams-Novikov Spectral Sequence for exotic element $Z \in \kappa_2$ has a horizontal vanishing line at $s = 13$. This is implicit in known results, especially work of Shimomura and his coauthors. For the sphere itself, see [21]. For a general Z , the result can be deduced after a bit of work from [16]. We include here a proof that uses our technology.

Theorem 4.2. *Let $Z \in \kappa_2$. Then the Adams-Novikov Spectral Sequence*

$$E_2^{s,t} = H^s(\mathbb{G}_2, (E_2)_t) \implies \pi_{t-s} Z$$

has a horizontal vanishing line

$$E_{10}^{s,t} = 0, \quad s \geq 13.$$

This implies, among other things, that the Adams-Novikov Spectral Sequence has only d_5 and d_9 . The proof will occupy the remainder of the section.

Let $\{X_q\}$ be the tower (4.4) with layers F_q refining the topological resolution of $L_{K(2)} S^0$. Let $C_0 = L_{K(2)} S^0$ and define spectra C_q by the cofiber sequence

$$L_{K(2)} S^0 \rightarrow X_{q-1} \rightarrow \Sigma^{-q+1} C_q.$$

Then there are cofiber sequences

$$C_q \rightarrow F_q \rightarrow C_{q+1}$$

which induce short exact sequences in $(E_2)_*$ -homology. This can be summarized in the diagram

$$(4.5) \quad \begin{array}{ccccccccccc} L_{K(2)} S^0 & \leftarrow & \cdots & \leftarrow & C_1 & \leftarrow & \cdots & \leftarrow & C_2 & \leftarrow & \cdots & \leftarrow & C_3 & \leftarrow & \cdots & \leftarrow & C_4 & \xrightarrow{\cong} & F_4 \\ & \searrow & & \nearrow & & \searrow & & \nearrow \\ & & F_0 & & & & F_1 & & & & F_2 & & & & F_3 & & & & & & F_4 \end{array}$$

where the dotted arrows are maps $C_q \rightarrow \Sigma C_{q-1}$.

For any spectrum X write $E_r^{s,t} X$ for the terms in the Adams-Novikov Spectral Sequence

$$E_2^{s,t} X = H^s(\mathbb{G}_2, (E_2)_t X) \implies \pi_{t-s} L_{K(2)} X.$$

Write $C_q X$ for $L_{K(2)}(C_q \wedge X)$ and $F_q X$ for $L_{K(2)}(F_q \wedge X)$.

Using the cofiber sequences of (4.5) we then obtain a spectral sequence

$$(4.6) \quad E_1^{p,q,*} = H^p(\mathbb{G}_2, (E_2)_* F_q X) \implies H^{p+q}(\mathbb{G}_2, (E_2)_* X)$$

from the exact couple

$$\begin{array}{ccc} E_2^{*,*} C_q X & \leftarrow & \cdots & \leftarrow & E_2^{*,*} C_{q+1} X \\ & \searrow & & \nearrow & \\ & & E_2^{*,*} F_q X & & \end{array}$$

The dashed arrows raise cohomological degree by 1. This is isomorphic to the spectral sequence obtained from the resolution of Remark 2.14; compare Corollary 2.15.

Lemma 4.3. *Let $Z \in \kappa_2$. The spectral sequence*

$$(4.7) \quad E_1^{p,q,*} = H^p(\mathbb{G}_2, (E_2)_* F_q Z) \implies H^{p+q}(\mathbb{G}_2, (E_2)_* Z)$$

collapses at the first term for $p + q > 4$.

Proof. It is sufficient to prove this for $Z = L_{K(2)}S^0$, as the spectral sequence (4.7) doesn't depend on $Z \in \kappa_2$. Let $V(1)$ be the cofiber of the Adams map from the Moore spectrum to itself. Then for all q there is an injection

$$H^p(\mathbb{G}_2, (E_2)_*F_q) \rightarrow H^p(\mathbb{G}_2, (E_2)_*F_qV(1))$$

for $p > 1$. To see this, note that we need only show that

$$H^p(G_{24}, (E_2)_*) \rightarrow H^p(G_{24}, (E_2)_*/(3, u_1))$$

is an injection for $p > 1$ and this is a simple calculation. See, for example, Theorem A.2 of [9].

To complete the proof, we see note that the spectral sequence(4.6) for $V(1)$ collapses at the second page, with only differentials d_1 on the $p = 0$ line. This follows from Theorem A.3 of [9]. From this we deduce that in the spectral sequence for S^0 , the longest potentially non-zero differential is from $E_4^{0,3,*}$ to $E_4^{4,0,*}$. The result then follows. \square

We will prove Theorem 4.2 by comparing the Adams-Novikov Spectral Sequence with another spectral sequence which agrees with the Adams-Novikov Spectral Sequence in high degrees, but which is much more regular.

Let $\beta_1 \in \pi_{10}S^0$ be the usual element of order 3. Define $\beta_1^{-1}S^0$ to be the homotopy colimit of

$$S^0 \xrightarrow{\beta_1} S^{-10} \xrightarrow{\beta_1} S^{-20} \xrightarrow{\beta_1} \dots$$

As β_1 is nilpotent, the spectrum $\beta_1^{-1}S^0$ is contractible. If X is any other spectrum, let

$$\beta_1^{-1}X = \beta_1^{-1}S^0 \wedge X.$$

Since β_1 is detected by $\beta \in E_2^{2,12}S^0$, we then obtain a localized spectral

$$\beta^{-1}E_2^{*,*}(X) \implies \pi_*\beta^{-1}L_{K(2)}X = 0$$

which, since colimits and limits do not always commute, does not obviously converge. In all our examples it will converge as we will have $\beta^{-1}E_{10}^{*,*}(X) = 0$.

Example 4.4. As a warm-up, let's consider the case of $\beta_1^{-1}E_2^{hG_{24}}$. Theorem 2.18 gives an isomorphism

$$\mathbb{F}_3[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha) \cong \beta^{-1}H^*(G_{24}, (E_2)_*).$$

Then the differential $d_5(\Delta) = \pm\alpha\beta^2$ (see Theorem 2.19) gives

$$\beta^{-1}E_6^{*,*} = \mathbb{F}_3[\Delta^{\pm 3}, \beta^{\pm 1}] \otimes \Lambda(\alpha\Delta^2)$$

and the differential $d_9(\alpha\Delta^2) = \pm\beta^5$ gives $\beta^{-1}E_{10}^{*,*} = 0$. The equivariantly minded reader will recognize $\beta_1^{-1}E_2^{hG_{24}}$ as the Tate spectrum of the G_{24} -action on E_2 .

Remark 4.5. We now examine the spectral sequence beginning with $\beta^{-1}E_2^{*,*}S^0$ for the sphere itself. First we combine Lemma 4.3 and Theorem 2.18 to deduce that $\beta^{-1}E_2^{*,*}S^0$ has a filtration with associated graded of the form

$$\mathbb{F}_3[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha, \zeta, e)$$

The elements α , β and ζ come from the sphere and have canonical lifts. We chose lifts of Δ and of e and, at the risk of causing a great deal of confusion, call the lifts by the same names as their residue classes. In this way we obtain an isomorphism of filtered differential graded algebras

$$\mathbb{F}_3[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha, \zeta, e) \cong \beta^{-1}E_2^{*,*}Z.$$

The differential is given by d_5 . Recall that F_3 and F_4 have copies of $\Sigma^{48}E_2^{hG_{24}}$; therefore, applying Theorem 2.19, we deduce that this differential satisfies

$$\begin{aligned} d_5(\Delta) &\equiv \pm\alpha\beta^2 \\ d_5(e) &\equiv \pm\alpha\beta^2\Delta^{-1}e \end{aligned}$$

where \equiv means “modulo elements of higher filtration”. All other generators x have $d_5(x) = 0$, since they detect permanent cycles in $E_2^{*,*}S^0$.

We now have a spectral sequence for $\beta^{-1}E_6^{*,*}S^0$ which initial term

$$\mathbb{F}_3[\Delta^{\pm 3}, \beta^{\pm 1}]\{1, \alpha\Delta^2\} \otimes \Lambda(\zeta, \Delta^2 e).$$

The differential d_r of this auxiliary spectral sequence changes degree as follows:

$$(p, q, t) \mapsto (p + 5 - r, q + r, t + 4).$$

As a result, the spectral sequence collapses for degree reasons and hence $\beta^{-1}E_6^{*,*}S^0$ is isomorphic to this algebra, with the choices above. The only other differential remaining is d_9 , which is determined by

$$d_9(\alpha\Delta^2) \equiv \pm\beta^5.$$

The differential on all other generators is either zero outright or zero modulo elements of higher filtration. We have $\beta^{-1}E_{10}^{*,*}S^0 = 0$.

Remark 4.6. Now let $Z \in \kappa_2$ be any exotic element of the Picard group. By similar reasoning to that of Remark 4.5, we deduce that $\beta^{-1}E_2^{*,*}Z$ is a filtered differential module over $\beta^{-1}E_2^{*,*}S^0$ which is free of rank 1 on a generator ι_Z of bidegree $(0, 0)$. The differentials in $\beta^{-1}E_2^{*,*}Z$ are determined by the differentials in $\beta^{-1}E_2^{*,*}S^0$ and the differential

$$d_5(\iota_Z) \equiv \mp k\alpha\beta^2\Delta^{-1}\iota_Z$$

where k , $0 \leq k \leq 2$, is the integer so that $E_2^{hG_{24}} \wedge Z = \Sigma^{24k}E_2^{hG_{24}}$.

Again arguing as in Remark 4.5, we see that $\beta^{-1}E_6^{*,*}Z$ is free of rank 1 over $\beta^{-1}E_6^{*,*}S^0$ on $\Delta^k\iota_Z$; again we have $\beta^{-1}E_{10}^{*,*}Z = 0$.

We now begin an analysis of the localization map of spectral sequences. From Lemma 4.3, we know

$$E_2^{*,*}Z \rightarrow \beta^{-1}E_2^{*,*}Z \cong [\mathbb{F}_3[\Delta^{\pm 1}, \beta^{\pm 1}] \otimes \Lambda(\alpha, \zeta, e)]\iota_Z$$

is an isomorphism in homological degrees $s > 4$. To get some hold on the image define a sub-differential graded module

$$A \stackrel{\text{def}}{=} \mathbb{F}_3[\Delta^{\pm 1}, \beta]\{\beta^2, \alpha\beta^2, \beta^2\zeta, \alpha\beta\zeta, \beta e, \alpha e, \beta\zeta e, \alpha\zeta e\} \subseteq \beta^{-1}E_2^{*,*}S^0.$$

This inclusion is also an isomorphism in degrees $s > 4$. Since the longest differential in the algebraic spectral sequence (4.6) is a d_4 , the inclusion of $A \rightarrow \beta^{-1}E_2^{*,*}S^0$ factors through $E_2^{*,*}S^0 \rightarrow \beta^{-1}E_2^{*,*}S^0$. This factoring is unique in degrees $s > 4$, which implies that the factoring automatically commutes with d_5 . Choose a factoring and identify A with its image in $E_2^{*,*}S^0$.

Next observe that $A\iota_Z \subseteq E_2^{*,*}Z$ is an inclusion of a differential submodule and an isomorphism in degree $s > 4$. From this we deduce that the map

$$H^s(A\iota_Z, d_5) \rightarrow E_6^{s,*}Z$$

is onto for $s > 4$ and an isomorphism for $s > 9$. This implies that the map

$$E_6^{*,*}Z \rightarrow \beta^{-1}E_6^{*,*}Z$$

is an isomorphism in degrees $s > 9$.

We now move on to the calculation of d_9 . The inclusion of A , as above, defines an inclusion

$$B \stackrel{\text{def}}{=} \mathbb{F}_3[\Delta^{\pm 3}, \beta]\{\beta^2, \alpha\beta^2\Delta^2, \beta^2\zeta, \alpha\beta\zeta\Delta^2, \beta\Delta^2e, \alpha\Delta^4e, \beta\Delta^2\zeta e, \alpha\Delta^4\zeta e\} \subseteq H^*A \rightarrow E_6^{*,*}S^0$$

which maps injectively through to $\beta^{-1}E_6^{*,*}S^0$. Then

$$B\Delta^k\iota_Z \rightarrow \beta^{-1}E_9^{s,*}Z$$

is onto in degrees $s > 4$ and we have that the inclusion $B\Delta^k\iota_Z \rightarrow E_6^{*,*}Z$ is an isomorphism in degrees $s > 9$. Since B has no elements of degree $s = 0$, B is closed under d_9 . We then have

$$H^*(B\Delta^k\iota_Z, d_9) \cong E_{10}^{*,*}Z$$

is an isomorphism for degree $s > 10$. The d_9 differentials in B are all induced from the formula $d_9(\alpha\Delta^2) = \pm\beta^5$; hence, $H^*(B, d_9) = 0$ in degrees $s > 12$ and we can deduce Theorem 4.2.

There are non-zero elements of degree $s = 12$; for example, the elements

$$e\zeta\Delta^2\Delta^{3k}\beta^4$$

are of s filtration 12 and non-zero in $E_{10}^{*,*}S^0$. Many of these detect permanent cycles in $\pi_*L_{K(2)}V(1)$; see [3].

Corollary 4.7. *Let Z be an element of κ_2 so that $d_5(\iota_Z) = 0$. Then ι_Z is a permanent cycle and $L_{K(2)}S^0 \simeq Z$.*

Proof. By the vanishing result, we need only check that $d_9(\iota_Z) = 0$. In the spectral sequence of Remark 2.16 for computing $H^*(\mathbb{G}_2, (E_2)_8)$ all groups in total degree 9 vanish, hence $H^9(\mathbb{G}_2, (E_2)_8) = 0$. \square

5. THE DECOMPOSITION OF THE GROUP OF EXOTIC ELEMENTS

In order to prove Theorem 3.5 and complete the calculation of κ_2 we discuss which exotic elements in κ_2 can be detected by $E_2^{hG_{24}}$. This will leave a subgroup of order 3 which cannot be seen by $E_2^{hG_{24}}$; this subgroup is discussed in the last subsection.

Here is an outline of the arguments of this section. Let κ_2^1 denote the set of weak equivalence classes of $E_2^{hG_{24}}$ -modules X so that

$$(E_2)_*X \cong (E_2)_*E_2^{hG_{24}}$$

as twisted \mathbb{G}_2 -modules. Then κ_2^1 is a group under smash product over $E_2^{hG_{24}}$. There is a homomorphism

$$\tau^1 : \kappa_2^1 \longrightarrow H^5(G_{24}, (E_2)_4) \cong \mathbb{Z}/3$$

defined, as in Construction 3.3, using d_5 in the Adams-Novikov Spectral Sequence and the Schapiro isomorphism

$$H^*(\mathbb{G}_2, (E_2)_*X) \cong H^*(G_{24}, (E_2)_*).$$

We will see in Proposition 5.4 that this map is a surjection. It is also an injection, for if $d_5(\iota_X) = 0$, then ι_X is a permanent cycle and the resulting homotopy class can be extended to an equivalence $E_2^{hG_{24}} \simeq X$.

There is a map $\kappa_2 \rightarrow \kappa_2^1$ sending Z to $E_2^{hG_{24}} \wedge Z$ and a commutative diagram

$$\begin{array}{ccc} \kappa_2 & \xrightarrow{\tau} & H^5(\mathbb{G}_2, (E_2)_4) \\ \downarrow & & \downarrow \\ \kappa_2^1 & \xrightarrow[\tau^1]{\cong} & H^5(G_{24}, (E_2)_4) \end{array}$$

where the map on group cohomology is the restriction. Theorem 5.5 below shows that the map $\kappa_2 \rightarrow \kappa_2^1$ is onto. We now let κ_2^0 be the kernel. From Proposition 3.1 we then get an induced map

$$\tau^0 : \kappa_2^0 \longrightarrow H^1(G_{24}, (E_2)_4) \cong \mathbb{Z}/3$$

and we must show that this map is an isomorphism. This is accomplished in Propositions 5.7 and 5.9.

5.1. Exotic elements detected by $E_2^{hG_{24}}$. Let's begin by revisiting the case $n = 1$ and $p = 2$. See Remark 2.10.

Example 5.1. At $p = 2$ and $n = 1$, there is very short resolution

$$(5.1) \quad K_* \longrightarrow K_*KO \xrightarrow{\psi^3-1} K_*KO \longrightarrow 0$$

of $K_* = K_*S^0$ as a continuous module over the Adams operations. This can be realized by the fiber sequence

$$L_{K(1)}S^0 \longrightarrow KO \xrightarrow{\psi^3-1} KO.$$

However, there is another topological realization of the resolution (5.1) using the fact that the Bott periodicity isomorphism

$$\Sigma^4 K_*K \xrightarrow{\cong} K_*K$$

induces an isomorphism of \mathbb{G}_1 -modules

$$\Sigma^4 K_*KO \xrightarrow{\cong} K_*KO$$

which cannot be realized topologically. We then get a fiber sequence

$$X \longrightarrow \Sigma^4 KO \xrightarrow{\Sigma^4 3^{-2}\psi^3-1} \Sigma^4 KO$$

defining an exotic element $X \in \kappa_1$. The map $X \rightarrow \Sigma^4 KO$ extends to a weak equivalence $KO \wedge X \simeq \Sigma^4 KO$ of KO -modules. Since $\kappa_1 \cong \mathbb{Z}/2$, this implies $X \simeq L_{K(1)}DQ$, the non-zero element. Note this construction produces only $L_{K(1)}DQ$ and not the finite complex DQ itself.

We now do something similar, but slightly more elaborate, at the prime 3. Let $G \subseteq \mathbb{G}_2$ be a finite subgroup. The invariants $H^0(G, (E_2)_*)$ form a graded ring. Suppose we can choose an invertible element x in this ring of degree t . Then x defines an isomorphism of $(E_2)_*[G]$ -modules

$$x : \Sigma^t (E_2)_* \xrightarrow{\cong} (E_2)_*$$

which extends to an isomorphism of twisted \mathbb{G}_2 -modules

$$\alpha : \Sigma^t \text{map}(\mathbb{G}_2/G, (E_n)_*) \xrightarrow{\cong} \text{map}(\mathbb{G}_2/G, (E_2)_*)$$

given by

$$\alpha(\Sigma^t \phi)(g) = (gx)\phi(g).$$

We are using the diagonal \mathbb{G}_2 -action; compare Equation 2.3. This isomorphism of Equation 2.4 induces an isomorphism

$$(E_2)_* E_2^{hG} \cong \text{map}(\mathbb{G}_n/G, (E_2)_*)$$

and so x yields an algebraic isomorphism

$$(E_2)_* \Sigma^t E_2^{hG} \xrightarrow{\cong} (E_2)_* E_2^{hG}$$

which may not be realized topologically. If x is a unit of minimal positive degree, then we say that E_2^{hG} has *algebraic periodicity* t and x is a periodicity operator. This construction works, of course, at all primes p and all n . From §3 of [5] we have the following result. See also Theorem 2.18.

Lemma 5.2. 1.) *The fixed point spectrum $E_2^{hG_{24}}$ has algebraic periodicity 24 and $\Delta \in H^0(G_{24}, (E_2)_{24})$ is a periodicity operator.*

2.) *The fixed point spectrum $E_2^{hSD_{16}}$ has algebraic periodicity 16 and $v_2 = u^{-8}$ is a periodicity operator.*

Remark 5.3. The algebraic periodicity on $E_2^{hSD_{16}}$ can be realized topologically, because there can be no differentials in the fixed point spectral sequence. However, the algebraic periodicity of $E_2^{hG_{24}}$ cannot be realized topologically as Δ is not a permanent cycle. (See Theorem 2.19.) Topologically this spectrum has periodicity 72 with periodicity operator Δ^3 .

Proposition 5.4. *The homomorphism*

$$\tau^1 : \kappa_2^1 \rightarrow H^5(G_{24}, (E_2)_4) \cong \mathbb{Z}/3$$

is an isomorphism.

Proof. We already noted at the beginning of the section that the map is a monomorphism. Remark 5.3 implies that it is also an epimorphism. \square

The next result now says that the map $\kappa_2 \rightarrow \kappa_2^1$ sending Z to $E_2^{hG_{24}} \wedge Z$ is surjective.

Theorem 5.5. *There exists an exotic element $P \in \kappa_2$ so that*

$$E_2^{hG_{24}} \wedge P \simeq \Sigma^{48} E_2^{hG_{24}}.$$

Proof. We proceed as in Example 5.1 and twist the resolution of Theorem 4.1 in order to produce P . We use the isomorphisms

$$(E_2)_* \Sigma^{48} E_2^{hF} \cong (E_2)_* E_2^{hF}$$

and the resolution 4.3 to produce a new resolution

$$\begin{aligned} (E_2)_* &\rightarrow (E_2)_* \Sigma^{48} E_2^{hG_{24}} \rightarrow (E_2)_* \Sigma^{56} E_2^{hSD_{16}} \times (E_2)_* \Sigma^{48} E_2^{hG_{24}} \\ &\rightarrow (E_2)_* \Sigma^{88} E_2^{hSD_{16}} \times (E_2)_* \Sigma^{56} E_2^{hSD_{16}} \\ &\rightarrow (E_2)_* \Sigma^{96} E_2^{hG_{24}} \times (E_2)_* \Sigma^{88} E_2^{hSD_{16}} \rightarrow (E_2)_* \Sigma^{96} E_2^{hG_{24}} \rightarrow 0 \end{aligned}$$

This has a topological realization. In fact, the arguments of Theorem 5.5 of [5] go through verbatim to produce the sequence

$$\begin{aligned} \Sigma^{48} E_2^{hG_{24}} &\rightarrow \Sigma^{56} E_2^{hSD_{16}} \times \Sigma^{48} E_2^{hG_{24}} \rightarrow \Sigma^{88} E_2^{hSD_{16}} \times \Sigma^{56} E_2^{hSD_{16}} \\ &\rightarrow \Sigma^{96} E_2^{hG_{24}} \times \Sigma^{88} E_2^{hSD_{16}} \rightarrow \Sigma^{96} E_2^{hG_{24}} \end{aligned}$$

with all the necessary Toda brackets zero modulo indeterminacy. Notice that each of the spectra in this new resolution are spectra of the old resolution, suspended 48 times; however, the maps are not simple suspensions of the old maps. Then P is the top of the resulting

tower. The induced map $P \rightarrow \Sigma^{48} E_2^{hG_{24}}$ from the top of the tower to the bottom extends a map

$$E_2^{hG_{24}} \wedge P \rightarrow \Sigma^{48} E_2^{hG_{24}}$$

of $E_2^{hG_{24}}$ -modules and we'd like to see that this is an equivalence. To see this, we consider the following commutative diagram

$$\begin{array}{ccc} (E_2)_*(E_2^{hG_{24}} \wedge P) & \longrightarrow & (E_2)_*\Sigma^{48} E_2^{hG_{24}} \\ \downarrow & & \downarrow \\ \text{map}(\mathbb{G}_2/G_{24}, (E_2)_*X) & \longrightarrow & \text{map}(\mathbb{G}_2/G_{24}, \Sigma^{48}(E_2)_*). \end{array}$$

The vertical maps are the isomorphisms of (4.1) and the lower map is the isomorphism induced by algebraic periodicity. It follows that the upper map is an isomorphism, as needed. \square

Remark 5.6. The element P is not uniquely determined by the requirement that $E_2^{hG_{24}} \wedge P \simeq E_2^{hG_{24}}$. However, we show in [6] that P is the unique element in κ_2 so that

- (1) $E_2^{hG_{24}} \wedge P \simeq \Sigma^{48} E_2^{hG_{24}}$, and
- (2) $P \wedge V(1) \simeq \Sigma^{48} L_{K(2)}V(1)$, where $V(1)$ is the cofiber of the Adams map on the mod-3 Moore spectrum.

5.2. The truly exotic elements. Recall that κ_2^0 is the subgroup of κ_2 consisting of the exotic elements in $Z \in \text{Pic}(\mathcal{K}_2)$ so that there is an equivalence of $E_2^{hG_{24}}$ -modules

$$E_2^{hG_{24}} \wedge Z \simeq E_2^{hG_{24}}.$$

We must now show that $\kappa_2^0 \cong H^1(G_{24}, (E_2)_4)$.

We prove the injectivity statement of Theorem 3.5.

Proposition 5.7. *The homomorphism*

$$\tau^0 : \kappa_2^0 \rightarrow H^1(G_{24}, (E_2)_4) \subseteq H^5(\mathbb{G}_2, (E_2)_4)$$

is injective.

Proof. This is a restatement of Proposition 3.1 and Corollary 4.7. \square

It remains to show that $\tau : \kappa_2 \rightarrow H^5(\mathbb{G}_2, (E_2)_4)$ is surjective. We will prove this by constructing explicit elements in κ_2 .

To prepare for the argument, consider the tower realizing the resolution of Equation (4.2)

$$(5.2) \quad \begin{array}{ccccccc} E_2^{h\mathbb{G}_2^1} & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & E^{hG_{24}} = F_0^1 \\ \uparrow & & \uparrow & & \uparrow & & \\ \Sigma^{-3}F_3^1 & & \Sigma^{-2}F_2^1 & & \Sigma^{-1}F_1^1 & & \end{array}$$

Lemma 5.8. *Let $Z \in \kappa_2^0$. Then there is a weak equivalence of $E_2^{h\mathbb{G}_2^1}$ -modules*

$$E_2^{h\mathbb{G}_2^1} \simeq E_2^{h\mathbb{G}_2^1} \wedge Z.$$

Proof. There is no obstruction to lifting the map $S^0 \rightarrow E_2^{hG_{24}} \wedge Z$ detected by ι_Z up the tower obtained from (5.2) by applying $(-) \wedge Z$. \square

There is a short fiber sequence

$$(5.3) \quad L_{K(2)}S^0 \longrightarrow E_2^{h\mathbb{G}_2^1} \xrightarrow{\psi^4-1} E_2^{h\mathbb{G}_2^1}$$

where $\psi^4 = \psi^{p+1}$ is a generator for the central $\mathbb{Z}_3 \cong \mathbb{Z}_3^\times / \{\pm 1\}$ acting on $E_2^{h\mathbb{G}_2^1}$. We apply $(-)\wedge Z$ and the equivalence of Lemma 5.8 to obtain a fiber sequence

$$Z \longrightarrow E_2^{h\mathbb{G}_2^1} \xrightarrow{f_Z} E_2^{h\mathbb{G}_2^1}.$$

Let $\iota \in \pi_0 E_2^{h\mathbb{G}_2^1}$ be the unit element. Then $Z \simeq L_{K(2)}S^0$ if and only if

$$(f_Z)_*(\iota) = 0.$$

An analysis of the homotopy groups of $E_2^{h\mathbb{G}_2^1}$ using the tower (5.2) shows there is an exact sequence

$$0 \rightarrow \mathbb{Z}/3 = \pi_{-45} E_2^{hG_{24}} \rightarrow \pi_0 E_2^{h\mathbb{G}_2^1} \rightarrow \pi_0 E_2^{hG_{24}}$$

and the E_2 -Hurewicz homomorphism shows that $(f_Z)_*(\iota)$ must land in $\pi_{-45} E_2^{hG_{24}}$. Thus, to construct the non-trivial exotic Z needed to show τ^0 is a surjection, we need only produce enough self-maps for $E_2^{h\mathbb{G}_2^1}$ to realize this obstruction. This we do in the proof of the following result.

Theorem 5.9. *Let κ_2^0 be the subgroup of κ_2 consisting of those elements with $E_2^{hG_{24}} \wedge Z \simeq E_2^{hG_{24}}$. The homomorphism*

$$\tau^0 : \kappa_2^0 \longrightarrow H^1(G_{24}, (E_2)_4) \subseteq H^5(\mathbb{G}_2, (E_2)_4)$$

is surjective.

Proof. We begin with a calculation of $[E_2^{h\mathbb{G}_2^1}, E_2^{h\mathbb{G}_2^1}]$. Apply the functor $[E_2^{h\mathbb{G}_2^1}, -]$ to the tower (5.2), then use Proposition 2.6 of [5] and the calculations of Theorem 2.19 to get an exact sequence

$$\begin{aligned} 0 \rightarrow \pi_3(\Sigma^{48} E_2^{hG_{24}} [[\mathbb{G}_2/\mathbb{G}_2^1]]) &\rightarrow [E_2^{h\mathbb{G}_2^1}, E_2^{h\mathbb{G}_2^1}] \\ &\rightarrow \pi_0(E_2^{hG_{24}} [[\mathbb{G}_2/\mathbb{G}_2^1]]) \rightarrow \pi_0(\Sigma^8 E_2^{hSD_{16}} [[\mathbb{G}_2/\mathbb{G}_2^1]]). \end{aligned}$$

The last map is induced by $d_1; E_2^{hG_{24}} \rightarrow \Sigma^8 E_2^{hSD_{16}}$ at the beginning of the topological resolution realizing (4.2). In Theorem 4.3 of [4] we proved that the kernel of $(d_1)_* : \pi_0 E_2^{hG_{24}} \rightarrow \pi_0 \Sigma^8 E_2^{hSD_{16}}$ is \mathbb{Z}_3 generated by the multiplicative identity. Thus we have a short exact sequence

$$(5.4) \quad 0 \rightarrow \pi_3(\Sigma^{48} E_2^{hG_{24}} [[\mathbb{G}_2/\mathbb{G}_2^1]]) \rightarrow [E_2^{h\mathbb{G}_2^1}, E_2^{h\mathbb{G}_2^1}] \rightarrow \mathbb{Z}_3 [[\mathbb{G}_2/\mathbb{G}_2^1]] \rightarrow 0.$$

Since $\mathbb{G}_2/\mathbb{G}_2^1 \cong \mathbb{Z}_3$, there is an isomorphism of complete local rings

$$\mathbb{Z}_3[[T]] \xrightarrow{\cong} \mathbb{Z}_3[[\mathbb{G}_2/\mathbb{G}_2^1]]$$

sending T to $\psi - 1$. Again from Theorem 2.19 we then have

$$\pi_3(\Sigma^{48} E_2^{hG_{24}} [[\mathbb{G}_2/\mathbb{G}_2^1]]) \cong \mathbb{Z}/3[[T]],$$

We also have

$$(5.5) \quad 0 \rightarrow \mathbb{Z}/3 \cong \pi_3(\Sigma^{48} E_2^{hG_{24}}) \rightarrow \pi_0(E_2^{h\mathbb{G}_2^1}) \rightarrow \mathbb{Z}_3 \rightarrow 0.$$

The final \mathbb{Z}_3 is the kernel of the map $(d_1)_* : \pi_0 E_2^{hG_{24}} \rightarrow \pi_0 \Sigma^8 E_2^{hSD_{16}}$, as above. We can obtain the sequence (5.5) by direct calculation or from Equation 5.4 by killing the action of T with the fibration sequence of Equation (5.3). Combining the exact sequences of Equations

5.4 and 5.5 we obtain a commutative diagram with the vertical maps induced by the unit map $S^0 \rightarrow E_2^{h\mathbb{G}_2^1}$.

$$(5.6) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/3[[T]] & \longrightarrow & [E_2^{h\mathbb{G}_2^1}, E_2^{h\mathbb{G}_2^1}] & \longrightarrow & \mathbb{Z}_3[[T]] \longrightarrow 0 \\ & & \downarrow 0=T & & \downarrow & & \downarrow T=0 \\ 0 & \longrightarrow & \mathbb{Z}/3 & \longrightarrow & \pi_0 E_2^{h\mathbb{G}_2^1} & \longrightarrow & \mathbb{Z}_3 \longrightarrow 0. \end{array}$$

Let $b \in \pi_3(\Sigma^{48} E_2^{h\mathbb{G}_2^{24}}[[\mathbb{G}_2/\mathbb{G}_2^1]]) \cong \mathbb{Z}/3[[T]]$ and define spectra Q_b by the fibration sequences

$$(5.7) \quad Q_b \longrightarrow E_2^{h\mathbb{G}_2^1} \xrightarrow{T+b} E_2^{h\mathbb{G}_2^1}.$$

Recall $T = \psi - 1$. Since $(E_2)_* b = 0$, we have $(E_2)_* Q_b \cong (E_2)_*$ as twisted \mathbb{G}_2 -modules; hence $Q_b \in \kappa_2^0$. Furthermore, from the diagram of Equation 5.6, we see that the unit element $\pi_0 E_2^{h\mathbb{G}_2^1}$ factors through Q_b if and only if b is divisible by T . In particular if $b \not\equiv 0$ modulo T , then

$$\tau(Q_b) \neq 0 \in \pi_{-45} E_2^{h\mathbb{G}_2^{24}}.$$

□

Remark 5.10. We constructed all the elements of κ_2 in Theorem 5.5 and Theorem 5.9. However, these constructions write the exotic elements as homotopy inverse limits of diagrams built from the infinite spectra E_2^{hG} , with G finite. What we have not done is construct finite CW spectra whose $K(n)$ -localizations realize these elements. This is in contrast to the case $n = 1$ and $p = 2$, where the exotic element is the localization of an explicit three-cell complex.

6. THE COMPUTATION OF $\text{Pic}(\mathcal{L}_2)$

We begin with some generalities. Recall that \mathcal{L}_n denotes the category of L_n -local spectra and \mathcal{K}_n the category of $K(n)$ -local spectra.

Lemma 6.1. *Localization defines a homomorphism*

$$L_{K(n)} : \text{Pic}(\mathcal{L}_n) \longrightarrow \text{Pic}_n$$

which restricts to a homomorphism $\kappa(\mathcal{L}_n) \rightarrow \kappa_n$.

Proof. If X and Y are spectra, then the natural map $X \wedge Y \rightarrow L_{K(n)} X \wedge L_{K(n)} Y$ induces an equivalence

$$L_{K(n)}(X \wedge Y) \rightarrow L_{K(n)}(L_{K(n)} X \wedge L_{K(n)} Y).$$

The result follows from the definition of the Picard group. □

A first calculation is due to Hovey and Sadofsky [14]:

Proposition 6.2. *Let $X \in \text{Pic}(\mathcal{L}_n)$. Then there is an integer k so that $E(n)_* X \cong E(n)_* S^k$ as $E(n)_* E(n)_*$ -comodules. Furthermore there is a short exact sequence*

$$0 \longrightarrow \kappa(\mathcal{L}_n) \longrightarrow \text{Pic}(\mathcal{L}_n) \xrightarrow{\dim} \mathbb{Z} \longrightarrow 0$$

where $\dim(X) = k$.

Now let $n = 2$ and $p = 3$. We then have the following result of Kamiya and Shimomora [17].

Proposition 6.3. *Let $p = 3$. There is a non-trivial element of order 3 in $\kappa(\mathcal{L}_2)$ and $\kappa(\mathcal{L}_2)$ is contained in $\mathbb{Z}/3 \oplus \mathbb{Z}/3$; that is, there are inclusions*

$$\mathbb{Z}/3 \subset \kappa(\mathcal{L}_2) \subset \mathbb{Z}/3 \oplus \mathbb{Z}/3 .$$

We can now sharpen this result. With Proposition 6.2 the next result completes the calculation of $\text{Pic}(\mathcal{L}_2)$.

Theorem 6.4. *Let $p = 3$. Then $\kappa(\mathcal{L}_2)$ maps isomorphically to κ_2 .*

The proof will be supplied below after some preliminaries.

Remark 6.5. We will show that for every $X \in \kappa_2$ there is an element $Z \in \text{Pic}(\mathcal{L}_2)$ so that $L_{K(n)}Z \simeq X$. By Theorem 2.11, $(E_2)_*(-)$ also detects dimensions of spheres. This fact and Proposition 6.2 imply that $Z \in \kappa(\mathcal{L}_2)$. We can then conclude that the map $\kappa(\mathcal{L}_2) \rightarrow \kappa_2$ of Lemma 6.1 is surjective; with Proposition 6.3, this will imply Theorem 6.4.

We define $\mathcal{L}_{n-1} \times_{\mathcal{L}_{n-1}\mathcal{K}_n} \mathcal{K}_n$ to be the category of triples (X, Y, f) with $X \in \mathcal{L}_{n-1}$, $Y \in \mathcal{K}_n$, and f a homotopy class of maps $f : X \rightarrow L_{n-1}Y$. A morphism from $(X_1, Y_1, f_1) \rightarrow (X_2, Y_2, f_2)$ are homotopy classes of morphism $X_1 \rightarrow X_2$ and $Y_1 \rightarrow Y_2$ so that

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{f_2} & Y_2 \end{array}$$

commutes up to homotopy. We give $\mathcal{L}_{n-1} \times_{\mathcal{L}_{n-1}\mathcal{K}_n} \mathcal{K}_n$ an internal smash product with

$$(X_1, Y_1, f_1) \wedge (X_2, Y_2, f_2) = (X_1 \wedge X_2, L_{K(n)}(Y_1 \wedge Y_2), g)$$

where g is defined by the following diagram

$$\begin{array}{ccc} X_1 \wedge X_2 & \xrightarrow{f_1 \wedge f_2} & L_{n-1}Y_1 \wedge L_{n-1}Y_2 \\ & & \uparrow \simeq \\ & & L_{n-1}(Y_1 \wedge Y_2) \longrightarrow L_{n-1}(L_{K(n)}(Y_1 \wedge Y_2)). \end{array}$$

We will write $f_1 * f_2$ for g .

If Z is any L_n -local spectrum, there is a *chromatic fracture square*:

$$(6.1) \quad \begin{array}{ccc} Z & \xrightarrow{\eta_Z} & L_{K(n)}Z \\ \downarrow & & \downarrow \\ L_{n-1}Z & \xrightarrow{L_{n-1}\eta_Z} & L_{n-1}L_nZ \end{array}$$

where the vertical maps and η_Z are the obvious localization maps. By Theorem 6.19 of [15] this is a homotopy pull-back square.

Proposition 6.6. *The functor*

$$\mathcal{L}_n \longrightarrow \mathcal{L}_{n-1} \times_{\mathcal{L}_{n-1}\mathcal{K}_n} \mathcal{K}_n$$

$$Z \longmapsto (L_{n-1}Z, L_{K(n)}Z, L_{n-1}(\eta_Z))$$

is bijective on isomorphism classes of objects and commutes with smash products.

Proof. The chromatic fracture square (6.1) gives that this functor is one-to-one on isomorphism classes of objects. To show that it is onto, let (X, Y, f) be an object in the target and consider the homotopy pull-back square

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & L_{n-1}Y. \end{array}$$

Another application of Theorem 6.19 of [15] shows that any L_{n-1} -local spectrum is $K(n)$ -acyclic. Thus $g : Z \rightarrow Y$ is $K(n)$ -localization. Since the map $Y \rightarrow L_{n-1}Y$ is an $E(n-1)_*$ isomorphism, so is $Z \rightarrow X$; hence f is homotopic to $L_{n-1}g$.

That the functor commutes with smash product is a matter of definitions. \square

Remark 6.7. By Proposition 6.6, the Picard group $\text{Pic}(\mathcal{L}_n)$ gets identified with the set of isomorphism classes of triples (X, Y, f) such that

- (1) $X \in \text{Pic}(\mathcal{L}_{n-1})$ with inverse X_1 ;
- (2) $Y \in \text{Pic}_n$ with inverse Y_1 ; and,
- (3) there is a map $f_1 : X_1 \rightarrow L_{n-1}Y_1$ so that $f * f_1 = L_{n-1}\eta : L_{n-1}S^0 \rightarrow L_{n-1}L_{K(n)}S^0$.

We will be interested in the case where $Y = L_1S^0$. The next result is Theorem 5.10 of [4].

Theorem 6.8. *Let $X \in \kappa_2$. Then the localized Hurewicz homomorphism*

$$\pi_0 L_1 X \longrightarrow \pi_0 L_1 L_{K(2)}(E_2 \wedge X)$$

is injective. Any choice of isomorphism $f : (E_2)_ \rightarrow (E_2)_* X$ of twisted \mathbb{G}_2 -modules defines a generator of $\pi_0 L_1 X \cong \mathbb{Z}_3$. This generator determines a weak equivalence of $L_1 L_{K(2)} S^0$ -modules*

$$L_1 L_{K(2)} S^0 \simeq L_1 X.$$

The Proof of Theorem 6.4. Let $X \in \kappa_2$ and choose an isomorphism $\phi : (E_2)_* \rightarrow (E_2)_* X$ of twisted \mathbb{G}_2 -modules. Theorem 6.8 now gives an object

$$(L_1 S^0, X, f) \in \mathcal{L}_{n-1} \times_{\mathcal{L}_{n-1} \kappa_n} \mathcal{K}_n$$

with f the composition

$$L_1 S^0 \xrightarrow{L_1 \eta} L_1 L_{K(2)} S^0 \longrightarrow L_1 X.$$

To construct an inverse of $(L_1 S^0, X, f)$, let $Y \in \kappa_2$ be an inverse for X and let $\psi : (E_2)_* \rightarrow (E_2)_* Y$ be the isomorphism of twisted \mathbb{G}_2 -modules determined requiring the following composition to be the identity

$$(E_2)_* \cong (E_2)_* \otimes_{(E_2)_*} (E_2)_* \xrightarrow[\cong]{\phi \otimes \psi} (E_2)_* X \otimes_{(E_2)_*} (E_2)_* Y \xrightarrow[\cong]{} (E_2)_*(X \wedge Y) \cong (E_2)_*.$$

Construct $g : L_1 S^0 \rightarrow L_1 Y$ as we constructed the map f . Then the resulting triple $(L_1 S^0, Y, g)$ is the needed inverse. For this we need to check that the map

$$f * g : L_1 S^0 \rightarrow L_1 L_{K(2)}(X \wedge Y) \simeq L_1 L_{K(2)} S^0$$

is the standard map $L_1 \eta$. But this follows from the choice of ψ .

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, U.S.A.

INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, C.N.R.S. - UNIVERSITÉ DE STRASBOURG, F-67084 STRASBOURG, FRANCE

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, U.S.A.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA CHAMPAIGN,
IL 61801, U.S.A.