K-Theory **4**: 1–27, 1990.

Calculus I: The First Derivative of Pseudoisotopy Theory

Dedicated to Alexander Grothendieck

THOMAS G. GOODWILLIE*
Department of Mathematics, Box 1917, Brown University, Providence, RI 02912, U.S.A.

(Received: August 1988)

Abstract. Let $\mathscr{P}(X)$ be the (stable, smooth) pseudoisotopy space of the space X. For any map $f: Y \to X$ of spaces, we identify the homotopy type of the fiber of $\mathscr{P}(f): \mathscr{P}(Y) \to \mathscr{P}(X)$ in a stable range, roughly twice the connectivity of the map $Y \to X$. We establish some language for discussing and manipulating such stable-range relative calculations for any homotopy functor. The theorem about $\mathscr{P}$ has a corollary about Waldhausen's A.

Key words. Pseudoisotopy, concordance, Waldhausen K-theory of spaces, manifold, homotopy functor.

0. Introduction

Let $\mathscr{P}(X)$ be the (stable, smooth) pseudoisotopy space of the space X. Any k-connected map $Y \to X$ induces a $(k-2)$-connected map $\mathscr{P}(Y) \to \mathscr{P}(X)$. We describe the $(2k-3)$-homotopy type of the fiber of $\mathscr{P}(Y) \to \mathscr{P}(X)$. This stable-range calculation of relative $\mathscr{P}$-theory implies a corresponding result about relative A-theory, where $A(X)$ is the Waldhausen K-theory of X.

The theorem is proved in Section 3. In Section 1, we establish some language for systematically discussing stable-range calculations, and in Section 2 we work out an important class of examples.

Here is a sketch of the contents of Section 1:

A homotopy functor F from spaces to spaces is called *excisive* if it takes (homotopy-)co-Cartesian square diagrams to (homotopy-)Cartesian square diagrams. In this case the *reduced* functor $\tilde{F}(Y)$, the fiber of $F(Y) \to F(*)$ (where $*$ is a one-point space), behaves like a homology theory: the spaces $\tilde{F}(S^i)$ form a spectrum and the homotopy groups of $\tilde{F}(Y)$ are the homology groups of Y with coefficients in that spectrum, at least if Y is a finite complex. If F is reduced ($F(*) \sim *$) and excisive then it is said to be *linear* and $\{F(S^i)\}$ is called its *coefficient spectrum.*

If F is not excisive, then the spaces $\tilde{F}(S^i)$ still form a prespectrum. The associated spectrum is written $\partial F(*)$ and called the *derivative* of F at $*$. The *differential* of F

* Research supported in part by NSF grant DMS-8806444 and by a Sloan Fellowship. Research at MSRI supported in part by NSF grant DMS-8505550.

at $*$, written D_*F, is the functor taking Y to the (homotopy co-)limit of

$$\tilde{F}(Y) \to \Omega\tilde{F}(SY) \to \Omega^2\tilde{F}(S^2Y) \to \cdots.$$

Most homotopy functors F occurring in nature are *stably excisive* in the sense that when the maps $X \to X_1$ and $X \to X_2$ in the co-Cartesian diagram

$$\begin{array}{ccc} X & \to & X_1 \\ \downarrow & & \downarrow \\ X_2 & \to & X_{12} \end{array}$$

are, respectively, k_1-connected and k_2-connected, then F gives a $(k_1 + k_2 - c)$-Cartesian diagram. (Here, c is some constant depending on F, and we may have to assume that the k_i are not too small.) Stable excision for F implies that D_*F is linear and that $\partial F(*)$ is its coefficient spectrum. It also implies that $\tilde{F}$ and D_*F *agree to first order* at $*$ in the sense that there is a natural map $\tilde{F}(Y) \to (D_*F)(Y)$ whose connectivity is about twice that of Y. In fact, D_*F is the only linear functor agreeing with $\tilde{F}$ to first order. It may be thought of as the best linear approximation to $\tilde{F}$. (In the notation of [7] and [8], $(D_*F)(Y)$ would be $F^S(Y_+)$.)

We generalize the construction above by replacing $*$ by any space X. The derivative of F at the based space (X, x) is defined to be the spectrum $\partial_x F(X)$ associated to the pre-spectrum

$$i \mapsto \text{fiber}(F(X \vee S^i) \to F(X)).$$

The differential $D_X F$ is a certain functor from spaces over X to spaces. It is the 'best linear approximation' to the functor $Y \mapsto \text{fiber}(F(Y) \to F(X))$.

Our conclusion about $A(X)$ is that its derivative is (naturally equivalent to) the unreduced suspension spectrum of the loop space:

$$\partial_x A(X) \sim \sum^{\infty} (\Omega_x X_+).$$

Knowledge of the derivative of a functor becomes truly useful if the functor is *analytic*. This is explained in the sequel [4], where the ideas of Section 1 are extended from square to cubical diagrams. Diagrams in the shape of $(n+1)$-dimensional cubes yield the concepts *n-excisive* and *stably n-excisive*. *Analytic* means stably n-excisive for all n, more or less.

Analyticity has strong consequences. For example, with suitable analyticity the statement $\partial_x F(X) \sim *$ for all (X, x) implies that F is 'locally constant': $F(X)$ depends only on the ρ-homotopy type of X for some small number ρ. (This principle has been successfully applied when F is the fiber of a natural map between analytic functors. See [4] for references.)

As another example, an analytic functor has a 'Taylor series' in the following sense: To each map $Y \to X$ of spaces is associated a tower $\{(P_nF)(Y \to X)\}$. At the bottom is $P_0F \sim F(X)$, and if $Y \to X$ is $(\rho + 1)$-connected then the tower converges to $F(Y)$. As a functor of Y for fixed X, P_nF is n-excisive. The first layer (fiber of the map from P_1F to P_0F) is the differential D_XF defined here. In many cases the nth layer (the fiber of $P_nF \to P_{n-1}F$, which as a functor of Y is *homogeneous of degree n*) can be identified. In many interesting examples the 'radius of convergence' ρ is 1. For all of this see [4].

CONVENTIONS. A *space* is a compactly generated topological space and products and function spaces are formed in that category in the usual way. *Basepoints* are always nondegenerate unless we say otherwise. An *equivalence* of (based or unbased) spaces is a weak homotopy equivalence.

Homotopy limits and colimits of diagrams are defined as in [1], except that we work with diagrams of topological spaces rather than diagrams of simplicial sets. Thus, for a diagram $\alpha \mapsto X(\alpha)$ of spaces (i.e. a functor from a small category A to spaces) the hocolim is the realization of a simplicial space whose space of p-simplices is the coproduct, over all p-simplices $\alpha_0 \to \cdots \to \alpha_p$ in the nerve of A, of $X(\alpha_0)$. 'Coproduct' may be disjoint union or wedge, according as the spaces are unbased or based. The holim is the realization ('Tot') of a cosimplicial space whose space of p-simplices is the product over $\alpha_0 \to \cdots \to \alpha_p$ of $X(\alpha_p)$.

A *prespectrum* is a sequence of based spaces $\{C_i \mid i \geqslant 0\}$ and based maps $C_i \to \Omega C_{i+1}$. A map of prespectra is a collection of maps $C_i \to D_i$ strictly respecting the structure maps. A *spectrum* is a prespectrum in which the structure maps are equivalences. The associated spectrum of a prespectrum $\{C_i\}$ is $\{\mathrm{hocolim}_j\, \Omega^j C_{j+i}\}$. (Homotopy colimit is needed here instead of colimit because we have placed no special conditions on the structure maps of a prespectrum.) An *equivalence* of spectra is a map which gives an equivalence of spaces for each i. An equivalence of prespectra is a map which induces an equivalence of the associated spectra.

It will sometimes happen, when trying to make a (pre)spectrum out of some based spaces C_j, that instead of maps $C_j \to \Omega C_{j+1}$ we only have diagrams

$$C_j \to D_j \xleftarrow{\simeq} E_j \to \Omega C_{j+1},$$

in which the backwards map is an equivalence. This situation can be rectified in the following way:

(0.1) Let F_j be the homotopy limit (not colimit) of the diagram:

$$C_j \to D_j \xleftarrow{\simeq} E_j \to \Omega C_{j+1} \to \Omega D_{j+1} \xleftarrow{\simeq} \Omega E_{j+1} \to \Omega^2 C_{j+2} \to \cdots.$$

Then F_j is equivalent to C_j and the $\{F_j\}$ form a prespectrum.

1. Differentiation of Functors

We will define the differential, at any space X, of any suitable functor $F : \mathscr{U} \to \mathscr{T}$ from spaces to based spaces. This will be a 'linear' functor $D_X F : \mathscr{U}_X \to \mathscr{T}$ from the category of spaces over X to the category of based spaces.

An object in $\mathscr{U}_X$ is a pair (Y, f) where $f: Y \to X$ is a map in $\mathscr{U}$. A map from (Y, f) to (Y', f') is a map $Y \to Y'$ in $\mathscr{U}$ whose composition with f' is f. When no confusion can result, we will sometimes omit to mention the structure map f, and just say 'Y is an object of $\mathscr{U}_X$' or 'Y is a space over X'.

We always assume that F is a homotopy functor in the sense that it takes (weak homotopy) equivalences to (weak homotopy) equivalences.

1.1. DEFINITION. A functor $L : \mathscr{U}_X \to \mathscr{T}$ is *linear* if it satisfies the following three conditions:

(i) It is a *homotopy functor*. This means that it preserves equivalences. By definition, a map $(Y,f) \to (Y',f')$ in $\mathscr{U}_X$ is an equivalence if the underlying map $Y \to Y'$ of spaces is an equivalence.

(ii) It is *excisive*. This means that F takes every co-Cartesian square diagram in $\mathscr{U}_X$ to a Cartesian square diagram in $\mathscr{T}$.

(iii) It is *reduced*. This means that it 'vanishes' at the final object $X = (X, 1 : X \to X) \in \mathscr{U}_X$, in the sense that $L(X)$ is a (weakly) contractible space.

(Co)Cartesian is short for homotopy (co)Cartesian. That is, these terms are used here in the following sense:

1.2. DEFINITION. A (commutative) square diagram of spaces

$$\begin{array}{ccc} Y & \to & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \to & Y_{12} \end{array}$$

is *Cartesian* if the associated map from Y to $\operatorname{holim}(Y_1 \to Y_{12} \leftarrow Y_2)$ is an equivalence. (The target, the 'homotopy fiber product' of Y_1 and Y_2 over Y_{12}, is the fiber product of the associated path fibrations.) The diagram is *co-Cartesian* if the associated map from $\operatorname{hocolim}(Y_1 \leftarrow Y \to Y_2)$ to Y_{12} is an equivalence. (The source is the union along Y of the mapping cylinders of $Y \to Y_1$ and $Y \to Y_2$.)

('Cartesian' implies that (for every basepoint in Y) the relative homotopy groups of $Y \to Y_1$ map isomorphically to those of $Y_2 \to Y_{12}$. Similarly 'co-Cartesian' implies that the relative homology groups of $Y \to Y_1$ map isomorphically to those of $Y_2 \to Y_{12}$.)

A square diagram of spaces over X is called (co)Cartesian if the underlying diagram of spaces is (co)Cartesian.

In the case $X = *$, so that $\mathscr{U}_X = \mathscr{U}$, it is well known that a linear functor $L : \mathscr{U} \to \mathscr{T}$ gives rise to a (generalized, reduced) homology theory. Indeed, for any space Y there is a co-Cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & CY \\ \downarrow & & \downarrow \\ CY & \longrightarrow & SY \end{array}$$

where CY and SY are the (unreduced) cone and suspension. By 1.1(ii) this yields a Cartesian diagram

$$\begin{array}{ccc} L(Y) & \longrightarrow & L(CY) \\ \downarrow & & \downarrow \\ L(CY) & \to & L(SY) \end{array}$$

Since $L(CY)$ is contractible by 1.1(i) and 1.1(iii), this makes $L(Y)$ equivalent to $\Omega L(SY)$. In particular the homotopy groups $h_* Y = \pi_{*+i} L(S^i Y)$ are independent of i as soon as they are defined, i.e. when $i > -*$. The axioms for homology follow easily from 1.1.

We will not need the classical fact that every homology theory arises from some linear functor of spaces. (This can be deduced easily from the theorem of G. Whitehead [11] which says that every homology theory on based finite CW complexes has the form $h_i(Y) = \pi_i(\mathbf{C} \wedge Y)$ for some spectrum $\mathbf{C}$.) We will, however, need some (well known) ideas related to this fact:

1.3. *Remarks* (on spectra and linear functors):

(i) A linear functor L from spaces to based spaces naturally takes values in infinite loopspaces. More precisely $L(Y)$ is naturally equivalent to the zeroth space in a spectrum $\mathbf{L}(Y)$ whose jth space $L_j(Y)$ is equivalent to $L(S^j Y)$. ($S^j Y$ is the j-fold unreduced suspension of Y. We need 0.1 here, because there is not literally a map from $L(Y)$ to $\Omega L(SY)$.) The functor L_j is again linear from spaces to based spaces, and $\mathbf{L}$ is linear from spaces to spectra with the obvious extension of Definition 1.1. (A square diagram of spectra is called Cartesian if it consists of Cartesian square diagrams of spaces.)

(ii) Conversely any linear functor $\mathbf{L} = \{L_j\}$ from spaces to spectra can be recovered from the functor L_0 as in (i). That is, the spectrum $\{L_j(Y)\}$ and the spectrum $\{L_0(S^i Y)\}$ are related by a (chain of) natural (weak) equivalence(s). This is proved using the bispectrum $\{L_j(S^i Y)\}$.

(iii) A linear functor L from spaces to based spaces is determined up to equivalence, at least on finite complexes, by its *coefficient spectrum* $\{L(S^j)\}$. This is a consequence of the corresponding fact about spectrum-valued functors, namely:

(iv) A linear functor $\mathbf{L}$ from spaces to spectra is determined, on finite complexes Y, by the spectrum $\mathbf{L}(S^0)$. In fact, (a) any spectrum $\mathbf{C}$ determines a linear functor $\mathbf{L}_\mathbf{C}$, and (b) in the case $\mathbf{C} = \mathbf{L}(S^0)$ there is a comparison map $\mathbf{L}_\mathbf{C} \to \mathbf{L}$. Here are the essential details:

(a) Given $\mathbf{C}$ and a based set Y, let $\mathbf{L}_\mathbf{C}(Y)$ be the spectrum associated to the pre-spectrum $\{C_j \wedge Y\}$, namely $\{\text{hocolim } \Omega^i(C_{i+j} \wedge Y)\}$. Extend this construction to based simplicial sets Y by $\mathbf{L}_\mathbf{C}(Y) = |[l] \mapsto \mathbf{L}_\mathbf{C}(Y_l)|$. Extend it to the unbased case by redefining $\mathbf{L}_\mathbf{C}(Y)$ to be what was formerly the homotopy fiber of $\mathbf{L}_\mathbf{C}(Y_+) \to \mathbf{L}_\mathbf{C}(S^0)$. (In the based case, this only changes $\mathbf{L}_\mathbf{C}(Y)$ by a natural equivalence.) Define it on spaces by evaluating $\mathbf{L}_\mathbf{C}$ on the singular complex. (In the case of the realization of a simplicial set, this only changes it by an equivalence.)

(b) Now let $\mathbf{C}$ be $\mathbf{L}(S^0)$. For a based set Y there is an obvious map of prespectra

$$C_j \wedge Y = L_j(S^0) \wedge Y \to L_j(Y)/L_j(*) \xleftarrow{\sim} L_j(Y).$$

Passing to the associated spectra yields a (chain of) map(s) $\mathbf{L}_\mathbf{C}(Y) \to \mathbf{L}(Y)$ when Y is a based set, and up to natural equivalence this extends to the case of an unbased space. By construction, the map is an equivalence when $Y = S^0$. The same conclusion now follows when Y is a finite simplicial set (or a finite CW complex), by a Mayer–Vietoris argument using the linearity of the two functors. (A map between two Cartesian squares of *spectra* must be an equivalence in all four corners of the square if it is so in any three. This is not quite true for spaces.)

(v) It often happens that the particular linear functor L under consideration satisfies the *limit axiom*: For every CW complex Y the direct limit of $\pi_* L(Y')$ over finite subcomplexes $Y' \subset Y$ is $\pi_* L(Y)$. In this case, (iii) or (iv) is valid for all spaces.

A linear functor $L : \mathscr{U}_X \to \mathscr{T}$ can be thought of as a homology theory on spaces over X. If (Y, f) is a space over X, then the diagram of spaces over X

$$(1.4)\qquad \begin{array}{ccc} Y & \longrightarrow & C_X Y \\ \downarrow & & \downarrow \\ C_X Y & \longrightarrow & S_X Y \end{array}$$

is co-Cartesian, where the *fiberwise cone* $C_X Y$ is the mapping cylinder of f and the *fiberwise suspension* $S_X Y$ is the union of two such cones along X. Thus, again LY is naturally equivalent to $\Omega L(S_X Y)$ and the groups $\pi_{*+i} L(S_X^i Y)$, $* > -i$, satisfy analogues of the usual axioms. (Of course, 1.1(i) yields a very strong 'homotopy axiom', since the notion of equivalence in $\mathscr{U}_X$ is so weak. In particular, up to isomorphism the homology groups of Y depend only on the homotopy class of $f : Y \to X$.)

The most fundamental example of a linear functor is the one corresponding to stable homotopy. For a based space Y we write as usual $Q(Y)$ for the limit of $\Omega^i(S^i \wedge Y)$ and $\Sigma^\infty Y$ for the corresponding spectrum. For an unbased space Y we write Y_+ for Y with a disjoint basepoint added, and we sometimes write $Q_+(Y)$ for $Q(Y_+)$ and $\Sigma_+^\infty Y$ for $\Sigma^\infty(Y_+)$. Thus in the language used here Q_+ is an excisive homotopy functor from $\mathscr{U}$ to $\mathscr{T}$ which when reduced yields the linear functor $Y \mapsto \text{fiber}(Q(Y) \to Q(*))$, whose coefficient spectrum is the sphere spectrum $\Sigma^\infty S^0 = \Sigma_+^\infty(*)$.

More generally there is the following class of linear functors:

1.5. CONSTRUCTION. Let $E \to X$ be a fibration. If Y is a space over X, write $E \times_X Y$ for the fiber product. Then $Y \mapsto Q_+(E \times_X Y)$ is an excisive homotopy functor from $\mathscr{U}_X$ to $\mathscr{T}$, so it yields a linear functor $Y \mapsto \text{fiber}(Q_+(E \times_X Y) \to Q_+(E))$.

1.6. *Remark.* Parts (i) and (ii) of 1.3 apply equally well to functors defined on spaces over X. A linear functor L from $\mathscr{U}_X$ to $\mathscr{T}$ has a *coefficient spectrum* for each point x in X, namely $\{L(X \vee_x S^i)\}$. (In 1.5, the coefficient spectrum is $\mathbf{Q}(E_x)$, where E_x is the fiber of $E \to X$.) This governs the behavior of L on spaces over X of the form $Y = X \vee_x Z$, where the structure map $Y \to X$ is the retraction sending Z to x. For different points x in the same path-component of X the coefficient spectra are equivalent, but the equivalence depends on a choice of path. Informally, L corresponds to a locally trivial coefficient system of spectra over X (which may be globally nontrivial even if X is 1-connected). We will not need to make this precise. We will, however, need that in a weak sense any linear functor is determined by its coefficient spectra:

1.7. PROPOSITION. *Assume that a map $L \to M$ between two linear functors from $\mathscr{U}_X$ to $\mathscr{T}$ induces an equivalence of coefficient spectra for each point in X. If L and M satisfy the limit axiom, then $L(Y) \to M(Y)$ is an equivalence for every Y over X. (In general, it is an equivalence for every Y which satisfies a suitable finiteness condition, for example the following: the map $Y \to X$ can be factored $Y \to X' \to X$ with (X', Y) a finite relative CW complex and $X' \to X$ an equivalence.)*

Proof. By 1.3(i) and 1.3(ii) we may work instead with the corresponding spectrum-valued functors $\mathbf{L}$ and $\mathbf{M}$. Consider the homotopy fiber of $\mathbf{L}(Y) \to \mathbf{M}(Y)$. By assumption this functor of $Y \in \mathscr{U}_X$ vanishes (gives a contractible spectrum) when Y is $X \amalg \{\text{point}\}$. Since it is a homotopy functor it also vanishes when Y is $X \amalg \{\text{disk}\}$ (with any map from the disk to X). Being reduced, it vanishes when Y is X. A Mayer–Vietoris argument as in 1.3(iv.b) now implies that it vanishes, first, when Y is $X \amalg \{\text{sphere}\}$ (with any map from the sphere to X), and then whenever Y satisfies the finiteness condition. In the presence of the limit axiom this implies that it vanishes in the general case. □

For a functor $F: \mathscr{U} \to \mathscr{T}$ we can now give an informal definition of $D_X F$. Namely, restrict F to $\mathscr{U}_X$ and let $\Phi: \mathscr{U}_X \to \mathscr{T}$ be the reduced functor:

$$\Phi(Y) = \text{fiber}(F(Y) \to F(X)).$$

Define $(D_X F)(Y)$ as the direct limit of $\Omega^i \Phi(S^i_X Y)$. (The $(i+1)$st map in the limit system is made by evaluating the reduced functor $Y \mapsto \Omega^i \Phi(S^i_X Y)$ on 1.4. In trying to say this carefully, one would need 0.1; the actual definition of $D_X F$ will avoid that difficulty.)

In order that this process should actually yield an excisive functor in the limit, it seems necessary to make some hypothesis about F. The following mild one will suffice; it is satisfied in every interesting example.

1.8. HYPOTHESIS (*stable excision*). F has the following property for some integers c and κ:

$E(c, \kappa)$: If $\mathscr{Y}$ is any co-Cartesian square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_{12} \end{tikzcd}$$

of spaces in which the map $Y \to Y_i$ is k_i-connected and $k_i \geqslant \kappa$ for $i = 1, 2$, then the square diagram $F(\mathscr{Y})$ is $(k_1 + k_2 - c)$-Cartesian. That is, the resulting map $F(Y) \to \text{holim}(F(Y_1) \to F(Y_{12}) \leftarrow F(Y_2))$ is (at least) $(k_1 + k_2 - c)$-connected.

1.9. EXAMPLES

(i) The Blakers–Massey triad connectivity theorem says that the identity functor from spaces to spaces satisfies $E(1, \kappa)$ for any κ.

(ii) Waldhausen's functor A satisfies $E(1, 2)$, according to ([7], 2.4).

(iii) The functor $X \mapsto F(X) = Q\,\mathrm{Map}(K, X)$ where K is a d-dimensional finite complex satisfies $E(2\dim(K), \kappa)$ for any κ. The proof uses the Blakers–Massey theorem. For details and a more general statement, see [4].

1.10. DEFINITION. If $\Phi: \mathscr{U}_X \to \mathscr{T}$ is a homotopy functor, then the functor $T\Phi: \mathscr{U}_X \to \mathscr{T}$ is given by

$$(T\Phi)(Y) = \mathrm{holim}(\Phi(C_X Y) \to \Phi(S_X Y) \leftarrow \Phi(C_X Y)),$$

and $t(\Phi): \Phi \to T\Phi$ is induced by 1.4. The functor $P\Phi: \mathscr{U}_X \to \mathscr{T}$ is defined by making a homotopy colimit of iterates of T:

$$(P\Phi)(Y) = \mathrm{hocolim}(\Phi(Y) \to (T\Phi)(Y) \to (TT\Phi)(Y) \to \cdots).$$

(The map from $T^i\Phi$ to $T^{i+1}\Phi$ in this diagram is $t(T^i\Phi)$.) The inclusion map $\Phi \to P\Phi$ is called $p(\Phi)$. The *linearization* of Φ is the functor $D\Phi: \mathscr{U}_X \to \mathscr{T}$ obtained by reducing $P\Phi$:

$$(D\Phi)(Y) = \mathrm{fiber}((P\Phi)(Y) \to (P\Phi)(X)).$$

Of course, if Φ is excisive then $t(\Phi)$ and $p(\Phi)$ are equivalences.

1.11. DEFINITION. Let $F: \mathscr{U} \to \mathscr{T}$ be a homotopy functor and $X \in \mathscr{U}$ a space. If $\Phi: \mathscr{U}_X \mapsto \mathscr{T}$ is the restriction of F (composition with the forgetful functor) then:

(i) The functor $P_X F = P\Phi: \mathscr{U}_X \to \mathscr{T}$ is the 1-*jet* of F at X,
(ii) The functor $D_X F = D\Phi: \mathscr{U}_X \to \mathscr{T}$ is the *differential* of F at X.

It is immediate from the definitions that $P_X F$ and $D_X F$ are again homotopy functors. Stable excision for F implies stable excision for Φ because the forgetful functor $\mathscr{U}_X \to \mathscr{U}$ preserves co-Cartesian squares and connectivity of maps. We will see that stable excision for Φ implies excision for $P_X F$ and $D_X F$.

1.12. REMARK. Let $\Phi \mapsto \bar{\Phi}$ denote the operation of reducing a functor, so that $\bar{\Phi}(Y) = \mathrm{fiber}(\Phi(Y) \to \Phi(X))$. We have defined $D\Phi$ as $\overline{P\Phi}$. It could just as well have been defined as $P\bar{\Phi}$. Indeed, the diagram

$$\begin{array}{ccc} (P\bar{\Phi})(Y) & \longrightarrow & (P\Phi)(Y) \\ \downarrow & & \downarrow \\ (P\bar{\Phi})(X) & \longrightarrow & (P\Phi)(X) \end{array}$$

is Cartesian and $(P\bar{\Phi})(X)$ is equivalent to a point.

Stable excision for Φ implies stable excision for $T\Phi$, but with the improved constants $c - 1$ and $\kappa - 1$. (This uses the fact that the fiberwise suspension functor

$S_X : \mathscr{U}_X \to \mathscr{U}_X$ preserves co-Cartesian squares and raises connectivity of maps by one.) Thus, $T^i\Phi$ satisfies $E(c-i, \kappa-i)$ and in the limit $P\Phi$ is actually excisive.

$E(c, \kappa)$ for Φ also immediately implies that the map $t(\Phi) : \Phi(Y) \to (T\Phi)(Y)$ is $(2k-c)$-connected whenever the structure map $Y \to X$ is k-connected and $k \geqslant \kappa$. This state of affairs deserves a name:

1.13. DEFINITION. Two functors Φ and Ψ from $\mathscr{U}_X$ to $\mathscr{T}$ *agree to first order* via a natural map $f : \Phi \to \Psi$ if f satisfies the following condition:

> $\mathcal{O}(c, \kappa)$: there are constants c and κ such that, whenever $Y \to X$ is k-connected and $k \geqslant \kappa$, then $f : \Phi(Y) \to \Psi(Y)$ is $(2k-c)$-connected.

$E(c-i, \kappa-i)$ for $T^i\Phi$ implies that the maps in the telescope are better and better: the map $t(T^i\Phi)$ from $T^i\Phi$ to $T^{i+1}\Phi$ satisfies $\mathcal{O}(c-i, \kappa-i)$. In particular, $p(\Phi)$ satisfies $\mathcal{O}(c, \kappa)$, because each $t(T^i\Phi)$ does so. We conclude:

1.14. PROPOSITION. *If Φ is stably excisive, then*

(i) *$P\Phi$ is excisive and $D\Phi$ is linear.*
(ii) *Φ agrees with $P\Phi$, and $\bar{\Phi}$ agrees with $D\Phi$, to first order.*

(To obtain the statement about $\bar{\Phi}$ just note that in the diagram

$$\begin{array}{ccc} \Phi(Y) & \longrightarrow & (P\Phi)(Y) \\ \downarrow & & \downarrow \\ \Phi(X) & \longrightarrow & (P\Phi)(X) \end{array}$$

the lower map is an equivalence.)

Of course as a special case of Proposition 1.14 we have:

1.15. PROPOSITION. *If $F : \mathscr{U} \to \mathscr{T}$ is any stably excisive homotopy functor, then for every space X:*

(i) *P_XF is excisive and D_XF is linear.*
(ii) *As functors of $Y \in \mathscr{U}_X$, $F(Y)$ agrees to first order with $(P_XF)(Y)$ and fiber $(F(Y) \to F(X))$ agrees to first order with $(D_XF)(Y)$.*

1.16. DEFINITION. For each point $x \in X$, the coefficient spectrum of D_XF at $x \in X$ is the *derivative* of F at (X, x), denoted $\partial_x F(X)$.

The general idea behind D_XF is that it should be the linear functor of $Y \in \mathscr{U}_X$ which 'best approximates' the fiber of $F(Y) \to F(X)$. Proposition 1.18 below will show that this is the case in two precise senses: D_XF is both the universal example of a linear functor of Y with a map from the fiber of $F(Y) \to F(X)$ and the unique example of a linear functor with which the latter agrees to first order.

The proof requires the following simple result:

1.17. PROPOSITION. *Suppose that two homotopy functors Φ and Ψ agree to first order ($\mathcal{O}(c, \kappa)$) via a map $g : \Phi \to \Psi$. Then*

(i) *$T\Phi$ and $T\Psi$ agree to first order via Tg, with the improved constants $c-1$ and $\kappa-1$.*
(ii) *$Pg: P\Phi \to P\Psi$ is an equivalence.*
(iii) *If Φ and Ψ are excisive, then g itself must have been an equivalence.*

Proof. For (i) use the diagram

$$\begin{tikzcd} \Phi(C_X Y) \arrow[r] \arrow[d] & \Phi(S_X Y) \arrow[d] & \Phi(C_X Y) \arrow[l] \arrow[d] \\ \Psi(C_X Y) \arrow[r] & \Psi(S_X Y) & \Psi(C_X Y) \arrow[l] \end{tikzcd}$$

noting that the first and third vertical maps are equivalences. ($g: \Phi(X) \to \Psi(X)$ must be an equivalence if g satisfies 1.13.)

(i) implies that $T^i g$ satisfies $\mathcal{O}_n(c-i, \kappa-i)$. This implies (ii).

(iii) follows from (ii) using the comment following 1.10. □

The statement of 1.18 involves a homotopy category. Let $h\mathcal{F}$ be the homotopy category associated to the category $\mathcal{F}$ of homotopy functors from $\mathcal{U}_X$ to $\mathcal{T}$. In $\mathcal{F}$, an object is any homotopy functor and a map is any natural transformation. Such a map is an equivalence if it gives an equivalence in $\mathcal{T}$ for each object of $\mathcal{U}_X$. The category $h\mathcal{F}$ has the same objects; it is obtained from $\mathcal{F}$ by formally inverting the equivalences. A map in $h\mathcal{F}$ is called a weak map (of homotopy functors).

Note that P is a functor from $\mathcal{F}$ to $\mathcal{F}$ and induces a functor from $h\mathcal{F}$ to $h\mathcal{F}$. Likewise $p(F)$ is a natural transformation between functors from $h\mathcal{F}$ to $h\mathcal{F}$, from the identity to P. Moreover, it makes sense to speak of two functors agreeing to first order via a weak map.

1.18. PROPOSITION. *Let $\Phi: \mathcal{U}_X \to \mathcal{T}$ satisfy stable excision, and suppose that $f: \Phi \to \Psi$ is a weak map to an excisive functor. Then*:

(i) *In $h\mathcal{F}$ there is a unique map $g: P\Phi \to \Psi$ such that f is the composition $g \circ p(\Phi)$.*
(ii) *If Φ agrees to first order with Ψ via f, then g is an equivalence (i.e. an isomorphism in $h\mathcal{F}$).*

Proof. For the existence of g in (i) use the diagram (in $h\mathcal{F}$)

$$\begin{tikzcd} \Phi \arrow[r, "f"] \arrow[d, "p(\Phi)"'] & \Psi \arrow[d, "p(\Psi)"] \\ P\Phi \arrow[r, "Pf"'] & P\Psi \end{tikzcd}$$

noting that $p(\Psi)$ is invertible. For uniqueness let g be any weak map from $P\Phi$ to Ψ and consider the diagram

$$\begin{tikzcd} \Phi \arrow[r, "p(\Phi)"] \arrow[d, "p(\Phi)"'] & P\Phi \arrow[r, "g"] \arrow[d, "p(P\Phi)"'] & \Psi \arrow[d, "p(\Psi)"] \\ P\Phi \arrow[r, "Pp(\Phi)"'] & PP\Phi \arrow[r, "Pg"'] & P\Psi \end{tikzcd}$$

The maps $p(P\Phi)$ and $p(\Psi)$ are invertible because $P\Phi$ and Ψ are excisive. In addition,

$Pp(\Phi)$ is invertible because Φ and $P\Phi$ agree to first order via $p(\Phi)$. Now since $g \circ p(\Phi)$ determines $P(g \circ p(\Phi)) = (Pg) \circ (Pp(g))$, which determines Pg, which determines g, the proof of (i) is complete.

(ii) follows easily from (i) and 1.17. □

Incidentally, it follows that the maps $p(P\Phi)$ and $Pp(\Phi)$ were equal (in the homotopy category) although we could not assert that until we had proved 1.18.

1.19. EXAMPLES

(i) The derivative of any excisive functor is a constant functor. In particular $\partial_x Q(X)$, the derivative of Q, is the sphere spectrum.

(ii) The derivative of $Q_+(X^n)$ is $\partial_x Q_+(X^n) \sim \Sigma_+^\infty(\{1, \ldots, n\} \times X^{n-1})$. To see this, write $\text{fiber}(Q_+((X \vee Z)^n) \to Q_+(X^n)) \sim Q((X \vee Z)^n/X^n)$ and note that as a functor of Z the latter agrees to first order with $Q((\cup_{1 \leqslant i \leqslant n} X^{i-1} \times Z \times X^{n-i})/X^n)$, a linear functor of Z which can be written $Q(\bigvee_{1 \leqslant i \leqslant n}(X_+^{i-1} \wedge Z \wedge X_+^{n-i})) \sim Q((\{1, \ldots, n\} \times X^{n-1})_+ \wedge Z)$.

(iii) More generally, if G is a subgroup of the symmetric group Σ_n, then $\partial_x Q_+(X^n \times_G EG) \sim \Sigma_+^\infty([\{1, \ldots, n\} \times X^{n-1}] \times_G EG)$.

If F is a functor to unbased spaces then the definitions of $D_X F$ and $\partial_x F(X)$ do not make sense (although that of $P_X F: \mathscr{U} \to \mathscr{U}$ does), because we cannot speak of *the* fiber of a map to $F(X)$. Thus, for example, the derivative of the identity functor $\mathscr{U} \to \mathscr{U}$ is undefined. However, we can speak of the derivative and differential of a functor F from $\mathscr{T}$ to $\mathscr{T}$ by a slight adaptation of our conventions. Namely, if $X = (X, \xi)$ is a based space then $D_X F = D_{(X,\xi)} F$ is a linear functor from $\mathscr{U}_X$ to $\mathscr{T}$ which depends on ξ, and $\partial_x F(X, \xi)$ is a spectrum which depends on both points ξ and x. (It is the coefficient spectrum of the linear functor $Z \mapsto (D_{(X,\xi)}F)(X \vee_x Z)$, where Z is joined to X at x.)

1.20. EXAMPLE. The differential of the identity functor $1: \mathscr{T} \to \mathscr{T}$ at a space (X, ξ) is, almost by definition, the functor

$$D_{(X,\xi)} 1 : (Y, f) \mapsto Q(\text{fiber}_\xi(f: Y \to X)).$$

This functor is an example of 1.5; the linear functor is determined by a space E fibered over X, namely the (contractible) space of all paths in X which end at the point ξ. In particular, for the derivative we have

$$\partial_x(X, \xi) \sim \sum_+^\infty P_{x,\xi}(X),$$

where $P_{x,\xi}(X)$ is the space of paths in X from x to ξ.

Curiously, our result (Corollary 3.3) about Waldhausen's $A(X)$ looks very similar to this. Namely, $D_X A$ is determined by the free loopspace $\Lambda X = \text{Map}(S^1, X)$, fibered over X by evaluation at a point in S^1, so that for the derivative we have

$$\partial_x A(X) \sim \sum_+^\infty \Omega_x(X).$$

2. Example: Stable Homotopy of Function Spaces

We now determine the differential (and in particular the derivative) of the functor $X \mapsto Q_+ \operatorname{Map}(K, X)$, where K is any finite complex and $\operatorname{Map}(K, X)$ is the space of continuous maps from K to X. A specific reason for paying special attention to this class of examples is that $Q_+ \operatorname{Map}(S^1, X)$ turns out to be closely related to algebraic K-theory. A more general reason is that the functors $Q_+ \operatorname{Map}(K, X)$ appear to play a central role in the 'calculus of functors', perhaps analogous to the role of the rational functions in ordinary calculus. (Functors like those in 1.19 are analogous to polynomials.)

The statement of the result will require a slightly more general method than 1.5 for writing down excisive functors on the category of spaces over X.

2.1. CONSTRUCTION. If $E \to K$ is a (Serre) fibration over a finite complex K, then for each $i \geqslant 0$ let E_i be the colimit (pushout) of

$$K \leftarrow \partial D^i \times E \to D^i \times E.$$

The evident map $E_i \to K$ is a fibration with a distinguished section. It is obtained from $E \to K$ by, fiberwise over K, adding a basepoint to E and suspending i times; its fiber over $k \in K$ is $S^i(E(k)_+)$, where $E(k)$ is the fiber of $E \to K$. (We had to use unreduced suspension here to guarantee a fibration.) Let $\Gamma_{i,K}(E)$ be the based space of all sections of $E_i \to K$. These spaces form a prespectrum (after rectification by 0.1). Let $\mathbf{\Gamma}_K(E)$ be the associated spectrum and let $\Gamma_K(E)$ be its zeroth space, the homotopy colimit over i of $\Omega^i \Gamma_{i,K}(E)$.

2.2. CONSTRUCTION. More generally, suppose X is a space, K a finite complex, and $E \to K \times X$ a fibration. We define an excisive functor $Y \mapsto \Gamma_K(E_Y)$ from $\mathscr{U}_X$ to $\mathscr{T}$. Given $Y \to X$, let E_Y be the fiber product of E with $K \times Y$ over $K \times X$. Viewing E_Y as a space fibered over K, make $\Gamma_K(E_Y)$ as in 2.1.

In the case of a trivial bundle $E = Z \times K \times X$ this amounts to taking

$$\begin{aligned} \Gamma_{i,K}(E_Y) &= \operatorname{Map}(K, S^i((Z \times Y)_+)), \\ \Gamma_K(E_Y) &\sim \operatorname{hocolim} \operatorname{Map}(K, \Omega^i S^i((Z \times Y)_+)) \\ &\sim \operatorname{Map}(K, Q_+(Z \times Y)), \end{aligned}$$

where the last step uses the finiteness of K. In particular, $\Gamma_K(E_Y)$ is an excisive functor of $Y \in \mathscr{U}_X$ in this case.

To verify excision (of $\Gamma_K(E_Y)$ as a functor of Y) in general, note that construction 2.2 is contravariantly functorial over subcomplexes of K. Excision holds when K is a point, or more generally, a cell, and also trivially when K is empty. A Mayer–Vietoris argument over subcomplexes of K yields the general case.

The fibration to which we need to apply 2.2 is

$$\text{(2.3)} \quad \begin{aligned} E(K, X) = K \times \operatorname{Map}(K, X) &\to K \times X \\ (k, f) &\mapsto (k, f(k)). \end{aligned}$$

Write $\Theta_K(Y \to X) = \Gamma_K(E_Y(K, X)) = \operatorname{hocolim} \Omega^i \Gamma_K(E_{i,Y}(K, X))$. This is a functor of maps $Y \to X$ of spaces, and for fixed X it is an excisive functor of $Y \in \mathcal{U}_X$. Write $\Theta_K(X)$ for $\Theta_K(X \to X) = \Gamma_K(E(K, X))$. The reduced part

$$\bar{\Theta}_K(Y \to X) = \text{fiber}(\Theta_K(Y \to X) \to \Theta_K(X))$$

is a linear functor of Y. Its coefficient spectrum at $x \in X$ is given, in the notation of 2.1, by $\Gamma_K(E_x(K, X))$, where $E_x(K, X)$ is the space

$$E_x(K, X) = \{(k, f) \in E(K, X) \mid f(k) = x\} \to K$$

fibered over K with fiber $\text{Map}(K, k; X, x)$.

2.4. THEOREM. *Let $F(X) = Q_+ \text{Map}(K, X)$, K a finite complex. Then the differential of F is $\bar{\Theta}_K$ above; there is a natural equivalence*

$$(D_X F)(Y \to X) \sim \bar{\Theta}_K(Y \to X).$$

In particular the derivative is given by

$$\partial_x Q \text{ Map}(K, X) \sim \Gamma_K(E_x(K, X))$$

2.5. COROLLARY. *Write ΛX for the free loopspace* $\text{Map}(S^1, X)$. *We have*

$$\begin{aligned}\partial_x Q_+ \Lambda X &\sim \text{Map}(S^1, \Sigma_+^\infty \Omega_x X)\\ &\sim (\Sigma_+^\infty \Omega_x X) \times (\Omega \Sigma_+^\infty \Omega_x X).\end{aligned}$$

Proof of 2.5. $E_x(S^1, X)$ is a trivial bundle over S^1.

Proof of 2.4. The plan is to write down a natural map from the fiber of $F(Y) \to F(X)$ to $\bar{\Theta}_K(Y \to X)$ and then to check that when Y is $X \vee S^j$ this map is approximately $2j$-connected.

There is a tautological map φ from $F(X)$ to $\Theta_K(X)$; $\text{Map}(K, X)$ is mapped into the space of sections of $K \times \text{Map}(K, X) \to K$ and $S^i(\text{Map}(K, X)_+)$ is mapped into $\Gamma_{i,K}(E(K, X))$. Now in the diagram

$$\begin{tikzcd}
F(Y) \arrow[r] \arrow[d, "\varphi"'] & F(X) \arrow[dd, "\varphi"] \\
\Theta_K(Y) \arrow[d] & \\
\Theta_K(Y \to X) \arrow[r] & \Theta_K(X)
\end{tikzcd}$$

compare homotopy fibers of horizontal maps to obtain a natural map

$$\text{fiber}(F(Y) \longrightarrow F(X)) \xrightarrow{\bar{\varphi}} \bar{\Theta}_K(Y \to X).$$

Linearize both sides in the sense of 1.10. Since the right-hand side is already linear this yields a (chain of) natural map(s)

$$(D_X F)(Y) \to \bar{\Theta}_K(Y \to X).$$

To show that this is an equivalence it will be enough, by 1.7, to compare coefficient spectra. This means proving:

2.6. LEMMA. *When Y is $X \vee S^j$, then $\bar{\varphi}$ is $(2j-c)$-connected for some constant c which depends only on K.*

Proof. Probably any proof of this would use something like configuration spaces; this proof uses framed bordism.

Without loss of generality K is a smooth compact framed manifold with boundary, because changing K by a homotopy equivalence only changes F and Θ_K by homotopy equivalences. The constant c will be related to $d = \dim(K)$. We take K to be framed in the strong sense that its tangent bundle has a chosen trivialization (not just stably).

The homotopy groups of fiber$(F(X \vee S^j) \to F(X))$ are the relative stable homotopy groups of the pair $(\mathrm{Map}(K, X \vee S^j), \mathrm{Map}(K, X))$. An element of π_{i+j} corresponds to a bordism class of pairs (M, h), where M^{i+j} is a smooth framed compact manifold with boundary and h is a continuous map of pairs

$$(M \times K, \partial M \times K) \to (X \vee S^j, X).$$

Let $0 \in S^j$ be a point different from the basepoint. We may assume that h is smooth near $h^{-1}(0)$ and that both h and its restriction to $M \times \partial K$ are transverse to 0. Then the set $W = h^{-1}(0)$ is a smooth compact framed manifold of dimension $i + d$, with boundary $\partial W = W \cap (M \times \partial K)$. It is equipped with a map of pairs $s : (W, \partial W) \to (E_x, \partial E_x)$, where E_x is above and we have written

$$\partial E_x = E_x \cap (\partial K \times \mathrm{Map}(K, X)).$$

Namely, for $(m, k) \in W \subset M \times K$ define

$$s(m, k) = (k, s_m) \in E_x \subset K \times \mathrm{Map}(K, X),$$

$$s_m(k') = r(h(m, k')),$$

where r is the retraction $X \vee S^j \to X$. This construction gives a map of framed bordism groups

$$\bar{\varphi}_* : \mathscr{F}r_{i+j}(\mathrm{Map}(K, X \vee S^j), \mathrm{Map}(K, X)) \to \mathscr{F}r_{i+d}(E_x, \partial E_x)$$

which we may identify with the effect of $\bar{\varphi}$ on π_{i+j}. Therefore, 2.6 will follow from:

2.7. CLAIM. *$\bar{\varphi}_*$ is surjective if $j \geqslant d$. It is injective if $j \geqslant d$ and $i < j - 2d$.*

Proof. For surjectivity, let (W, s) represent an element of the right-hand side. Thus $(W, \partial W)$ has a stable framing and $s : (W, \partial W) \to (E_x, \partial E_x)$ is a map of pairs. Write $s(w) = (t(w), u_w)$ and $u(k, w) = u_w(k)$, so that $t : (W, \partial W) \to (K, \partial K)$ is a map of pairs and for each $w \in W$ the map $u_w : K \to X$ sends $t(w)$ to x. We are free to alter t by any homotopy of pairs, since any such homotopy can be lifted to a homotopy of s. Choose t to be smooth, and also proper in the sense that ∂W is the transverse preimage of ∂K. Let $\bar{W}$ be W with an external boundary collar attached. Let Γ be the graph of t, a proper submanifold of $K \times \mathrm{int}(\bar{W})$. Choose a continuous function $u : K \times \bar{W} \to X$ extending the given $u : K \times W \to X$. Note that $u(k, w) = x$ for all

$(k, w) \in \Gamma$. Alter u by a homotopy fixed on Γ so as to make $u(k, w) = x$ for all (k, w) in a neighborhood of Γ, say $\mathcal{N}$. The restriction to $K \times \bar{W}$ still represents the same element of $\mathscr{F}r_{i+d}$ because s has only been altered by a homotopy of pairs again. Note that the normal bundle of Γ in $K \times \bar{W}$ is (unstably) trivial because the tangent bundle of K is trivial. We obtain a trivialization of the normal bundle of $\Gamma \times 0$ in $K \times \bar{W} \times D^{j-d}$. Use it to make a product tubular neighborhood $T \cong D^j \times \Gamma$ of $\Gamma \times 0$ in $K \times \bar{W} \times D^{j-d}$. Choose T to be small:

(2.8) $T \subset \mathcal{N} \times \operatorname{int}(D^{j-d})$.

The Thom–Pontryagin construction now yields a map

$$K \times \bar{W} \times D^{j-d} \to T/\partial T \cong S^j \wedge \Gamma_+ \to S^j.$$

This together with the map

$$K \times \bar{W} \times D^{j-d} \to K \times \bar{W} \xrightarrow{u} X$$

makes a map $h: K \times M \to X \times S^j$. The image of h lies in $X \vee S^j$ by 2.8. Putting $M = \bar{W} \times D^{j-d}$, we have a map of pairs

$$h: (K \times M, K \times \partial M) \to (X \vee S^j, X).$$

It is easy to see that the class of h is sent to the class of (W, s) by $\bar{\varphi}_*$. (In particular, the transverse preimage $h^{-1}(0)$ is $\Gamma \times 0 \cong W$.)

The main step in proving injectivity is to show that every element of $\mathscr{F}r_{i+j}(\operatorname{Map}(K, X \vee S^j), \operatorname{Map}(K, X))$ has a representative of the kind which was constructed during the proof of surjectivity. Any element is represented by some framed $(M^{i+j}, \partial M)$ and some map

$$h: (K \times M, K \times \partial M) \to (X \vee S^j, X)$$

with h transverse to $0 \in S^j$. Consider the manifold

$$\Gamma^{i+d} = h^{-1}(0) \subset K \times \operatorname{int}(M).$$

Using the dimensional hypothesis $i + j > 2(i + d)$, arrange (by an isotopy of $K \times M$) that the projection $K \times M \to M$ embeds Γ in $\operatorname{int}(M)$. Let $W \subset \operatorname{int}(M)$ be the image of Γ. Use the hypothesis again to see that every stable trivialization of the relative normal bundle of W in M exists unstably. In particular, this is so for the stable trivialization which results from comparing the given framing of M with the stable trivialization of the tangent bundle Γ which arises from the definition of Γ as a transverse preimage. Attach a collar to W in M and call the result $\bar{W}$. Extend to $\bar{W}$ the trivialization of the normal bundle of W. Use this to make a product tube $M_0 \cong \bar{W} \times D^{j-d}$ for $\bar{W}$ in M. For a small disk D_ϵ about 0 in S^j the preimage $h^{-1}(D_\epsilon)$ will be a product tube for $\Gamma \times 0$ in $K \times M_0$, say $T \cong \Gamma \times D^j$. By a homotopy of h which is fixed near $h^{-1}(0)$ we can arrange for $\operatorname{int}(T)$ to be $h^{-1}(S^j - *)$ rather than $h^{-1}(\operatorname{int}(D_\epsilon))$. Now since h sends all of $K \times (M - M_0)$ to X we can replace M by M_0 and still have a representative for the same bordism class.

If we write M for M_0, then the new representative is of the familiar special kind: $M = \bar{W} \times D^{j-d}$, where $\bar{W} = W \cup [\text{collar}]$ and h is given by the Thom–Pontryagin construction on a product tube around the graph of a proper map $(W, \partial W) \to (K, \partial K)$.

The rest of the injectivity proof is a copy of the surjectivity proof. As we used a W to construct an M, we can use a null-cobordism of W to construct a null-cobordism of M.

This completes the proof of 2.7, 2.6, and 2.4. □

3. The Derivative of Pseudoisotopy Theory

We now identify the derivative of the functor $X \mapsto \mathscr{P}(X)$, stable smooth pseudoisotopy (=concordance) theory. As a corollary we also identify the derivative of $X \mapsto A(X)$, the algebraic K-theory of spaces. The answers are related to the free loopspace $\Lambda X = \mathrm{Map}(S^1, X)$, considered as a fibration over X by evaluation at a point in S^1.

For any map $Y \to X$ of spaces let $\Lambda(Y \to X)$ be the fiber product of Y with ΛX over X. Thus $Q_+\Lambda(Y \to X)$ is excisive as a functor of $Y \in \mathscr{U}_X$. The reduced part of this (a linear functor of $Y \in \mathscr{U}_X$) turns out to be the differential $D_X A$. Its coefficient spectrum at $x \in X$ is $\Sigma^\infty_+(\Omega_x X)$.

The formula for $D_X\mathscr{P}$ looks a bit more complicated; we have to take away the constant loops and shift by two dimensions. Let $\Psi(Y \to X)$ be $\Omega^2 Q(\Lambda(Y \to X)/Y)$, where $\Lambda(Y \to X)/Y$ is the cofiber of the inclusion

(3.1) $\quad Y \to \Lambda(Y \to X)$

of the space of constant loops. Again Ψ is an excisive functor of Y. Write $\Psi(X)$ for $\Psi(X \to X)$ and $\bar{\Psi}(Y \to X)$ for the reduced part:

$$\bar{\Psi}(Y \to X) = \mathrm{fiber}(\Psi(Y \to X) \to \Psi(X)).$$

3.2. THEOREM. *The differential $D_X\mathscr{P}$ is (naturally equivalent to) the linear functor $Y \mapsto \bar{\Psi}(Y \to X)$ defined above. In particular, the derivative is given by $\partial_x\mathscr{P}(X) \sim \Omega^2\Sigma^\infty(\Omega_x X))$.*

3.3. COROLLARY. *The differential $D_X A$ is the reduced part of the excisive functor $Y \mapsto Q\Lambda(Y \to X)$. In particular, the derivative is given by*

$$\partial_x A(X) \sim \sum_+^\infty (\Omega_x X).$$

Thus there is a natural map

$$\mathrm{fiber}(A(Y) \to A(X)) \to \mathrm{fiber}(Q_+\Lambda(Y \to X) \to Q_x \Lambda X)$$

which is $(2k-1)$-connected if $Y \to X$ is k-connected and $k \geqslant 2$.

Proof of 3.3. *from* 3.2. By a theorem of Waldhausen [10], there is a homotopy functor $\mathrm{Wh} = \mathrm{Wh}^{\mathrm{diff}}$ related to $\mathscr{P}$ and A by natural equivalences

$$\Omega^2\mathrm{Wh}(X) \sim \mathscr{P}(X), \qquad A(X) \sim Q_+(X) \times \mathrm{Wh}(X).$$

It follows from 3.2 that $\Omega^2 D_X \mathrm{Wh}$ is the reduced part of

$$Y \mapsto \Omega^2 Q(\Lambda(Y \to X)/Y)$$

and this implies (by 1.3(i) and 1.3(ii)) that $D_X \mathrm{Wh}$ is the reduced part of

$$Y \mapsto Q(\Lambda(Y \to X)/Y).$$

Therefore, $D_X A$ is the reduced part of

$$Y \mapsto Q_+ Y \times Q(\Lambda(Y \to X)/Y) \sim Q_+ \Lambda(Y \to X).$$

(The last equivalence uses a retraction of 3.1.)

The last statement in 3.3 follows from 1.9(ii) and 1.15(ii). □

3.4. *Remarks.* Of course stable excision for A implies the same for Wh and $\mathscr{P}$. There is also a direct proof, due to Morlet, of stable excision for $\mathscr{P}$; see 3.9 below.

Recall the definition of pseudoisotopy:

3.5. DEFINITION. Let M be a smooth compact manifold (possibly with corners). A pseudoisotopy of M is a diffeomorphism F from $M \times I$ to itself such that $F(x, t) = (x, t)$ whenever x is near ∂M or t is 0. The space of all pseudoisotopies of M, with the C^∞ topology, is denoted $P(M)$. The same symbol denotes the simplicial model for $P(M)$ which is its 'smooth singular complex'. (A p-simplex is a fibered pseudoisotopy over Δ^p; see ([2], App. I).)

P is functorial with respect to codimension-zero embeddings, because a pseudoisotopy of $N \subset M$ can be extended from $N \times I$ to $M \times I$ by the identity. There is also a *suspension map*

$$P(M) \to P(M \times [-1, 1]).$$

(For example, see [6].) According to Igusa's stability theorem [6], the suspension is k-connected where k is roughly $\dim(M)/3$.

The direct limit with respect to suspension $\mathrm{hocolim}_n\, P(M \times [-1, 1]^n)$ is the *stable pseudoisotopy space.* It is rather easy to see that its homotopy type depends only on that of M. A better statement is that there is a homotopy functor $\mathscr{P}$ from spaces to spaces such that for compact manifolds M the spaces $\mathrm{hocolim}_n\, P(M \times [-1, 1]^n)$ and $\mathscr{P}(M)$ are equivalent. Such a functor $\mathscr{P}$ can be constructed by something like a left Kan extension. We use the following construction, described by Waldhausen in [9] (where he applies it not to P itself but to a double delooping of it): First fix $m \geqslant 0$. For a space X make the homotopy colimit of $P(M)$ where M runs through the category of m-dimensional parallelized smooth manifolds over X, with embeddings as the morphisms. Now $\mathscr{P}(X)$ is the homotopy colimit with respect to m.

The proof of 3.2 will follow the same overall pattern as the proof of 2.4. We will define a natural map $\tau : \mathscr{P}(X) \to \Psi(X)$. Then for every $Y \to X$ the diagram

$$(3.6)\qquad \begin{tikzcd} \mathscr{P}(Y) \arrow[r] \arrow[d, "\tau"'] & \mathscr{P}(X) \arrow[dd, "\tau"] \\ \Psi(Y) \arrow[d] & \\ \Psi(Y \to X) \arrow[r] & \Psi(X) \end{tikzcd}$$

will yield a map

$$\mathrm{fiber}(\mathscr{P}(Y) \to \mathscr{P}(X)) \xrightarrow{\bar{\tau}} \Psi(Y \to X) = \mathrm{fiber}(\Psi(Y \to X) \to \Psi(X)).$$

Finally, for certain k-connected maps $Y \to X$, we will verify that $\bar{\tau}$ is an equivalence in a stable range (roughly $2k$).

We begin with that last step, which is the geometric crux of the proof. We first identify the fiber of $P(N) \to P(M)$ in a stable range when $N \to M$ is a (very special) highly-connected codimension-zero embedding, and afterwards address the problem of fitting what we have learned into the framework described above. Let the m-dimensional compact manifold M be obtained from the (codimension-zero) submanifold N by attaching a handle of index $k \geqslant 3$. Let $H \subset M$ be the handle and let $D = D^{m-k} \subset H$ be its cocore, a proper disk in M of codimension k.

3.7. PROPOSITION. *The homotopy fiber of $P(N) \to P(M)$ is equivalent to $\Omega PE(D, M)$ where $PE(D, M)$ is the space of pseudoisotopy embeddings.*

Proof and definition. A point in $PE(D, M)$ is an embedding of $D \times I$ in $M \times I$ which carries $D \times 1$ into $M \times 1$ and which fixes pointwise both $D \times 0$ and a neighborhood of $(D \cap \partial M) \times I$. There is also the space $PG(D, M)$ of germs of pseudoisotopies; this is the direct limit, over neighborhoods U of D in M, of $PE(U, M)$. The restriction maps

$$\begin{tikzcd} PG(H, M) \arrow[r] \arrow[d] & PG(D, M) \arrow[d] \\ PE(H, M) \arrow[r] & PE(D, M) \end{tikzcd}$$

are clearly equivalences ([2], App. I). Finally, the restriction map $P(M) \to PG(H, M)$ is (simplicially, say) a fibration with fiber $P(N)$. (Actually some fibers might be empty in general, but $PE(D^{m-k}, M)$ is connected if $k \geqslant 3$. This is a theorem of Hudson [5]. It will also come out in the proof of 3.19 below.) □

Thus, our task is to identify the homotopy type, up to dimension roughly $2k$, of $PE(D, M)$ when $D \subset M$ is a proper disk of large codimension k. It turns out, by an argument of Morlet (used below in the proof of 3.19), that it is enough to do this when D is a point. The key to the argument is the following result. For the proof see [2] (but the statement there has $k_1 + k_2 - 5$; for the sharp version see [3]).

3.8. LEMMA (Morlet). *Let D_1 and D_2 be disjoint proper disks in M of codimensions k_1 and k_2. Assume $k_1 \geqslant 3$ and $k_2 \geqslant 3$. Then the inclusion map $PE(D_2, M - D_1) \to PE(D_2, M)$ is $(k_1 + k_2 - 4)$-connected.*

3.9. *Remark.* This 'disjunction lemma' has the following consequence: If M is obtained from N by attaching two disjoint handles H_1 and H_2 of indices $k_1 \geqslant 3$ and $k_2 \geqslant 3$, then the diagram

$$\begin{tikzcd} P(N) \arrow[r] \arrow[d] & P(N \cup H_1) \arrow[d] \\ P(N \cup H_2) \arrow[r] & P(M) \end{tikzcd}$$

is $(k_1 + k_2 - 5)$-Cartesian. Thus, the result can be viewed as a version of the statement that $\mathscr{P}$ satisfies stable excision. (It is a strong version, because it applies to P rather than just $\mathscr{P}$.)

We now examine $PE(x, M)$, where x is an interior point of M. We will find that a certain map

$$\sigma : PE(x, M) \to \Omega^2 Q \Sigma^m \Omega M$$

is $(2m - 5)$-connected. (ΩM means the space of loops based at x. Recall that m is the dimension of M.)

Before defining σ we describe the map

$$\sigma_* : \pi_i PE(x, M) \to \pi_i \Omega^2 Q \Sigma^m \Omega M$$

which it induces. The description uses transversality; the target of σ_* is the framed bordism group $\mathscr{F}r_{i+2-m}(\Omega M, x)$.

An element of $\pi_i PE(x, M)$ can be represented by a smooth map $F : D^i \times I \to M \times I$ such that for each $z \in D^i$ the map $F_z : t \mapsto F(z, t)$ is a smooth proper embedding of I in $M \times I$ sending 0 to $(x, 0)$, sending 1 into $M \times 1$, and in case $z \in \partial D^i$ sending t to (x, t) for all t. Write $F_z(t) = (f_z(t), g_z(t))$, so that

$$f_z : I \to M, \qquad g_z : I \to I.$$

We will say that the point $(s, t) \in I \times I$ is a *crossing* for F_z if one of the following conditions holds:

(3.10) $\quad s < t$ and $f_z(s) = f_z(t)$ and $g_z(s) > g_z(t)$ $\quad$ (*ordinary crossing*),
(3.11) $\quad s = t$ and $Df_z(t) = 0$ and $Dg_z(t) < 0$ $\quad$ (*infinitesimal crossing*).

Here $Df_z(t)$ and $Dg_z(t)$ are derivatives with respect to t. (They are a tangent vector of M and a number, respectively.) The representative (f, g) can always be chosen in such a way that any crossings which occur satisfy a general-position hypothesis:

3.12. HYPOTHESIS. At an ordinary crossing the map

$$(z, s, t) \quad \mapsto (f_z(s), f_z(t)),$$
$$D^i \times I \times I \to M \times M$$

is transverse to the diagonal, and at an infinitesimal crossing the map

$$(z, t) \mapsto Df_z(t),$$
$$D^i \times I \to TM$$

is transverse to the space of zero-vectors.

In that case the set of all (z, s, t) at which crossings occur is a compact $(i+2-m)$-manifold

$$W \subset D^i \times I^2$$

with boundary. Set $\Delta = \{(s, t) \mid 0 \leqslant s \leqslant t \leqslant 1\}$ and $\mathscr{E} = \{(t, t)\} \subset \Delta$; W is contained in $D^i \times \Delta$ and the only intersection of W with $\partial(D^i \times \Delta)$ is the set $\partial W = W \cap (D^i \times \mathscr{E})$ of infinitesimal crossings. W is (stably) framed, for the normal bundle of $W - \partial W$ in $D^i \times \Delta$ may be identified with the pullback of the tangent bundle of M by the map $(z, s, t) \mapsto f_z(s) = f_z(t)$, a map which is homotopic

(3.13) $\quad f_z(us), \quad 0 \leqslant u \leqslant 1$

to the constant function x.

To obtain a bordism class of the required kind we now need a map of pairs $(W, \partial W) \to (\Omega M, x)$. A point $(z, s, t) \in W$ determines a loop

(3.14) $\quad v \mapsto f_z(s + v(t-s))$

$$I \to M$$

and the loop is constant if $w \in \partial W$. The loop is based at $f_z(s) = f_z(t)$ rather than at x, but because of the homotopy 3.13 this is good enough. (Rather than being mapped into the actual fiber ΩM of $\Lambda M \to M$, W is mapped into the homotopy fiber, say $h\Omega M$. The subspace ∂W is mapped into the homotopy fiber hx of $M \to M$. The inclusion $(\Omega M, x) \to (h\Omega M, hx)$ is an equivalence of pairs.)

It is easy to see that we now have a well-defined map σ_*.

We next make a map of spaces σ inducing σ_*. The first step is to reformulate the definition of σ_* as follows: Write $\mathscr{M}$ for $PE(x, M)$ and think of this as an infinite-dimensional manifold. Inside $\mathscr{M} \times \Delta$ define the set of crossings $\mathscr{W}$ just as above (replacing the parameter space D^i by $\mathscr{M}$). Thus, $\mathscr{W}$ is a closed subset of $\mathscr{M} \times \Delta$ and is a submanifold of codimension m with trivial normal bundle. $\mathscr{W}$ has a boundary $\partial\mathscr{W} = \mathscr{W} \cap (\mathscr{M} \times \mathscr{E}) = \mathscr{W} \cap (\mathscr{M} \times \partial\Delta)$. The intersection of $\mathscr{W}$ with $\mathscr{M} \times \mathscr{E}$ is transverse. The image of the projection $\mathscr{W} \to \mathscr{M}$ does not contain the base point $\mu \in \mathscr{M}$ (which is the inclusion $x \times I \to M \times I$).

A map F as above corresponds to a smooth map $\varphi : (D^i, \partial D^i) \to (\mathscr{M}, \mu)$. Hypothesis 3.12 says that the maps $\varphi \times \mathrm{id}: D^i \times \Delta \to \mathscr{M} \times \Delta$ and $\varphi \times \mathrm{id}: D^i \times \mathscr{E} \to \mathscr{M} \times \mathscr{E}$ are transverse to $\mathscr{W}$ and $\partial\mathscr{W}$, respectively, and we have $(W, \partial W) = (\varphi \times \mathrm{id})^{-1}(\mathscr{W}, \partial\mathscr{W})$. It is clear that σ_* is a composition

$$\pi_i PE(x, M) \to \mathscr{F}r_{i+2-m}(\mathscr{W}, \partial\mathscr{W}) \to \mathscr{F}r_{i+2-m}(\Omega M, x),$$

where the first map is defined by transversality and the second is induced by a map of pairs $\omega : (\mathscr{W}, \partial\mathscr{W}) \to (h\Omega M, hx)$ defined by 3.13 and 3.14.

To define σ itself we must trade the transversality for a Thom space construction. This means identifying the Thom space $T(v)$ of the normal bundle v of $\mathscr{W}$ in $\mathscr{M} \times \Delta$ with the homotopy cofiber of the inclusion $(\mathscr{M} \times \Delta) - \mathscr{W} \to \mathscr{M} \times \Delta$, and similarly for the normal bundle of $\partial\mathscr{W}$ in $\mathscr{M} \times \mathscr{E}$. Writing $X//Y$ for homotopy cofiber of an inclusion $Y \to X$, we have:

$$T(v) \sim \mathscr{M} \times \Delta//\mathscr{M} \times \Delta - \mathscr{W},$$

$$T(v \mid \partial) \sim \mathscr{M} \times \mathscr{E}//\mathscr{M} \times \mathscr{E} - \partial\mathscr{W}.$$

(This step will be justified below.)

Define σ by letting its adjoint be the composition of the following maps:

$$\begin{array}{lll} (3.15)\ \text{(i)} & \Sigma^2\mathscr{M} & \to \Sigma^m(\mathscr{W}/\partial\mathscr{W}) \\ \quad\quad\ \ \text{(ii)} & & \to \Sigma^m\Omega M \\ \quad\quad\ \ \text{(iii)} & & \to Q\Sigma^m\Omega M. \end{array}$$

Here (iii) is the usual inclusion, (ii) is induced by ω, and (i) is the Thom–Pontryagin map, the quotient map

$$(\mathscr{M} \times \Delta)/(\mathscr{M} \times \mathscr{E} \cup \mu \times \Delta) \to T(v)/T(v \mid \partial).$$

3.16. LEMMA. *Let* $\mathscr{M} = PE(x, M), x \in \operatorname{int}(M), m = \dim(M)$. *Then the map* $\sigma : \mathscr{M} \to \Omega^2 Q\Sigma^m\Omega M$ *is* $(2m - 5)$*-connected.*

Proof. First of all, $\mathscr{M}$ is $(m-3)$-connected. To see this, let $\bar{\mathscr{M}}$ be the space of all smooth maps $F : (I, 0) \to (M \times I, x \times 0)$, not necessarily embeddings, transverse to $M \times \partial I$, such that $F^{-1}(M \times 0) = \{0\}$ and $F^{-1}(M \times 1) = \{1\}$. Thus $\bar{\mathscr{M}}$ is contractible and by a general-position argument the inclusion $\mathscr{M} \to \bar{\mathscr{M}}$ is $(m-2)$-connected. (When $i < m - 1$ an i-parameter family of maps $I \to M \times I$ is usually a family of embeddings.)

A consequence is that the double suspension map $\mathscr{M} \to \Omega^2\Sigma^2\mathscr{M}$ is $(2m-5)$-connected. To finish proving 3.16, we have to show that (i), (ii), and (iii) are all $(2m-3)$-connected.

Certainly (iii) is $(2m-1)$-connected.

(ii) is $(2m-2)$-connected if the map $\mathscr{W}/\partial\mathscr{W} \to h\Omega M/hx$ induced by ω is $(m-2)$-connected. Extend ω to a larger space $\bar{\mathscr{W}}/\partial\bar{\mathscr{W}}$ as follows: Let $\bar{\mathscr{W}} \subset \bar{\mathscr{M}} \times \Delta$ be defined like $\mathscr{W} \subset \mathscr{M} \times \Delta$. Thus, (F, s, t) belongs to $\bar{\mathscr{W}}$ if F has a crossing at (s, t) in the sense of 3.10 and 3.11. Let $\partial\bar{\mathscr{W}}$ be $\bar{\mathscr{W}} \cap \bar{\mathscr{M}} \times \mathscr{E}$. We have an obvious extension $\bar{\omega} : (\bar{\mathscr{W}}, \partial\bar{\mathscr{W}}) \to (h\Omega M, hx)$ of ω. The inclusion $\mathscr{W}/\partial\mathscr{W} \to \bar{\mathscr{W}}/\partial\bar{\mathscr{W}}$ is $(m-2)$-connected, because this is so for each of the inclusions $\mathscr{W} \to \bar{\mathscr{W}}$ and $\partial\mathscr{W} \to \partial\bar{\mathscr{W}}$ by a general-position argument.

The map $\bar{\omega}$ is an equivalence. To see this, factor it

$$(\bar{\mathscr{W}}, \partial\bar{\mathscr{W}}) \to (\mathscr{V}, \partial\mathscr{V}) \to (h\Omega M, hx)$$

$$(f, g, s, t) \mapsto (f, s, t).$$

Here $\mathcal{V}$ is the space of all triples (f, s, t) such that $f: (I, 0) \to (M, x)$ is smooth, $0 < s \leqslant t < 1$, and either

$s < t$ and $f(s) = f(t)$, in which case we say $(f, s, t) \in \mathcal{V} - \partial\mathcal{V}$, or

$s = t$ and $Df(t) = 0$, in which case we say $(f, s, t) \in \partial\mathcal{V}$.

The projections $\mathcal{W} - \partial\mathcal{W} \to \mathcal{V} - \partial\mathcal{V}$ and $\partial\mathcal{W} \to \partial\mathcal{V}$ are both fiber bundles with contractible fibers. The inclusions $\mathcal{W} - \partial\mathcal{W} \to \mathcal{W}$ and $\mathcal{V} - \partial\mathcal{V} \to \mathcal{V}$ are both equivalences. The maps $\mathcal{V} \to \Omega M$ and $\partial\mathcal{V} \to x$ are equivalences. It follows that $\mathcal{W}/\partial\mathcal{W} \to h\Omega M \sim M$ is an equivalence.

For (i) let $\mathcal{W}' \subset \mathcal{W}$ be the set of all (f, g, s, t) such that (s, t) is the only crossing of $F = (f, g)$, and such that the projection $\pi: \mathcal{M} \times \Delta \to \mathcal{M}$ when restricted to $\mathcal{W}$ is an injective immersion at that point. For an ordinary crossing this means that the vectors $Df(s)$ and $Df(t)$ are independent, and for an infinitesimal crossing it means that the vectors D^2f and D^3f are independent. (Strictly speaking D^2f is only defined because Df is zero, and D^3f is only defined modulo D^2f.) The set $\mathcal{W} - \mathcal{W}'$ is closed in $\mathcal{W}$, and the set $\pi(\mathcal{W} - \mathcal{W}')$ is closed in $\mathcal{M}$. Let $\mathcal{M}' = \pi\mathcal{W}' \cup (\mathcal{M} - \pi\mathcal{W})$ be its complement. Writing $\partial\mathcal{W}'$ for $\mathcal{W}' \cap \partial\mathcal{W}$, we have a diagram

$$\begin{array}{ccc} \Sigma^2\mathcal{M}' & \longrightarrow & \Sigma^m(\mathcal{W}'/\partial\mathcal{W}') \\ \downarrow & & \downarrow \\ \Sigma^2\mathcal{M} & \longrightarrow & \Sigma^m(\mathcal{W}/\partial\mathcal{W}) \end{array}$$

in which the lower map is 3.15(i). By general position the inclusion $\mathcal{M}' \to \mathcal{M}$ is $(2m-3)$-connected while $\mathcal{W}' \to \mathcal{W}$ and $\partial\mathcal{W}' \to \partial\mathcal{W}$ are $(m-3)$-connected. To finish proving the lemma we show that the upper map is an equivalence.

The map is the Thom–Pontryagin map for the submanifold-with-boundary $\mathcal{W}' \subset \mathcal{M}' \times \Delta$. Note that $\mathcal{W}'$ projects diffeomorphically to a submanifold $\pi\mathcal{W}' \subset \mathcal{M}'$. This is a closed subset of $\mathcal{M}'$, and it has a boundary although $\mathcal{M}'$ itself does not. It follows that the map can be identified with the double suspension of the quotient map $\mathcal{M}' \to \mathcal{M}'//(\mathcal{M}' - \pi\mathcal{W}')$. But $\mathcal{M}' - \pi\mathcal{W}' = \mathcal{M} - \pi\mathcal{W}$ (the space of all elements of $\mathcal{M}$ without crossings) is contractible: a deformation to the contractible subspace $\{(f, g) \in \mathcal{M} \mid \forall_t g(t) = t\}$ is given by the straight-line homotopy

$$\begin{aligned} I \times (\mathcal{M} - \pi\mathcal{W}) &\to \mathcal{M} - \pi\mathcal{W} \\ (v, (f, g)) &\mapsto (f, g_v) \\ g_v(t) &= (1-v)g(t) + vt. \end{aligned}$$

□

3.17. JUSTIFICATION (For the use of Thom spaces of normal bundles in an infinite-dimensional setting). Suppose $\mathcal{X}$ is an infinite-dimensional smooth manifold (in some sense which will not be made precise – the argument here applies in the cases we need). Let $\mathcal{Y} \subset \mathcal{X}$ be both a closed subset and a submanifold of finite codimension m. Let ν be the normal bundle. For simplicity assume that both $\mathcal{X}$ and $\mathcal{Y}$ are without boundary. We claim that the homotopy cofiber $\mathcal{X}//(\mathcal{X} - \mathcal{Y})$ is (weakly homotopy-) equivalent to the Thom space $T(v)$. In the finite-dimensional case, one proves this using tubular neighborhoods. We outline a proof in the general case using a combination of bordism and simplicial methods. Transversality shows that $\mathscr{X}//(\mathscr{X}-\mathscr{Y})$ and $T(v)$ have isomorphic homotopy groups: in each case π_p may be identified with the set of bordism classes of triples (K, f, Φ), $K \subset \mathbb{R}^p$ a $(p-m)$-dimensional closed manifold, $f: K \to \mathscr{Y}$ a continuous map, and $\Phi: f^*v \to v(K)$ a bundle isomorphism to the normal bundle of K in $\mathbb{R}^p$. Introduce a simplicial set $\mathscr{T}(v)$ as follows: A p-simplex is a compact manifold $K \subset \Delta^p$ of dimension $p-m$, transverse to all faces and corner sets, with f and Φ as above. (If $p < m$, there is only the basepoint simplex $K = \emptyset$.) This is a Kan complex, so its homotopy groups are the bordism sets described above. We construct an equivalence from a model for $\mathscr{X}//\mathscr{X} \to \mathscr{Y}$ to $\mathscr{T}(v)$. For $\mathscr{X}$ use the smooth singular simplices $\varphi: \Delta^p \to \mathscr{X}$ which are transverse to $\mathscr{Y}$ on Δ^p and on all of its faces and corner sets. There is a map $\varphi \mapsto \varphi^{-1}(\mathscr{Y})$ from this model of $\mathscr{X}$ to $\mathscr{T}(v)$; it takes the smooth singular complex of $\mathscr{Y}$ to the basepoint and so induces a map from (a model of) $\mathscr{X}//(\mathscr{X}-\mathscr{Y})$ to $\mathscr{T}(v)$. This is an equivalence by examining homotopy groups. The same construction, with $\mathscr{X}$ and $\mathscr{X}-\mathscr{Y}$ replaced by $T(v)$ and the complement of the zero-section, gives an equivalence $T(v) \to \mathscr{T}(v)$.

Lemma 3.16 gave a stable-range description of the fiber of $P(N) \to P(M)$ in the very special case when N was the complement of an interior m-disk in M. Before extending it to the more general case where M is the union of N with a handle of index $k \geqslant 3$, we define the map

$$\tau: P(M) \to \Psi(M) = \Omega^2\tilde{Q}(\Lambda M/M).$$

As with σ, the definition uses manifolds of crossings. By a *crossing* for $F = (f, g): M \times I \to M \times I$ we mean a crossing for the restriction of F to $x \times I$ for some $x \in M$. Thus, an ordinary crossing is a point $(x, s, t) \in M \times \Delta$ where $f(x, s) = f(x, t)$ and $g(x, s) > g(x, t)$; it occurs when there are two points $p = (x, s)$ and $q = (x, t)$ in $M \times I$ such that p is directly below q and $F(p)$ is directly above $F(q)$. An infinitesimal crossing occurs when at some point p the upward vertical tangent ray is mapped by DF to the downward vertical tangent ray.

Let $\mathscr{W} \subset P(M) \times M \times \Delta$ be the set of all (F, x, s, t) such that (x, s, t) is a crossing for F. This is a closed submanifold of codimension m, which meets the boundary only in its own boundary $\partial\mathscr{W} = \mathscr{W} \cap (P(M) \times M \times \mathscr{E})$, and which does not meet $\mathrm{id} \times M \times \Delta$ at all. The normal bundle of $\mathscr{W}$ is isomorphic to the tangent bundle of M (pulled back to $\mathscr{W}$ by the projection $\mathscr{W} \to M$).

As with σ, we first describe the effect of τ on homotopy groups. Represent an element of $\pi_i P(M)$ by a smooth map $\varphi: (D^i, \partial D^i) \to (P(M), \mathrm{id})$, and choose φ so that it satisfies:

3.18. HYPOTHESIS. $\varphi \times \mathrm{id}: D^i \times M \times \Delta \to P(M) \times M \times \Delta$ as well as $\varphi \times \mathrm{id}: D^i \times \mathscr{E} \to P(M) \times \mathscr{E}$ are transverse to $\mathscr{W}$ and $\partial\mathscr{W}$, respectively.

The preimage $W = (\varphi \times \mathrm{id})^{-1}(\mathscr{W})$ is then an $(i+2)$-dimensional smooth compact manifold with boundary and has a map $(W, \partial W) \to (\Lambda M, M)$ given by 3.13. It is stably framed, because its normal bundle in $D^i \times M \times \Delta$ is isomorphic to the tangent bundle of M (pulled back by the projection). Thus we have an element of the framed bordism group $\mathscr{F}r_{i+2}(\Lambda M, M)$.

To define τ itself we use a combination of cobordism and simplicial methods as before. For any space X the following *cobordism model* is equivalent to Q_+X:A p-simplex is a manifold P, smooth, compact, p-dimensional, with a continuous map to X and a smooth proper map to Δ^p. 'Proper' means transverse to the j-codimensional corner set for all j and such that the preimage of that set is the j-codimensional corner set of P. 'Framed' means stably framed, and two stable framings are considered the same if they are homotopic. This is a Kan complex, so that its homotopy groups are the framed bordism groups of X. Inside the total singular complex of Q_+X define a subcomplex by allowing only those maps $\varphi : \Delta^p \to \Omega^n\Sigma^n(X_+)$ which correspond to maps $\hat{\varphi} : \Sigma^n(\Delta^p_+) \to \Sigma^n(X_+)$ that are transverse to $0 \times X \subset \mathbb{R}^n \times X$ (in the sense that

$$\hat{\varphi}^{-1}(\mathbb{R}^n \times X) \to \mathbb{R}^n \times X \to \mathbb{R}^n$$

is transverse to 0 and in particular smooth in a neighborhood of $\hat{\varphi}^{-1}(0 \times X)$). We also require this for the restriction of φ to each corner set. This subcomplex has the homotopy type of the full complex, and of course it has a map $\varphi \mapsto \hat{\varphi}^{-1}(0)$ to the cobordism model. The map is an equivalence.

By a little variation on the construction above we can make a model for $Q(X//Y)$ when (X, Y) is a pair. One way to do it is to say that a p-simplex is a (smooth, framed, compact) manifold P with a proper map to $\Delta^p \times [0, 1)$ and a map of pairs (P, dP), where dP means preimage of $\Delta^p \times 0$. Another little variation gives a model for $\Omega^j Q(X/Y)$; just say that now $\dim(P) = p \times j$ for a p-simplex. We omit the details.

It is the cobordism model for $\Omega^2 Q(\Lambda M/M)$ which will be the target for a map from $P(M)$, or rather from a model for $P(M)$, namely the subcomplex of the smooth singular complex consisting of all those maps $\Delta^p \to P(M)$ which (together with their restrictions to corner sets) satisfy 3.18. With these models there is a map

$$\tau : P(M) \to \Omega^2 Q(\Lambda M/M) = \Psi(M)$$

defined by the same geometry which defined τ_*.

For any codimension-zero submanifold $N \subset M$ we have a diagram

$$\begin{tikzcd}
P(N) \arrow[r] \arrow[d, "\tau"'] & P(M) \arrow[dd, "\tau"] \\
\Psi(N) \arrow[d] & \\
\Psi(N \to M) \arrow[r] & \Psi(M)
\end{tikzcd}$$

and so a map $\bar{\tau}$ from $\bar{P}(N) = \text{fiber}(P(N) \to P(M))$ to $\Psi(M \to N)$.

3.19. LEMMA. *If M is obtained from N by attaching a handle of index $k \geqslant 3$, then the map $\bar{\tau}$ is $(2k-5)$-connected if $k < m$ and $(2k-6)$-connected if $k = m$.*

Proof. We use 3.16 in the case when $k = m$ and use the disjunction lemma to pass from $k+1$ to k.

First assume $k = m$. Call the handle (an interior m-disk) H. Let x be a point inside H. The following diagram commutes (up to sign):

$$\begin{array}{ccccccc}
\cdots \to \pi_{i+1}PE(x, M) & \to & \pi_i P(N) & \longrightarrow & \pi_i P(M) & \to & \pi_i PE(x, M) \to \cdots \\
 & & \big\downarrow{\scriptstyle \tau_*} & & & & \\
\big\downarrow{\scriptstyle \sigma_*} & & \pi_i \Psi(N) & & \big\downarrow{\scriptstyle \tau_*} & & \big\downarrow{\scriptstyle \sigma_*} \\
 & & \big\downarrow & & & & \\
\cdots \to \pi_{i+3} Q\Sigma^m \Omega M & \to & \pi_i \bar{\Psi}(N \to M) & \to & \pi_i \Psi(M) & \to & \pi_{i+2} Q\Sigma^m \Omega M \to \cdots
\end{array}$$

Here, the upper exact sequence comes from 3.7 and the lower one uses the excisiveness of $Y \mapsto \Psi(Y \to M)$:

$$\begin{aligned}
\bar{\Psi}(N \to M) &= \text{fiber}(\Psi(N \to M) \to \Psi(M \to M)) \\
&\sim \text{fiber}(\Psi(\partial H \to M) \to \Psi(H \to M)) \\
&\sim \text{fiber}(\Omega^2 Q(\partial H_+ \wedge \Omega M) \to \Omega^2 Q(H_+ \wedge \Omega M)) \\
&\sim \Omega^3 Q \Sigma^m \Omega M.
\end{aligned}$$

(The fact that the diagram commutes is checked using cobordism groups.) The conclusion follows from 3.16.

Now assume that the handle H has index $k < m$. We use an argument similar to the one in ([2], pp. 23–25). View M as N plus two k-handles H_1 and H_2, disjoint from each other, plus a $(k+1)$-handle H_0 (by writing the cocore of H, an $(m-k)$-disk, as the union of two half-disks along an $(m-k-1)$-disk). This leads to a square diagram

$$(3.20)\qquad \begin{array}{ccc}
PE(H, M) & \to & PE(H_0 \cup H_2, M) \\
\big\downarrow & & \big\downarrow \\
PE(H_0 \cup H_1, M) & \to & PE(H_0, M).
\end{array}$$

The maps are all fibrations. The upper right and lower left spaces are contractible by a simple geometric argument. The map between the fibers of the vertical maps is the inclusion

$$PE(H_2, N \cup H_2) \to PE(H_2, N \cup H_1 \cup H_2)$$

and so is $(2k-4)$-connected by 3.8.

This immediately yields a stable-range description of $PE(D^{m-k}, M)$, namely a $(2k-4)$-connected map $[2k-5$ if $k = m]$ from it to $\Omega^2 Q \Sigma^k \Omega M$. Indeed, we have a $(2k-4)$-connected map from $PE(D^{m-k}, M)$, the upper left space in 3.20, to $\Omega PE(D^{m-k-1}, M)$, the loopspace of the lower right space, and this provides a downward induction with respect to k. (In particular it proves that $PE(D^{m-k}, M)$

is connected, as mentioned in the proof of 3.7.) However, we need a statement involving τ and so we must say a few more words.

The square 3.20 is $(2k-4)$-Cartesian. The resulting $(2k-5)$-Cartesian square of loop spaces may be written

$$\begin{array}{ccc} \bar{P}(N) & \longrightarrow & \bar{P}(N \cup H_1) \\ \downarrow & & \downarrow \\ \bar{P}(N \cup H_2) & \to & \bar{P}(N \cup H_1 \cup H_2) \end{array}$$

by 3.7. Map it by $\bar{\tau}$ to the Cartesian square

$$\begin{array}{ccc} \bar{\Psi}(N \to M) & \to & \bar{\Psi}(N \cup H_1 \to M) \\ \downarrow & & \downarrow \\ \bar{\Psi}(N \cup H_2 \to M) & \to & \bar{\Psi}(N \cup H_1 \cup H_2 \to M). \end{array}$$

Again the upper right and lower left spaces are contractible. The inductive hypothesis that $\bar{\tau}$ in the lower right is $(2(k+1)-6)$-connected $[2(k+1)-5)$-connected if $k=m]$ implies that $\bar{\tau}$ in the upper left is $(2k-5)$-connected, as it is equivalent to the composition of $(2k-5)$-connected maps

$$\bar{P}(N) \to \Omega\bar{P}(N \cup H_1 \cup H_2) \to \Omega\bar{\Psi}(N \cup H_1 \cup H_2 \to M).$$

This completes the proof of 3.19. □

To prove 3.2, we must extend the definition of τ from unstable to stable pseudoisotopy theory. Observe that whenever we changed models for $P(M)$ or $\Psi(M)$ in constructing τ it was by equivalences natural with respect to codimension-zero embeddings, and τ is natural with respect to these. (This property of τ was needed already in the proof of 3.19.) Observe also that $\Psi(X)$ is defined for all spaces X and that the equivalence between it and the bordism model is natural for continuous maps.

$\mathscr{P}(X)$ was built in a certain way using the spaces $P(M)$, where M is a manifold over X. The construction required P to be functorial with respect to codimension-zero embeddings and also required suspension maps $P(M) \to P(M \times [-1, 1])$, natural with respect to such embeddings. Carry out the same construction now with Ψ in place of P, letting the 'suspension' $\Psi(M) \to \Psi(M \times [-1, 1])$ be the equivalence given by $x \mapsto x \times 0$. The resulting functor of spaces is just Ψ again, up to natural equivalence; this would be true for any homotopy functor from spaces to spaces which, like Ψ, satisfies a limit axiom on infinite complexes.

Thus $\tau : P(M) \to \Psi(M)$ leads to a new map from $\mathscr{P}(X)$ to $\Psi(X)$. Call it τ again. Diagram 3.6 is now defined. For any map $Y \to X$ it induces a map $\bar{\tau}$ from $(D_X\mathscr{P})(Y)$ to $\bar{\Psi}(Y \to X)$.

3.21. CLAIM. *If (X, Y) is a finite CW pair with $X = Y \cup e^k, k \geqslant 3$, then $\bar{\tau}$ is $(2k-5)$-connected.*

Proof. This is almost immediate from 3.19, using an equivalence of pairs $(M, N) \to (X, Y)$ where M and N are compact manifolds of some dimension and M is N with a k-handle attached. $\square$

By 1.7, 3.21 suffices to prove 3.2 if X is finite CW. The general case follows by a colimit argument. $\square$

Acknowledgements

Some of the writing of both this paper and [4] took place during visits to Edinburgh, Bielefeld, and MSRI. I am grateful to those institutions for their hospitality.

References

1. Bousfield, D. and Kan, D.: *Homotopy Limits, Localizations, and Completions*, Lecture Notes in Math. **304**, Springer (1972).
2. Burghelea, D., Lashof, R., and Rothenberg, M.: *Groups of Automorphisms of Manifolds*, Lecture Notes in Math. **473**, Springer (1975).
3. Goodwillie, T. G.: A multiple disjunction lemma for smooth concordance embeddings, to appear in *AMS Memoirs*.
4. Goodwillie, T. G.: Calculus II: Analytic functors, preprint.
5. Hudson, J. F. P.: Embeddings of bounded manifolds, *Proc. Camb. Phil. Soc.* **72** (1972), 11–25.
6. Igusa, K.: The stability theorem for smooth pseudoisotopies, *K-Theory* **2** (1988), 1–355.
7. Waldhausen, F.: Algebraic K-theory of topological spaces I, *Proc. Symp. Pure. Math.* **32**, Part 1, AMS (1978), 35–60.
8. Waldhausen, F.: Algebraic K-theory of topological spaces II, *Algebraic Topology, Aarhus 1978*, Lecture Notes in Math. **763**, Springer (1979), 356–394.
9. Waldhausen, F.: Algebraic K-theory of spaces, a manifold approach, *Canadian Math. Soc. Conf. Proc.* **2**, Part 1, AMS (1982) 141–184.
10. Waldhausen, F.: Algebraic K-theory of spaces, concordance, and stable homotopy, *Ann. of Math. Studies* **113** (1987), 392–417.
11. Whitehead, G.: Generalized homology theories, *Trans. AMS* **102** (1962), 227–283.