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Relative algebraic K-theory and cyclic homology

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Introduction

0.1. *The Theorem*. This paper is concerned with the algebraic K-theory and cyclic homology of simplicial rings. It relies heavily on two earlier papers [G1]

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and [G2]. The main result is the following:

MAIN THEOREM. *Suppose $f\colon R \to S$ is a homomorphism of simplicial rings such that the induced ring homomorphism $\pi_0 R \to \pi_0 S$ is a surjection with nilpotent kernel. Then rationally the relative algebraic K-theory is the same as the relative cyclic homology*:

$$K_n(f) \otimes \mathbf{Q} \cong HC_{n-1}(f) \otimes \mathbf{Q}.$$

A simplicial ring is a simplicial object in the category of (associative) rings (with 1). The K-theory $K_* R$ of a simplicial ring is defined as in [W1]. The relative K-groups $K_*(f)$ are defined so as to fit into an exact sequence

$$\dots \to K_* R \to K_* S \to K_* f \to K_{*-1} R \to \dots .$$

The cyclic homology $HC_* R$ of a simplicial ring is defined by a straightforward generalization of one of the usual definitions for ordinary (discrete) rings. The relative groups $HC_* f$ are defined so as to fit into an exact sequence

$$\dots \to HC_* R \to HC_* S \to HC_* f \to HC_{*-1} R \to \dots .$$

From a philosophical viewpoint the theorem may be compared with the result of Loday and Quillen [L-Q] and Tsygan [T] which asserts that, just as $K_*(R) \otimes \mathbf{Q}$ is the primitive part of the Hopf algebra $H_*(\mathrm{GL}(R); \mathbf{Q})$, $HC_{*-1}(R) \otimes \mathbf{Q}$ is the primitive part of the Hopf algebra $H_*(\mathfrak{gl}(R \otimes \mathbf{Q}); \mathbf{Q})$. From a practical viewpoint the theorem may be viewed as a computation in K-theory, since in general algebraic K-theory is much harder to compute than cyclic homology.

Even for those interested in rings (as opposed to simplicial rings) the theorem has some content, since any ring can be viewed as a constant simplicial ring. The theorem provides a rational computation of the relative K-theory associated with any nilpotent ideal.

Perhaps more interesting is the application to Waldhausen's algebraic K-theory of topological spaces (and hence to pseudoisotopy theory). For the simplicial group ring $\mathbf{Z}G$ of a simplicial group G, both algebraic K-theory and cyclic homology have geometric interpretations in terms of the classifying space $X = BG$. On the one hand $K_*\mathbf{Z}G$ is rationally isomorphic to $\pi_* A(X) \otimes \mathbf{Q}$ (see [W2], Corollary 2.3.8. and Theorem 2.2.1). On the other hand $HC_*\mathbf{Z}G$ is isomorphic to the homology of the space

$$\Lambda(X) \underset{\mathrm{SO}(2)}{\times} E\,\mathrm{SO}(2)$$

where $\Lambda(X)$ (the "free loop space" of X) is the space of all continuous maps $S^1 \to X$ and the group SO(2) acts on ΛX in the obvious way (see [G1], Theorem V.1.1. or [Bu5], Theorem A). This leads to the corollary below. One proves it first in the case when $X \to Y$ is the map $BG \to BH$ induced by a 1-connected homomorphism $G \to H$ of simplicial groups; then more generally for any 2-connected map $X \to Y$ of path-connected pointed spaces, using the natural weak equivalence between such a space and the classifying space of its Kan loop group, and finally in the general case by noting that both $A(X)$ and $H_*(\Lambda(X) \times_{SO(2)} E\,SO(2))$ are additive with respect to disjoint union.

COROLLARY. *If $X \to Y$ is any 2-connected map of spaces, then*

$$\pi_*(A(X) \to A(Y)) \otimes \mathbf{Q}$$
$$\cong H_{*-1}\Big(\Lambda(X) \underset{SO(2)}{\times} E\,SO(2) \to \Lambda(Y) \underset{SO(2)}{\times} E\,SO(2); \mathbf{Q}\Big).$$

0.2. *History.* For discrete rings it seems that Soulé [So] had the first result of this kind. He computed the rational relative K-theory of the map $\mathcal{O}_k[\varepsilon] \to \mathcal{O}_k$, where $\mathcal{O}_k$ is the ring of integers in a finite extension field k of $\mathbf{Q}$ and $\mathcal{O}_k[\varepsilon] = \mathcal{O}_k[x]/(x^2)$ is the ring of dual numbers.

For simplicial rings and applications to $A(X)$ the story begins with work of Dwyer, Hsiang, and Staffeldt [D-H-St1]. They considered the problem of computing the "reduced" rational Waldhausen K-theory $\pi_{*-1}\bar{A}(X) \otimes \mathbf{Q} \cong \pi_*(A(X) \to A(*)) \otimes \mathbf{Q}$ of a 1-connected space X, and succeeded in reducing this to an algebraic problem. They also had similar success with $\pi_*(A(X) \to A(K(\pi, 1))) \otimes \mathbf{Q}$, when $\pi = \pi_1 X$ was a finite group [D-H-St2]. Hsiang and Staffeldt [H-St1][H-St2] then solved the algebraic problem for many 1-connected spaces. Burghelea [Bu1], [Bu2], [Bu3] also had some overlapping results at about the same time.

Meanwhile some (unpublished) work of mine on relative pseudoisotopy theory, in which the subject was approached directly and not through $A(X)$ or any kind of K-theory, indicated that $\pi_*(A(X) \to A(Y)) \otimes \mathbf{Q}$ should be accessible to computation for *any* 2-connected map of spaces $X \to Y$, with no restriction on the fundamental group $\pi_1 X = \pi_1 Y$.

The subject received new impetus when Connes' cyclic homology appeared on the scene. It was clear to a number of people that this was a powerful new tool which should prove useful in organizing one's computations. Staffeldt [St1], generalizing the result of Soulé, proved that

$$K_*(R \to \mathcal{O}_k) \otimes \mathbf{Q} \cong HC_{*-1}(R \to \mathcal{O}_K) \otimes \mathbf{Q}$$

when R is a ring, $I \subset R$ a nilpotent ideal, and $\mathcal{O}_k \cong R/I$ the ring of integers in a number field. (More precisely he obtained a natural isomorphism of associated graded objects.) Burghelea [Bu4], [Bu5] proved the special case of our Main Theorem which is required for applications to $\pi_* \bar{A}(X) \otimes \mathbf{Q}$ of 1-connected spaces X, namely the case of (certain) maps $R \to \mathbf{Z}$ where R is a simplicial ring with $\pi_0 R \cong \mathbf{Z}$. At about the same time Staffeldt [St2] independently proved a closely related theorem: If X is 1-connected then $\pi_* \bar{A}(X) \otimes \mathbf{Q}$ is *dual* to $\overline{HC}_*(\mathscr{M})$ where $\mathscr{M}$ is a minimal model (or any model) for the rational homotopy type of X.

All of these relative K-theory computations applied only to certain special kinds of homomorphisms $R \to S$ of rings or simplicial rings. In particular S, or its component ring $\pi_* S$, was never allowed to be very different from $\mathbf{Z}$; it was always either a ring of algebraic integers or the group ring of a finite group. The reason for this restriction was that one needed to calculate the homology of the group $\mathrm{GL}(S)$ with coefficients in various representations, especially the adjoint representation $V = M(I \otimes \mathbf{Q})$ associated with an S-bimodule I, in terms of homology with constant coefficients. To do so one appealed to work of Borel which strongly used the arithmetic nature of S.

In our Main Theorem the restrictions on S are completely removed. Accordingly the most important new ingredient here is a new and more general method for computing $H_*(\mathrm{GL}(S); V)$. The method was introduced in [G2], where it was applied to adjoint representations only. In Section IV.3 of the present paper it is extended to tensor products and exterior powers of adjoint representations. (This more general case was not included in the paper [G2] because when that was being written we did not think we would need it here; see Remark 2 in 0.4 below.)

0.3. *The Proof.* Here is an outline of the proof of the Main Theorem.

In order to relate K_* to HC_{*-1} we introduce another series of groups HC_*^-. While $HC_*(R)$ is defined as the homology of the "positive half" of a certain 2-periodic double chain complex and $HP_*(R)$ ("periodic homology," denoted $HC_*^{\mathrm{per}}(R)$ in [G1]) is the homology of the entire double complex, $HC_*^-(R)$ is the homology of the negative half. The homology of the "axis" where the two halves meet is the Hochschild homology $H_*(R)$. These four kinds of homology groups fit together in a diagram (with exact rows)

$$(0.3.1)\qquad \begin{tikzcd} \arrow[r] & HC_{*-1} \arrow[r, "\beta"] \arrow[d, equal] & HC_*^- \arrow[r] \arrow[d, "\pi"] & HP_* \arrow[r] \arrow[d] & HC_{*-2} \arrow[r] \arrow[d, equal] & {} \\ \arrow[r] & HC_{*-1} \arrow[r, "B"] & H_* \arrow[r, "I"] & HC_* \arrow[r, "S"] & HC_{*-2} \arrow[r] & {} . \end{tikzcd}$$

Since HC_*^- is neither periodic (like HP_*) nor bounded below (like H_* and HC_*), it is in some ways the least appealing of these four homology functors.

Nevertheless it has its uses. In fact it comes equipped with natural maps

$$K_*(R) \xrightarrow{\alpha} HC_*^-(R) \xleftarrow{\beta} HC_{*-1}(R)$$

and similarly in the relative case with "f" instead of "R". We tensor with $\mathbf{Q}$ in the following sense:

$$K_*(f) \otimes \mathbf{Q} \xrightarrow{\alpha_{\mathbf{Q}}(f)} HC_*^-(f \otimes \mathbf{Q}) \xleftarrow{\beta_{\mathbf{Q}}(f)} HC_{*-1}(f \otimes \mathbf{Q})$$

(it is important to write "$\otimes \mathbf{Q}$" outside the bracket on the left and inside the bracket in the center; on the right side it may be written inside or outside). Then assuming that f satisfies the hypothesis of the Main Theorem we prove that $\alpha_{\mathbf{Q}}(f)$ and $\beta_{\mathbf{Q}}(f)$ are isomorphisms.

In the case of $\beta_{\mathbf{Q}}(f)$ this is relatively easy, given [G1]. The map β is part of the upper exact sequence in 0.3.1., so that an equivalent assertion is that $HP_*(f) = 0$ when f is a map of simplicial $\mathbf{Q}$-algebras satisfying the hypothesis of the Main Theorem. This follows easily from the results of [G1].

The case of $\alpha_{\mathbf{Q}}(f)$ is more involved. First α must be defined. We define α in Section II so as to make the diagram commute

$$\begin{tikzcd} & HC_*^- \arrow[d, "\pi"] \\ K_* \arrow[ur, "\alpha"] \arrow[r, "\tau"'] & H_* \end{tikzcd}$$

where π is as in 0.3.1 and τ is Dennis' trace map from K-theory to Hochschild homology [I]. The construction of α uses the method of acyclic models. (See, however, Remark 1 in 0.4 below.)

Having defined $\alpha_{\mathbf{Q}}(f)$, we prove that it is an isomorphism (assuming the hypothesis). The first step is to reduce (rather easily) to the special case of a quotient map $R \to R/I$ of simplicial rings with $I^2 = 0$. The second is to reduce further to the case of a *split* surjection of *discrete* rings $R \to R/I = S$ where $I^2 = 0$ and I is a *free* S-bimodule. This step uses a lemma (proved in I.2) to the effect that the relative K-theory of a square-zero simplicial ideal can be computed "dimensionwise". These two reductions are carried out in Section III.

Finally in Section IV we compute relative rational K-theory in this very special case and check that $\alpha_{\mathbf{Q}}(f)$ is an isomorphism. The computation uses the fact that (in this case) the rational homology of fiber $(\mathrm{BGL}(R)^+ \to \mathrm{BGL}(S)^+)$ is the same as the $\mathrm{GL}(S)$-coinvariant part of the rational homology of fiber $(\mathrm{BGL}(R) \to \mathrm{BGL}(S))$. This in turn uses the computation of $H_*(\mathrm{GL}(S);\ \Lambda^r M(I \otimes \mathbf{Q}))$.

0.4. *Miscellaneous Remarks.* (1) Hood and Jones ([J], [HJ]) have also defined a map from K_* to HC^-_*, by refining a construction of Connes and Karoubi [Kar]. They prove that is is multiplicative with respect to a product structure which they define on HC^-_*. It can be shown that their map (they call it a Chern character) coincides with our α; in effect, they have written down a natural chain map $a(G)$ satisfying our Lemma II.3.2 below.

(2) The Main Theorem, or even the main result of [G2], can be used to compute rational stable K-theory (see [Kas] for the definition). One obtains

$$K^S_*(R; I) \otimes \mathbf{Q} \cong H_*(R; I) \otimes \mathbf{Q}$$

for any ring R and bimodule I. (Here $H_*(\ ;\ )$ is Hochschild homology with coefficients.) On the other hand it is relatively easy to show (by the computation in IV.2. below) that "stable cyclic homology" (with the evident definition) is rationally given by the same formula:

$$HC^S_{*-1}(R; I) \otimes \mathbf{Q} \cong H_*(R; I) \otimes \mathbf{Q}.$$

There was an earlier plan for proving the Main Theorem. The idea was to use the "calculus of functors" to obtain the theorem from the fact that rational K-theory and rational cyclic homology have the same stabilization (or rather from the fact that there are maps

$$K_*(\ ) \otimes \mathbf{Q} \to HC^-_*(\otimes \mathbf{Q}) \leftarrow HC_{*-1}(\otimes \mathbf{Q})$$

inducing isomorphisms of stabilizations). This plan was abandoned because it apparently required the stronger hypothesis: f is a 1-connected map of simplicial rings.

(3) I am very grateful to Paul Selick for pointing out a significant error in a preprint of [G2]. (I forgot to thank him in print at the time.)

I. Preliminaries

I.1. *Simplicial Tools.* We begin by recalling or proving some facts concerning simplicial and multisimplicial sets, simplicial spaces, and simplicial and multisimplicial objects in various algebraic categories, namely rings, groups, monoids, abelian groups and modules or bimodules over a fixed ring. By the homotopy groups of such an object we mean the homotopy groups of the realization of the underlying pointed (multi) simplicial set. A map of, say, simplicial rings is called k-connected if the map of realizations is k-connected, i.e. if it induces a surjection in π_i for $i \leq k$ and an injection in π_i for $i < k$ for all basepoints. An ∞-connected map is called an equivalence.

I.1.1. *Simplicial Abelian Groups.* A simplicial abelian group $A = \{A_p | p \geq 0\}$ gives rise to two chain complexes, the non-normalized complex $\mathrm{Ch}(A)$ given by

$$\mathrm{Ch}_p(A) = A_p,$$
$$da = \sum_{i=0}^{p} (-1)^i d_i a, \qquad a \in \mathrm{Ch}_p(A)$$

and the normalized complex $N(A)$. The latter can be viewed either as the quotient of $\mathrm{Ch}(A)$ by the subcomplex generated by degenerate simplices, or as the subcomplex

$$N_p(A) = \bigcap_{i=1}^{p} \ker(d_i\colon A_p \to A_{p-1}).$$

The subcomplex is a direct summand and has the same homology as $\mathrm{Ch}(A)$. The functor N is an equivalence of categories from simplicial abelian groups to chain complexes. (See [DP], 3.6.) The homology groups of $\mathrm{Ch}(A)$ or of $N(A)$ are naturally isomorphic to the homotopy groups of A. (See [L], VII.5.2.)

I.1.2. *Multisimplicial Objects.* If a simplicial object in a category $\mathscr{C}$ is a contravariant functor from Δ to $\mathscr{C}$ then an r-multisimplicial object is a contravariant functor from the product of r copies of Δ to $\mathscr{C}$. It may be described as a collection of objects

$$X_{p_1, \dots, p_r}, \qquad p_j \geq 0$$

and morphisms

$$d_i^j\colon X_{p_1,\dots,p_r} \to X_{p_1,\dots,p_j-1,\dots,p_r} \qquad \begin{cases} 1 \leq j \leq r,\ p_j > 0 \\ \quad 0 \leq i \leq p_j, \end{cases}$$
$$s_i^j\colon X_{p_1,\dots,p_r} \to X_{p_1,\dots,p_j+1,\dots,p_r}, \qquad 1 \leq j \leq r, 0 \leq i \leq p_j$$

satisfying certain identities. Associated with a multisimplicial object X is its diagonal simplicial object $\mathrm{Diag}(X)$, the composition with the diagonal inclusion of Δ into the product $\Delta \times \cdots \times \Delta$. A basic fact about multisimplicial sets is that several kinds of "realization" are naturally homeomorphic, including the realization of the diagonal. (See [Q], p. 94, lemma, for a precise statement and proof in the bisimplicial case.)

Observe that any permutation ϕ of $\{1, \dots, r\}$ gives a way of making the r-multisimplicial object X into a new object of the same kind

$$X^\phi = \left\{X^\phi_{p_1,\dots,p_r}\right\} = \left\{X_{\phi(p_1),\dots,\phi(p_r)}\right\}.$$

The new object has the same diagonal as the old.

I.1.3. *Simplicial Spaces*. We will need the following fact. Suppose $X \to Y$ is a map of simplicial spaces and Y is pointed. We can take homotopy fibers and then realize, or we can realize and then take the homotopy fiber. There is an evident map of spaces

$$\left\|\left\{\text{fiber}\left(X_p \to Y_p\right)\right\}\right\| \to \text{fiber}(\|X\| \to \|Y\|),$$

where $\| \ \|$ is the realization as in [Se]. The fact is that this is an equivalence if all the spaces X_p and Y_p are 0-connected. This follows from [B-F], Theorem B.4, p. 121.

A related but easier statement is that if a map $X \to Y$ of simplicial spaces is such that each $X_p \to Y_p$ is an equivalence then the map of realizations $\|X\| \to \|Y\|$ is an equivalence. ([Se], Prop. A.1.).

Both of these statements have obvious analogues involving bisimplicial sets instead of simplicial spaces.

I.1.4. *Multisimplicial Abelian Groups*. An r-multisimplicial abelian group A determines a multigraded chain complex $\mathrm{Ch}(A)$ in an obvious way; the chain groups are

$$\mathrm{Ch}_n(A) = \bigoplus_{\sum_{i=1}^r p_j = n} A_{p_1, \ldots, p_r}$$

and the boundary of $a \in A_{p_1, \ldots, p_r}$ is

$$da = \sum_{j=1}^{r} (-1)^{p(j)} \sum_{i=0}^{p_j} (-1)^i d_i^j a \quad \text{with } p(j) = \sum_{k<j} p_k.$$

In view of the result stated in I.1.1 it is not surprising that $\mathrm{Ch}(A)$ has the same homology as $\mathrm{Ch}(\mathrm{Diag}(A))$. In fact a natural quasi-isomorphism from $\mathrm{Ch}(A)$ to $\mathrm{Ch}(\mathrm{Diag}(A))$ is given by a sort of "shuffle" formula. (For details in the case $r = 2$ see [D-P], p. 217.) Also, for any $\phi \in \Sigma_r$ the diagram of chain complexes

$$\begin{tikzcd}
\mathrm{Ch}(A) \arrow[r, "\text{shuffle}"] \arrow[d] & \mathrm{Ch}(\mathrm{Diag}(A)) \arrow[d, equal] \\
\mathrm{Ch}(A^\phi) \arrow[r, "\text{shuffle}"] & \mathrm{Ch}(\mathrm{Diag}(A^\phi))
\end{tikzcd}$$

commutes, where the left-hand arrow sends

$$A_{\phi(p_1), \ldots, \phi(p_r)} \subset \mathrm{Ch}_{\Sigma p_j}(A)$$

to

$$A^\phi_{p_1, \ldots, p_r} \subset \mathrm{Ch}_{\Sigma p_j}(A^\phi)$$

by the sign $(-1)^{\Sigma p_j p_k}$, the sum being taken over all j and k such that $j < k$ and $\phi(j) > \phi(k)$.

I.1.5. *Simplicial Rings and Chain Rings.* Let R be a simplicial ring. In particular R is a simplicial abelian group and so gives rise to a chain complex $\mathrm{Ch}(R)$ as in I.1.1. Write $R \otimes R$ for the simplicial abelian group $\{R_p \otimes R_p\}$ and $R \boxtimes R$ for the bisimplicial abelian group $\{R_p \otimes R_q\}$. With the convention that a tensor product of chain complexes (X, d) and (Y, d) has boundary map

$$d(x \otimes y) = dx \otimes y + (-1)^p x \otimes dy, \qquad x \in X_p,\, y \in Y_q,$$

we have maps of chain complexes

$$\begin{aligned}\mathrm{Ch}(R) \otimes \mathrm{Ch}(R) = \mathrm{Ch}(R \boxtimes R) &\xrightarrow{\text{shuffle}} \mathrm{Ch}(\mathrm{Diag}(R \boxtimes R)) \\ &= \mathrm{Ch}(R \otimes R) \to \mathrm{Ch}(R),\end{aligned}$$

the last map being induced by the multiplication in R. This multiplication in $\mathrm{Ch}(R)$ is associative and has a unit $1 \in \mathrm{Ch}_0(R)$; we say that $\mathrm{Ch}(R)$ is a chain ring (or a chain algebra over $\mathbf{Z}$). The homology of the underlying chain complex of a chain ring is always a graded ring. In the case at hand we have

$$H_n \mathrm{Ch}(R) \cong \pi_n(R) \qquad \text{by I.1.1.}$$

I.1.6. *Free Resolutions.* The following discussion applies equally to any of the algebraic categories mentioned at the beginning of I.1; we use rings as an example.

The forgetful functor G from rings to sets has a left adjoint F. Call a ring "free" if it is isomorphic to $F(S)$ for some set S. Call a simplicial ring $R = \{R_p\}$ "free" if (i) each ring R_p is free and (ii) bases for the R_p can be chosen in such a way that every degeneracy map $R_p \to R_{p+1}$ carries the basis into the basis.

To any simplicial ring R one can functorially associate a free simplicial ring ΦR and a natural equivalence $\Phi R \to R$. This can be done either by "attaching cells" (as in the proof of the lemma below) or as follows.

Given any ring R, form the augmented simplicial ring.

$$R \leftarrow FGR \begin{smallmatrix}\leftarrow\\ \rightarrow\\ \leftarrow\end{smallmatrix} (FG)^2 R \begin{smallmatrix}\leftrightarrows\\ \leftarrow\\ \rightleftarrows\end{smallmatrix} (FG)^3 R \ldots$$

with face maps defined by the adjunction $FG \to 1$ and degeneracies defined by the adjunction $1 \to GF$. View this as a map from a simplicial ring $\Phi R = \{(FG)^{p+1} R\}$ to the (constant) simplicial ring R. It is an equivalence (that is, the underlying map of simplicial sets is an equivalence) because after applying the functor G to the augmented simplicial object above one has an "extra degeneracy." More generally if $R = \{R_q\}$ is a simplicial ring then the procedure above yields a bisimplicial ring $\{(FG)^{p+1}R_q\}$ with an equivalence to the bisimplicial ring $\{R_q\}$. Passing to diagonals yields an equivalence of simplicial rings $\Phi R = \{(FG)^{p+1}R_p\} \to R$, with ΦR free.

I.1.7. *A Lemma.* Recall that a CW-complex, or a simplicial set, is called "k-reduced" if its $(k-1)$ skeleton is a point. Of course k-reduced implies $(k-1)$-connected, and conversely it is usually possible to replace a $(k-1)$-connected object by an equivalent k-reduced object. We will need to be able to do this for simplicial ideals in the following sense.

LEMMA. *Suppose $I \subset R$ is a $(k-1)$-connected simplicial ideal in a simplicial ring. Then there exist a simplicial ring S, a k-reduced ideal $J \subset S$, and a map $(S, J) \to (R, I)$ of simplicial ring-ideal pairs such that both $S \to R$ and $J \to I$ are equivalences.*

Proof. (This is just like the proof of IV.2.3 in [G1], except that there we had chain rings instead of simplicial rings and k was zero.) We will inductively prove a sequence of statements

$$\mathrm{A}(-1), \mathrm{B}(0), \mathrm{A}(0), \mathrm{B}(1), \mathrm{A}(1), \dots .$$

A(n): There exist a simplicial ring-ideal pair $(S(n), J(n))$ with $J(n)$ k-reduced and a map $(S(n), J(n)) \to (R, I)$ such that the maps $J(n) \to I$ and $S(n)/J(n) \to R/I$ are both n-connected (and hence the map $S(n) \to R$ is n-connected as well).

B(n): There exist a simplicial ring-ideal pair $(S'(n), J'(n) \to (R, I)$ with $J'(n)$ k-reduced such that $S'(n) \to R$ is n-connected and $J'(n) \to I$ is $(n-1)$-connected (and hence $S'(n)/J'(n) \to R/I$ is n-connected).

First observe that for A(-1) we can take $(S(-1), J(-1)) = (\mathbf{Z}, 0)$.

Now assume A$(n-1)$. Make $S'(n)$ by "attaching n-cells" to $S(n\text{-}1)$ to kill the relative homotopy group $\pi_n(S(n-1) \to R)$. That is, for each relative homotopy class pick a representative diagram of simplicial sets

$$\begin{array}{ccc} \partial\Delta^n & \to & GS(n-1) \\ \downarrow & & \downarrow \\ \Delta^n & \to & GR \end{array}$$

(this is possible because simplicial abelian groups are fibrant simplicial sets), and put these all together to make a diagram of simplicial sets

$$\begin{array}{ccc} \coprod \partial\Delta^n & \to & GS(n-1) \\ \downarrow & & \downarrow \\ \coprod \Delta^n & \to & GR \end{array}$$

This corresponds to a diagram of simplicial rings

$$\begin{array}{ccc} F(\coprod \partial\Delta^n) & \to & S(n-1) \\ \downarrow & & \downarrow \\ F(\coprod \Delta^n) & \to & R \end{array}$$

Let $S'(n)$ be the pushout in this square. It comes with a map to R. The map is n-connected, as one sees by examining the sequence of homotopy groups

$$\dots \to \pi_n(S(n-1) \to S'(n)) \xrightarrow{\alpha} \pi_n(S(n-1) \to R) \to \pi_n(S'(n) \to R) \dots$$

using the hypothesis $A(n-1)$, and noting that $S'(n)$ contains $S(n-1)$ and has the same $(n-1)$-skeleton as $S(n-1)$ and that α is surjective. Let $J'(n) \subset S'(n)$ be the smallest simplicial ideal which contains $J(n-1)$. Then $J'(n)$ is mapped into I. Since $J'(n)$ has the same $(n-1)$-skeleton as $J(n-1)$ another exact sequence argument shows that $J'(n) \to I$ is $(n-1)$-connected. Clearly $J'(n)$ is k-reduced if $J(n-1)$ was.

Next assume $B(n)$. If $n < k$ then take $(S(n), J(n)) = (S'(n), J'(n))$ and check $A(n)$; the n-connectedness of $J(n) \to I$ is automatic because both I and $J(n)$ are $(k-1)$-connected and hence n-connected. If $n \geq k$ then attach n-cells as before, except this time attach them to $S'(n)$ using representatives for $\pi_n(J'(n) \to I)$. Call the pushout $S(n)$ and let $J(n)$ be the simplicial ideal generated by $J'(n)$ and the attached cells. Then we have a map $(S(n), J(n)) \to (R, I)$. The induced map $S(n)/J(n) \to R/I$ is n-connected because $S(n)/J(n) = S'(n)/J'(n)$. By an exact sequence argument using

$$J'(n) \to J(n) \to I$$

one sees that $J(n) \to I$ is n-connected.

Finally take S and J to be the increasing unions $\bigcup_n S(n)$ and $\bigcup_n J(n)$. □

I.1.8. *Simplicial Groups and Monoids*. If G is a simplicial group, or more generally a simplicial monoid, then BG is the diagonal of the bisimplicial set which results from applying the standard nerve construction to G in each simplicial dimension. The basic fact that $|G|$ is equivalent to $\Omega|BG|$ for a simplicial group G fails for general monoids. However, it holds if G is a *group-like* simplicial monoid, i.e. if the monoid $\pi_0(G)$ is a group. (This follows for example from [B-F], Theorem B.4, by consideration of a diagram

$$\begin{array}{ccc} G_q & \to E_p G_q & = G_q^{p+1} \\ \downarrow & \downarrow & \\ * & \to B_p G_q & = G_q^p \end{array}$$

of bisimplicial sets.)

Using this fact we can functorially construct a simplicial group equivalent to a given grouplike simplicial monoid, as follows. Let ΦG be the free resolution of the simplicial monoid G as in I.1.6., so that there is an equivalence of simplicial monoids

$$G \leftarrow \Phi G.$$

Let $\langle \Phi G \rangle$ be the simplicial group obtained from ΦG by adjoining inverses. If G is grouplike then the natural map

$$\Phi G \to \langle \Phi G \rangle$$

is also an equivalence. In fact, for this it suffices to prove that

$$B\Phi G \to B\langle \Phi G \rangle$$

is an equivalence, and this follows from the simple fact that if F is a free (discrete) monoid then $BF \to B\langle F \rangle$ is an equivalence.

I.2. *Algebraic K-theory of Simplicial Rings.*

Let $M_n(R)$ be the "ring of $n \times n$ matrices" over the simplicial ring R, i.e. the simplicial ring defined by applying the functor M_n (from rings to rings) in every simplicial dimension. Considered as a simplicial set it is the product of n^2 copies of R. Its graded ring of homotopy groups is

$$\pi_* M_n(R) = M_n(\pi_* R).$$

In particular M_n preserves equivalences: If a map $R \to S$ of simplicial rings is an equivalence then so is the induced map $M_n R \to M_n S$. Let $M(R)$ be $\bigcup_n M_n(R)$, a simplicial ring-without-1.

Following Waldhausen [W1], define the "space of matrices invertible up to homotopy" $\widehat{\mathrm{GL}}(R)$. Namely, define the simplicial monoid $\widehat{\mathrm{GL}}_n(R)$ by the pull-back diagram

$$\begin{array}{ccc} \widehat{\mathrm{GL}}_n(R) & \to M_n(R) \\ \downarrow & \downarrow \\ \mathrm{GL}_n(\pi_0 R) & \to M_n(\pi_0 R) \end{array}$$

and let $\widehat{\mathrm{GL}}(R)$ be the increasing union $\bigcup_n \widehat{\mathrm{GL}}_n(R)$. Notice that $\pi_0\widehat{\mathrm{GL}}(R) = \mathrm{GL}(\pi_0 R)$, while for $i > 0$, $\pi_i\widehat{\mathrm{GL}}(R) = M(\pi_i R)$. In particular the functor $\widehat{\mathrm{GL}}$ preserves equivalences.

The simplicial monoid $\widehat{\mathrm{GL}}(R)$ is grouplike, so the classifying space $B\widehat{\mathrm{GL}}(R)$ is a delooping of it (I.1.8). The K-groups of R are defined as the homotopy groups of the plus construction $\pi_i B\widehat{\mathrm{GL}}(R)^+$.

It is convenient to make $B\widehat{\mathrm{GL}}(R)^+$, and not just its homotopy groups, functorial in R. One way to do this is to use Volodin's construction of K-theory. Thus for each partial ordering σ of the set $\{1, \ldots, n\}$, introduce the simplicial group of "triangular" matrices $T^\sigma(R) \subset \widehat{\mathrm{GL}}_n(R) \subset \widehat{\mathrm{GL}}(R)$;

$$T^\sigma(R)_p = T^\sigma(R_p) = \left\{X \in \mathrm{GL}_n(R_p) | X_{ij} = \delta_{ij} \text{ unless } i \overset{\sigma}{<} j\right\}.$$

Let $X(R) = \bigcup_\sigma BT^\sigma(R) \subset B\widehat{\mathrm{GL}}(R)$. One sees that $X(R)$ is connected and that

$$\begin{aligned}\pi_1 X(R) &= \varinjlim \pi_1 BT^\sigma(R) = \varinjlim \pi_0 T^\sigma(R) = \varinjlim T^\sigma(\pi_0 R)\\ &= \pi_1 X(\pi_0 R) = \mathrm{St}(\pi_0 R),\end{aligned}$$

the Steinberg group of $\pi_0 R$. In particular the image of $\pi_1 X(R)$ in $\pi_1 B\widehat{\mathrm{GL}}(R) = \mathrm{GL}(\pi_0 R)$ is the commutator subgroup $E(\pi_0 R)$. According to [Su] each $X(R_p)$ is acyclic, and it follows easily that $X(R)$ is acyclic. Therefore the quotient map $B\widehat{\mathrm{GL}}(R) \to B\widehat{\mathrm{GL}}(R)/X(R)$ is a plus construction. Furthermore the square

$$\begin{array}{ccc} X(R) & \to & B\widehat{\mathrm{GL}}(R) \\ \downarrow & & \downarrow \\ * & \to & B\widehat{\mathrm{GL}}(R)/X(R) \end{array}$$

is homotopy-cartesian.

Definition. For any simplicial ring R, $K(R)$ is the pointed simplicial set $B\widehat{\mathrm{GL}}(R)/X(R)$. For $i \geq 1$, $K_i(R)$ is the abelian group $\pi_i K(R)$. For any map $f\colon R \to S$ of simplicial rings and for $i \geq 2$, $K_i(f)$ is the abelian group $\pi_i(K(R) \to K(S)) = \pi_{i-1}(\mathrm{fiber}(K(R) \to K(S)))$.

Remarks. (1) $K_1(R) = (\pi_0\widehat{\mathrm{GL}}(R))^{\mathrm{ab}} = K_1(\pi_0 R)$.

(2) The fact that $K(R)$ has a natural H-space structure ([W2], Theorem 2.3.2.) insures that $K_2(f)$ is abelian.

(3) There is an evident long exact sequence

$$\ldots \to K_2(S) \to K_2(f) \to K_1(R) \to K_1(S)$$

for any $f\colon R \to S$.

(4) In general, of course, the map $K_1(R) \to K_1(S)$ is not surjective. One can extend the exact sequence to the right if necessary, setting $K_i(R) = K_i(\pi_0 R)$ for

$i \leq 0$, but that need not concern us here because we will only consider $K_i(f)$ in cases where f induces a surjection $\pi_0(R) \to \pi_0(S)$ with nilpotent kernel, and hence a surjection $K_1(R) \to K_1(S)$.

I.2.1. LEMMA. *If f: $R \to S$ is a k-connected map of simplicial rings, $k \geq 1$, then the map $K(R) \to K(S)$ is $(k + 1)$-connected. In particular the functor K preserves equivalences.*

Proof. [W1], Prop. 1.1. □

In general it is not true in any sense that the K-theory of a simplicial ring R is determined by the K-theory of the discrete rings R_p. However, in certain relative cases there is a true statement of this kind (Lemma I.2.2. below). Before stating and proving it we will show that $K(R)$ can in any case be *compared* with something which is built out of the $K(R_p)$'s.

Let R be a simplicial ring. For each $p \geq 0$ let R^{Δ^p} be the following simplicial ring. As a simplicial set it is defined by the universal property

$$\mathrm{Map}(X, R^{\Delta^p}) = \mathrm{Map}(\Delta^p \times X, R)$$

(in particular its q-simplices are the simplicial maps $\Delta^p \times \Delta^q \to R$). The ring structure is the obvious one. As p varies, the simplicial rings R^{Δ^p} constitute a bisimplicial ring. There is a map $R \to R^{\Delta^p}$ of simplicial rings (induced by $\Delta^p \to$ point) and as p varies this gives a map

$$\{R\} \neq \{R^{\Delta^p}\}$$

of bisimplicial rings, where the first one is "constant in the p-direction." Reversing the roles of the two simplicial directions and using the evident isomorphism $(R^{\Delta^p})_q \cong (R^{\Delta^q})_p$ one obtains another map of bisimplicial rings

$$\{R_p\} \to \{R^{\Delta^p}\}$$

where this time the first is "constant in the q-direction." We thus have for each p a diagram of simplicial rings (and taking all p at once a diagram of bisimplicial rings)

$$R_p \to R^{\Delta^p} \leftarrow R.$$

Notice that the arrow on the right is an equivalence for each p. The arrow on the left does not have this property, but at least it is an equivalence of bisimplicial rings, since it differs from the right-hand arrow only by interchanging the simplicial directions.

Now apply the functor K to obtain

$$K(R_p) \to K(R^{\Delta^p}) \leftarrow K(R)$$

a diagram of simplicial sets for each p, and in all a diagram of bisimplicial sets. The arrow on the right becomes an equivalence after realization (since this is so even with p fixed), but this is usually false for the other arrow.

The lemma that we need is concerned with the following relative situation. Let $R \to S$ be a map of simplicial rings. It determines maps of simplicial spaces, given in dimension p by

$$\begin{aligned} \text{fiber}\big(|K(R_p)| \to |K(S_p)|\big) &\to \text{fiber}\big(|K(R^{\Delta^p})| \to |K(S^{\Delta^p})|\big) \\ &\leftarrow \text{fiber}\big(|K(R)| \to |K(S)|\big). \end{aligned}$$

(Here "fiber" denotes homotopy fiber.) Of course the arrow on the right is an equivalence for each p.

I.2.2. Lemma. *If $R \to S$ is a surjection of simplicial rings whose kernel I satisfies $I^2 = 0$, then the map of simplicial spaces*

$$\big\{\text{fiber}\big(|K(R_p)| \to |K(S_p)|\big)\big\} \to \big\{\text{fiber}\big(|K(R^{\Delta^p})| \to |K(S^{\Delta^p})|\big)\big\}$$

is an equivalence (i.e. is an equivalence after realization).

Proof. By I.1.3 it will suffice if the square of bisimplicial sets

$$\begin{array}{ccc} K(R_p) & \to & K(R^{\Delta^p}) \\ \downarrow & & \downarrow \\ K(S_p) & \to & K(S^{\Delta^p}) \end{array}$$

is homotopy-cartesian, i.e., becomes homotopy-cartesian after (total) realization. Using the equivalence $\Omega|K(\ )| \simeq \text{fiber}(|X(\ )| \to |B\widehat{\mathrm{GL}}(\ )|)$ and invoking I.1.3. again, we see that it will suffice if both of the squares

$$\begin{array}{ccc} X(R_p) & \to & X(R^{\Delta^p}) \\ \downarrow & & \downarrow \\ X(S_p) & \to & X(S^{\Delta^p}), \end{array} \tag{I.2.3}$$

$$\begin{array}{ccc} B\widehat{\mathrm{GL}}(R_p) & \to & B\widehat{\mathrm{GL}}(R^{\Delta^p}) \\ \downarrow & & \downarrow \\ B\widehat{\mathrm{GL}}(S_p) & \to & B\widehat{\mathrm{GL}}(S^{\Delta^p}) \end{array} \tag{I.2.4}$$

are homotopy cartesian.

Actually in I.2.3. the horizontal maps are equivalences. This follows from the fact that the map of bisimplicial rings $\{R_p\} \to \{R^{\Delta^p}\}$ is an equivalence, and from two properties of the functor

$$X\colon \text{Simplicial rings} \to \text{Simplicial sets};$$

namely, X *preserves equivalences* and *is defined dimensionwise* (i.e. one applies X to a simplicial ring R by first applying it to the discrete rings R_p and then diagonalizing the resulting bisimplicial set).

I.2.4 requires a little more effort, since $B\widehat{\mathrm{GL}}$ is not defined dimensionwise. (Incidentally $B\,\mathrm{GL}$, which is defined dimensionwise, does not preserve equivalences.) Consider the diagram of bisimplicial sets

$$\text{(I.2.5)} \qquad \begin{array}{ccc} M(I_p) & \to & M(I^{\Delta^p}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GL}}(R_p) & \to & \widehat{\mathrm{GL}}(R^{\Delta^p}) \\ \downarrow & & \downarrow \\ \widehat{\mathrm{GL}}(S_p) & \to & \widehat{\mathrm{GL}}(S^{\Delta^p}). \end{array}$$

Here as in I.2.4 the symbols "^" on the left are superfluous. The map $M(\ ) \to \widehat{\mathrm{GL}}(\ )$ sends X to $1 + X$.

The left side of I.2.5 is a fibration sequence. In fact its diagonal is a short exact sequence of simplicial groups and hence a Kan fibration. (This uses that $I_p^2 = 0$.)

The right side is also a fibration sequence, for a similar reason: For each n the diagonal of

$$M_n(I^{\Delta^p}) \to \widehat{\mathrm{GL}}_n(R^{\Delta^p}) \to \widehat{\mathrm{GL}}_n(S^{\Delta^p})$$

is a Kan fibration because it is obtained by restricting the short exact sequence

$$M_n(I^{\Delta^p}) \to M_n(R^{\Delta^p}) \to M_n(S^{\Delta^p})$$

(of additive groups) to the subset $\widehat{\mathrm{GL}}_n(S^{\Delta^p}) \subset M_n(S^{\Delta^p})$. (This again uses that $I^2 = 0$.) Also, the top horizontal arrow in I.2.5 is an equivalence.

Apply the functor B to the diagram of grouplike bisimplicial monoids I.2.5. The resulting diagram still is a map of fibration sequences and an equivalence on fibers. It follows that its lower square I.2.4 is homotopy-cartesian. $\square$

I.3. *Cyclic Homology of Simplicial Rings.*

The purpose of this section is to establish some notation and to adapt some results of [G1] concerning chain rings to simplicial rings.

A ring R and a bimodule B determine a simplicial abelian group (the "cyclic bar construction"), which we will denote by $\mathrm{Cyc}(R; B)$; namely,

$$\mathrm{Cyc}_n(R; B) = B \otimes R^{\otimes n},$$

$$d_i(b \otimes r_1 \otimes \cdots \otimes r_n) = \begin{cases} br_1 \otimes \cdots \otimes r_n & i = 0 \\ b \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & 0 < i < n \\ r_n b \otimes r_1 \otimes \cdots \otimes r_{n-1} & i = n, \end{cases}$$

$$s_i(b \otimes r_1 \otimes \cdots \otimes r_n) = b \otimes r_1 \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n.$$

In the important special case when the bimodule is the ring itself the simplicial abelian group $\mathrm{Cyc}(R; R)$ acquires a little extra structure, namely an action of $\mathbf{Z}/n+1$ on $\mathrm{Cyc}_n(R, R)$ for all $n \geq 0$, and becomes a cyclic abelian group, which we will denote $Z(R)$ as in [G1], Section II.1. The action of a generator $t_{n+1} \in \mathbf{Z}/n+1$ is given by

$$t_{n+1}(r_0 \otimes r_1 \otimes \cdots \otimes r_n) = r_n \otimes r_0 \otimes \cdots \otimes r_{n-1}.$$

With any cyclic abelian group X (in particular $Z(R)$) we will associate four kinds of homology groups, called $H_*(X)$, $HC_*(X)$, $HP_*(X)$, and $HC_*^-(X)$. The first two of these are just as in [G1], Section II.2. Thus $H_*(X)$ is π_* of the underlying simplicial abelian group of X, and in particular when $X = Z(R)$ it is the Hochschild homology $H_*(R; R)$, while $HC_*(X)$ is cyclic homology. The groups $HP_*(X)$ are the periodic homology groups which were called $HC_*^{\mathrm{per}}(X)$ in [G1]. The groups $HC_*^-(X)$ are new. In order to give a uniform description of all four families of groups let us adopt the following notation.

As in [G1], II.2, define the 2-periodic double chain complex $\overline{B}_{**}(X)$. Thus the chain groups are

$$\overline{B}_{p,q}(X) = \begin{cases} \overline{X}_{q-p} & +\infty > q \geq p > -\infty \\ 0 & q < p \end{cases}$$

where $\overline{X}_n = X_n/\text{degeneracies}$. The "vertical" boundary is given in $\overline{B}_{p,q}(X) = \overline{X}_{q-p}$ by

$$b = \sum_{i=0}^{q-p} (-1)^i d_i$$

so that each "column" $\overline{B}_{p,*}(X)$ is a (shifted) copy of the normalized chain complex associated with the underlying simplicial abelian group of X. Thus in particular when $X = Z(R)$ we have

$$\overline{B}_{p,*}(X) \simeq N_{*-p}\mathrm{Cyc}(R;R),$$
$$H_q\overline{B}_{p,*}(X) = H_{q-p}(R;R).$$

The "horizontal" differential B is given in $\overline{B}_{p,q}(X) = \overline{X}_n$ $(n = q - p)$ by

$$B = t_{n+2}s_n \sum_{k=1}^{n+1} \left((-1)^n t_{n+1}\right)^k.$$

Now for $-\infty \le \alpha \le \beta \le +\infty$ let $T_*^{\alpha,\beta}(X)$ be the chain complex obtained from the double complex $\overline{B}_{**}(X)$ by using only the columns numbered α through β. Thus

$$T_n^{\alpha,\beta}(X) = \prod_{\alpha \le p \le \beta} \overline{B}_{p,n-p}(X).$$

Note that when $\alpha = -\infty$ this is an infinite product, not an infinite sum. Then we have

$$H_n(X) = H_nT_*^{0,0}(X),$$
$$HC_n(X) = H_nT_*^{0,+\infty}(X),$$
$$HP_n(X) = H_nT_*^{-\infty,+\infty}(X),$$
$$HC_n^-(X) = H_nT_*^{-\infty,0}(X).$$

In addition to the relations among these groups which were written down in [G1] there are some new ones involving HC_*^-. For example there is a diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & T_*^{-\infty,0}(X) \arrow[r] \arrow[d, "\pi"] & T_*^{-\infty,+\infty}(X) \arrow[r] \arrow[d] & T_*^{1,+\infty}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_*^{0,0}(X) \arrow[r] & T_*^{0,+\infty}(X) \arrow[r] & T_*^{1,+\infty}(X) \arrow[r] & 0.
\end{tikzcd}$$

In view of the isomorphism $T_*^{1,+\infty}(X) \cong T_{*-2}^{0,+\infty}(X)$ this gives rise to a map of long exact sequences

(I.3.1)
$$\begin{tikzcd}
\arrow[r] & HC_{*-1}(X) \arrow[r, "\beta"] \arrow[d, equal] & HC_*^-(X) \arrow[r] \arrow[d, "\pi"] & HP_*(X) \arrow[r] \arrow[d] & HC_{*-2}(X) \arrow[r] \arrow[d, equal] & \cdots \\
\arrow[r] & HC_{*-1}(X) \arrow[r, "B"] & H_*(X) \arrow[r, "I"] & HC_*(X) \arrow[r, "S"] & HC_{*-2}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where the lower sequence is the fundamental "Connes-Gysin sequence" (II.2.2 of [G1]).

When $X = ZR$ and the ring R is flat over $\mathbf{Z}$, we write HC_*R instead of HC_*ZR and likewise for the other functors.

Recall ([L-Q], Prop. 1.2.) that when X is a cyclic vector space over $\mathbf{Q}$ there is another complex with homology $HC_*(X)$, namely the one used in Connes' original definition. This is a quotient complex of the non-normalized chain complex of the underlying simplicial vector space. Denoting it by $C^\lambda_*(X)$, we have

$$C^\lambda_n(X) = \operatorname{coker}\left(X_n \xrightarrow{1-(-1)^n t_{n+1}} X_n\right).$$

In this case the map I of I.3.1 can be viewed as being induced by the quotient map of chain complexes.

If R is a *simplicial* ring (flat over $\mathbf{Z}$) then, applying the functor Z in very simplicial dimension, we obtain a simplicial cyclic abelian group ZR. View it as a cyclic simplicial abelian group and apply the functor Ch to make a cyclic chain complex $\mathrm{Ch}(ZR)$. Now given a cyclic chain complex X (with no nonzero chains in negative dimensions) we can define "Hochschild," cyclic, and periodic homology groups as in [G1], Section III, and also HC^--groups. This means forming a triple complex for each choice of $-\infty \le \alpha \le \beta \le +\infty$ (with axes labelled by p, q, and r (say): $\alpha \le p \le \beta$, $q \ge p$, and $r \ge 0$). Call the total complex of this triple complex $T^{\alpha,\beta}_*(X)$ as before; again, in forming the total complex we use products rather than sums when $\alpha = -\infty$. Then define H_*, HC_*, HP_*, and HC^-_* as before. (The use of products rather than sums insures that

$$T^{-\infty,\beta}_*(X) = \varprojlim_\alpha T^{\alpha,\beta}_*(X),$$

so that there is, for example, an exact sequence

$$0 \to \varprojlim_n{}^1 HC_{*+1+2n}(X) \to HP_*(X) \to \varprojlim_n{}^0 HC_{*+2n}(X) \to 0.$$

as in [G1], I.3.) All the familiar relations, in particular I.3.1, still hold. When this construction is applied to $\mathrm{Ch}(ZR)$, R a simplicial ring, we call the resulting groups $H_*(R)$, $HC_*(R)$, $HP_*(R)$, and $HC^-_*(R)$. Note that $H_*(R)$ can be identified with π_* of the bisimplicial abelian group $\{\mathrm{Cyc}_p(R_q; R_q)\}$. It is easy to extend these constructions to the relative case by using algebraic mapping cylinders as in [G1], p. 200. Thus for example if $f\colon R \to S$ is a map of simplicial rings there are groups $HC_*(f)$ (or $HC_*(R \to S)$) which fit into a long exact sequence

$$\text{(I.3.2.)} \quad \cdots \to HC_*(R) \to HC_*(S) \to HC_*(f) \to HC_{*-1}(R) \to \cdots.$$

It is easy to check that the notation is consistent with identification of a ring with the "discrete" (i.e. constant) simplicial ring that it determines. It should also be noted that the various homology groups defined here for a simplicial ring R are really the same as those which were defined for the underlying chain ring $\mathrm{Ch}(R)$ in [G1]. The point is that for each $n \geq 0$ there is a "shuffle" quasi-isomorphism

$$\mathrm{Ch}(R)^{\otimes(n+1)} \to \mathrm{Ch}(R^{\otimes(n+1)})$$

as in I.1.4, and these fit together to give a map of cyclic chain complexes

$$Z(\mathrm{Ch}(R)) \to \mathrm{Ch}(Z(R)).$$

This is an equivalence in the sense of [G1], Def. III.2.6, and so induces isomorphisms of H_* and HC_* (by [G1], Prop. III.2.7), HP_* (by the $\lim^0 - \lim^1$ sequence), and HC_*^- (by I.3.1).

In view of this last observation the main results of [G1] imply a statement about simplicial rings:

I.3.3. LEMMA. *If f: $R \to S$ is a map of simplicial* $\mathbf{Q}$*-algebras such that the ring homomorphism $\pi_0(R) \to \pi_0(S)$ is a surjection with nilpotent kernel, then f induces an isomorphism $HP_*(R) \to HP_*(S)$ and the map $\beta(f)$: $HC_{*-1}(f) \to HC_*^-(f)$ is an isomorphism.*

Proof. For the first statement it is enough to obtain the corresponding conclusion for each of the other three sides of the square

$$\begin{array}{ccc} R & \to & S \\ \downarrow & & \downarrow \\ \pi_0(R) & \to & \pi_0(S). \end{array}$$

In the case of the lower horizontal arrow this follows from [G1], Theorem II.5.1. Each vertical arrow is one-connected, and so induces a one-connected map of chain algebras. Then Theorem IV.2.6 of [G1], together with the $\lim^0 - \lim^1$ exact sequence relating HP_* to HC_*, shows that the relative HP_* vanishes.

The second conclusion is equivalent to the first, since they are both equivalent to the vanishing of $HP_*(f)$. □

Here are two simple results which will be needed below. We have already seen their K-theoretic analogues.

I.3.4. LEMMA. *If f: $R \to S$ is a k-connected map of simplicial rings, both flat over* $\mathbf{Z}$*, then $H_*(f)$ and $HC_*(f)$ vanish for $* \leq k$.*

Proof. It is enough to do the case of $H_*(f)$, in view of the spectral sequence ([G1], III.2.3). Each tensor power of f

$$f^{\otimes(n+1)}\colon R^{\otimes(n+1)} \to S^{\otimes(n+1)}$$

is k-connected by the Künneth formula (we have used flatness here), and this easily implies the result, for example by using a spectral sequence

$$E^1_{p,q} = \pi_q\big(Z_p(R) \to Z_p(S)\big) \Rightarrow H_{p+q}(R \to S). \qquad \square$$

I.3.5. Lemma. *If $f\colon R \to S$ is an equivalence of simplicial rings flat over* $\mathbf{Z}$ *then the groups $H_*(f)$, $HC_*(f)$, $HP_*(f)$, and $HC^-_*(f)$ all vanish.*

Proof. Use the preceding lemma (or III.2.9 of [G1]) for H_* and HC_*. The exact sequences III.3.2 of [G1] and I.3.1 of the present paper then extend the result to HP_* and HC^-_*, respectively. $\square$

Incidentally Lemma I.3.5 is false without the flatness hypothesis. If one has to deal with rings or simplicial rings which have torsion in their additive groups, the best approach is to make a functorial free resolution ΦR (as in I.1.6) and define $H_*(R) = H_*(\Phi R)$, etc. Then I.3.5 implies that in the flat case this coincides with the old definition (up to a natural isomorphism), and it is clear that I.3.4 and I.3.5 are now true in general.

I.3.6. Lemma. *If Hochschild homology vanishes in dimensions $> k$ then the homology of $T^{\alpha,\beta}_*$ vanishes in dimensions $> k + 2\beta$.*

Proof. The conclusion holds for $\alpha = 0 = \beta$ by hypothesis, and for $\alpha = \beta$ because $T^{\alpha,\beta}_* \cong T^{0,\beta-\alpha}_{*-2\alpha}$. It holds for $-\infty < \alpha \le \beta < +\infty$ by induction on $\beta - \alpha$, using the short exact sequences

$$0 \to T^{\alpha,\beta-1}_* \to T^{\alpha,\beta}_* \to T^{\beta,\beta}_* \to 0.$$

It holds for $-\infty = \alpha < \beta < +\infty$ because of the exact sequence

$$0 \to \varprojlim_{\gamma}{}^1 H_{i+1} T^{\gamma,\beta}_* \to H_i T^{-\infty,\beta}_* \to \varprojlim_{\gamma}{}^0 H_i T^{\gamma,\beta}_* \to 0.$$

Finally we recall the *Morita-invariance* of Hochschild homology. (See for example [G2], p. 402). This says that for $k \ge 1$ the matrix ring M_kR has the same Hochschild homology as the ring R. More precisely the "trace" map of cyclic abelian groups $ZM_kR \to ZR$ given by

(I.3.7) $$Z_pM_kR \to Z_pR$$

$$r_0 \otimes \cdots \otimes r_p \to \sum r_0(i_0, i_1) \otimes \cdots \otimes r_p(i_p, i_0)$$

induces an isomorphism in Hochschild homology. This immediately extends to simplicial rings (the map of bisimplicial sets

$$\mathrm{Cyc}_p(M_kR_q; M_kR_q) \to \mathrm{Cyc}_p(R_q; R_q)$$

gives an equivalence for each p, so is itself an equivalence). Also, the same trace map induces isomorphisms in HC_*, HP_*, and HC_*^- by the usual spectral sequence and exact sequence arguments.

II. A map from K-theory to cyclic homology

Our goal here is to define for any simplicial ring R a natural map

$$K_i(R) \xrightarrow{\alpha(R)} HC_i^-(R)$$

which together with the map

$$HC_{i-1}(R) \xrightarrow{\beta(R)} HC_i^-(R)$$

of I.3.1 will yield the isomorphism

$$K_i(f) \otimes \mathbf{Q} \cong HC_{i-1}(f) \otimes \mathbf{Q}$$

of the Main Theorem. More precisely if $f\colon R \to S$ satisfies the hypothesis of the Main Theorem then each of two maps

$$K_i(f) \otimes \mathbf{Q} \xrightarrow{\alpha_{\mathbf{Q}}(f)} HC_i^-(f \otimes \mathbf{Q}) \xleftarrow{\beta_{\mathbf{Q}}(f)} HC_{i-1}(f) \otimes \mathbf{Q}$$

will be an isomorphism. Here $\beta_{\mathbf{Q}}(f)$ is the map

$$HC_{i-1}(f) \otimes \mathbf{Q} = HC_{i-1}(f \otimes \mathbf{Q}) \xrightarrow{\beta(f \otimes \mathbf{Q})} HC_i^-(f \otimes \mathbf{Q}),$$

that is, the relative version of β for the homomorphism of simplicial rings $f \otimes \mathbf{Q}\colon R \otimes \mathbf{Q} \to S \otimes \mathbf{Q}$; it is an isomorphism by I.3.3. Likewise $\alpha_{\mathbf{Q}}(f)$ is the composition

$$\begin{aligned} K_i(f) \otimes \mathbf{Q} &\longrightarrow K_i(f \otimes \mathbf{Q}) \otimes \mathbf{Q} \\ &\xrightarrow{\alpha(f \otimes \mathbf{Q}) \otimes \mathbf{Q}} HC_i^-(f \otimes \mathbf{Q}) \otimes \mathbf{Q} \\ &= \quad HC_i^-(f \otimes \mathbf{Q}), \end{aligned}$$

where $\alpha(f \otimes \mathbf{Q})$ is a relative version of α and the first map is induced by the diagram of simplicial rings

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"] & R \otimes \mathbf{Q} \arrow[d, "f \otimes \mathbf{Q}"] \\ S \arrow[r] & S \otimes \mathbf{Q}. \end{tikzcd}$$

In this section we will define α. In Sections III and IV we will carry out the proof that $\alpha_{\mathbf{Q}}(f)$, or equivalently the composite $\beta_{\mathbf{Q}}(f)^{-1} \cdot \alpha_{\mathbf{Q}}(f)$, is an isomorphism assuming the hypothesis of the Main Theorem.

The map α will be defined so as to make the diagram commute

$$\begin{tikzcd} & HC_i^- \arrow[d, "\pi"] \\ K_i \arrow[ur, "\alpha"] \arrow[r, "\tau"] & H_i \end{tikzcd},$$

where π is as in I.3.1 and τ is Dennis' trace map ([I]) from K-theory to Hochschild homology suitably generalized from discrete rings to simplicial rings. (Incidentally in using α we will hardly need to know anything about it except that it makes that diagram commute.) We therefore first recall the definition of τ for discrete rings, then extend it to simplicial rings, then obtain the lifting α.

II.1. *The Dennis Trace for Discrete Rings.*

For any (discrete) group G there is a natural injection from the simplicial abelian group $\mathbf{Z}BG$ (i.e. the free abelian group generated by the simplicial set BG) to the cyclic bar construction $\mathrm{Cyc}(\mathbf{Z}G; \mathbf{Z}G)$ of the group ring $\mathbf{Z}G$. On the standard basis for $\mathbf{Z}B_pG$, namely $B_pG = G^p$, it is given by

$$(g_1, \dots, g_p) \to \left(g_p^{-1} \dots g_1^{-1}\right) \otimes g_1 \otimes \cdots \otimes g_p$$

Applying the functor Ch and the natural quotient map $\mathrm{Ch} \to N$ one gets a natural chain map

$$C_* BG = \mathrm{Ch}_*(\mathbf{Z}BG) \xrightarrow{t(G)} N_* \mathrm{Cyc}(\mathbf{Z}G; \mathbf{Z}G) = T_*^{0,0} \mathbf{Z}G$$

from the (non-normalized) chains on BG to the (normalized) Hochschild chains of $\mathbf{Z}G$.

For any (discrete) ring define a chain map

$$T_*^{0,0} \mathbf{Z}\,\mathrm{GL}_n R \xrightarrow{\varepsilon} T_*^{0,0} R$$

by combining the evident ring homomorphism

$$\mathbf{Z}\mathrm{GL}_n(R) = \mathbf{Z}\mathrm{GL}_1 M_n(R) \to M_n(R)$$

with the trace $T_*^{0,0} M_n(R) \to T_*^{0,0} R$ of I.3.8. Because we are using normalized chains the map ε is compatible with the standard inclusions $\mathrm{GL}_n R \to \mathrm{GL}_{n+1} R$ and so yields a chain map

$$T_*^{0,0}\mathbf{Z}\mathrm{GL}(R) \xrightarrow{\varepsilon} T_*^{0,0} R.$$

The map of homology groups induced by $\varepsilon \circ t(\mathrm{GL}(R))$, composed with the Hurewicz map

$$K_i R = \pi_i K(R) \to H_i K(R) \cong H_i B\,\mathrm{GL}(R)$$

is by definition the Dennis trace map

$$K_i R \xrightarrow{\tau} H_i R, \qquad i \geq 1,$$

at least if R is flat as a $\mathbf{Z}$-module.

II.2. *The Dennis Trace for Simplicial Rings.*

If R is a simplicial ring then the construction above runs into some difficulty because $\widehat{\mathrm{GL}}(R)$ is not a simplicial group. We will get around this by systematically replacing grouplike simplicial monoids by simplicial groups as in I.1.8.

For any simplicial ring R which is flat over $\mathbf{Z}$ let $T_*^{0,0} R$ be the "Hochschild double complex" $\{T_p^{0,0} R_q\}$. Likewise for any simplicial monoid G let $C_* BG$ be the double complex $\{C_p BG_q\}$. If G happens to be a simplicial group then there is a map of double complexes

$$C_* BG \xrightarrow{t(G)} T_*^{0,0}\mathbf{Z}G$$

consisting of the maps $t(G_q)$ of II.1.

Now let R be a simplicial ring (flat over $\mathbf{Z}$) and write G for $\widehat{\mathrm{GL}}(R)$. Consider the diagram of double complexes

$$\begin{array}{ccccc}
C_* BG & \longleftarrow & C_* B\Phi G & \longrightarrow & C_* B\langle \Phi G\rangle \\
 & & & & \downarrow{\scriptstyle t\langle\Phi G\rangle} \\
T_*^{0,0}\mathbf{Z}G & \leftarrow & T_*^{0,0}\mathbf{Z}\Phi G & \rightarrow & T_*^{0,0}\mathbf{Z}\langle \Phi G\rangle. \\
{\scriptstyle\varepsilon}\downarrow & & & & \\
T_*^{0,0} R & & & &
\end{array}$$

Here ε: $T_p^{0,0}\mathbf{Z}\widehat{\mathrm{GL}}(R)_q \to T_p^{0,0} R_q$ is defined as in II.1. The horizontal maps are induced by the simplicial monoid maps

$$G \leftarrow \Phi G \rightarrow \langle \Phi G \rangle$$

of I.1.8; they are quasi-isomorphisms because the monoid maps are equivalences.

In homology this yields a map

$$H_i B\widehat{\mathrm{GL}}(R) \rightarrow H_i R$$

and so as in the discrete case a map

$$K_i R \rightarrow H_i R$$

which will again be called τ or the Dennis trace.

In the special case of a discrete ring R (still flat over $\mathbf{Z}$) the τ defined here coincides with the one defined in II.1. To see this observe that a natural dotted arrow exists in

$$\begin{tikzcd} & \Phi G \arrow[dl] \arrow[dr] & \\ G & & \langle \Phi G \rangle \arrow[ll, dotted] \end{tikzcd}$$

if G happens to be a simplicial group, in particular a discrete group, in particular $\mathrm{GL}(R)$. The commutative diagram

$$\begin{tikzcd} & C_* B\Phi G \arrow[dl] \arrow[dr] & \\ C_* BG \arrow[dd, "t(G)"'] & & C_* B\langle \Phi G \rangle \arrow[ll] \arrow[dd, "t\langle \Phi G \rangle"] \\ & T_*^{0,0}\mathbf{Z}\Phi G \arrow[dl] \arrow[dr] & \\ T_*^{0,0}\mathbf{Z}G \arrow[d, "\varepsilon"'] & & T_*^{0,0}\mathbf{Z}\langle \Phi G \rangle \arrow[ll] \\ T_*^{0,0}R & & \end{tikzcd}$$

then proves the assertion.

For the sake of completeness the definition of τ should be extended to the non-flat case (even though it is only really needed here for $\mathbf{Q}$-algebras). Of course one proceeds as in the definition of Hochschild homology (discussion following

I.3.5), using a functorial free resolution $\Phi R \to R$ and the isomorphisms

$$K_iR \underset{\text{I.2.1}}{\cong} K_i\Phi R,$$
$$H_iR \underset{\text{def}}{\cong} H_i\Phi R.$$

II.3. *The Lifting of τ to HC_i^-.*

II.3.1. THEOREM. *For any simplicial ring R there is a natural map α making the diagram commute*

$$\begin{tikzcd} & HC_i^-(R) \arrow[d, "\pi"] \\ K_i(R) \arrow[ur, "\alpha"] \arrow[r, "\tau"] & H_i(R) \end{tikzcd}$$

(Here π is as in I.3.1.)

Proof. The map τ was constructed using the natural chain map $t(G)$. Construct α in the same way, but substituting for $t(G)$ the map $a(G)$ provided by the following lemma.

II.3.2. LEMMA. *For any group G there is a natural chain map $a(G)$ making the diagram commute*

$$\begin{tikzcd} & T_*^{-\infty,0}\mathbf{Z}G \arrow[d, "\pi"] \\ C_*BG \arrow[ur, "a(G)"] \arrow[r, "t(G)"] & T_*^{0,0}\mathbf{Z}G. \end{tikzcd}$$

In fact $a(G)$ is unique up to natural chain homotopy.

Proof. This uses the method of acyclic models. For each $p \geq 0$ the abelian group C_pBG has a basis which may be naturally identified with the set G^p, or equivalently with the set of all homomorphisms $F_p \to G$ from the free group F_p on letters $g_1, \dots g_p$ to G. Therefore natural homomorphisms $C_pBG \to T_p^{-\infty,0}\mathbf{Z}G$ correspond to elements of $T_p^{-\infty,0}\mathbf{Z}F_p$. Suppose natural maps $a_i(G)$, $0 \leq i < p$, have already been chosen, satisfying $\pi a_i = t_i$ and $\partial a_i = a_{i-1}\partial$. To define a_p we have to choose an element $x \in T_p^{-\infty,0}\mathbf{Z}F_p$ (which will be a_pg for $g = (g_1, \dots g_p) \in C_pBF_p$) satisfying $\pi x = t_pg$ and $\partial x = a_{p-1}\partial g$. Since π is surjective there exists y such that $\pi y = t_pg$. Since $\pi a_{p-1}\partial g = \partial t_pg$ by induction, the chain $\partial y - a_{p-1}\partial g$ is a cycle in the chain complex $\ker(\pi) = T_*^{-\infty,-1}\mathbf{Z}F_p$. If it is a boundary in $\ker(\pi)$, say ∂z, then we can set $x = y - z$. Thus the existence statement in the lemma will follow if $T_*^{-\infty,-1}\mathbf{Z}F_p$ has zero homology in dimension $p-1$ for all $p \geq 0$. Likewise the uniqueness will follow if the same is true in dimension p. In fact we will prove $H_i T_*^{-\infty,-1}\mathbf{Z}F_p = 0$ for $i \geq p-1$, which in a certain sense proves that $a(G)$ is unique up to a "contractible choice."

By I.3.7 it will be enough if $H_i\mathbf{Z}F_p = 0$ for $i > p$. But for any (discrete, or even simplicial) group G the Hochschild homology of $\mathbf{Z}G$ is the homology of the free loop space Map $(S^1, |BG|)$ (see [G1], pp. 209–211). For discrete G the free loop space is homotopy equivalent to a disjoint union of spaces $|B\Gamma|$ for certain subgroups $\Gamma \subset G$, namely the centralizers of representatives for conjugacy classes of G. When $G = F_p$ these are subgroups of a free group, so are themselves free, so that the homology vanishes in dimensions > 1. When $p = 0$ it vanishes in dimensions > 0. □

II.3.3. *Addendum.* Maps τ and α can also be constructed in the relative case: If $f\colon R \to S$ is a map of simplicial rings then there are natural maps

$$\begin{tikzcd} & HC_i^-(f) \arrow[d, "\pi"] \\ K_i(f) \arrow[ur, "\alpha"] \arrow[r, "\tau"'] & H_i(f) \end{tikzcd}$$

making the squares commute (up to sign) in

$$\begin{tikzcd} \cdots \arrow[r] & K_i(S) \arrow[r] \arrow[d, "\alpha"] & K_i(f) \arrow[r] \arrow[d, "\alpha"] & K_{i-1}(R) \arrow[r] \arrow[d, "\alpha"] & \cdots \\ \cdots \arrow[r] & H_i(S) \arrow[r] & H_i(f) \arrow[r] & H_{i-1}(R) \arrow[r] & \cdots \end{tikzcd}$$

and similarly for τ.

Proof. This is just a routine use of algebraic mapping cones and homotopy fibers. One needs the relative Hurewicz homomorphism and commutativity of

$$\begin{tikzcd} \pi_i K(S) \arrow[r] \arrow[d] & \pi_i(K(R) \to K(S)) \arrow[r] \arrow[d] & \pi_{i-1}K(R) \arrow[d] \\ H_i K(S) \arrow[r] & H_i(K(R) \to K(S)) \arrow[r] & H_{i-1}K(R). \end{tikzcd}$$

We omit the details. □

Now we define $\alpha_{\mathbf{Q}}(f)$ as indicated at the beginning of Section II. To prove the Main Theorem stated in the introduction it remains to prove:

II.3.4. THEOREM. *If f: $R \to S$ is a map of simplicial rings such that $\pi_0 R \to \pi_0 S$ is a surjection with nilpotent kernel then*

$$\alpha_{\mathbf{Q}}(f) \colon K_i(f) \otimes \mathbf{Q} \to HC_i^-(f \otimes \mathbf{Q})$$

is an isomorphism.

III. Reduction to a special case

In this section we will use the existence and naturality of the map $\alpha_{\mathbf{Q}}(f)$ to minimize the work which goes into proving Theorem II.3.4. It turns out to be enough to prove the theorem under these special conditions: (i) f: $R \to R/I = S$ is a split surjection of discrete rings, (ii) $I^2 = 0$, and (iii) as an S-bimodule, I is free. In Section IV we will handle the special case.

III.1. *Reduction to the Square-Zero Case.*

It is easy to see that for any diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & S \arrow[dl] \\ & T & \end{tikzcd}$$

of simplicial rings in which all three maps satisfy the hypothesis of II.3.4 the conclusion of II.3.4 must hold for any one of the three maps if it holds for both of the other two. (The proof is a five-lemma argument using the commutativity in Addendum II.3.3 and the surjectivity of $K_1(R) \to K_1(S)$.)

III.1.1. LEMMA. *If Theorem* II.3.4 *holds for every surjection* (*of simplicial rings*) *with square-zero kernel then it holds in general.*

Proof. Assume that the theorem holds for surjections $R \to R/I$ with $I^2 = 0$. then it also holds for surjections with $I^n = 0$, for any $n \geq 2$. This follows by induction on n by applying the principle above to the diagram

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & R/I^n \arrow[dl] \\ & R/I^{n-1} & \end{tikzcd}$$

Now let f: $R \to S$ be any map satisfying the hypothesis of II.3.4. Consider the diagram of simplicial rings

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ \pi_0 R \arrow[r] & \pi_0 S \end{tikzcd}$$

and use the same principle again; to obtain the conclusion of II.3.4 for f it is enough to do so for the other three maps in the square. The lower arrow is a surjection with nilpotent kernel, so we have only to consider the vertical maps. Each of these is a surjection with connected kernel.

Therefore consider the case of a simplicial ring R with a connected ideal $I \subset R$ and f: $R \to S = R/I$ the quotient map. We will show that

$$\alpha_{\mathbf{Q}}(f)\colon K_j(f) \otimes \mathbf{Q} \to HC_j^-(f \otimes \mathbf{Q})$$

is an isomorphism by using a downward induction on the connectivity k of the ideal I.

For $k \geq \max(j-2, 0)$ both $K_j(f)$ and $HC_j^-(f \otimes \mathbf{Q})$ are zero by I.2.1, I.3.3, and I.3.4. Now let $k \geq 0$, assume I is k-connected, and suppose that $\alpha_{\mathbf{Q}}(g)$ is known to be an isomorphism in dimension j whenever g is a surjection with $(k+1)$-connected kernel. We want to show that $\alpha_{\mathbf{Q}}(f)$ is an isomorphism in dimension j.

By I.1.7 and the fact that K-theory and cyclic homology both respect equivalences (I.2.1 and I.3.5) we can assume that I is not just k-connected but $(k+1)$-reduced. Now consider the diagram

$$\begin{tikzcd} R \arrow[rr, "g"] \arrow[dr, "f"'] & & R/I^2 \arrow[dl, "h"] \\ & R/I & \end{tikzcd}$$

By assumption $\alpha_{\mathbf{Q}}(h)$ is an isomorphism in all dimensions, so that we will be done if $\ker(g)$ is $(k+1)$-connected. The next result shows that it is.

III.1.2. Lemma. *If the simplicial ideal I is r-reduced for some $r \geq 1$ then I^2 is r-connected.*

Proof. Consider the short exact sequence of simplicial abelian groups

$$0 \to \ker(m) \to I \otimes I \xrightarrow{m} I^2 \to 0$$

where m is multiplication. Since I is r-reduced the same is true of $\ker(m)$. In particular $\ker(m)$ is $(r-1)$-connected. Thus I^2 will be r-connected if $I \otimes I$ is

r-connected. In fact $I \otimes I$ is $(2r-1)$-connected. To see this, view it as the diagonal of the bisimplicial abelian group $\{I_p \otimes I_q\}$. The realization of the diagonal is homeomorphic to the bisimplicial realization. The latter is a $2r$-reduced CW complex because $I_p \otimes I_q = 0$ if $p < r$ or $q < r$.

This concludes the proof of III.1.1. □

III.2. *A Further Reduction.*

We have seen that it is enough to prove Theorem II.3.4 for extensions of simplicial rings by square-zero simplicial ideals.

III.2.1. LEMMA. *Theorem* II.3.4 *is true for all extensions* $I \to R \xrightarrow{f} S$ *of simplicial rings by square-zero ideals, provided it is true in the special case when* R *and* S *are discrete, the extension is split* (*i.e.* f *has a right inverse*), *and* I *is a free* S-*bimodule.*

The key to III.2.1 is the following.

III.2.2. LEMMA. *Let* $I \to R \xrightarrow{f} S$ *be an extension of simplicial rings with* $I^2 = 0$. *If for each of the surjections* f_p: $R_p \to S_p$ *of discrete rings the map* $\alpha_{\mathrm{Q}}(f_p)$ *is an isomorphism then* $\alpha_{\mathrm{Q}}(f)$ *is an isomorphism.*

Proof that III.2.2 implies III.2.1. Let $I \to R \xrightarrow{f} S$ be an extension of simplicial rings with $I^2 = 0$.

In proving that $\alpha_{\mathrm{Q}}(f)$ is an isomorphism we may assume that S is free. In fact, make a free resolution $\Phi S \to S$ (I.1.6) and form the fiber product

$$\begin{array}{ccc} R' & \xrightarrow{g} & \Phi S \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & S. \end{array}$$

Since the vertical maps are equivalences $\alpha_{\mathrm{Q}}(f)$ will be an isomorphism if $\alpha_{\mathrm{Q}}(g)$ is. (Note that $\ker(g) \cong \ker(f) = I$ still has square zero.)

Now assume S is free. By III.2.2 it will be enough if for each f_p: $R_p \to S_p$ $(p \geq 0)$ $\alpha_{\mathrm{Q}}(f_p)$ is an isomorphism. But each f_p is a split surjection of discrete rings with square-zero kernel. (The splitting exists because each S_p is a free ring.) Therefore we may assume that R and S are discrete and that the extension is split.

Any ideal $I \subset R$ is an R-bimodule. If $I^2 = 0$ then I is an (R/I)-bimodule. If in addition $I \to R \to S$ is a split extension then it is isomorphic to the extension

$$I \to I \times S \to S$$

where $I \times S$ has the ring structure

$$(i_1, s_1) + (i_2, s_2) = (i_1 + i_2, s_1 + s_2),$$
$$(i_1, s_1)(i_2, s_2) = (i_1 s_2 + s_1 i_2, s_1 s_2).$$

In fact this construction obviously gives an equivalence of categories between S-bimodules and split extensions of S by square-zero ideals (where morphisms in the latter category are required to preserve the splitting).

Now to finish proving III.2.1. use the fact (I.1.6.) that any S-bimodule I has a free resolution $\Phi I = \{\Phi_p I\}$. Apply III.2.2. again, this time to the extension of simplicial rings

$$\{\Phi_p I\} \to \{(\Phi_p I) \times S\} \to \{S\}. \qquad \square$$

Proof of III.2.2. The main point here is that the relative K-theory associated with a square-zero simplicial ideal "can be computed dimensionwise" (Lemma I.2.2.).

Since K-groups are defined as homotopy groups of certain spaces while HC^--groups are defined as homology groups of certain chain complexes, it will be convenient to interpret the latter as homotopy groups. By I.1.1 the homology groups of any chain complex can be so interpreted, provided the complex is zero in negative dimensions. Our complex $T_*^{-\infty,0}$ is not. Presumably one could get around this by constructing a *spectrum* which is a functor of R and whose homotopy groups are $HC_*^-(R)$. We will take an easier route, replacing the chain complex by another one which is zero in negative dimensions and which is as good as the original complex for purposes of the relative calculation being made here.

III.2.3. *Definition.* If T_* is any chain complex then $\tilde{T}_*$ is the subcomplex defined by $\tilde{T}_n = T_n$ for $n > 0$, $\tilde{T}_n = 0$ for $n < 0$, $\tilde{T}_0 = \ker(d\colon T_0 \to T_{-1})$.

Thus $H_n\tilde{T}_* = H_n T_*$ for $n \geq 0$, $H_n\tilde{T}_* = 0$ for $n < 0$. In the case at hand we claim that the diagram

$$\begin{array}{ccc} \tilde{T}_*^{-\infty,0}(R \otimes \mathbf{Q}) & \rightarrowtail & T_*^{-\infty,0}(R \otimes \mathbf{Q}) \\ \downarrow & & \downarrow \\ \tilde{T}_*^{-\infty,0}(S \otimes \mathbf{Q}) & \rightarrowtail & T_*^{-\infty,0}(S \otimes \mathbf{Q}) \end{array}$$

induces an isomorphism from $H_n(\tilde{T}_*^{-\infty,0}(R \otimes \mathbf{Q}) \to \tilde{T}_*^{-\infty,0}(S \otimes \mathbf{Q}))$ to $HC_n^-(f \otimes \mathbf{Q})$. In fact the cokernel of the upper map in the square has n-th homology

$$\begin{cases} HC_n^-(R \otimes \mathbf{Q}) = HP_n(R \otimes \mathbf{Q}), & n < 0 \\ 0, & n \geq 0 \end{cases}$$

by I.3.1, since $HC_n = 0$ for $n < 0$; by I.3.3 this maps isomorphically to the homology of the lower cokernel, and this proves the claim.

Now, part of the construction of the map α was what might be called a "weak chain map"

$$\mathrm{Ch}_*\mathbf{Z}B\widehat{\mathrm{GL}}(R) \xrightarrow{w} T_*^{-\infty,0}(R),$$

that is, a diagram of actual chain maps

$$\begin{aligned} \mathrm{Ch}_*\mathbf{Z}B\widehat{\mathrm{GL}}(R) &= C_*B\widehat{\mathrm{GL}}(R) \\ &\xleftarrow{\sim} C_*B\Phi\widehat{\mathrm{GL}}(R) \\ &\to C_*B\langle\Phi\widehat{\mathrm{GL}}(R)\rangle \\ &\xrightarrow{a} T_*^{-\infty,0}\mathbf{Z}\langle\Phi\widehat{\mathrm{GL}}(R)\rangle \\ &\xleftarrow{\sim} T_*^{-\infty,0}\mathbf{Z}\Phi\widehat{\mathrm{GL}}(R) \\ &\to T_*^{-\infty,0}\mathbf{Z}\widehat{\mathrm{GL}}(R) \\ &\xrightarrow{\varepsilon} T_*^{-\infty,0}(R) \end{aligned}$$

in which every left-pointing arrow is a quasi-isomorphism. Note that w factors through the subcomplex $\tilde{T}_*^{-\infty,0}(R) \subset T_*^{-\infty,0}(R)$, or more precisely w is obtained by composing a weak map

$$\mathrm{Ch}_*\mathbf{Z}B\widehat{\mathrm{GL}}(R) \xrightarrow{\tilde{w}} \tilde{T}_*^{-\infty,0}(R)$$

and the inclusion of the subcomplex.

Consider the diagram of simplicial abelian groups

$$\begin{aligned} \mathbf{Z}K(R) \xleftarrow{\sim}\ & \mathbf{Z}B\widehat{\mathrm{GL}}(R) \\ \cong\ & N^{-1}N\mathbf{Z}B\widehat{\mathrm{GL}}(R) \\ \xrightarrow{\sim}\ & N^{-1}\mathrm{Ch}\mathbf{Z}B\widehat{\mathrm{GL}}(R) \\ \xrightarrow{N^{-1}\tilde{w}}\ & N^{-1}\tilde{T}^{-\infty,0}(R) \underset{\mathrm{def}}{=} F(R). \end{aligned}$$

Here the first map is induced by the quotient map of simplicial sets $B\widehat{\mathrm{GL}}(R) \to K(R)$; it is an equivalence because the map of simplicial sets is a homology isomorphism. The functor N^{-1} is an inverse (up to isomorphism) of the equivalence of categories N. (I.1.1) The last map (actually only a weak map) is induced by $\tilde{w}$. This diagram, combined with the "Hurewicz map" (of simplicial sets)

$$K(R) \to \mathbf{Z}K(R)$$

gives a map of homotopy groups

$$K_i(R) = \pi_i K(R) \to \pi_i F(R) = H_i \tilde{T}_*^{-\infty,0}(R).$$

Its composition with the map induced by inclusion

$$H_i \tilde{T}_*^{-\infty,0}(R) \to H_i T_*^{-\infty,0}(R) = HC_i^-(R)$$

is, by definition, α.

The same is true in the relative setting: For $f\colon R \to S$ there is a weak map of pairs of simplicial sets

$$(K(R) \to K(S)) \to (F(R) \to F(S))$$

inducing a map from $K_i(f)$ to $H_i(\tilde{T}_*^{-\infty,0}(R) \to \tilde{T}_*^{-\infty,0}(S))$ which, combined with the inclusion, yields the map $\alpha(f)$. By the claim following Definition III.2.3, the map $\alpha_{\mathbf{Q}}(f)$ can be identified with the map of rational relative homotopy groups induced by the (weak) map of pairs

$$(K(R) \to K(S)) \to (F(R \otimes \mathbf{Q}) \to F(S \otimes \mathbf{Q})).$$

Now recall Lemma I.2.2 and the discussion preceding it. We have a diagram of simplicial spaces given in dimension p by

$$\begin{aligned} \operatorname{fiber}\big(K(R_p) \to K(S_p)\big) &\to \operatorname{fiber}\big(|K(R^{\Delta^p})| \to |K(S^{\Delta^p})|\big) \\ &\leftarrow \operatorname{fiber}\big(|K(R)| \to |K(S)|\big). \end{aligned}$$

By I.2.2 both of the maps in this diagram become equivalences after realization. There is also an analogous diagram with F in place of K, and it is easy to see that it again becomes a diagram of equivalences after realization. (In fact each of the maps in the diagram

$$F(R_p) \to F(R^{\Delta^p}) \leftarrow F(R)$$

is an equivalence after realization, by a spectral sequence argument.)

By the hypothesis of III.2.2 we know that

$$\operatorname{fiber}\big(|K(R_p)| \to |K(S_p)|\big) \xrightarrow{\tilde{w}} \operatorname{fiber}\big(|F(R_p)| \to |F(S_p)|\big)$$

is a rational homotopy equivalence for each p. It follows that it yields a rational equivalence after realization with respect to p. Therefore the same is true for the map of (constant) simplicial spaces

$$\text{fiber}(|K(R)| \to |K(S)|) \xrightarrow{\tilde{w}} \text{fiber}(|F(R)| \to |F(S)|).$$

This proves III.2.2. $\square$

IV. Calculation in the special case

We will calculate both $K_*(f) \otimes \mathbf{Q}$ and $HC_{*-1}(f) \otimes \mathbf{Q}$ under the assumption that $I \to R \xrightarrow{f} S$ is a split extension of discrete rings with $I^2 = 0$ and I is a free S-bimodule. They turn out to be isomorphic; more precisely the map $\alpha_{\mathbf{Q}}(f)^{-1} \circ \beta_{\mathbf{Q}}(f)$ is an isomorphism between them. This will complete the proof of Theorem II.3.4.

IV.1. *The Bimodules $D(\rho)$.*

Let S be any (discrete) ring. If $B_1, \ldots, B_r$ are $(S \otimes \mathbf{Q})$-bimodules then $B_1 \otimes \cdots \otimes B_r$ can be made into an $(S \otimes \mathbf{Q})^{\otimes r}$-bimodule in various ways. In particular if $\rho \in \Sigma_r$ is any permutation then let $D(\rho; B_1, \ldots, B_r)$ be $B_1 \otimes \cdots \otimes B_r$ with the bimodule structure

$$(s_1 \otimes \cdots \otimes s_r)(x_1 \otimes \cdots \otimes x_r) = s_1 x_1 \otimes \cdots \otimes s_r x_r, \qquad s_i \in S,\ x_i \in B_i,$$

$$(x_1 \otimes \cdots \otimes x_r)(s_1 \otimes \cdots \otimes s_r) = x_1 s_{\rho(1)} \otimes \cdots \otimes x_r s_{\rho(r)}, \qquad s_i \in S,\ x_i \in B_i.$$

Denote the Hochschild homology $H_*((S \otimes \mathbf{Q})^{\otimes r}; D(\rho; B_1, \ldots, B_r))$ by $W_*(\rho; B_1, \ldots, B_r)$.

We will be particularly interested in $W_0(\rho; B_1, \ldots, B_r)$; this is the quotient of $B_1 \otimes \cdots \otimes B_r$ by the relations

(IV.1.1)

$$x_1 \otimes \cdots \otimes x_i s \otimes \cdots \otimes x_r \sim x_1 \otimes \cdots \otimes s x_{\rho(i)} \otimes \cdots \otimes x_r, \qquad 1 \le i \le r.$$

Denote the class of $x_1 \otimes \cdots \otimes x_r$ in $W_0(\rho; B_1, \ldots, B_r)$ by $[x_1, \ldots, x_r]_\rho$.

Notice the effect of conjugation: a permutation $\lambda \in \Sigma_r$ gives rise to an isomorphism of ring-bimodule pairs

(IV.1.2)

$$\left((S \otimes \mathbf{Q})^{\otimes r}, D(\rho; B_1, \ldots, B_r)\right) \cong \left((S \otimes \mathbf{Q})^{\otimes r}, D\left(\lambda^{-1}\rho\lambda; B_{\lambda(1)}, \ldots, B_{\lambda(r)}\right)\right)$$

$$s_1 \otimes \cdots \otimes s_r \mapsto s_{\lambda(1)} \otimes \cdots \otimes s_{\lambda(r)},$$

$$x_1 \otimes \cdots \otimes x_r \mapsto x_{\lambda(1)} \otimes \cdots \otimes x_{\lambda(r)}.$$

This in turn gives an isomorphism

(IV.1.3) $$W_*(\rho; B_1, \dots, B_r) \cong W_*\big(\lambda^{-1}\rho\lambda; B_{\lambda(1)}, \dots, B_{\lambda(r)}\big)$$

which in W_0 is given by

$$[x_1, \dots, x_r]_\rho \mapsto [x_{\lambda(1)}, \dots, x_{\lambda(r)}]_{\lambda^{-1}\rho\lambda}.$$

Notice also what happens when $r = s + t$ and ρ belongs to the standard copy of $\Sigma_s \times \Sigma_t$ in Σ_r. If ρ corresponds to $(\sigma, \tau) \in \Sigma_s \times \Sigma_t$, write $\rho = \sigma \cup \tau$. Then $D(\sigma \cup \tau; B_1, \dots, B_{s+t})$ is isomorphic to the tensor product (over $\mathbf{Q}$) of the $(S \otimes \mathbf{Q})^{\otimes s}$-bimodule $D(\sigma; B_1, \dots, B_s)$ and the $(S \otimes \mathbf{Q})^{\otimes t}$-bimodule $D(\tau; B_{s+1}, \dots, B_{s+t})$. This implies that

$$W_n(\sigma \cup \tau; B_1, \dots, B_{s+t}) \cong \bigoplus_{p+q=n} W_p(\sigma; B_1, \dots, B_s) \otimes W_q(\tau; B_{s+1}, \dots, B_t).$$

In particular when $n = 0$ we have

(IV.1.4)
$$W_0(\sigma \cup \tau; B_1, \dots, B_{s+t}) \cong W_0(\sigma; B_1, \dots, B_s) \otimes W_0(\tau; B_{s+1}, \dots, B_{s+t}),$$
$$[x_1, \dots, x_{s+t}]_{\sigma\cup\tau} \leftrightarrow [x_1, \dots, x_s]_\sigma \otimes [x_{s+1}, \dots, x_{s+t}]_\tau.$$

The special case $B_1 = \cdots = B_r = B$ is particularly important. In this case we abbreviate

$$D(\rho; B) = D(\rho; B, \dots, B),$$
$$W_*(\rho; B) = W_*(\rho; B, \dots, B).$$

By IV.1.3, the centralizer of ρ in Σ_r, which we will denote by $C(\rho)$, acts on $W_*(\rho; B)$. Let $\overline{W}_*(\rho; B)$ be the antisymmetrization of $W_*(\rho; B)$ with respect to this action. Write $\overline{[x_1, \dots, x_r]}_\rho$ for the image of $[x_1, \dots, x_r]_\rho \in W_0(\rho; B)$ in $\overline{W}_0(\rho; B)$.

We will find in IV.2 that for a split extension $I \to R \xrightarrow{f} S$ with $I^2 = 0$ the relative rational cyclic homology is given by

(IV.1.5) $$HC_*(f) \otimes \mathbf{Q} \cong \bigoplus_{r \geq 1} \overline{W}_{*-r}(\rho_r; I \otimes \mathbf{Q})$$

where $\rho_r \in \Sigma_r$ is the basic transitive permutation $1 \mapsto 2 \mapsto \cdots \mapsto r \mapsto 1$. Note that in the special case when I is a free S-bimodule, so that $D(\rho_r; I \otimes \mathbf{Q})$ is a free $(S \otimes \mathbf{Q})^{\otimes r}$-bimodule, IV.1.5 simply says that

$$HC_n(f) \otimes \mathbf{Q} \cong \overline{W}_0(\rho_n; I \otimes \mathbf{Q}).$$

In IV.4, we will find that in this special case the same formula is valid for K-theory:

(IV.1.6) $$K_{n+1}(f) \otimes \mathbf{Q} \cong \overline{W}_0(\rho_n; I \otimes \mathbf{Q}).$$

IV.2. *A Cyclic Homology Calculation.*

We want to prove IV.1.5 for every split extension of discrete rings $I \to R \xrightarrow{f} S$ with $I^2 = 0$. Since $HC_*(-\otimes \mathbf{Q}) = HC_*(-) \otimes \mathbf{Q}$ we may replace R, S, and I by $R \otimes \mathbf{Q}$, $S \otimes \mathbf{Q}$, and $I \otimes \mathbf{Q}$. Thus we have to compute $HC_*(f)$ when $I \to R \xrightarrow{f} S$ is a split extension of discrete $\mathbf{Q}$-algebras with $I^2 = 0$.

The splitting of f implies that

$$HC_*(R) = HC_*(S) \oplus HC_{*+1}(f).$$

In fact more is true. View R as a graded ring with $\mathrm{gr}_0 R = S$, $\mathrm{gr}_1 R = I$, $\mathrm{gr}_j R = 0$ for $j \geq 2$. This puts a grading on each tensor power of R, and in fact makes the cyclic vector space ZR a cyclic graded vector space. In particular each cyclic homology group is graded

$$HC_* R = \bigoplus_{r \geq 0} \mathrm{gr}_r HC_* R$$

with $\mathrm{gr}_0 HC_* R = HC_* S$. The conclusion IV.1.5 will follow from

$$\text{(IV.2.1)} \qquad \mathrm{gr}_r HC_{*-1} R \cong \overline{W}_{*-r}(\rho_r; I), \qquad r > 0,$$

To prove IV.2.1, fix $r > 0$ and consider the cyclic vector space $\mathrm{gr}_r ZR$. In dimension n it consists of all those terms in $R^{\otimes(n+1)} = (S \oplus I)^{\otimes(n+1)}$ which have exactly r copies of I, i.e. the direct sum of all the tensor products

$$S^{\otimes n_0} \otimes I \otimes S^{\otimes n_1} \otimes \cdots \otimes I \otimes S^{\otimes n_r}$$

where $n_0, \ldots, n_r$ are natural numbers such that $r + \Sigma n_i = n + 1$.

We next introduce a sort of "r-fold cover" $V = V(I, \ldots, I)$ of $\mathrm{gr}_r ZR$; this will be a cyclic vector space with an action of the group $\mathbf{Z}/r\mathbf{Z}$ such that $\mathrm{gr}_r ZR$ is isomorphic (as cyclic vector space) to the space of coinvariants for the action. To define V we need to label the "r copies of I" in $\mathrm{gr}_r ZR$. Thus for the moment let $I_1, \ldots, I_r$ be any S-bimodules and consider for each $n \geq 0$ the vector space

$$\text{(IV.2.2)} \quad V_n(I_1, \ldots, I_r) = \oplus S^{\otimes n_0} \otimes I_{j_1} \otimes S^{\otimes n_1} \otimes \cdots \otimes I_{j_r} \otimes S^{\otimes n_r}$$

where the direct sum is over all $\{n_i \geq 0 | 0 \leq i \leq r\}$ such that $r + \Sigma n_i = n + 1$ and all *cyclic* permutations $(j_1, \ldots, j_r)$ of $(1, \ldots, r)$. These constitute a cyclic vector space $V(I_1, \ldots, I_r)$ in an evident way, the structure maps being defined essentially by the same rules used in defining ZR([G1], p. 189). (More precisely $V(I_1, \ldots, I_r)$ can be viewed as a subobject of the cyclic vector space ZA where A is the split extension of S by the bimodule $\oplus I_j$.)

There is an obvious isomorphism of cyclic vector spaces $V(I_1,\dots,I_r) \cong V(I_2,\dots,I_r,I_1)$; in the special case $V = V(I,\dots,I)$ this is an automorphism of period r. The space of coinvariants for the resulting $(\mathbf{Z}/r\mathbf{Z})$-action is clearly isomorphic to $\mathrm{gr}_r ZR$.

We now compute $HC_* V(I_1,\dots,I_r)$. Let $V_n'(I_1,\dots,I_r) \subset V_n(I_1,\dots,I_r)$ be the subspace obtained by allowing only those terms in IV.2.2. which have $n_0 = 0$ and $(j_1,\dots,j_r) = (1,\dots,r)$. Thus $V' \subset V$ is not a cyclic subobject, but it is a simplicial subobject. In fact it is clear that V is the free cyclic vector space generated by the simplicial vector space V'. It follows easily that the non-normalized chain complex $\mathrm{Ch}_*(V')$ is isomorphic to the complex $C_*^\lambda(V)$, and in particular has homology $HC_*(V)$.

On the other hand $\mathrm{Ch}_*(V')$ is related to a multisimplicial object. Let $\mathscr{V}(\rho) = \mathscr{V}(\rho; I_1,\dots,I_r)$ be the r-multisimplicial vector space

$$\mathscr{V}_{n_1,\dots,n_r}(\rho) = I_1 \otimes S^{\otimes n_1} \otimes \cdots \otimes I_r \otimes S^{\otimes n_r}$$

with face and degeneracy maps d_i^j and s_i^j $(1 \le j \le r)$ given by

$$d_i^j(x(1) \otimes s(1,1) \otimes \cdots \otimes s(1,n_1) \otimes x(2) \otimes s(2,1) \otimes \cdots \otimes s(r,n_r))$$
$$= \begin{cases} x(1) \otimes \cdots \otimes x(j)s(j,1) \otimes \cdots \otimes s(r,n_r) & \text{if } i = 0 \\ x(1) \otimes \cdots \otimes s(j,i)s(j,i+1) \otimes \cdots \otimes s(r,n_r) & \text{if } 0 < i < n_j \\ x(1) \otimes \cdots \otimes s(j,n_j)x(\rho(j)) \otimes \cdots \otimes s(r,n_r) & \text{if } i = n_j; \end{cases}$$

$$s_i^j(x(1) \otimes s(1,1) \otimes \cdots \otimes s(r,n_r))$$
$$= x(1) \otimes s(1,1) \otimes \cdots s(j,i) \otimes 1 \otimes s(j,i+1) \otimes \cdots s(r,n_r).$$

Consider the total chain complex $\mathrm{Ch}_* \mathscr{V}(\rho_r)$ as in I.1.4. It has the same chain groups as $\mathrm{Ch}_*(V')$ except for a dimension shift: both $\mathrm{Ch}_n(\mathscr{V}(\rho_r))$ and $\mathrm{Ch}_{n+r-1}(V')$ are equal to

$$\bigoplus_{\Sigma n_j = n} \mathscr{V}_{n_1,\dots,n_r}(\rho_r).$$

In fact the following is an isomorphism of chain complexes

(IV.2.3)

$$\mathrm{Ch}_*(\mathscr{V}(\rho_r)) \cong \mathrm{Ch}_{*+r-1}(V'),$$

$$x \leftrightarrow (-1)^\nu x, \qquad \nu = \nu(n_1,\dots,n_r) = \sum_{j \text{ even}} n_j, \qquad x \in \mathscr{V}_{n_1,\dots,n_r}.$$

Notice also that the simplicial vector space $\mathrm{Diag}(\mathscr{V}(\rho_r))$ can be identified with the cyclic bar construction $\mathrm{Cyc}(S^{\otimes r}; D(\rho_r; I_1,\dots,I_r))$.

Putting the last few observations together we obtain for any $I_1, \dots, I_r$ an isomorphism

(IV.2.4)
$$\begin{aligned} HC_{*-1}(V) &\cong H_{*-1}\mathrm{Ch}(V') \\ &\cong H_{*-r}\mathrm{Ch}(\mathscr{V}(\rho_r)) \\ &\cong H_{*-r}\mathrm{Ch}(\mathrm{Diag}(\mathscr{V}(\rho_r))) \\ &\cong W_{*-r}(\rho_r; I_1, \dots, I_r). \end{aligned}$$

Now consider the effect of a cyclic permutation of $I_1, \dots, I_r$ on IV.2.4.

IV.2.5. CLAIM. *The following diagram commutes up to the sign* $(-1)^{r+1}$:

$$\begin{array}{ccc} HC_{*-1}V(I_1, \dots, I_r) & \cong & HC_{*-1}V(I_r, I_1, \dots, I_{r-1}) \\ \scriptstyle\text{IV.2.4.}\ \wr\parallel & & \scriptstyle\text{IV.2.4.}\ \wr\parallel \\ W_{*-r}(\rho_r; I_1, \dots, I_r) & \underset{\text{IV.1.3.}}{\cong} & W_{*-r}(\rho_r; I_r, I_1, \dots, I_{r-1}). \end{array}$$

Proof of Claim. The cyclic objects $V(I_1, \dots, I_r)$ and $V(I_r, I_1, \dots, I_{r-1})$ may be considered identical. The simplicial objects $V'(I_1, \dots, I_r)$ and $V'(I_r, I_1, \dots, I_{r-1})$ are not even isomorphic, but their chain complexes are: The diagram

$$\begin{array}{ccc} C^{\lambda}_{n+r-1}V(I_1, \dots, I_r) & = & C^{\lambda}_{n+r-1}V(I_r, \dots, I_{r-1}) \\ \wr\parallel & & \wr\parallel \\ \mathrm{Ch}_{n+r-1}V'(I_1, \dots, I_r) & \cong & \mathrm{Ch}_{n+r-1}V'(I_r, \dots, I_{r-1}) \end{array}$$

commutes if the lower isomorphism is given by

(IV.2.6) $x(1) \otimes s(1,1) \otimes \cdots \otimes s(r, n_r)$

$$\leftrightarrow (-1)^{(n_r+1)(1+n+r)} x(r) \otimes s(1,r) \otimes \cdots \otimes s(r-1, n_{r-1}).$$

The multisimplicial objects satisfy

(IV.2.7)
$$\mathscr{V}(\rho_r; I_1, \dots, I_r)^{\rho_r} \cong \mathscr{V}(\rho_r; I_r, \dots, I_{r-1})$$

in the notation of I.1.2; the resulting chain isomorphism between $\mathrm{Ch}_*\mathscr{V}(\rho_r; I_1, \dots, I_r)$ and $\mathrm{Ch}_*\mathscr{V}(\rho_r; I_r, \dots, I_{r-1})$ (see I.1.4) is given in $\mathscr{V}(\rho_r; I_1, \dots, I_r)_{n_1, \dots, n_r}$ by

(IV.2.8) $x(1) \otimes \cdots \otimes s(r, n_r) \mapsto (-1)^{n_r(n-n_r)} x(r) \otimes \cdots \otimes s(r-1, n_{r-1}).$

Comparing various formulas one now finds that the diagram

$$\begin{array}{ccc} \mathrm{Ch}_{n+r-1}V'(I_1,\dots,I_r) & \underset{\text{IV.2.6.}}{\cong} & \mathrm{Ch}_{n+r-1}V'(I_r,\dots,I_{r-1}) \\ \text{IV.2.3.} \wr\Vert & & \text{IV.2.3.} \wr\Vert \\ \mathrm{Ch}_n\mathscr{V}(\rho_r; I_1,\dots,I_r) & \underset{\text{IV.2.8.}}{\cong} & \mathrm{Ch}_n\mathscr{V}(\rho_r; I_r,\dots,I_{r-1}) \end{array}$$

commutes up to the sign $(-1)^{r+1}$. (One checks that modulo 2

$(n_r+1)(n+r+1)+\nu(n_r,n_1,\dots,n_{r-1})+\nu(n_1,\dots,n_r)+n_r(n-r)\equiv r+1.)$

The simplicial objects Diag $\mathscr{V}(\rho_r; I_1,\dots,I_r)$ and Diag $\mathscr{V}(\rho_r; I_r,\dots,I_{r-1})$ are isomorphic because of IV.2.7, and the rest of the proof of the claim is now clear. □

In the case $I_1=\cdots=I_r=I$, IV.2.5 says that the isomorphism IV.2.4 is equivariant with respect to $\mathbf{Z}/r\mathbf{Z}$, provided the sign $(-1)^{r+1}$ is inserted in the action on W. Now IV.2.1 follows by passing to coinvariants.

We will need an explicit formula on the chain level for the isomorphism

$$\overline{W}_0(\rho_r; I)\cong \mathrm{gr}_r HC_{r-1}(R).$$

It is easy to see that, up to sign, the correct formula is

$$\left[\overline{x_1,\dots,x_r}\right]_{\rho_r}\mapsto\left\{\overline{x_1\otimes\cdots\otimes x_r}\right\}.$$

Here $x_i\in I\subset R$, $x_1\otimes\cdots\otimes x_r\in Z_{r-1}R$, $\overline{x_1\otimes\cdots\otimes x_r}$ is its image in $C^{\lambda}_{r-1}(R)$ (a cycle) and $\{\ \}$ denotes homology class in $H_{r-1}C^{\lambda}_*(R)=HC_{r-1}(R)$. In particular the map B in the Connes-Gysin sequence takes this to the Hochschild homology class

$$\text{(IV.2.9)}\qquad \left\{\sum_{j=0}^{r-1}(-1)^{(r+1)j}1\otimes x_j\otimes\cdots\otimes x_{j-1}\right\}\in \mathrm{gr}_r H_r(R)$$

(see [L-Q], Prop. 1.11).

IV.3. *Homology of Tensor Products of Adjoint Representations.*

Let S be a ring. If $B_1,\dots,B_r$ are $(S\otimes\mathbf{Q})$-bimodules then the group $\mathrm{GL}(S)$ acts by conjugation on each of the $\mathbf{Q}$-vector spaces $M(B_j)$ and so acts diagonally on the tensor product

$$T^r=T^r(B_1,\dots,B_r)=\bigotimes_{1\le j\le r}M(B_j).$$

We will compute the homology of $\mathrm{GL}(S)$ with coefficients in T^r; it turns out to be given by

$$H_*(\mathrm{GL}(S);T^r) = H_*(\mathrm{GL}(S);\mathbf{Q}) \otimes \bigoplus_{\rho\in\Sigma_r} W_*(\rho),$$

where the $W_*(\rho) = W_*(\rho; B_1, \dots, B_r)$ are the Hochschild homology groups introduced in IV.1. By specializing to the case $B_1 = \cdots = B_r = B$ and taking antisymmetric parts we will obtain a calculation of $H_*(\mathrm{GL}(S); \Lambda^r M(B))$ and use it (in IV.4) to make the K-theory calculation IV.1.6.

We assume familiarity with [G2], where the special case $r = 1$ was treated.

For each $\rho \in \Sigma_r$ let $\mathscr{C}_*(\rho)$ be the chain complex

$$\varinjlim_k \mathrm{Ch}_*\mathrm{Cyc}\Big(M_k\big((S \otimes \mathbf{Q})^{\otimes r}\big); M_kD(\rho; B_1, \dots, B_r)\Big),$$

a direct limit of non-normalized Hochschild complexes of matrix rings as in [G2], p. 402. By Morita invariance (see Section I.3), $W_*(\rho)$ can be identified with the homology of $\mathscr{C}_*(\rho)$. On the other hand the group $\mathrm{GL}(S)$ acts, through $\mathrm{GL}(S \otimes \mathbf{Q})$, on the chain complex $\mathscr{C}_*(\rho)$. (The group $\mathrm{GL}(S \otimes \mathbf{Q})$ maps diagonally into $\mathrm{GL}(S \otimes \mathbf{Q})^r$. This group maps into $\mathrm{GL}((S \otimes \mathbf{Q})^{\otimes r})$ because the r different ring maps $S \otimes \mathbf{Q} \to (S \otimes \mathbf{Q})^{\otimes r}$ have commuting images. The group $\mathrm{GL}((S \otimes \mathbf{Q})^{\otimes r})$ acts on the chain group

$$\mathscr{C}_n(\rho) = M(D(\rho)) \otimes M\big((S \otimes \mathbf{Q})^{\otimes r}\big) \otimes \cdots \otimes M\big((S \otimes \mathbf{Q})^{\otimes r}\big)$$

by conjugating in each factor.) The map

$$T^r = \bigotimes_j M(B_j) \to M\Big(\bigotimes_j B_j\Big) = \mathscr{C}_0(\rho)$$

given by matrix multiplication is $\mathrm{GL}(S)$-equivariant. Using all $\rho \in \Sigma_r$ at once we get an equivariant map from T^r to the group of 0-chains in the complex $\oplus_\rho \mathscr{C}_*(\rho)$. This in turn induces a map

(IV.3.1) $$H_n(\mathrm{GL}(S); T^r) \to \bigoplus_\rho \mathbf{H}_n(\mathrm{GL}(S); \mathscr{C}_*(\rho))$$

into hyperhomology.

IV.3.2. THEOREM. *For any ring S and any $(S \otimes \mathbf{Q})$-bimodules $B_1, \dots, B_r$ the map* IV.3.1 *is an isomorphism. Moreover*

$$\mathbf{H}_n(\mathrm{GL}(S); \mathscr{C}_*(\rho)) \cong \bigoplus_{p+q=n} H_p(\mathrm{GL}(S);\mathbf{Q}) \otimes W_q(\rho).$$

Proof. For the second statement the proof is exactly as in the special case $r = 1$ ([G2], Prop.V.2). We omit the details.

For the first statement the proof is only slightly more complicated than it was in the case $r = 1$ ([G2], Theorem V.3). We proceed in several steps.

Step 1. (Reduce to the free case.) Resolving by free bimodules if necessary, we may assume that each B_j is free. In this case each $D(\rho)$ is a free

$(S \otimes \mathbf{Q})^{\otimes r}$-bimodule, so the Hochschild homology $W_q(\rho)$ vanishes for $q > 0$. View $W_0(\rho)$ as a chain complex (with zero chain groups except in dimension zero) equipped with (trivial) $\mathrm{GL}(S)$-action. We then have an equivariant quasi-isomorphism

$$\mathscr{C}_*(\rho) \to W_0(\rho).$$

Since this induces an isomorphism on hyperhomology, the desired conclusion can be restated as follows: The composed map

$$\mathrm{tr}\colon T^r \to \bigoplus_\rho \mathscr{C}_0(\rho) \to \bigoplus_\rho W_0(\rho)$$

induces an isomorphism in $H_*(\mathrm{GL}(S); -)$. Notice that tr is a sort of trace; we have

$$\mathrm{tr}(X_1 \otimes \cdots \otimes X_r) = \sum_{\rho \in \Sigma_r} \sum_{\{i_j\}} [X_1(i_1, i_2), \ldots, X_r(i_r, i_1)]_\rho.$$

Step 2. (Replace $\mathrm{BGL}(S)$ by the subcomplex $X(S)$.) According to Proposition I.3 of [G2], it is enough if tr induces an isomorphism

$$H_*(X(S); T^r) \cong \bigoplus_\rho H_*(X(S); W_0(\rho)).$$

Step 3. (Replace groups by Lie algebras.) Each vector space upon which we have made $\mathrm{GL}(S)$ act has an analogous action of the Lie algebra $\mathfrak{gl}(S \otimes \mathbf{Q})$. For example the action on T^r is given by

$$[X_1 \otimes \cdots \otimes X_r, u] = \sum_{j=1}^r X_1 \otimes \cdots \otimes (X_j u - u X_j) \otimes \cdots \otimes X_r,$$
$$u \in \mathfrak{gl}(S \otimes \mathbf{Q}),\ X_j \in M(B_j),$$

while the action on $W_0(\rho)$ is trivial. The map tr is a map of $(\mathrm{GL}, \mathfrak{gl})$-modules ([G2], Def. III.3). Thus by [G2], Prop.III.5 it is enough if tr induces isomorphisms

$$H_n X_*(S \otimes \mathbf{Q}; T^r) \cong \bigoplus_\rho H_n X_*(S \otimes \mathbf{Q}; W_0(\rho)).$$

Here X_* is the Lie analogue of Volodin's construction defined in [G]. We recall the definition. If A is a $\mathbf{Q}$-algebra and V is a module for the Lie algebra $\mathfrak{gl}(A) = \bigcup_n \mathfrak{gl}_n(A)$ of matrices then $X_*(A; V)$ is defined to be the chain subcomplex of the Koszul complex $C_*(\mathfrak{gl}(A); V)$ generated by the Koszul complexes $C_*(t^\sigma(A); V)$, where $t^\sigma(A)$ ranges over all "triangular" Lie algebras in $\mathfrak{gl}(A)$. From now on we may as well assume $S = S \otimes \mathbf{Q}$.

Step 4. (Pass to the full general linear Lie algebra.) It is enough if tr induces isomorphisms

$$H_n(\mathfrak{gl}(S); T^r) \to \bigoplus_p H_n(\mathfrak{gl}(S); W_0(\rho)). \tag{IV.3.3}$$

To see this we must generalize Theorem II.3 of [G2]. Recall the statement: For any **Q**-algebra A and $\mathfrak{gl}(A)$-module V, there is a spectral sequence with

$$E^2_{p,q} = H_p(\mathfrak{gl}(A); H_q X_*(A; V)) \Rightarrow H_{p+q}(\mathfrak{gl}(A); V),$$

for a certain abelian action of $\mathfrak{gl}(A)$ on $H_q X_*(A; V)$. We claim that the same conclusion holds even when V is a chain complex of $\mathfrak{gl}(A)$-modules $\{V_r\}$. In this case the (hyper)homology $H_*(\mathfrak{gl}(A); V)$ is defined to be the homology of the double complex $C_n(\mathfrak{gl}(A); V_r)$, while $X_*(A; V)$ is the double complex $X_n(A; V_r)$. The proof is a routine modification of the proof in [G2], Section IV: Define an increasing filtration F_p of $C_n(\mathfrak{gl}(A); V_q)$ with $F_0 = X_n(\mathfrak{gl}(A); V_q)$, and use the spectral sequence associated with this filtration.

Apply this result with $A = S$, and with V equal to the 2-term chain complex $T^r \to \bigoplus_\rho W_0(\rho)$. It yields a spectral sequence with

$$E^2_{p,q} = H_p\Big(\mathfrak{gl}(S); H_q X_*\Big(S; T^r \to \bigoplus_\rho W_0(\rho)\Big)\Big)$$
$$\Rightarrow H_{p+q}\Big(\mathfrak{gl}(S); T^r \to \bigoplus_\rho W_0(\rho)\Big).$$

The relevant action of $\mathfrak{gl}(S)$ on $H_q X_*(S; T^r \to \bigoplus_\rho W_0(\rho))$ is trivial, for the same reason as in the case $r = 1$ ([G2], p. 404). Thus E^2 is a tensor product and in the usual way the vanishing of E^∞ implies the vanishing of E^2.

Step 5. (Pass to the complex of coinvariants.) The Koszul complex $C_*(\mathfrak{gl}(S); T^r)$ has an action of the Lie algebra $\mathfrak{gl}(S)$ and hence an action of $\mathfrak{gl}(\mathbf{Q})$. We can take coinvariants either on the chain level or on the level of homology, and there are obvious maps

$$H_n(\mathfrak{gl}(S); T^r) \to H_n(\mathfrak{gl}(S); T^r)_{\mathfrak{gl}(\mathbf{Q})}$$
$$\to H_n(C_*(\mathfrak{gl}(S); T^r)_{\mathfrak{gl}(\mathbf{Q})}).$$

Both of these are isomorphisms, exactly as in the case $r = 1$. (This statement is valid for any **Q**-algebra S and any S-bimodules $B_1, \dots, B_r$, not necessarily free.) Since details were not given in [G2] we give them here.

That the first map is an isomorphism—i.e., that $\mathfrak{gl}(\mathbf{Q})$ acts trivially on $H_*(\mathfrak{gl}(S); T^r)$—is true for general reasons; a Lie algebra $\mathfrak{g}$ always acts trivially on $H_*(\mathfrak{g}; V)$ for any $\mathfrak{g}$-module V. (See for example [G2], p. 398.)

The proof that the second map is an isomorphism relies on a semisimplicity (or "complete reducibility") argument. We will actually prove the analogous statement with $\mathfrak{gl}_n(\mathbf{Q})$ instead of $\mathfrak{gl}(\mathbf{Q})$; this suffices by taking a direct limit with respect to n. From general facts about semisimple modules it follows that if a Lie algebra g acts on a chain complex C_* in such a way that each chain group C_n is a semisimple $\mathfrak{g}$-module (= completely reducible representation) then the natural map

$$(H_nC_*)_{\mathfrak{g}} \to H_n(C_{*\mathfrak{g}})$$

is an isomorphism. Apply this with $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{Q})$ and $C_* = C_*(\mathfrak{gl}(\mathbf{Q}); T^r)$. As a $\mathfrak{gl}(\mathbf{Q})$-representation $C_n = T^r \otimes \Lambda^n \mathfrak{gl}(S)$ is a quotient of

$$T^r \otimes \mathfrak{gl}(S)^{\otimes n} = \Big(\bigotimes_j B_j \Big) \otimes S^{\otimes n} \otimes \mathfrak{gl}(\mathbf{Q})^{\otimes (r+n)},$$

the tensor product of a trivial representation and several copies of the adjoint representation. Since semisimplicity is inherited by quotients and (infinite) direct sums, we have only to show that every tensor power of $\mathfrak{gl}(\mathbf{Q})$ is semisimple as a $\mathfrak{gl}_n(\mathbf{Q})$-representation. Let V be the standard n-dimensional representation of $\mathfrak{gl}_n(\mathbf{Q})$. As a $\mathfrak{gl}_n(\mathbf{Q})$-representation

$$\mathfrak{gl}(\mathbf{Q}) \cong (V \oplus \text{trivial}) \otimes (V^* \oplus \text{trivial}).$$

Therefore we have only to show that $V^{\otimes a} \otimes V^{*\otimes b}$ is semisimple for every $a \geq 0$ and $b \geq 0$. Note that the identity matrix in $\mathfrak{gl}_n(\mathbf{Q})$ acts via the scalar $a - b$; thus every $\mathfrak{sl}_n(\mathbf{Q})$-subrepresentation is a $\mathfrak{gl}_n(\mathbf{Q})$-subrepresentation. But *any* finite-dimensional representation of $\mathfrak{sl}_n(\mathbf{Q})$ is semisimple. ([Bo], I.6.2, Theorem 2).

Step 6. (Analyze the complex of coinvariants.) For any $\mathbf{Q}$-algebra S and S-bimodules $B_1, \dots, B_r$ we will obtain an isomorphism of chain complexes

$$\text{(IV.3.4)} \quad C_*(\mathfrak{gl}(S); T^r)_{\mathfrak{gl}(\mathbf{Q})} \cong \bigoplus_{\rho \in \Sigma_r} \mathrm{Ch}_* \mathscr{V}(\rho) \otimes C_*(\mathfrak{gl}(S); \mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})}$$

where $\mathscr{V}(\rho) = \mathscr{V}(\rho; B_1, \dots, B_r)$ is the multisimplicial vector space encountered in IV.2. This will imply an isomorphism in homology

$$\text{(IV.3.5)} \qquad H_*(\mathfrak{gl}(S); T^r) \cong \bigoplus_{\rho} W_*(\rho) \otimes H_*(\mathfrak{gl}(S); \mathbf{Q})$$

and in particular in the special case when each B_j is free it will imply IV.3.3, thus finishing the proof of Theorem IV.3.2.

As preparation for analyzing the $\mathfrak{gl}(\mathbf{Q})$-coinvariants of $C_n(\mathfrak{gl}(S); T^r) = T^r \otimes \Lambda^n \mathfrak{gl}(S)$, consider

$$T^r \otimes \mathfrak{gl}(S)^{\otimes n} = M(B_1) \otimes \cdots \otimes M(B_r) \otimes M(S) \otimes \cdots \otimes M(S).$$

Its space of $\mathfrak{gl}(\mathbf{Q})$-coinvariants is

$$\bigoplus_{\lambda \in \Sigma_{r+n}} B_1 \otimes \cdots \otimes B_r \otimes S^{\otimes n},$$

the projection to the λ factor being given by

$$\text{(IV.3.6)} \quad X_1 \otimes \cdots \otimes X_{r+n} \mapsto \sum X_1(i, \lambda(i_1)) \otimes \cdots \otimes X_{r+n}(i_{r+n}, \lambda(i_{r+n})).$$

(This follows from the description of $(\mathfrak{gl}(\mathbf{Q})^{\otimes n})_{\mathfrak{gl}(\mathbf{Q})}$ given in [G2], pp. 404–405, with $r+n$ replacing n. That in turn follows from the fact that the same description holds for $(\mathfrak{gl}_k(\mathbf{Q})^{\otimes n})_{\mathfrak{gl}_k(\mathbf{Q})}$ when $k \geq n$. This description of coinvariants follows by semisimplicity from Weyl's description of the $\mathfrak{gl}_k(\mathbf{Q})$-*invariants* of $\mathfrak{gl}_k(\mathbf{Q})^{\otimes n}$, as quoted in [L-Q], p. 584.)

The space of $\mathfrak{gl}(\mathbf{Q})$-coinvariants for $C_n(\mathfrak{gl}(S); T^r)$ is obtained from this by antisymmetrizing with respect to the action of $\Sigma_n = \mathrm{Aut}\{r+1, \ldots, r+n\} \subset \Sigma_{r+n}$. It can be written

$$C_n(\mathfrak{gl}(S); T^r)_{\mathfrak{gl}(\mathbf{Q})} = \bigoplus_{\lambda} B_1 \otimes \cdots \otimes B_r \otimes \Lambda^n_\lambda(S),$$

where λ runs through a system of representatives for the orbits of the conjugation action of Σ_n on Σ_{r+n} and $\Lambda^n_\lambda(S)$ denotes the antisymmetrization of $S^{\otimes n}$ with respect to the largest subgroup of Σ_n which centralizes λ.

Any $\lambda \in \Sigma_{r+n}$ determines integers $\{p_j \geq 0 \mid 1 \leq j \leq r\}$ and a permutation $\rho \in \Sigma_r$ as follows: $p_j + 1$ is the least positive integer k such that $\lambda^k(j) \in \{1, \ldots, r\}$, and $\rho(j) = \lambda^{p_j+1}(j)$. These $\{p_j\}$ and ρ depend only on the Σ_n-orbit of λ. Set $p = \Sigma p_j$ and $q = n - p$. We can arrange for the chosen representative λ of each orbit to act like

$$\text{(IV.3.7)} \quad j \mapsto r + 1 + \sum_{k<j} p_k \mapsto r + 2 + \sum_{k<j} p_k \mapsto \cdots \mapsto r + \sum_{k \leq j} p_k \mapsto \rho(j)$$

for all $j \in \{1, \ldots, r\}$. In particular λ then belongs to the subgroup $\Sigma_{r+p} \times \Sigma_q \subset \Sigma_{r+n}$. Write $\lambda = \mu \cup \xi$, $\mu \in \Sigma_{r+p}$, $\xi \in \Sigma_q$. Then μ is determined by the orbit of λ, in fact by $(p_1, \ldots, p_r)$, but ξ is determined only up to conjugation (in Σ_q). Also, the orbit of λ is determined by ρ, p_j, and the conjugacy class of ξ. This leads to new expressions for the coinvariants:

$$\begin{aligned}
\text{(IV.3.8)} \quad C_n(\mathfrak{gl}(S); T^r)_{\mathfrak{gl}(\mathbf{Q})} &= \bigoplus_{\substack{\rho \in \Sigma_r \\ p+q=n \\ \Sigma p_j = p}} \left(\bigotimes_{j=1}^{r} B_j \right) \otimes S^{\otimes p} \otimes \bigoplus_{\xi} \Lambda^q_\xi(S) \\
&= \oplus \bigotimes_{j=1}^{r} \left(B_j \otimes S^{\otimes p_j} \right) \otimes \bigoplus_{\xi} \Lambda^q_\xi(S) \\
&= \oplus \mathscr{V}(\rho)_{p_1, \ldots, p_r} \otimes \bigoplus_{\xi} \Lambda^q_\xi(S) \\
&= \bigoplus_{\substack{\rho \in \Sigma_r \\ p+q=n}} \mathrm{Ch}_p \mathscr{V}(\rho) \otimes C_q(\mathfrak{gl}(S); \mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})}.
\end{aligned}$$

(Here ξ runs through representatives for conjugacy classes in Σ_q and $\Lambda^q_\xi(S)$ is the antisymmetrization of $S^{\otimes q}$ by the centralizer of ξ in Σ_q.) This is the desired isomorphism IV.3.4, but at the moment it is only an isomorphism of graded vector spaces.

To prove that it is a chain isomorphism we will in effect write down its inverse and prove that that is a chain map. Choose any n, ρ, p, q, and $\{p_j\}$ as above and consider any element

$$x \otimes \{y\} \in \mathrm{Ch}_p \mathscr{V}(\rho) \otimes C_q(\mathfrak{gl}(S); \mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})},$$

where

$$\begin{aligned} x &= x_1 \otimes s_{1,1} \otimes \cdots \otimes s_{1,p_1} \otimes x_2 \otimes s_{2,1} \otimes \cdots \otimes s_{r,p_r} \\ &\in \mathscr{V}(\rho)_{p_1,\ldots,p_r} \subset \mathrm{Ch}_p \mathscr{V}(\rho), \end{aligned}$$

$$x_j \in B_j,\ 1 \le j \le r,$$

$$s_{j,i} \in S,\ 1 \le j \le r,\ 1 \le i \le p_j,$$

$$y = (1|y_1|\ldots|y_q) \in C_q(\mathfrak{gl}(S); \mathbf{Q}), \qquad y_k \in \mathfrak{gl}(S),\ 1 \le k \le q.$$

(As in [G2] we use the notation $(v|u_1|\ldots u_n)$ for the element

$$v \otimes (u_1 \wedge \cdots \wedge u_n) \in V \otimes \Lambda^n \mathfrak{g} = C_n(\mathfrak{g}; V)$$

of the Koszul complex, when a Lie algebra $\mathfrak{g}$ acts on a vector space V.)

We will write down an element $z \in C_n(\mathfrak{gl}(S); T^r)$ and show that under the given isomorphism IV.3.8, $\{z\}$ corresponds to $x \otimes \{y\}$ and $d\{z\}$ corresponds to $d(x \otimes \{y\})$.

Choose $r + p$ distinct natural numbers $\{l_{j,i} | 1 \le j \le r,\ 0 \le i \le p_j\}$, such that none of them is a number l such that any matrix y_k has a nonzero l-th row or column. Define ordered pairs $\varepsilon_{j,i}$ of natural numbers by

$$\varepsilon_{j,i} = (l_{j,i}, l_{j,i+1}), \qquad 0 \le i \le p_j$$
$$\text{where } l_{j,p_j+1} = l_{\rho(j),0}$$

If $\varepsilon = (\mu, \nu)$ is an ordered pair of natural numbers and x is an element of an abelian group, let us write $E(\varepsilon, x)$ for the matrix having x as the entry in the μ-th row and ν-th column and all other entries zero. Write

$$X_j = E(\varepsilon_{j,0}, x_j) \in M(B_j),\ 1 \le j \le r,$$
$$S_{j,i} = E(\varepsilon_{j,i}, s_{j,i}) \in \mathfrak{gl}(S),\ 1 \le j \le r,\ 1 \le i \le p_j.$$

Set

$$z = \left(\bigotimes_j X_j | S_{1,1} | S_{1,2} | \ldots | S_{r,p_r} | y_1 | \ldots | y_q \right) \in C_n(\mathfrak{gl}(S); T^r).$$

We now check that the coinvariant class $\{z\}$ corresponds to $x \otimes \{y\}$ under IV.3.8. This means computing traces. Consider the element

$$\left(\bigotimes_j X_j \right) \otimes \left(\bigotimes_j \bigotimes_i S_{j,i} \right) \otimes \left(\bigotimes_k y_k \right) \in \bigotimes_j M(B_j) \otimes M(S)^{\otimes n},$$

which maps to z under antisymmetrization. For each $\lambda \in \Sigma_{r+n}$ apply the map

IV.3.6. The result (an element of $(\otimes_j B_j) \otimes S^{\otimes n}$) will be zero except in cases where λ belongs to $\Sigma_{r+p} \times \Sigma_q$ and satisfies IV.3.7, because of the way in which the numbers $l_{j,i}$ were chosen. In these cases it will be

$$\left(\bigotimes_j x_j\right) \otimes \left(\bigotimes_j \bigotimes_i s_{j,i}\right) \otimes \sum \bigotimes_k y_k\left(i_k, i_{\xi(k)}\right),$$

where $\lambda = \mu \cup \xi$ and the sum is over all $(i_1, \ldots, i_q) \in \mathbf{N}^q$. Upon antisymmetrizing and using the identification IV.3.8 this becomes $x \otimes \{y\}$.

Finally we check that $d\{z\}$ corresponds to $d(x \otimes \{y\})$. By definition these are respectively $\{dz\}$ and $dx \otimes \{y\} + (-1)^p x \otimes \{dy\}$. First notice that the following matrix products all vanish, by the choice of the numbers $l_{j,i}$:

$$X_j y_k,\ y_k X_j,$$

$$S_{j,i} y_k,\ y_k S_{j,i}$$

$$\begin{array}{ll} S_{j,i}S_{j',i'} & \text{except when } j' = j \text{ and } i' = i+1, \\ X_j S_{j',i'} & \text{except when } j' = j \text{ and } i' = 1, \\ S_{j,i}X_{j'} & \text{except when } j' = \rho(j) \text{ and } i = p_j. \end{array}$$

In view of this the formula for the Koszul differential ([G2], p. 388) boils down to

(IV.3.9)

$$\begin{aligned} dz = & \sum_{p_j \geq 1} (-1)^{p(j)} \\ & \times\left(X_1 \otimes \cdots \otimes X_j S_{j,1} \otimes \cdots \otimes X_r | S_{1,1} | \ldots |\widehat{S_{j,1}}| \ldots | S_{r,p_r} | y_1 | \ldots | y_q\right) \\ & - \sum_{p_j \geq 1} (-1)^{p(j)+p_j-1} \\ & \quad \times\left(X_1 \otimes \cdots \otimes S_{j,p_j} X_{\rho(j)} \otimes \cdots \otimes X_r | S_{1,1} | \ldots |\widehat{S_{j,p_j}}| \ldots | S_{r,p_r} | y_1 | \ldots | y_q\right) \\ & + \sum_j \sum_{1 \leq i < p_j} (-1)^{(p(j)+i)+(p(j)+i+1)} \\ & \quad \times\left(X_1 \otimes \cdots \otimes X_r | S_{j,i} S_{j,i+1} | S_{1,1} | \ldots |\widehat{S_{j,i}}|\widehat{S_{j,i+1}}| \ldots | S_{r,p_r} | y_1 | \ldots | y_q\right) \\ & + \sum_{k<k'} (-1)^{(p+k)+(p+k')} \\ & \quad \times\left(X_1 \otimes \cdots \otimes X_r | [y_k, y_{k'}] | S_{1,1} | \ldots | S_{r,p_r} | \ldots y_1 | \ldots |\widehat{y_k}| \ldots |\widehat{y_{k'}}| \ldots | y_q\right), \end{aligned}$$

where we have written $p(j)$ for $\sum_{1 \leq j' < j} p_{j'}$. The fourth term here can be rewritten

$$(-1)^p \sum_{k<k'} (-1)^{k+k'}\left(\bigotimes_j X_j | S_{1,1} | \ldots | S_{r,p_r} | [y_k, y_{k'}] | y_1 | \ldots |\widehat{y_k}| \ldots |\widehat{y_{k'}}| \ldots | y_q\right);$$

its class corresponds to $(-1)^p x \otimes \{dy\}$ by the same reasoning which showed that $\{z\}$ corresponds to $x \otimes \{y\}$. The third term is the same as

$$\sum_j \sum_{1 \le i < p_j} (-1)^{p(j)+i}$$
$$\times \left(X_1 \otimes \cdots \otimes X_r | S_{1,1} | \ldots | S_{j,i-1} | S_{j,i} S_{j,i+1} | \ldots | S_{r,p_r} | y_1 | \ldots | y_q\right).$$

Noting that $S_{j,i} S_{j,i+1} = E((l_{j,i}, l_{j,i+2}), s_{j,i} s_{j,i+1})$ and using the same reasoning a third time, we see that the class of this element corresponds to

$$\sum_{1 \le j \le r} \sum_{0 < i < p_j} (-1)^{p(j)+i} \left(d_i^j x\right) \otimes \{y\}.$$

(Here d_i^j is a face map of $\mathscr{V}(\rho)$.) Similarly, since

$$X_j S_{j,1} = E\left(\left(l_{j,0}, l_{j,2}\right), x_j s_{j,1}\right)$$

and

$$S_{j,p_j} X_{\rho(j)} = E\left(\left(l_{j,p_j}, l_{\rho(j),1}\right), s_{j,p_j} x_{\rho(j)}\right),$$

the first and second terms of IV.3.9 become respectively

$$\sum_{\substack{1 \le j \le r \\ p_j \ge 1}} (-1)^{p(j)} \left(d_0^j x\right) \otimes \{y\}$$

and

$$\sum_{\substack{1 \le j \le r \\ p_j \ge 1}} (-1)^{p(j)+p_j} \left(d_{p_j}^j x\right) \otimes \{y\}.$$

Thus the total of the first three terms is

$$\left(\sum_{1 \le j \le r} \sum_{0 \le i \le p_j} (-1)^{p(j)+i} d_i^j x\right) \otimes \{y\} = dx \otimes \{y\}.$$

(We appear to be missing the term containing d_0^j for each j such that $p_j = 0$, but these terms vanish anyway.)

This completes the proof of the isomorphism IV.3.5. One checks easily that the projection

$$H_n(\mathfrak{gl}(S); T^r) \to \bigoplus_\rho W_0(\rho) \otimes H_n(\mathfrak{gl}(S); \mathbf{Q})$$

is the same as the map IV.3.3 induced by tr, so that in particular when the B_j are free bimodules IV.3.3 is an isomorphism. □

IV.3.10. COROLLARY. *If S is a ring and $B_1, \ldots, B_r$ are free $(S \otimes \mathbf{Q})$-bimodules then the* $\mathrm{GL}(S)$*-module* $T^r = M(B_1) \otimes \cdots \otimes M(B_r)$ *has the property* *that the natural surjection $T^r \twoheadrightarrow T^r_{\mathrm{GL}(S)}$ to the coinvariants induces an isomorphism*

$$H_*(\mathrm{GL}(S); T^r) \to H_*\big(\mathrm{GL}(S); T^r_{\mathrm{GL}(S)}\big)$$

Proof. It will suffice if $\overline{\mathrm{tr}}$ in the commutative diagram

$$\begin{tikzcd} T^r \arrow[rr, two heads] \arrow[dr, "\mathrm{tr}"'] & & T^r_{\mathrm{GL}(S)} \arrow[dl, "\overline{\mathrm{tr}}"] \\ & \bigoplus_{\rho \in \Sigma_r} W_0(\rho) & \end{tikzcd}$$

is an isomorphism. Consider this as a diagram of $\mathrm{GL}(S)$-modules, where only T^r has a nontrivial action. The fact that tr induces a surjection on $H_0(\mathrm{GL}(S); -)$ implies that tr is itself surjective, and hence that $\overline{\mathrm{tr}}$ is surjective. The fact that tr induces an injection on $H_0(\mathrm{GL}(S); -)$ and a surjection on $H_1(\mathrm{GL}(S); -)$ now implies the vanishing of $H_0(\mathrm{GL}(S); \ker(\mathrm{tr}))$, and hence the vanishing of its quotient $H_0(\mathrm{GL}(S); \ker(\overline{\mathrm{tr}})) = \ker(\overline{\mathrm{tr}})$. □

IV.3.11. Corollary. *If S is a ring and B is a free $(S \otimes \mathbf{Q})$-bimodule, then the $\mathrm{GL}(S)$-module $\Lambda^r M(B)$ has the property that the natural surjection $\Lambda^r M(B) \twoheadrightarrow (\Lambda^r M(B))_{\mathrm{GL}(S)}$ induces an isomorphism on $H_*(\mathrm{GL}(S); -)$.*

Proof. Apply IV.3.10 with $B_1 = \cdots = B_r = B$. Passing to Σ_r-coinvariants commutes with taking $\mathrm{GL}(S)$-homology since the Σ_r-action and the $\mathrm{GL}(S)$-action commute (and since we are working with $\mathbf{Q}$-vector spaces). □

IV.4. *A K-theory Calculation.*

Let $I \to R \xrightarrow{f} S$ be as in the introduction to Section IV. Consider the diagram of spaces

(IV.4.1)
$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & F_2 \arrow[d] \\ B\,\mathrm{GL}(R) \arrow[r] \arrow[d] & K(R) \arrow[d] \\ B\,\mathrm{GL}(S) \arrow[r] & K(S) \end{tikzcd}$$

where the lower vertical maps are induced by f and the F_i are the homotopy fibers. We will use the rational homology of F_1 to compute the rational relative K-groups $\pi_*(F_2) \otimes \mathbf{Q}$.

Let $H_*(\ )$ denote rational homology. In the Serre rational homology spectral sequence for $K(R) \to K(S)$ the E^2 term is a tensor product; the action of $\pi_1 K(S) = \mathrm{GL}(S)^{\mathrm{ab}}$ on $H_*(F_2)$ is trivial because $K(R) \to K(S)$ is an H-space map. The corresponding action of $\pi_1 B\,\mathrm{GL}(S) = \mathrm{GL}(S)$ on $H_*(F_1)$ is highly nontrivial. In fact F_1 is a $K(\pi, 1)$ with $\pi = \ker(\mathrm{GL}(R) \to \mathrm{GL}(S)) \cong M(I)$ an abelian group (this uses the fact that $I^2 = 0$). Thus $H_j(F_1) \cong \Lambda^j M(I \otimes \mathbf{Q})$, with the action of $\mathrm{GL}(S)$ being given by the adjoint action of $\mathrm{GL}(S)$ on $M(I \otimes \mathbf{Q})$.

The fact that the two lower horizontal arrows in IV.4.1 are plus constructions implies that there is a spectral sequence with

$$\text{(IV.4.2)} \qquad \begin{aligned} E^2_{i,j} &= H_i\big(B\,\mathrm{GL}(S); H_j(F_1 \to F_2)\big) \\ E^\infty_{i,j} &= 0. \end{aligned}$$

If the action of $\mathrm{GL}(S)$ on $H_j(F_1 \to F_2)$ were trivial (which it is not) we could conclude that E^2 is a tensor product and hence that E^2 vanishes and that $H_j(F_1 \to F_2)$ vanishes (which it does not). Instead we introduce the coinvariant homology $H_j(F_1)_{\mathrm{GL}(S)}$ and the maps

$$\begin{tikzcd} H_j(F_1) \arrow[rr, "\phi_j"] \arrow[dr, "\psi_j"'] & & H_j(F_2) \\ & H_j(F_1)_{\mathrm{GL}(S)} \arrow[ur, "\bar{\phi}_j"'] & \end{tikzcd}$$

(Here ϕ_j is induced by IV.4.1, ψ_j is the quotient map, and $\bar{\phi}_j$ makes the triangle commute; such a $\bar{\phi}_j$ exists because $\mathrm{GL}(S)$ acts trivially on $H_j(F_2)$.)

IV.4.3. LEMMA. *$\bar{\phi}_j$ is an isomorphism for all j.*

Proof. The key fact here is Corollary IV.3.11; it says that the map ψ_j induces an isomorphism of $H_*(\mathrm{GL}(S); -)$.

Assume that for some j, $\bar{\phi}_j$ is not an isomorphism. Choose the smallest j such that either $\bar{\phi}_j$ fails to surject or $\bar{\phi}_{j-1}$ fails to inject. Since $\mathrm{GL}(S)$ is acting trivially on both $H_j(F_1)_{\mathrm{GL}(S)}$ and $H_j(F_2)$, this is also the smallest j such that for some $i \geq 0$ either $H_i(\mathrm{GL}(S); H_j(F_1)_{\mathrm{GL}(S)}) \to H_i(\mathrm{GL}(S); H_j(F_2))$ fails to surject or the corresponding map for $j - 1$ fails to inject; and moreover this failure occurs already when $i = 0$. By IV.3.11, the subscript $\mathrm{GL}(S)$ can be omitted in this last statement. Thus in the spectral sequence IV.4.2 the group $E^2_{0,j}$ is nontrivial and j is the least integer such that any $E^2_{i,j}$ is nontrivial. This implies that $E^\infty_{0,j} = E^2_{0,j} \neq 0$, a contradiction. □

We have now computed the rational homology of F_2 as a graded vector space:

$$H_nF_2 \cong (H_nF_1)_{\mathrm{GL}(S)} \cong \Lambda^n M(I \otimes \mathbf{Q})_{\mathrm{GL}(S)}.$$

We need to compute the coalgebra structure as well, in order to find the primitive part. Of course the surjection $\phi\colon H_*F_1 \to H_*F_2$ is a coalgebra map since it is induced by a map of spaces; this is what will allow us to make the computation.

Let A_* (resp. Λ_*) be the free graded (resp. free commutative graded) $\mathbf{Q}$-algebra generated by a copy of $M(I \otimes \mathbf{Q})$ in dimension one. This is a Hopf algebra in a unique way, namely with comultiplication Δ given by

$$\Delta x = x \otimes 1 + 1 \otimes x \quad \text{for } x \in A_1 \,(\text{resp. } \Lambda_1).$$

The group $\mathrm{GL}(S)$ acts on $M(I \otimes \mathbf{Q})$, hence on both of the Hopf algebras. When a group acts on a coalgebra the coinvariants form a coalgebra. We get a diagram of coalgebras and coalgebra maps

$$\begin{tikzcd} A_* \arrow[r] \arrow[d] & \Lambda_* \arrow[d] & \cong H_*F_1 \arrow[d, "\phi"] \\ (A_*)_{\mathrm{GL}(S)} \arrow[r] & (\Lambda_*)_{\mathrm{GL}(S)} & \cong H_*F_2 \end{tikzcd}$$

(Note: ϕ is not an algebra map, and $(A_*)_{\mathrm{GL}(S)}$ is not even being given an algebra structure.)

We next determine the structure of the coalgebra $(A_*)_{\mathrm{GL}(S)}$. By the proof of Corollary IV.3.10 we can write

$$\text{(IV.4.4)} \qquad (A_n)_{\mathrm{GL}(S)} = \bigoplus_{\nu \in \Sigma_n} W_0(\nu; I \otimes \mathbf{Q}).$$

To determine the comultiplication, consider any element $\prod_{i=1}^n E_i \in A_n$, $E_i \in A_1$. We have

$$\begin{aligned} \Delta\Big(\prod_i E_i\Big) &= \prod_i \Delta E_i \\ &= \prod_i (E_i \otimes 1 + 1 \otimes E_i) \\ &= \sum_{\substack{p+q=n \\ \rho}} \operatorname{sgn}(\rho)\left(\prod_{\alpha=1}^{p} E_{\rho(\alpha)} \otimes \prod_{\beta=p+1}^{n} E_{\rho(\beta)}\right) \end{aligned}$$

where ρ runs through all (p, q)-shuffles in Σ_n. In particular suppose that for some $\nu \in \Sigma_n$ and $x_1, \dots, x_n \in I \otimes \mathbf{Q}$ we take E_i to be $E((i, \nu(i)), x_i)$, using the notation of IV.3. Then the element $\{\prod E_i\} \in (A_n)_{\mathrm{GL}(S)}$ corresponds to

$[x_1, \ldots, x_n]_\nu \in W_0(\nu)$ under the isomorphism IV.4.4. Now let us see what the element $\{\prod_{\alpha=1}^{p} E_{\rho(\alpha)}\}$ corresponds to under

$$(A_p)_{\mathrm{GL}(S)} = \bigoplus_{\pi \in \Sigma_p} W_0(\pi).$$

The answer is zero unless the set $\rho\{1, \ldots, p\}$ is preserved by ν, in which case it is

$$[x_{\rho(1)}, \ldots, x_{\rho(p)}]_\pi \in W_0(\pi),$$

where $\pi \in \Sigma_p$ is defined by $\pi(\alpha) = \rho^{-1}\nu\rho(\alpha)$, $1 \le \alpha \le p$. Likewise $\{\prod_{\beta=p+1}^{n} E_{\rho(\beta)}\} = 0$ unless $\rho\{p+1, \ldots, n\}$ is preserved by ν, in which case it corresponds to

$$[x_{\rho(p+1)}, \ldots, x_{\rho(n)}]_\xi \in W_0(\xi),$$

where $\xi \in \Sigma_q$ is defined by $\xi(\beta - p) = \rho^{-1}\nu\rho(\beta) - p$, $p + 1 \le \beta \le n$.

In other words if we identify $(A_n)_{\mathrm{GL}(S)}$ with $\oplus_{\nu \in \Sigma_n} W_0(\nu)$ according to IV.4.4, then the comultiplication is given by

$$\text{(IV.4.5)} \qquad \Delta[x_1, \ldots, x_n]_\nu = \sum_{\substack{p+q=n \\ \rho^{-1}\nu\rho \in \Sigma_p \times \Sigma_q}} \operatorname{sgn}(\rho)[x_{\rho(1)}, \ldots, x_{\rho(p)}]_\pi \otimes [x_{\rho(p+1)}, \ldots, x_{\rho(n)}]_\xi$$

where ρ runs through all (p, q)-shuffles such that $\rho^{-1}\nu\rho$ belongs to $\Sigma_p \times \Sigma_q$, and where π and ξ are such that $\rho^{-1}\nu\rho = \pi \cup \xi$.

Now we pass from $(A_*)_{\mathrm{GL}(S)}$ to $(\Lambda_*)_{\mathrm{GL}(S)}$. Antisymmetrizing IV.4.4, we can write

$$\text{(IV.4.6)} \qquad (\Lambda_n)_{\mathrm{GL}(S)} = \bigoplus_{\nu} \overline{W}_0(\nu),$$

where ν ranges over representatives for conjugacy classes in Σ_n. However, we prefer not to choose actual representatives, but rather make the convention that *all* the $\overline{W}_0(\nu)$ are contained in $(\Lambda_n)_{\mathrm{GL}(S)}$, with

$$\text{(IV.4.7)} \qquad \overline{W}_0(\rho^{-1}\nu\rho) = \overline{W}_0(\nu)$$

$$[\overline{x_{\rho(1)}, \ldots, x_{\rho(n)}}]_{\rho^{-1}\nu\rho} = \operatorname{sgn}(\rho)[\overline{x_1, \ldots, x_n}]_\nu$$

Now from IV.4.5 we can read off a formula for the comultiplication in $(\Lambda_*)_{\mathrm{GL}(S)}$:

$$\text{(IV.4.8)} \qquad \Delta[\overline{x_1, \ldots, x_n}]_\nu = \sum_{\substack{p+q=n \\ \rho^{-1}\nu\rho \in \Sigma_p \times \Sigma_q}} \operatorname{sgn}(\rho)[\overline{x_{\rho(1)}, \ldots, x_{\rho(p)}}]_\pi \otimes [\overline{x_{\rho(p+1)}, \ldots, x_{\rho(n)}}]_\xi$$

just as before. The set of all $\rho \in \Sigma_n$ such that $\rho^{-1}\nu\rho \in \Sigma_p \times \Sigma_q$ is a union of right cosets of $\Sigma_p \times \Sigma_q$, and each coset is represented by a unique (p, q)-shuffle. Note that by the convention just established (IV.4.7), equation IV.4.8 remains valid even if we use some other set of coset representatives instead of the shuffles.

IV.4.9. Claim. *The primitive part of* $(\Lambda_n)_{\mathrm{GL}(S)}$ *is the summand* $\overline{W}_0(\rho_n)$ *in* IV.4.6.

Proof. Consider, for any $\nu \in \Sigma_n$, $\pi \in \Sigma_p$, and $\xi \in \Sigma_q$, the composition

$$\overline{W}_0(\nu) \rightarrowtail (\Lambda_n)_{\mathrm{GL}(S)} \xrightarrow{\Delta} \left((\Lambda_*)_{\mathrm{GL}(S)} \otimes (\Lambda_*)_{\mathrm{GL}(S)}\right)_n \twoheadrightarrow \overline{W}_0(\pi) \otimes \overline{W}_0(\xi).$$

Clearly by IV.4.8 if ν is not conjugate to $\pi \cup \xi$ in Σ_n then the composition is zero. If ν is conjugate to $\pi \cup \xi$ then we will show that the composition is injective. This easily implies the claim.

We may assume $\nu = \pi \cup \xi$. Write

$$W_0(\pi \cup \xi) = W_0(\pi) \otimes W_0(\xi),$$

using IV.1.4. The centralizer $G = C(\pi \cup \xi)$ acts on $W_0(\pi \cup \xi)$. Insert a sign in the action, so that the space of coinvariants is $\overline{W}_0(\pi \cup \xi)$. For the subgroup $H = C(\pi) \times C(\xi) \subset G$ the space of coinvariants may be identified with $\overline{W}_0(\pi) \otimes \overline{W}_0(\xi)$. In these terms the map

$$\overline{W}_0(\pi \cup \xi) \to \overline{W}_0(\pi) \otimes \overline{W}_0(\xi)$$

under investigation is the "transfer"

$$W_0(\pi \cup \xi)_G \to W_0(\pi \cup \xi)_H.$$

That is, it takes the element represented by

$$[x_1, \ldots, x_n]_{\pi \cup \xi} \in W_0(\pi \cup \xi)$$

to the element represented by the sum

$$\sum_\rho \mathrm{sgn}(\rho)\left[x_{\rho(1)}, \ldots, x_{\rho(n)}\right]_{\pi \cup \xi} \in W_0(\pi \cup \xi)$$

where ρ runs through coset representatives for G/H. Indeed by IV.4.8, it takes it to the sum

$$\sum_\rho \mathrm{sgn}(\rho)\left[\overline{x_{\rho(1)}, \ldots, x_{\rho(p)}}\right]_{\pi'} \otimes \left[\overline{x_{\rho(p+1)}, \ldots, x_{\rho(n)}}\right]_{\xi'}$$

where ρ runs through any system of representatives for those cosets of $\Sigma_p \times \Sigma_q$ in Σ_n such that $\rho^{-1}\nu\rho$ has the form $\pi' \cup \xi'$ with π' conjugate to π in Σ_p and

ξ' conjugate to ξ in Σ_q. In particular we may choose each ρ such that $\pi' = \pi$ and $\xi' = \xi$, and in this case ρ runs through representatives for G/H.

The transfer is injective by a well-known principle: The composition with the quotient map

$$W_0(\pi \cup \xi)_G \to W_0(\pi \cup \xi)_H \to W_0(\pi \cup \xi)_G$$

is an isomorphism, namely multiplication by the index $(G : H)$. □

This completes the K-theory computation IV.1.6, since

$$\begin{aligned} K_{r+1}(f) \otimes \mathbf{Q} &= \pi_{r+1}(K(R) \to K(S)) \otimes \mathbf{Q} \\ &= \pi_r(F_2) \otimes \mathbf{Q} \\ &\cong \operatorname{Prim}_r H_*(F_2) \\ &\cong \operatorname{Prim}_r(\Lambda_{*\mathrm{GL}(S)}) \\ &\cong \overline{W}_0(\rho_r; I \otimes \mathbf{Q}). \end{aligned}$$

To finish the proof of Theorem II.3.4. we still have to check that the upper portion of the diagram

$$\begin{tikzcd} & \overline{W}_0(\rho_n; I \otimes \mathbf{Q}) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ K_{n+1}(f) \otimes \mathbf{Q} \arrow[r, "\alpha_{\mathbf{Q}}"] \arrow[dr, "\tau_{\mathbf{Q}}"'] & HC^-_{n+1}(f \otimes \mathbf{Q}) \arrow[d, "\pi"] & HC_n(f \otimes \mathbf{Q}) \arrow[l, "\beta_{\mathbf{Q}}"'] \arrow[dl, "B"] \\ & H_{n+1}(f \otimes \mathbf{Q}) & \end{tikzcd}$$

commutes, perhaps up to a sign depending on n. Fortunately we can immediately bypass the groups HC^- and use the outer diamond-shaped diagram instead, thanks to the following fact.

IV.4.10. Claim. *The map* π: $HC^-_{n+1}(f \otimes \mathbf{Q}) \to H_{n+1}(f \otimes \mathbf{Q})$ *is injective for all* n.

Proof. This is equivalent to saying that B: $HC_n(f \otimes \mathbf{Q}) \to H_{n+1}(f \otimes \mathbf{Q})$ is injective, or equivalently that S: $HC_{n+2}(f \otimes \mathbf{Q}) \to HC_n(f \otimes \mathbf{Q})$ is zero. The latter is true because $f \otimes \mathbf{Q}$ is the projection from a graded $\mathbf{Q}$-algebra to its degree zero part ([G1], p. 197 proof of Claim 1). □

Of course we can also replace $H_{n+1}(f \otimes \mathbf{Q})$ by $H_n(R \otimes \mathbf{Q})$, into which it injects. The composition

$$\overline{W}_0(\rho_n; I \otimes \mathbf{Q}) \cong HC_n(f \otimes \mathbf{Q}) \xrightarrow{B} H_{n+1}(f \otimes \mathbf{Q}) \rightarrowtail H_n(R \otimes \mathbf{Q})$$

has already been computed, in IV.2.9. We have to compute the composition

$$\overline{W}_0(\rho_n; I \otimes \mathbf{Q}) \cong K_{n+1}(f) \otimes \mathbf{Q} \xrightarrow{\tau_{\mathbf{Q}}} H_{n+1}(f \otimes \mathbf{Q}) \rightarrowtail H_n(R \otimes \mathbf{Q}).$$

To do so we use the diagram

$$\begin{tikzcd}
\overline{W}_0(\rho_n; I \otimes \mathbf{Q}) & \cong K_{n+1}(f) \otimes \mathbf{Q} \arrow[r] \arrow[d, equal] & K_n(R) \otimes \mathbf{Q} \arrow[d, equal] & \\
\mathrm{Prim}_n((\Lambda_*)_{\mathrm{GL}(S)}) \arrow[d] & \cong \pi_n(F_2) \otimes \mathbf{Q} \arrow[d] & = \pi_n K(R) \otimes \mathbf{Q} \arrow[d] & \\
(\Lambda_n)_{\mathrm{GL}(S)} & \cong H_n(F_2) \arrow[r] & H_n K(R) & \\
\Lambda_n \arrow[u] & \cong H_n(F_1) \arrow[r] \arrow[u] & H_n B\,\mathrm{GL}(R) \arrow[u, "\cong"'] \arrow[r] & H_n(R \otimes \mathbf{Q}).
\end{tikzcd}$$

Let $x_i \in I$, $1 \leq i \leq n$. We want to chase the element $[\overline{x_1, \ldots, x_n}]_{\rho_n}$ around the top and right edges of the diagram above. Instead we chase it around the left and lower edges. Chasing it to the lower left corner we obtain the element

$$e_1 \ldots e_n \in \Lambda_n$$

where $e_i \in \Lambda_1 = M(I \otimes \mathbf{Q})$ is the matrix

$$\begin{aligned} &E((i, i+1), x_i), && 1 \leq i < n, \\ &E((n, 1), x_n), && i = n. \end{aligned}$$

Recall that F_1 is (equivalent to) $BM(I)$. Using the standard chain complex $C_* B\pi$ for the classifying space of a group, we then get the class of the cycle

$$\sum_{\rho \in \Sigma_n} \mathrm{sgn}(\rho)(e_{\rho(1)}, \ldots, e_{\rho(n)}) \in C_n BM(I).$$

Applying the group homomorphism $e \to 1 + e$ from $M(I)$ to $\mathrm{GL}(R)$ we then get the cycle

$$\sum_{\rho \in \Sigma_n} \mathrm{sgn}(\rho)(1 + e_{\rho(1)}, \ldots, 1 + e_{\rho(n)}) \in C_n B\,\mathrm{GL}(R).$$

Next apply the map $t(\mathrm{GL}(R))$ of II.1 to get the cycle

$$\sum_{\rho \in \Sigma_n} \mathrm{sgn}(\rho)\left(1 - \sum_{i=1}^{n} e_i\right) \otimes (1 + e_{\rho(1)}) \otimes \cdots \otimes (1 + e_{\rho(n)})$$
$$\in T_n^{0,0}(\mathbf{Z}\,\mathrm{GL}(R)).$$

(Recall that $e_i e_i = 0$ since $I^2 = 0$.) Since we are using the *normalized* Hochschild complex this can be written

$$\sum_{\rho} \operatorname{sgn}(\rho)\left(1 - \sum_{i=1}^{n} e_i\right) \otimes e_{\rho(1)} \otimes \cdots \otimes e_{\rho(n)}.$$

The class of this cycle in $H_n(R \otimes \mathbf{Q})$ is in fact in $\operatorname{gr}_n H_n(R \otimes \mathbf{Q})$, since by IV.2.1 we have

$$HC_{n-1}(R \otimes \mathbf{Q}) = \operatorname{gr}_0 HC_{n-1}(R \otimes \mathbf{Q}) \oplus \operatorname{gr}_n HC_{n-1}(R \otimes \mathbf{Q}).$$

Therefore we may use the n-th graded part of the cycle:

$$\sum_{\rho \in \Sigma_n} \operatorname{sgn}(\rho) 1 \otimes e_{\rho(1)} \otimes \cdots \otimes e_{\rho(n)}.$$

Apply the trace map ε of II.1. There are contributions only from those ρ which are powers of ρ_n:

$$\sum_{j=0}^{n-1} (-1)^{j(n+1)} 1 \otimes x_{j+1} \otimes \cdots \otimes x_j.$$

This agrees with IV.2.9. □

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