ON THE MOTIVIC SEGAL CONJECTURE

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Abstract. We establish motivic versions of the theorems of Lin and Gunawardena, thereby confirming the motivic Segal conjecture for the algebraic group μ_ℓ of ℓ-th roots of unity, where ℓ is any prime. To achieve this we develop motivic Singer constructions associated to the symmetric group S_ℓ and to μ_ℓ, and introduce a delayed limit Adams spectral sequence.

Contents

1. Introduction

Let $\gamma^1 \downarrow \mathbb{R}P^\infty \simeq BC_2$ be the tautological line bundle over infinite-dimensional real projective space, let $\mathbb{R}P^\infty_{-m} = Th(-m\gamma^1)$ be the Thom spectrum of the negative of m times γ^1, and let $\mathbb{R}P^\infty_{-\infty} = \operatorname{holim}_m \mathbb{R}P^\infty_{-m}$. Mahowald conjectured that there is a 2-adic equivalence $\mathbb{R}P^\infty_{-\infty} \simeq S^{-1}$, see [Ada74, p. 5]. More generally, Segal conjectured for finite groups G that there is an $I(G)$-adic equivalence $(S_G)^G \simeq (S_G)^{hG}$ from the fixed points to the homotopy fixed points of the G-equivariant sphere spectrum. Here $I(G)$ denotes the augmentation ideal in the Burnside ring of G.

Mahowald's conjecture, which is equivalent to Segal's Burnside ring conjecture for C_2, was proved by Lin in [Lin80, Thm. 1.2]. For odd primes ℓ, Gunawardena [Gun81] proved Segal's conjecture for C_ℓ, obtaining an ℓ-adic equivalence $L^\infty_{-\infty} \simeq S^{-1}$. Here $L^\infty_{-\infty}$ denotes a homotopy limit of Thom spectra over the infinite-dimensional lens space $L^\infty \simeq BC_\ell$. Segal's conjecture was later affirmed for all finite groups by Carlsson [Car84], building on [MM82], [AGM85] and [CMP87].

In this paper we promote the classical theorems of Lin and Gunawardena to the motivic setting, obtaining $\pi_{*,*}$-isomorphisms $\mathbb{S} \simeq \Sigma^{1,0} L^\infty_{-\infty}$, after (ℓ, η)-adic completion, for all primes ℓ. Here $\mathbb{S}$ denotes the motivic sphere spectrum, and now

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$L^\infty_{-\infty} = \operatorname{holim}_m L^\infty_{-2m}$, where L^∞_{-2m} is the Thom spectrum of a virtual algebraic vector bundle over the geometric classifying space $L^\infty = B\mu_\ell$ of the algebraic group μ_ℓ of ℓ-th roots of unity. More precisely, $L^\infty_{-2m} = \operatorname{hocolim}_n L^{2n-2m-1}_{-2m}$, where $L^{2n-2m-1}_{-2m} = Th(-m\gamma_n^* \downarrow L^{2n-1})$ and γ_n^* is the dual of the tautological algebraic line bundle over $L^{2n-1} = (\mathbb{A}^n \setminus \{0\})/\mu_\ell$.

Theorem 1.1. *Let S be a finite dimensional Noetherian scheme, essentially smooth over a field or Dedekind domain containing $1/\ell$. There is a $\pi_{*,*}$-isomorphism*

$$e^\wedge_{\ell,\eta} \colon \mathbb{S}^\wedge_{\ell,\eta} \longrightarrow (\Sigma^{1,0} L^\infty_{-\infty})^\wedge_{\ell,\eta}$$

in the stable motivic homotopy category $SH(S)$. If $S = \operatorname{Spec} k$ for k a field, then $e^\wedge_{\ell,\eta}$ is a motivic equivalence.

In other words, we prove the motivic Segal conjecture in its non-equivariant form, in the case of the algebraic group μ_ℓ, for any prime ℓ. For $\ell = 2$ this is the motivic version of Mahowald's conjecture and Lin's theorem. For ℓ odd it is the motivic version of Gunawardena's theorem.

Already for $S = \operatorname{Spec} k$ in the algebraically closed case $k = \mathbb{C}$, the additional information about motivic weight has proved to be a valuable new tool for calculational purposes [Isa19], [IWX20]. In the real case $k = \mathbb{R}$, many new phenomena arise [Hil11], [DI17], [BI22]. Our results are valid even in the arithmetically most substantial cases of (rings of ℓ-integers in) number fields. In particular, we have made an effort to not have to assume that the mod ℓ motivic cohomology groups $H^{*,*} = H^{*,*}(S; \mathbb{Z}/\ell)$ are finite in each bidegree. Our results enable an analysis of $\pi_{*,*}(\mathbb{S})$ by comparison with the homotopy spectral sequence associated to the tower $\{L^\infty_{-2m}\}_m$, i.e., the motivic Mahowald root invariants, refining [Mah85], [MR93]. Such applications have already appeared in [Qui21a], [Qui21b]. We expect the interplay between the motivic cohomology of number fields and the Mahowald root invariants to be very rich.

In Section 2 we review from Voevodsky's article [Voe03a] the Hopf algebroid structure of the motivic dual Steenrod algebra $\mathscr{A}_{*,*}$, and of its quotients $A(n)_{*,*} = \mathscr{A}_{*,*}/I(n)$. In Section 3 we generalize the approach of Adams–Gunawardena–Miller from [AGM85, §2], and introduce the $A(n)_{*,*}$-$A(n-1)_{*,*}$ bicomodule algebras $C(n)_{*,*} = \mathscr{A}_{*,*}/J(n)$ and their localizations $B(n)_{*,*}$ away from ξ_1. In Section 4 we dualize these constructions, following Boardman [Boa82, §3], obtaining the motivic Steenrod algebra $\mathscr{A}$, its finite subalgebras $A(n)$, and the $A(n)$-$A(n-1)$-bimodules $C(n)$ and $B(n)$. In Section 5 we generalize the (small) Singer construction of [Sin81] and [LS82], obtaining an $\mathscr{A}$-module $R_S(M) = \operatorname{colim}_n B(n) \otimes_{A(n-1)} M$ and a natural homomorphism $\epsilon \colon R_S(M) \to M$ for each $\mathscr{A}$-module M.

We prove in Theorem 5.8 that $R_S(H^{*,*}) \cong \Sigma^{1,0} H^{*,*}(BS_\ell)_{\text{loc}}$ is a shifted localization of the motivic cohomology of the geometric classifying space of the symmetric group S_ℓ on ℓ letters. In Section 6 we recast [AGM85, §5] and construct a (large) Singer construction $R_\mu(M)$ and a natural $\mathscr{A}$-module homomorphism $\epsilon \colon R_\mu(M) \to M$. We show in Corollary 6.6 that $R_\mu(H^{*,*}) \cong \Sigma^{1,0} H^{*,*}(B\mu_\ell)_{\text{loc}}$ is a shifted localization of the motivic cohomology of the infinite lens space $B\mu_\ell$. In Section 7 we prove that the evaluation homomorphisms ϵ are Ext-equivalences. Here we deviate from the Tor-equivalence approach of [AGM85, §2], due to the two-sided nature of Hopf algebroids.

In Section 10 we construct the tower $\{L^\infty_{-2m}\}_m$ of motivic spectra, and the map $e \colon \mathbb{S} \to \Sigma^{1,0} L^\infty_{-\infty}$ to the suspension of their homotopy limit. We show in Proposition 10.10 that the continuous cohomology $H^{*,*}_c(L^\infty_{-\infty}) = \operatorname{colim}_m H^{*,*}(L^\infty_{-2m})$ is isomorphic as an $\mathscr{A}$-module to $H^{*,*}(B\mu_\ell)_{\text{loc}}$, and that e induces the Ext-equivalence ϵ, via the identifications above. The plan is now to compare the motivic mod ℓ Adams spectral sequence for $\mathbb{S}$ with the tower of Adams spectral sequences associated to the L_{-2m}^{∞}. This works fine in the presence of sufficient finiteness to ensure that the algebraic limit of these Adams spectral sequences is again a spectral sequence, as is the case in the classical setting of [CMP87]. However, for base schemes S such that $H^{*,*}$ is not finite in each bidegree, this approach can fail. Instead, we form a modified Adams spectral sequence, called the delayed limit Adams spectral sequence, where any $\lim^1$-classes arising from non-exactness are shifted up into the next filtration degree.

In Section 8 we prepare for this construction by introducing some terminology for motivic generalized Eilenberg–MacLane spectra, and formulate a finiteness condition, called bifinite type, which lets us identify the E_1- and E_2-terms of motivic Adams spectral sequences in algebraic terms. In Section 9 we introduce the delayed limit Adams spectral sequence in Definition 9.1, and identify its E_2-term as Ext for a continuous cohomology $\mathscr{A}$-module in Proposition 9.2. In Proposition 9.6 we show that the delayed limit Adams spectral sequence converges conditionally, and in Proposition 9.7 we adapt a comparison theorem from [Boa99] for morphisms of conditionally convergent spectral sequences. In Section 11 the threads are brought together. See Theorem 11.1 for the proof of Theorem 1.1.

This article is based on the first author's PhD thesis [Gre13], guided by the second author.

2. The motivic Steenrod algebra and its dual

Let S be a Noetherian (separated) scheme of finite (Krull) dimension d, essentially smooth over a field or a Dedekind domain, and let ℓ be a prime that is invertible on S.

Let $SH(S)$ be Voevodsky's motivic stable homotopy category [Voe98, Def. 5.7], [Jar00] associated to smooth schemes over S. It is triangulated, and has a compatible closed symmetric monoidal structure given by the motivic sphere spectrum $\mathbb{S} = \Sigma^\infty S_+$, the smash product pairing $- \wedge -$, the twist isomorphism γ and the function spectrum $F(-,-)$. Let $H = H\mathbb{Z}/\ell$ be the motivic Eilenberg–MacLane spectrum representing motivic cohomology with coefficients in $\mathbb{Z}/\ell$. It is a commutative ring spectrum, with unit map $\eta\colon \mathbb{S} \to H$ and product $\mu\colon H \wedge H \to H$. Moreover, H is known to be cellular [Hoy15, Prop. 8.1], [Spi18, Cor. 10.4], i.e., an iterated homotopy colimit of stable motivic spheres.

Let $H_{*,*} = \pi_{*,*}(H) = H^{-*,-*}$ denote the motivic homology and cohomology groups of the base scheme S. Then $H^{p,q} = 0$ unless $0 \le p \le \min\{q+d, 2q\}$, cf. [Gei04, Cor. 4.4], [Hoy15, Cor. 4.26]. For $x \in \pi_{t,u}(X)$, where X is any motivic spectrum, we refer to t and u as the topological degree and weight of x, respectively. We write $|x| = \deg(x) = t$, $\mathrm{wt}(x) = u$ and $\|x\| = (t,u)$. The cup product induced by μ gives $H^{*,*} = H_{-*,-*}$ the structure of a bigraded commutative $\mathbb{Z}/\ell$-algebra. Only the parity of the topological degree plays a role in bigraded commutativity.

Let $\mathscr{A} = H^{*,*}(H) = \pi_{-*,-*}F(H,H)$ denote the motivic Steenrod algebra, and let $\mathscr{A}_{*,*} = H_{*,*}(H) = \pi_{*,*}(H \wedge H)$ denote its dual. Then $\mathscr{A}_{*,*}$ is free as a left $H_{*,*}$-module, cf. Lemma 2.1, so the pair $(H_{*,*}, \mathscr{A}_{*,*})$ admits the structure of a bigraded Hopf algebroid [Ada69, Lec. 3], [MR77, §1], [Rav86, Def. A1.1.1]. Its structure maps are the following $\mathbb{Z}/\ell$-algebra homomorphisms:

(1) the left unit $\eta_L\colon H_{*,*} \to \mathscr{A}_{*,*}$ induced by $1 \wedge \eta\colon H = H \wedge \mathbb{S} \to H \wedge H$;
(2) the right unit $\eta_R\colon H_{*,*} \to \mathscr{A}_{*,*}$ induced by $\eta \wedge 1\colon H = \mathbb{S} \wedge H \to H \wedge H$;
(3) the product $\phi\colon \mathscr{A}_{*,*} \otimes \mathscr{A}_{*,*} \to \mathscr{A}_{*,*}$ induced by $(\mu \wedge \mu)(1 \wedge \gamma \wedge 1)\colon H \wedge H \wedge H \wedge H \to H \wedge H$;
(4) the counit $\epsilon\colon \mathscr{A}_{*,*} \to H_{*,*}$ induced by $\mu\colon H \wedge H \to H$;
(5) the coproduct $\psi\colon \mathscr{A}_{*,*} \to \mathscr{A}_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*}$ induced by $1 \wedge \eta \wedge 1\colon H \wedge H = H \wedge \mathbb{S} \wedge H \to H \wedge H \wedge H \cong (H \wedge H) \wedge_H (H \wedge H)$;

(6) the conjugation $\chi\colon \mathscr{A}_{*,*} \to \mathscr{A}_{*,*}$ induced by $\gamma\colon H \wedge H \to H \wedge H$.

We use the left and right units to view $\mathscr{A}_{*,*}$ as an $H_{*,*}$-$H_{*,*}$-bimodule, and $-\otimes_{H_{*,*}} -$ in (5) denotes the bimodule tensor product.

More explicitly,

$$\mathscr{A}_{*,*} = H_{*,*}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 - T_i \mid i \ge 0)$$

is a bigraded commutative $H_{*,*}$-algebra generated by classes τ_i in bidegree $\|\tau_i\| = (2\ell^i - 1, \ell^i - 1)$ and ξ_i in bidegree $\|\xi_i\| = (2\ell^i - 2, \ell^i - 1)$, where

$$T_i = \begin{cases} \tau\xi_{i+1} + \rho\tau_{i+1} + \rho\tau_0\xi_{i+1} & \text{for } \ell = 2, \\ 0 & \text{for } \ell \text{ odd.} \end{cases}$$

Here the elements $\rho \in H^{1,1} = H_{-1,-1}$ and $\tau \in H^{0,1} = H_{0,-1}$ are specified for $\ell = 2$ in [Voe03a, Thm. 6.10]. They shall be interpreted to be zero for ℓ odd. In these terms,

(1) the algebra unit is η_L;
(2) $\eta_R = \chi\eta_L$ satisfies $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \rho\tau_0$;
(3) the algebra product is ϕ;
(4) the counit ϵ maps each τ_i and ξ_i to 0;
(5) the coproduct ψ satisfies

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{\ell^j} \otimes \tau_j \qquad \text{and} \qquad \psi(\xi_k) = \sum_{i+j=k} \xi_i^{\ell^j} \otimes \xi_j \,,$$

where $\xi_0 = 1$;
(6) the conjugation χ satisfies

$$\tau_k + \sum_{i+j=k} \xi_i^{\ell^j} \chi(\tau_j) = 0 \qquad \text{and} \qquad \sum_{i+j=k} \xi_i^{\ell^j} \chi(\xi_j) = 0 \,,$$

and $\chi^2 = 1$.

See [Voe03a, Thm. 12.6, Lem. 12.11, Rem. 12.12], [Voe10, Thm. 3.49], [HKØ17, Thm. 5.6] and [Spi18, Thm. 10.26] for proofs.

Lemma 2.1. *The monomials*

$$\tau^E \xi^R = \tau_0^{e_0} \tau_1^{e_1} \cdots \xi_1^{r_1} \xi_2^{r_2} \cdots \,,$$

where $E = (e_0, e_1, \dots)$ and $R = (r_1, r_2, \dots)$ range through the finite length integer sequences with $e_s \in \{0, 1\}$ and $r_s \ge 0$, form a basis for

$$\mathscr{A}_{*,*} = \frac{H_{*,*}[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]}{(\tau_i^2 - T_i \mid i \ge 0)}$$

as a free left $H_{*,*}$-module.

Proof. For ℓ odd this is clear. The claim for $\ell = 2$ follows from the form of the relations $\tau_i^2 = T_i$, since ξ_{i+1}, τ_{i+1} and $\tau_0\xi_{i+1}$ have higher weight than τ_i^2. □

Lemma 2.2. *The same monomials $\tau^E \xi^R$ as in Lemma 2.1 form a basis for $\mathscr{A}_{*,*}$ as a free right $H_{*,*}$-module.*

Proof. For $t \ge 0$ let

$$F^t \mathscr{A}_{*,*} = \langle \tau^E \xi^R \mid \deg(\tau^E \xi^R) \ge t \rangle \subset \mathscr{A}_{*,*}$$

be the left $H_{*,*}$-submodule generated by the monomials from Lemma 2.1 of topological degree $\ge t$. These are also right $H_{*,*}$-submodules, since $\epsilon\eta_L = \mathrm{id} = \epsilon\eta_R$ implies $\eta_L \equiv \eta_R \mod F^1\mathscr{A}_{*,*} = \ker(\epsilon)$, and $F^t\mathscr{A}_{*,*} \cdot F^1\mathscr{A}_{*,*} \subset F^{t+1}\mathscr{A}_{*,*} \subset F^t\mathscr{A}_{*,*}$. (The first inclusion uses that $\tau_i^2 = T_i$ has topological degree less than or equal to that of ξ_{i+1}, τ_{i+1} and $\tau_0\xi_{i+1}$.) This defines a decreasing filtration of $\mathscr{A}_{*,*}$ by $H_{*,*}$-$H_{*,*}$-bimodules, such that the left and right $H_{*,*}$-module actions agree on each filtration quotient

$$\mathrm{gr}^t \mathscr{A}_{*,*} = \frac{F^t\mathscr{A}_{*,*}}{F^{t+1}\mathscr{A}_{*,*}} .$$

The (cosets of the) degree $= t$ monomials $\tau^E\xi^R$ from Lemma 2.1 freely generate this quotient as a left $H_{*,*}$-module, hence also as a right $H_{*,*}$-module. It follows that the degree ≥ 0 monomials $\tau^E\xi^R$ freely generate $\mathscr{A}_{*,*}$ as a right $H_{*,*}$-module, since in any given bidegree $F^t\mathscr{A}_{*,*} = 0$ for all sufficiently large t. □

The classical definitions of [Ste62, §II.3, §VI.4] generalize to the motivic setting.

Definition 2.3. For $n \geq -1$, let $I(n) \subset \mathscr{A}_{*,*}$ be the ideal

$$I(n) = (\tau_{n+1}, \tau_{n+2}, \dots, \xi_1^{\ell^n}, \xi_2^{\ell^{n-1}}, \dots, \xi_n^{\ell}, \xi_{n+1}, \xi_{n+2}, \dots)$$

generated by τ_k for $k \geq n+1$ and by $\xi_i^{\ell^j}$ for $i \geq 1$, $j \geq 0$ and $i + j \geq n+1$. Note that $T_i \in I(n)$ for $i \geq n$. Let

$$A(n)_{*,*} = \mathscr{A}_{*,*}/I(n) = \frac{H_{*,*}[\tau_0, \dots, \tau_n, \xi_1, \xi_2, \dots, \xi_n]}{(\tau_0^2 - T_0, \dots, \tau_{n-1}^2 - T_{n-1}, \tau_n^2, \xi_1^{\ell^n}, \xi_2^{\ell^{n-1}}, \dots, \xi_n^{\ell})}$$

be the quotient algebra.

Example 2.4.

$$\begin{aligned} I(-1) &= (\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots) \\ I(0) &= (\tau_1, \tau_2, \dots, \xi_1, \xi_2, \dots) \\ I(1) &= (\tau_2, \tau_3, \dots, \xi_1^{\ell}, \xi_2, \xi_3, \dots) \end{aligned}$$

so

$$\begin{aligned} A(-1)_{*,*} &= H_{*,*} \\ A(0)_{*,*} &= H_{*,*}[\tau_0]/(\tau_0^2) \\ A(1)_{*,*} &= H_{*,*}[\tau_0, \tau_1, \xi_1]/(\tau_0^2 - T_0, \tau_1^2, \xi_1^{\ell}) . \end{aligned}$$

Lemma 2.5. *There is a unique Hopf algebroid structure on $(H_{*,*}, A(n)_{*,*})$ making the canonical projection $\pi_n \colon \mathscr{A}_{*,*} \to \mathscr{A}_{*,*}/I(n) = A(n)_{*,*}$ a Hopf algebroid homomorphism.*

Proof. The Hopf algebroid structure maps of $(H_{*,*}, A(n)_{*,*})$ are $\mathbb{Z}/\ell$-algebra homomorphisms, determined as follows:

(1) The left unit $\eta_{L,n} \colon H_{*,*} \to A(n)_{*,*}$ is the composite $\pi_n \circ \eta_L$.
(2) The right unit $\eta_{R,n} \colon H_{*,*} \to A(n)_{*,*}$ is the composite $\pi_n \circ \eta_R$.
(3) The algebra product $\phi_n \colon A(n)_{*,*} \otimes A(n)_{*,*} \to A(n)_{*,*}$ is characterized by $\phi_n \circ (\pi_n \otimes \pi_n) = \pi_n \circ \phi$, and exists because $I(n) \subset \mathscr{A}_{*,*}$ is an ideal.
(4) The counit $\epsilon_n \colon A(n)_{*,*} \to H_{*,*}$ is characterized by $\epsilon_n \circ \pi_n = \epsilon$, and exists because $\epsilon(x) = 0$ for each generator x of $I(n)$.
(5) The coproduct $\psi_n \colon A(n)_{*,*} \to A(n)_{*,*} \otimes_{H_{*,*}} A(n)_{*,*}$ is characterized by $\psi_n \circ \pi_n = (\pi_n \otimes \pi_n)\psi$, and exists because $(\pi_n \otimes \pi_n)\psi(x) = 0$ for each generator x of $I(n)$.
(6) The conjugation $\chi_n \colon A(n)_{*,*} \to A(n)_{*,*}$ is characterized by $\chi_n \circ \pi_n = \pi_n \circ \chi$, and exists because $\chi(x) \in I(n)$ for each generator x of $I(n)$.

In more detail, the explicit formulas for the coproduct show that $\psi(\tau_k)$, for all $k \geq n+1$, and $\psi(\xi_i^{\ell^j}) = \psi(\xi_i)^{\ell^j}$, for all $i \geq 1$, $j \geq 0$ with $i + j \geq n+1$, are in the image of

$$I(n) \otimes_{H_{*,*}} \mathscr{A}_{*,*} \;\oplus\; \mathscr{A}_{*,*} \otimes_{H_{*,*}} I(n) \longrightarrow \mathscr{A}_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*} ,$$

so that $\psi(I(n))$ is contained in this image. Likewise, the recursive formulas for the conjugation show that $\chi(\tau_k)$ and $\chi(\xi_i^{\ell^j}) = \chi(\xi_i)^{\ell^j}$ are in $I(n)$, for the same k, i and j, so that $\chi(I(n)) = I(n)$. The verification that these structure maps make $(H_{*,*}, A(n)_{*,*})$ a Hopf algebroid, with (id, π_n) a Hopf algebroid homomorphism, follows formally from the fact that $(H_{*,*}, \mathscr{A}_{*,*})$ is a Hopf algebroid. $\square$

Lemma 2.6. *The monomials $\tau^E\xi^R$, where $E = (e_0, \dots, e_n)$ and $R = (r_1, \dots, r_n)$ range through the integer sequences with $e_s \in \{0, 1\}$ and $0 \le r_s < \ell^{n+1-s}$, form a basis for $A(n)_{*,*}$ as a finitely generated free left $H_{*,*}$-module.*

Proof. The ideal $I(n)$ equals the free left $H_{*,*}$-submodule of $\mathscr{A}_{*,*}$ generated by the monomials $\tau^E\xi^R$ from Lemma 2.1 for which $e_s = 1$ for some $s \ge n+1$ or $r_s \ge \ell^{n+1-s}$ for some $s \ge 1$. This implies the claim. $\square$

Lemma 2.7. *The same monomials $\tau^E\xi^R$ as in Lemma 2.6 form a basis for $A(n)_{*,*}$ as a free right $H_{*,*}$-module.*

Proof. Replace $\mathscr{A}_{*,*}$ and Lemma 2.1 in the proof of Lemma 2.2 by $A(n)_{*,*}$ and Lemma 2.6. $\square$

The inclusions $I(n) \subset I(n-1)$ induce a tower of surjective Hopf algebroid homomorphisms

$$\mathscr{A}_{*,*} \longrightarrow \dots \longrightarrow A(n)_{*,*} \longrightarrow A(n-1)_{*,*} \longrightarrow \dots \longrightarrow H_{*,*} . \tag{2.1}$$

The composites

$$\lambda \colon \mathscr{A}_{*,*} \xrightarrow{\psi} \mathscr{A}_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*} \xrightarrow{\pi_n \otimes \mathrm{id}} A(n)_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*}$$

and

$$\rho \colon \mathscr{A}_{*,*} \xrightarrow{\psi} \mathscr{A}_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*} \xrightarrow{\mathrm{id} \otimes \pi_{n-1}} \mathscr{A}_{*,*} \otimes_{H_{*,*}} A(n-1)_{*,*}$$

are $\mathbb{Z}/\ell$-algebra homomorphisms giving $\mathscr{A}_{*,*}$ the structure of an $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule algebra, and the projection $\pi_n \colon \mathscr{A}_{*,*} \to A(n)_{*,*}$ is a morphism in the category of such bicomodule algebras.

Definition 2.8. Let

$$X(n)_{*,*} = H_{*,*}\{\tau^E\xi^R \mid e_0 = \dots = e_n = 0, \ell^n \mid r_1, \dots, \ell \mid r_n\}$$

be the free left $H_{*,*}$-module generated by the monomials $\tau^E\xi^R$ with $E = (e_0, e_1, \dots)$ and $R = (r_1, r_2, \dots)$ satisfying $e_s = 0$ for $0 \le s \le n$ and $\ell^{n+1-s} \mid r_s$ for $1 \le s \le n$. Let $\alpha_n \colon \mathscr{A}_{*,*} \to X(n)_{*,*}$ be the left $H_{*,*}$-module homomorphism mapping $\tau^E\xi^R$ to the same monomial if $e_0 = \dots = e_n = 0$ and $\ell^n \mid r_1, \dots, \ell \mid r_n$, and to 0 otherwise.

Lemma 2.9. *The composite*

$$\mathscr{A}_{*,*} \xrightarrow{\lambda} A(n)_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*} \xrightarrow{\mathrm{id} \otimes \alpha_n} A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*}$$

is a left $A(n)_{*,*}$-comodule isomorphism.

Proof. Both λ and $\mathrm{id} \otimes \alpha_n$ respect the left $A(n)_{*,*}$-coactions, so it suffices to show that their composite is a left $H_{*,*}$-module isomorphism. Each monomial in the basis from Lemma 2.1 for $\mathscr{A}_{*,*}$ factors uniquely as $\tau^E\xi^R = \tau^{E'}\xi^{R'} \cdot \tau^{E''}\xi^{R''}$ with

$$\begin{cases} E' = (e_0, \dots, e_n, 0, \dots) & \\ R' = (r_1, \dots, r_n, 0, \dots) & \text{where } r_s < \ell^{n+1-s} \text{ for } 1 \le s \le n, \\ E'' = (0, \dots, 0, e_{n+1}, \dots) & \\ R'' = (r_1, \dots, r_n, r_{n+1}, \dots) & \text{where } \ell^{n+1-s} \mid r_s \text{ for } 1 \le s \le n, \end{cases}$$

and $E = E' + E''$, $R = R' + R''$. Hence the restricted multiplication

$$\begin{aligned} A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*} &\xrightarrow{\phi} \mathscr{A}_{*,*} \\ \tau^{E'}\xi^{R'} \otimes \tau^{E''}\xi^{R''} &\longmapsto \tau^{E'}\xi^{R'} \cdot \tau^{E''}\xi^{R''} \end{aligned}$$

defines a left $H_{*,*}$-module isomorphism. We show that the composite

$$(\mathrm{id} \otimes \alpha_n)\lambda\phi \colon A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*} \longrightarrow A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*}$$

is bijective. For $t \geq 0$ let $F^t X(n)_{*,*}$ be the free left $H_{*,*}$-submodule generated by the monomials from Definition 2.8 that have topological degree $\geq t$. These define a decreasing filtration of $X(n)_{*,*}$, with associated graded modules $\mathrm{gr}^t X(n)_{*,*} = F^t X(n)_{*,*}/F^{t+1}X(n)_{*,*}$. Direct calculation of $\lambda = (\pi_n \otimes \mathrm{id})\psi$ shows that

$$\lambda(\tau^{E'}\xi^{R'}) \equiv \tau^{E'}\xi^{R'} \otimes 1 \mod A(n)_{*,*} \otimes_{H_{*,*}} F^1\mathscr{A}_{*,*} \,,$$

where $F^1\mathscr{A}_{*,*} = \ker(\epsilon)$ as before, and

$$\lambda(\tau^{E''}\xi^{R''}) = 1 \otimes \tau^{E''}\xi^{R''} \,,$$

since each τ_k for $k \geq n+1$ and each $\xi_i^{\ell^j}$ for $i + j \geq n+1$ is left $A(n)_{*,*}$-comodule primitive. It follows that for $\tau^{E'}\xi^{R'} \in A(n)_{*,*}$ and $\tau^{E''}\xi^{R''} \in F^t X(n)_{*,*}$ we have

$$(\mathrm{id} \otimes \alpha_n)\lambda(\tau^{E'}\xi^{R'} \cdot \tau^{E''}\xi^{R''}) \equiv \tau^{E'}\xi^{R'} \otimes \tau^{E''}\xi^{R''} \mod A(n)_{*,*} \otimes_{H_{*,*}} F^{t+1}X(n)_{*,*} \,.$$

Hence $(\mathrm{id} \otimes \alpha_n)\lambda\phi$ maps $A(n)_{*,*} \otimes_{H_{*,*}} F^t X(n)_{*,*}$ to itself, for each $t \geq 0$, and the induced homomorphism

$$A(n)_{*,*} \otimes_{H_{*,*}} \mathrm{gr}^t X(n)_{*,*} \longrightarrow A(n)_{*,*} \otimes_{H_{*,*}} \mathrm{gr}^t X(n)_{*,*}$$

is the identity. The lemma follows, since $A(n)_{*,*} \otimes_{H_{*,*}} F^t X(n)_{*,*}$ is eventually zero in any given bidegree. □

3. Some bicomodule algebras

The classical definitions of [AGM85, §2] also generalize to the motivic setting.

Definition 3.1. For $n \geq 0$, let $J(n) \subset \mathscr{A}_{*,*}$ be the ideal

$$J(n) = (\tau_{n+1}, \tau_{n+2}, \dots, \xi_2^{\ell^{n-1}}, \xi_3^{\ell^{n-2}}, \dots, \xi_n^{\ell}, \xi_{n+1}, \xi_{n+2}, \dots)$$

generated by τ_k for $k \geq n+1$ and by $\xi_i^{\ell^j}$ for $i \geq 2$, $j \geq 0$ and $i + j \geq n+1$. Note that $I(n) = J(n) + (\xi_1^{\ell^n})$. Let

$$C(n)_{*,*} = \mathscr{A}_{*,*}/J(n) = \frac{H_{*,*}[\tau_0, \dots, \tau_n, \xi_1, \xi_2, \dots, \xi_n]}{(\tau_0^2 - T_0, \dots, \tau_n^2 - T_n, \xi_2^{\ell^{n-1}}, \dots, \xi_n^{\ell})}$$

be the quotient algebra. Let

$$B(n)_{*,*} = C(n)_{*,*}[1/\xi_1] = \frac{H_{*,*}[\tau_0, \dots, \tau_n, \xi_1^{\pm 1}, \xi_2, \dots, \xi_n]}{(\tau_0^2 - T_0, \dots, \tau_n^2 - T_n, \xi_2^{\ell^{n-1}}, \dots, \xi_n^{\ell})}$$

be the localization of $C(n)_{*,*}$ away from ξ_1.

Example 3.2.

$$\begin{aligned} J(0) &= (\tau_1, \tau_2, \dots, \xi_2, \xi_3, \dots) \\ J(1) &= (\tau_2, \tau_3, \dots, \xi_2, \xi_3, \dots) \\ J(2) &= (\tau_3, \tau_4, \dots, \xi_2^{\ell}, \xi_3, \dots) \end{aligned}$$

so

$$\begin{aligned}
C(0)_{*,*} &= H_{*,*}[\tau_0, \xi_1]/(\tau_0^2 - T_0) \\
C(1)_{*,*} &= H_{*,*}[\tau_0, \tau_1, \xi_1]/(\tau_0^2 - T_0, \tau_1^2) \\
C(2)_{*,*} &= H_{*,*}[\tau_0, \tau_1, \tau_2, \xi_1, \xi_2]/(\tau_0^2 - T_0, \tau_1^2 - T_1, \tau_2^2, \xi_2^\ell)
\end{aligned}$$

and

$$\begin{aligned}
B(0)_{*,*} &= H_{*,*}[\tau_0, \xi_1^{\pm 1}]/(\tau_0^2 - T_0) \\
B(1)_{*,*} &= H_{*,*}[\tau_0, \tau_1, \xi_1^{\pm 1}]/(\tau_0^2 - T_0, \tau_1^2) \\
B(2)_{*,*} &= H_{*,*}[\tau_0, \tau_1, \tau_2, \xi_1^{\pm 1}, \xi_2]/(\tau_0^2 - T_0, \tau_1^2 - T_1, \tau_2^2, \xi_2^\ell)\,.
\end{aligned}$$

Lemma 3.3. *(a) The monomials $\tau^E \xi^R$, where $E = (e_0, \dots, e_n)$ and $R = (r_1, \dots, r_n)$ range through all sequences with $e_s \in \{0, 1\}$ for $0 \le s \le n$, $r_1 \ge 0$ and $0 \le r_s < \ell^{n+1-s}$ for $2 \le s \le n$, form a basis for $C(n)_{*,*}$ as a free left $H_{*,*}$-module.*

(b) The monomials

$$\tau^E \xi^R = \tau_0^{e_0} \cdots \tau_n^{e_n} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}$$

with (E, R) as in (a), except that r_1 can now be any integer, form a basis for $B(n)_{*,*}$ as a free left $H_{*,*}$-module.

Proof. The ideal $J(n)$ equals the free left $H_{*,*}$-submodule of $\mathscr{A}_{*,*}$ generated by the monomials $\tau^E \xi^R$ from Lemma 2.1 for which $e_s = 1$ for some $s \ge n+1$ or $r_s \ge \ell^{n+1-s}$ for some $s \ge 2$. This implies part (a). Part (b) follows by inverting ξ_1. □

Lemma 3.4. *(a) The same monomials $\tau^E \xi^R$ as in Lemma 3.3(a) form a basis for $C(n)_{*,*}$ as a free right $H_{*,*}$-module.*

(b) The same monomials $\tau^E \xi^R$ as in Lemma 3.3(b) form a basis for $B(n)_{*,*}$ as a free right $H_{*,*}$-module.

Proof. For part (a), replace $\mathscr{A}_{*,*}$ and Lemma 2.1 in the proof of Lemma 2.2 by $C(n)_{*,*}$ and Lemma 3.3(a).

For part (b), instead replace these by $B(n)_{*,*}$ and Lemma 3.3(b), and allow the filtration index t in the proof of Lemma 2.2 to run over all integers, noting that in any given bidegree $F^t B(n)_{*,*} = B(n)_{*,*}$ for all sufficiently negative t. (Alternatively, part (b) can be deduced from part (a) by inverting ξ_1, but the given proof also ensures that the left and right $H_{*,*}$-actions on $\mathrm{gr}^t B(n)_{*,*}$ agree, which will be needed in Lemma 4.16(b).) □

Example 3.5. (a) The monomials

$$\{\tau_0^e \xi_1^r \mid e \in \{0, 1\}, r \ge 0\}$$

form a basis for $C(0)_{*,*}$, both as a left $H_{*,*}$-module and as a right $H_{*,*}$-module.

(b) The monomials

$$\{\tau_0^e \xi_1^r \mid e \in \{0, 1\}, r \in \mathbb{Z}\}$$

form a basis for $B(0)_{*,*}$, both as a left $H_{*,*}$-module and as a right $H_{*,*}$-module. The homological bidegree of $\tau_0^e \xi_1^r$ is $(e + (2\ell - 2)r, (\ell - 1)r)$.

The inclusions $J(n) \subset I(n)$ and the localization homomorphisms yield a commutative diagram of $\mathbb{Z}/\ell$-algebras and algebra homomorphisms

$$
\begin{tikzcd}
\mathscr{A}_{*,*} \arrow[r, two heads, "\pi_n"] \arrow[d, equal] & A(n)_{*,*} \arrow[r, two heads] & A(n-1)_{*,*} \arrow[r, two heads] & A(0)_{*,*} \arrow[r, two heads] & H_{*,*} \\
\mathscr{A}_{*,*} \arrow[r, two heads, "\pi'_n"] \arrow[d, tail] & C(n)_{*,*} \arrow[u, two heads] \arrow[rr, two heads, "\gamma_n"] \arrow[d, tail] & & C(0)_{*,*} \arrow[u, two heads] \arrow[d, tail] & \\
\mathscr{A}_{*,*}[1/\xi_1] \arrow[r, two heads] & B(n)_{*,*} \arrow[rr, two heads, "\beta_n"] & & B(0)_{*,*} &
\end{tikzcd}
\tag{3.1}
$$

Lemma 3.6. *There is a unique $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule algebra structure on $C(n)_{*,*}$ making the canonical projection $\pi'_n \colon \mathscr{A}_{*,*} \to \mathscr{A}_{*,*}/J(n) = C(n)_{*,*}$ an $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule algebra homomorphism.*

Proof. The bicomodule structure maps are $\mathbb{Z}/\ell$-algebra homomorphisms, determined as follows:

(1) The left coaction $\lambda_n \colon C(n)_{*,*} \to A(n)_{*,*} \otimes_{H_{*,*}} C(n)_{*,*}$ is characterized by $\lambda_n \circ \pi'_n = (\mathrm{id} \otimes \pi'_n) \circ \lambda$, and exists because $(\pi_n \otimes \pi'_n)\psi(x) = 0$ for each generator x of $J(n)$.

(2) The right coaction $\rho_n \colon C(n)_{*,*} \to C(n)_{*,*} \otimes_{H_{*,*}} A(n-1)_{*,*}$ is characterized by $\rho_n \circ \pi'_n = (\pi'_n \otimes \mathrm{id}) \circ \rho$, and exists because $(\pi'_n \otimes \pi_{n-1})\psi(x) = 0$ for each generator x of $J(n)$.

More explicitly, $\psi(\tau_k)$ and $\psi(\xi_i^{\ell^j})$ are in the image of both

$$I(n) \otimes_{H_{*,*}} \mathscr{A}_{*,*} \ \oplus \ \mathscr{A}_{*,*} \otimes_{H_{*,*}} J(n) \longrightarrow \mathscr{A}_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*}$$

and

$$J(n) \otimes_{H_{*,*}} \mathscr{A}_{*,*} \ \oplus \ \mathscr{A}_{*,*} \otimes_{H_{*,*}} I(n-1) \longrightarrow \mathscr{A}_{*,*} \otimes_{H_{*,*}} \mathscr{A}_{*,*}$$

for each $k \geq n+1$ and each $i \geq 2$, $j \geq 0$ and $i+j \geq n+1$, respectively. The verification that the algebra homomorphisms λ_n and ρ_n define coactions, and that they commute, follows formally from the fact that $\mathscr{A}_{*,*}$ is an $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule. □

Lemma 3.7. *Let $\|\xi_1^{\ell^n}\| = ((2\ell-2)\ell^n, (\ell-1)\ell^n)$ denote the bidegree of $\xi_1^{\ell^n}$. There is a short exact sequence*

$$0 \to \Sigma^{\|\xi_1^{\ell^n}\|} C(n)_{*,*} \xrightarrow{\cdot \xi_1^{\ell^n}} C(n)_{*,*} \longrightarrow A(n)_{*,*} \to 0$$

of $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodules, where $\cdot\xi_1^{\ell^n}$ denotes $x \mapsto x \cdot \xi_1^{\ell^n}$.

Proof. From the definition of $I(n)$ and $J(n)$ it is clear that multiplication by $\xi_1^{\ell^n}$ acts injectively on $C(n)_{*,*}$ with cokernel $A(n)_{*,*}$. It remains to verify that $\cdot\xi_1^{\ell^n}$ is an $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule homomorphism, i.e., that it commutes with the left $A(n)_{*,*}$-coaction and the right $A(n-1)_{*,*}$-coaction. This is equivalent to $\xi_1^{\ell^n}$ being left $A(n)_{*,*}$-comodule primitive and right $A(n-1)_{*,*}$-comodule primitive, which follows from the observations that

$$\psi(\xi_1^{\ell^n}) \equiv 1 \otimes \xi_1^{\ell^n} \mod I(n) \otimes_{H_{*,*}} \mathscr{A}_{*,*}$$

and

$$\psi(\xi_1^{\ell^n}) \equiv \xi_1^{\ell^n} \otimes 1 \mod \mathscr{A}_{*,*} \otimes_{H_{*,*}} I(n-1)\,.$$

□

Definition 3.8. We assign to

$$B(n)_{*,*} = C(n)_{*,*}[1/\xi_1^{\ell^n}] = \operatorname*{colim}_j \big(\Sigma^{-j\|\xi_1^{\ell^n}\|} C(n)_{*,*}\big)$$

the $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule structure given by the colimit of the diagram

$$C(n)_{*,*} \xrightarrow{\cdot \xi_1^{\ell^n}} \Sigma^{-\|\xi_1^{\ell^n}\|} C(n)_{*,*} \xrightarrow{\cdot \xi_1^{\ell^n}} \Sigma^{-2\|\xi_1^{\ell^n}\|} C(n)_{*,*} \xrightarrow{\cdot \xi_1^{\ell^n}} \dots .$$

Lemma 3.9. *$B(n)_{*,*}$ is an $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule algebra, and the canonical morphism $C(n)_{*,*} \to B(n)_{*,*}$ is an $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule algebra homomorphism.*

Proof. The left $A(n)_{*,*}$-coaction

$$\lambda_n \colon B(n)_{*,*} \longrightarrow A(n)_{*,*} \otimes_{H_{*,*}} B(n)_{*,*}$$

is obtained from the left coaction

$$\lambda_n \colon C(n)_{*,*} \longrightarrow A(n)_{*,*} \otimes_{H_{*,*}} C(n)_{*,*}$$

by inverting (a positive power of) ξ_1. Since the latter coaction is an algebra homomorphism, so is the former. The case of right $A(n-1)_{*,*}$-coactions is entirely similar. $\square$

Definition 3.10. Let $\gamma_n \colon C(n)_{*,*} \to C(0)_{*,*}$ and $\beta_n \colon B(n)_{*,*} \to B(0)_{*,*}$ be the $\mathbb{Z}/\ell$-algebra homomorphisms shown in (3.1). Let $\gamma'_n \colon C(n)_{*,*} \to H_{*,*}[\xi_1^{\ell^n}]$ be the composite of γ_n and the left $H_{*,*}$-module homomorphism

$$C(0)_{*,*} = H_{*,*}[\tau_0, \xi_1]/(\tau_0^2 - T_0) \longrightarrow H_{*,*}[\xi_1^{\ell^n}]$$

given for $e \in \{0,1\}$ and $r \geq 0$ by

$$\tau_0^e \xi_1^r \longmapsto \begin{cases} \xi_1^r & \text{if } e = 0 \text{ and } \ell^n \mid r, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\beta'_n \colon B(n)_{*,*} \to H_{*,*}[\xi_1^{\pm \ell^n}]$ be its localization.

Note that $\tau_0 \cdot \eta_R(\tau)$ in $C(0)_{*,*}$ maps by γ'_0 to $\rho\tau\xi_1$. Hence γ'_n is sometimes not right $H_{*,*}$-linear.

Proposition 3.11. *The composites*

$$C(n)_{*,*} \xrightarrow{\lambda_n} A(n)_{*,*} \otimes_{H_{*,*}} C(n)_{*,*} \xrightarrow{\mathrm{id} \otimes \gamma'_n} A(n)_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\ell^n}]$$

and

$$B(n)_{*,*} \xrightarrow{\lambda_n} A(n)_{*,*} \otimes_{H_{*,*}} B(n)_{*,*} \xrightarrow{\mathrm{id} \otimes \beta'_n} A(n)_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\pm \ell^n}]$$

are left $A(n)_{*,*}$-comodule isomorphisms.

Proof. The $\mathbb{Z}/\ell$-algebra homomorphism

$$(\mathrm{id} \otimes \gamma_n)\lambda_n \colon C(n)_{*,*} \longrightarrow A(n)_{*,*} \otimes_{H_{*,*}} C(0)_{*,*}$$

is left $H_{*,*}$-linear and maps the remaining algebra generators by

$$\begin{cases} \tau_0 \longmapsto \tau_0 \otimes 1 + 1 \otimes \tau_0 & \\ \tau_k \longmapsto \tau_k \otimes 1 + \xi_k \otimes \tau_0 & \text{for } 1 \leq k \leq n, \\ \xi_1 \longmapsto \xi_1 \otimes 1 + 1 \otimes \xi_1 & \\ \xi_k \longmapsto \xi_k \otimes 1 + \xi_{k-1}^{\ell} \otimes \xi_1 & \text{for } 2 \leq k \leq n. \end{cases}$$

In particular, it and $(\mathrm{id} \otimes \gamma'_n)\lambda_n$ respect the decreasing $(\xi_1^{\ell^n})$-adic filtrations defined (internally to this proof) for $m \geq 0$ by

$$\begin{aligned} F^m C(n)_{*,*} &= C(n)_{*,*} \cdot (\xi_1^{\ell^n})^m \\ F^m C(0)_{*,*} &= C(0)_{*,*} \cdot (\xi_1^{\ell^n})^m \\ F^m H_{*,*}[\xi_1^{\ell^n}] &= H_{*,*}[\xi_1^{\ell^n}] \cdot (\xi_1^{\ell^n})^m \,. \end{aligned}$$

The induced homomorphism

$$\frac{F^m C(n)_{*,*}}{F^{m+1} C(n)_{*,*}} \longrightarrow A(n)_{*,*} \otimes_{H_{*,*}} \frac{F^m H_{*,*}[\xi_1^{\ell^n}]}{F^{m+1} H_{*,*}[\xi_1^{\ell^n}]}$$

of associated graded left $A(n)_{*,*}$-comodules is the isomorphism given by

$$\tau^E \xi^R \cdot (\xi_1^{\ell^n})^m \longmapsto \tau^E \xi^R \otimes (\xi_1^{\ell^n})^m \,,$$

where $e_s \in \{0,1\}$ for $0 \leq s \leq n$ and $0 \leq r_s < \ell^{n+1-s}$ for $1 \leq s \leq n$. Each filtration is eventually zero in each bidegree, so this implies that $(\mathrm{id} \otimes \gamma'_n)\lambda_n$ is an isomorphism. Inverting $\xi_1^{\ell^n}$, it follows that

$$(\mathrm{id} \otimes \beta'_n)\lambda_n \colon B(n)_{*,*} \xrightarrow{\cong} A(n)_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\pm \ell^n}]$$

is also an isomorphism. □

Proposition 3.12. *The composite*

$$B(n)_{*,*} \xrightarrow{\rho_n} B(n)_{*,*} \otimes_{H_{*,*}} A(n-1)_{*,*} \xrightarrow{\beta_n \otimes \mathrm{id}} B(0)_{*,*} \otimes_{H_{*,*}} A(n-1)_{*,*}$$

is a right $A(n-1)_{*,*}$-comodule algebra isomorphism.

Proof. The $\mathbb{Z}/\ell$-algebra homomorphism $(\beta_n \otimes \mathrm{id})\rho_n$ is left $H_{*,*}$-linear and maps the remaining algebra generators by

$$\begin{cases} \tau_0 \longmapsto \tau_0 \otimes 1 + 1 \otimes \tau_0 \\ \tau_k \longmapsto \xi_1^{\ell^{k-1}} \otimes \tau_{k-1} + 1 \otimes \tau_k & \text{for } 1 \leq k \leq n, \\ \xi_1 \longmapsto \xi_1 \otimes 1 + 1 \otimes \xi_1 \\ \xi_k \longmapsto \xi_1^{\ell^{k-1}} \otimes \xi_{k-1} + 1 \otimes \xi_k & \text{for } 2 \leq k \leq n, \\ \xi_1^{-\ell^n} \longmapsto \xi_1^{-\ell^n} \otimes 1 \,. \end{cases}$$

Letting

$$\begin{cases} \check{\tau}_k = \tau_k \cdot \xi_1^{-\ell^{k-1}} & \text{for } 1 \leq k \leq n, \\ \check{\xi}_k = \xi_k \cdot \xi_1^{-\ell^{k-1}} & \text{for } 2 \leq k \leq n, \end{cases}$$

we can rewrite the presentation in Definition 3.1 as

$$B(n)_{*,*} = \frac{H_{*,*}[\tau_0, \check{\tau}_1, \dots, \check{\tau}_n, \xi_1^{\pm 1}, \check{\xi}_2, \dots, \check{\xi}_n]}{(\tau_0^2 - T_0, \check{\tau}_1^2 - \check{T}_1, \dots, \check{\tau}_n^2 - \check{T}_n, \check{\xi}_2^{\ell^{n-1}}, \dots, \check{\xi}_n^{\ell})} \,,$$

where for $1 \leq i \leq n$ we use the notation

$$\check{T}_i = \begin{cases} \tau \check{\xi}_{i+1} + \rho \check{\tau}_{i+1} + \rho \tau_0 \check{\xi}_{i+1} & \text{for } \ell = 2, \\ 0 & \text{for } \ell \text{ odd.} \end{cases}$$

Note that $\check{\tau}_k = \tau_k \cdot \xi_1^{\ell^n - \ell^{k-1}} \cdot \xi_1^{-\ell^n}$ and $\check{\xi}_k = \xi_k \cdot \xi_1^{\ell^n - \ell^{k-1}} \cdot \xi_1^{-\ell^n}$. Hence $(\beta_n \otimes \mathrm{id})\rho_n$ satisfies

$$\begin{aligned} \check{\tau}_k \longmapsto 1 \otimes \tau_{k-1} + \xi_1^{-\ell^{k-1}} \otimes \tau_k + \dots \\ + \xi_1^{\ell^{k-1} - \ell^n} \otimes \tau_{k-1} \xi_1^{\ell^n - \ell^{k-1}} + \xi_1^{-\ell^n} \otimes \tau_k \xi_1^{\ell^n - \ell^{k-1}} \end{aligned}$$

for $1 \le k \le n$, and

$$\check\xi_k \longmapsto 1 \otimes \xi_{k-1} + \xi_1^{-\ell^{k-1}} \otimes \xi_k + \dots \\ + \xi_1^{\ell^{k-1}-\ell^n} \otimes \xi_1^{\ell^n-\ell^{k-1}} \xi_{k-1} + \xi_1^{-\ell^n} \otimes \xi_1^{\ell^n-\ell^{k-1}} \xi_k$$

for $2 \le k \le n$. The omitted summands involve binomial coefficients, and each summand after the first has a negative power of ξ_1 as its left hand tensor factor. Hence $(\beta_n \otimes \mathrm{id})\rho_n$ respects the increasing filtrations defined (internally to this proof) for $m \in \mathbb{Z}$ by

$$F_m B(n)_{*,*} = H_{*,*}\{\tau_0^{e_0} \check\tau_1^{e_1} \cdots \check\tau_n^{e_n} \xi_1^{r_1} \check\xi_2^{r_2} \cdots \check\xi_n^{r_n} \mid e_0 + 2r_1 \le m\}$$
$$F_m B(0)_{*,*} = H_{*,*}\{\tau_0^{e_0} \xi_1^{r_1} \mid e_0 + 2r_1 \le m\},$$

where $e_s \in \{0,1\}$ for $0 \le s \le n$, $r_1 \in \mathbb{Z}$ and $0 \le r_s < \ell^{n+1-s}$ for $2 \le s \le n$ as in Lemma 3.3(b). The induced homomorphism

$$\frac{F_m B(n)_{*,*}}{F_{m-1} B(n)_{*,*}} \longrightarrow \frac{F_m B(0)_{*,*}}{F_{m-1} B(0)_{*,*}} \otimes_{H_{*,*}} A(n-1)_{*,*}$$

of associated graded right $A(n-1)_{*,*}$-comodules is the left $H_{*,*}$-module isomorphism given by

$$\tau_0^{e_0} \check\tau_1^{e_1} \cdots \check\tau_n^{e_n} \xi_1^{r_1} \check\xi_2^{r_2} \cdots \check\xi_n^{r_n} \longmapsto \tau_0^{e_0} \xi_1^{r_1} \otimes \tau_0^{e_1} \cdots \tau_{n-1}^{e_n} \xi_1^{r_2} \cdots \xi_{n-1}^{r_n}$$

for $e_0 + 2r_1 = m$. In particular, $\check\tau_k \mapsto \tau_{k-1}$ and $\check\xi_k \mapsto \xi_{k-1}$. Each filtration is exhaustive and eventually zero in each bidegree, so this implies that $(\beta_n \otimes \mathrm{id})\rho_n$ is an isomorphism. □

4. ... and their dual bimodules

We now dualize the results of the previous section, following [Boa82].

Definition 4.1 ([Boa82, Def. 3.2]). Given a left $H_{*,*}$-module M we define the *dual* left $H_{*,*}$-module to be

$$M^\vee = \mathrm{Hom}_{H_{*,*}}(M, H_{*,*}).$$

The left action of $h \in H_{*,*}$ on $f \in M^\vee$ is given by

$$(hf)(m) = h\, f(m) = (-1)^{|h||f|} f(hm)$$

for $m \in M$, where $|h|$ and $|f|$ are the topological degrees of h and f, respectively. If M is an $H_{*,*}$-$H_{*,*}$-bimodule then $M^\vee$ is also a bimodule, with right action defined by

$$(fh)(m) = (-1)^{|h||m|} f(mh).$$

Example 4.2. The canonical isomorphism $H_{*,*}^\vee \cong H_{*,*} = H^{-*,-*}$, taking f to $f(1)$, is $H_{*,*}$-$H_{*,*}$-bilinear.

Lemma 4.3 ([Boa82, Lem. 3.3]). *Let M be an $H_{*,*}$-$H_{*,*}$-bimodule and let N be a left $H_{*,*}$-module.*

(a) There is a natural homomorphism $\theta \colon M^\vee \otimes_{H_{*,*}} N^\vee \longrightarrow (M \otimes_{H_{*,*}} N)^\vee$ of left $H_{*,*}$-modules (or of $H_{*,*}$-$H_{*,*}$-bimodules, if N is a bimodule), given by

$$\theta(f \otimes g)(m \otimes n) = (-1)^{|g||m|} f(m\, g(n))$$

for $f \in M^\vee$, $g \in N^\vee$, $m \in M$ and $n \in N$.

(b) If L is another bimodule, the diagram

$$\begin{tikzcd} L^\vee \otimes_{H_{*,*}} M^\vee \otimes_{H_{*,*}} N^\vee \arrow[r, "\theta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\theta"'] & (L \otimes_{H_{*,*}} M)^\vee \otimes_{H_{*,*}} N^\vee \arrow[d, "\theta"] \\ L^\vee \otimes_{H_{*,*}} (M \otimes_{H_{*,*}} N)^\vee \arrow[r, "\theta"] & (L \otimes_{H_{*,*}} M \otimes_{H_{*,*}} N)^\vee \end{tikzcd}$$

commutes.

(c) Both composites $M^\vee \cong M^\vee \otimes_{H_{*,*}} H^\vee_{*,*} \xrightarrow{\theta} (M \otimes_{H_{*,*}} H_{*,*})^\vee = M^\vee$ *and* $M^\vee \cong H^\vee_{*,*} \otimes_{H_{*,*}} M^\vee \xrightarrow{\theta} (H_{*,*} \otimes_{H_{*,*}} M)^\vee = M^\vee$ *are the identity homomorphism.*

Lemma 4.4 ([Boa82, Lem. 3.4])**.** *(a) Let* $(H_{*,*}, \Gamma)$ *be a Hopf algebroid. The dual* $\Gamma^\vee$ *is a bigraded* $\mathbb{Z}/\ell$*-algebra, containing* $H^\vee_{*,*}$ *as a subalgebra.*

(b) Let M *be a left* Γ*-comodule. The dual* $M^\vee$ *is a left* $\Gamma^\vee$*-module.*

(c) Let $(H_{*,*}, \Sigma)$ *be a second Hopf algebroid, and let* N *be a* Γ*-*Σ*-bicomodule. The dual* $N^\vee$ *is a* $\Gamma^\vee$*-*$\Sigma^\vee$*-bimodule.*

Proof. Let $\psi\colon \Gamma \to \Gamma \otimes_{H_{*,*}} \Gamma$ be the coproduct, and let $\lambda\colon M \to \Gamma \otimes_{H_{*,*}} M$ be the left coaction. Boardman uses Lemma 4.3 to define the multiplication on $\Gamma^\vee$ as the composite

$$\Gamma^\vee \otimes \Gamma^\vee \longrightarrow \Gamma^\vee \otimes_{H_{*,*}} \Gamma^\vee \xrightarrow{\theta} (\Gamma \otimes_{H_{*,*}} \Gamma)^\vee \xrightarrow{\psi^\vee} \Gamma^\vee$$

and to define the left action on $M^\vee$ as the composite

$$\Gamma^\vee \otimes M^\vee \longrightarrow \Gamma^\vee \otimes_{H_{*,*}} M^\vee \xrightarrow{\theta} (\Gamma \otimes_{H_{*,*}} M)^\vee \xrightarrow{\lambda^\vee} M^\vee .$$

Likewise, we define the bimodule action on $N^\vee$ as the now evident composite

$$\Gamma^\vee \otimes N^\vee \otimes \Sigma^\vee \longrightarrow \Gamma^\vee \otimes_{H_{*,*}} N^\vee \otimes_{H_{*,*}} \Sigma^\vee \longrightarrow (\Gamma \otimes_{H_{*,*}} N \otimes_{H_{*,*}} \Sigma)^\vee \longrightarrow N^\vee .$$

The dual $\epsilon^\vee\colon H^\vee_{*,*} \to \Gamma^\vee$ of the Hopf algebroid counit is split by $\eta_L^\vee$ (and by $\eta_R^\vee$), and exhibits $H^\vee_{*,*}$ as a subalgebra of $\Gamma^\vee$. □

The dual $\mathbb{Z}/\ell$-algebra $\Gamma^\vee$ is usually non-commutative. Switching to cohomological grading, we now refer to the duals of (left or right) $H_{*,*}$-module actions as (left or right) $H^{*,*}$-module actions.

Notation 4.5. The motivic Steenrod algebra $\mathscr{A} = \mathscr{A}^\vee_{*,*}$ is the dual of the Hopf algebroid $(H_{*,*}, \mathscr{A}_{*,*})$, cf. [Voe03a, §13], and contains $H^{*,*}$ as a subalgebra. It is freely generated as a left $H^{*,*}$-module by the Milnor basis $\{\rho(E,R)\}_{E,R}$, defined to be dual to the monomial basis $\{\tau^E \xi^R\}_{E,R}$ of Lemma 2.1. The cohomological bidegree of $\rho(E,R)$ is equal to the homological bidegree of $\tau^E\xi^R$. In particular, the Steenrod operation $\beta^e P^r$ is dual to $\tau_0^e \xi_1^r$, for $e \in \{0,1\}$ and $r \ge 0$, cf. [Voe03a, Lem. 13.1, Lem. 13.5]. By [Voe03a, Lem. 11.1, Cor. 12.5] and the Adem relations [Voe03a, Thm. 10.3], [Rio12, Thm. 4.5.1] the operations $\beta, P^1, P^\ell, P^{\ell^2}, \dots,$ together with the elements of $H^{*,*}$, generate $\mathscr{A}$ as a $\mathbb{Z}/\ell$-algebra. When $\ell = 2$ we write Sq^{2r} for P^r in cohomological bidegree $(2r, r)$ and Sq^{2r+1} for βP^r in cohomological bidegree $(2r+1, r)$.

Lemma 4.6. *The operations* $\rho(E,R)$*, for* (E,R) *as in Lemma 2.1, also form a basis for* $\mathscr{A}$ *as a right* $H^{*,*}$*-module.*

Proof. Recall the decreasing $H_{*,*}$-$H_{*,*}$-bimodule filtration $F^t\mathscr{A}_{*,*}$ of $\mathscr{A}_{*,*}$ from the proof of Lemma 2.2. For $t \ge 0$ let

$$F_{t-1}\mathscr{A} = \langle \rho(E,R) \mid \deg(\tau^E\xi^R) < t \rangle \subset \mathscr{A}$$

be the left $H^{*,*}$-submodule generated by the operations $\rho(E,R)$ of cohomological topological degree $< t$. This is also a right $H^{*,*}$-submodule, in view of the short exact sequence

$$0 \to F_{t-1}\mathscr{A} \longrightarrow \mathscr{A}^\vee_{*,*} \longrightarrow (F^t\mathscr{A}_{*,*})^\vee \to 0 .$$

Hence $\{F_t\mathscr{A}\}_t$ is an increasing filtration of $\mathscr{A}$ by $H^{*,*}$-$H^{*,*}$-bimodules, with filtration quotients

$$\mathrm{gr}_t \mathscr{A} = \frac{F_t\mathscr{A}}{F_{t-1}\mathscr{A}} \cong (\mathrm{gr}^t \mathscr{A}_{*,*})^\vee .$$

Since the left and right $H_{*,*}$-module actions agree on $\mathrm{gr}^t \mathscr{A}_{*,*}$, the dual left and right $H^{*,*}$-module actions on $\mathrm{gr}_t \mathscr{A}$ are also equal. Hence the (cosets of the) operations $\rho(E,R)$ of degree $= t$ freely generate $\mathrm{gr}_t \mathscr{A}$ as a right $H^{*,*}$-module. Since the filtration is exhaustive, the set of degree ≥ 0 operations is a right $H^{*,*}$-module basis for $\mathscr{A}$. ☐

Definition 4.7. For $n \geq -1$ let the $\mathbb{Z}/\ell$-algebra $A(n) = A(n)^\vee_{*,*} \subset \mathscr{A}$ be the dual of the Hopf algebroid $(H_{*,*}, A(n)_{*,*})$.

Lemma 4.8. *The operations $\rho(E,R)$, for (E,R) as in Lemma 2.6, form a basis for $A(n)$ as a finitely generated free left $H^{*,*}$-module. In particular, there is an exhaustive sequence of $\mathbb{Z}/\ell$-algebra homomorphisms*

$$H^{*,*} \subset \cdots \subset A(n-1) \subset A(n) \subset \cdots \subset \mathscr{A}\,.$$

Proof. This follows from (the proof of) Lemma 2.6, since $I(n) \subset \mathscr{A}_{*,*}$ is a monomial ideal. The sequence is dual to the tower (2.1). ☐

Lemma 4.9. *The operations $\rho(E,R)$, for (E,R) as in Lemma 2.6, also form a basis for $A(n)$ as a free right $H^{*,*}$-module.*

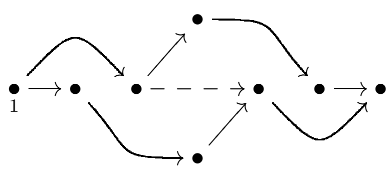
Proof. Replace $\mathscr{A}_{*,*}$ and Lemma 2.2 in the proof of Lemma 4.6 by $A(n)_{*,*}$ and Lemma 2.7. ☐

Example 4.10. (a) $A(0) = H^{*,*}\langle\beta\rangle/(\beta^2)$ with $[\beta, x] = \beta(x)$ for $x \in H^{*,*}$, where $[\beta, x] = \beta x - (-1)^{|x|} x\beta$ denotes the graded commutator.

(b) For $\ell = 2$,

$$A(1) = \frac{H^{*,*}\langle\beta, P^1\rangle}{(\beta^2, P^1P^1 = \tau\beta P^1\beta, (\beta P^1)^2 = (P^1\beta)^2)}$$

with $[\beta, x] = \beta(x)$ and $[P^1, x] = P^1(x)$ for $x \in H^{*,*}$. In the figure below, each bullet represents a copy of $H^{*,*}$, the operations $\beta = Sq^1$ and $P^1 = Sq^2$ map one and two columns to the right, respectively, and the dashed arrow indicates that $P^1P^1 = Sq^2Sq^2$ is τ times the generator $\beta P^1\beta = Sq^3Sq^1$.



The following property is sometimes taken as the definition of $A(n)$.

Lemma 4.11. *For $n \geq 0$ the operations $\beta, P^1, P^\ell, \dots, P^{\ell^{n-1}}$, together with the elements of $H^{*,*}$, generate $A(n)$ as a $\mathbb{Z}/\ell$-algebra.*

Proof. For ℓ odd, the Adem relations [Voe03a, Thm. 10.3] show that the subalgebra of $A(n)$ generated by $\beta, P^1, P^\ell, \dots, P^{\ell^{n-1}}$ is isomorphic to the classical finite subalgebra $A(n)^{\mathrm{cl}}$ of the classical Steenrod algebra. By [Mil58, Prop. 2] it has $\mathbb{Z}/\ell$-module basis equal to the $H^{*,*}$-module basis for $A(n)$ of Lemma 4.8.

For $\ell = 2$, the τ- and ρ-coefficients in the Adem relations [Rio12, Thm. 4.5.1] (correcting [Voe03a, Thm. 10.2]) mean that Milnor's product formula [Mil58, Thm. 4b] requires adjustment in the motivic setting. For $i \geq 0$ let Q_i be the Milnor basis element dual to τ_i, and for $i \geq 1$ and $j \geq 0$ let P^j_i be the Milnor basis element dual to $\xi_i^{\ell^j}$. In particular, $Q_0 = \beta$ and $P_1^j = P^{\ell^j}$. The arrays

$$\begin{array}{lllll} \xi_1^{\ell^{n-1}} & & & & \\ \xi_1^{\ell^{n-2}} & \xi_2^{\ell^{n-2}} & & & \\ \vdots & \vdots & \ddots & & \\ \xi_1 & \xi_2 & \dots & \xi_n & \\ \tau_0 & \tau_1 & \dots & \tau_{n-1} & \tau_n \end{array} \qquad \text{and} \qquad \begin{array}{lllll} P_1^{n-1} & & & & \\ P_1^{n-2} & P_2^{n-2} & & & \\ \vdots & \vdots & \ddots & & \\ P_1^0 & P_2^0 & \dots & P_n^0 & \\ Q_0 & Q_1 & \dots & Q_{n-1} & Q_n \end{array}$$

may be helpful, cf. [Mar83, p. 232]. Let $n \geq 1$ and suppose, by induction, that the lemma holds for $A(n-1)$. We show that the inclusions

$$\begin{aligned} A(n-1)\langle P_1^{n-1}\rangle \subset A(n-1)\langle P_1^{n-1}, P_2^{n-2}\rangle \subset \dots \\ \subset A(n-1)\langle P_1^{n-1}, \dots, P_n^0, Q_n\rangle \subset A(n) \end{aligned}$$

are all equalities. Here we write $A(n-1)\langle P_1^{n-1}, \dots, P_k^{n-k}\rangle$ to denote the subalgebra of $A(n)$ generated by $A(n-1)$ and the P_t^{n-t} with $1 \leq t \leq k$, and similarly in the case with Q_n. This will complete the inductive step, since $A(n-1)\langle P_1^{n-1}\rangle$ is generated by $\beta, P^1, \dots, P^{\ell^{n-2}}, P^{\ell^{n-1}}$ and the elements of $H^{*,*}$. Consider $2 \leq k \leq n$. We claim that

$$[P_{k-1}^{n+1-k}, P_1^{n-k}] = P_k^{n-k} \mod A(n-1)\,. \tag{4.1}$$

The left hand commutator is an $H^{*,*}$-linear combination of Milnor basis elements $\rho(E,R)$ in $A(n)$, as in Lemma 4.8. The $H^{*,*}$-coefficient of $\rho(E,R)$ is the sum of the $H_{*,*}$-coefficients of

$$\xi_{k-1}^{\ell^{n+1-k}} \otimes \xi_1^{\ell^{n-k}} \qquad \text{and} \qquad \xi_1^{\ell^{n-k}} \otimes \xi_{k-1}^{\ell^{n+1-k}} \tag{4.2}$$

in $\psi(\tau^E\xi^R)$, where we can ignore signs since $\ell = 2$. The basis element P_k^{n-k} appears with coefficient 1, due to the term $\xi_{k-1}^{\ell^{n+1-k}} \otimes \xi_1^{\ell^{n-k}}$ in $\psi(\xi_k^{\ell^{n-k}})$.

For other $\rho(E,R)$ not in $A(n-1)$, degree considerations show that exactly one of $\xi_1^{\ell^{n-1}}, \dots, \xi_{k-1}^{\ell^{n+1-k}}$ must divide $\tau^E\xi^R$. When $1 \leq t \leq k-2$, no term of the coproduct

$$\psi(\xi_t^{\ell^{n-t}}) = \sum_{i+j=t} \xi_i^{\ell^{n-i}} \otimes \xi_j^{\ell^{n-t}}$$

divides either one of the tensor products in (4.2). Hence the $\rho(E,R)$ with these $\xi_t^{\ell^{n-t}}$ dividing $\tau^E\xi^R$ do not contribute to the commutator in (4.1). In the one remaining case, $t = k-1$, the coproduct $\psi(\xi_{k-1}^{\ell^{n+1-k}})$ contains two terms dividing those in (4.2), namely $1 \otimes \xi_{k-1}^{\ell^{n+1-k}}$ and $\xi_{k-1}^{\ell^{n+1-k}} \otimes 1$. The complementary factors $\xi_1^{\ell^{n-k}} \otimes 1$ and $1 \otimes \xi_1^{\ell^{n-k}}$ only appear in

$$\psi(\xi_1^{\ell^{n-k}}) = 1 \otimes \xi_1^{\ell^{n-k}} + \xi_1^{\ell^{n-k}} \otimes 1\,,$$

so the last possible contribution to (4.1) is $\rho(E,R)$ dual to $\xi_1^{\ell^{n-k}} \cdot \xi_{k-1}^{\ell^{n+1-k}}$, with $H^{*,*}$-coefficient the sum of the $H_{*,*}$-coefficients in

$$\psi(\xi_1^{\ell^{n-k}} \cdot \xi_{k-1}^{\ell^{n+1-k}}) = \psi(\xi_1^{\ell^{n-k}}) \cdot \psi(\xi_{k-1}^{\ell^{n+1-k}})\,.$$

Since each of $\xi_{k-1}^{\ell^{n+1-k}} \otimes \xi_1^{\ell^{n-k}}$ and $\xi_1^{\ell^{n-k}} \otimes \xi_{k-1}^{\ell^{n+1-k}}$ occurs twice in this product, this last contribution is $0 \mod \ell$. This establishes claim (4.1). The analogous formula

$$[P_n^0, Q_0] = Q_n \tag{4.3}$$

holds strictly in $A(n)$, and was already proved in [Voe03a, Prop. 13.6]. It follows by induction on k that

$$A(n-1)\langle P_1^{n-1}\rangle = A(n-1)\langle P_1^{n-1}, \dots, P_n^0\rangle = A(n-1)\langle P_1^{n-1}, \dots, P_n^0, Q_n\rangle\,.$$

Finally, the identity

$$A(n-1)\langle P_1^{n-1}, \dots, P_n^0, Q_n\rangle = A(n)$$

follows by classical filtration-by-excess considerations, as in [Mar83, Prop. 15.8], where the excess of $\rho(E, R)$ is defined to be $\sum_s e_s + 2\sum_s r_s$. □

Lemma 4.12. *The operations $\rho(E, R)$ for (E, R) as in Definition 2.8 form a basis for $\mathscr{A}$ as a free left $A(n)$-module.*

Proof. This follows by dualization from Lemma 2.9. □

Lemma 4.13. *Let $E_n = (1, \dots, 1)$ and $R_n = (\ell^n - 1, \dots, \ell - 1)$, so that $t_n = \deg(\tau^{E_n}\xi^{R_n})$ is the highest topological degree of a monomial in $A(n)_{*,*}$. Then $A(n)^{p,q} = 0$ unless $0 \le p \le q + d + t_n$. Hence the subset*

$$\{(e, r) \mid A(n)^{p-e-(2\ell-2)r, q-(\ell-1)r} \neq 0\} \subset \{0, 1\} \times \mathbb{Z}$$

is finite, for each given cohomological bidegree (p, q).

Proof. The $H_{*,*}$-module generators of $A(n)_{*,*}$ lie in homological bidegrees (t, u) with $0 \le t \le t_n$ and $u \ge 0$. Hence the $H^{*,*}$-module generators of $A(n)$ lie in cohomological bidegrees (p, q) with $0 \le p \le t_n$ and $q \ge 0$. Since $H^{*,*}$ is concentrated in bidegrees with $0 \le p \le q + d$, it follows that $A(n)$ is concentrated in the infinite triangular region where $0 \le p \le q+d+t_n$. Each line of slope $1/2$ in the (p, q)-plane intersects this triangular region in a bounded interval, which implies the finiteness assertion. □

Definition 4.14. For $n \ge 0$ let the $A(n)$-$A(n-1)$-bimodules $C(n) = C(n)^\vee_{*,*} \subset \mathscr{A}$ and $B(n) = B(n)^\vee_{*,*}$ be the duals of the $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodules $C(n)_{*,*}$ and $B(n)_{*,*}$, respectively. Let the symbol

$$\rho(E, R) \in B(n)$$

be dual to $\tau^E\xi^R$ in the monomial left $H_{*,*}$-module basis for $B(n)_{*,*}$. The dual of the localization monomorphism $C(n)_{*,*} \to B(n)_{*,*}$ is a canonical $A(n)$-$A(n-1)$-bimodule epimorphism $B(n) \to C(n)$.

Lemma 4.15. *(a) The operations $\rho(E, R)$, for (E, R) as in Lemma 3.3(a), form a basis for $C(n)$ as a free left $H^{*,*}$-module.*

(b) The symbols $\rho(E, R)$, for (E, R) as in Lemma 3.3(b), form a basis for $B(n)$ as a free left $H^{*,*}$-module.

(c) The canonical epimorphism $B(n) \to C(n)$ satisfies

$$\rho(E, R) \longmapsto \begin{cases} \rho(E, R) & \text{for } r_1 \ge 0, \\ 0 & \text{for } r_1 < 0. \end{cases}$$

Proof. Part (a) follows from (the proof of) Lemma 3.3(a), since $J(n) \subset \mathscr{A}_{*,*}$ is a monomial ideal. Part (b) likewise follows from Lemma 3.3(b). The restriction of $\rho(E, R)$ to $C(n)_{*,*}$ is then dual to $\tau^E\xi^R$ if $r_1 \ge 0$, and zero otherwise, proving (c). □

Lemma 4.16. *(a) The operations $\rho(E, R)$, for (E, R) as in Lemma 3.3(a), also form a basis for $C(n)$ as a free right $H^{*,*}$-module.*

(b) The symbols $\rho(E, R)$, for (E, R) as in Lemma 3.3(b), also form a basis for $B(n)$ as a free right $H^{*,*}$-module.

Proof. For part (a), replace $\mathscr{A}_{*,*}$ and Lemma 2.2 in the proof of Lemma 4.6 by $C(n)_{*,*}$ and Lemma 3.4(a).

For part (b), instead replace these by $B(n)_{*,*}$ and Lemma 3.4(b), and allow the filtration index t in the proof of Lemma 4.6 to run over all integers, noting that in any given bidegree $F_tB(n) = 0$ for all sufficiently negative t. □

Example 4.17. (a) The Steenrod operations

$$\{\beta^e P^r \mid e \in \{0,1\}, r \geq 0\}$$

form a basis for $C(0) \subset \mathscr{A}$ as a left $H^{*,*}$-module, and as a right $H^{*,*}$-module. When $\ell = 2$, these are the Steenrod operations Sq^k for $k \geq 0$.

(b) The symbols

$$\{\beta^e P^r \mid e \in \{0,1\}, r \in \mathbb{Z}\}\,,$$

with $\beta^e P^r$ dual to $\tau_0^e \xi_1^r$, form a basis for $B(0)$ as a left $H^{*,*}$-module, and as a right $H^{*,*}$-module. When $\ell = 2$, these are the symbols Sq^k for $k \in \mathbb{Z}$. The homomorphism $B(0) \to C(0)$ maps $\beta^e P^r$ to the corresponding Steenrod operation for $r \geq 0$, and to zero for $r < 0$.

Lemma 4.18. *For each $n \geq 0$ there is a commutative diagram of $A(n)$-$A(n-1)$-bimodules*

$$\begin{tikzcd}
A(n) \arrow[r, tail] \arrow[d, tail] & A(n+1) \arrow[d, tail] \\
C(n) \arrow[r, tail] & C(n+1) \\
B(n) \arrow[r, tail] \arrow[u, two heads] & B(n+1)\,, \arrow[u, two heads]
\end{tikzcd}$$

where the bimodule structures on the right hand side are obtained by restriction from the inherent $A(n+1)$-$A(n)$-bimodule structures.

Proof. This is readily obtained by comparing diagram (3.1) to its analogue with n replaced by $n+1$, and dualizing. □

Proposition 4.19. *(a) The inclusion $H_{*,*}[\xi_1^{\ell^n}]^\vee \subset C(n)$ extends to an isomorphism*

$$A(n) \otimes_{H^{*,*}} H_{*,*}[\xi_1^{\ell^n}]^\vee \xrightarrow{\cong} C(n)$$

of left $A(n)$-modules. Hence the Steenrod operations

$$\{P^r \mid r \geq 0 \text{ with } \ell^n \mid r\}$$

form a basis for $C(n)$ as a free left $A(n)$-module.

(b) The inclusion $H_{*,*}[\xi_1^{\pm\ell^n}]^\vee \subset B(n)$ extends to an isomorphism

$$A(n) \otimes_{H^{*,*}} H_{*,*}[\xi_1^{\pm\ell^n}]^\vee \xrightarrow{\cong} B(n)$$

of left $A(n)$-modules. Hence the symbols

$$\{P^r \mid r \in \mathbb{Z} \text{ with } \ell^n \mid r\}$$

form a basis for $B(n)$ as a free left $A(n)$-module. The cohomological bidegree of P^r is $((2\ell-2)r, (\ell-1)r)$.

Proof. The homomorphisms

$$\theta \colon A(n)^\vee_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\ell^n}]^\vee \longrightarrow (A(n)_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\ell^n}])^\vee$$

and

$$\theta \colon A(n)^\vee_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\pm\ell^n}]^\vee \longrightarrow (A(n)_{*,*} \otimes_{H_{*,*}} H_{*,*}[\xi_1^{\pm\ell^n}])^\vee$$

are isomorphisms. This follows from Lemmas 4.3(c) and 4.13, since in each case the source of θ is a direct sum of shifted copies of $A(n)$, the target of θ is the corresponding product, and in each bidegree (p,q) only finitely many of the factors in the product are nonzero.

The claims in (a) and (b) then follow by dualization from Proposition 3.11. □

Proposition 4.20. *The inclusion $B(0) \subset B(n)$ extends to an isomorphism*

$$B(0) \otimes_{H^{*,*}} A(n-1) \xrightarrow{\cong} B(n)$$

of right $A(n-1)$-modules. Hence the symbols

$$\{\beta^e P^r \mid e \in \{0,1\}, r \in \mathbb{Z}\}$$

form a basis for $B(n)$ as a free right $A(n-1)$-module. The cohomological bidegree of $\beta^e P^r$ is $(e + (2\ell - 2)r, (\ell - 1)r)$.

Proof. The homomorphism

$$\theta \colon B(0)^\vee_{*,*} \otimes_{H_{*,*}} A(n-1)^\vee_{*,*} \longrightarrow (B(0)_{*,*} \otimes_{H_{*,*}} A(n-1)_{*,*})^\vee$$

is an isomorphism, by Lemmas 4.3(c) and 4.13. Thus the claim follows by dualization from Proposition 3.12 and Example 4.17(b). □

5. The small motivic Singer construction

In this section and the next, we generalize the classical Singer construction $R_+(M)$ of [Sin81] and [LS82] to the motivic context, following the strategy of [AGM85]. We shall write $R_S(M)$ for the (small) construction associated to the symmetric group S_ℓ, which is denoted $R_+(M)$ in [LS82] and $T''(M)$ in [AGM85], and whose desuspension $\Sigma^{-1}R_S(M)$ is denoted $R_+(M)$ in [Sin80] and [Sin81] and $T'(M)$ in [AGM85]. We shall write $R_\mu(M)$ for the (large) construction associated to the cyclic group C_ℓ and the algebraic group μ_ℓ of ℓ-th roots of unity, which is denoted $T(M)$ in [AGM85] and $R_+(M)$ in [LNR12]. For $\ell = 2$ the two constructions agree.

Lemma 5.1. *Let $n \geq 0$.*

(a) For each left $A(n-1)$-module M, the tensor product $B(n) \otimes_{A(n-1)} M$ is a left $A(n)$-module. The inclusion $B(0) \subset B(n)$ induces an isomorphism

$$B(0) \otimes_{H^{*,*}} M \xrightarrow{\cong} B(n) \otimes_{A(n-1)} M\,.$$

(b) If M is a left $A(n)$-module, then the inclusion $B(n) \subset B(n+1)$ induces an isomorphism

$$B(n) \otimes_{A(n-1)} M \xrightarrow{\cong} B(n+1) \otimes_{A(n)} M$$

of left $A(n)$-modules.

(c) If M is a left $\mathscr{A}$-module, then the composition $B(n) \to C(n) \subset \mathscr{A}$ induces a left $A(n)$-module homomorphism

$$\epsilon_n \colon B(n) \otimes_{A(n-1)} M \longrightarrow \mathscr{A} \otimes_{A(n-1)} M \longrightarrow M\,,$$

and these are compatible for varying n.

Proof. (a) This is clear from the $A(n)$-$A(n-1)$-bimodule structure of $B(n)$ and Proposition 4.20.

(b) The morphism exists because $B(n) \subset B(n+1)$ is an $A(n)$-$A(n-1)$-bimodule homomorphism, with respect to the restricted bimodule structure on the target. It is an isomorphism by comparison with the isomorphisms of part (a) for n and $n+1$.

(c) This follows because the inclusions $C(n) \subset C(n+1) \subset \mathscr{A}$ are $A(n)$-$A(n-1)$-bimodule homomorphisms. In each case the morphism $\mathscr{A} \otimes_{A(n-1)} M \longrightarrow M$ is induced by the left module action $\mathscr{A} \otimes M \to M$. □

Definition 5.2. Let M be any left $\mathscr{A}$-module.

(a) Let the *small motivic Singer construction*

$$R_S(M) = \operatorname*{colim}_n \, (B(n) \otimes_{A(n-1)} M)$$

be the colimit of the sequence of isomorphisms

$$B(0) \otimes_{H^{*,*}} M \xrightarrow{\cong} \dots \xrightarrow{\cong} B(n) \otimes_{A(n-1)} M \xrightarrow{\cong} B(n+1) \otimes_{A(n)} M \xrightarrow{\cong} \dots ,$$

equipped with the unique left $\mathscr{A}$-module structure for which the canonical map $B(n) \otimes_{A(n-1)} M \to R_S(M)$ is an isomorphism of $A(n)$-modules, for each $n \geq 0$.

(b) Let the *small evaluation homomorphism*

$$\epsilon \colon R_S(M) \longrightarrow M$$

be the left $\mathscr{A}$-module homomorphism such that its restriction to $B(n) \otimes_{A(n-1)} M$ is equal to the $A(n)$-module homomorphism ϵ_n of Lemma 5.1(c), for each $n \geq 0$.

Evidently, R_S is an exact and colimit-preserving endofunctor of left $\mathscr{A}$-modules, and $\epsilon \colon R_S \to \mathrm{id}$ is a natural transformation.

Lemma 5.3. *As a left $A(0)$-module, the small motivic Singer construction is given by the tensor product*

$$\begin{aligned} R_S(M) &\cong B(0) \otimes_{H^{*,*}} M \\ &= H^{*,*}\{\beta^e P^r \mid e \in \{0,1\}, r \in \mathbb{Z}\} \otimes_{H^{*,*}} M , \end{aligned}$$

with the $A(0)$-action from $B(0)$. Each element of $R_S(M)$ is thus a finite sum of terms $\beta^e P^r \otimes m$, with $e \in \{0,1\}$, $r \in \mathbb{Z}$ and $m \in M$, where $\beta(P^r \otimes m) = \beta P^r \otimes m$ and $\beta(\beta P^r \otimes m) = 0$. The small evaluation homomorphism is given by

$$\epsilon(\beta^e P^r \otimes m) = \begin{cases} \beta^e P^r(m) & \text{for } r \geq 0, \\ 0 & \text{for } r < 0. \end{cases}$$

Proof. Clear. □

The following formulas generalize the one of Singer [Sin80, (2.1)] for $\ell = 2$ and a rewriting of the those of Li–Singer [LS82, §3] for ℓ odd. By $\tau^{j \bmod 2}$ we mean $\tau^0 = 1$ for j even and $\tau^1 = \tau$ for j odd.

Proposition 5.4. *For $\ell = 2$ and $a \geq 0$ even the action of Sq^a on $R_S(M)$ is given by*

$$Sq^a(Sq^b \otimes m) = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \tau^{j \bmod 2} \cdot Sq^{a+b-j} \otimes Sq^j(m)$$

for $b \in \mathbb{Z}$ even, and

$$\begin{aligned} Sq^a(Sq^b \otimes m) = &\sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} \otimes Sq^j(m) \\ &+ \sum_{\substack{j=1 \\ odd}}^{[a/2]} \binom{b-1-j}{a-2j} \rho \cdot Sq^{a+b-j-1} \otimes Sq^j(m) \end{aligned}$$

for $b \in \mathbb{Z}$ odd.

For ℓ odd and $a \geq 0$ the action of P^a on $R_S(M)$ is given by

$$P^a(P^b \otimes m) = \sum_{j=0}^{[a/\ell]} (-1)^{a+j} \binom{(\ell-1)(b-j)-1}{a-\ell j} P^{a+b-j} \otimes P^j(m)$$

and

$$P^a(\beta P^b \otimes m) = \sum_{j=0}^{[a/\ell]} (-1)^{a+j} \binom{(\ell-1)(b-j)}{a-\ell j} \beta P^{a+b-j} \otimes P^j(m)$$
$$+ \sum_{j=0}^{[(a-1)/\ell]} (-1)^{a+j-1} \binom{(\ell-1)(b-j)-1}{a-\ell j-1} P^{a+b-j} \otimes \beta P^j(m)$$

for all $b \in \mathbb{Z}$.

Proof. For $a = 0$ the formulas confirm that Sq^0 and P^0 are the identity operations.

For $\ell = 2$ and $a > 0$ even, choose n so that $Sq^a \in A(n)$. Then $Sq^j \in A(n-1)$ for all $0 \le j \le [a/2]$, and $Sq^i Sq^j \otimes m = Sq^i \otimes Sq^j(m)$ in $B(n) \otimes_{A(n-1)} M$. When $a < 2b$ the formulas for $Sq^a(Sq^b \otimes m)$ then follow from the Adem relations [Voe03a, Thm. 10.2] for $Sq^a Sq^b$, as corrected in [Rio12, Thm. 4.5.1].

Similarly, for ℓ odd and $a > 0$, choose n so that $P^a \in A(n)$. Then $P^j, \beta P^j \in A(n-1)$ for all $0 \le j \le [a/\ell]$, and $P^i P^j \otimes m = P^i \otimes P^j(m)$ and $P^i \beta P^j \otimes m = P^i \otimes \beta P^j(m)$ in $B(n) \otimes_{A(n-1)} M$. When $a < \ell b$ the formulas for $P^a(P^b \otimes m)$ and $P^a(\beta P^b \otimes m)$ then follow from the Adem relations [Voe03a, Thm. 10.3] for $P^a P^b$ and $P^a \beta P^b$. (The last Adem relation is valid for $0 < a \le \ell b$, cf. [Rio12, Thm. 4.5.2].)

For the rest of the argument, ℓ can be even or odd. By Definition 3.8, the left $A(n)_{*,*}$-coaction on $B(n)_{*,*}$ commutes with multiplication by $\xi_1^{\ell^n}$, so the left $A(n)$-action on $B(n)$ commutes with the operation $\beta^e P^r \mapsto \beta^e P^{r+\ell^n}$. All mod ℓ binomial coefficients in sight also repeat ℓ^n-periodically in b. Hence the formulas for $a \ge \ell b$ follow from those for $a < \ell b$. □

Corollary 5.5. *For $\ell = 2$ and $a \ge 0$ even the action of Sq^a on $R_S(H^{*,*}) \cong B(0)$ is given by*

$$Sq^a(Sq^b) = \binom{b-1}{a} Sq^{a+b}$$

for $b \in \mathbb{Z}$.

For ℓ odd and $a \ge 0$ the action of P^a on $R_S(H^{*,*}) \cong B(0)$ is given by

$$P^a(P^b) = (-1)^a \binom{(\ell-1)b-1}{a} P^{a+b}$$

and

$$P^a(\beta P^b) = (-1)^a \binom{(\ell-1)b}{a} \beta P^{a+b}$$

for $b \in \mathbb{Z}$.

Proof. This is the special case $M = H^{*,*}$ of Proposition 5.4, where we identify $R_S(H^{*,*}) \cong B(0) \otimes_{H^{*,*}} H^{*,*} = B(0)$ and note that $Sq^j(1) = 0$ and $P^j(1) = 0$ in $H^{*,*}$ for all $j > 0$. When $\ell = 2$, the formulas for $P^a(P^b)$ and $P^a(\beta P^b)$ agree with the given formulas for $Sq^{2a}(Sq^{2b})$ and $Sq^{2a}(Sq^{2b+1})$, since $\binom{b-1}{a} \equiv \binom{2b-1}{2a}$ and $\binom{b}{a} \equiv \binom{2b}{2a} \mod 2$. □

Notation 5.6. Let $B\mu_\ell$ and BS_ℓ be the geometric classifying spaces of the linear algebraic groups μ_ℓ and S_ℓ, respectively. Recall from [Voe03a, Thm. 6.10, Thm. 6.16] that

$$H^{*,*}(B\mu_\ell) = H^{*,*}[u, v]/(u^2 = \tau v + \rho u)$$

with $\beta(u) = v$, and

$$H^{*,*}(BS_\ell) = H^{*,*}[c, d]/(c^2 = \tau d + \rho c)$$

with $\beta(c) = d$, as graded commutative $\mathscr{A}$-module $H^{*,*}$-algebras. The cohomological bidegrees of u, v, c and d are $(1, 1)$, $(2, 1)$, $(2\ell-3, \ell-1)$ and $(2\ell-2, \ell-1)$, respectively.

The coefficients τ and ρ are interpreted as 0 when ℓ is odd. Any choice of a primitive ℓ-th root of unity ζ defines a map $p_\zeta \colon B\mu_\ell \to BS_\ell$ inducing

$$p_\zeta^* \colon c \longmapsto -uv^{\ell-2}$$
$$p_\zeta^* \colon d \longmapsto -v^{\ell-1} .$$

We suppress p_ζ^* from the notation, viewing $H^{*,*}(BS_\ell)$ as an $\mathscr{A}$-module subalgebra of $H^{*,*}(B\mu_\ell)$. The natural left $\mathscr{A}$-module structure on $H^{*,*}(B\mu_\ell)$ is determined by the cases

$$\beta^e P^r(u) = \begin{cases} u & \text{for } (e,r) = (0,0), \\ v & \text{for } (e,r) = (1,0), \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta^e P^r(v) = \begin{cases} v & \text{for } (e,r) = (0,0), \\ v^\ell & \text{for } (e,r) = (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and the Cartan formula [Voe03a, Prop. 9.7], leading to the expressions

$$\begin{aligned} P^r(uv^k) &= \binom{k}{r} uv^{(\ell-1)r+k} \\ \beta P^r(uv^k) &= \binom{k}{r} v^{(\ell-1)r+1+k} \\ P^r(v^k) &= \binom{k}{r} v^{(\ell-1)r+k} \\ \beta P^r(v^k) &= 0 . \end{aligned}$$

The restricted $\mathscr{A}$-module action on $H^{*,*}(BS_\ell)$ is given by

$$\begin{aligned} P^r(cd^k) &= (-1)^r \binom{(\ell-1)(k+1)-1}{r} cd^{r+k} \\ \beta P^r(cd^k) &= (-1)^r \binom{(\ell-1)(k+1)-1}{r} d^{r+1+k} \\ P^r(d^k) &= (-1)^r \binom{(\ell-1)k}{r} d^{r+k} \\ \beta P^r(d^k) &= 0 , \end{aligned}$$

for $r \geq 0$ and $k \geq 0$, cf. [Rio12, Prop. 4.4.6].

In particular, $\beta(v^{\ell^n}) = 0$ and $P^r(v^{\ell^n}) = 0$ for all $0 < r < \ell^n$, so multiplication by v^{ℓ^n} acts left $A(n)$-linearly on $H^{*,*}(B\mu_\ell)$, cf. Lemma 4.11. Likewise, multiplication by d^{ℓ^n} acts left $A(n)$-linearly on $H^{*,*}(BS_\ell)$. Hence the following two localizations inherit compatible left $A(n)$-module structures for all $n \geq 0$. These combine to well-defined left $\mathscr{A}$-module structures, such that the localization homomorphisms are maps of $\mathscr{A}$-module $H^{*,*}$-algebras.

Definition 5.7. Let

$$\begin{aligned} H^{*,*}(B\mu_\ell)_{\mathrm{loc}} = H^{*,*}(B\mu_\ell)[1/v] &= H^{*,*}[u, v^{\pm 1}]/(u^2 = \tau v + \rho u) \\ &= H^{*,*}\{u^i v^k \mid i \in \{0,1\}, k \in \mathbb{Z}\} \end{aligned}$$

and

$$\begin{aligned} H^{*,*}(BS_\ell)_{\mathrm{loc}} = H^{*,*}(BS_\ell)[1/d] &= H^{*,*}[c, d^{\pm 1}]/(c^2 = \tau d + \rho c) \\ &= H^{*,*}\{c^i d^k \mid i \in \{0,1\}, k \in \mathbb{Z}\} \end{aligned}$$

denote the localizations away from v and d, respectively.

Theorem 5.8. *Let $\Sigma = \Sigma^{1,0}$. There is a left $\mathscr{A}$-module isomorphism*

$$R_S(H^{*,*}) \xrightarrow{\cong} \Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}}$$

defined by

$$P^k \longmapsto \Sigma c d^{k-1} \qquad \text{and} \qquad \beta P^k \longmapsto -\Sigma d^k$$

for $k \in \mathbb{Z}$. The composite $\Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}} \cong R_S(H^{*,*}) \xrightarrow{\epsilon} H^{*,*}$ is the left $\mathscr{A}$-linear homomorphism given by

$$\Sigma c d^{-1} \longmapsto 1 \qquad \text{and} \qquad \Sigma c^i d^k \longmapsto 0$$

for $(i,k) \neq (1,-1)$, where $i \in \{0,1\}$ and $k \in \mathbb{Z}$.

Proof. By Corollary 5.5, Notation 5.6 and Definition 5.7 the indicated $H^{*,*}$-module isomorphism maps $P^r(P^k)$ and $P^r(\beta P^k)$ to $P^r(\Sigma c d^{k-1})$ and $P^r(-\Sigma d^k)$, respectively, for all $r \geq 0$ and $k \in \mathbb{Z}$. Moreover, $\beta(\Sigma c d^{k-1}) = -\Sigma d^k$ and $\beta(-\Sigma d^k) = 0$. Hence the isomorphism is $\mathscr{A}$-linear. The calculation of the composite follows by noting that $\beta^e P^r(1) = 0$ in $H^{*,*}$ unless $(e,r) = (0,0)$. □

6. The large motivic Singer construction

Our next aim, following [AGM85, §5], is to construct the large Singer construction $R_\mu(M)$ as an extension of $R_S(M)$, with $R_\mu(H^{*,*}) \cong \Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}}$. We first note that $H^{*,*}(BS_\ell)_{\mathrm{loc}} \subset H^{*,*}(B\mu_\ell)_{\mathrm{loc}}$ is a pair of graded Frobenius algebras.

Definition 6.1. Let the *residue homomorphisms*

$$\mathrm{res}\colon \Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$
$$\mathrm{res}\colon \Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$

be the left $H^{*,*}$-linear Frobenius forms defined for $i \in \{0,1\}$ and $k \in \mathbb{Z}$ by

$$\mathrm{res}(\Sigma u^i v^k) = \mathrm{res}(\Sigma c^i d^k) = \begin{cases} 1 & \text{for } (i,k) = (1,-1), \\ 0 & \text{otherwise.} \end{cases}$$

The associated Frobenius pairings

$$\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \otimes_{H^{*,*}} H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$
$$\Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}} \otimes_{H^{*,*}} H^{*,*}(BS_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$

map $\Sigma x \otimes y$ to $\mathrm{res}(\Sigma xy)$, and the adjoint $H^{*,*}$-linear homomorphisms

$$\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \xrightarrow{\cong} (H^{-*,-*}(B\mu_\ell)_{\mathrm{loc}})^\vee$$
$$\Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}} \xrightarrow{\cong} (H^{-*,-*}(BS_\ell)_{\mathrm{loc}})^\vee$$

are the isomorphisms given by

$$\Sigma v^k \longmapsto (uv^{-k-1})^\vee$$
$$\Sigma uv^{k-1} \longmapsto (v^{-k})^\vee + \rho \cdot (uv^{-k})^\vee$$

and

$$\Sigma d^k \longmapsto (cd^{-k-1})^\vee$$
$$\Sigma cd^{k-1} \longmapsto (d^{-k})^\vee + \rho \cdot (cd^{-k})^\vee$$

for $k \in \mathbb{Z}$.

Lemma 6.2. *The Frobenius forms, the associated Frobenius pairings, and the adjoint isomorphisms, are all left $\mathscr{A}$-linear.*

Proof. The residue homomorphism in the case of $B\mu_\ell$ is $\mathscr{A}$-linear, because for $r > 0$ we have $P^r(uv^k) = 0$ whenever $(\ell-1)r + k = -1$, since

$$\binom{(\ell-1)(-r)-1}{r} = (-1)^r \binom{\ell r}{r} \equiv 0 \mod \ell\,.$$

The case of BS_ℓ follows from this, or from the second part of Theorem 5.8. The $\mathscr{A}$-linearity of the remaining homomorphisms follows formally. □

Recall the cotensor product $\square$ of comodules, e.g. from [EM66, §2].

Definition 6.3. Let

$$R^S(H_{*,*}) = \lim_n (B(n)_{*,*} \square_{A(n-1)_{*,*}} H_{*,*}) \cong B(0)_{*,*}$$

be the (achieved) limit of the right $A(n-1)_{*,*}$-comodule primitives in $B(n)_{*,*}$. It is a left $A(n)_{*,*}$-comodule algebra for each $n \geq 0$, and these coactions combine to a completed left $\mathscr{A}_{*,*}$-comodule algebra structure. We write

$$R^S(H_{*,*}) = H^{*,*}[\tilde\tau, \tilde\xi^{\pm 1}]/(\tilde\tau^2 = \tau\tilde\xi + \rho\tilde\tau\tilde\xi)\,,$$

with $\tilde\tau$ and $\tilde\xi$ mapping to τ_0 and ξ_1 in $B(0)_{*,*}$, respectively. Note that $R_S(H^{*,*}) \cong R^S(H_{*,*})^\vee$, with $\beta^e P^r$ dual to $\tilde\tau^e\tilde\xi^r$ in the monomial basis.

Lemma 6.4. *The composite left $\mathscr{A}$-module isomorphism*

$$R_S(H^{*,*}) \xrightarrow{\cong} \Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}} \xrightarrow{\cong} (H^{-*,-*}(BS_\ell)_{\mathrm{loc}})^\vee$$

is the dual of the $H^{*,*}$-algebra isomorphism

$$\Phi\colon H^{-*,-*}(BS_\ell)_{\mathrm{loc}} \xrightarrow{\cong} R^S(H_{*,*})$$

given by

$$c \longmapsto -\tilde\tau\tilde\xi^{-1} + \rho\cdot 1 \qquad \text{and} \qquad d \longmapsto \tilde\xi^{-1}\,.$$

Proof. The composite isomorphism maps P^k to $(d^{-k})^\vee + \rho\cdot(cd^{-k})^\vee$ and maps βP^k to $-(cd^{-k-1})^\vee$, hence is dual to the $H^{*,*}$-linear homomorphism mapping d^{-k} to $(P^k)^\vee = \tilde\xi^k$ and mapping cd^{-k} to $-(\beta P^{k-1})^\vee + \rho\cdot(P^k)^\vee = -\tilde\tau\tilde\xi^{k-1} + \rho\cdot\tilde\xi^k$. This is indeed an algebra isomorphism. □

For a left $\mathscr{A}_{*,*}$-comodule $M_{*,*}$, the $A(n)_{*,*}$-$A(n-1)_{*,*}$-bicomodule algebra product ϕ on $B(n)_{*,*}$ induces a pairing

$$(B(n)_{*,*} \square_{A(n-1)_{*,*}} H_{*,*}) \otimes_{H_{*,*}} (B(n)_{*,*} \square_{A(n-1)_{*,*}} M_{*,*}) \longrightarrow (B(n)_{*,*} \square_{A(n-1)_{*,*}} M_{*,*})$$

of left $A(n)_{*,*}$-comodules for each $n \geq 0$, making

$$R^S(M_{*,*}) = \lim_n (B(n)_{*,*} \square_{A(n-1)_{*,*}} M_{*,*})$$

an $R^S(H_{*,*})$-module in completed left $\mathscr{A}_{*,*}$-comodules. Viewing $R^S(M_{*,*})$ as an $H^{-*,-*}(BS_\ell)_{\mathrm{loc}}$-module via the algebra isomorphism Φ, we can form the induced $H^{-*,-*}(B\mu_\ell)_{\mathrm{loc}}$-module

$$R^\mu(M_{*,*}) = H^{-*,-*}(B\mu_\ell)_{\mathrm{loc}} \underset{H^{-*,-*}(BS_\ell)_{\mathrm{loc}}}{\otimes} R^S(M_{*,*})\,.$$

As a left $A(n)_{*,*}$-comodule, it is isomorphic to a finite direct sum

$$R^\mu(M_{*,*}) \cong H^{*,*}\{1, v^{\ell^n}, \dots, v^{\ell^n(\ell-2)}\} \otimes_{H^{*,*}} R^S(M_{*,*})\,,$$

where each power of v^{ℓ^n} is $A(n)_{*,*}$-comodule primitive.

Dually, for a left $\mathscr{A}$-module M the completed $A(n)$-$A(n-1)$-bimodule coproduct

$$B(n) = B(n)^\vee_{*,*} \xrightarrow{\phi^\vee} (B(n)_{*,*} \otimes_{H_{*,*}} B(n)_{*,*})^\vee = B(n) \mathbin{\widehat\otimes}_{H^{*,*}} B(n)$$

induces a "copairing"

$$B(n) \otimes_{A(n-1)} M \longrightarrow (B(n) \otimes_{A(n-1)} H^{*,*}) \widehat{\otimes}_{H^{*,*}} (B(n) \otimes_{A(n-1)} M)$$

of left $A(n)$-modules for each $n \geq 0$, making the small Singer construction

$$R_S(M) = \operatorname*{colim}_n (B(n) \otimes_{A(n-1)} M)$$

a completed $R_S(H^{*,*})$-comodule in left $\mathscr{A}$-modules. Here $R_S(H^{*,*})$ has the completed $H^{*,*}$-coalgebra structure dual to the $H_{*,*}$-algebra structure on $R^S(H_{*,*})$ that appears in Lemma 6.4. It corresponds via the isomorphism in Theorem 5.8 to a completed $H^{*,*}$-coalgebra structure on $\Sigma H^{*,*}(BS_\ell)_{\text{loc}}$. Moreover, the algebra inclusion $H^{-*,-*}(BS_\ell)_{\text{loc}} \subset H^{-*,-*}(B\mu_\ell)_{\text{loc}}$ in left $\mathscr{A}$-modules corresponds under duality and the Frobenius isomorphisms from Definition 6.1 to a completed $H^{*,*}$-coalgebra epimorphism

$$\pi \colon \Sigma H^{*,*}(B\mu_\ell)_{\text{loc}} \twoheadrightarrow \Sigma H^{*,*}(BS_\ell)_{\text{loc}}$$

in left $\mathscr{A}$-modules, given by

$$\Sigma v^{(\ell-1)k} \longmapsto (-1)^k \Sigma d^k \qquad \text{and} \qquad \Sigma u v^{(\ell-1)k-1} \longmapsto (-1)^k \Sigma c d^{k-1},$$

while the remaining $H^{*,*}$-module generators $\Sigma u^i v^k$ with $i \in \{0,1\}$ and $k \in \mathbb{Z}$ map to zero. This discussion motivates the following definition.

Definition 6.5. Let M be any left $\mathscr{A}$-module.

(a) Let the *large motivic Singer construction*

$$R_\mu(M) = \Sigma H^{*,*}(B\mu_\ell)_{\text{loc}} \mathop{\square}_{\Sigma H^{*,*}(BS_\ell)_{\text{loc}}} R_S(M)$$

be the left $\mathscr{A}$-module coinduced from $R_S(M)$ along the completed $H^{*,*}$-coalgebra epimorphism π. As a left $A(n)$-module it is isomorphic to the finite direct sum

$$R_\mu(M) \cong H^{*,*}\{1, v^{\ell^n}, \dots, v^{\ell^n(\ell-2)}\} \otimes_{H^{*,*}} R_S(M)$$

where $A(n)$ acts trivially, i.e., via $\eta^\vee_{L,n} \colon A(n) \to H^{*,*}$, on each power of v^{ℓ^n}.

(b) Let the *large evaluation homomorphism*

$$\epsilon \colon R_\mu(M) \longrightarrow M$$

be the composite $\epsilon(\pi \,\square\, 1) \colon R_\mu(M) \to R_S(M) \to M$.

Corollary 6.6. *There is a left $\mathscr{A}$-module isomorphism*

$$R_\mu(H^{*,*}) \cong \Sigma H^{*,*}(B\mu_\ell)_{\text{loc}}.$$

The composite $\Sigma H^{*,*}(B\mu_\ell)_{\text{loc}} \cong R_\mu(H^{*,*}) \xrightarrow{\epsilon} H^{*,*}$ equals the residue homomorphism for $B\mu_\ell$.

Proof. This follows directly from Theorem 5.8. □

Lemma 6.7. *As a left $A(0)$-module, the large motivic Singer construction is given by the tensor product*

$$R_\mu(M) \cong H^{*,*}\{\Sigma u^i v^k \mid i \in \{0,1\}, k \in \mathbb{Z}\} \otimes_{H^{*,*}} M$$

with the $A(0)$-action from $\Sigma H^{*,*}(B\mu_\ell)_{\text{loc}}$. Each element of $R_\mu(M)$ is thus a finite sum of terms $\Sigma u^i v^k \otimes m$, with $i \in \{0,1\}$, $k \in \mathbb{Z}$ and $m \in M$, where $\beta(\Sigma u v^k \otimes m) = -\Sigma v^{k+1} \otimes m$ and $\beta(\Sigma v^k \otimes m) = 0$.

Proof. Clear. □

The following formulas generalize the classical one of Singer [Sin81, (3.2)] for $\ell = 2$, and of Lunøe–Nielsen and the second author [LNR12, Def. 3.1] for ℓ odd. The latter two formulas were surely known to the authors of [AGM85].

Proposition 6.8. *For $\ell = 2$ and $r \geq 0$ the action of Sq^{2r} on $R_\mu(M) \cong R_S(M)$ is given by*

$$Sq^{2r}(\Sigma uv^k \otimes m) = \sum_{j=0}^{[r/2]} \binom{k-j}{r-2j} \Sigma uv^{r+k-j} \otimes Sq^{2j}(m) + \sum_{j=0}^{[(r-1)/2]} \binom{k-j}{r-2j-1} \tau \cdot \Sigma v^{r+k-j} \otimes Sq^{2j+1}(m)$$

and

$$Sq^{2r}(\Sigma v^k \otimes m) = \sum_{j=0}^{[r/2]} \binom{k-j}{r-2j} \Sigma v^{r+k-j} \otimes Sq^{2j}(m) + \sum_{j=0}^{[(r-1)/2]} \binom{k-j-1}{r-2j-1} \Sigma(u+\rho) v^{r+k-j-1} \otimes Sq^{2j+1}(m) .$$

Here $\Sigma(u+\rho)v^{r+k-j-1} = \Sigma uv^{r+k-j-1} + \rho \cdot \Sigma v^{r+k-j-1}$.

For ℓ odd and $r \geq 0$ the action of P^r on $R_\mu(M)$ is given by

$$P^r(\Sigma uv^{k-1} \otimes m) = \sum_{j=0}^{[r/\ell]} \binom{k-(\ell-1)j-1}{r-\ell j} \Sigma uv^{k+(\ell-1)(r-j)-1} \otimes P^j(m)$$

and

$$P^r(\Sigma v^k \otimes m) = \sum_{j=0}^{[r/\ell]} \binom{k-(\ell-1)j}{r-\ell j} \Sigma v^{k+(\ell-1)(r-j)} \otimes P^j(m) + \sum_{j=0}^{[(r-1)/\ell]} \binom{k-(\ell-1)j-1}{r-\ell j-1} \Sigma uv^{k+(\ell-1)(r-j)-1} \otimes \beta P^j(m) .$$

Proof. For $\ell = 2$, the formulas are obtained from Proposition 5.4 by replacing Sq^{2k} and Sq^{2k+1} by $\Sigma cd^{k-1} = \Sigma uv^{k-1}$ and $-\Sigma d^k = \Sigma v^k$, respectively, as in Theorem 5.8. The summations over $0 \leq j \leq [a/2]$ split into two cases, according to the parity of j, and the resulting terms can be collected as shown.

For ℓ odd, we first rewrite $R_S(M)$ as

$$\Sigma H^{*,*}(BS_\ell)_{\text{loc}} \otimes_{H^{*,*}} M = H^{*,*}\{\Sigma c^i d^k \mid i \in \{0,1\}, k \in \mathbb{Z}\} \otimes_{H^{*,*}} M ,$$

replacing P^k and βP^k by Σcd^{k-1} and $-\Sigma d^k$, respectively. For $r \geq 0$ the action of P^r on $R_S(M)$ is then given by

$$P^r(\Sigma(cd^{-1})(-d)^b \otimes m) = \sum_{j=0}^{[r/\ell]} \binom{(\ell-1)(b-j)-1}{r-\ell j} \Sigma(cd^{-1})(-d)^{r+b-j} \otimes P^j(m)$$

and

$$P^r(\Sigma(-d)^b \otimes m) = \sum_{j=0}^{[r/\ell]} \binom{(\ell-1)(b-j)}{r-\ell j} \Sigma(-d)^{r+b-j} \otimes P^j(m) + \sum_{j=0}^{[(r-1)/\ell]} \binom{(\ell-1)(b-j)-1}{r-\ell j-1} \Sigma(cd^{-1})(-d)^{r+b-j} \otimes \beta P^j(m)$$

for $b \in \mathbb{Z}$. Substituting $cd^{-1} = uv^{-1}$, $-d = v^{\ell-1}$ and $k = (\ell-1)b$ we obtain the claimed formulas, in the cases where k is a multiple of $\ell - 1$. The general cases follow, since for n so large that $0 < r < \ell^n$ the action of P^r on $R_\mu(M)$ commutes with multiplication by v^{ℓ^n}, and ℓ^n is relatively prime to $\ell - 1$. All mod ℓ binomial coefficients in sight are ℓ^n-periodic as functions of $k \in \mathbb{Z}$. $\square$

7. The evaluations are Ext-equivalences

We can now adapt [AGM85, Lem. 2.2] to the motivic setting. We sidestep their use of $\mathbb{F}_p \otimes_{\mathscr{A}} (-)$ and Tor-equivalences, since $H^{*,*}$ is not naturally a right $\mathscr{A}$-module, and pass directly to $\mathrm{Hom}_{\mathscr{A}}(-, H^{*,*})$ and Ext-equivalences.

Lemma 7.1. *Let M be a free left $\mathscr{A}$-module. The Singer constructions $R_S(M)$ and $R_\mu(M)$ are free as left $A(n)$-modules, for each n, and flat as left $\mathscr{A}$-modules.*

Proof. Since M is left $\mathscr{A}$-free, it is left $A(n-1)$-free by Lemma 4.12. Hence the left $A(n)$-module $R_S(M) = B(n) \otimes_{A(n-1)} M$ is a direct sum of (suitably suspended) copies of $B(n)$, each of which is left $A(n)$-free by Proposition 4.19(b). Therefore $R_S(M)$ is left $A(n)$-free for each n, so that

$$\mathrm{Tor}_s^{\mathscr{A}}(K, R_S(M)) \cong \operatorname*{colim}_n \mathrm{Tor}_s^{A(n)}(K, R_S(M)) = 0$$

for each right $\mathscr{A}$-module K and $s \geq 1$. Equivalently, $R_S(M)$ is left $\mathscr{A}$-flat.

Moreover, $R_\mu(M)$ is a direct sum, as a left $A(n)$-module, of copies of $R_S(M)$. Hence it is also left $A(n)$-free for each n, and therefore left $\mathscr{A}$-flat, by the same argument as before. $\square$

Proposition 7.2. *Let M be a free left $\mathscr{A}$-module. The evaluation homomorphisms induce isomorphisms*

$$\mathrm{Hom}(\epsilon, \mathrm{id}) \colon \mathrm{Hom}_{\mathscr{A}}(M, H^{*,*}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(R_S(M), H^{*,*})$$
$$\mathrm{Hom}(\epsilon, \mathrm{id}) \colon \mathrm{Hom}_{\mathscr{A}}(M, H^{*,*}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(R_\mu(M), H^{*,*}) .$$

Proof. It suffices to consider the case $M = \mathscr{A} = \mathrm{colim}_n A(n-1)$. Then

$$\begin{aligned}
\mathrm{Hom}_{A(n)}(R_S(A(n-1)), H^{*,*}) &= \mathrm{Hom}_{A(n)}(B(n) \otimes_{A(n-1)} A(n-1), H^{*,*}) \\
&= \mathrm{Hom}_{A(n)}(B(n), H^{*,*}) \\
&\cong \mathrm{Hom}_{H^{*,*}}(H^{*,*}\{P^r \mid r \in \mathbb{Z} \text{ with } \ell^n \mid r\}, H^{*,*}) \\
&\cong \prod_{\ell^n \mid r} H^{*,*}\{\xi_1^r\}
\end{aligned}$$

by Proposition 4.19(b), where we identify ξ_1^r with the dual of the left $A(n)$-module generator $P^r \in B(n)$. The bidegrees of the classes ξ_1^r with $\ell^n \mid r$ are integer multiples of $((2\ell-2)\ell^n, (\ell-1)\ell^n)$. In any fixed bidegree, only the factor generated by ξ_1^0 can make a nonzero contribution to this product when n is sufficiently large. Hence

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{A}}(R_S(\mathscr{A}), H^{*,*}) &\cong \lim_n \mathrm{Hom}_{A(n)}(R_S(A(n-1)), H^{*,*}) \\
&\cong \lim_n \prod_{\ell^n \mid r} H^{*,*}\{\xi_1^r\} \cong H^{*,*}\{\xi_1^0\} .
\end{aligned}$$

Since ϵ maps $P^0 \otimes 1$ to $P^0(1) = 1$, it follows that

$$H^{*,*} \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}, H^{*,*}) \xrightarrow{\mathrm{Hom}(\epsilon,1)} \mathrm{Hom}_{\mathscr{A}}(R_S(\mathscr{A}), H^{*,*}) \cong H^{*,*}\{\xi_1^0\}$$

is an isomorphism.

Similarly,

$$\begin{aligned}
&\mathrm{Hom}_{A(n)}(R_\mu(A(n-1)), H^{*,*}) \\
&\qquad \cong H^{*,*}\{1, v^{\ell^n}, \dots, v^{\ell^n(\ell-2)}\} \otimes_{H^{*,*}} \prod_{\ell^n \mid r} H^{*,*}\{\xi_1^r\} .
\end{aligned}$$

The bidegrees of the classes $v^{\ell^n j} \otimes \xi_1^r$ with $0 \le j \le \ell - 2$ and $\ell^n \mid r$ are integer multiples of $(2\ell^n, \ell^n)$. Again, in any fixed bidegree, only the factor generated by $v^0 \otimes \xi_1^0$ can make a nonzero contribution for n sufficiently large. Hence

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}}(R_\mu(\mathscr{A}), H^{*,*}) &\cong \lim_n \mathrm{Hom}_{A(n)}(R_\mu(A(n-1)), H^{*,*}) \\ &\cong H^{*,*}\{v^0 \otimes \xi_1^0\}\,. \end{aligned}$$

Since $\epsilon(\pi \,\square\, 1)$ maps $v^0 \otimes P^0$ to $P^0(1) = 1$, it follows that

$$H^{*,*} \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}, H^{*,*}) \xrightarrow{\mathrm{Hom}(\epsilon,1)} \mathrm{Hom}_{\mathscr{A}}(R_\mu(\mathscr{A}), H^{*,*}) \cong H^{*,*}\{v^0 \otimes \xi_1^0\}$$

is an isomorphism. $\square$

The Ext-groups for modules over $\mathscr{A}$ or $A(n)$ are trigraded. In the case of an $\mathscr{A}$-module M we write $\mathrm{Ext}_{\mathscr{A}}^{s,t,u}(M, H^{*,*})$ for the group in tridegree (s,t,u), where s is the cohomological degree and (t,u) is the internal bidegree.

Definition 7.3. An $\mathscr{A}$-module homomorphism $\theta\colon L \to M$ will be said to be an Ext-*equivalence* if the induced homomorphism

$$\theta^*\colon \mathrm{Ext}_{\mathscr{A}}^{*,*,*}(M, H^{*,*}) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*,*}(L, H^{*,*})$$

is an isomorphism.

We can now generalize part of [AGM85, Prop. 1.2, Thm. 1.3].

Theorem 7.4. *Let M be any left $\mathscr{A}$-module. The (small and large) evaluation homomorphisms*

$$\epsilon\colon R_S(M) \longrightarrow M \qquad \text{and} \qquad \epsilon\colon R_\mu(M) \longrightarrow M$$

are Ext-*equivalences.*

Proof. Let

$$\ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \to 0$$

be a free $\mathscr{A}$-module resolution of M. Then

$$\ldots \longrightarrow R_S(F_1) \longrightarrow R_S(F_0) \longrightarrow R_S(M) \to 0$$

is a flat $\mathscr{A}$-module resolution, and a free $A(n)$-module resolution for each n, by Lemma 7.1. By Proposition 7.2 the evaluation homomorphism induces an isomorphism

$$\mathrm{Hom}(\epsilon, 1)\colon \mathrm{Hom}_{\mathscr{A}}(F_s, H^{*,*}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(R_S(F_s), H^{*,*})$$

for each $s \ge 0$. Passing to cohomology, it also induces isomorphisms

$$\epsilon'\colon \mathrm{Ext}^s_{\mathscr{A}}(M, H^{*,*}) \xrightarrow{\cong} H^s(\mathrm{Hom}_{\mathscr{A}}(R_S(F_*), H^{*,*}))$$

for all $s \ge 0$. Let

$$\ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R_S(M) \to 0$$

be a free $\mathscr{A}$-module resolution of $R_S(M)$, and choose an $\mathscr{A}$-module chain map $\zeta\colon E_* \to R_S(F_*)$ over $R_S(M)$. There is then an induced map ζ^* of lim-lim^1 short exact sequences, from

$$\begin{aligned} 0 \to \lim_n{}^1 H^{s-1}(\mathrm{Hom}_{A(n)}(R_S(F_*), H^{*,*})) &\longrightarrow H^s(\mathrm{Hom}_{\mathscr{A}}(R_S(F_*), H^{*,*})) \\ &\longrightarrow \lim_n H^s(\mathrm{Hom}_{A(n)}(R_S(F_*), H^{*,*})) \to 0 \end{aligned}$$

to

$$\begin{aligned} 0 \to \lim_n{}^1 \mathrm{Ext}^{s-1}_{A(n)}(R_S(M), H^{*,*}) &\longrightarrow \mathrm{Ext}^s_{\mathscr{A}}(R_S(M), H^{*,*}) \\ &\longrightarrow \lim_n \mathrm{Ext}^s_{A(n)}(R_S(M), H^{*,*}) \to 0\,, \end{aligned}$$

cf. [Mar83, Prop. 11.9, Prop. 13.4]. When viewed as an $A(n)$-module chain map, ζ becomes a chain homotopy equivalence, hence induces isomorphisms

$$\zeta^*\colon H^s(\mathrm{Hom}_{A(n)}(R_S(F_*), H^{*,*})) \xrightarrow{\cong} \mathrm{Ext}^s_{A(n)}(R_S(M), H^{*,*})$$

for all n and s. Applying $\lim_n$ and $\lim^1{}_n$, we deduce that

$$\zeta^*\colon H^s(\mathrm{Hom}_{\mathscr{A}}(R_S(F_*), H^{*,*})) \xrightarrow{\cong} \mathrm{Ext}^s_{\mathscr{A}}(R_S(M), H^{*,*})$$

is an isomorphism. Hence the composite $\epsilon^* = \zeta^*\epsilon'$ is also an isomorphism, as claimed.

The proof for R_μ in place of R_S is identical. □

Corollary 7.5. *The residue homomorphisms*

$$\mathrm{res}\colon \Sigma H^{*,*}(BS_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*} \qquad \textit{and} \qquad \mathrm{res}\colon \Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$

are Ext-*equivalences.*

Proof. In view of Theorem 5.8 and Corollary 6.6, this is the case $M = H^{*,*}$ of Theorem 7.4. □

8. Generalized Eilenberg–MacLane spectra

Since H is cellular, the monomial basis $\{\tau^E\xi^R\}_{(E,R)}$ for $\mathscr{A}_{*,*}$ as a right (or left) $H_{*,*}$-module determines an equivalence $H \wedge H \simeq \bigvee_{(E,R)} \Sigma^{\|E,R\|} H$ of right (or left) H-module spectra. Here (E, R) ranges over the sequences in Lemma 2.1, and $\|E, R\| = \|\tau^E\xi^R\|$. It follows that the natural homomorphism

$$\mathscr{A}_{*,*} \otimes_{H_{*,*}} H_{*,*}(X) \xrightarrow{\cong} \pi_{*,*}((H \wedge H) \wedge_H (H \wedge X)) = \pi_{*,*}(H \wedge H \wedge X)$$

is an isomorphism for any motivic spectrum X, and that $1 \wedge \eta \wedge 1\colon H \wedge X = H \wedge \mathbb{S} \wedge X \to H \wedge H \wedge X$ induces a natural left $\mathscr{A}_{*,*}$-coaction on $H_{*,*}(X)$.

If $M \simeq H \wedge X$ for some motivic spectrum X, then the fork

$$\pi_{*,*}M \xrightarrow{\ \eta\ } \pi_{*,*}(H \wedge M) \underset{1\wedge\eta}{\overset{\eta\wedge 1}{\rightrightarrows}} \pi_{*,*}(H \wedge H \wedge M)$$

is split by $\mu\colon \pi_{*,*}(H \wedge M) \cong \pi_{*,*}(H \wedge H \wedge X) \to \pi_{*,*}(H \wedge X) \cong \pi_{*,*}M$ and $\mu\colon \pi_{*,*}(H\wedge H\wedge M) \to \pi_{*,*}(H\wedge M)$, hence exhibits $\pi_{*,*}M$ as a split equalizer [ML98, §VI.6]. Under the identifications $\pi_{*,*}(H \wedge M) = H_{*,*}(M)$ and $\pi_{*,*}(H \wedge H \wedge M) \cong \mathscr{A}_{*,*} \otimes_{H_{*,*}} H_{*,*}(M)$, this provides an isomorphism

$$\pi_{*,*}M \cong H_{*,*} \,\square_{\mathscr{A}_{*,*}}\, H_{*,*}(M) \cong \mathrm{Hom}_{\mathscr{A}_{*,*}}(H_{*,*}, H_{*,*}(M)) \tag{8.1}$$

of $\pi_{*,*}M$ with the left $\mathscr{A}_{*,*}$-comodule primitives in $H_{*,*}(M)$.

Definition 8.1. By a *motivic GEM* (short for generalized Eilenberg–MacLane spectrum) we shall mean a left H-module spectrum

$$M \simeq \bigvee_{\alpha \in J} \Sigma^{p_\alpha, q_\alpha} H$$

that is equivalent to a wedge sum of bigraded suspensions of H.

These are precisely the H-cellular module spectra M, in the sense of [DI05, §7.9], with the property that $\pi_{*,*}(M)$ is free as a left $H_{*,*}$-module. This generalizes the split Tate objects of [Voe10, §2.4], in that we allow arbitrary bigraded suspensions.

If M is a motivic GEM, we can write $M \simeq H \wedge X$ with $X = \bigvee_{\alpha\in J} S^{p_\alpha,q_\alpha}$. Then

$$H_{*,*}(X) \cong \bigoplus_{\alpha\in J} \Sigma^{p_\alpha,q_\alpha} H_{*,*}$$

as left $\mathscr{A}_{*,*}$-comodules, and the natural homomorphism

$$\pi_{*,*}F(X,H) = \pi_{*,*}F_H(H \wedge X, H) \xrightarrow{\cong} \mathrm{Hom}_{H_{*,*}}(H_{*,*}(X), H_{*,*}) = H_{*,*}(X)^\vee$$

is an isomorphism, so that

$$H^{*,*}(X) \cong \prod_{\alpha \in J} \Sigma^{p_\alpha, q_\alpha} H^{*,*}$$

as left $\mathscr{A}$-modules.

Definition 8.2. With these notations we say that $H \wedge X$ (or $H_{*,*}(X)$, or $H^{*,*}(X)$) has *bifinite type* if for each bidegree (p,q) there are only finitely many $\alpha \in J$ for which

$$p_\alpha \le p \qquad \text{or} \qquad p_\alpha - q_\alpha \le p - q\,.$$

This condition is more restrictive than the notions of motivically finite type from [DI10, Def. 2.11, Def. 2.12] and of finite type from [HKO11, §2]. It ensures that both inclusions

$$\bigoplus_{\alpha \in J} \Sigma^{p_\alpha, q_\alpha} H_{*,*} \xrightarrow{\cong} \prod_{\alpha \in J} \Sigma^{p_\alpha, q_\alpha} H_{*,*}$$
$$\bigoplus_{\alpha \in J} \Sigma^{p_\alpha, q_\alpha} H^{*,*} \xrightarrow{\cong} \prod_{\alpha \in J} \Sigma^{p_\alpha, q_\alpha} H^{*,*}$$

are isomorphisms, so that the canonical homomorphism

$$H_{*,*}(X) \xrightarrow{\cong} \mathrm{Hom}_{H^{*,*}}(H^{*,*}(X), H^{*,*}) = H^{*,*}(X)^\vee$$

is an isomorphism. Moreover, if $H \wedge X$ has bifinite type, then so does $H \wedge H \wedge X \simeq \bigvee_{(E,R)} \Sigma^{\|E,R\|} H \wedge X$. Hence $f \mapsto f^\vee$ defines an isomorphism

$$\mathrm{Hom}_{\mathscr{A}_{*,*}}(H_{*,*}, H_{*,*}(X)) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(H^{*,*}(X), H^{*,*}) \tag{8.2}$$

from the $\mathscr{A}_{*,*}$-comodule homomorphisms $H_{*,*} \to H_{*,*}(X)$ to the $\mathscr{A}$-module homomorphisms $H^{*,*}(X) \to H^{*,*}$.

9. A delayed limit Adams spectral sequence

Let

$$\dots \to Y(m+1) \xrightarrow{f_{m+1}} Y(m) \xrightarrow{f_m} Y(m-1) \to \dots \tag{9.1}$$

be any tower of motivic spectra. Its homotopy limit $Y = \mathrm{holim}_m Y(m)$ sits in a homotopy cofiber sequence

$$\Sigma^{-1} \prod_m Y(m) \xrightarrow{i} Y \xrightarrow{j} \prod_m Y(m) \xrightarrow{k} \prod_m Y(m)\,,$$

where k is the difference between the identity map and the product of the maps f_m. Let

$$\begin{tikzcd} \dots \arrow[r] & \mathbb{S}_2 \arrow[rr, "\alpha"] & & \mathbb{S}_1 \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & \mathbb{S}_0 \arrow[dl, "\beta"] \\ & & H \wedge \mathbb{S}_1 \arrow[ul, dashed, "\gamma"] & & H \wedge \mathbb{S}_0 \arrow[ul, dashed, "\gamma"] & \end{tikzcd}$$

be the canonical mod ℓ Adams resolution of the motivic sphere spectrum $\mathbb{S} = \mathbb{S}_0$, inductively defined by the homotopy cofiber sequences

$$\mathbb{S}_{s+1} \xrightarrow{\alpha} \mathbb{S}_s \xrightarrow{\beta} H \wedge \mathbb{S}_s \xrightarrow{\gamma} \Sigma\mathbb{S}_{s+1}$$

where $\beta = \eta \wedge 1$. (Dashed arrows indicate morphisms of degree -1.) Form the smash products

$$Y_s(m) = \mathbb{S}_s \wedge Y(m)$$
$$K_s(m) = H \wedge \mathbb{S}_s \wedge Y(m)$$

so as to obtain a tower of canonical Adams resolutions
(9.2)

$$\begin{tikzcd}
& \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\dots \arrow[r] & Y_2(m{+}1) \arrow[rr] \arrow[dd] & & Y_2(m) \arrow[rr] \arrow[dd] & & Y_2(m{-}1) \arrow[r] \arrow[dd] & \dots \\
& \dots \arrow[r] & K_1(m{+}1) \arrow[rr] \arrow[ul, dashed] & & K_1(m) \arrow[rr] \arrow[ul, dashed] & & K_1(m{-}1) \arrow[r] \arrow[ul, dashed] & \dots \\
\dots \arrow[r] & Y_1(m{+}1) \arrow[rr] \arrow[dd] \arrow[ur] & & Y_1(m) \arrow[rr] \arrow[dd] \arrow[ur] & & Y_1(m{-}1) \arrow[r] \arrow[dd] \arrow[ur] & \dots \\
& \dots \arrow[r] & K_0(m{+}1) \arrow[rr] \arrow[ul, dashed] & & K_0(m) \arrow[rr] \arrow[ul, dashed] & & K_0(m{-}1) \arrow[r] \arrow[ul, dashed] & \dots \\
\dots \arrow[r] & Y_0(m{+}1) \arrow[rr] \arrow[ur] & & Y_0(m) \arrow[rr] \arrow[ur] & & Y_0(m{-}1) \arrow[r] \arrow[ur] & \dots\,.
\end{tikzcd}$$

Let

$$Y_s = \operatorname*{holim}_m Y_s(m)$$
$$K_s = \operatorname*{holim}_m K_s(m)$$

be the homotopy limits of the terms in these Adams resolutions. These fit in a commutative diagram
(9.3)

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\Sigma^{-1}\prod_m Y_2(m) \arrow[rr, "i"] \arrow[dd, "\alpha"'] & & Y_2 \arrow[rr, "j"] \arrow[dd, "\alpha"'] & & \prod_m Y_2(m) \arrow[dd, "\alpha"'] & \\
& \Sigma^{-1}\prod_m K_1(m) \arrow[rr, "i"] \arrow[ul, dashed, "\gamma"'] & & K_1 \arrow[rr, "j"] \arrow[ul, dashed, "\gamma"'] & & \prod_m K_1(m) \arrow[ul, dashed, "\gamma"'] \\
\Sigma^{-1}\prod_m Y_1(m) \arrow[rr, "i"] \arrow[dd, "\alpha"'] \arrow[ur, "\beta"'] & & Y_1 \arrow[rr, "j"] \arrow[dd, "\alpha"'] \arrow[ur, "\beta"'] & & \prod_m Y_1(m) \arrow[dd, "\alpha"'] \arrow[ur, "\beta"'] & \\
& \Sigma^{-1}\prod_m K_0(m) \arrow[rr, "i"] \arrow[ul, dashed, "\gamma"'] & & K_0 \arrow[rr, "j"] \arrow[ul, dashed, "\gamma"'] & & \prod_m K_0(m) \arrow[ul, dashed, "\gamma"'] \\
\Sigma^{-1}\prod_m Y_0(m) \arrow[rr, "i"] \arrow[ur, "\beta"'] & & Y_0 \arrow[rr, "j"] \arrow[ur, "\beta"'] & & \prod_m Y_0(m) \arrow[ur, "\beta"'] &
\end{tikzcd}$$

with horizontal homotopy cofiber sequences extending to

$$\Sigma^{-1}\prod_m Y_s(m) \xrightarrow{i} Y_s \xrightarrow{j} \prod_m Y_s(m) \xrightarrow{k} \prod_m Y_s(m)$$
$$\Sigma^{-1}\prod_m K_s(m) \xrightarrow{i} K_s \xrightarrow{j} \prod_m K_s(m) \xrightarrow{k} \prod_m K_s(m)\,.$$

The subdiagram

$$\begin{tikzcd}
\dots \arrow[r] & Y_2 \arrow[rr, "\alpha"] & & Y_1 \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & Y_0 \arrow[dl, "\beta"] \\
& & K_1 \arrow[ul, dashed, "\gamma"] & & K_0 \arrow[ul, dashed, "\gamma"] &
\end{tikzcd} \tag{9.4}$$

is not generally an Adams resolution. Nonetheless, one may consider its associated homotopy spectral sequence, with E_1-term

$$ {}^{\lim}E_1^{s,*,*} = \pi_{*,*}(K_s) $$

and abutment $\pi_{*,*}(Y_0) = \pi_{*,*}(Y)$. Under finiteness hypotheses which ensure that all limits in sight are exact, the E_2-term of this *limit Adams spectral sequence* was described in [CMP87, Prop. 7.1] and [LNR12, Prop. 2.2]. In our motivic context, these finiteness hypotheses are only realistic if $H^{*,*}$ is finite in each bidegree, which excludes some very interesting base schemes S, such as Spec of a global field.

To avoid this restrictive hypothesis, we shall instead show that there is a modified Adams spectral sequence, in the style of [Mil72, Lem. 5.3.1] and [Bru82], with the same abutment as before, whose E_2-term is recognizable under more flexible finiteness conditions. This kind of modification is referred to in [BR21, §12.6] as a delayed Adams spectral sequence, to distinguish it from another kind of modified Adams spectral sequence in current usage (the hastened one).

To construct the delayed limit Adams spectral sequence we may assume that the maps i and α in (9.3) are all cofibrations, let $W_0 = Y_0$, and form the pushouts

$$ W_s = Y_s \cup \Sigma^{-1} \prod_m Y_{s-1}(m) $$

along $\Sigma^{-1}\prod_m Y_s(m)$, for all $s \geq 1$. There are then homotopy cofiber sequences

$$ W_1 \longrightarrow W_0 \xrightarrow{\beta j} L_0 = \prod_m K_0(m) \longrightarrow \Sigma W_1 $$

and

$$ W_{s+1} \longrightarrow W_s \xrightarrow{\beta j \cup \beta} L_s = \prod_m K_s(m) \vee \Sigma^{-1} \prod_m K_{s-1}(m) \longrightarrow \Sigma W_{s+1} $$

for all $s \geq 1$, defining the spectra L_s. This produces a delayed resolution

(9.5)
$$ \begin{tikzcd} \cdots \arrow[r] & W_2 \arrow[rr, "\alpha"] & & W_1 \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & W_0 \arrow[dl, "\beta"] \\ & & L_1 \arrow[ul, dashed, "\gamma"] & & L_0 \arrow[ul, dashed, "\gamma"] & \end{tikzcd} $$

of $W_0 = Y_0 \simeq Y$. The inclusions $Y_s \subset W_s$ induce a map of diagrams from (9.4) to (9.5).

Definition 9.1. The *delayed limit Adams spectral sequence* of the tower (9.1) is the homotopy spectral sequence associated to the resolution (9.5), with E_1-term

$$ {}^{\mathrm{del}}E_1^{s,*,*} = \pi_{*,*}(L_s) $$

and d_1-differential induced by the composite

$$ \beta\gamma \colon L_s \longrightarrow \Sigma W_{s+1} \longrightarrow \Sigma L_{s+1} \, . $$

We now make the assumption that each $H \wedge Y(m)$ is a motivic GEM of bifinite type. For example, this is the case if each $Y(m)$ is cellular with $H_{*,*}(Y(m))$ free of bifinite type as a left $H_{*,*}$-module. It follows by induction on s that each $K_s(m)$ is a motivic GEM of bifinite type. Hence the isomorphisms (8.1) and (8.2) identify the (E_1, d_1)-term

$$ \cdots \longleftarrow \pi_{*,*}K_2(m) \longleftarrow \pi_{*,*}K_1(m) \longleftarrow \pi_{*,*}K_0(m) \leftarrow 0 $$

of the Adams spectral sequence for $Y(m)$ with $\mathrm{Hom}_{\mathscr{A}}(-, H^{*,*})$ applied to the free $\mathscr{A}$-module resolution

$$ \cdots \longrightarrow H^{*,*}(K_2(m)) \xrightarrow{\partial} H^{*,*}(K_1(m)) \xrightarrow{\partial} H^{*,*}(K_0(m)) \to 0 $$

of $H^{*,*}(Y(m))$. In view of (9.2), these resolutions are compatible for varying m. Passing to colimits over m, we obtain a flat $\mathscr{A}$-module resolution

$$\dots \longrightarrow H_c^{*,*}(K_2) \xrightarrow{\partial} H_c^{*,*}(K_1) \xrightarrow{\partial} H_c^{*,*}(K_0) \to 0$$

of $H_c^{*,*}(Y)$, where we write

$$H_c^{*,*}(Y) = \operatorname*{colim}_m H^{*,*}(Y(m))$$
$$H_c^{*,*}(K_s) = \operatorname*{colim}_m H^{*,*}(K_s(m))$$

for the "continuous" cohomology groups of the towers $\{Y(m)\}_m$ and $\{K_s(m)\}_m$, respectively. The $\mathscr{A}$-module $H_c^{*,*}(K_s)$ might not be free, but remains flat, since such modules are preserved under filtered colimits. These colimits can also be written as cokernels, as in the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \bigoplus_m H^{*,*}(K_1(m)) \arrow[r, "k^*"] \arrow[d, "\partial"] & \bigoplus_m H^{*,*}(K_1(m)) \arrow[r, "j^*"] \arrow[d, "\partial"] & H_c^{*,*}(K_1) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & \bigoplus_m H^{*,*}(K_0(m)) \arrow[r, "k^*"] \arrow[d, "\epsilon"] & \bigoplus_m H^{*,*}(K_0(m)) \arrow[r, "j^*"] \arrow[d, "\epsilon"] & H_c^{*,*}(K_0) \arrow[r] \arrow[d, "\epsilon"] & 0 \\
0 \arrow[r] & \bigoplus_m H^{*,*}(Y(m)) \arrow[r, "k^*"] \arrow[d] & \bigoplus_m H^{*,*}(Y(m)) \arrow[r, "j^*"] \arrow[d] & H_c^{*,*}(Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Omitting the bottom row and the right hand column, we have a bicomplex of free $\mathscr{A}$-modules, whose total complex (F_*, ∂) is a free resolution of $H_c^{*,*}(Y)$. Here

$$F_0 = \bigoplus_m H^{*,*}(K_0(m))$$

and

$$F_s = \bigoplus_m H^{*,*}(K_s(m)) \, \oplus \, \bigoplus_m H^{*,*}(\Sigma^{-1} K_{s-1}(m))$$

for $s \ge 1$. Hence we can recognize

$$\begin{aligned}
{}^{\mathrm{del}}E_1^{0,*,*} = \pi_{*,*} L_0 = \prod_m \pi_{*,*} K_0(m) &\cong \prod_m \operatorname{Hom}_{\mathscr{A}}(H^{*,*}(K_0(m)), H^{*,*}) \\
&\cong \operatorname{Hom}_{\mathscr{A}}(\bigoplus_m H^{*,*}(K_0(m)), H^{*,*}) = \operatorname{Hom}_{\mathscr{A}}(F_0, H^{*,*})
\end{aligned}$$

and

$$\begin{aligned}
{}^{\mathrm{del}}E_1^{s,*,*} &= \pi_{*,*} L_s \cong \prod_m \pi_{*,*} K_s(m) \oplus \prod_m \pi_{*,*} \Sigma^{-1} K_{s-1}(m) \\
&\cong \prod_m \operatorname{Hom}_{\mathscr{A}}(H^{*,*}(K_s(m)), H^{*,*}) \oplus \prod_m \operatorname{Hom}_{\mathscr{A}}(H^{*,*}(\Sigma^{-1} K_{s-1}(m)), H^{*,*}) \\
&\cong \operatorname{Hom}_{\mathscr{A}}(\bigoplus_m H^{*,*}(K_s(m)) \, \oplus \, \bigoplus_m H^{*,*}(\Sigma^{-1} K_{s-1}(m)), H^{*,*}) \\
&= \operatorname{Hom}_{\mathscr{A}}(F_s, H^{*,*})
\end{aligned}$$

for $s \geq 1$. Moreover, $d_1 \colon {}^{\mathrm{del}}E_1^{s,*,*} \to {}^{\mathrm{del}}E_1^{s+1,*,*}$ is induced by the boundary operator $\partial \colon F_{s+1} \to F_s$ in the total complex, as can be verified by tracing through the definitions. Since (F_*, ∂) is a free $\mathscr{A}$-module resolution of $H_c^{*,*}(Y)$, we obtain the desired isomorphism

$$ {}^{\mathrm{del}}E_2^{s,*,*} \cong \mathrm{Ext}_{\mathscr{A}}^{s,*,*}(H_c^{*,*}(Y), H^{*,*}) $$

for each $s \geq 0$.

Proposition 9.2. *Let $\cdots \to Y(m+1) \to Y(m) \to \dots$ be a tower of motivic spectra, with each $H \wedge Y(m)$ a motivic GEM of bifinite type. Let $Y = \mathrm{holim}_m Y(m)$ and $H_c^{*,*}(Y) = \mathrm{colim}_m H^{*,*}(Y(m))$. The delayed limit Adams spectral sequence*

$$ {}^{\mathrm{del}}E_1^{s,*,*} \Longrightarrow \pi_{*,*}(Y) $$

has E_2-term

$$ {}^{\mathrm{del}}E_2^{s,*,*} = \mathrm{Ext}_{\mathscr{A}}^{s,*,*}(H_c^{*,*}(Y), H^{*,*}) \,, $$

with Ext *calculated in the category of $\mathscr{A}$-modules.*

Proof. This summarizes the discussion so far in this section. □

Let $Y_\infty(m) = \mathrm{holim}_s Y_s(m)$. The Adams spectral sequence for $Y(m)$ is conditionally convergent [Boa99, Def. 5.10] to $\pi_{*,*}Y(m)$ if and only if $\pi_{*,*}Y_\infty(m) = 0$. This fails in many interesting examples, but often becomes true after (ℓ, η)-adic completion in the following sense.

Definition 9.3. Let $\ell \in \pi_{0,0}(\mathbb{S})$ be ℓ times the class of the identity map $\mathbb{S} \to \mathbb{S}$, and let $\eta \in \pi_{1,1}(\mathbb{S})$ be the class of the Hopf fibration $S^{3,2} \simeq \mathbb{A}^2 - \{0\} \to \mathbb{P}^1 \simeq S^{2,1}$. For any motivic spectrum X let

$$ \begin{aligned} X_\ell^\wedge &= \operatorname*{holim}_n X/\ell^n \\ X_{\ell,\eta}^\wedge &= \operatorname*{holim}_n (X_\ell^\wedge)/\eta^n \simeq \operatorname*{holim}_n X/(\ell^n, \eta^n) \end{aligned} $$

be the *ℓ- and (ℓ, η)-adic completions* of X, respectively, as in [HKO11, p. 574] and [Man21, Def. 3.2.9]. There are canonical completion maps $X \to X_\ell^\wedge \to X_{\ell,\eta}^\wedge$.

Lemma 9.4. *For each H-module spectrum M the completion map $M \to M_{\ell,\eta}^\wedge$ is a $\pi_{*,*}$-isomorphism.*

Proof. Multiplication by ℓ and by η act trivially on $H_{*,*}$, hence also on $\pi_{*,*}M$. The tower of short exact sequences

$$ \begin{tikzcd} & \vdots \arrow[d, "\mathrm{id}"'] & \vdots \arrow[d] & \vdots \arrow[d, "0"'] & \\ 0 \arrow[r] & \pi_{*,*}(M) \arrow[r] \arrow[d, "\mathrm{id}"'] & \pi_{*,*}(M/\ell^{n+1}) \arrow[r] \arrow[d] & \pi_{*,*}(\Sigma M) \arrow[r] \arrow[d, "0"'] & 0 \\ 0 \arrow[r] & \pi_{*,*}(M) \arrow[r] \arrow[d, "\mathrm{id}"'] & \pi_{*,*}(M/\ell^n) \arrow[r] \arrow[d] & \pi_{*,*}(\Sigma M) \arrow[r] \arrow[d, "0"'] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd} $$

and the lim-lim^1 sequence imply that $M \to M^\wedge_\ell$ is a $\pi_{*,*}$-isomorphism. Likewise, the tower of short exact sequences

$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathrm{id}"'] & \vdots \arrow[d] & \vdots \arrow[d, "0"'] & \\
0 \arrow[r] & \pi_{*,*}(M^\wedge_\ell) \arrow[r] \arrow[d, "\mathrm{id}"'] & \pi_{*,*}(M^\wedge_\ell/\eta^{n+1}) \arrow[r] \arrow[d] & \pi_{*,*}(\Sigma^{n+2,n+1}M^\wedge_\ell) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & \pi_{*,*}(M^\wedge_\ell) \arrow[r] \arrow[d, "\mathrm{id}"'] & \pi_{*,*}(M^\wedge_\ell/\eta^{n}) \arrow[r] \arrow[d] & \pi_{*,*}(\Sigma^{n+1,n}M^\wedge_\ell) \arrow[r] \arrow[d, "0"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and the lim-lim^1 sequence imply that $M^\wedge_\ell \to M^\wedge_{\ell,\eta}$ is a $\pi_{*,*}$-isomorphism. □

Applying (ℓ,η)-adic completion to the tower of Adams resolutions (9.2) yields another diagram of the same shape. At its lower edge, the tower of spectra

$$\cdots \to Y(m+1)^\wedge_{\ell,\eta} \longrightarrow Y(m)^\wedge_{\ell,\eta} \longrightarrow Y(m-1)^\wedge_{\ell,\eta} \to \dots$$

has homotopy limit $Y^\wedge_{\ell,\eta}$. For each m the Adams resolution of $Y(m)$ maps to the diagram

$$\begin{tikzcd}
\dots \arrow[r] & Y_2(m)^\wedge_{\ell,\eta} \arrow[rr, "\alpha"] & & Y_1(m)^\wedge_{\ell,\eta} \arrow[rr, "\alpha"] \arrow[dl, "\beta"] & & Y_0(m)^\wedge_{\ell,\eta} \arrow[dl, "\beta"] \\
 & & K_1(m)^\wedge_{\ell,\eta} \arrow[ul, dashed, "\gamma"] & & K_0(m)^\wedge_{\ell,\eta} \arrow[ul, dashed, "\gamma"] &
\end{tikzcd}$$

of homotopy cofiber sequences, and by Lemma 9.4 the induced map of homotopy spectral sequences is an isomorphism from the E_1-term and onward. The new spectral sequence has abutment $\pi_{*,*}(Y(m)^\wedge_{\ell,\eta})$, and is conditionally convergent to this target if and only if $\pi_{*,*}(Y_\infty(m)^\wedge_{\ell,\eta}) = 0$.

We now make the additional assumption, for each m, that the Adams spectral sequence for $Y(m)$ converges conditionally to $\pi_{*,*}(Y(m)^\wedge_{\ell,\eta})$. The following theorem was proved by Hu–Kriz–Ormsby [HKO11, Thm. 1] in the case of a cellular spectrum X of finite type over $\operatorname{Spec} k$ for k a field of characteristic 0. It was generalized to bounded below spectra X over S, in our generality, by Mantovani [Man21].

The homotopy t-structure on $SH(S)$ is defined as in [Hoy15, §2.1], and a motivic spectrum is bounded below if it lies in $SH(S)_{\ge -m}$ for some finite m. (When $S = \operatorname{Spec} k$ for a field k, Morel's stable $\mathbb{A}^1$-connectivity theorem [Mor05, Thm. 3] shows that X lies in $SH(S)_{\ge -m}$ if and only if the homotopy sheaves $\underline{\pi}_{t,u}(X)$ vanish whenever $t - u < -m$.)

Theorem 9.5. *Suppose that X in $SH(S)$ is bounded below in the homotopy t-structure. Then the mod ℓ Adams spectral sequence for X is conditionally convergent to $\pi_{*,*}(X^\wedge_{\ell,\eta})$.*

Proof. As reviewed in [Man21, §5], this is an application of [Man21, Thm. 1.0.2, Thm. 1.0.4] in the case $E = H$, which satisfies Mantovani's hypotheses because of [Hoy15, Thm. 3.8, Thm. 7.12] and [Spi18, Thm. 10.3]. □

Proposition 9.6. *Let $Y = \operatorname{holim}_m Y(m)$ be the homotopy limit of a tower of motivic spectra. Suppose, for each m, that the mod ℓ Adams spectral sequence for*

$Y(m)$ converges conditionally to $\pi_{*,*}(Y(m)^\wedge_{\ell,\eta})$. Then the limit and delayed limit Adams spectral sequences

$$
\begin{aligned}
{}^{\lim}E_1^{s,*,*} &= \pi_{*,*}(K_s) \Longrightarrow \pi_{*,*}(Y^\wedge_{\ell,\eta}) \\
{}^{\mathrm{del}}E_1^{s,*,*} &= \pi_{*,*}(L_s) \Longrightarrow \pi_{*,*}(Y^\wedge_{\ell,\eta})
\end{aligned}
$$

are both conditionally convergent to the bigraded homotopy groups of the (ℓ,η)-adic completion of Y.

Proof. Let $Y_\infty = \operatorname{holim}_s Y_s$ and $W_\infty = \operatorname{holim}_s W_s$. The inclusions

$$W_{s+1} \subset Y_s \subset W_s \subset Y_{s-1}$$

imply that $Y_\infty \simeq W_\infty$. Granting that $\pi_{*,*}(Y_\infty(m)^\wedge_{\ell,\eta}) = 0$ for all m, the short exact lim-lim[1] sequence shows that $\pi_{*,*}((Y_\infty)^\wedge_{\ell,\eta}) \cong \pi_{*,*}((W_\infty)^\wedge_{\ell,\eta}) = 0$, so that both the limit Adams spectral sequence and the delayed limit Adams spectral sequence are conditionally convergent to $\pi_{*,*}(Y^\wedge_{\ell,\eta})$. □

Let $g\colon X \to Y = \operatorname{holim}_m Y(m)$ be a map of motivic spectra. The resulting compatible maps from the canonical Adams resolution of X to the canonical Adams resolutions of the $Y(m)$ induce a map from the former to the diagram (9.4), which can be naturally continued to map to the delayed limit Adams resolution (9.5). Applying (ℓ,η)-adic completion, and passing to the associated homotopy spectral sequences, we obtain morphisms of spectral sequences

$$g\colon E_1^{*,*,*}(X) \longrightarrow {}^{\lim}E_1^{*,*,*}(Y) \longrightarrow {}^{\mathrm{del}}E_1^{*,*,*}(Y)$$

with abutment

$$g\colon \pi_{*,*}(X^\wedge_{\ell,\eta}) \longrightarrow \pi_{*,*}(Y^\wedge_{\ell,\eta}) \xrightarrow{=} \pi_{*,*}(Y^\wedge_{\ell,\eta})\,.$$

We can now appeal to a special case of Boardman's comparison theorem [Boa99, Thm. 7.2] for conditionally convergent spectral sequences. This version of the comparison theorem is particularly convenient, in view of the failure of strong convergence for the motivic Adams spectral sequence for the sphere spectrum over a number field, demonstrated by Kylling–Wilson in [KW19, Cor. 7.8].

Proposition 9.7. *Let $g\colon X \to Y = \operatorname{holim}_m Y(m)$, with $H \wedge X$ and each $H \wedge Y(m)$ a motivic GEM of bifinite type. Suppose that the mod ℓ Adams spectral sequences for X and the $Y(m)$ are conditionally convergent to $\pi_{*,*}(X^\wedge_{\ell,\eta})$ and $\pi_{*,*}(Y(m)^\wedge_{\ell,\eta})$, respectively. If the $\mathscr{A}$-module homomorphism*

$$g^*\colon H_c^{*,*}(Y) \longrightarrow H^{*,*}(X)$$

is an Ext*-isomorphism, so that*

$$
\begin{aligned}
g\colon E_2^{s,*,*}(X) = \operatorname{Ext}_{\mathscr{A}}^{s,*,*}(H^{*,*}(X), H^{*,*}) \\
\xrightarrow{\cong} {}^{\mathrm{del}}E_2^{s,*,*}(Y) = \operatorname{Ext}_{\mathscr{A}}^{s,*,*}(H_c^{*,*}(Y), H^{*,*})
\end{aligned}
$$

is an isomorphism, then g induces an isomorphism

$$g\colon \pi_{*,*}(X^\wedge_{\ell,\eta}) \xrightarrow{\cong} \pi_{*,*}(Y^\wedge_{\ell,\eta})\,.$$

Proof. The identification of the (E_1- and) E_2-term for X follows as usual from (8.1) and (8.2), and the delayed limit E_2-term for Y is given by Proposition 9.2. We now apply [Boa99, Thm. 7.2(i)]. If g induces an isomorphism of E_2-terms, then it certainly also induces isomorphisms of E_∞- and RE_∞-terms. Hence g induces the stated isomorphism of (filtered) abutments. □

10. A tower of Thom spectra

Our next aim is to construct a diagram

$$\begin{tikzcd}
 & & L^{\infty}_{-\infty} \arrow[d] \\
Th(-(m+1)\gamma_n^*) \arrow[r, "i"] \arrow[d, "j"] & Th(-(m+1)\gamma_{n+1}^*) \arrow[r] \arrow[d, "j"] & L^{\infty}_{-2m-2} \arrow[d, "j"] \\
Th(-m\gamma_n^*) \arrow[r, "i"] & Th(-m\gamma_{n+1}^*) \arrow[r] & L^{\infty}_{-2m}
\end{tikzcd}$$

of motivic Thom spectra, with $H^{*,*}_c(L^{\infty}_{-\infty})$ realizing $H^{*,*}(B\mu_\ell)_{\mathrm{loc}}$. The left hand square will commute up to a generalized sign, and replacing i by i^4 will give a strictly commuting diagram in the stable homotopy category $SH(S)$.

For an algebraic vector bundle $\alpha \downarrow X$ over a smooth scheme X (over our base scheme S) we let $E(\alpha)$ denote its total space, let $E_0(\alpha) = E(\alpha) \setminus z(X)$ be the complement of its zero section, and let the Thom space $Th(\alpha) = E(\alpha)/E_0(\alpha)$ be the motivic quotient space, formed as in [Voe03a, §4].

For $n \geq 1$ let μ_ℓ act diagonally on $\mathbb{A}^n$, let $\mathbb{A}^n_0 = \mathbb{A}^n \setminus \{0\}$, and let $L^{2n-1} = \mathbb{A}^n_0/\mu_\ell$ denote the n-th algebraic lens space, which is smooth and quasi-projective. Write $[x_1, \dots, x_n]$ in L^{2n-1} for the image of $(x_1, \dots, x_n)$ in $\mathbb{A}^n_0$. The inclusion $\mu_\ell \subset \mathbb{G}_m$ induces a projection $L^{2n-1} \to \mathbb{P}^{n-1}$. Let $\gamma_n = \gamma_n^1 \downarrow L^{2n-1}$ be the pullback of the tautological line bundle $\mathscr{O}(-1)$ over $\mathbb{P}^{n-1}$, with total space given by the balanced product

$$E(\gamma_n) = \mathbb{A}^n_0 \times_{\mu_\ell} \mathbb{A}^1$$

and bundle projection mapping $[x_1, \dots, x_n; y]$ to $[x_1, \dots, x_n]$. Let $\epsilon_n = \epsilon_n^n \downarrow L^{2n-1}$ be the trivial rank n bundle over L^{2n-1}, with total space $E(\epsilon_n) = L^{2n-1} \times \mathbb{A}^n$ and bundle projection to the first factor. There is a canonical embedding $\gamma_n \to \epsilon_n$, given in coordinates by

$$[x_1, \dots, x_n; y] \longmapsto ([x_1, \dots, x_n], x_1 y, \dots, x_n y)\,.$$

Let $\zeta_n = \zeta_n^{n-1} \downarrow L^{2n-1}$ be its cokernel, so that there is a short exact sequence

$$0 \to \gamma_n \longrightarrow \epsilon_n \longrightarrow \zeta_n \to 0$$

of algebraic vector bundles over L^{2n-1}. Let γ_n^*, ϵ_n^* and ζ_n^* denote the dual bundles, fitting in a short exact sequence

$$0 \to \zeta_n^* \xrightarrow{j} \epsilon_n^* \longrightarrow \gamma_n^* \to 0\,. \tag{10.1}$$

Here the total space of γ_n^* is given by the orbit space $E(\gamma_n^*) = (\mathbb{A}^n_0 \times \mathbb{A}^1)/\mu_\ell$, where μ_ℓ acts diagonally.

More generally, the total space of the k-fold direct sum $k\gamma_n^* = \gamma_n^* \oplus \cdots \oplus \gamma_n^*$, where $k \geq 0$, is

$$E(k\gamma_n^*) = (\mathbb{A}^n_0 \times \mathbb{A}^k)/\mu_\ell\,,$$

which comes with a canonical map to $\mathbb{A}^{n+k}_0/\mu_\ell = L^{2n+2k-1}$. By [MV99, Lem. 3.1.6], the inclusion

$$Th(k\gamma_n^*) = \frac{E(k\gamma_n^*)}{E_0(k\gamma_n^*)} = \frac{(\mathbb{A}^n_0 \times \mathbb{A}^k)/\mu_\ell}{(\mathbb{A}^n_0 \times \mathbb{A}^k_0)/\mu_\ell} \longrightarrow \frac{(\mathbb{A}^n \times \mathbb{A}^k)_0/\mu_\ell}{(\mathbb{A}^n \times \mathbb{A}^k_0)/\mu_\ell}$$

is an equivalence of Nisnevich sheaves, and by [MV99, Ex. 3.2.2] the projection $(\mathbb{A}^n \times \mathbb{A}^k_0)/\mu_\ell \to \mathbb{A}^k_0/\mu_\ell = L^{2k-1}$ is an $\mathbb{A}^1$-homotopy equivalence, so there is a homotopy cofiber sequence

$$L^{2k-1} \xrightarrow{i^n} L^{2n+2k-1} \longrightarrow Th(k\gamma_n^*) \tag{10.2}$$

of motivic spaces. Following James [Jam59, p. 117], Atiyah [Ati61, Prop. 4.3] and Kambe–Matsunaga–Toda [KMT66, Thm. 1] we may therefore write $Th(k\gamma_n^*) = L_{2k}^{2n+2k-1}$ for the Thom space of $k\gamma_n^*$ over L^{2n-1}, and refer to it as a motivic stunted lens space.

Following Mahowald and Adams [Ada74, p. 4], we are, however, more interested in the cases where $k = -m$ is negative, corresponding to Thom spectra $Th(-m\gamma_n^*) = L_{-2m}^{2n-2m-1}$ of virtual bundles $-m\gamma_n^*$. In view of (10.1), $-m\gamma_n^* \equiv m\zeta_n^* - m\epsilon_n^*$ as virtual bundles over L^{2n-1}, where $m\epsilon_n^*$ is trivial of rank mn, which leads to the following definition.

Definition 10.1. For $m \geq 0$ let

$$L_{-2m}^{2n-2m-1} = Th(-m\gamma_n^*) = \Sigma^{-2mn,-mn} Th(m\zeta_n^*)$$

denote a (finite, motivic) stunted lens spectrum.

Consider the inclusion $i\colon L^{2n-1} \to L^{2n+1}$. There are natural isomorphisms $i^*(\gamma_{n+1}) \cong \gamma_n$, $i^*(\epsilon_{n+1}) \cong \epsilon_n \oplus \epsilon_n^1$ and $i^*(\zeta_{n+1}) \cong \zeta_n \oplus \epsilon_n^1$, where ϵ_n^1 is trivial of rank 1. Dually, $i^*(\gamma_{n+1}^*) \cong \gamma_n^*$, $i^*(\epsilon_{n+1}^*) \cong \epsilon_n^* \oplus (\epsilon_n^1)^*$ and $i^*(\zeta_{n+1}^*) \cong \zeta_n^* \oplus (\epsilon_n^1)^*$.

Consider also the inclusion $j\colon \zeta_n^* \to \epsilon_n^*$ of bundles over L^{2n-1}.

Definition 10.2. Let

$$i\colon L_{-2m}^{2n-2m-1} = Th(-m\gamma_n^*) \longrightarrow Th(-m\gamma_{n+1}^*) = L_{-2m}^{2n-2m+1}$$

and

$$j\colon L_{-2m-2}^{2n-2m-3} = Th(-(m+1)\gamma_n^*) \longrightarrow Th(-m\gamma_n^*) = L_{-2m}^{2n-2n-1}$$

be the maps obtained by applying $\Sigma^{-2m(n+1),-m(n+1)}$ and $\Sigma^{-2(m+1)n,-(m+1)n}$ to

$$\Sigma^{2m,m} Th(m\zeta_n^*) \overset{\mathrm{sh}}{\cong} Th(m(\zeta_n^* \oplus (\epsilon_n^1)^*)) \xrightarrow{mi} Th(m\zeta_{n+1}^*)$$

and

$$Th((m+1)\zeta_n^*) \xrightarrow{m\,\mathrm{id}\oplus j} Th(m\zeta_n^* \oplus \epsilon_n^*) = \Sigma^{2n,n} Th(m\zeta_n^*)\,,$$

respectively. Here sh denotes the isomorphism of spectra induced by the shuffle $m\zeta_n^* \oplus m(\epsilon_n^1)^* \cong m(\zeta_n^* \oplus (\epsilon_n^1)^*)$.

Definition 10.3. Let $-\epsilon \in \pi_{0,0}(\mathbb{S})$ be the class of the symmetry isomorphism $\gamma\colon S^{2,1} \wedge S^{2,1} \cong S^{2,1} \wedge S^{2,1}$. It satisfies $(-\epsilon)^2 = 1$, since $\gamma^2 = \mathrm{id}$.

Lemma 10.4. *The rectangle*

$$\begin{tikzcd}
L_{-2m-2}^{2n-2m-3} = Th(-(m+1)\gamma_n^*) \arrow[r, "i"] \arrow[d, "j"'] & Th(-(m+1)\gamma_{n+1}^*) = L_{-2m-2}^{2n-2m-1} \arrow[dd, "j"] \\
L_{-2m}^{2n-2m-1} = Th(-m\gamma_n^*) \arrow[d, "(-\epsilon)^{mn}"', "\simeq"] & \\
L_{-2m}^{2n-2m-1} = Th(-m\gamma_n^*) \arrow[r, "i"] & Th(-m\gamma_{n+1}^*) = L_{-2m}^{2n-2m+1}
\end{tikzcd}$$

commutes up to homotopy.

Proof. The diagrams

$$\begin{tikzcd}
Th((m+1)(\zeta_n^* \oplus (\epsilon_n^1)^*)) \arrow[r, "(m+1)i"] \arrow[d, "m\,\mathrm{id}\oplus j\oplus \mathrm{id}"'] & Th((m+1)\zeta_{n+1}^*) \arrow[d, "m\,\mathrm{id}\oplus j"] \\
Th(m(\zeta_n^* \oplus (\epsilon_n^1)^*) \oplus \epsilon_n^* \oplus (\epsilon_n^1)^*) \arrow[r, "mi\oplus i"] & Th(m\zeta_{n+1}^* \oplus \epsilon_{n+1}^*)
\end{tikzcd}$$

and

$$\begin{tikzcd}
Th((m+1)\zeta_n^* \oplus (m+1)(\epsilon_n^1)^*) \arrow[r, "\mathrm{sh}"] \arrow[d, "m\,\mathrm{id}\oplus j\oplus \mathrm{id}"'] & Th((m+1)(\zeta_n^* \oplus (\epsilon_n^1)^*)) \arrow[dd, "m\,\mathrm{id}\oplus j\oplus \mathrm{id}"] \\
Th(m\zeta_n^* \oplus \epsilon_n^* \oplus (m+1)(\epsilon_n^1)^*) \arrow[d, "\mathrm{id}\oplus\chi_{m,n}\oplus\mathrm{id}"', "\cong"] & \\
Th(m\zeta_n^* \oplus m(\epsilon_n^1)^* \oplus \epsilon_n^* \oplus (\epsilon_n^1)^*) \arrow[r, "\mathrm{sh}\oplus\mathrm{id}"] & Th(m(\zeta_n^* \oplus (\epsilon_n^1)^*) \oplus \epsilon_n^* \oplus (\epsilon_n^1)^*)
\end{tikzcd}$$

commute strictly, where $\chi_{m,n}$ is induced by the symmetry isomorphism

$$\epsilon_n^* \oplus m(\epsilon_n^1)^* \cong m(\epsilon_n^1)^* \oplus \epsilon_n^* ,$$

hence is homotopic to multiplication by $(-\epsilon)^{mn}$. Applying

$$\Sigma^{-2(m+1)(n+1),-(m+1)(n+1)}$$

yields the stated homotopy commutative rectangle. □

Corollary 10.5. *The square*

$$\begin{tikzcd}
L_{-2m-2}^{2n-2m-3} = Th(-(m+1)\gamma_n^*) \arrow[r, "i^4"] \arrow[d, "j"'] & Th(-(m+1)\gamma_{n+4}^*) = L_{-2m-2}^{2n-2m+5} \arrow[d, "j"] \\
L_{-2m}^{2n-2m-1} = Th(-m\gamma_n^*) \arrow[r, "i^4"] & Th(-m\gamma_{n+4}^*) = L_{-2m}^{2n-2m+7}
\end{tikzcd}$$

commutes up to homotopy.

Proof. This follows from Lemma 10.4, since $mn + m(n+1) + m(n+2) + m(n+3)$ is always even. □

Definition 10.6. Let the (infinite, motivic) stunted lens spectrum

$$L_{-2m}^{\infty} = \operatorname*{hocolim}_n L_{-2m}^{2n-2m-1} = \operatorname*{hocolim}_n Th(-m\gamma_n^*)$$

be the homotopy colimit of the maps

$$\cdots \to L_{-2m}^{2n-2m-1} \xrightarrow{i} L_{-2m}^{2n-2m+1} \to \cdots .$$

For a fixed choice of commuting homotopies in Corollary 10.5, let $j\colon L_{-2m-2}^{\infty} \longrightarrow L_{-2m}^{\infty}$ be the induced map. Let

$$L_{-\infty}^{\infty} = \operatorname*{holim}_m L_{-2m}^{\infty}$$

be the homotopy limit of the resulting tower

$$\cdots \to L_{-2m-2}^{\infty} \xrightarrow{j} L_{-2m}^{\infty} \to \cdots .$$

Recall Notation 5.6.

Lemma 10.7.

$$H^{*,*}(L^{2n-1}) = H^{*,*}[u,v]/(u^2 = \tau v + \rho u, v^n) ,$$

where v is the mod ℓ Euler class of $\gamma_n^* \downarrow L^{2n-1}$ and $\beta(u) = v$.

Proof. This follows by the same argument as for [Voe03a, Thm. 6.10], working with $L^{2n-1} \to \mathbb{P}^{n-1}$ in place of $B\mu_\ell \to \mathbb{P}^\infty$. □

Definition 10.8. Let $U_{m\zeta_n^*} \in H^{2m(n-1),m(n-1)}(Th(m\zeta_n^*))$ be the mod ℓ Thom class of $m\zeta_n^* \downarrow L^{2n-1}$. Let

$$U_{-m\gamma_n^*} \in H^{-2m,-m}(L_{-2m}^{2n-2m-1}) = H^{-2m,-m}(Th(-m\gamma_n^*))$$

be its image under the (de-)suspension isomorphism. We write $x \mapsto x \cdot U_{m\zeta_n^*}$ and $x \mapsto x \cdot U_{-m\gamma_n^*}$ for the Thom isomorphisms

$$H^{*,*}(L^{2n-1}) \cong H^{*+2m(n-1),*+m(n-1)}(Th(m\zeta_n^*))$$
$$H^{*,*}(L^{2n-1}) \cong H^{*-2m,*-m}(Th(-m\gamma_n^*)),$$

cf. [Voe03a, Prop. 4.3].

Lemma 10.9. *The homomorphisms*

$$i^*\colon H^{*,*}(Th(-m\gamma_{n+1}^*)) \longrightarrow H^{*,*}(Th(-m\gamma_n^*))$$
$$j^*\colon H^{*,*}(Th(-m\gamma_n^*)) \longrightarrow H^{*,*}(Th(-(m+1)\gamma_n^*))$$

are given by

$$i^*(x \cdot U_{-m\gamma_{n+1}^*}) = i^*(x) \cdot U_{-m\gamma_n^*}$$
$$j^*(x \cdot U_{-m\gamma_n^*}) = xv \cdot U_{-(m+1)\gamma_n^*}.$$

Proof. The Thom class of $m\zeta_{n+1}^*$ maps under i^* to the Thom class of $i^*(m\zeta_{n+1}^*) = m(\zeta_n^* \oplus (\epsilon_n^1)^*)$, which corresponds under the suspension isomorphism to the Thom class of $m\zeta_n^*$. This proves the first formula, where $x \in H^{*,*}(L^{2n+1})$.

The Thom class of $m\zeta_n^*$ corresponds under the suspension isomorphism to the Thom class of $m\zeta_n^* \oplus \epsilon_n^*$. By the Jouanolou trick [Voe03a, Lem. 4.7] it maps under j^* to the Euler class v of γ_n^* times the Thom class of $(m+1)\zeta_n^*$. This proves the second formula, where $x \in H^{*,*}(L^{2n-1})$. □

Proposition 10.10. *The structure maps $L_{-2m}^{2n-2m-1} \to L_{-2m}^{\infty}$ and $L_{-\infty}^{\infty} \to L_{-2m}^{\infty}$ induce $\mathscr{A}$-module isomorphisms*

$$H^{*,*}(L_{-2m}^{\infty}) \cong H^{*,*}(B\mu_\ell)\{U_{-m\gamma^*}\} \cong H^{*,*}(B\mu_\ell)\{v^{-m}\}$$

and

$$H_c^{*,*}(L_{-\infty}^{\infty}) = \operatorname*{colim}_m H^{*,*}(L_{-2m}^{\infty}) \cong H^{*,*}(B\mu_\ell)_{\mathrm{loc}}.$$

Proof. Since each $i^*\colon H^{*,*}(Th(-m\gamma_{n+1}^*)) \to H^{*,*}(Th(-m\gamma_n^*))$ is surjective, the lim-lim^1 sequence gives an isomorphism

$$H^{*,*}(L_{-2m}^{\infty}) \cong \lim_n H^{*,*}(Th(-m\gamma_n^*)) \cong \lim_n H^{*,*}(L^{2n-1})\{U_{-m\gamma_n^*}\}.$$

Letting $U_{-m\gamma^*}$ correspond to the compatible sequence $(U_{-m\gamma_n^*})_n$ gives the first isomorphism. The second isomorphism sends $U_{-m\gamma^*}$ to v^{-m}. The induced homomorphism $j^*\colon H^{*,*}(B\mu_\ell)\{U_{-m\gamma^*}\} \to H^{*,*}(B\mu_\ell)\{U_{-(m+1)\gamma^*}\}$ maps $U_{-m\gamma^*}$ to $v \cdot U_{-(m+1)\gamma^*}$, hence corresponds to the homomorphism

$$H^{*,*}(B\mu_\ell)\{v^{-m}\} \longrightarrow H^{*,*}(B\mu_\ell)\{v^{-m-1}\}$$

sending v^{-m} to $v \cdot v^{-m-1}$. It follows that $\operatorname{colim}_m H^{*,*}(L_{-2m}^{\infty})$, i.e., the continuous cohomology $H_c^{*,*}(L_{-\infty}^{\infty})$, is isomorphic to the localization $H^{*,*}(B\mu_\ell)[1/v] = H^{*,*}(B\mu_\ell)_{\mathrm{loc}}$.

It remains to justify that these isomorphisms are compatible with the Steenrod operations. The short exact sequence

$$0 \to H^{*,*}(Th(m\gamma_n^*)) \longrightarrow H^{*,*}(L^{2n+2m-1}) \xrightarrow{i^{n*}} H^{*,*}(L^{2m-1}) \to 0$$

induced from (10.2) shows that the Steenrod action on $U_{m\gamma_n^*}$ matches that on v^m in $H^{*,*}(L^{2n-1})\{v^m\}$. By another application of the Jouanolou trick, and the Cartan formula, it follows that the Steenrod action on $U_{m\zeta_n^*}$ is compatible with that on

$\Sigma^{2mn,mn}v^{-m}$. Hence, by stability, the Steenrod action on $U_{-m\gamma_n^*}$ matches that on v^{-m} in $H^{*,*}(L^{2n-1})\{v^{-m}\}$. Passing to the limit over n and the colimit over m completes the argument. □

The next two lemmas confirm the assumptions required (for recognition of the E_2-term and conditional convergence) of the delayed limit Adams spectral sequence for $\Sigma L_{-\infty}^{\infty}$.

Lemma 10.11. *The spectra $L_{-2m}^{2n-2m-1}$ are cellular of finite type. The spectra L_{-2m}^{∞} are cellular and bounded below.*

Proof. The Zariski cover of L^{2n-1} by the affines $(\mathbb{A}^{i-1}\times\mathbb{A}_0^1\times\mathbb{A}^{n-i})/\mu_\ell$, with $1 \le i \le n$, is completely stably cellular in the sense of [DI05, Def. 3.7]. It trivializes γ_n, hence also $m\zeta_n^*$. It follows as in [DI05, Cor. 3.10] that $L_{-2m}^{2n-2m-1} = \Sigma^{-2mn,-mn}Th(m\zeta_n^*)$ is cellular. Inspection of the argument shows that it admits a cell structure with finitely many cells, all in bidegrees (p_α, q_α) satisfying $p_\alpha - q_\alpha \ge -m$.

Since this bound is uniform, it follows by passage to the homotopy colimit that L_{-2m}^{∞} is also cellular, with cells in the same range of bidegrees. Hence L_{-2m}^{∞} lies in $SH(S)_{\ge -m}$ of the homotopy t-structure. □

Recall Definitions 8.1 and 8.2.

Lemma 10.12. *The H-module spectra $H \wedge L_{-2m}^{2n-2m-1}$ and $H \wedge L_{-2m}^{\infty}$ are motivic GEMs of bifinite type.*

Proof. These spectra are H-cellular by Lemma 10.11. The homology version of Lemma 10.7 shows that $H_{*,*}(L^{2n-1})$ is finitely generated and free over $H_{*,*}$ on generators in bidegrees $(i + 2k, i + k)$ for $i \in \{0, 1\}$ and $0 \le k < n$. It then follows from the Thom isomorphism in motivic homology that $H_{*,*}(L_{-2m}^{2n-2m-1})$ is finitely generated and free on similar generators for $i \in \{0, 1\}$ and $-m \le k < n - m$, and that $H_{*,*}(L_{-2m}^{\infty})$ is free on one generator in each bidegree $(i+2k, i+k)$ for $i \in \{0, 1\}$ and $k \ge -m$. In particular, $H_{*,*}(L_{-2m}^{\infty})$ is of bifinite type. □

Finally, we construct maps

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "c"] & \Sigma^{2,1}\mathbb{P}_{-\infty}^{-1} \arrow[r, "d"] \arrow[d] & \Sigma L_{-\infty}^{-1} \arrow[d] \\
& \Sigma^{2,1}\mathbb{P}_{-\infty}^{\infty} & \Sigma L_{-\infty}^{\infty} ,
\end{tikzcd}
\tag{10.3}
$$

whose composite induces the (large) residue homomorphism from Definition 6.1 in motivic cohomology.

To define $\mathbb{P}_{-\infty}^{\infty}$, let the (finite, motivic) stunted projective spectrum $\mathbb{P}_{-m}^{n-m-1} = Th(-m\gamma_n^* \downarrow \mathbb{P}^{n-1})$ be the Thom spectrum of the negative of $m\gamma_n^* = \mathscr{O}(1)^m$ over $\mathbb{P}^{n-1}$. We have maps $i\colon \mathbb{P}_{-m}^{n-m-1} \to \mathbb{P}_{-m}^{n-m}$ and $j\colon \mathbb{P}_{-m-1}^{n-m-2} \to \mathbb{P}_{-m}^{n-m-1}$ as in Definition 10.2, and let $\mathbb{P}_{-m}^{\infty} = \operatorname{hocolim}_n \mathbb{P}_{-m}^{n-m-1}$ and $\mathbb{P}_{-\infty}^{\infty} = \operatorname{holim}_m \mathbb{P}_{-m}^{\infty}$ as in Definition 10.6. We obtain $H_c^{*,*}(\mathbb{P}_{-\infty}^{\infty}) \cong H^{*,*}[v^{\pm 1}]$, by the same arguments as for lens spectra.

Proposition 10.13. *There is a map $c\colon \mathbb{S} \to \Sigma^{2,1}\mathbb{P}_{-\infty}^{-1}$ inducing*

$$\Sigma^{2,1}v^{-1} \longmapsto 1$$

in cohomology. The $H^{*,*}$-module generators $\Sigma^{2,1}v^k$ for $k \le -2$ map to zero.

Proof. We use that $\mathbb{P}^{n-1}$ is a smooth projective variety, with stable normal bundle

$$\nu = \epsilon_n^1 - n\gamma_n^* = \mathscr{O} - \mathscr{O}(1)^n .$$

By the construction leading to algebraic Atiyah duality, see [Voe03b, Prop. 2.7], [Hu05, Cl. 2] and [Hoy17, §5.3], there is a Pontryagin–Thom collapse map

$$c_n \colon \mathbb{S} \longrightarrow Th(\nu \downarrow \mathbb{P}^{n-1}) = \Sigma^{2,1}\mathbb{P}^{-1}_{-n}$$

inducing the homomorphism $\Sigma^{2,1}v^{-1} \mapsto 1$ in cohomology. The generators $\Sigma^{2,1}v^k$ for $-n \le k \le -2$ map to zero for bidegree reasons. When combined with the Thom diagonal and an adjunction, this leads to the Atiyah duality equivalence $Th(\nu \downarrow \mathbb{P}^{n-1}) \simeq D(\mathbb{P}^{n-1}_+) = F(\mathbb{P}^{n-1}_+, \mathbb{S})$, under which c_n is functionally dual to the collapse map $\mathbb{P}^{n-1}_+ \to S^0$. In particular, these maps are compatible up to homotopy for varying n, and combine to define a map $c \colon \mathbb{S} \to \Sigma^{2,1}\mathbb{P}^{-1}_{-\infty} \simeq D(\mathbb{P}^\infty_+)$, as required. □

Since L^{2n-1} is only quasi-projective, we need a different method to obtain the second map of (10.3).

Proposition 10.14. *There is a map $d \colon \Sigma^{2,1}\mathbb{P}^{-1}_{-\infty} \to \Sigma L^{-1}_{-\infty}$ inducing*

$$\Sigma uv^{-1} \longmapsto \Sigma^{2,1}v^{-1}$$

in cohomology, modulo $H^{*,*}$-multiples of $\Sigma^{2,1}v^k$ for $k \le -2$.

Proof. For algebraic vector bundles $\alpha, \beta \downarrow X$ over the same smooth scheme X, where β may be virtual, there is a homotopy cofiber sequence

$$Th(p^*\beta \downarrow E_0(\alpha)) \longrightarrow Th(\beta \downarrow X) \longrightarrow Th(\alpha \oplus \beta \downarrow X)$$

of motivic spectra. Here $p \colon E_0(\alpha) \to X$ denotes the projection, and $p^*\beta$ is the pullback of β along p. We apply this with $X = \mathbb{P}^{n-1}$, $\alpha = \gamma_n^{\otimes \ell} = \mathscr{O}(-\ell)$ and $\alpha \oplus \beta = \nu$, the stable normal bundle of $\mathbb{P}^{n-1}$. We identify $E_0(\gamma_n^{\otimes\ell}) \cong L^{2n-1}$, as in [Voe03a, Lem. 6.3]. Moreover, $p^*\gamma_n^{\otimes\ell} \cong \epsilon_n^1$ over L^{2n-1}, i.e., $\mathscr{O}(-\ell)$ and $\mathscr{O}$ pull back to the same bundle, which implies that $p^*\beta \cong -n\gamma_n^*$ as a stable bundle over L^{2n-1}. This leads to the homotopy cofiber sequence

$$L^{-1}_{-2n} \longrightarrow Th(\beta \downarrow \mathbb{P}^{n-1}) \longrightarrow \Sigma^{2,1}\mathbb{P}^{-1}_{-n} \xrightarrow{d_n} \Sigma L^{-1}_{-2n} .$$

The long exact sequence in cohomology shows that the connecting map d_n induces a homomorphism mapping Σuv^{-1} to $\Sigma^{2,1}v^{-1}$, modulo $H^{*,*}$-multiples of $\Sigma^{2,1}v^k$ for $-n \le k \le -2$. Again, these maps are compatible up to homotopy for varying n, and combine to define the required map d. □

Proposition 10.15. *There is a map $e \colon \mathbb{S} \to \Sigma L^\infty_{-\infty}$ of motivic spectra, inducing the residue homomorphism*

$$e^* = \mathrm{res} \colon \Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$

in cohomology.

Proof. We take e to be dc followed by the inclusion $\Sigma L^{-1}_{-\infty} \to \Sigma L^\infty_{-\infty}$ (of the homotopy fiber of $\Sigma L^\infty_{-\infty} \to \Sigma L^\infty_0$). To check that e induces the residue homomorphism, we use Corollary 7.5 in cohomological degree 0, giving an isomorphism

$$\mathrm{res} \colon \mathrm{Hom}_{\mathscr{A}}(H^{*,*}, H^{*,*}) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}}, H^{*,*}) .$$

In other words, any $\mathscr{A}$-module homomorphism $\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \to H^{*,*}$ is characterized by its value on Σuv^{-1}. Since e^* and res agree on this element, they are equal. □

11. The motivic Lin and Gunawardena theorems

We can now prove a motivic refinement of the classical theorems of Lin [Lin80] (for $\ell = 2$) and Gunawardena [Gun81] (for ℓ an odd prime).

Recall that μ_ℓ denotes the algebraic group of ℓ-th roots of unity, $L^{2n-1} = (\mathbb{A}^n \setminus \{0\})/\mu_\ell$ is an algebraic lens space, $\gamma_n^* \downarrow L^{2n-1}$ is the dual of the tautological line bundle, $L_{-2m}^{2n-2m-1} = Th(-m\gamma_n^*)$ is a stunted lens spectrum, and $L_{-2m}^\infty = \operatorname{hocolim}_n L_{-2m}^{2n-2m-1}$ and $L_{-\infty}^\infty = \operatorname{holim}_m L_{-2m}^\infty$ are infinite lens spectra. The continuous mod ℓ cohomology $H_c^{*,*}(L_{-\infty}^\infty) = \operatorname{colim}_m H^{*,*}(L_{-2m}^\infty)$ is isomorphic to the localization $H^{*,*}(B\mu_\ell)_{\mathrm{loc}} = H^{*,*}[u, v^{\pm 1}]/(u^2 = \tau v + \rho u)$. The map $e\colon \mathbb{S} \to \Sigma L_{-\infty}^\infty$ induces the Ext-equivalence $\mathrm{res}\colon \Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \to H^{*,*}$.

Theorem 11.1. *Let S be a Noetherian scheme of finite dimension d, essentially smooth over a field or Dedekind domain, and let ℓ be a prime that is invertible on S. The (ℓ,η)-completed map*

$$e_{\ell,\eta}^\wedge\colon \mathbb{S}_{\ell,\eta}^\wedge \longrightarrow (\Sigma L_{-\infty}^\infty)_{\ell,\eta}^\wedge$$

is a $\pi_{*,*}$-isomorphism. If $S = \operatorname{Spec} k$ for k a field, then $e_{\ell,\eta}^\wedge$ is a motivic equivalence.

Proof. We apply Proposition 9.7 with $X = \mathbb{S}$, $Y = \Sigma L_{-\infty}^\infty$, $Y(m) = \Sigma L_{-2m}^\infty$ and $g = e$.

$$\mathbb{S} \xrightarrow{e} \Sigma L_{-\infty}^\infty \to \cdots \to \Sigma L_{-2m}^\infty \to \dots .$$

The H-modules $H \wedge \mathbb{S}$ and $H \wedge \Sigma L_{-2m}^\infty$ are motivic GEMs of bifinite type by Lemma 10.12. Moreover, $\mathbb{S}$ and each ΣL_{-2m}^∞ is bounded below in the homotopy t-structure on $SH(S)$ by Lemma 10.11. By Theorem 9.5 the Adams spectral sequences for $\mathbb{S}$ and the ΣL_{-2m}^∞ are conditionally convergent to the (ℓ,η)-adic completions. By Proposition 9.6 the delayed limit Adams spectral sequence for $\Sigma L_{-\infty}^\infty$ is also conditionally convergent to the (ℓ,η)-adic completion. The $\mathscr{A}$-module homomorphism

$$e^*\colon H_c^{*,*}(\Sigma L_{-\infty}^\infty) \longrightarrow H^{*,*}(\mathbb{S})$$

agrees with

$$\mathrm{res}\colon \Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \longrightarrow H^{*,*}$$

via the isomorphism of Proposition 10.10, by Proposition 10.15. Finally, res is an Ext-equivalence by Corollary 7.5. Hence the induced map of spectral sequences

$$\begin{aligned} e\colon E_2^{s,*,*}(\mathbb{S}) &= \operatorname{Ext}_{\mathscr{A}}^{s,*,*}(H^{*,*}, H^{*,*}) \\ &\xrightarrow{\cong} {}^{\mathrm{del}}E_2^{s,*,*}(\Sigma L_{-\infty}^\infty) = \operatorname{Ext}_{\mathscr{A}}^{s,*,*}(\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}}, H^{*,*}) \end{aligned}$$

is an isomorphism, from the E_2-term and onward, and the map of abutments

$$e\colon \pi_{*,*}(\mathbb{S}_{\ell,\eta}^\wedge) \xrightarrow{\cong} \pi_{*,*}((\Sigma L_{-\infty}^\infty)_{\ell,\eta}^\wedge)$$

is an isomorphism of (filtered) bigraded abelian groups.

Let $Ce_{\ell,\eta}^\wedge$ denote the homotopy cofiber of $e_{\ell,\eta}^\wedge$. If $S = \operatorname{Spec} k$ for a field k, then the homotopy sheaves $\underline{\pi}_{*,*}(Ce_{\ell,\eta}^\wedge)$ are pure in the sense of [Mor05, Def. 6.4.9], hence unramified in the sense of [Mor12, Def. 2.1], by [Mor05, Lem. 6.4.11] and [Mor12, Thm. 1.9]. For any (irreducible) smooth k-scheme U with function field $k(U)$, the vanishing of $\pi_{*,*}(Ce_{\eta,\ell}^\wedge)$ over $\operatorname{Spec} k(U)$ implies the vanishing of $\pi_{*,*}(Ce_{\eta,\ell}^\wedge)$ over U, so that $\underline{\pi}_{*,*}(Ce_{\eta,\ell}^\wedge) = 0$ and $e_{\eta,\ell}^\wedge$ is a motivic equivalence. This application of Morel's theorems also appears in [Hem22, Prop. 4]. □

Remark 11.2. As an alternative to our fairly explicit construction of $e\colon \mathbb{S} \to \Sigma L_{-\infty}^\infty$ in Proposition 10.15, one might appeal to the weight 0 part of the delayed limit Adams spectral sequence

$${}^{\mathrm{del}}E_2^{s,t,0} = \operatorname{Ext}_{\mathscr{A}}^{s,t,0}(\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}}, H^{*,*}) \Longrightarrow_s \pi_{t-s,0}((\Sigma L_{-\infty}^\infty)_{\ell,\eta}^\wedge)$$

for $\Sigma L^{\infty}_{-\infty}$ to show the existence of a homotopy class $e' \in \pi_{0,0}((\Sigma L^{\infty}_{-\infty})^{\wedge}_{\ell,\eta})$ detected in Adams filtration $s = 0$ by res: $\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}} \to H^{*,*}$ in

$$^{\mathrm{del}}E_{\infty}^{0,0,0} \subset {}^{\mathrm{del}}E_2^{0,0,0} = \mathrm{Hom}_{\mathscr{A}}(\Sigma H^{*,*}(B\mu_\ell)_{\mathrm{loc}}, H^{*,*})\,.$$

By Corollary 7.5

$$^{\mathrm{del}}E_2^{s,t,0} \cong \mathrm{Ext}_{\mathscr{A}}^{s,t,0}(H^{*,*}, H^{*,*})\,,$$

and res corresponds to id: $H^{*,*} \to H^{*,*}$ in $\mathrm{Hom}_{\mathscr{A}}(H^{*,*}, H^{*,*})$.

If $d < \ell - 1$, which is always the case for $S = \operatorname{Spec} k$, the canonical (or normalized cobar) $\mathscr{A}$-module resolution of $H^{*,*}$ shows that $^{\mathrm{del}}E_2^{s,t,0} \cong \mathrm{Ext}_{\mathscr{A}}^{s,t,0}(H^{*,*}, H^{*,*}) = 0$ whenever $t - s = -1$. Hence $d_r^{0,0,0} = 0$ for all $r \geq 2$, so that $^{\mathrm{del}}E_{\infty}^{0,0,0} = {}^{\mathrm{del}}E_2^{0,0,0}$ and res is an infinite cycle.

To ensure that res detects a homotopy class, we also need strong convergence in its bidegree. By [Boa99, Thm. 7.3], see also [HR19, Thm. 3.9], it suffices to know that $RE_{\infty}^{s,t,0} = 0$ whenever $t - s = +1$. By [KW19, Cor. 6.1] this condition is satisfied for $S = \operatorname{Spec} k$, subject to the additional hypothesis for $\ell = 2$ that k has finite virtual cohomological dimension. The class e' is then represented by a map $e' \colon \mathbb{S} \to (\Sigma L^{\infty}_{-\infty})^{\wedge}_{\ell,\eta}$, which can be used in place of e.

References

[Ada69] J. F. Adams, *Lectures on generalised cohomology*, Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three), Springer, Berlin, 1969, pp. 1–138. MR0251716

[Ada74] ———, *Operations of the nth kind in K-theory, and what we don't know about RP^{∞}*, New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London, 1974, pp. 1–9. MR0339178

[AGM85] J. F. Adams, J. H. Gunawardena, and H. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology **24** (1985), no. 4, 435–460, DOI 10.1016/0040-9383(85)90014-X. MR816524

[Ati61] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310, DOI 10.1112/plms/s3-11.1.291. MR131880

[BI22] Eva Belmont and Daniel C. Isaksen, *$\mathbb{R}$-motivic stable stems*, J. Topol. **15** (2022), no. 4, 1755–1793, DOI 10.1112/topo.12256. MR4461846

[Boa82] J. M. Boardman, *The eightfold way to BP-operations or E_*E and all that*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 187–226. MR686116

[Boa99] ———, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (Baltimore, MD, 1998), Contemp. Math., vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84, DOI 10.1090/conm/239/03597. MR1718076

[Bru82] Robert R. Bruner, *Two generalizations of the Adams spectral sequence*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 275–287. MR686121

[BR21] Robert R. Bruner and John Rognes, *The Adams spectral sequence for topological modular forms*, Mathematical Surveys and Monographs, vol. 253, American Mathematical Society, Providence, RI, 2021. MR4284897

[Car84] Gunnar Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Ann. of Math. (2) **120** (1984), no. 2, 189–224, DOI 10.2307/2006940. MR763905

[CMP87] J. Caruso, J. P. May, and S. B. Priddy, *The Segal conjecture for elementary abelian p-groups. II. p-adic completion in equivariant cohomology*, Topology **26** (1987), no. 4, 413–433, DOI 10.1016/0040-9383(87)90040-1. MR919728

[DI05] Daniel Dugger and Daniel C. Isaksen, *Motivic cell structures*, Algebr. Geom. Topol. **5** (2005), 615–652, DOI 10.2140/agt.2005.5.615. MR2153114

[DI10] ———, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010), no. 2, 967–1014, DOI 10.2140/gt.2010.14.967. MR2629898

[DI17] ———, *Low-dimensional Milnor-Witt stems over $\mathbb{R}$*, Ann. K-Theory **2** (2017), no. 2, 175–210, DOI 10.2140/akt.2017.2.175. MR3590344

[EM66] Samuel Eilenberg and John C. Moore, *Homology and fibrations. I. Coalgebras, cotensor product and its derived functors*, Comment. Math. Helv. **40** (1966), 199–236, DOI 10.1007/BF02564371. MR203730

[Gei04] Thomas Geisser, *Motivic cohomology over Dedekind rings*, Math. Z. **248** (2004), no. 4, 773–794, DOI 10.1007/s00209-004-0680-x. MR2103541

[Gre13] Thomas Gregersen, *A Singer construction in motivic homotopy theory* (2013), available at `http://urn.nb.no/URN:NBN:no-38690`. PhD thesis, University of Oslo.

[GI15] Bertrand J. Guillou and Daniel C. Isaksen, *The η-local motivic sphere*, J. Pure Appl. Algebra **219** (2015), no. 10, 4728–4756, DOI 10.1016/j.jpaa.2015.03.004. MR3346515

[Gun81] J. H. Gunawardena, *Cohomotopy of some classifying spaces* (1981). Cambridge thesis.

[HR19] Gard Olav Helle and John Rognes, *Boardman's whole-plane obstruction group for Cartan-Eilenberg systems*, Doc. Math. **24** (2019), 1855–1878. MR4369361

[Hem22] David Hemminger, *Detecting motivic equivalences with motivic homology*, Proc. Amer. Math. Soc. Ser. B **9** (2022), 180–185, DOI 10.1090/bproc/82. MR4410405

[Hil11] Michael A. Hill, *Ext and the motivic Steenrod algebra over $\mathbb{R}$*, J. Pure Appl. Algebra **215** (2011), no. 5, 715–727, DOI 10.1016/j.jpaa.2010.06.017. MR2747214

[Hoy15] Marc Hoyois, *From algebraic cobordism to motivic cohomology*, J. Reine Angew. Math. **702** (2015), 173–226. MR3341470

[Hoy17] ———, *The six operations in equivariant motivic homotopy theory*, Adv. Math. **305** (2017), 197–279, DOI 10.1016/j.aim.2016.09.031. MR3570135

[HKØ17] Marc Hoyois, Shane Kelly, and Paul Arne Østvær, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 12, 3813–3849. MR3730515

[Hu05] Po Hu, *On the Picard group of the stable $\mathbb{A}^1$-homotopy category*, Topology **44** (2005), no. 3, 609–640, DOI 10.1016/j.top.2004.12.001. MR2122218

[HKO11] P. Hu, I. Kriz, and K. Ormsby, *Convergence of the motivic Adams spectral sequence*, J. K-Theory **7** (2011), no. 3, 573–596, DOI 10.1017/is011003012jkt150. MR2811716

[Isa19] Daniel C. Isaksen, *Stable stems*, Mem. Amer. Math. Soc. **262** (2019), no. 1269, viii+159, DOI 10.1090/memo/1269. MR4046815

[IWX20] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, *Stable homotopy groups of spheres*, Proc. Natl. Acad. Sci. USA **117** (2020), no. 40, 24757–24763, DOI 10.1073/pnas.2012335117. MR4250190

[Jam59] I. M. James, *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. (3) **9** (1959), 115–140, DOI 10.1112/plms/s3-9.1.115. MR102810

[Jar00] J. F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552. MR1787949

[KMT66] Tsunekazu Kambe, Hiromichi Matsunaga, and Hirosi Toda, *A note on stunted lens space*, J. Math. Kyoto Univ. **5** (1966), 143–149, DOI 10.1215/kjm/1250524531. MR190929

[KW19] Jonas Irgens Kylling and Glen Matthew Wilson, *Strong convergence in the motivic Adams spectral sequence* (2019), 29 pp., available at `https://arxiv.org/abs/1901.03399`.

[LS82] Hu Hsiung Li and William M. Singer, *Resolutions of modules over the Steenrod algebra and the classical theory of invariants*, Math. Z. **181** (1982), no. 2, 269–286, DOI 10.1007/BF01215024. MR674277

[Lin80] Wen Hsiung Lin, *On conjectures of Mahowald, Segal and Sullivan*, Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 449–458, DOI 10.1017/S0305004100056887. MR556925

[LDMA80] W. H. Lin, D. M. Davis, M. E. Mahowald, and J. F. Adams, *Calculation of Lin's Ext groups*, Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 459–469, DOI 10.1017/S0305004100056899. MR569195

[LNR12] Sverre Lunøe-Nielsen and John Rognes, *The topological Singer construction*, Doc. Math. **17** (2012), 861–909. MR3007679

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR1712872

[Mah85] Mark Mahowald, *Lin's theorem and the EHP sequence*, Conference on algebraic topology in honor of Peter Hilton (Saint John's, Nfld., 1983), Contemp. Math., vol. 37, Amer. Math. Soc., Providence, RI, 1985, pp. 115–119, DOI 10.1090/conm/037/789799. MR789799

[MR93] Mark E. Mahowald and Douglas C. Ravenel, *The root invariant in homotopy theory*, Topology **32** (1993), no. 4, 865–898, DOI 10.1016/0040-9383(93)90055-Z. MR1241877

[Man21] Lorenzo Mantovani, *Localizations and completions of stable ∞-categories* (2021), 41 pp., available at `https://arxiv.org/abs/2105.02285`.

[Mar83] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, vol. 29, North-Holland Publishing Co., Amsterdam, 1983. Modules over the Steenrod algebra and the stable homotopy category. MR738973

[MM82] J. P. May and J. E. McClure, *A reduction of the Segal conjecture*, Current trends in algebraic topology, Part 2 (London, Ont., 1981), CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 209–222. MR686147

[Mil72] R. James Milgram, *Group representations and the Adams spectral sequence*, Pacific J. Math. **41** (1972), 157–182. MR304463

[MR77] Haynes R. Miller and Douglas C. Ravenel, *Morava stabilizer algebras and the localization of Novikov's E_2-term*, Duke Math. J. **44** (1977), no. 2, 433–447. MR0458410

[Mil58] John Milnor, *The Steenrod algebra and its dual*, Ann. of Math. (2) **67** (1958), 150–171, DOI 10.2307/1969932. MR99653

[MM65] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264, DOI 10.2307/1970615. MR0174052

[Mor05] Fabien Morel, *The stable $\mathbb{A}^1$-connectivity theorems*, K-Theory **35** (2005), no. 1-2, 1–68, DOI 10.1007/s10977-005-1562-7. MR2240215

[Mor12] ———, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012. MR2934577

[MV99] Fabien Morel and Vladimir Voevodsky, *$\mathbf{A}^1$-homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001). MR1813224

[Qui21a] J. D. Quigley, *Real motivic and C_2-equivariant Mahowald invariants*, J. Topol. **14** (2021), no. 2, 369–418, DOI 10.1112/topo.12185. MR4235014

[Qui21b] ———, *Motivic Mahowald invariants over general base fields*, Doc. Math. **26** (2021), 561–582. MR4282796

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986. MR860042

[Rio12] Joël Riou, *Opérations de Steenrod motiviques* (2012), available at `https://arxiv.org/abs/1207.3121`.

[Sin80] William M. Singer, *On the localization of modules over the Steenrod algebra*, J. Pure Appl. Algebra **16** (1980), no. 1, 75–84, DOI 10.1016/0022-4049(80)90043-2. MR549705

[Sin81] ———, *A new chain complex for the homology of the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 2, 279–292, DOI 10.1017/S0305004100058746. MR620738

[Spi18] Markus Spitzweck, *A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. Fr. (N.S.) **157** (2018), 110, DOI 10.24033/msmf.465. MR3865569

[Ste62] N. E. Steenrod, *Cohomology operations*, Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50, Princeton University Press, Princeton, N.J., 1962. MR0145525

[Voe98] Vladimir Voevodsky, *$\mathbf{A}^1$-homotopy theory*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), 1998, pp. 579–604. MR1648048

[Voe03a] ———, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57. MR2031198

[Voe03b] ———, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104. MR2031199

[Voe10] ———, *Motivic Eilenberg–MacLane spaces*, Publ. Math. Inst. Hautes Études Sci. **112** (2010), 1–99. MR2737977

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