GUIDO’S BOOK OF CONJECTURES

A GIFT TO

GUIDO MISLIN

ON THE OCCASION OF HIS RETIREMENT FROM ETHZ

JUNE 2006

Edited by Indira Chatterji, with the help of Mike Davis, Henry Glover, Tadeusz Januszkiewicz, Ian Leary and tons of enthusiastic contributors.

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Foreword

This book containing conjectures is meant to occupy my husband, Guido Mislin, during the long years of his retirement. I view this project with appreciation, since I was wondering how that mission was to be accomplished. In the thirty-five years of our acquaintance, Guido has usually kept busy with his several jobs, our children and occasional, but highly successful projects that he has undertaken around the house. The prospect of a Guido unleashed from the ETH; unfettered by professional duties of any sort, wandering around the world as a free agent with, in fact, nothing to do, is a prospect that would frighten nations if they knew it was imminent. I find it a little scary myself, so I am in a position to appreciate the existence of this project from the bottom of my heart.

Of course, the book is much more than the sum of its parts. It wouldn't take Guido long to read a single page in a book, but a page containing a conjecture, particularly a good one, might take him years. This would, of course, be a very good thing.

Though not any sort of mathematician, I have observed the field long enough to know that mathematicians do not share our mundane reality. They breathe a more rarefied air. For example, in the regular world one might say, "There is a dead chicken on that table." or "There is no dead chicken on that table." In this example, one would have little trouble proving the point either way because a close examination of the table would quickly reveal whether it held a dead chicken or not. Now, in the world of the mathematician, these two alternatives simply did not offer enough scope, so in the last century, a third alternative was provided. Currently we have the case of the table with a dead chicken; the table with no dead chicken and the table where it will never be proven whether there is a dead chicken on it or not. In that case, one would declare the problem undecidable, which means that it probably cannot be proven one way or the other so don't bother. Impressive, isn't it? When one hits a wall in mathematics, the wall simply gets redefined or reinvented. I only wish it were that simple for the rest of us.

Guido is fortunate to have the promise this book offers of interesting conjectures in his future and he is also very fortunate to have the interest and the efforts of his friends and colleagues who contributed, particularly Indira Chatterji whose idea and hard work this book represents. As for me, Guido is my own wondrous undecidable conjecture and that has been the greatest good fortune of my life.

Gwynyth Mislin
June 2006

Contents

1. **Alejandro Adem**
A Conjecture About Group Actions

Dear Guido,

Here is my favorite conjecture, which is easy to state but can be quite dangerous. If anyone can solve it in an elegant way it will be you!

Best wishes and thanks for everything,

Alejandro Adem.

Conjecture 1.1. *If G is a finite group of rank equal to r, then it acts freely on a finite complex with the homotopy type of a product of k spheres if and only if $r \leq k$.*

2. **Dominique Arlettaz**

A vanishing conjecture for the homology of congruence subgroups

Dear Guido,

It was a pleasure to be your student, it is just great to be your friend. I hope you'll have a very nice and active retirement.

With my best wishes,
Dominique

For any prime number p, let $\Gamma_{n,p}$ denote the congruence subgroup of $SL_n(\mathbb{Z})$ of level p, i.e., the kernel of the surjective homomorphism $f_p : SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_p)$ induced by the reduction mod p, and let us write $\Gamma_p = \varinjlim_n \Gamma_{n,p}$, where the limit is defined by upper left inclusions.

If p is odd, then the group Γ_p is torsion-free. Therefore, it is of particular interest to detect torsion classes in the integral (co)homology of Γ_p. It turns out that $H^*(\Gamma_p;\mathbb{Z})$ contains 2-torsion elements in arbitrarily large dimensions (see Corollary 1.10 of [Ar]). Groups like this are called groups with "very strange torsion" by S. Weintraub in 1986.

However, vanishing results for the (co)homology of Γ_p are also extremely useful. Let us propose the following

Conjecture 2.1. *For an odd integer n and an odd prime p, the homology group $H_n(\Gamma_p;\mathbb{Z})$ contains no q-torsion if q is a sufficiently large prime (in comparison with n), $q \neq p$.*

As far as I know, this problem is not solved, but one should notice its relationship with the study of the Dwyer-Friedlander map $\varphi_{\mathbb{Z}} : (K_n(\mathbb{Z}))_q \to K_n^{\text{ét}}(\mathbb{Z}[\frac{1}{q}])$ relating the q-torsion of algebraic K-theory to étale K-theory. This map is known to be surjective and it is conjecturally an isomorphism (this is a version of the Quillen-Lichtenbaum conjecture, see [DF], Theorem 8.7 and Remark 8.8).

For $q \neq p$, the Dwyer-Friedlander map and the reduction mod p induce the commutative diagram

$$\begin{tikzcd}
(K_n(\mathbb{Z}))_q \arrow[r, "\varphi_{\mathbb{Z}}"] \arrow[d, "(f_p)_*"] & K_n^{\text{ét}}(\mathbb{Z}[\frac{1}{q}]) \arrow[d] \\
(K_n(\mathbb{F}_p))_q \arrow[r, "\varphi_{\mathbb{F}_p}"] & K_n^{\text{ét}}(\mathbb{F}_p)
\end{tikzcd}$$

The map $\varphi_{\mathbb{F}_p}$ is an isomorphism, since the K-theory of finite fields is completely known. If we define $A_n = \ker \varphi_{\mathbb{Z}}$ and $B_n = \ker (f_p)_*$, this implies that A_n is contained in B_n.

On the other hand, one can show by using Postnikov decompositions that B_n is a direct summand of $(H_n(\Gamma_p;\mathbb{Z}))_q$ for large enough primes $q \neq p$ (see [Ar], Introduction and Theorem 2.1).

Consequently, the proof of the above conjecture for n odd and q a large enough prime would imply the vanishing of B_n and therefore the vanishing of A_n which provides the assertion that the Dwyer-Friedlander map $\varphi_{\mathbb{Z}}$ is an isomorphism.

References

[Ar] D. Arlettaz: Torsion classes in the cohomology of congruence subgroups. Math. Proc. Cambridge Philos. Soc. 105 (1989), 214-248.

[DF] W.C. Dwyer and E.M. Friedlander: Algebraic and étale K-theory. Trans. Amer. Math. Soc. 292 (1985), 247-280.

3. **Goulnara N. Arzhantseva**

The uniform Kazhdan property for $SL_n(\mathbb{Z}), n \geq 3$.

Dear Guido,
You were the first mathematician I have met in Zurich. I felt very honored that I was introduced to the ETHZ mathematical life by you. It is my great pleasure to contribute to this volume. All my best wishes for the retirement !

Let Γ be a discrete group, and let S be a finite subset of Γ. For a unitary representation π of Γ in a separable Hilbert space $\mathcal{H}$ we define the number

$$K(\pi, \Gamma, S) = \inf_{0 \neq u \in \mathcal{H}} \max_{s \in S} \frac{\|\pi(s)u - u\|}{\|u\|}.$$

Then the *Kazhdan constant* of Γ with respect to S is defined as

$$K(\Gamma, S) = \inf_{\pi} K(\pi, \Gamma, S),$$

where the infimum is taken over unitary representations π having no invariant vectors. We also define the *uniform Kazhdan constant* of Γ as

$$K(\Gamma) = \inf_{S} K(\Gamma, S),$$

where the infimum is taken over all finite generating sets S of Γ.

A group Γ is said to have *Kazhdan property* (T) (or to be a Kazhdan group) if there exists a finite subset S of Γ with $K(\Gamma, S) > 0$. A group Γ is *uniform Kazhdan* if $K(\Gamma) > 0$.

Shortly after its introduction by David Kazhdan in the mid 60's, property (T) was used by Gregory Margulis to give a first explicit construction of infinite families of expander graphs of bounded degree. In particular, a major problem of practical application in the design of efficient communication networks was solved.

A classical example of a Kazhdan group is the group $SL_n(\mathbb{Z})$ for $n \geq 3$ (for more details and a general context of locally compact groups see a recent book [1]). Surprisingly, the following question is still open.

Question 3.1. *Is the group $SL_n(\mathbb{Z})$, for $n \geq 3$, uniform Kazhdan ?*

Infinite finitely generated uniform Kazhdan groups were discovered very recently [2,3]. However, these groups are neither finitely presented

[1] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T)*, http://www.unige.ch/math/biblio/preprint/2006/KazhdansPropertyT.pdf.

[2] D. Osin, D. Sonkin, *Uniform Kazhdan groups*, arxiv: math.GR/0606012.

[3] G. Arzhantseva, T. Delzant, *A random group with uniform Kazhdan property*, preliminary version.

nor residually finite. The latter construction provides an infinite uniform Kazhdan group that weakly [4] contains an infinite family of expanders in its Cayley graph.

An affirmative answer to the above question would give, in particular, the first example of a residually finite (and, in addition, finitely presented) infinite uniform Kazhdan group. It is crucial for applications: infinite families of expanders could be constructed independently of the choice of the group generating set.

A negative answer would be interesting as well. In that case, this classical group would belong to the class of non-uniform Kazhdan groups. First examples of such groups were obtained using Lie groups [5]. Then, all word hyperbolic groups were also shown to have zero uniform Kazhdan constant [6].

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[4]M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. 13 (2003), no. 1, 73–146.

[5]T. Gelander and A.Zuk, *Dependence of Kazhdan constants on generating subsets*, Israel J. Math. 129 (2002), 93–98.

[6]D. Osin, *Kazhdan constants of hyperbolic groups*, Funktsional. Anal. i Prilozhen. 36 (2002), no. 4, 46–54; translation in Funct. Anal. Appl. 36 (2002), no. 4, 290–297.

4. **Christian Ausoni and John Rognes**

The Chromatic Red-Shift in algebraic K-theory

Dear Guido,

The algebraic K-theory of the sphere spectrum $\mathbb{S}$ is of interest in geometric topology by Waldhausen's Stable Parametrised h-Cobordism Theorem (2006), and we would like to understand it like we understand $K\mathbb{Z}$, via Galois descent. In a first approximation, the algebraic K-theory of the Bousfield localization $L_{K(n)}\mathbb{S}$ of $\mathbb{S}$ with respect to the n-th Morava K-theory $K(n)$ might be more accessible. John has developed a theory of Galois extensions for $\mathbb{S}$-algebras, and in this framework he has stated conjectures extending the ordinary Lichtenbaum-Quillen Conjectures. Their precise formulation is distilled from the clues provided by our computations of the algebraic K-theory of topological K-theory and related spectra, and it is to be expected that they will keep maturing in a cask of scepticism for a few years.

We first recall the relevant definitions. Let G be a finite group, and if G acts on a spectrum X we denote by X^{hG} the homotopy fixed-point spectrum.

Definition 4.1. A map $A \to B$ of commutative $\mathbb{S}$-algebras is a $K(n)$-local G-Galois extension if G acts on B through commutative A-algebra maps and the canonical maps $A \to B^{hG}$ and $B \wedge_A B \to \prod_G B$ are $K(n)$-equivalences.

Let V be a finite CW-spectrum of chromatic type $n+1$. It admits an essentially unique v_{n+1}-self-map, and let $T = v_{n+1}^{-1}V$ be its mapping telescope. For example, if $n = 0$ we take $V = V(0) = \mathbb{S}/p$ the Moore spectrum, and for $n = 1$ and $p \geq 3$ we take $V = V(1) = V(0)/v_1$.

Conjecture 4.2. *Let $A \to B$ be a $K(n)$-local G-Galois extension. Then there is a homotopy equivalence*

$$T \wedge KA \to T \wedge (KB)^{hG}.$$

If $n = 0$, then $A \to B$ is a G-Galois extension of commutative $\mathbb{Q}$-algebras, say number fields, and Conjecture 4.2 is the Galois Descent Conjecture of Lichtenbaum-Quillen (1973). In the case $n = 1$, Conjecture 4.2 holds for the $K(1)$-local $\mathbb{F}_p^\times$-Galois extension $L_p \to KU_p$, where KU_p is the p-complete periodic complex K-theory spectrum, and L_p is the Adams summand.

Conjecture 4.3. *Let B be a suitably finite $K(n)$-local commutative $\mathbb{S}$-algebra (for example $L_{K(n)}\mathbb{S} \to B$ could be a G-Galois extension). Then the map*

$$V \wedge KB \to T \wedge KB$$

induces an isomorphism on homotopy groups in sufficiently high degrees.

If $n = 0$ and $B = HF$ is the Eilenberg-Mac Lane spectrum of a number field F, then there is a homotopy equivalence $T \wedge KF \simeq K^{\text{ét}}(F;\mathbb{Z}/p)$ by Thomason's Theorem (1985), and the map

$$V(0) \wedge KF = K(F;\mathbb{Z}/p) \to K^{\text{ét}}(F;\mathbb{Z}/p)$$

induces an isomorphism on homotopy groups in sufficiently high degrees. For $n = 1$ and $p \geq 5$, and for $B = L_p, KU_p$ or their connective versions ℓ_p and ku_p, it is known that $V(1)_*KB$ is a finitely generated free $\mathbb{F}_p[v_2]$-module in high degrees, hence Conjecture 4.3 holds for these $\mathbb{S}$-algebras. This is evidence for the "Red-Shift Conjecture", which, in a less precise formulation than Conjecture 4.3, asserts that algebraic K-theory increases chromatic complexity by one.

In the case of a ring of integers $\mathcal{O}_F$ in a number field F, $K(\mathcal{O}_F;\mathbb{Z}/p)$ can be computed from $K(F;\mathbb{Z}/p)$ and the K-theory of the residue fields by the localization sequence. For computing $K(F;\mathbb{Z}/p)$, one then relies on Suslin's Theorem (1983) that $K(\bar{F};\mathbb{Z}/p) \simeq V(0) \wedge ku$ and uses descent with respect to the absolute Galois group G_F.

To generalize this program we have to make sense of the $\mathbb{S}$-algebraic fraction field of $L_{K(n)}\mathbb{S}$, construct a separable closure Ω_n, and evaluate its K-theory. Let E_{n+1} be Morava's E-theory associated to the universal deformation of a height $n+1$ formal group law over $\mathbb{F}_{p^{n+1}}$, with coefficients $(E_{n+1})_* = W(\mathbb{F}_{p^{n+1}})[[u_1, \ldots, u_n]][u^{\pm 1}]$.

Conjecture 4.4. *If Ω_n is a separably closed extension of the $\mathbb{S}$-algebraic fraction field of $L_{K(n)}\mathbb{S}_p$, then there is a homotopy equivalence*

$$L_{K(n+1)}K(\Omega_n) \simeq E_{n+1}.$$

For $n = 0$ this reduces to $L_{K(1)}K(\bar{\mathbb{Q}}_p) \simeq E_1 \simeq KU_p$, a weaker formulation of Suslin's Theorem.

We did some computations aimed at better understanding what the fraction field $\mathcal{F}$ of KU_p might be. We defined $K\mathcal{F}$ as the cofibre of the transfer map for $KU_p \to KU_p/p$, to sit in a hypothetical localization sequence $K(KU_p/p) \to KKU_p \to K\mathcal{F}$. The result is that $V(1)_*K\mathcal{F}$ is, in high enough degrees, a free $\mathbb{F}_p[v_2]$-module on $2(p^2+3)(p-1)$ generators. In particular $\mathcal{F}$ cannot be the $H\mathbb{Q}_p$-algebra $KU_p[1/p]$. We rather believe that $\mathcal{F}$ is an $\mathbb{S}$-algebraic analogue of a two-dimensional local field. For example, there appears to be a perfect arithmetic duality pairing in the Galois cohomology of $\mathcal{F}$, analogous to Tate-Poitou Duality (1963) for local number fields.

With our best wishes for the future !

5. **Angela Barnhill and Indira Chatterji**
Property (T) versus Property FW

Recall (e.g. from de la Harpe and Valette's book on property (T)) that a group G has property (T) if and only if every continuous affine action on a real Hilbert space has a global fixed point. Niblo and Reeves[7] showed that for a group satisfying Kazhdan's property (T), every cellular action on a finite dimensional CAT(0) cube complex has a global fixed point. We will look at the following:

Definition 5.1. A group G has **property FW$_n$** if every cellular action of G on every n-dimensional CAT(0) cube complex has a global fixed point. The group G has **property FW** if G has FW$_n$ for all n.

So, according to Niblo and Reeves, if G has Kazhdan's property (T) then G has property FW. Note that the abbreviation FW stands for "fix" and "walls". Recall the following:

Definition 5.2 (Haglund and Paulin[8]). A **wall space** is a set Y together with a nonempty collection $\mathcal{H} \subseteq \mathcal{P}(Y)$ of *half-spaces* such that $h \in \mathcal{H} \implies h^C \in \mathcal{H}$ and $\#\{h \in \mathcal{H} : x \in h, y \in h^c\} < \infty$ for every $(x, y) \in Y \times Y$. A wall structure endows Y with a pseudo-metric (by counting how many walls separate two points) and yields a metric on a quotient of Y. An action of a group G on the wall space Y is an action of G on Y that preserves the wall structure, i.e. an action such that $g(h) \in \mathcal{H}$ for every $g \in G$ and $h \in \mathcal{H}$.

It turns out that acting on a wall space is very similar to acting isometrically on a CAT(0) cube complex: It is well-known[9] that an isometric action on a CAT(0) cube complex gives an isometric action on a wall space, and the converse holds as well, as shown by Nica[10] and by the second author with Niblo[11]. Moreover, the distance between a point x and gx is the same in the CAT(0) complex as in the corresponding wall space.

Recently Cherix, Martin, and Valette[12] showed that a finitely generated group has property (T) if and only if every action on a space with *measured* walls has a global fixed point. A natural question, then, is the following:

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[7] *Groups acting on CAT(0) cube complexes*, Geom. Topol. 1 (1997), 1-7.

[8] *Simplicité de groupes d'automorphismes d'espaces à courbure négative.*, Geom. Topol. Monogr., 1, Geom. Topol. Publ., Coventry, 1998.

[9] unclear who stated it first in that form, but it is reminiscent in M. Sageev's *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. 71 (1995)

[10] *Cubulating spaces with walls*, AGT 4 (2004), 297–309.

[11] *From wall spaces to CAT(0) cube complexes*, IJAC 15 (2005), 875-885.

[12] *Spaces with measured walls, Haagerup property and property (T)*, Ergodic Theory Dynam. Systems 24 (2004).

Question 5.3. *Is FW equivalant to (T), or*[13] *does there exist a group G such that G does not have property (T), but G and all its finite index subgroups have property FW?*

Remark 5.4. Groups with Kazhdan's property (T) are also known[14] to have Serre's property FA, but many groups with FA do not have (T). The following generalization of property FA was introduced by Farb: A group is said to have *property FA_n* if every cellular action of the group on an n-dimensional CAT(0) (piecewise-Euclidean or piecewise-hyperbolic) complex has a global fixed point. In particular, FA_n implies FW_n. However, $\mathrm{SL}_m(\mathbb{Z}[\frac{1}{p}])$ has FA_{m-2}[15] but $\mathrm{SL}_m(\mathbb{Z}[\frac{1}{p}])$ acts without a global fixed point on the Bruhat-Tits building for $\mathrm{SL}_m(\mathbb{Q}_p)$, an $(m-1)$-dimensional CAT(0) complex, so $\mathrm{SL}_m(\mathbb{Z}[\frac{1}{p}])$ does not have FA_{m-1}. Hence FA_n distinguishes between these property (T) groups whereas every property (T) group has FW_n for all n.

Indira's Appendix. Let $C^*(G)$ be the maximal C*-algebra of G and $\mathbf{C}G_f \subseteq C^*(G)$ the $\mathbf{C}$-vector space with basis the elements of finite order in G. In (one of our many) joint work with you Guido[16], an element $x \in HH^{top}(C^*(G)) = C^*(G)/\overline{[C^*(G), C^*(G)]}$ was called *f-supported* if it is in the image of $\mathbf{C}G_f \subseteq C^*(G)$ under the natural quotient map $C^*(G) \to HH^{top}(C^*(G))$. We showed that for a property (T) group there is always an element in $K_0(C^*(G))$ whose image in $HH^{top}(C^*(G))$ under the Hattori-Stallings trace is not f-supported, so a natural question is the following:

Question 5.5. *Let G be a group such that every finite index subgroup of G has property FW. Are there elements in $K_0(C^*(G))$ whose image in $HH^{top}(C^*(G))$ under the Hattori-Stallings trace is not f-supported?*

[13] as hinted by Cherix, Martin, and Valette on the second page of their paper

[14] Y. Watatani. *Property (T) of Kazhdan implies property FA of Serre*, Math. Japon. 27 (1982), no. 1, 97–103.

[15] B. Farb. *Group actions and Helly's theorem*, in preparation.

[16] *Traces and reduced group C*-algebras*, Contemporary mathematics 399

6. **Laurent Bartholdi**
Growth and Amenability

Dear Guido,
I know you're very fond of growth of discrete groups and amenability – and it's a bit upsetting that amenability and subexponential growth are not equivalent. Well, I propose to make this a true statement!

Let G be a finitely generated, residually-p group. Let $\{G_n\}$ be its p-lower central series, defined by $G_1 = G$ and $G_{n+1} = G_n^p[G_n, G]$ for example. Write $|G : G_n| = p^{b_n}$.

Conjecture 6.1. *A residually-p group is amenable if and only if its growth series $\{b_n\}$ grows subexponentially, that is,* $\lim(b_n)^{1/n} = 1$.

A little motivation: consider the augmentation ideal ϖ in the group ring $\mathbb{F}_pG$. Then the sequence $\{b_n\}$ grows subexponentially if and only if the sequence $\dim \varpi^n/\varpi^{n+1}$ grows subexponentially. This means that the associated graded $\bigoplus_{n\geq 0} \varpi^n/\varpi^{n+1}$ is amenable in the sense of [M. Gromov: *Topological invariants of dynamical systems and spaces of holomorphic maps. I*, Math. Phys. Anal. Geom. **2** (1999) 323–415; MR1742309 (2001j:37037)]

Questions related to this were already asked in:
[A. Vershik: *Amenability and approximation of infinite groups*, Selecta Math. Soviet. **2** (1982) 311–330; MR0721030 (86g:43006)]
[L. Bartholdi and R. Grigorchuk: *Lie methods in growth of groups and groups of finite width*, Computational and Geometric Aspects of Modern Algebra (Michael Atkinson et al., ed.), London Math. Soc. Lect. Note Ser., vol. 275, Cambridge Univ. Press, Cambridge, 2000, pp. 1–27; MR1776763 (2001h:20046)].

7. **Oliver Baues**

Two problems concerning polycyclic spaces

A finite space (CW-complex) X is called a *nilpotent space* if the fundamental group $\pi_1(X)$ acts nilpotently on the homotopy groups of X. In particular, $\pi_1(X)$ is finitely generated nilpotent itself. Sullivan proved that the group of *homotopy equivalences* of a nilpotent space is, modulo finite kernels, commensurable with an *arithmetic group*.

Natural examples for nilpotent spaces are aspherical spaces X with (torsion-free) finitely generated nilpotent fundamental group. In this case, the statement about the homotopy-equivalences corresponds to a purely group theoretic result on the *outer automorphism group* of $\pi_1(X)$.

For aspherical spaces with finitely generated nilpotent fundamental group natural *compact smooth model spaces* exist. These spaces are traditionally called *nilmanifolds*. Among all smooth manifolds representing a given nilpotent aspherical homotopy type, nilmanifolds are characterised by their distinctive geometric properties, for example, the existence of almost flat Riemannian metrics. Surprisingly, there do exist also *exotic smooth models* in a nilpotent aspherical homotopy type, which are then *not* diffeomorphic to any nilmanifold.

Quite close to nilmanifolds, but less well understood, are *solvmanifolds* and their finite *geometric* quotients, which are called *infrasolvmanifolds*. By definition, a solvmanifold is a homogeneous space for a solvable Lie group. Generalising nilmanifolds (which admit a transitive action of a nilpotent Lie group), these smooth manifolds do provide natural compact smooth models for aspherical manifolds with a (torsion-free) *polycyclic by-finite* fundamental group.

Here come two problems, which are in the realm of the above ideas.

The first concerns the existence of "good" geometric structures on smooth aspherical compact manifolds with solvable fundamental group. (Note that, in this case, the fundamental group is necessarily a polycyclic group.)

Problem 1).
A recent result states that a compact aspherical *Kähler-manifold* with solvable fundamental group is (diffeomorphic) to an infra-nilmanifold, which is finitely covered by a smooth *standard torus*. On the other hand, it is well known that there exist many solv- and nilmanifolds

(not necessarily covered by a torus) which admit a complex manifold structure. Problem: *Given any aspherical compact complex manifold with solvable fundamental group, is it diffeomorphic to an infra-solvmanifold?*

The second problem concerns Sullivan's arithmeticity result for nilpotent spaces.

Problem 2).
As proved recently, the outer automorphism group of any polycyclic by finite group, and, hence, also the group of homotopy equivalences of any aspherical space with a polycyclic by finite fundamental group *is* an arithmetic group. Hence, we ask: *Does Sullivan's arithmeticity result for nilpotent spaces carry over to a (suitable) more general class of* polycyclic spaces*?*

For more background on problem 1), see Baues, Cortes, Aspherical Kähler manifolds with solvable fundamental group, math.DG/0601616, and the references therein. For problem 2), see Baues, Grunewald, Automorphism groups of polycyclic-by-finite groups and arithmetic groups, math.GR/0511624.

8. **Gilbert Baumslag**

Groups with the same lower central sequences

Two groups G and H are said to have the same lower central sequences if

$$G/\gamma_n(G) \cong H/\gamma_n(H)$$

for every n, where $\gamma_n(G)$ denotes the n^{th} term of the lower central series of G.

Suppose that G and H are residually nilpotent, i.e., suppose that the intersection of their lower central series is the identity. The basic question then is how much do two residually nilpotent groups with the same lower centrals series have in common? So, for example,

- If G and H are both finitely generated and one is finitely presented, is the other also finitely presented?
- If G and H are both finitely generated and one has finitely generated H_2 with integral coefficients, does the other?
- If G is finitely generated and has the same lower central series as a free group, is $H_2(G, \mathbb{Z}) = 0$? So G is a so-called parafree group. This question has been tackled by many people and an incorrect proof has even been published. Bousfield and Kan have proved that the pronilpotent completion of a residually nilpotent group has the same lower central sequence as any given finitely generated residually nilpotent group. These completions turn up in homotopy theory, one of Guido's interests. However they do not, for the most part, reflect the properties of a given residually nilpotent group. It should be noted that the pronilpotent completion of a finitely generated, residually nilpotent group is finitely generated only if the group itself is nilpotent. In the case of a non–abelian, finitely generated free group, Bousfield and Kan have shown that the second homology group with integral coefficients of its pronilpotent completion has as many elements as the reals. So it is definitely not 0.
- If G and H are finitely generated nilpotent groups and have the same finite images, do they have the same homology?

The last of these questions is especially formulated for Guido who has been interested from time to time in the so-called genus of finitely generated nilpotent groups.

And Happy Retirement, Guido.

9. **Paul Baum**
The Extended Quotient

Let Γ be a finite group acting on a (topological space X or) an affine variety X.

$$\Gamma \times X \to X$$

The quotient variety (or quotient topological space) X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x.

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x.

The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$.

This is done as follows:

Set $\tilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

$\tilde{X} \subset \Gamma \times X$

$\tilde{X}$ is an affine variety and is a sub-variety of $\Gamma \times X$.

Γ acts on $\tilde{X}$.

$\Gamma \times \tilde{X} \to \tilde{X}$

$g(\gamma, x) = (g\gamma g^{-1}, gx) \quad g \in \Gamma \quad (\gamma, x) \in \tilde{X}$

The extended quotient, denoted $X//\Gamma$, is $\tilde{X}/\Gamma$.

i.e. The extended quotient $X//\Gamma$ is the ordinary quotient for the action of Γ on $\tilde{X}$.

The extended quotient is an affine variety (or a topological space). The evident projection $\tilde{X} \to X \quad (\gamma, x) \mapsto x$ passes to quotient spaces to give a map $\rho : X//\Gamma \to X/\Gamma$. ρ is the projection of the extended quotient onto the ordinary quotient.

Let G be a reductive p-adic group. Examples are:

$GL(n, F) \qquad SL(n, F) \qquad$ where F is any finite extension of the p-adic numbers $\mathbb{Q}_p$

Let V be a vector space over the complex numbers $\mathbb{C}$.

Definition 9.1. A representation

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G.

$\widehat{G}$ denotes the set of equivalence classes of smooth irreducible representations of G.

One of the main problems in the representation theory of p-adic groups (which is closely related to the local Langlands conjecture) is to describe $\widehat{G}$.

The Hecke algebra of G, denoted $\mathcal{H}G$ is the convolution algebra of all complex-valued locally-constant compactly-supported functions $f : G \to \mathbb{C}$. $\widehat{G}$ is in bijection with the set of primitive ideals in $\mathcal{H}G$. On the set of primitive ideals in $\mathcal{H}G$ there is the Jacobson topology. Hence we may consider each connected component of the primitive ideal space. Typically there will be countably many of these connected components.

$\mathbb{C}^\times$ denotes the (complex) affine variety $\mathbb{C} - \{0\}$.

Definition 9.2. A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.$$

J. Bernstein assigns to each $\alpha \in \pi_o \mathrm{Prim}\mathcal{H}G$ a complex torus T_α and a finite group Γ_α acting on T_α.

He then forms the quotient variety T_α/Γ_α and proves that there is a surjective map (the infinitesimal character) π_α mapping X_α onto T_α/Γ_α.

$$X_\alpha \xrightarrow{\pi_\alpha} T_\alpha/\Gamma_\alpha$$

X_α is the connected component of $\mathrm{Prim}(\mathcal{H}G)$ corresponding to α. In Bernstein's work X_α is a set (i.e. is only a set) so π_α is a map of sets. π_α is surjective, finite-to-one and generically one-to-one.

Conjecture. There is a certain resemblance between

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & X_\alpha \\ \Big\downarrow{\scriptstyle \rho_\alpha} & \text{and} & \Big\downarrow{\scriptstyle \pi_\alpha} \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Here ρ_α is (as above) the projection of the extended quotient onto the ordinary quotient. How can this conjecture be made precise?

For the precise conjecture see papers of A-M. Aubert, P. Baum and R. Plymen.

10. **Dave Benson**

The regularity conjecture in the cohomology of groups

Dear Guido,

Here is a conjecture I'm rather fond of. It's about the cohomology of finite groups. Let k be a field of characteristic p and let G be a finite group.

Conjecture 10.1. *The Castelnuovo–Mumford regularity of the cohomology ring is equal to zero:*

$$\mathsf{Reg}\, H^*(G,k) = 0.$$

This conjecture was first announced at the opening workshop of the MSRI commutative algebra year (1), as a refinement of a conjecture of Benson and Carlson (4). Subsequent work on the conjecture was reported in (2) and (3).

We begin with the definitions. Let H be a finitely generated graded commutative k-algebra, with $H^0 = k$ and $H^i = 0$ for $i < 0$ (e.g., $H = H^*(G,k)$). Write $\mathfrak{m}$ for the maximal ideal generated by the elements of positive degree. If M is a graded H-module then the local cohomology is doubly graded: $H^{i,j}_{\mathfrak{m}} M$ denotes the part in local cohomological degree i and internal degree j. Local cohomology can either be regarded as the right derived functors of the $\mathfrak{m}$-torsion functor $\Gamma_{\mathfrak{m}}(M) = \{ x \in M \mid \exists\, n \geq 0,\ \mathfrak{m}^n.x = 0 \}$, or as the cohomology of the stable Koszul complex (see for example Theorem 3.5.6 of Bruns and Herzog 6). The a-invariants of M are defined to be

$$a^i_{\mathfrak{m}}(M) = \max\{ j \in \mathbb{Z} \mid H^{i,j}_{\mathfrak{m}} M \neq 0 \}$$

(or $a^i_{\mathfrak{m}}(M) = -\infty$ if $H^i_{\mathfrak{m}} M = 0$). The Castelnuovo–Mumford regularity of M is then defined as

$$\mathsf{Reg}\, M = \max_{i \geq 0} \{ a^i_{\mathfrak{m}}(M) + i \}.$$

Of particular interest is the regularity of the ring itself, $\mathsf{Reg}\, H$.

The reason for the interest in local cohomology of group cohomology comes from the Greenlees version (7) of Benson–Carlson duality (4), in the form of a spectral sequence

$$H^{i,j}_{\mathfrak{m}} H^*(G,k) \Rightarrow H_{-i-j}(G,k).$$

In particular, the existence of the "last survivor" of (4) shows the following (2):

Theorem 10.2. $\mathsf{Reg}\, H^*(G,k) \geq 0$.

The regularity conjecture is known to hold in the following situations:

- $H^*(G,k)$ is Cohen–Macaulay (e.g., groups with abelian Sylow p-subgroups; groups with extraspecial Sylow 2-subgroups with $p = 2$; groups of Lie type with characteristic coprime to p) (1)

- Krull dimension minus depth at most two (e.g., 2-groups of order ≤ 64) (2)
- Symmetric and alternating groups in any characteristic (these are examples where Krull dimension minus depth is arbitrarily large) (3)

There is also a corresponding conjecture for compact Lie groups. Let G be a compact Lie group of dimension d, and suppose that the adjoint action of G on Lie(G) preserves orientation. Then there is a spectral sequence (Benson–Greenlees (5))

$$H^{i,j}_{\mathfrak{m}} H^*(BG; k) \Rightarrow H_{-i-j-d}(BG; k).$$

Conjecture 10.3. $\mathsf{Reg}\, H^*(BG; k) = -d$.

To explain the orientation condition, let $G = T^3 \rtimes \mathbb{Z}/2$, a semidirect product of a 3-torus by an involution acting through inversion, and k be a field of characteristic $\neq 2$. Then $H^*(BG; k) = H^*(BT; k)^{\mathbb{Z}/2}$ is Cohen–Macaulay but not Gorenstein, and $\mathsf{Reg}\, H^*(BG; k) = -5$. The appropriate modification in this situation is that if ε denotes the orientation character, then $\mathsf{Reg}\, H^*(BG; \varepsilon) = -d$.

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11. **Vitaly Bergelson**
Questions on amenability

Dear Guido,

As you will -hopefully- discover very soon, retirement means more freedom. In particular, more freedom to think about *anything* you want. I hope that some of the following questions may entertain you.

Fondly, Vitaly

I. Some questions about amenable groups.

Question 11.1. *Is it true that any infinite amenable group contains an infinite abelian subgroup (this is of course of interest only for torsion groups)?*

Question 11.2. *For solvable non-virtually nilpotent groups, is there a canonical way of constructing a Følner sequence ? (Say, in terms of generators or judiciously chosen neighborhoods of the identity.)*

Question 11.3. *Is there a nice characterization of amenability of a group G via the topological algebra in βG, the Stone-Cech compactification of G? (Here the term "topological algebra" refers to properties of left or right ideals, idempotents, etc.)*

Definition 11.4. A set $R \subseteq G \setminus \{e\}$ is said to have *property TR* (for Topological Recurrence) if for every minimal action of G by homeomorphisms $T_g, g \in G$ of a compact metric space X and any open non-empty set $U \subseteq X$ there exists $g \in R$ such that $U \cap T_g U \neq \emptyset$. Here *minimal* means that, for any $x \in X$, $\overline{\{T_g x, g \in G\}} = X$. A set $R \subseteq G \setminus \{e\}$ is said to have *property MR* (for Measurable Recurrence) if for any action of G by measure preserving transformations $T_g, g \in G$ on a probability space $(X, \mathcal{B}, \mu)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $g \in R$ such that $\mu(A \cap T_g A) > 0$.

Conjecture 11.5. *A countable discrete group is amenable if and only if property MR implies property TR (that is, every set of measurable recurrence is a set of topological recurrence).*

There are amenable groups which are minimally almost periodic, i.e. have no non-trivial finite-dimensional unitary representations. In the language of ergodic theory this simply means that any *ergodic* finite measure preserving action of such group is automatically *weakly mixing.* One more equivalent formulation of this property is the following:

Definition 11.6. An amenable group G is minimally almost periodic if for any unitary representation $(U_g)_{g \in G}$ on a Hilbert space $\mathcal{H}$, one has a G-invariant splitting $\mathcal{H} = \mathcal{H}_{inv} \oplus \mathcal{H}_{wm}$, where $\mathcal{H}_{inv} = \{f \in \mathcal{H} : U_g f = f \forall g \in G\}$ and $\mathcal{H}_{wm} = \{f \in \mathcal{H} : \forall \epsilon > 0$ the set $\{g \in G : |\langle U_g f, f\rangle| < \epsilon\}$ is neglectable $\}$, where a set is called *neglectable* if for any Følner sequence $(F_n)_{n \in \mathbf{N}}$ one has $\frac{|S \cap F_n|}{|F_n|} \to 0$ as $n \to \infty$.

Question 11.7. *Are there countable discrete amenable groups for which the neglectable set appearing in the above splitting is always finite? In other words, are there amenable groups G possessing (similarly to, say, $SL(2,\mathbf{R})$) the property of "decay of matrix coefficients" meaning that for any unitary action $U_g : \mathcal{H} \to \mathcal{H}, g \in G$ which has no invariant vectors, one has, for all $f \in \mathcal{H}$, $\langle U_g f, f\rangle \to 0$ as $g \to \infty$.*

II. Some questions on invariant means. One of the many equivalent definitions of amenability for discrete groups is the postulation of the existence of invariant means on the Banach space $B_{\mathbf{R}}(G)$ of bounded real-valued functions on the group G. But even when G is non-amenable, certain important classes of functions on G possess an invariant and even unique mean. For example, by Ryll-Nardzewsky theorem [R-N], if G is any locally compact group, the space $WAP(G)$ of weakly almost periodic functions on G has a unique invariant mean. Since positive definite functions are weakly almost periodic, this implies that there exists a unique on the algebra of functions of the form $\varphi(g) = \langle U_g f_1, f_2\rangle$, where $U_g : \mathcal{H} \to \mathcal{H}, g \in G$ is a unitary representation of G on a Hilbert space $\mathcal{H}$ and $f_1, f_2 \in \mathcal{H}$. One can show (see for example [S]) that any such function φ can also be represented as $\varphi(g) = \int f_1(T_g x) f_2(x)\, d\mu(x)$ where $(T_g)_{g\in G}$ is a measure-preserving action of G on a probability space $(X, \mathcal{B}, \mu)$ and $f_1, f_2 \in L^\infty(X, \mathcal{B}, \mu)$. This makes natural the following question:

Question 11.8. *Let G be a locally compact group, let $k \in \mathbf{N}$ and let $(T_g^{(1)})_{g\in G}, (T_g^{(2)})_{g\in G}, \ldots, (T_g^{(k)})_{g\in G}$ be k commuting measure-preserving actions of G on a probability space $(X, \mathcal{B}, \mu)$. ("Commuting" means that $T_g^{(i)} T_h^{(j)} = T_h^{(j)} T_g^{(i)}$.) Is it true that there exists a unique invariant mean on the algebra of functions on G generated by the functions of the form*

$$\varphi(g) = \int f_0(x) f_1(T_g^{(1)} x) f_2(T_g^{(2)} T_g^{(1)} x) \ldots f_k(T_g^{(k)} \ldots T_g^{(2)} T_g^{(1)} x)\, d\mu,$$

where $f_i \in L^\infty(X, \mathcal{B}, \mu)$, $i = 0, 1, \ldots, k$.

Remark 11.9. The answer is YES for $k = 1$ (as explained above) and, if G is amenable, for $k = 2$ (follows from [BMZ]).

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12. **A. Jon Berrick and Jonathan A. Hillman**

The Whitehead Conjecture and $L^{(2)}$-Betti numbers

In (8), J. H. C. Whitehead asked whether any subcomplex of an aspherical 2-dimensional complex must be aspherical. An affirmative answer to this question is widely known as the Whitehead Conjecture.

Failure of the Whitehead Conjecture implies either

(a) there is a finite 2-complex X with $\pi_2(X) \neq 0$ and a map $f : S^1 \to X$ such that $Y = X \cup_f e^2$ is contractible; or
(b) there is an infinite ascending chain $K_n \subset K_{n+1}$ of finite 2-complexes with $\pi_2(K_n) \neq 0$ for all n and such that $\cup_{n\geq 1} K_n$ is aspherical. (4)

Moreover, a counterexample of the first type implies the existence of a counterexample of the second type (6).

Our concern here is with the finite case of the conjecture, namely the assertion that any subcomplex of a finite aspherical 2-complex is also aspherical. This assertion implies that (a) above does not hold. An interesting question is whether the negation of (a) above is actually equivalent to the finite case of the conjecture.

If X is a finite 2-complex such that $Y = X \cup_f e^2$ is contractible then $\pi = \pi_1(X)$ has a presentation of deficiency 1 (since $\chi(X) = 0$), and π is the normal closure of the element represented by the attaching map f; so π has weight 1. Conversely, the usual 2-complex of any deficiency 1 presentation of a group π of weight 1 is such a finite subcomplex X of a contractible 2-complex (where f corresponds to a normal generator of π). (Every such group π is the group of a 2-knot (5).)

We now introduce L^2-Betti numbers $\beta_i^{(2)}$ (6) into this situation. Relevant facts here include that for a finite 2-complex X,

$$\chi(X) = \chi^{(2)}(X) = \beta_0^{(2)}(X) - \beta_1^{(2)}(X) + \beta_2^{(2)}(X)$$

and $\beta_i^{(2)}(X) = \beta_i^{(2)}(\pi_1(X))$ for $i = 0, 1$. (The question of whether the equality of the two Euler characteristics still holds for finitely dominated X relates to the weak Bass conjecture for $\pi_1(X)$ (1).)

A finite 2-complex X is aspherical if $\chi(X) = 0$ and $\beta_1^{(2)}(X) = 0$ (3). From (2), the L^2-Betti number condition is satisfied if, for instance, $\pi_1(X)$ has an infinite subgroup that is

(i) amenable and ascendant; or
(ii) finitely generated, subnormal and of infinite index.

In fact, in the presence of (i), $\chi(X) = 0$ is necessary and sufficient for asphericity. On the other hand, in the presence of (ii) instead, $\chi(X) = 0$ is not necessary for asphericity. To see this, take for X the classifying space of the group $F(2) \times F(2)$, where $F(2)$ denotes the free group of rank 2; here $\chi(X) = 1$.

If πK is the group of a tame classical knot $K \subset S^3$ then $\beta_1^{(2)}(\pi K) = 0$ (see §4.3 of (7)), and so the 2-complex associated to any deficiency 1 presentation of a classical knot group is aspherical.

On the basis of this modest evidence, we suggest that a better understanding of L^2-Betti numbers may contribute to the finite case of the Whitehead Conjecture.

Question. *If a group π has weight 1 and a finite presentation of deficiency 1, is $\beta_1^{(2)}(\pi) = 0$?*

Neither deficiency 1 alone nor weight 1 alone is enough. The free product $Z * Z/2Z$ has deficiency 1, but it is also a semidirect product $Z * Z/2Z \cong F(2) \rtimes Z/2Z$ and so $\beta_1^{(2)}(Z * Z/2Z) = \frac{1}{2}\beta_1^{(2)}(F(2)) > 0$. The free product $Z/2Z * Z/3Z$ has weight 1 (since equating generators for the free factors kills the group), but its commutator subgroup is free of rank 2 and has index 6, so $\beta_1^{(2)}(Z/2Z * Z/3Z) = \frac{1}{6}\beta_1^{(2)}(F(2)) > 0$. (Neither of these groups is π_1 of a finite aspherical complex.)

On the other hand, Z^2 has deficiency 1 and $\beta_1^{(2)}(Z^2) = 0$, but clearly Z^2 has weight 2. The semidirect product $(Z/3Z) \rtimes_{-1} Z$ has weight 1 and $\beta_1^{(2)}((Z/3Z) \rtimes_{-1} Z) = 0$, but this group has deficiency 0. Thus neither hypothesis "deficiency 1" nor "weight 1" is implied by the conjunction of the other with the condition $\beta_1^{(2)}(\pi) = 0$.

We wish you an entertaining retirement, Guido, and hope you seize opportunities to visit our part of the world.

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13. **Martin R. Bridson and Michael Tweedale**
PUTATIVE RELATION GAPS

Dear Guido,

We thought that you might enjoy the following concrete speculations concerning the efficiency of finite group-presentations.

Given $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$, the action of the free group $F(\mathcal{A})$ by conjugation on $R = \langle\langle \mathcal{R} \rangle\rangle$ induces an action of Γ on the abelian group $M = R/[R,R]$. It is obvious that the rank of M as a $\mathbf{Z}\Gamma$-module serves as a lower bound on the minimal number of relators that one requires to present Γ on the generators $\mathcal{A}$. This lower bound seems so crude that one cannot imagine it would be sharp in general. And yet, despite sustained attack over many years, not a single example has been established to lend substance to this intuition. The question of whether or not there exists such an example has become known as the *relation gap problem.*

It belongs to a circle of notoriously hard problems concerning the homotopy properties of finite 2-complexes — the Andrews–Curtis conjecture, Whitehead's asphericity conjecture, the Eilenberg–Ganea conjecture, and the question (resolved by Bestvina and Brady) of finite presentability versus FP_2.

To clarify our convictions regarding the relation gap problem we state:

Conjecture 13.1. *There exist finitely presented groups with arbitrarily large relation gaps.*

This conjecture is closely related to the $D(2)$ conjecture: if a group Γ with $H^3(\Gamma; \mathbf{Z}\Gamma) = 0$ admits a presentation that both realizes the group's deficiency and has a relation gap, then the $D(2)$ conjecture is false, i.e. there exists a finite 3-complex that looks homologically like a 2-complex, in the sense that it possesses Wall's property $D(2)$, but that does not have the homotopy type of a 2-complex[17].

In the remainder of this note, we'll describe two families of groups and indicate why we think that they ought to have relation gaps, making explicit conjectures to that effect. The two families are of a very different nature: the first consists of groups with finite classifying spaces, based on the Bestvina–Brady construction; the second is comprised of virtually free groups and it is the nature of the torsion that dictates the key features of the relation module that we believe lead to a relation gap.

Cyclic coverings and right-angled Artin groups. Let Σ be a connected flag complex with non-trivial, perfect fundamental group, and let G be the associated right-angled Artin group. This group has a presentation with generating set the vertices v_i of Σ, and defining

[17]The result is due to M. Dyer; for a proof, see J. Harlander (MR1751092).

relations asserting that two generators commute if and only if the corresponding vertices in Σ are joined by an edge. Let $\pi : G \to \mathbf{Z}$ be the homomorphism sending each v_i to a fixed generator. The kernel of π is the Bestvina–Brady group H_Σ, which is FP_2 but not finitely presented.

Let $\Gamma_n \subset G$ be the index n subgroup $\pi^{-1}(n\mathbf{Z})$. Notice that H_Σ is the intersection of the Γ_n. We construct a presentation for Γ_n where the generators S_n are indexed by the vertices of Σ and the $\mathbf{Z}\Gamma$-rank of the relation module is bounded independently of n.

Conjecture 13.2. *The number of relators needed to present Γ_n on the generators S_n goes to infinity as $n \to \infty$, so Γ_n has a relation gap for n sufficiently large.*

Some virtually free examples. We consider groups similar in spirit to ones considered by D. Epstein, C. Hog-Angeloni, W. Metzler and M. Lustig, and more recently by K. Gruenberg and P. Linnell.

Given letters x_m and t_m, let ρ_m be the word

$$\rho_m = (t_m x_m t_m^{-1}) x_m (t_m x_m^{-1} t_m^{-1}) x_m^{-m}.$$

We look at the groups $\Gamma_{m,n} = Q_m * Q_n$, where

$$Q_m = \left\langle x_m, t_m \;\middle|\; \rho_m, x_m^{m-1} \right\rangle$$

and $(m^{m-1} - 1)$ and $(n^{n-1} - 1)$ are coprime.

One of the main attractions we see in these new examples is that one can give a short, transparent and natural proof that the relation module of the obvious presentation of $\Gamma_{m,n}$ can be generated by three elements, namely the images of ρ_m, ρ_n and $x_m^{m-1} x_n^{n-1}$.

The groups $\Gamma = \Gamma_{m,n}$ are virtually free, so $H^3(\Gamma; \mathbf{Z}\Gamma) = 0$. Thus one could find both a relation gap and a counterexample to the $D(2)$ conjecture simply by solving the following concrete problem:

Conjecture 13.3. *The kernel of the map $F_4 \to \Gamma$ associated to the presentation*

$$\Gamma = \left\langle x_m, t_m, x_n, t_m \;\middle|\; \rho_m, x_m^{m-1}, \rho_n, x_n^{n-1} \right\rangle$$

is not the normal closure of three elements.

14. **Jacek Brodzki, Graham A. Niblo and Nick Wright**
Property A and exactness of the uniform Roe algebra

It has long been established that certain properties of groups can be described through properties of suitably chosen C^*-algebras associated with them. A model result in this direction is a theorem of Lance that a discrete group is amenable if and only if its reduced C^*-algebra $C^*_r(G)$ is nuclear.

Property A was introduced by Yu as a geometric analogue of the Følner criterion that describes amenability of a group. It implies many of the interesting consequences of amenability for a discrete group, for example, property A implies uniform embeddability in Hilbert space, which in turn gives the Coarse Baum-Connes conjecture and therefore the Novikov conjecture [18].

Property A and the uniform Roe algebra can be defined for arbitrary metric spaces. Let us recall the main definitions.

A uniformly discrete metric space (X, d) has *property A* if for all $R, \epsilon > 0$ there exists a family of finite non-empty subsets A_x of $X \times \mathbb{N}$, indexed by x in X, such that

- for all x, y with $d(x, y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \epsilon$;
- there exists S such that for all x and $(y, n) \in A_x$ we have $d(x, y) \leq S$.

The *uniform Roe algebra*, $C^*_u(X)$, is the C^*-algebra completion of the algebra of bounded operators on $l^2(X)$ which have finite propagation. The details are as follows. A kernel $u\colon X \times X \to \mathbb{C}$ has *finite propagation* if there exists $R \geq 0$ such that $u(x, y) = 0$ for $d(x, y) > R$. If X is a proper discrete metric space, and $u\colon X \times X \to \mathbb{C}$ is a finite propagation kernel then for each x there are only finitely many y with $u(x, y) \neq 0$. Thus u defines a linear map from $l^2(X)$ to itself, $u * \xi(x) = \sum_{y \in X} u(x, y)\xi(y)$. Note that if additionally X has bounded geometry, then every bounded finite propagation kernel gives rise to a *bounded* operator on $l^2(X)$. The uniform Roe algebra is the completion of the algebra generated by bounded linear operators arising from bounded propagation kernels.

For a discrete group G, Yu's property A is equivalent both to the nuclearity of the uniform Roe algebra $C^*_u(G)$ and to the exactness of the reduced C^*-algebra $C^*_r(G)$. This follows from the results of

[18] G.L. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into a Hilbert space*, Inventiones Mathematicae 138 (2000), 201–240.

Anantharaman-Delaroche and Renault [19], Higson and Roe [20], Guentner and Kaminker [21], and Ozawa [22].

It is natural to state the following conjecture.

Conjecture 14.1. *A uniformly discrete bounded geometry metric space X has property A if and only if the uniform Roe algebra $C^*_u(X)$ is exact.*

In evidence for the conjecture we offer the following. The conjecture is true for any countable discrete group equipped with its natural coarse structure. It is then an easy exercise to show that the conjecture holds for any metric space which admits a proper co-compact action by a group of isometries.

We proved recently [23] that the conjecture also holds if the space is sufficiently group-like in the following sense.

One of the key ingredients in the proof of the conjecture for groups is the interplay between the left and the right action of a group on itself. By convention, the left action is by isometries while the right action has the curious property that each point is moved by the same distance by a given element of the group. By analogy with Euclidean geometry we call such transformations *translations* even though they are not in general isometries. It is often overlooked that it is the translation action, rather than the isometric action, that allows one to identify $C^*_r(G)$ with a subalgebra of the uniform Roe algebra $C^*_u(G)$. One may abstract from this the notion of a translation structure for a space. We can then say that a space is more or less group-like depending on how much this structure resembles the natural left-right multiplication structure on a group.

When the space X is sufficiently group-like in this sense, the conjecture holds for X. For example, this is the case when X embeds uniformly in a group.

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15. **Michelle Bucher-Karlsson and Anders Karlsson**
Volumes of ideal simplices in Hilbert's geometry and symmetric spaces

Dear Guido,

We, the two authors of this note, first met each other in a weekly seminar that you organized at ETHZ and our first joint publication was in fact an outgrowth of one of those very appreciated seminars. It is therefore an extra pleasure for us to jointly contribute to the present volume. We learnt a lot from you in our years in Zurich and maybe you can help us once more with the following geometric question we are interested in:

Let X be a bounded convex domain in $\mathbb{R}^n$ endowed with its natural Hilbert's metric d, where $d(x,y)$ equals the logarithm of the projective cross-ratio of x and y, and the endpoints of the chord through x and y. It is also a Finslerian metric and as such there is a natural notion of volume.

The question is to study the volume of ideal simplices in this geometry. Of particular interest is to find out which admit the minimal or maximal volume, and whether the volume at all is bounded. Two examples:

- Let X be a ball in $\mathbb{R}^n$. Then X with the Hilbert metric is nothing but the n-dimensional hyperbolic space. It is a well known result of Haagerup and Munkholm that the volume of an ideal geodesic simplex is maximal if and only if the simplex is regular, i.e. any permutation of its vertices can be realized by an isometry.
- Let X be triangle in $\mathbb{R}^2$. To avoid degeneracies, we restrict our attention to simplices which have their three vertices on the three different 1-faces of X. Colbois, Vernicos and Verovic showed that, while the volume (or area) of ideal triangles is unbounded, its minimum is attained by the simplex having as vertices the midpoints of the 1-faces of X - and its isometric copies of course. This can be given an elementary proof without any computation and without knowing the exact definition of the invariant volume, once it is observed that those simplices are here again, the regular ones.

Thus, the slogan we would like to advocate here, is that "most regular" simplices have extremal volume.

The symmetric space $SL(n,\mathbb{R})/SO(n)$ has a natural model as a bounded, convex domain in $\mathbb{R}^{n(n+1)/2}$, namely as the positive definite matrices normalized to have trace equal to 1. The action of $SL(n,\mathbb{R})$ on $SL(n,\mathbb{R})/SO(n)$ is then given by the projective transformations $S \mapsto g \cdot S = (1/\mathrm{tr}(gSg^t))gSg^t$. Note that the Riemannian symmetric metric does not coincide with the Hilbert metric. The Hilbert volume or the Riemannian volume can be used to define a cocycle

$$V : (g_0, ..., g_d) \longmapsto \mathrm{Vol}\langle g_0 \cdot x, ..., g_d \cdot x\rangle$$

mapping the $(d+1)$-tuple of elements of $SL(n,\mathbb{R})$, where d denotes the dimension of $SL(n,\mathbb{R})/SO(n)$, to the volume of the convex simplex with vertices the orbit of a fixed point x in $SL(n,\mathbb{R})/SO(n)$. The cocycle V represents the top dimensional generator of the real valued continuous cohomology $H_c^*(SL(n,\mathbb{R}))$ of $SL(n,\mathbb{R})$.

Question 15.1. *Is the cocycle V uniformly bounded?*

A positive answer to this question would give a new unified proof of the fact that compact manifold whose universal cover is isometric to the symmetric space $SL(n,\mathbb{R})/SO(n)$ have strictly positive simplicial volume, proven by Thurston for $n = 2$, by the first-named author of this note for $n = 3$ and by Lafont and Schmidt for $n \geq 4$.

The behavior of the Hilbert volume in $SL(n,\mathbb{R})/SO(n)$ is mixed, since the latter space contains both isometric copies of the hyperbolic space and of triangles.

Question 15.2. *Which convex simplices in $SL(n,\mathbb{R})/SO(n)$ have extremal volume?*

The understanding of these questions also in the cases of lower-dimensional simplices and volumes may furthermore yield insights, or the solution of, the conjecture on the surjectivity of the comparison map

$$H_{c,b}^*(SL(n,\mathbb{R})) \to H_c^*(SL(n,\mathbb{R}))$$

for $SL(n,\mathbb{R})$ discussed in this volume by Burger, Iozzi, Monod and Wienhard.

16. **Marc Burger, Alessandra Iozzi, Nicolas Monod and Anna Wienhard**

Bounds for cohomology classes

Let G be a simple Lie group (connected and with finite centre). Consider the continuous cohomology $\mathrm{H}^*(G, \mathbf{R})$ of G, which can be defined for instance with the familiar bar-resolutions of the Eilenberg–MacLane cohomology, except that the cochains are required to be *continuous* maps on G (or equivalently smooth or just measurable).

Conjecture 16.1. *Every cohomology class of* $\mathrm{H}^*(G, \mathbf{R})$ *is* bounded*, i.e. is represented by a bounded cocycle.*

Recall that $\mathrm{H}^*(G, \mathbf{R})$ is isomorphic to the algebra of invariant differential forms on the symmetric space associated to G, hence to a relative cohomology of Lie algebras and thus moreover to the cohomology of the *compact dual space* associated to G. It is however not understood how these isomorphisms interact with boundedness of cochains (compare Dupont[24]).

We emphasise also that, unlike for discrete groups, $\mathrm{H}^*(G, \mathbf{R})$ does not coincide with the cohomology of the classifying space BG. There is however a natural transformation $\mathrm{H}^*(BG, \mathbf{R}) \to \mathrm{H}^*(G, \mathbf{R})$ and we refer to its image as the *primary characteristic classes*. By a difficult result of M. Gromov[25], the latter are indeed bounded; M. Bucher-Karlsson gave a simpler proof of this fact in her thesis[26].

In order to prove the above conjecture, it would suffice to establish the boundedness of the *secondary invariants* of Cheeger–Simons; indeed, Dupont–Kamber proved that the latter together with the primary classes generate $\mathrm{H}^*(G, \mathbf{R})$ as an algebra.

An important example where boundedness was established very recently is the class of the *volume form* of the associated symmetric space. Using estimates by Connell–Farb[27], Lafont–Schmidt[28] provided bounded cocycles in all cases except $\mathrm{SL}_3(\mathbf{R})$, the latter case being settled by M. Bucher-Karlsson[29] (a previous proof of R. Savage is incorrect). It follows that the fundamental class of closed locally symmetric spaces is bounded; as explained by M. Gromov, this provides a nonzero lower bound for the *minimal volume* of such a manifold, i.e. a non-trivial lower bound for its volume with respect to *any* (suitably normalised) Riemannian metric.

[24] in *Algebraic topology, Aarhus 1978*, 109–119, Springer 1979.

[25] *Publ. Math. IHÉS*, (56):5–99, 1982.

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Many more questions are related to the above conjecture via the following steps listed in increasing order of refinement: (i) find a bounded cocycle representing a given class; (ii) establish a sharp numerical bound for that class; (iii) determine the equivalence class of the cocycle up to boundaries of bounded cochains only.

The latter point leads to introduce the (continuous) *bounded cohomology* $\mathrm{H}^*_{\mathrm{b}}$ of groups or spaces, where all cochains are required to be bounded. There is then an obvious natural transformation

$$\mathrm{H}^*_{\mathrm{b}}(-, \mathbf{R}) \longrightarrow \mathrm{H}^*(-, \mathbf{R}) \qquad (*)$$

and the above conjecture amounts to the surjectivity of that map for a connected simple Lie group with finite centre. As of now, there is not a single simple Lie group for which $\mathrm{H}^*_{\mathrm{b}}(G, \mathbf{R})$ is known; all the partial results are however consistent with a positive answer to the following:

Question 16.2. *Is the map* $(*)$ *an isomorphism?*

For instance, the answer is yes in degree two[30] (and trivially yes in degrees $0, 1$); for $G = \mathbf{SL}_n(\mathbf{R})$, it is also yes in degree three[31].

The functor $\mathrm{H}^*_{\mathrm{b}}$ is quite interesting for discrete groups as well and has found applications notably to representation theory, dynamics, geometry and ergodic theory. This notwithstanding, *there is not a single countable group for which* $\mathrm{H}^*_{\mathrm{b}}(-, \mathbf{R})$ *is known,* unless it is known to vanish in all degrees (e.g. for amenable groups). In any case, the map $(*)$ fails dramatically either to be injective or surjective in many examples. Most known results regard the degree two, with for instance a large supply of groups having an infinite-dimensional $\mathrm{H}^2_{\mathrm{b}}(-, \mathbf{R})$, including the non-Abelian free group F_2. Interestingly, the surjectivity of the map $(*)$ (with more general coefficients) in degree two *characterises non-elementary Gromov-hyperbolic groups* (Mineyev[32]).

It appears that new techniques are required in higher degree. Here is a test on which to try them:

Question 16.3. *For which degrees n is* $\mathrm{H}^n_{\mathrm{b}}(F_2, \mathbf{R})$ *non-trivial?*

It is known to be non-trivial for $n = 2, 3$. (Triviality for $n = 1$ and non-triviality for $n = 0$ are elementary to check for any group.)

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[30]see Burger–Monod, *Geom. Funct. Anal.*, **12**(2):219–280, 2002.

[31]For $n = 2$: Burger–Monod, in *Rigidity in dynamics and geometry*, 19–37, Springer 2002. For $n \geq 3$: N. Monod, *Contemp. Math.* **347**:191–202, AMS 2004.

[32]*Q. J. Math.*, 53(1):59–73, 2002.

17. **Cornelia M. Busch**

Worry about primes in Farrell cohomology

Dear Guido,

we suffered headache while I was thinking about the following question.

Question 17.1. *Let p be any odd prime, and let $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$ denote the group of symplectic $(p-1) \times (p-1)$-matrices over $\mathbb{Z}[1/n]$, where $0 \neq n \in \mathbb{Z}$ is any nonzero integer. What is the p-period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z})\,?$$

Here are some partial answers.

The symplectic group $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$ has finite virtual cohomological dimension and moreover each elementary abelian subgroup has rank ≤ 1. There are nontrivial subgroups of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$. The following property was proven by K. S. Brown[33]:

Let G be a group with finite virtual cohomological dimension and such that each elementary abelian p-subgroup of G has rank ≤ 1. Then

$$\widehat{\mathrm{H}}^*(G, \mathbb{Z})_{(p)} \cong \prod_{P \in \mathfrak{P}} \widehat{\mathrm{H}}^*(N(P), \mathbb{Z})_{(p)},$$

where $\mathfrak{P}$ is a set of representatives of conjugacy classes of subgroups P of order p in G and $N(P)$ is the normalizer of P. Here $\widehat{\mathrm{H}}^*(G, \mathbb{Z})_{(p)}$ denotes the p-primary part of the Farrell cohomology of G with coefficients in $\mathbb{Z}$.

In order to use this property, we analyze the structure of the normalizer of subgroups of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$. The conjugacy classes of elements of order p in $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$ are related to the ideal classes in $\mathbb{Z}[1/n][\xi]$, where ξ is a primitive pth root of unity. First we consider the case $n = 1$. The number of conjugacy classes of elements of order p in $\mathrm{Sp}(p-1, \mathbb{Z})$ depends on h^-, the relative class number of the cyclotomic field $\mathbb{Q}(\xi)$. We get the following result[34].

Theorem 17.2. *Let p be an odd prime for which the relative class number h^- is odd and let y be such that $p - 1 = 2^r y$ with y odd. Then the period of $\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbb{Z}), \mathbb{Z})_{(p)}$ equals $2y$.*

The smallest prime p for which h^- is even is $p = 29$. In fact it isn't known if the statement of this theorem is also true for primes with even relative class number. We can avoid the class number by replacing the ring $\mathbb{Z}$ with $\mathbb{Z}[1/n]$, $0 \neq n \in \mathbb{Z}$, because it is possible to choose the

[33]K. S. Brown, *Cohomology of Groups*, GTM 87, Springer 1982.

[34]C. M. Busch, *The Farrell cohomology of* $\mathrm{Sp}(p-1, \mathbb{Z})$, Documenta Mathematica 7 (2002), 239-254.

integer n such that $\mathbb{Z}[1/n][\xi]$ is a principal ideal domain. Then we get the following theorem[35].

Theorem 17.3. *Let p be an odd prime. Let n be such that $\mathbb{Z}[1/n][\xi]$ and $\mathbb{Z}[1/n][\xi+\xi^{-1}]$ are principal ideal domains and moreover $p \mid n$. Let y be the greatest odd divisor of $p-1$. Then the p-period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z})$$

is y if and only if for each $j \mid y$ a prime $q \mid n$ exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$. If for some j no such q exists, the p-period is $2y$.

For every $f \neq p$ an infinite number of primes q exist with $q^f \equiv 1 \mod p$ because of the theorem of arithmetic progression. Therefore it is possible to choose n such that for every $f_q \mid p-1$ a prime q exists with $q \mid n$ and f_q is the inertia degree of q.

Now we can try to guess the answer to Question 17.1, but there is still a long way to go to get the proof.

Conjecture 17.4. *Let p be an odd prime and let y be the greatest odd divisor of $p-1$. The p-period of the Farrell cohomology ring*

$$\widehat{\mathrm{H}}^*(\mathrm{Sp}(p-1, \mathbb{Z}[1/n]), \mathbb{Z})$$

is y if and only if for each $j \mid y$ a prime $q \mid n$ exists with inertia degree f_q such that $j \mid \frac{p-1}{2f_q}$. If for some j no such q exists, the p-period is $2y$.

[35]C. M. Busch, *On p-periodicity in the Farrell cohomology of* $\mathrm{Sp}(p-1, \mathbb{Z}[1/n])$, arXiv math.GR/0509693 (2005).

18. **Kai-Uwe Bux**
An FP_m-Conjecture for Nilpotent-by-Abelian Groups

Let G be a finitely generated metabelian group, i.e., we have short exact sequence

$$N \longrightarrow G \longrightarrow Q$$

of Abelian groups, wherein the quotient Q in finitely generated and the kernel N is finitely generated as an $\mathbf{Z}Q$-module. For any homomorphism $\chi : Q \to \mathbf{R}$, let $Q_\chi := \{q \in Q | \chi(q) \geq 0\}$ be the monoid of elements in Q non-negative with respect to χ. R. Bieri and R. Strebel defined the geometric invariant of G as

$$\Sigma_Q(N) := \{\chi \in \mathrm{Hom}(Q, \mathbf{R}) | N \text{ is finitely generated over } \mathbf{Z}Q_\chi\}$$

Note that homomorphisms that are positive scalar multiples of one another define the same non-negative sub-monoid of Q. Thus, the geometric invariant is a conical subset of the real vector space $\mathrm{Hom}(Q, \mathrm{R})$. Also note that $Q_0 = Q$, whence the geometric invariant contains 0 since G is finitely generated.

Bieri-Strebel showed that $\Sigma_Q(N)$ determines whether G is finitely presented. However, this information is more easily extracted from the complement

$$\Sigma_Q^c(N) := \Sigma_Q(N) - \mathrm{Hom}(Q, \mathrm{R}).$$

Theorem 18.1. (Bieri-Strebel, [BiSt80]) *The following are equivalent:*

(1) *G is finitely presented.*
(2) *G is of type FP_2.*
(3) *The complement $\Sigma_Q^c(N)$ does not contain two antipodal points, i.e., whenever $\chi \in \Sigma_Q^c(N)$, then $-\chi \notin \Sigma_Q^c(N)$.*

Bieri conjectured that the information about higher finiteness properties of G is also encoded in $\Sigma_Q^c(N)$. Recall that a group G is of type FP_m if there is a partial resolution

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \twoheadrightarrow \mathbf{Z}$$

of $\mathbf{Z}$, regarded as the trivial $\mathbf{Z}G$-module, by finitely generated projective $\mathbf{Z}G$-modules.

Conjecture 18.2 (Bieri)**.** *For any $m \geq 2$, the following are equivalent:*

(1) *G is of type FP_m.*
(2) *The complement $\Sigma_Q^c(N)$ is m-tame.*

Here, we call a conical subset U of a real vector space *m-tame* if

$$0 \notin \underbrace{U + U + \cdots + U}_{m \text{ summands}}$$

Evidence for this conjecture is mounting. It has been proved for many special cases. In particular, H.Åberg settled the case when N is virtually torsion free of finite rank [Åb86], and the case $m = 3$ was settled by R. Bieri and J. Harlander for the case of split extensions [BiHa98].

Now, let G be nilpotent-by-Abelian, i.e., suppose G fits into a short exact sequence

$$N \longrightarrow G \longrightarrow Q$$

where N is nilpotent and Q is Abelian. Again, we assume that G is finitely generated. In that case, every Abelian factor $M_i := N_i/N_{i+1}$ along the lower central series $N = N_1 > N_2 > N_3 > \cdots$ is a finitely generated $\mathbf{Z}Q$-module to which we can associate, as above, a geometric invariant $\Sigma_Q(M_i)$ and a complement denoted by $\Sigma_Q^c(M_i)$.

Note that a necessary condition for G to be of type FP_m is that the homology groups $\mathrm{H}_i(G;\mathbf{Z})$ are finitely generated in dimensions up to m. Therefore, the most optimistic and most straight forward generalization of the FP_m-conjecture to the class of nilpotent-by-Abelian groups would be that the metabelian quotient of G contains all of the relevant information needed besides the obvious homological restrictions. We thus arrive at:

Conjecture 18.3. *For $m \geq 2$, the following are equivalent:*

(1) *G is of type FP_m.*
(2) *The complement $\Sigma_Q^c(M_1)$ is m-tame and the homology groups $\mathrm{Hom}_i(N;\mathbf{Z})$ are finitely generated as $\mathbf{Z}Q$-modules for all dimensions $i \in \{1, 2, \ldots, m\}$.*

Surprisingly, this very optimistic conjecture has some support: By results of H. Abels, the conjecture holds for $m = 2$ if G is a solvable S-arithmetic group over a number field [Ab87]. My own results on solvable S-arithmetic groups over function fields are also compatible with the conjecture [Bu04]. However, the conjecture appears too optimistic, so a better question might be: is there a way to characterize the higher FP_m-properties of a nilpotent-by-Abelian group G in terms of its homology and the geometric invariants of the modules M_i?

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19. **Carles Casacuberta and Manuel Castellet**
Postnikov pieces of finite dimension

Dear Guido,

Perhaps you know the answer to the following question. *Can the higher homotopy groups of a finite CW-complex X be nonzero vector spaces over the field $\mathbb{Q}$ of rationals?*

Of course you know that the answer is negative if X is nilpotent. On the other hand, if X is not nilpotent, then the higher homotopy groups of X need not be finitely generated, not even as modules over the integral group ring of the fundamental group of X (Stallings gave a counterexample in 1963). Hence, the question does make sense.

What is your intuition about it? At first we thought that it is very unlikely that anybody finds a finite CW-complex X such that, say, $\pi_2(X) \cong \mathbb{Q}$. However, we have been unable to discard this possibility in spite of considerable effort, so we would not be surprised if such an example existed.

Let us now explain why this question is relevant and give a reason why we would like the answer to be negative.

Finite-dimensional Postnikov pieces

Are there CW-complexes of finite dimension with only a finite number of nonzero homotopy groups? Yes, indeed: a wedge of circles, a torus or a Klein bottle are examples, as well as any compact surface of higher genus. Note that each of these examples is a $K(G,1)$. Rational spheres and rational complex projective spaces are examples with nontrivial higher homotopy groups. These admit finite-dimensional models, yet surely not finite.

The following theorem appeared in [C. Casacuberta, On Postnikov pieces of finite dimension, *Collect. Math.* **49** (1998), 257–263]: *If X is a CW-complex of finite dimension with only a finite number of nonzero Postnikov invariants, then X is a Postnikov piece and its higher homotopy groups $\pi_n(X)$ are $\mathbb{Q}$-vector spaces for $n \geq 2$.*

The proof uses homotopical localization with respect to the map from $B\mathbb{Z}/p$ to a point, where p is any prime, together with Miller's famous proof of the Sullivan conjecture. Alternatively, it follows from a result in [C. A. McGibbon and J. A. Neisendorfer, On the homotopy groups of a finite-dimensional space, *Comment. Math. Helv.* **59** (1984), 253–257], which relies on a similar line of argument. Thus, it is not surprising that our question has implications around Serre's theorem about the higher homotopy groups of 1-connected finite CW-complexes. Now let us be optimistic and state the following.

Conjecture 19.1. *If X is a finite CW-complex that is a Postnikov piece, then X is a $K(G,1)$.*

By the theorem mentioned above, if X is a finite CW-complex that is a Postnikov piece, then the higher homotopy groups of X are $\mathbb{Q}$-vector spaces. Hence, it seems that the fate of this conjecture will depend on our ability to find finite CW-complexes whose higher homotopy groups are nonzero $\mathbb{Q}$-vector spaces. Do you know any? It would be very exciting to generalize Serre's old theorem along these lines, if the conjecture were true. So far we can only report that the conjecture holds (rather easily) if the dimension of X is less than or equal to 3.

We are very pleased to include this problem in your gift book, since it is much related to a number of topics that we learnt from your beautiful monograph with Peter Hilton and Joe Roitberg many years ago.

We wish you all the best, most sincerely, on your retirement.

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20. **Natàlia Castellana, Juan A. Crespo and Jérôme Scherer**
Cohomological finiteness conditions: spaces versus H-spaces

We wish to ask a very naive and classically flavored question. Consider a finite complex X and an integer n. Does its n-connected cover $X\langle n\rangle$ satisfy any cohomological finiteness property? When X is an H-space we have an answer.

Theorem 20.1. *Let X be a finite H-space and n an integer. Then $H^*(X\langle n\rangle; \mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra.*

This leads naturally to ask whether the same statement holds for arbitrary spaces.

Question 20.2. *Let X be a finite space and n an integer. Is it true that $H^*(X\langle n\rangle; \mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra?*

Because the "difference" between a finite complex and its n-connected cover is a finite Postnikov piece, a first step towards a solution to this question would be to understand the cohomological finiteness properties of finite type Postnikov pieces.

Question 20.3. *Is the cohomology of a finite type Postnikov piece finitely generated as an algebra over the Steenrod algebra?*

Again, we know the answer is yes if the Postnikov piece is an H-space. The proof of Theorem 20.1 is based on the analysis of the Eilenberg-Moore spectral sequence, and the following algebraic result, whose proof relies deeply on the Borel-Hopf structure theorem for Hopf algebras.

Theorem 20.4. *Let A be an unstable Hopf algebra which is finitely generated as an algebra over the Steenrod algebra. Then so is any unstable Hopf subalgebra $B \subset A$.*

For plain unstable algebras, this is false, as pointed out to us by Hans-Werner Henn. Consider indeed the unstable algebra

$$H^*(\mathbf{C}P^\infty \times S^2; \mathbf{F}_p) \cong \mathbf{F}_p[x] \otimes E(y)$$

where both x and y have degree 2. Take the ideal generated by y, and add 1 to turn it into an unstable subalgebra. Since $y^2 = 0$, this is isomorphic, as an unstable algebra, to $\mathbf{F}_p \oplus \Sigma^2\mathbf{F}_p \oplus \Sigma^2\widetilde{H}^*(\mathbf{C}P^\infty; \mathbf{F}_p)$, which is not finitely generated. Theorem 20.4 is the only result where we fully understand the general situation!

From where do these questions come from? The condition that $H^*(X; \mathbf{F}_p)$ is finitely generated as an algebra over the Steenrod algebra is equivalent to the fact that the unstable module of indecomposable elements $QH^*(X;\mathbf{F}_p)$ is finitely generated as module over the Steenrod algebra. This guarantees that $QH^*(X;\mathbf{F}_p)$ lives in the Krull filtration of the category $\mathcal{U}$ of unstable modules, which was introduced by Lionel Schwartz. An unstable module M lives in $\mathcal{U}_n$ if and only if $\bar{T}^{n+1}M = 0$, where $\bar{T}$ denotes Lannes' reduced T functor. This algebraic filtration can be compared with Bousfield's nullification filtration with respect to $B\mathbf{Z}/p$.

Theorem 20.5. *Let X be a connected H-space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V. Then $QH^*(X)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbf{Z}/p$-local.*

Bill Dwyer and Clarence Wilkerson have shown that the case $n = 0$ holds for arbitrary spaces. However, our methods rely so deeply on the H-structure that we still don't know if one should look for a positive or negative answer to our last question.

Question 20.6. *Let X be a connected space such that $T_V H^*(X)$ is of finite type for any elementary abelian p-group V, and let $n \geq 1$. Is it true that $QH^*(X)$ is in $\mathcal{U}_n$ if and only if $\Omega^{n+1}X$ is $B\mathbf{Z}/p$-local?*

21. **Ruth Charney and Karen Vogtmann**

Automorphism groups of right-angled Artin groups

Dear Guido,

We have been thinking lately about outer automorphism groups of right-angled Artin groups ("RAAGs"). Since F_n and Z^n are examples of RAAGs, it is tempting to view outer automorphism groups of general right-angled Artin groups as interpolating between $Out(F_n)$ and $GL(n, Z)$, and to ask to what extent properties common to both $Out(F_n)$ and $GL(n, Z)$ are true in general.

Recall that a RAAG A_Γ based on a simplicial graph Γ is generated by the vertices of Γ, and the only relations are that v commutes with w if v and w are joined by an edge of Γ. Laurence [L] gave a set of generators for $Aut(A_\Gamma)$, but not much else is known about this group except in the cases when Γ is discrete (so A_Γ is free) and Γ is the complete graph (so A_Γ is free abelian). Laurence's generators are of four types: (1) inner automorphisms, (2) symmetries of Γ and inversions of the generators v, (3) *partial conjugations*, which conjugate everything in some connected component of $\Gamma - st(v)$ by v, and (4) *transvections*, which multiply v by w (on the right or left) if $lk(v) \subseteq st(w)$. Thus every automorphism of A_Γ lifts to an automorphism of the free group on the vertices of Γ, but the natural map from $Out(A_\Gamma)$ to $GL(n, Z)$ is not usually surjective.

There is a CAT(0) cube complex associated to any RAAG, whose 1-skeleton is the Cayley graph of the group, and which has a k-dimensional cube whenever the 1-skeleton of the cube appears (kind of a "cube-flag" complex). The RAAG acts freely on this; the quotient has a loop for each generator and a k-torus for each complete subgraph on k vertices in Γ. This cube complex is 2-dimensional if and only if the graph Γ has no triangles. In this case, we have constructed an "outer space" for $Out(A_\Gamma)$, which is a contractible space on which $Out(A_\Gamma)$ acts with finite stabilizers. Points in this outer space are morally actions of A_Γ on 2-dimensional CAT(0) complexes, though the actual description is in terms of products of trees. This outer space is finite-dimensional, and we obtain

Theorem 21.1. *If Γ is connected and triangle-free, then $Out(A_\Gamma)$ contains a torsion-free subgroup of finite index and it has finite virtual cohomological dimension.*

Although our outer space is finite-dimensional, its dimension is quite large, and is unlikely to be the actual virtual cohomological dimension, so we ask

Question 21.2. *What is the exact virtual cohomological dimension of the outer automorphism group of a 2-dimensional right-angled Artin group?*

The no-triangles condition on Γ was very convenient, but of course we would like to know what happens for any Γ:

Question 21.3. *Do the outer automorphism groups of all right-angled Artin groups have torsion-free subgroups of finite index?*

Question 21.4. *Calculate the virtual cohomological dimension of the automorphism group of any right-angled Artin group.*

Best wishes on your retirement,
Ruth and Karen

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22. **Yuqing Chen**
Planar 2-cocycles of finite groups

Let G be a finite group and A a G-module. Recall that a (normalized) 2-cocycle of G with coefficients in A is a function

$$f : G \times G \to A$$

satisfying

(i) $f(g, 1) = f(1, g) = 0$, for all $g \in G$;
(ii) $f(g, h) + f(gh, k) = gf(h, k) + f(g, hk)$ for all $g, h, k \in G$.

Definition 22.1. A 2-cocycle of G with coefficients in A is called planar (the extension group acts on a finite projective plane as a collineation group) if

(i) $|G| = |A|$;
(ii) for every $1 \neq g \in G$, the maps

$$f(g, \quad) : G \to A$$

and

$$f(\quad, g) : G \to A$$

are bijections.

Example 22.2. Let $\mathbb{F}$ be a finite field. We can regard $\mathbb{F}$ as a trivial $\mathbb{F}$-module. For any Galois automorphism σ, we define $f_\sigma : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ by

$$f_\sigma(x, y) = x\sigma(y).$$

The function f_σ is a planar 2-cocycle of the additive group of $\mathbb{F}$ with coefficients in the same group.

Conjecture 22.3. *If G has a planar 2-cocycle, then G is a p-group.*

Conjecture 22.4. (*Stronger version*) *If G has a planar 2-cocycle with coefficients in a G-module A, then G and A are elementary abelian groups.*

23. **Fred Cohen and Ran Levi**

THE HOMOTOPY EXPONENT CONJECTURE FOR p-COMPLETED CLASSIFYING SPACES OF FINITE GROUPS

Conjecture 23.1. *Let π be a finite group, and let $B\pi^\wedge_p$ denote its p-completed classifying space. Then the homotopy groups of $B\pi^\wedge_p$ have a bounded exponent, i.e., there exists an integer r (depending on π) such that*

$$p^r \cdot \pi_*((B\pi)^\wedge_p)) = \{0\}.$$

Since π is finite, the fundamental group of $B\pi^\wedge_p$ is the finite p-group given by the quotient of G by its minimal normal subgroup of p-power index, denoted by $O^p(\pi)$. If the order of $O^p(\pi)$ is not divisible by p, then $B\pi^\wedge_p$ is homotopy equivalent to $B(\pi/O^p(\pi))$, and the conjecture reduces to a triviality. If p divides the order of $O^p(\pi)$, then $B\pi^\wedge_p$ has infinitely many nonvanishing homotopy groups, all of which are finite p-groups. It is therefore natural to ask whether there is an upper bound on the exponent of these homotopy groups.

A particular case of the Moore finite exponent conjecture is the statement that if X is a finite simply-connected CW complex whose homotopy groups are all finite, then $\pi_*(X)$ has an exponent. One can show that if π is a finite group, then the component of the constant loop in $\Omega B\pi^\wedge_p$ is a retract of the loop space of a finite simply-connected torsion complex. Therefore our conjecture would follow at once if the much stronger Moore conjecture were true.

For any finite group π, $\pi_i(B\pi^\wedge_p) \cong \pi_i(BO^p(\pi)^\wedge_p)$ for all $i \geq 2$. Furthermore, for each $i \geq 3$ $\pi_i(B\pi^\wedge_p) \cong \pi_i(BU^p(\pi)^\wedge_p)$, where $U^p(\pi)$ is the p-universal central extension of $O^p(\pi)$. In all known examples for the conjecture, the order of the Sylow p-subgroup of $U^p(\pi)$ is an upper bound for the order of torsion in $\pi_*(B\pi^\wedge_p)$. There are examples where this bound is sharp.

It is known that for any finite group π, the p-torsion in the homology of the loop space $\Omega(B\pi)^\wedge_p$ is bounded above by the order of the Sylow p-subgroup of $O^p(\pi)$.

Example 23.2. Some examples are known. A few are given by

$$\pi = A_5, A_6, A_7, J_4, M_{11}$$

at $p = 2$. A few more at the prime 2 are given by those finite simple groups of 2-rank 2 (including M_{11}) with the possible exception of $U(3, \mathbb{F}_4)$. The finite simple groups of classical Lie type over the field $\mathbb{F}_q$, where q is a the power of a prime different from p provide a large family of examples at the prime p.

Finite simple groups of Lie type at the defining characteristic: Almost no examples are known of the behavior of $\pi_i(BG(\mathbb{F}_{p^k})^\wedge_p)$, where G is a finite simple groups of Lie type.

Finite groups with Abelian Sylow p-subgroups: For a finite group π with cyclic Sylow p-subgroup and $p > 2$, $B\pi^\wedge_p$ is known to have a homotopy exponent. Furthermore, the best possible upper bound for this exponent is given by the order of the Sylow p-subgroup. On the other hand, for a finite group π with an abelian Sylow p-subgroup of rank larger than 1 neither one of the above statements is known to hold.

Alternating groups: Further natural open cases are A_n with $n > 7$ at the prime 2, and at all primes p in the cases where the Sylow p-subgroup is not cyclic.

The examples mentioned above are obtained by considering the structure of the loop space of $(B\pi)^\wedge_p$. This space sometimes admits a non-trivial product decomposition or is finitely resolvable by fibrations involving more recognizable spaces, which are known to have homotopy exponents by the work of Cohen-Moore-Neisendorfer.

A survey of many of the known results on spaces of type $B\pi^\wedge_p$ for π finite, including examples of homotopy exponents, is the paper (Fred Cohen and Ran Levi; *On the homotopy theory of p-completed classifying spaces*; Group representations: cohomology, group actions and topology (Seattle, WA, 1996), 157–182, Proc. Sympos. Pure Math., 63, Amer. Math. Soc., Providence, RI, 1998.)

Guido, we wish you all the best on your retirement from ETH.

Fred and Ran

24. **Jim Davis**

Isomorphism conjectures for the mapping class group

Dear Guido,

I enjoyed your lecture in Münster where you showed that Teichmüller space was a classifying space for actions of the mapping class group whose isotropy groups are all finite. I will pose a question and then remind you of one I posed at that time.

Let Σ_g be a closed surface of genus g, let Γ_g be its mapping class group, and $\tau_g \cong \mathbb{R}^{6g-6}$ its Teichmüller space.

Question 1 (Borel Conjecture): Let Γ be a torsion free subgroup of the mapping class group, for example the Torelli group. Is the action of Γ on τ_g topologically rigid? That is, is any proper homotopy equivalence $h : M \to \tau_g/\Gamma$, which is a homeomorphism outside of a compact set, properly homotopic to a homeomorphism.

Question 2: (Isomorphism conjecture injectivity) Is there an contractible compactification of Teichmüller space which is small at infinity and equivariant with respect to the action of the mapping class group?

Discussion: A solution to Question 1 would likely involve carrying out the program of Farrell-Jones in the mapping class group case. The terms in Question 2 are defined in the thesis of David Rosenthal. A positive solution to Question 2 would likely lead to a proof of the injectivity map of the assembly map in K- and L-theory with respect to the family of finite subgroups, and thereby a new proof of the Novikov conjecture in this case. (It seems that the Novikov conjecture in the mapping class group case has been recently proved by Ursula Hamenstaedt.)

Best of luck in all your endeavors, mathematically and otherwise,

Jim

25. **Michael W. Davis**
The Hopf Conjecture and the Singer Conjecture

Dear Guido,

As you know, I have been thinking about the following question for a long time.

Conjecture 25.1. *Suppose M^{2k} is a closed, aspherical manifold of dimension $2k$. Then $(-1)^k\chi(M^{2k}) \geq 0$.*

The conjecture is true in dimension 2 since the only surfaces which have positive Euler characteristic are S^2 and $\mathbf{R}P^2$ and they are the only two which are not aspherical. In the special case where M^{2k} is a nonpositively curved Riemannian manifold this conjecture is usually attributed to Hopf by topologists and either to Chern or to both Chern and Hopf by differential geometers.

When I first heard about this conjecture in 1981, I thought I could come up with a counterexample by using right–angled Coxeter groups. Given a finite simplicial complex L which is a flag complex, there is an associated right-angled Coxeter group W. Its Euler characteristic is given by the formula

$$\chi(W) = 1 + \sum_{i=0}^{\dim L} \left(\frac{1}{2}\right)^{i+1} f_i, \tag{1}$$

where f_i denotes the number of i-simplices in L. If L is a triangulation of S^{n-1}, then W acts properly and cocompactly on a contractible n-manifold. The quotient of this contractible manifold by any finite index, torsion-free subgroup $\Gamma \subset W$ is a closed aspherical n-manifold M^n. Since $\chi(M^n)$ is a positive multiple of $\chi(W)$ (by $[W : \Gamma]$), they have the same sign. So, this looked like a good way to come up with counterexamples to 25.1. On the other hand, if you believe Conjecture 1.1, then you must also believe the following.

Conjecture 25.2. *If L is any flag triangulation of S^{2k-1}, then*

$$(-1)^k \kappa(L) \geq 0,$$

where $\kappa(L)$ is the quantity defined by the right hand side of (1).

Ruth Charney and I published this conjecture (*Pac. J. Math.* **171** (1995), 117-137). It is sometimes called the Charney–Davis Conjecture.

In the 1970's Atiyah introduced L^2 methods into topology. If a discrete group Γ acts properly and cocompactly on a smooth manifold or CW complex Y, then one can define the reduced L^2-cohomology spaces of Y and their "dimensions" with respect to Γ, the so-called "L^2-Betti numbers." Let $L^2b_i(Y;\Gamma)$ be the Γ-dimension of the L^2-cohomology of Y in dimension i. It is a *nonnegative* real number. If $Y \to X$ is a regular covering of a finite CW complex X with group of deck transformations Γ, the Euler characteristic of X can be calculated from the L^2-Betti numbers of Y by the formula:

$$\chi(X) = \sum (-1)^i L^2 b_i(Y;\Gamma). \tag{2}$$

Shortly after Atiyah described this formula, Dodziuk and Singer realized that there is a conjecture about L^2-Betti numbers which is stronger than 25.1. It is usually called the Singer Conjecture. Beno Eckmann also discusses it in his note to you.

Conjecture 25.3. *Suppose M^n is an aspherical manifold with fundamental group π and universal cover $\widetilde{M}^n$. Then $L^2 b_i(\widetilde{M}^n;\pi) = 0$ for all $i \neq \frac{n}{2}$. (In particular, when n is odd this means all its L^2-Betti numbers vanish.)*

This implies Conjecture 25.1 since, when $n = 2k$, formula (2) gives: $(-1)^k \chi(M^{2k}) = L^2 b_k(\widetilde{M}^{2k};\pi) \geq 0$.

Of course, there is also the following version of 25.3 for Coxeter groups.

Conjecture 25.4. *Suppose that L is a triangulation of S^{n-1} as a flag complex, that W is the associated right–angled Coxeter group and that Σ is the contractible n-manifold on which W acts. Then $L^2 b_i(\Sigma;W) = 0$ for all $i \neq \frac{n}{2}$.*

Boris Okun and I discussed this conjecture in a paper (*Geometry & Topology* **5** (2001), 7–74) and we proved it for $n \leq 4$. The result for $n = 4$ implies Conjecture 25.2 when L is a flag triangulation of S^3. So, Conjecture 25.2 is true in the first dimension for which it is not obvious.

Have a great retirement,
Mike

26. **Johan Dupont and Walter Neumann**

Rigidity and realizability for $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$

Conjecture 26.1. (Rigidity Conjecture for $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$): $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$ *is countable (the* $^\delta$ *means discrete topology).*

This conjecture[36] is equivalent to the conjecture that the map

$$H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}})^\delta;\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$$

is an an isomorphism. It is also equivalent to the corresponding rigidity conjecture for $K_3^{ind}(\mathbb{C})$, which has been formulated in greater generality by Suslin, and it is implied by some much more far-reaching conjectures of Ramakrishnan in algebraic K-theory, and of Zagier in number theory. It is thus a little drop in a big bucket. However, the latter conjectures seem currently unapproachable, so this drop is worth pursuing. Moreover, it has beautiful geometry attached, so it represents a combination very appropriate to our honoree, Guido Mislin.

One aspect of the geometry is scissors congruence. The "Dehn-Sydler theorem" gave closure to Hilbert's 3rd problem by showing that volume $vol(P)$ and Dehn invariant $\delta(P)$ determine the scissors congruence class of an Euclidean polytope P. Here, $\delta(P) \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\pi\mathbb{Q}$ is defined as the sum of (length)$\otimes$(dihedral angle) over the edges of P.

The corresponding result for polytopes in $\mathbb{H}^3$ or $\mathbb{S}^3$ remains conjectural. If, for $\mathbb{X} = \mathbb{H}^3$ or $\mathbb{S}^3$, we denote by $\mathcal{D}(\mathbb{X})$ the kernel of Dehn invariant δ on the Grothendieck group of $\mathbb{X}$–polytopes modulo scissors congruence, then asking if vol and δ classify $\mathbb{X}$–polytopes up to scissors congruence becomes the question whether

$$vol\colon \mathcal{D}(\mathbb{X}) \to \mathbb{R}$$

is injective. This map has countable image, so its injectivity would imply countability of $\mathcal{D}(\mathbb{X})$. On the other hand, there is a natural isomorphism:

$$H_3(\mathrm{PSL}(2,\mathbb{C})^\delta) \cong \mathcal{D}(\mathbb{S}^3)/\mathbb{Z} \oplus \mathcal{D}(\mathbb{H}^3) \qquad (*)$$

So countability of both $\mathcal{D}(\mathbb{H}^3)$ and $\mathcal{D}(\mathbb{S}^3)$ is equivalent to Conjecture 26.1. In fact, countability of either one suffices. (In particular truth of the "Dehn-Sydler theorem" for $\mathbb{H}^3$–scissors congruence would imply Conjecture 26.1. But this is injectivity of $vol : \mathcal{D}(\mathbb{H}^3) \to \mathbb{R}$, which seems currently no more approachable than Zagier's conjecture, which wildly generalized it.)

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[36]We do not have space for detailed references in the following discussion, which collates work of Bloch, Bökstedt, Brun, Parry, Sah, Suslin, Wigner, Yang, ourselves, and others. For more details see Dupont's book "Scissors Congruences, Group Homology, and Characteristic Classes" (World Scientific 2001) or Neumann's survey "Hilbert's 3rd problem and invariants of 3-manifolds" in "The Epstein Birthday Schrift" (Geometry and Topology Monographs **1**, 1998).

Any compact hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ has a "fundamental class" $\beta(M) \in H_3(\mathrm{PSL}(2,\mathbb{C})^\delta;\mathbb{Z})$: the image of the fundamental class $[M] \in H_3(M) = H_3(\Gamma)$ under the map induced by the inclusion $\Gamma \to \mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$. The image of $\beta(M)$ in $\mathcal{D}(\mathbb{H}^3)$ for the above splitting $(*)$ is just the scissors congruence class of M, but the image in $D(\mathbb{S}^3)/\mathbb{Z}$ is more mysterious. It is orientation sensitive and its volume gives the Chern-Simons invariant of M.

The class $\beta(M)$ is defined more generally for any finite volume $M = \mathbb{H}^3/\Gamma$ (using a natural splitting $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta, P) \cong H_3(\mathrm{PSL}(2,\mathbb{C})^\delta) \oplus H_2(P)$ where P is the parabolic subgroup), and lies in $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}})^\delta)$.

The validity of the following rather wild conjecture would clearly imply Rigidity Conjecture 26.1.

Conjecture 26.2. (Realizability Conjecture): $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta$ *is generated by fundamental classes of hyperbolic 3-manifolds.*

The torsion of $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta)$ is $\mathbb{Q}/\mathbb{Z}$ (it is in the summand $\mathcal{D}(\mathbb{S}^3)/\mathbb{Z}$, where is generated by lens spaces), while $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta)/\mathrm{Torsion}$ is, amazingly, a $\mathbb{Q}$–vector-space (of infinite dimension). So a slightly less wild version of Conjecture 26.2 is

Conjecture 26.3. (Realizability over $\mathbb{Q}$): $H_3(\mathrm{PSL}(2,\mathbb{C})^\delta)/\mathrm{Torsion}$ *is generated over* $\mathbb{Q}$ *by fundamental classes of hyperbolic 3-manifolds.*

Neither version is likely to be useful for Conjecture 26.1: — each is equivalent to the same conjecture restricted to $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}})^\delta)$ together with Conjecture 26.1, which look like rather independent conjectures.

There is no strong evidence for Conjecture 26.2 or the weaker 26.3. The only justification for going out so far on a limb is that the conjecture is enticing, and there is some very weak experimental evidence for the $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}})^\delta)$ version of the conjecture (and the Rigidity Conjecture is widely believed). One could formulate the conjecture just for the first summand in $(*)$ — scissors congruence — but computational evidence suggests that this is no more or less likely to be true than the full conjecture.

27. **Beno Eckmann**
Conjecture or question?

If the answer to the following problem is YES then it is a conjecture; if it is NO then it is a question. We suggest that the problem be solved immediately after June 26, 2006.

Here is the problem. If M is a closed aspherical Riemannian manifold, are the ℓ_2-Betti numbers of its universal covering $\tilde{M}$ all $= 0$ except possibly for the middle dimension? If M has non-positive curvature the problem is known as the Singer conjecture (answer not known).

[Some remarks about "aspherical": It has the following equivalent meanings
1) All homotopy groups in dimensions > 1 are $= 0$.
2) All homology groups of $\tilde{M}$ are $= 0$.
3) $\tilde{M}$ is contractible.
It is a fact known long ago that if M has non-positive sectional curvature then it is aspherical.]

28. **Emmanuel Dror Farjoun**
Nine Problems about (co-)Localizations

To Guido, a guide and a friend.

Introduction

In the forgoing we denote by X, Y etc. either groups or pointed spaces, assuming them to be either CW-complexes or simplicial set satisfying the usual Kan conditions when needed. It is well-known that a map $\phi : X \to Y$ has the form $X \to L_f Y$ of the canonical co-augmentation map for *some* localization functor L_f if and only if it induces an equivalence of mapping objects: (i.e. sets of maps with the appropriate extra structure on them.)

$$map_*(Y, Y) \to map_*(X, Y).$$

Such a map ϕ will be called here a localization map. The notation map_* denotes either the set of all group-maps or the space of all pointed maps.

Similarly, a map $\psi : V \to W$ is called cellular if it is of the form of the canonical augmentation map for the cellularization functor in the relevant category: $cell_A W \to W$. It is not hard to see that a map is cellular if and only if it induces an equivalence $map_*(V, V) \to map_*(V, W)$. In that case $V = cell_A W$ is equivalent to $cell_V W$.

Given the above concepts, most of the following problems-conjectures are elementary in their formulations. But some have proven surprisingly difficult to confirm or negate. I will not dwell here on their implications, the problems seem sufficiently simple minded and attractive as they stand. Most of them can be generalized in various ways, to yield statements in other categories. Some progress and results of similar nature on these and related problems is indicated in the references cited below. On the basis of special cases, one may expect a positive answers for these questions, maybe under mild additional assumptions, save maybe questions number 3. and 4.

Nine open problems

Problem 1. Prove that any localization $P \to L_f P$ of a finite p-group P is a surjective map, in particular the localization is a finite p-group. This is known for groups of nilpotent class 3, by a result of M. Aschbacher.

Problem 2. More generally any localization $L_f N$ of a nilpotent group is a nilpotent group.

Problem 3. Is it true that for any map f and any A the composite functors $cell_A \circ L_f$ and $L_f \circ cell_A$ are idempotent functors?

Problem 4. The localization or cellularization of a principal fibration sequence $G \to E \to B$, *with a connected fibre G,* is a principal fibration sequence. In general the fibration is not preserved, however, its principal nature is supposed to be preserved under mild restrictions. If true this would be in line with the well-known Bousfield-Kan fibre lemma about R_∞ where the fibration is actually preserved and with the observation that it hold for the Postnikov section and n-connected cover functors. For localizations, connectivity of the fibre is an essential condition.

Problem 5. The localization of any 1-connected space is 1-connected. A weaker version: A universal covering projection $\widetilde{X} \to X$ is a localization map only if is it an equivalence, namely, X is 1-connected.

Problem 6. Any localization of a polyGEM is a polyGEM. The localization of an n-connected Postnikov stage is an n-connected Postnikov stage.

Problem 7. A version of problem 3. Suppose $X = cell_A\, M$ where M is an f-local space, for some f,— say a $K(\pi, 1)$. Then $X \cong cell_A L_f\, X$. This is true in several special cases, say for finitely generated abelian groups.

Problem 8. Any localization and cellularization of a space X all whose homotopy groups are p-torsion is also such a space.

Problem 9. The cellularization of a finite Postnikov stage with finite homotopy groups, is a space whose homotopy groups are finite groups. This is true in the category of groups, for finite groups. But is not clear even for a $K(\pi, 1)$ with π a finite group.

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29. **Zig Fiedorowicz**
The Smith Filtration on the Universal Principal Bundle of a Coxeter Group

Definition 29.1. If G is any discrete group, let EG denote the standard construction of a universal principal G-bundle. That is, EG is the geometric realization of the simplicial set E_*G whose set of k-simplices is G^{k+1}, with faces given by deletion of coordinates and degeneracies given by repetition of coordinates and with G acting diagonally on the coordinates.

In his 1981 thesis, J. Smith constructed a natural filtration on the universal principal bundle $E\Sigma_n$ of the symmetric group Σ_n as follows. For each $m \geq 1$, let $F^{(m)}E_*\Sigma_n$ denote the simplicial subset of $E_*\Sigma_n$ consisting of those simplices $(g_0, g_1, \ldots, g_k)$, where for each pair $1 \leq i < j \leq n$, the number of times the pair gets reversed by the sequence of permutations $g_0g_1^{-1}$, $g_1g_2^{-1}$, $\ldots g_{k-1}g_k^{-1}$ is at most $m-1$ times. Let $F^{(m)}E\Sigma_n$ denote the geometric realization of this simplicial set. Smith conjectured the following result, which was later proved by C. Berger.

Theorem 29.2. *$F^{(m)}E\Sigma_n$ has the homotopy type of the configuration space of n-tuples of distinct points in $\mathbb{R}^{mn}$.*

This result can be reformulated in the context of Coxeter groups as follows. Let G be a finite Coxeter group generated by a collection of hyperplanes $\{H_i\}$ in $\mathbb{R}^n$. Then we can define the Smith filtration on E_*G by counting the number of times a generic point in $\mathbb{R}^n$ gets flipped around any one of the generating hyperplanes by the sequence $g_0g_1^{-1}$, $g_1g_2^{-1}$, $\ldots g_{k-1}g_k^{-1}$ corresponding to a k-simplex $(g_0, g_1, \ldots, g_k)$.

A natural generalization of the above result seems to be

Conjecture 29.3. *$F^{(m)}EG$ has the homotopy type of the complement of $\cup_i H_i \otimes \mathbb{R}^m$ in $\mathbb{R}^{mn}$.*

Moreover a considerable portion of Berger's proof carries through in this context. Berger's proof consists of constructing a certain poset and then decomposing the configuration space as a colimit of contractible subspaces indexed by this poset. He then shows that on the one hand the colimit has the same homotopy type as the homotopy colimit, and thus the same homotopy type as the nerve of the poset. On the other hand, he shows by a Quillen Theorem A argument that the nerve of the poset has the same homotopy type as $F^{(m)}E\Sigma_n$. Berger's poset has a natural interpretation in the Coxeter context. However there are certain technical difficulties in carrying out the complete proof.

It may also be the case that there are further generalizations possible for more general classes of reflection groups.

30. **Eric M. Friedlander**

The Friedlander-Milnor Conjecture

To my good friend, **Guido Mislin**

The conjecture of the title of this note has resisted 40 years of effort and remains not only unsolved but also lacking in a plausible means of either proof or counter-example.

The original form of this conjecture is one I struggled with during my days at Princeton in the early 1970's:

Conjecture 30.1. *Let $G(\mathbb{C})$ be a complex reductive algebraic group and let $G(\mathbb{C})^\delta$ denote this group viewed as a discrete group. Then the map on classifying spaces of the continuous (identity) group homomorphism*

$$i : G(\mathbb{C})^\delta \ \rightarrow \ G(\mathbb{C})$$

induces an isomorphism in cohomology with finite coefficients $\mathbb{Z}/n$ for any $n \geq 0$:

$$i^* : H^*(BG(\mathbb{C}), \mathbb{Z}/n) \overset{i^*}{\simeq} H^*(G(\mathbb{C})^\delta, \mathbb{Z}/n).$$

Conjecture 30.1 is easily seen to be true for a torus (i.e., $G = \mathbb{G}_m^{\times r}$ for some $r > 0$), but even the simplest non-trivial case (that of $G = SL_2$) remains inaccessible.

Guido and I published 5 papers together, all in some sense connected with this conjecture. We used the integral form $G_\mathbb{Z}$ of a complex reductive algebraic group (which is a group scheme over $\operatorname{Spec}\mathbb{Z}$) in order to form the group $G(F)$ of points of G with values in a field F. Most of our joint work investigated various relations between $G(\mathbb{C})$ and $G(F)$, the case $F = \overline{\mathbb{F}}_p$ (the algebraic closure of a prime field $\mathbb{F}_p$) being of special interest.

One knows from considerations of etale cohomology that the cohomology of $BG(\mathbb{C})$ with $\mathbb{Z}/n$ coefficients is naturally isomorphic to that of the etale homotopy classifying space of the algebraic group G_F for F algebraically closed of characteristic $p \geq 0$:

$$H^*(BG(\mathbb{C}), \mathbb{Z}/n) \ \simeq \ H^*((BG)_e t, \mathbb{Z}/n), \quad \text{provided that } (p, n) = 1.$$

This enables one to construct a map $H^*(BG(\mathbb{C}), \mathbb{Z}/n) \ \rightarrow \ H^*(G(F), \mathbb{Z}/n)$ relating the cohomology with mod-n coefficients of the classifying space of $G(\mathbb{C})$ with the cohomology with mod-n coefficients of the discrete group $G(F)$ for any field F.

The following is a generalization of Conjecture 30.1, one that appears likely to be true if and only if Conjecture 30.1 is valid.

Conjecture 30.2. *Let $G(\mathbb{C})$ be a complex reductive algebraic group, let $n > 0$ be a positive integer, and let p denote either 0 or a prime which does not divide n. Then for any algebraically closed field F of* *characteristic p, the comparison of the cohomology of $BG(\mathbb{C})$ and $G(F)$ determines an isomorphism*

$$i^* : H^*(G(F), \mathbb{Z}/n) \;\simeq\; H^*(BG, \mathbb{Z}/n).$$

In our first paper together (1), Guido and I began our investigation of "locally finite approximations" of Lie groups. We also formulated the following conjecture and proved it equivalent to Conjecture 30.2.

Conjecture 30.3. *Let F be an algebraically closed field of characteristic $p \geq 0$ and let $n > 0$ be a positive integer not divisible by p if $p > 0$. Then Conjecture 30.2 is valid for $G(F)$ if and only for every $0 \neq x \in H^*(G(F), \mathbb{Z}/n)$, there exists some finite subgroup $\pi \subset G(F)$ such that x restricts non-trivially to $H^*(\pi, \mathbb{Z}/n)$.*

The most familiar form of the "Friedlander-Milnor Conjecture" is that formulated by John Milnor in (2). In that paper, Milnor verifies this conjecture for solvable groups.

Conjecture 30.4. *Let G be a Lie group with finitely many components and let G^δ denote the same group now viewed as a discrete group. Then for any integer $n > 0$, the continuous (identity) map $i : G^\delta \to G$ induces an isomorphism on cohomology with mod-n coefficients:*

$$i^* : H^*(BG, \mathbb{Z}/n) \;\overset{i^*}{\simeq}\; H^*(G^\delta, \mathbb{Z}/n).$$

We remark that the most substantial progress to date on these conjectures is due to Andrei Suslin who proves a "stable" version of Conjectures 30.1 and 30.2 in (3).

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31. **Ross Geoghegan**
The fundamental group at infinity

Dear Guido: I do hope your retirement is only from receiving paychecks and not from mathematics. Here is a question about the fundamental group at infinity. Best wishes. Ross

Let G be a finitely presented group which has one end. There are three flavors of the question: homological, homotopical, and geometric.

1. The homological flavor

Question 1: Is it true that the abelian group $H^2(G, \mathbb{Z}G)$ is free?

Remarks 31.1.
(i) $H^0(G, \mathbb{Z}G)$ and $H^1(G, \mathbb{Z}G)$ are trivial.
(ii) $H^2(G, \mathbb{Z}G)$ is either trivial, or is infinite cyclic, or is an infinitely generated abelian group (Farrell).
(iii) $H^n(G, \mathbb{Z}G)$ need not be free abelian when $n > 2$ (Bestvina-Mess, Davis).
(iv) $H^2(G, \mathbb{Z}G)$ need not be free abelian when G is only finitely generated. Perhaps FP_2 could replace "finitely presented" here.

2. The homotopical flavor

Let X be any (one-ended) complex on which G acts cocompactly as a group of covering transformations.

Question 2: Is it true that the "fundamental group at infinity" of X is semistable (aka Mittag-Leffler)?

An inverse sequence of groups $\{G_r\}$ is *semistable* or *Mittag-Leffler* if, given any n, the sequence of images of the groups G_{n+k} in G_n is eventually constant. We choose a proper ray $\omega : [0, \infty) \to X$ and a filtration of X by finite subcomplexes K_n. By reparametrizing ω we can assume $\omega([r, \infty)) \subset X - K_r$ for all r. Let G_n denote the fundamental group of the complement of K_n based at $\omega(n)$, and let $f_n : G_{n+1} \to G_n$ be induced by inclusion using change of base point along ω. Question 2 asks if this $\{G_r\}$ is semistable.

Remarks 31.2.
(i) The answer only depends on G, not on X nor on the filtration nor on the base ray; so I can rephrase the homotopical question as

Question 2a: Is G semistable at infinity?

(ii) The answer is known to be YES for many classes of groups. For example: all of the following imply that G is semistable at infinity:

(*) G sits in the middle of a short exact sequence of infinite groups where the kernel is finitely generated (Mihalik).
(*) G is a one-relator group (Mihalik).
(*) G is the fundamental group of a graph of groups whose vertex groups are finitely presented and semistable at infinity, and whose edge groups are finitely generated (Mihalik-Tschantz).
(iii) There are positive answers coming from topology. Assume X admits a Z-set compactifying boundary. Then the answer is YES if and only if this (connected) boundary has semistable pro-π_1 in the sense of shape theory (the technical term is "pointed 1-movable"); examples are Coxeter groups (Davis). This π_1-condition holds if the boundary is locally connected; examples are hyperbolic groups (Bowditch, Swarup).
(iv) The answer is unknown for CAT(0) groups.

The homological Question 1 is equivalent to:

Question 1a: Is it true that the inverse sequence of integral first homology groups of the spaces $X - K_n$ is semistable?

Thus Question 1 is the abelianized version of Question 2, and is perhaps more likely to have a positive answer.

3. The geometric flavor

Question 3: Is it true that any two proper rays in X (one-ended!) are properly homotopic?

This is so deliciously simple and "right" that it hardly needs comment [37] except to say that it is EQUIVALENT to Question 2.

Final Remark: There are lots of contractible locally finite 2-dimensional complexes X whose fundamental groups at infinity are not semistable; for example the infinite inverse mapping telescope S associated with a dyadic solenoid (suitably coned off to make it contractible). The problem is to know if any of these admit a cocompact, free and properly discontinuous group action. We know that S does not admit such an action (G.-Mihalik).

[37] I have recently finished a book "Topological Methods in Group Theory" which contains a much more detailed account of what I summarize here.

32. **Henry Glover**

Metastable embedding, 2-equivalence and generic rigidity of flag manifolds

Guido, Here are two conjectures related to our previous work.

Conjecture 32.1. *Any two 2-equivalent manifolds embed in the same metastable dimension. I.e., let M^n and N^n be two simply connected closed differentiable manifolds such that their 2-localizations are homotopy equivalent. If M^n embeds in $\mathbb{R}^{n+k}$, $k > [n/2] + 2$, then N^n embeds in euclidean space of the same dimension.*

See H. Glover and G. Mislin, Metastable embedding and 2-localization, Lecture Notes in Math. 418, Springer 1974, for related work.

Conjecture 32.2. *All complex flag manifolds are generically rigid. We say that a simply connected space X is generically rigid if for all primes p and any simply connected space Y, the p-localizations of X and Y are homotopy equivalent implies the spaces X and Y are homotopy equivalent. A complex flag manifold is any space $U(n)/U(n_1) \times U(n_2) \times ... \times U(n_k)$, with $\Sigma_{i=1}^{k} n_i = n$.*

See H. Glover and G. Mislin, On the genus of generalized flag manifolds, L'Enseignement Mathématique 27 (1981), 211-219, for cases when the conjecture is known to be correct.

33. **Rostislav I. Grigorchuk and Volodya Nekrashevych**
Self-similar contracting groups

Let X be a finite set (alphabet) and let X^* be the free monoid generated by it. We imagine X^* as a rooted tree with the root equal to the empty word and a word v connected to every word of the form vx for $x \in X$.

Definition 33.1. A *self-similar group* is a group G acting faithfully on X^* such that for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xv) = yh(v)$$

for all $v \in X^*$.

It follows that for every $g \in G$ and $u \in X^*$ there exists $h \in G$ such that

$$g(uv) = g(u)h(v)$$

for all $v \in X^*$. The element h is denoted $g|_v$ and is called *restriction of* g in v.

Definition 33.2. A self-similar group G acting on X^* is called *contracting* if there exists a finite set $N \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in N$ for all $v \in X^*$ of length $|v| \geq n$.

Conjecture 33.3. *Finitely generated contracting groups have solvable conjugacy problem.*

Conjecture 33.4. *Finitely generated contracting groups have solvable membership problem.*

Remark 33.5. It is known that the word problem in contracting groups is solvable in polynomial time.

The next three conjectures are ordered by their strength (the last is the strongest).

Conjecture 33.6. *Contracting groups have no non-abelian free subgroups.*

Conjecture 33.7. *Contracting groups are amenable.*

Conjecture 33.8. *A simple random walk on a contracting group has zero entropy.*

Conjecture 33.9. *The group generated by the transformations a and b of $\{0,1\}^*$ defined by*

$$a(0w) = 1w, \quad a(1w) = 0a(w), \quad b(0w) = 0b(w), \quad b(1w) = 1a(w)$$

is amenable.

Remark 33.10. This group is not contracting, however it is known (due to a result of S. Sidki [38]) that this group does not contain a free

[38] S. Sidki, *Finite automata of polynomial growth do not generate a free group*, Geom. Dedicata **108** (2004), 193–204.

subgroup. The graphs of the action of this group on the boundary of the tree X^* have intermediate growth.

Conjecture 33.11. *The group H_k generated by the transformations a_{ij} of $\{1, 2, \ldots, k\}^*$, for $1 \leq i < j \leq k$, defined by*

$$a_{ij}(iw) = jw, \quad a_{ij}(jw) = iw, \quad a_{ij}(kw) = ka_{ij}(w) \text{ for } k \neq i, j$$

is non-amenable for $k \geq 4$.

Remark 33.12. The group H_k models the "Hanoi tower game" on k pegs. The graph of its action on the nth level of the tree $\{1, \ldots, k\}^*$ coincides with the graph of the game with n discs[39]. It is also not contracting, but its graphs of action on the boundary of the tree are of intermediate growth. The group H_3 is amenable and the graphs of the action on the boundary have polynomial growth.

We say that a group G is Tychonoff if it has a fixed ray for any affine action on a convex cone with compact base[40]. A definition of branch groups can be found in[41]. Every proper quotient of a branch group is virtually abelian.

Conjecture 33.13. *Branch group G is Tychonoff iff G is indicable and every proper non-trivial quotient is Tychonoff.*

[39] R. Grigorchuk and Z. Šunić, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups*, to appear in C. R. Acad. Sci. Paris. Sér. I Math., 2006

[40] R. I. Grigorchuk, *On Tychonoff groups*, in "Geometry and cohomology in group theory (Durham, 1994)", 170–187, Cambridge Univ. Press, Cambridge, 1998.

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34. **Pierre de la Harpe**

Piecewise isometries of hyperbolic surfaces

What does the group of piecewise isometries of a surface look like?

More precisely, let us consider compact Riemannian surfaces. Boundaries (if any) should be unions of finitely many geodesic segments; there is no reason to impose connectedness or orientability. For two surfaces M, N of this kind, a *piecewise isometry* from M to N is given by two partitions $M = \sqcup_{i=1}^{k} M_i$ and $N = \sqcup_{i=1}^{k} N_i$ in polygons, and a family $g_i : M_i \longrightarrow N_i$ of surjective isometries; two such piecewise isometries are identified if they coincide on the interiors of the pieces of finer polygonal partitions. When such a piecewise isometry exists, M and N are said to be *equidecomposable.* Piecewise isometries of a surface M to itself constitute the *group of piecewise isometries* $\mathcal{PI}(M)$. We want to stress that a piecewise isometry need not be continuous. The group $\mathcal{PI}(M)$ is a two–dimensional analogue of the group $\mathcal{PI}([0,1])$ of exchange transformations of the interval (the transformations themsleves have been studied by Keane, Sinai, and Veech, among others, and the group by Arnoux, Fathi, and Sah – see for example (5) and (1)).

It is well–known that two Euclidean polygons are equidecomposable if and only if their areas are equal (compare with Chapter IV in Hilbert's *Grundlagen der Geometrie* (7)). This carries over to polygons in the hyperbolic plane (see (3) for a proof). It follows that any orientable connected closed Riemannian surface M of genus $g \geq 2$ and of constant curvature -1 is piecewise isometric to any other, and in particular to a hyperbolic polygon of area $4\pi(g-1)$. In particular, viewed as an abstract group, $\mathcal{PI}(M)$ depends on g only, and can be denoted by $\mathcal{PI}_g$. There are many ways to check that it is an uncoutable group, containing torsion of any order and containing free abelian groups of arbitrary large ranks.

I would like to understand more of the groups $\mathcal{PI}_g$. For example, are $\mathcal{PI}_2$ and $\mathcal{PI}_3$ isomorphic?

Observe that $\mathcal{PI}_2$ is a subgroup of $\mathcal{PI}_3$ in many ways (think of a hyperbolic polygon of area 4π contained inside a hyperbolic polygon of area 6π). The isomorphism question can be phrased more generally for the group $\mathcal{PI}(P_t)$ of piecewise isometries of a hyperbolic polygon P_t of any area $t > 0$.

Are these groups acyclic? Simple? Or if not with simple commutator subgroups? (Arnoux-Fathi and Sah have defined a homomorphism from the analogous group $\mathcal{PI}([0,1])$ onto $\wedge^2_{\mathbf{Q}}\mathbf{R}$, reminiscent of the Dehn invariant for scissors congruences, and it is known that the kernel is a simple group; see (1)). Should they be regarded as topological groups? If yes for which topology? (two candidates: the topology of convergence in measure, and the weak topology discussed in (6)).

Similar questions make sense for other groups of piecewise isometries, for example related to polygons in a round sphere, or in a flat torus, or related to other spaces and appropriates pieces. The case of flat tori is usually phrased in terms of Euclidean spaces or polytopes; concerning this case, the little I am aware of ((2), (4), (8)) is about particular piecewise isometries and not about groups $\mathcal{PI}(M)$. One difficulty with other spaces is to choose an interesting class of pieces when "polygon" or "polytope" have no clear meaning.

A bijection of a finitely–generated group onto a subset of itself which is given piecewise by left multiplications can be viewed as a piecewise isometry. Bijections of this form are important ingredients in the theory of amenable groups (Tarski characterization of non–amenability by the existence of paradoxical decompositions, see e.g. (9)).

Piecewise isometries make sense for large classes of metric spaces, but the corresponding groups and pseudogroups seem to have been little explored so far in this generality. I am grateful to Pierre Arnoux for his comments on the first version of this short Note.

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35. **Hans-Werner Henn and Jean Lannes**
EXOTIC CLASSES IN THE MOD 2 COHOMOLOGY OF $\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}])$

Dear Guido,

Cohomology of linear groups and characteristic classes have played an important role in your work. That is why we hope that you will enjoy the following problem.

Let D_n denote the subgroup of the algebraic group GL_n consisting of diagonal matrices; let ι_n denote the inclusion of discrete groups $\mathrm{D}_n(\mathbf{Z}[\frac{1}{2}]) \hookrightarrow \mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}])$. S. Mitchell [M][42] has shown that the image of the restriction map $\iota_n^* : \mathrm{H}^*(\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}]); \mathbf{F}_2) \to \mathrm{H}^*(\mathrm{D}_n(\mathbf{Z}[\frac{1}{2}]; \mathbf{F}_2)$ is isomorphic to $\mathbf{F}_2[w_1, w_2, \ldots, w_n] \otimes \Lambda(e_1, e_3, \ldots, e_{2n-1})$. Here the w_i are the Stiefel-Whitney classes of the tautological representation $\mathrm{GL}_n(\mathbf{Z}[\frac{1}{2}]) \to \mathrm{GL}_n(\mathbf{R})$ and the e_{2i-1} are closely related to Quillen's odd-dimensional modular characteristic classes in the mod 2 cohomology of $\mathrm{GL}_n(\mathbf{F}_3)$. By explicit calculation the map ι_n^* is known to be injective for $n \leq 3$ ([M] for $n = 2$ and [H1][43] for $n = 3$). Work of Dwyer [D][44] shows that ι_n^* fails to be injective if $n \geq 32$ and still unpublished work of ours [HL] - which will hopefully see the light of the day sometime soon - shows that it already fails if $n \geq 14$. However, the reasoning in [D] as well as in [HL] is very indirect and does not produce any explicit elements in the kernel of ι_n^*. On the other hand one of us has shown[45] that the kernel of ι_n^* becomes very large as n grows.

Problem. *Construct explicit elements in the kernel of ι_n^*.*

COMMENTS.

1) Injectivity of ι_n^* amounts to the validity of an unstable Lichtenbaum-Quillen conjecture for $\mathbf{Z}[\frac{1}{2}]$ at the prime 2 (cf. [D]) so any (even partial) answer to the question would shed more light on why such an unstable conjecture fails.

2) The solution of the Milnor conjecture by Voevodsky led to the proof that the map ι_∞^* (ι_∞ being the colimit of the ι_n) is injective: in other words, the stable Lichtenbaum-Quillen conjecture for $\mathbf{Z}[\frac{1}{2}]$ at the prime 2 holds.

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3) Now let us come back to the ghost reference [HL]. Let O_n be the subgroup of GL_n consisting of orthogonal matrices (orthogonal for the euclidian metric) and ρ_n be the homomorphism of discrete groups $\mathrm{O}_n(\mathbf{Z}[\frac{1}{2}]) \to \mathrm{O}_n(\mathbf{F}_3)$ induced by mod 3 reduction. The reasoning in [HL] is as follows: first we show that if ι_n^* is injective then ρ_n^* is bijective and afterwards we prove that ρ_n^* is not bijective for $n \geq 15$.

Question. *Is ρ_∞^* bijective?*

A positive answer to this question amouts to the validity of an "orthogonal Lichtenbaum-Quillen conjecture for $\mathbf{Z}[\frac{1}{2}]$ at the prime 2."

Best wishes,

Hans-Werner and Jean

36. **Peter Hilton**

A CONJECTURE ON COMPLETE SYMBOLS

Dear Guido,

I do not have any exciting conjectures in homotopy theory at my disposal, so I offer you this conjecture in number theory, with my warmest good wishes to you for a happy and creative retirement.

Affectionately, Peter

A conjecture on complete symbols

Given positive integers b, a, odd, $a < \frac{b}{2}$, $\gcd(a,b) = 1$, we construct a *coach*

$$b \left| \begin{matrix} a_1 a_2 \cdots a_r \\ k_1 k_2 \cdots k_r \end{matrix} \right|, \quad a_1 = a, \tag{3}$$

where $b - a_i = 2^{k_i} a_{i+1}$, with k_i maximal positive, $i = 1, 2, \ldots, r$, $a_{r+1} = a_1$. It is then known that, if $k = \sum_i k_i$, k is the *quasi-order* of 2 mod b, that is, k is the smallest positive integer such that $2^k \equiv \pm 1$ mod b. In fact, $2^k \equiv (-1)^r$ mod b.

It is easy to see that, if S is the set of all positive integers satisfying the conditions on a above, then $a_i \in S$, $1 \leq i \leq r$. It is possible that (1) exhausts the set of integers a_i belonging to S. If not we may, of course, construct further coaches based on b. For example, with $b = 65$, there are 4 coaches

$$65 \left| \begin{matrix} 1 \\ 6 \end{matrix} \right| \begin{matrix} 3 & 31 & 17 \\ 1 & 1 & 4 \end{matrix} \left| \begin{matrix} 7 & 29 & 9 \\ 1 & 2 & 3 \end{matrix} \right| \begin{matrix} 11 & 27 & 19 & 23 & 21 \\ 1 & 1 & 1 & 1 & 2 \end{matrix} \right|, \tag{4}$$

forming what we call the *complete symbol* for $b = 65$. We write $c = c(b)$ for the number of coaches in a complete symbol.

We conjecture that it should be possible to determine if a complete symbol has only one coach without having to construct the coach and without having to determine the quasi-order of 2 mod b.

Remark 36.1. In fact, we know that $\Phi(b) = 2ck$, where Φ is the Euler totient function. Thus $c = 1$ if, and only if, $k = \frac{1}{2}\Phi(b)$.

37. **John Hubbuck**

A "dual" Dickson algebra

Let V be a finite dimensional vector space over a finite field. The ring of invariants of the polynomial algebra on V under the action of the general linear group $GL(V)$ is the well known *Dickson algebra.* With the standard gradings, the algebra structure of the Hopf algebra that is dual to the polynomial Hopf algebra is the divided polynomial algebra on the dual of V and again $GL(V)$ acts.

Question 37.1. *What is the ring of invariants?*

It is well known that the Dickson algebra is not a sub-Hopf algebra of the polynomial Hopf algebra.

My student David Salisbury has just finished his PhD thesis writing mainly on this topic, but progress has been limited and is very computational. In particular, the ring structure of the invariants seems indescribable without computer printouts. Classical invariant theory approaches appear to fail completely, usually because of the absence of finite generation and we need something new.

John Hubbuck

38. **Tadeusz Januszkiewicz**
SIMPLICAL NONPOSITIVE CURVATURE

Dear Guido, $\underline{E}G$ spaces that interested you for a long time often arise from geometric considerations. A prime example is the following situation: Let X be a proper $CAT(0)$ geodesic metric space, and let G be a properly discontinous isometric action. Then X is $\underline{E}G$. To see this one 1. proves a fixed point theorem for finite group actions on $CAT(0)$ spaces, 2. proves convexity properties, hence contractibility of fixed point sets.

Recently Jacek Świątkowski and I [46], studied a combinatorial analog of nonpositive curvature. Our motivation came from cube complexes which provide one of the richest sources of high dimensional CAT(0) spaces. Here CAT(0) condition on the geodesic metric for which every cube is a standard euclidean cube can be stated as a simple, checkable, combinatorial property of links: they should be flag simplicial complexes.

Then one tries to do the same for simplicial complexes. A condition equivalent to CAT(0) property of the geodesic metric for which every simplex is a standard equilateral euclidean simplex is unknown (and finding it is probably hard). However there is a simple condition we call systolicity, which implies many consequences of CAT(0), without actually implying it (and there are non-systolic triangulations for which geodesic metrics are CAT(0)).

The definition goes as follows: Suppose L is a flag simplicial complex. Define the systole $sys(L)$ to be the minimum of (lenghtγ), where γ is is a full subcomplex of L homeomorphic to S^1 and the length of γ is just the number of edges in γ. We say a simplicial complex X is k-systolic if it is simply connected and for any simplex σ, the systole of the link of σ is at least k. We say X is systolic if it is 6-systolic.

The point we are trying to make is that systolicity is indeed a good analog of CAT(0), as good as CAT(0) cubical complexes. We have proved significant parts of the CAT(0) package. Alas the fixed point theorem is still open.

Conjecture *A finite group F acting on a systolic complex X by simplicial automorphisms has a fixed point.*

We understand convexity well enough to be able to prove that fixed point sets X^F are contractible if nonempty. So if the Conjecture is true, systolic spaces provide geometric models for $\underline{E}G$ of systolic groups. Something you might like.

There are many examples of systolic spaces (and their compact quotients) in any dimension, but they are somewhat exotic from the usual standpoint. Three (related) examples of their strange properties are

[46] TJ +JŚ, Simplicial nonpositive curvature, to appear in Publ. Math. IHES

1. Systolic groups, that is fundamental groups of locally systolic spaces, do not contain fundamental groups of closed aspherical manifolds covered by R^n, $n \geq 3$.

2. Boundaries of Gromov hyperbolic systolic groups are hereditarily aspherical (every closed subset in ∂X is aspherical in appropriate Čech sense). Moreover the map induced by inclusion $A \subset X$ is injective on the Čech π_1[47].

3. A systolic space X is asymptotically hereditarily aspherical [48]. This means that for every $r \geq 0$ there exists $R \geq r$, such that for every subcomplex $A \subset X$ the inclusion of Rips complexes $R_r(A) \to R_R(A)$ induces the zero map on the homotopy groups $\pi_i, i \geq 2$)

Study of asymptotic properties of X rather than topological properties of a strange compactum ∂X looks more like a topology you like. And in a sense provides a more precise information about X.

One may think that these three properties point towards a definition of a "dimension", according to which systolic groups are 2-dimensional (it was Dani Wise who told me that those groups, some of which have large cohomological dimension are "two dimensional"). It is a speculation as of now, but still a useful guiding principle. And it motivates questions about non-systolic spaces. Here is an example.

Are there restrictions on "dimension" of the boundary of a CAT(-1) cubical complex? We do know that certain nice compact spaces (e.g. S^n, $n \geq 4$) are not boundaries of CAT(-1) cube complexes (this is related to Vinberg's theorem on the absence of Coxeter groups acting cocompactly on the classical hyperbolic space H^n for large n). Since the definition of "dimension" is lacking, I state the question conservatively.

Question What are topological restrictions on boundaries (or on asymptotic properties) of CAT(-1) cubical complexes? Can one find a restriction similar to (asymptotic) hereditary asphericity in case of systolic spaces.

Best regards, Tadeusz

[47] Various parts of 1. and 2. have been established by Jacek Świątkowski, Damian Osajda and myself

[48] TJ+ JŚ, Filling invariants in systolic complexes and groups, submitted

39. **Craig Jensen**
Cohomology of pure automorphism groups of free products of finite groups

Dear Guido,

Recently, John McCammond, John Meier and I verified the Brownstein-Lee conjecture (Alan Brownstein and Ronnie Lee, volume 150 of Contemp. Math. pages 51-61, 1993.) It concerns the cohomology of the pure symmetric automorphism group of a free group, $P\Sigma_n$, which is the group which has to take each of the preferred generators of free group F_n to a conjugate of itself.

Theorem 39.1 (The Brownstein-Lee Conjecture)**.** *The cohomology of $H^*(P\Sigma_n, \mathbb{Z})$ is generated by one-dimensional classes α^*_{ij} where $i \neq j$, subject to the relations*

(1) $\alpha^*_{ij} \wedge \alpha^*_{ij} = 0$
(2) $\alpha^*_{ij} \wedge \alpha^*_{ji} = 0$
(3) $\alpha^*_{kj} \wedge \alpha^*_{ji} = (\alpha^*_{kj} - \alpha^*_{ij}) \wedge \alpha^*_{ki}$

and the Poincaré series is $\mathfrak{p}(z) = (1 + nz)^{n-1}$.

What can you tell us about the cohomology of similar groups? A first question might be:

Question 39.2. *What is the cohomology of $PAut(\mathbb{Z}/p * \cdots * \mathbb{Z}/p)$? That is, what is the cohomology of the pure (meaning each $\mathbb{Z}/p$ in the free product has to be taken to a conjugate of itself) automorphism group of a free product of n copies of $\mathbb{Z}/p$?*

After you get this, a more generalized question would be:

Question 39.3. *Let $G_1, \ldots, G_n$ be finite abelian groups. What is the cohomology of $PAut(G_1 * \cdots * G_n)$?*

or perhaps

Question 39.4. *Let $G_1, \ldots, G_n$ be finite abelian groups or $\mathbb{Z}$. What is the cohomology of $PAut(G_1 * \cdots * G_n)$?*

or perhaps even

Question 39.5. *Let $G_1, \ldots, G_n$ be finite groups or $\mathbb{Z}$. What is the cohomology of $PAut(G_1 * \cdots * G_n)$?*

I wish you the very best, and hope that your dreams come true.

Best Regards,

Craig Jensen

40. **Radha Kessar and Markus Linckelmann**
Alperin's Weight Conjecture

Dear Guido,

Let p be a prime number. How much of a finite group is p-locally determined? For instance, consider the following inequalities.

Conjecture 40.1. *Let G be a finite group and let P be a Sylow p-subgroup of G.*

(i) The number of conjugacy classes of p'-elements of G is less than or equal to the number of conjugacy classes of $N_G(P)/P$.

(ii) If P is abelian, then the number of conjugacy classes of G is less than or equal to the number of conjugacy classes of $N_G(P)$.

The above inequalities would follow from Alperin's weight conjecture (J. L. Alperin, Weights for finite groups, The Arcata Conference on Finite Groups , Proc. Sympos. Pure Math., 47, 369–379, Amer. Math. Soc., Providence, R.I., 1987):

Let k be an algebraically closed field of characteristic p. For a finite group H denote by $l(kH)$ denote the number of isomorphism classes of simple kH-modules and denote by $w(kH)$ the number of isomorphism classes of simple projective kH-modules. The weight conjecture predicts the following:

Conjecture 40.2. *Let G be a finite group. Then*

$$l(kG) = \sum_{Q \in \mathcal{I}} w(kN_G(Q)/Q),$$

where I denotes a set of representatives of G-conjugacy classes of p-subgroups of G.

Conjecture 40.2 comes in a block-wise version as well and has been extended and reinterpreted in several ways. Despite having been verified for many families of finite groups, including finite p-solvable groups, symmetric groups, finite groups of Lie type and many sporadic simple groups, a true understanding of Conjecture 40.2 or indeed of Conjecture 40.1 remains elusive. In its original form stated above, Alperin's Weight Conjecture is a numerical equality interpreting the number of simple modules of a finite group or a p-block in terms of the involved p-local structure. In recent years a more structural approach to this and related conjectures in terms of cohomological invariants of functors over certain finite categories has emerged.

We wish you a very happy and healthy retirement.

Radha and Markus

PS (from Radha): Thank you for your Math 655 course in 1992/93 which I enjoyed very much.

41. **Kevin P. Knudson**
RELATIVE COMPLETIONS OF LINEAR GROUPS

Dear Guido,

We've never met in person, but I've always admired your work. I want to take this opportunity to wish you a long and happy retirement.

Here is a question that I've thought about a lot, but I can't seem to solve. I know you are familiar with the classical Malcev completion of a group. It has a universal mapping property that allows one to generalize the definition as follows. Let k be a field and let G be a group. The *unipotent k-completion* of G is a prounipotent k-group $\mathcal{U}$ that is universal among such groups admitting a map from G. The Malcev completion is the case $k = \mathbb{Q}$.

One possible problem with this construction is that it might be trivial; that is, the group $\mathcal{U}$ may consist of a single element. This happens, for example, when $H_1(G, k) = 0$. To get around this, there is a generalization (due to Deligne) called the *relative completion*. The set-up is the following. Suppose G is a discrete group and that $\rho : G \to S$ is a representation of G in a semisimple algebraic k-group S. Assume that the image of ρ is Zariski dense. The *completion of G relative to ρ* is a proalgebraic k-group $\mathcal{G}$ that is an extension of S by a prounipotent k-group $\mathcal{U}$:

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \longrightarrow S \longrightarrow 1,$$

along with a lift $\tilde{\rho} : G \to \mathcal{G}$ of ρ. The group $\mathcal{G}$ should satisfy the obvious universal mapping property. If S is the trivial group, then this reduces to the unipotent completion.

Consider the group $G = SL_n(k[t])$ with the map $\rho : SL_n(k[t]) \to SL_n(k)$ induced by setting $t = 0$.

Question 41.1. *What is the completion of G relative to ρ?*

There is an obvious guess, namely the group $SL_n(k[[T]])$, and this turns out to be correct sometimes. I proved this when k is a number field or a finite field. The proof goes like this. Let K be the kernel of ρ; this is the congruence subgroup of the ideal (t). Filter K by powers of (t): $K^i = \{A \in K : A \equiv I \mod t^i\}$. Then it is easy to see that for each i, $K^i/K^{i+1} \cong \mathfrak{sl}_n(k)$. Moreover, the filtration $K^\bullet$ turns out to be the lower central series in this case, and so it follows that the unipotent k-completion of K is $\varprojlim K/K^i = \ker\{SL_n(k[[T]]) \stackrel{T=0}{\to} SL_n(k)\}$. General properties of the relative completion then imply that the correct answer is $SL_n(k[[T]])$.

This approach fails for other fields, though. Here's why. Denote the lower central series of K by $\Gamma^\bullet$. For any field, there is a short exact sequence

$$1 \longrightarrow K^2/\Gamma^2 \longrightarrow H_1(K, \mathbb{Z}) \longrightarrow K/K^2 \longrightarrow 1.$$

The last group is $\mathfrak{sl}_n(k)$, and most of the time, the kernel K^2/Γ^2 surjects onto the module $\Omega^1_{k/\mathbb{Z}}$. For finite fields and number fields, this is no obstruction, but for $k = \mathbb{C}$, for example, we see that K^2/Γ^2 is very large. So $K^\bullet$ differs wildly from $\Gamma^\bullet$ and it is therefore not easy to compute the unipotent completion of K.

Still, I conjecture that $SL_n(k[[T]])$ is the correct answer all the time. In fact, I make the following, more ambitious, conjecture.

Conjecture 41.2. *Let k be a field and let C be a smooth affine curve over k. Denote the coordinate ring of C by A and assume that C has a k-rational point with associated maximal ideal $\mathfrak{m} \subset A$. Let $\rho : SL_n(A) \to SL_n(k)$ be induced by the isomorphism $A/\mathfrak{m} \to k$. Finally, let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. Then the completion of $SL_n(A)$ relative to ρ is the group $SL_n(\widehat{A})$.*

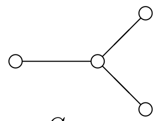
I proved that this is true if we replace A by the localization of A at $\mathfrak{m}$. And, not surprisingly, it is true when k is a number field.

That's all I know. Best wishes.

42. **Max - Albert Knus**

Outer automorphisms of order 3 of Lie algebras of type D_4

Simple Lie algebras over algebraically closed fields of characteristic zero are classified by their Dynkin diagrams. Moreover the group of automorphisms of the Lie algebra modulo the subgroup of inner automorphisms is isomorphic to the group of symmetries of the corresponding Dynkin diagram. In most cases this group of symmetries has at most two elements. The case of the Lie algebra of skew-symmetric 8×8-matrices is exceptional. The Dynkin diagram is D_4:



and has the permutation group S_3 as a group of automorphisms. The automorphisms of the Dynkin diagram can easily be extended to automorphisms of the Lie algebra using the root system. Thus over an algebraically closed field, the classes of automorphisms modulo inner automorphisms are explicitly known.

A complete list of conjugacy classes of outer automorphisms of order 3 over an algebraically closed field can be deduced from the classification of automorphisms of finite order of simple Lie algebras. Besides the conjugacy class of the automorphism constructed with help of the root system, whose fixed point algebra is of type G_2, there is one more conjugacy class in the full group of automorphisms, whose fixed point algebra is a simple Lie algebra of type A_2.

We consider outer automorphisms of order 3 of simple Lie algebras over an arbitrary field of characteristic zero. The orthogonal Lie algebra relative to the quadratic norm form of a Cayley algebra always admits such automorphisms. This is known as the "local triality principle" The converse also holds: if a Lie algebra of type D_4 admits an outer automorphism of order 3, then it is the orthogonal Lie algebra relative to the quadratic norm form of a Cayley algebra. Thus (local) triality and octonions are mutually "responsible" (Tits) for existence.

By descent, conjugacy classes of order 3 outer automorphisms must have Lie algebras of type G_2 or A_2 as fixed point algebras. We conjecture that conjugacy classes are essentially classified by the corresponding fixed point algebras and give a complete list of candidates for the classes. The main ingredient is the notion of a 8-dimensional symmetric composition:

Let S be a finite dimensional F-vector space with a bilinear multiplication $(x, y) \mapsto x \star y$. We say that a quadratic form n on S is *multiplicative* if $n(x \star y) = n(x)n(y)$ holds for all x, $y \in S$. A nonsingular multiplicative quadratic form can only occur in dimension $1, 2, 4$ and 8. A triple $(S, \star, n)$ with a nonsingular multiplicative quadratic form n is called a *symmetric composition* if the polar b of the norm form n satisfies the relation $b(x \star y, z) = b(x, y \star z)$ for $x,\ z,\ y \in S$. The norm form n of a 8-dimensional symmetric composition is always the norm of a (unique) Cayley algebra. However the multiplication of a Cayley algebra does not satisfy the axioms of a symmetric composition. It is known that there are two types of symmetric compositions in dimension 8:

1) Let $\mathcal{C}$ be a Cayley algebra with involution $x \mapsto \overline{x}$. The new multiplication $(x, y) \mapsto \overline{x} \cdot \overline{y}$ defines the structure of a symmetric composition on $\mathcal{C}$ ("Type G_2").

2) The other type is associated with a central simple algebra B of dimension 9 with an involution of second kind over the quadratic extension $K = F[x]/(x^2+3)$. Let $\mathrm{Sym}(B,\tau)$ be the set of symmetric elements in B and let

$$\mathrm{Sym}(B,\tau)^0 = \{x \in \mathrm{Sym}(B,\tau) \mid T_B(x) = 0\}.$$

the 8-dimensional subspace of reduced trace 0 elements. We define a multiplication $\star$ on $\mathrm{Sym}(B,\tau)^0$ by

$$x \star y = \mu xy + (1-\mu)yx - \tfrac{1}{3}T_B(yx)1.$$

where $\mu = \frac{1+\sqrt{-3}}{6}$ and $\sqrt{-3}$ is the class of x in K. Then $\big(\mathrm{Sym}(B,\tau)^0, \star\big)$ is a symmetric composition with norm $n(x) = \frac{1}{6}T_B(x^2)$ ("Type A_2"). We observe that the fact that n must be the norm form of some Cayley algebra implies the special choice of the center K. Details on symmetric compositions can (for example) be found in the Book of Involutions (Knus, Merkurjev, Rost, Tignol).

Let $\mathfrak{o}(n) \subset \mathrm{End}_F(S)$ be the orthogonal Lie algebra associated to the norm n of a symmetric composition. For any $f \in \mathfrak{o}(n)$ there are unique elements $g, h \in \mathfrak{o}(n)$ such that

$$f(x \star y) = g(x) \star y + x \star h(y)$$

and it can be shown that $\rho : f \mapsto g,\ \rho' : f \mapsto h$ are outer automorphisms of order 3 of $\mathfrak{o}(n)$ such that $\rho^2 = \rho'$. Moreover the fixed point Lie algebra under ρ is the Lie algebra of derivations of the algebra $(S, \star)$. This Lie algebra is of type G_2 if S is of type G_2 and of type A_2 if S is of type A_2. We believe that this construction of outer automorphisms of order three through symmetric compositions gives a complete set of representatives of outer automorphisms of order three up to conjugation. Details will hopefully be in a forthcoming paper.

43. **Peter Kropholler**
Classifying spaces for proper actions

It is an open problem to find good algebraic criteria for a group G to admit a finite dimensional model for $\underline{E}G$, the classifying space for proper actions. Guido Mislin introduced me to this variant of the classifying space some ten years ago and we together proved a theorem about it: namely that every $\mathbf{H}\mathfrak{F}$-group of type FP_∞ has a finite dimensional $\underline{E}$. This theorem was an improvement of my original conjecture that $\mathbf{H}\mathfrak{F}$-groups of type FP_∞ should belong to $\mathbf{H}_1\mathfrak{F}$. The class $\mathbf{H}_1\mathfrak{F}$ consists of all groups which admit a proper action on a finite dimensional contractible CW-complex. The proof of the Kropholler–Mislin theorem shows in addition that all $\mathbf{H}_1\mathfrak{F}$-groups for which there is a bound on the orders of the finite subgroups have finite dimensional models for $\underline{E}$. Therefore there is the following natural conjecture:

Conjecture 43.1. *Every* $\mathbf{H}_1\mathfrak{F}$*-group has a finite dimensional classifying space for proper actions.*

Examples which do not fall within the scope of the Kropholler–Mislin theorem include quasicyclic groups, the lamplighter group and many others. However, in all known cases of such examples it is always possible to verify the conjecture very easily. Therefore the conjecture is largely of theoretical interest but remains tantalizing.

Wolfgang Lück's work has greatly improved the dimension bounds on proper classifying spaces and subsequent authors including Leary, Martinez and Nucinkis have made further improvements. Lück brought Bredon cohomology to bear on the problem and in some sense this answers the original quest for an algebraic criterion. But we may regard the matter as open research territory while there is no simple proof or refutation of the above conjecture.

44. **Jean-François Lafont**

Construction of classifying spaces with isotropy in prescribed families of subgroups

Dear Guido,

It was a great pleasure to get to know you before your retirement. You will finally be free to spend time on the truly important things in life (your family and doing mathematics), without the hindrance of teaching classes, going to faculty meetings, and enduring the other administrative nonsense that is an essential part of our chosen career path. Here are a few questions that I have been thinking about in the last few months. I hope you enjoy them, and that at least one of them will turn out to be non-trivial. With best wishes,

Jean.

For an infinite group Γ, the Farrell-Jones Isomorphism conjecture [49] states that the algebraic K-theory $K_n(\mathbb{Z}\Gamma)$ of the integral group ring of Γ coincides with $H_n^\Gamma(E_{VC}\Gamma; \mathbb{KZ}^{-\infty})$, a certain equivariant generalized homology theory of the Γ-space $E_{VC}\Gamma$. This space is a model for the classifying space for Γ with isotropy in the family of virtually cyclic subgroups, i.e. a contractible Γ-CW-complex with the property that the fixed subset of a subgroup H is contractible if H is a virtually cyclic subgroup, and is empty otherwise. From such a classifying space, the homology $H_n^\Gamma(E_{VC}\Gamma; \mathbb{KZ}^{-\infty})$ can be computed via an Atiyah-Hirzebruch type spectral sequence discovered by Quinn [50]. The ingredients entering into the E^2-term of the spectral sequence are the algebraic K-theory of the various cell-stabilizers. In particular, for computational purposes, it is interesting to have a model for $E_{VC}\Gamma$ that is as "small" as possible. This motivates the first:

Question 44.1. *Find an efficient algebraic criterion that determines whether a finitely generated group Γ has a finite dimensional model for $E_{VC}\Gamma$.*

In general, given a family $\mathcal{F}$ of subgroups of Γ, one can define a model for the classifying space $E_{\mathcal{F}}\Gamma$ of Γ with isotropy in the family $\mathcal{F}$ [51]. For the family FIN consisting of finite subgroups, the classifying space $E_{FIN}\Gamma$ has been extensively studied, and explicit finite dimensional models are known for various classes of groups (δ-hyperbolic groups,

[49] F.T. Farrell and L. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. 6 (1993), 249-297.

[50] F. Quinn, *Ends of maps, II*, Invent. Math. 68 (1982), 353-424.

[51] W. Lück, *Survey on classifying spaces for families of subgroups*, Progr. Math. 248, pp. 269-322, Birkhauser, 2005.

groups acting by isometries on finite dimensional CAT(0) spaces, Coxeter groups, etc). In a paper with I. Ortiz [52], we defined the notion of a collection of subgroups to be *adapted* to a nested pair $\mathcal{F} \subset \overline{\mathcal{F}}$ of families of subgroups (for instance, one could take $FIN \subset VC$). This consists of a collection of subgroups $\{H_\alpha\}$ satisfying the following properties: (1) the collection is conjugacy closed, (2) the groups H_α are self-normalizing, (3) distinct groups in the collection intersect in elements of $\mathcal{F}$, and (4) every group in $\overline{\mathcal{F}} - \mathcal{F}$ is contained in one of the H_α. When there exists a collection of subgroups adapted to a pair $\mathcal{F} \subset \overline{\mathcal{F}}$, we explain how to modify a model for $E_{\mathcal{F}}\Gamma$ to obtain a model for $E_{\overline{\mathcal{F}}}\Gamma$. The modifications involve the collection of classifying spaces $E_{\overline{\mathcal{F}}(H_\alpha)}H_\alpha$, where $\overline{\mathcal{F}}(H_\alpha)$ is the restriction of the family $\overline{\mathcal{F}}$ to the subgroup H_α. In particular, when both the $E_{\mathcal{F}}\Gamma$ and the $E_{\overline{\mathcal{F}}(H_\alpha)}H_\alpha$ are finite dimensional, the construction yields a finite dimensional $E_{\overline{\mathcal{F}}}\Gamma$. This prompts the following:

Question 44.2. *Try to identify "natural" non-trivial collections of subgroups adapted to the pair $FIN \subset VC$ for various classical families of finitely generated groups.*

Since in our construction, the dimension of the $E_{VC}\Gamma$ is larger than the dimension of $E_{FIN}\Gamma$, one can also ask the following:

Question 44.3. *Find examples of finitely generated groups Γ for which there exists a finite dimensional model for $E_{FIN}\Gamma$, but there do not exist any finite dimensional models for $E_{VC}\Gamma$.*

And in fact, one might think that in general, one can find families of subgroups for which the classifying spaces can be arbitrarily complicated. For instance we can ask:

Question 44.4. *For Γ a (non-abelian) infinite group, does there always exist a family $\mathcal{F}$ of subgroups, with $FIN \subset \mathcal{F}$, and having the property that any model for $E_{\mathcal{F}}\Gamma$ is infinite dimensional?*

I will conclude with a problem which is a little bit removed from the types of questions one typically considers in this field, but which I nevertheless feel is of some interest.

Question 44.5. *For finitely presented groups, are the following two decision problems algorithmically unsolvable?*
(1) Does there exist a finite dimensional model for $E_{FIN}\Gamma$?
(2)Does there exist a finite dimensional model for $E_{VC}\Gamma$?

[52] J.-F. Lafont and I.J. Ortiz, *Relative hyperbolicity, classifying spaces, and lower algebraic K-theory*, preprint.

45. **Urs Lang and Stefan Wenger**

Isoperimetric inequalities and the asymptotic geometry of Hadamard spaces

The conjecture we describe here deals with isoperimetric fillings of k-cycles in a proper cocompact CAT(0)-space X, where it is assumed that k is greater than or equal to the Euclidean rank of X, i.e. the maximal $n \in \mathbb{N}$ for which $\mathbb{R}^n$ isometrically embeds into X. In order to state the conjecture let us fix the following notation. Given a complete metric space X and $k \in \mathbb{N}$ we define the filling volume function FV_{k+1} of X by

$$FV_{k+1}(s) := \sup \{\mathrm{FillVol}(T) : T \text{ is a } k\text{-cycle in } X \text{ with } \mathrm{Vol}(T) \leq s\},$$

where FillVol(T) is the least volume of a $(k+1)$-chain with boundary T. In this generality, a suitable chain complex is provided by the metric integral currents introduced by Ambrosio–Kirchheim in 2000. Alternatively, one may work with a simplicial approximation or thickening (e.g. a Rips complex) of X and then use Lipschitz chains or simplicial chains.

In his seminal paper *Filling Riemannian manifolds* Gromov proved that every Hadamard manifold, i.e. complete simply-connected Riemannian manifold of non-positive sectional curvature, admits a Euclidean isoperimetric inequality for k-cycles for every $k \geq 1$, thus

$$FV_{k+1}(s) \leq Cs^{\frac{k+1}{k}}$$

for all $s \geq 0$ and for some constant C. More generally, this holds true for CAT(0)-spaces, and even for metric spaces admitting cone type inequalities for l-cycles, $l = 1, \dots, k$, as was shown by S. Wenger in his thesis. The latter property is shared for example by all geodesic metric spaces with convex distance function and all Banach spaces.

If X is a CAT(κ)-space with $\kappa < 0$, i.e. has a strictly negative upper curvature bound, then it is not difficult to show that X admits a linear isoperimetric inequality for k-cycles for every $k \geq 1$, i.e.

$$FV_{k+1}(s) \leq Cs$$

for all s and for some constant C. Now, one of the rough guiding principles in the theory of non-positively curved spaces is that their asymptotic geometry should exhibit hyperbolic behavior in the dimensions above the rank. The following conjecture appears, though somewhat implicitly, in Gromov's book on asymptotic invariants of infinite groups.

Conjecture 45.1. *Every proper cocompact* CAT(0)*-space X of Euclidean rank r admits a linear isoperimetric inequality for k-cycles for every $k \geq r$.*

Instead of assuming X to be proper, cocompact and of Euclidean rank $\leq k$, one may also look at the larger class of CAT(0)-spaces all of whose asymptotic cones have geometric dimension at most k. For a proper cocompact CAT(0)-space X, the Euclidean rank r equals 1 if and only if X is hyperbolic in the sense of Gromov. Then, for $k = 1$, a linear isoperimetric inequality holds, as is well-known. More generally, in geodesic Gromov hyperbolic spaces satisfying suitable conditions on the geometry on small scales (not necessarily CAT(0)), linear isoperimetric inequalities for k-cycles hold for all $k \geq 1$. This was shown, in a simplicial setup, by U. Lang in 2000. In particular, the conjecture holds in the case $r = 1$.

As regards the case $r > 1$, the conjecture is known to hold for symmetric spaces of non-compact type. In fact, if X is a symmetric space of non-compact type and $F \subset X$ is a maximal flat of dimension r, the orthogonal projection onto F decreases r-dimensional volume exponentially with the distance from the flat. This can be used to produce fillings with a linear volume bound.

A consequence of the above conjecture would be that isoperimetric inequalities detect the Euclidean rank. This also follows from the following result, which has recently been proved by Wenger: Let $k \in \mathbb{N}$ and let X be a quasiconvex metric space admitting cone type inequalities for l-cycles for $l = 1, \ldots, k$. Then X admits a 'sub-Euclidean' isoperimetric inequality for k-cycles, i.e.

$$\limsup_{s\to\infty} \frac{FV_{k+1}(s)}{s^{\frac{k+1}{k}}} = 0,$$

if and only if every asymptotic cone of X has dimension at most k. As it stands the conjecture remains open for most cases even in the context of Hadamard manifolds.

46. **Ian Leary**
Proper actions on acyclic spaces

Dear Guido,

Here are a few questions about proper cellular actions of discrete groups G on acyclic spaces. I have deliberately avoided the classifying space for proper G-actions, $\underline{E}G$, partly because some of the questions have already been answered for this space, and partly because I know that some other people will write to you about questions concerning $\underline{E}G$. I start with a version of the classic question that was posed on p226 of Ken Brown's book 'Cohomology of Groups':

Question 46.1. *If G is of finite virtual cohomological dimension, does G act properly on some acyclic space of dimension equal to* $\mathrm{vcd}G$*?*

Remark 46.2. If $\mathrm{vcd}G$ is not equal to 2, then 'acyclic' in the above question can be replaced by 'contractible' without changing the question. The answer is 'yes' when $\mathrm{vcd}G = 1$ by a theorem of Martin Dunwoody, and Quillen's plus construction can be used to replace an acyclic space of dimension n by a contractible space of dimension equal to the maximum of n and 3.

Brita Nucinkis and I found examples to show that the dimension of the space $\underline{E}G$ can be strictly greater than $\mathrm{vcd}G$ [Leary-Nucinkis 'Some groups of type VF' 2003]. Incidentally, we used Bredon cohomology in our arguments, a subject that we both learned from you.

Secondly, a rather vague question. It is well-known that $\mathrm{vcd}G$ is finite if and only if G is virtually torsion-free and G acts properly on some finite dimensional contractible space [K. S. Brown, *loc. cit.*].

Question 46.3. *Are there any results concerning group cohomology where virtual torsion-freeness plays a role? For example, are there any results about $H^*(G;\mathbb{Z}G)$ that hold for groups of finite vcd, but do not hold for all groups in Peter Kropholler's class $\mathbf{H}_1\mathcal{F}$?*

Finally, a few questions concerning the connection between algebraic and topological finiteness conditions.

Question 46.4. *If G is of type FP over a ring R, does G act cellularly cocompactly on some R-acyclic CW-complex X with stabilizers whose orders are units in R?*

There is an algebraic version of this question too. Define a projective permutation module for the group algebra RG to be a direct sum of modules isomorphic to RG/H, where H ranges over the finite subgroups whose orders are units in R. Say that G is type FPP over R if there is a finite resolution of R over RG by finitely generated projective permutation modules.

Question 46.5. *If G is FP over R, is G necessarily of type FPP over R?*

For $R = \mathbb{Z}$, this question is equivalent to the famous question of whether every group of type FP is FL.

Question 46.6. *If G is FL over a prime field F, does G act freely cellularly cocompactly on some F-acyclic CW-complex?*

Remark 46.7. There are groups that are FP but not FL over $\mathbb{Q}$, and are FL over $\mathbb{C}$ ['The Euler class of a Poincaré duality group', 2002]. Such a group cannot act freely cellularly cocompactly on any $\mathbb{C}$-acyclic CW-complex. It is because of these examples that the previous question is stated only for the fields $\mathbb{Q}$ and $\mathbb{F}_p$.

47. **Peter A. Linnell**

ℓ^p-HOMOLOGY OF ONE-RELATOR GROUPS

Let G be a group and let

$$\dots \longrightarrow \mathbb{C}G^{e_{n+1}} \xrightarrow{d_n} \mathbb{C}G^{e_n} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathbb{C}G^{e_1} \xrightarrow{d_0} \mathbb{C}G \longrightarrow \mathbb{C} \longrightarrow 0$$

be a free $\mathbb{C}G$-resolution of $\mathbb{C}$. Let $1 \leq p < \infty$ and for $n \geq 0$, let

$$d_*^n \colon \mathbb{C}G^{e_{n+1}} \otimes_{\mathbb{C}G} \ell^p(G) \longrightarrow \mathbb{C}G^{e_n} \otimes_{\mathbb{C}G} \ell^p(G),$$
$$d_n^* \colon \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_n}, \ell^p(G)) \longrightarrow \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_{n+1}}, \ell^p(G))$$

be the maps induced by d_n; for convenience, we let $d_*^{-1} = d_{-1}^* = 0$. Then one has the usual homology and cohomology groups

$$H_n(G, \ell^p(G)) = \ker d_*^{n-1} / \operatorname{im} d_*^n,$$
$$H^n(G, \ell^p(G)) = \ker d_n^* / \operatorname{im} d_{n-1}^*,$$

which we shall call the (unreduced) ℓ^p-homology and cohomology groups of G respectively. In the case when all the e_n are finite,

$$\mathbb{C}G^{e_n} \otimes_{\mathbb{C}G} \ell^p(G) \cong \ell^p(G)^{e_n} \cong \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_n}, \ell^p(G)),$$

so one can also define the reduced ℓ^p-homology and cohomology groups of G:

$$\bar{H}_n(G, \ell^p(G)) = \ker d_*^{n-1} / \overline{\operatorname{im} d_*^n},$$
$$\bar{H}^n(G, \ell^p(G)) = \ker d_n^* / \overline{\operatorname{im} d_{n-1}^*},$$

where $^-$ indicates the closure in $\ell^p(G)^n$. The first ℓ^p-cohomology groups (reduced and unreduced) have been studied extensively recently by Bekka, Bourdon, Florian, Valette and others. Also Kappos has interesting results on general reduced homology and cohomology groups.

Let us concentrate now on the case G is a finitely generated torsion-free one-relator group. Let d denote the number of generators of G. Then we have a $\mathbb{C}G$-resolution of the form

$$0 \longrightarrow \mathbb{C}G \longrightarrow \mathbb{C}G^d \longrightarrow \mathbb{C}G \longrightarrow 0.$$

From this it is clear that homology groups $H_n(G, \ell^p(G))$, $H^n(G, \ell^p(G))$, $\bar{H}_n(G, \ell^p(G))$ and $\bar{H}^n(G, \ell^p(G))$ are all zero for $n \geq 3$. Warren Dicks and I determined the ℓ^2-Betti numbers of such groups[53]; an immediate consequence of this is that $H_2(G, \ell^p(G)) = \bar{H}_2(G, \ell^p(G)) = 0$ for $p \leq 2$. The proof of this depended on the results that a torsion-free group one-relator group is left orderable, and that if H is a left orderable group, $0 \neq \alpha \in \mathbb{C}H$ and $0 \neq \theta \in \ell^2(H)$, then $\alpha\theta \neq 0$.[54] However if $p > 2$, then we can have $0 \neq \alpha \in \mathbb{C}H$ and $0 \neq \theta \in \ell^p(H)$ with $\alpha\theta = 0$. Thus we have the following conjecture.

[53] ℓ^2-Betti numbers of one-relator groups, arxiv.org/abs/math.GR/0508370

[54] Peter A. Linnell. Zero divisors and $L^2(G)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 315(1):49–53, 1992.

Conjecture 47.1. *Let G be a finitely generated torsion-free one-relator group and let $1 \leq p < \infty$. Then $H_2(G, \ell^p(G)) = \bar{H}_2(G, \ell^p(G)) = 0$.*

This conjecture is true if G is a surface group, orientable or not.

If Conjecture 47.1 is true, then it follows from Kappos's Proposition 3.5[55] that $\bar{H}^2(G, \ell^p(G)) = 0$ for all $p > 1$. The situation for unreduced cohomology is less clear. Let us consider the special case $p = 2$. Recall that $\langle x, y \mid xyxy = 1\rangle$ is the Klein bottle group. Our next conjecture is

Conjecture 47.2. *Let G be a finitely generated torsion-free one-relator group. Then $H^2(G, \ell^2(G)) = 0$, provided G is neither $\mathbb{Z} \times \mathbb{Z}$ nor the Klein bottle group.*

This conjecture is also true if G is a surface group, orientable or not.

Finally we consider the first homology groups. A result of Guichardet (see Theorem A[56]) shows that if G is an arbitrary infinite group, then the natural epimorphism $H^1(G, \ell^2(G)) \to \bar{H}^1(G, \ell^2(G))$ is an isomorphism if and only if G is nonamenable. This leads to the following conjecture.

Conjecture 47.3. *Let G be a finitely generated torsion-free one-relator group which is neither $\mathbb{Z} \times \mathbb{Z}$ nor the Klein bottle group. Then the natural epimorphism $H_1(G, \ell^2(G)) \to \bar{H}_1(G, \ell^2(G))$ is an isomorphism.*

[55]Elias Kappos, ℓ^p-cohomology for groups of type FP_n. arxiv.org/abs/math/0511002

[56]Mohammed E. B. Bekka and Alain Valette. Group cohomology, harmonic functions and the first L^2-Betti number. *Potential Anal.*, 6(4):313–326, 1997.

48. **Wolfgang Lück**
The Farrell-Jones Conjecture in algebraic K- and L-theory

One of my favorite conjectures in geometric topology is the *Borel Conjecture*. It says that a homotopy equivalence between closed aspherical topological manifolds is homotopic to a heomeomorphism. This implies that two closed aspherical topological manifolds are homeomorphic if and only if their fundamental groups are isomorphic. (This is not true if one considers smooth manifolds and replaces homeomorphic by diffeomorphism). This is the topological analogue of Mostow rigidity. Another prominent conjecture is the *Novikov Conjecture*. It asserts that the higher signatures of a closed oriented smooth manifold are homotopy invariants. It is motivated by the signature formula of Hirzebruch. Furthermore the *Bass Conjecture* has gotten a lot of attention since it was formulated. It says that for a group G and an integral domain R of characteristic 0 the *Hattori-Stallings rank* of a finitely generated projective RG module evaluated at an element g of G is non-trivial only if g has finite order $|g|$ and $|g| \cdot 1_R$ is not a unit in R. Finally we mention the conjectures that for a torsionfree group G the *reduced projective class group* $\widetilde{K}_0(\mathbb{Z}G)$ and its *Whitehead group* $\mathrm{Wh}(G)$ vanish. The last two conjectures have equivalent geometric counterparts, if G is finitely presented, namely, that any *finitely dominated CW-complex* is homotopy equivalent to a finite CW-complex and that any *h-cobordism* over a closed manifold of dimension ≥ 5 is trivial.

It turns out the Farrell-Jones Conjecture for algebraic K- and L-theory does imply all of the conjectures above and gives a good understanding of the algebraic K and L-theory of group rings RG in terms of the algebraic K- and L-theory of the coefficient ring R and the homology of the group G. I will formulate it only for a torsionfree group G and a regular ring R. In this case the *Farrell-Jones Conjecture* predicts that the classical assembly maps

$$\begin{aligned} H_n(BG;\mathbb{K}_R) &\xrightarrow{\cong} K_n(RG); \\ H_n(BG;\mathbb{L}_R)[1/2] &\xrightarrow{\cong} L_n(RG)[1/2]; \end{aligned}$$

are bijective for all n. The sources of the assembly maps above are homology theories such that $H_n(\mathrm{pt.};\mathbb{K}_R) = K_n(R)$ and $H_n(\mathrm{pt.};\mathbb{L}_R) = L_n(R)$ hold. (In the L-theory case one needs to invert 2 for the version above but does not need regular). The general formulation uses more elaborate *equivariant homology theories* and *classifying spaces of families*.

It is fascinating how so many different important conjectures in topology, geometry and K-theory do follow from a single one. The Farrell-Jones Conjecture is still open but known for a good class of groups. Its non-commutative counterpart is the *Baum-Connes Conjecture*.

If one reads the text above and knows Guido's excellent and broad work, one realizes that Guido has studied and contributed to nearly all of the conjectures or notions mentioned above. And this does not cover all of his work! So I like to express my deepest respect to an excellent mathematician and a shining and very friendly person. I wish you all the best, Guido!

Wolfgang Lück

49. **Roman Mikhailov and Inder Bir S. Passi**
RESIDUALLY NILPOTENT GROUPS

Dear Guido,

In general, it is difficult to decide whether a given group is residually nilpotent; this is so even for rather simple looking one-relator groups. There is thus need to develop general methods for checking residual nilpotence. Apart from being of group-theoretic interest, such an investigation will also have impact on problems in topology; for example, in the context of Baumslag's parafree conjecture, Whitehead's asphericity conjecture.

We would like to list here for you two problems on residual nilpotence. You may also like to see Kourovka Notebook 2006, Problem 16.65.

Problem 1. *If F is a free group with finite basis $x_1, \dots, x_n$, and r a basic commutator, then is the group $\langle x_1, ..., x_n \,|\, r\rangle$ residually nilpotent?*

Let G be a residually nilpotent group. We say that G is *absolutely residually nilpotent* if for any k-central extension $1 \to N \to \tilde{G} \to G \to 1$, of G, i.e., a central extension satisfying $[N, \underbrace{\tilde{G}, \dots, \tilde{G}}_{k\ terms}] = 1$, the group $\tilde{G}$ is again residually nilpotent. As a first step, on examining absolutely residual nilpotence for one-relator groups, it turns out that *every central extension of a one-relator residually nilpotent group is again residually nilpotent*[57]. We are thus motivated to raise the following

Problem 2. *Is every one-relator residually nilpotent group absolutely residually nilpotent?*

It can be shown that the statement that *finitely-generated parafree groups are absolutely residually nilpotent* implies Baumslag's parafree conjecture: $H_2(G) = 0$ *for a finitely generated parafree group* G, and conversely.

We wish you a very healthy, peaceful and mathematically active retirement. Your contribution to the Bass' conjecture has been of great interest to us.

Best regards,

Roman & Inder Bir

[57]R. V. Mikhailov: Sbornik: Mathematics 196:11 1659-1695 (2005).

50. **Graham A. Niblo and Michah Sageev**
The Kropholler conjecture

A finitely generated group G is said to split over subgroup H if and only if G may be decomposed as an amalgamated free product $G = A \underset{C}{*} B$ (with $A \neq C \neq B$) or as an HNN extension $G = A \underset{C}{*}$. The Kropholler conjecture is concerned with the existence of such splittings.

Given a subgroup H of a finitely generated group G the invariant $e(G, H)$ is defined to be the number of Freudenthal (topological) ends of the quotient of the Cayley graph of G under the action of the subgroup H. This number does not depend on the (finite) generating set chosen for G [3] so it is an invariant of the pair (G, H). For example, if G is a free abelian group and H is an infinite cyclic subgroup then $e(G, H) = 0$, if G has rank 1, $e(G, H) = 2$ if G has rank 2 and $e(G, H) = 1$ if G has rank greater than or equal to 3. This invariant generalises Stallings' definition of the number of ends of the group G since if $H = \{1\}$ then $e(G, H) = e(G)$.

In [4] Stallings showed that the group G splits over some finite subgroup C if and only if $e(G) \geq 2$. There are several important generalisations of this fact, the most wide ranging being the algebraic torus theorem, established by Dunwoody and Swenson [1]. This states that, under suitable additional hypotheses, if G contains a polycyclic-by-finite subgroup H of Hirsch length n with $e(G, H) \geq 2$ then either

1. G is virtually polycyclic of Hirsch length $n + 1$,
2. G splits over a virtually polycyclic subgroup of Hirsch length n,
3. G is an extension of a virtually polycyclic group of Hirsch length $n - 1$ by a Fuchsian group.

This theorem generalises the classical torus theorem from low dimensional topology which asserts that a closed 3-manifold which admits an immersed incompressible torus either admits an embedded incompressible torus or has a Seifert fibration. These topological conclusions imply the algebraic conclusions for the fundamental group of the manifold. An important ingredient of the proof of the algebraic torus theorem is a special case of the so called Kropholler conjecture. Its original formulation relies on the following observation of Scott:

A subgroup H of a finitely generated group G satisfies $e(G, H) \geq 2$ if and only if G admits a subset A satisfying the following:

1. $A = HA$,
2. A is H-almost invariant, and
3. A is H-proper, i. e. , neither A nor $G - A$ is H-finite.

We will refer to the subset A as a proper H-almost invariant subset. In his proof of the algebraic torus theorem for Poincaré duality groups Kropholler observed that, under certain additional hypotheses, if G admits a proper H-almost invariant set A such that $A = AH$ then G

admits a splitting over some subgroup $C < G$ related to H (see [2] for an outline of the proof). He conjectured that the additional hypotheses were inessential. Specifically:

Conjecture 50.1 (The Kropholler conjecture)**.** *Let G be a finitely generated group and $H < G$. If G contains a proper H-almost invariant subset A such that $A = AH$ then G admits a non-trivial splitting over a subgroup C which is commensurable with a subgroup of H.*

The conjecture is known to hold when G is a Poincaré duality group or when G is word hyperbolic and H is a quasi-convex subgroup. In general it is known (for an arbitrary finitely generated group G) whenever H is a subgroup which satisfies the following descending chain condition:

Every descending chain of subgroups $H = H_0 \geq H_1 \geq H_2 \geq \dots$ such that H_{i+1} has infinite index in H_i eventually terminates.

This condition holds for example for the class of finitely generated polycyclic groups, in which class the Hirsch length is the factor controlling the length of such a chain. This is a key ingredient in the proof of the full algebraic torus theorem.

An alternative, more geometric, point of view on the conjecture is provided by the following characterisation:

Theorem 50.2. *Given a finitely generated subgroup G and a subgroup $H < G$ the invariant $e(G, H)$ is greater than or equal to 2 if and only if G acts with no global fixed point on a CAT(0) cubical complex with one orbit of hyperplanes, and so that H is a hyperplane stabiliser. H admits a right invariant, proper H-almost invariant subset if and only if the action can be chosen so that H has a fixed point in the complex.*

Bibliography

(1) M. J. Dunwoody and E. L. Swenson, 'The algebraic torus theorem', *Invent. Math.* **140** (2000), no. 3, pp. 605–637.

(2) P. H. Kropholler, A group theoretic proof of the Torus Theorem, in: (eds. G. A. Niblo and M. A. Roller) Geometric Group Theory, Sussex 1991, Volume 1, (Cambridge University Press 1993), pp. 138158.

(3) G. P. Scott, 'Ends of pairs of groups', *Journal of Pure and Applied Algebra***Vol. 11**, 1977, pp 179–198.

(4) J. R. Stallings, 'On torsion-free groups with infinitely many ends', *Ann. of Math.* **88**, (1968), pp. 312–334.

51. **Brita Nucinkis**
Soluble groups of type VF

Dear Guido,

Cohomological finiteness conditions for soluble groups are very well understood for torsion-free soluble groups, but there remains a gap for soluble groups with torsion. This can be summarised by the following conjecture:

Conjecture 51.1. *Every soluble group G of type VF admits a cocompact model for $\underline{E}G$.*

I believe it all began with Urs Stammbach's result [Stammbach 1970], that for a torsion-free soluble group the homological dimension $\mathrm{hd}G$ is equal to the Hirsch length $\mathrm{h}G$ of the group. It is well known that or polycylic groups the Hirsch length is equal to the cohomological dimension, $\mathrm{cd}G$ and it was Karl Gruenberg, who showed that a torsion-free nilpotent group is finitely generated if and only if $\mathrm{cd}G = \mathrm{h}G$. It was conjectured and proved in several cases by Gildenhuys and Strebel [On the cohomology of soluble groups II, 1982] that soluble groups are of type FP if and only if the cohomological dimension is equal to the Hirsch length, and finite. This conjecture was finally proved by Peter Kropholler [Cohomological dimension of soluble groups, 1986]. A later result [Kropholler, On groups of type FP_∞, 1993] means that this result can be phrased as follows:

Theorem 51.2. [Kropholler] *Let G be a soluble group. Then the following are equivalent*

(1) *G is of type* FP_∞,
(2) *G is virtually of type* FP,
(3) $\mathrm{vcd}G = \mathrm{h}G < \infty$,
(4) *G is virtually torsion-free and constructible.*

The fact that G is constructible implies that G is finitely presented, thus any torsion-free group satisfying the conditions of the Theorem is of type F.

As soon as the group has torsion we are in the realm of $\underline{E}G$ and Bredon (co)homology. As for the torsion-free case, for countable groups the Bredon cohomological dimension $\underline{\mathrm{cd}}G$ and the Bredon homological dimension $\underline{\mathrm{hd}}G$ differ by at most one. Ramon Flores and I [Bredon homology for elementary amenable groups, 2006] proved the analogue to Stammbach's result, namely that for soluble groups, $\underline{\mathrm{hd}}G = \mathrm{h}G$. This naturally leads to the following conjecture:

Conjecture 51.3. *Let G be a soluble group. Then the following are equivalent:*

(1) *G is of type* $\underline{\mathrm{FP}}_\infty$
(2) $\underline{\mathrm{cd}}G = \mathrm{h}G < \infty$
(3) *G is of type* FP_∞.

$\underline{\mathrm{FP}}_\infty$ denotes the Bredon analogue to FP_∞. It is not hard to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). In light of Kropholler's result it is obvious that Conjecture 51.1 implies Conjecture 51.3. Ian Leary and I, however, have examples of groups of type VF, which do not admit a cocompact model for $\underline{E}G$ [Some groups of type VF 2003], but all available evidence leads me to believe that Conjecture 51.1 still holds for soluble groups. Analogously to Lück's argument for $\underline{E}G$ one can show that a group is of type $\underline{\mathrm{FP}}_\infty$ if and only if it has finitely many conjugacy classes of finite subgroups and all centralisers of finite subgroups are of type FP_∞. I am confident that proving that soluble groups of type FP_∞ have finitely many conjugacy classes of finite subgroups is fairly straightforward. Hence, to prove (3) $\Rightarrow$ (1) in 51.3 it remains to show that centralisers of finite subgroups are of type FP_∞. To prove Conjecture 51.1 we will also need to show that these are finitely presented. These two things however, remain to this day frustratingly elusive.

52. **Tim R. Riley**
The Dehn function of $\mathrm{SL}_n(\mathbb{Z})$

Dear Guido, There's more than enough in this book to occupy a lifetime, let alone a retirement. All best wishes and many congratulations, Tim

For a word w on ${a_1}^{\pm 1}, \ldots, {a_m}^{\pm 1}$ representing 1 in a finite presentation $\mathcal{P} = \langle a_1, \ldots, a_m \mid \mathcal{R} \rangle$ of a group Γ, define $\mathrm{Area}(w)$ to be the minimal $A \in \mathbb{N}$ such that there is an equality $w = \prod_{i=1}^{A} {u_i}^{-1} {r_i}^{\varepsilon_i} u_i$ in the free group $F(a_1, \ldots, a_m)$ for some $\varepsilon_i = \pm 1$, some words u_i, and some $r_i \in \mathcal{R}$. Equivalently, $\mathrm{Area}(w)$ is the minimal A such that there is a van Kampen diagram for w over $\mathcal{P}$ with at most A 2-cells. Defining $\mathrm{Area}(n)$ to be the maximum of $\mathrm{Area}(w)$ over all w that have length at most n and represent 1 in Γ, gives the Dehn function $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$ of $\mathcal{P}$. Whilst $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$ is defined for $\mathcal{P}$, a different finite presentation $\mathcal{P}'$ for Γ will yield a Dehn function $\mathrm{Area}' : \mathbb{N} \to \mathbb{N}$ that is qualitatively the same — for example, $(\exists C > 1,\ \forall n,\ (1/C)n^2 \leq \mathrm{Area}'(n) \leq Cn^2)$ if and only if the same is true for $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$ (the C may differ).

Question 52.1. *Is the Dehn function of* $\mathrm{SL}_n(\mathbb{Z})$ *quadratic when* $n \geq 4$*?*

Presenting this as a question, rather that a claim, conjecture, or the like, may be unduly conservative. In his 1993 survey article[58], Gersten describes the quadratic Dehn function as an assertion of W.P.Thurston.

I am not even aware of a proof that the Dehn function of $\mathrm{SL}_n(\mathbb{Z})$ is bounded above by a polynomial when $n \geq 4$. By contrast, the Dehn function of $\mathrm{SL}_2(\mathbb{Z})$ is known to grow linearly – $\mathrm{SL}_2(\mathbb{Z})$ is hyperbolic – and that of $\mathrm{SL}_3(\mathbb{Z})$ grows like $n \mapsto \exp(n)$: Epstein & Thurston[59] proved an exponential lower bound and a result sketched by Gromov[60] gives the upper bound (an elementary proof would perhaps be a step towards 52.1).

Of course, 52.1 presupposes $\mathrm{SL}_n(\mathbb{Z})$ is finite presentable, but that has been long known. The $n^2 - n$ matrices e_{ij} with 1's on the diagonal, the off-diagonal ij-entry 1, and all others 0, generate $\mathrm{SL}_n(\mathbb{Z})$. Milnor[61], following J.R.Silvester and in turn Nielsen and Magnus, explains that the Steinberg relations $\{[e_{ij}, e_{kl}] = 1\}_{i \neq l, j \neq k}$ and $\{[e_{jk}, e_{kl}] = e_{jl}\}_{j \neq l}$ together with $\{(e_{ij} {e_{ji}}^{-1} e_{ij})^4 = 1\}_{i \neq j}$ are defining relations. A proof of 52.1 would be an exacting quantitative proof of finite presentability.

[58] Isoperimetric and isodiametric functions. In G.Niblo and M.Roller, eds., *Geometric group theory I*, no. 181 in LMS lecture notes, C.U.P., 1993.

[59] D.B.A.Epstein et al., *Word Processing in Groups*, Jones and Bartlett, 1992.

[60] Asymptotic invariants of infinite groups. In G. Niblo and M. Roller, eds., *Geometric group theory II*, no. 182 in LMS lecture notes, C.U.P., 1993. See §$2\mathrm{B}_1$, §$5A_7$, §5D(5)(c).

[61] *Introduction to algebraic K-theory*, vol. 72 of *Annals of Mathematical Studies*, Princeton University Press, 1971.

One can regard 52.1 as a higher dimensional version of the Lubotzky-Mozes-Raghunathan Theorem, establishing the existence of efficient words representing elements g of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$, that is, words of length like the log of the maximum of the absolute values of the matrix entries.[62] As a word representing g amounts to a path in the Cayley graph from 1 to g, the L.-M.-R. Theorem can be thought of as saying that 0-spheres admit efficient fillings by 1-discs. A word w representing 1 in a finite presentation $\mathcal{P}$ corresponds to a loop ρ_w in the Cayley graph; a van Kampen diagram for w can be regarded as a combinatorial homotopy disc for ρ_w in the Cayley 2-complex of $\mathcal{P}$. So 52.1 is, roughly speaking, the claim that 1-spheres admit efficient fillings by 2-discs in $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 4$. Gromov[60] takes this further and suggests that in $\mathrm{SL}_n(\mathbb{Z})$, Euclidean isoperimetric inequalities concerning filling k-spheres by $(k+1)$-discs persist up to $k = n-3$. (For $k = n-2$, the exponential lower bound of Epstein & Thurston[59] applies.)

One attack on 52.1 is that whilst $\mathrm{SL}_n(\mathbb{Z})$ is not a *cocompact* lattice in the symmetric space $X := \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, and so the quadratic isoperimetric inequality enjoyed by X does not immediately pass to $\mathrm{SL}_n(\mathbb{Z})$, open horoballs can be removed from X to give a space X_0 on which $\mathrm{SL}_n(\mathbb{Z})$ acts cocompactly. Druţu[63] and Leuzinger & Pittet[64] have made progress in this direction, including a quadratic isoperimetric inequality for the boundary horosphere of each removed horoball.

Chatterji has asked whether for $n \geq 4$, $\mathrm{SL}_n(\mathbb{Z})$ enjoys her property L_δ for some $\delta \geq 0$, which would imply a sub-cubic Dehn function[65].

The author's efforts towards 52.1 have, to date, yielded[66] a version of L.-M.-R. giving explicit efficient words. This may aid the construction of van Kampen diagrams, but that remains to be seen. However it has led to progress elsewhere.[67]

Finally, we mention that for $n > 3$, the Dehn functions of the cousins $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ of $\mathrm{SL}_n(\mathbb{Z})$ are also unknown.[68]

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[62] Cyclic subgroups of exponential growth and metrics on discrete groups, C.R. Acad. Sci. Paris, Série 1, 317:723–740, 1993. The word and Riemannian metrics on lattices of semisimple groups, *I.H.É.S. Publ. Math.*, 91:5–53, 2000.

[63] Filling in solvable groups and in lattices in semisimple groups, *Topology*, 43:983–1033, 2004.

[64] On quadratic Dehn functions, *Math. Z.*, 248(4):725—755, 2004.

[65] M.Elder, L_δ groups are almost convex and have a sub-cubic Dehn function, *Algebr. Geom. Topol.*, 4:23–29 (electronic), 2004.

[66] Navigating the Cayley graphs of $\mathrm{SL}_N(\mathbb{Z})$ and $\mathrm{SL}_N(\mathbb{F}_p)$, *Geometriae Dedicata*, 113(1):215–229, 2005.

[67] M.Kassabov and T.R.Riley, Diameters of Cayley graphs of Chevalley groups, to appear in Eur. J. Comb.

[68] M.R. Bridson and K. Vogtmann, Automorphism groups of free, surface, and free abelian groups, `arXiv:math.GR/0507612`.

53. **Kim Ruane**
My Two Favorite CAT(0) Group Questions

First, I would like to wish you a Happy Retirement Guido! I would also like to thank you for your kindness during my time in Zurich back in 1999/2000.

The conjectures discussed here are two of my favorites because I learned of them as a graduate student and they motivated me to study groups acting on CAT(0) spaces and boundaries.

We say a group G acts *geometrically* on a complete, proper, geodesic metric space X if G acts properly discontinuously and cocompactly by isometries on X. If G acts geometrically on a CAT(0) space, then G is called a *CAT(0) group*. Recall that if G acts geometrically on a δ-hyperbolic metric space, then G is word hyperbolic.

For G word hyperbolic, the following facts are well-known and can be found in the 1987 Gromov paper in Essays in Group Theory. Of course, many careful proofs have been written down in other places.

(1) Any Cayley graph of G is δ-hyperbolic using the corresponding word metric.
(2) The *boundary* of G, denoted ∂G, is well-defined up to homeomorphism - i.e., if G acts geometrically on spaces X and Y, then X and Y are quasi-isometric and this quasi-isometry extends to an (equivariant) homeomorphism of boundaries $\partial X \to \partial Y$.
(3) G acts as a convergence group on ∂G.
(4) G satisfies the Tits Alternative.
(5) Any finite index subgroup and any finite extension of G is again word hyperbolic.

The two conjectures I am interested in here involve the last two facts listed above, but for CAT(0) groups as opposed to word hyperbolic groups. If G is word hyperbolic, it is easy to see that any finite index subgroup or any finite extension of G is again word hyperbolic. Indeed, any such group is quasi-isometric to G and thus inherits word hyperbolicity via the quasi-isometry.

If G is a CAT(0) group acting on a CAT(0) space X and H is a finite index subgroup of G, then H is again a CAT(0) group. This is easy since H acts geometrically on X. Indeed, any subgroup of G will again act properly discontinuously and by isometries. Since H is finite index, H will also act cocompactly. But if K is a finite extension of G, then the question remains:

Question 53.1. *Suppose K is a finite extension of a CAT(0) group G. Is K also a CAT(0) group?*

The main problem here is that there is no geometric construction that models the group theoretic finite extension. It is still the case that K and G are quasi-isometric groups, but there is no natural candidate for a CAT(0) space for K to act on. Well, that isn't quite true...there is a candidate which comes from a construction of Serre. Suppose G is a finite index normal subgroup of K of index D and suppose G acts on a topological space X. Then Serre's construction gives an action of K on the direct product of D copies of X. In our setting, if G acts geometrically on X, then Serre's construction will produce a properly discontinuous and isometric action of K on the product of D copies of X (which is still CAT(0) using the product metric). The problem is finding a convex subspace on which K acts cocompactly.

The second conjecture is the Tits Alternative for CAT(0) groups. Recall that a group G satisfies the Tits Alternative if for every subgroup H of G, either H is virtually solvable or H contains a free subgroup of rank 2.

Question 53.2. *Does the Tits Alternative hold for G if G is a CAT(0) group?*

This question is still open even if the CAT(0) space is a manifold.

If G is word hyperbolic, then G satisfies the Tits Alternative. This fact was first observed by Gromov and there are many nice proofs written down in the literature. The beauty of this result (for me anyway!) is that the proof is quite simple if you use the action of the group G on its boundary ∂G. The proof goes like this: suppose H is an infinite subgroup of G and consider the closure $\overline{H}$ of H inside $G \cup \partial G$. The *limit set* of H, denoted $\mathcal{L}(H)$, is $\overline{H} \cap \partial G$. One first shows that $|\mathcal{L}(H)| \geq 2$. If it equals 2, then H is virtually $\mathbb{Z}$. If not, then there must be two infinite order elements $a, b \in H$ with $\mathcal{L}(\langle a \rangle) \cap \mathcal{L}(\langle b \rangle) = \emptyset$. Using the dynamics of the action on ∂G, one can do a ping-pong argument using carefully chosen open sets around the limit points of these two cyclic subgroups to show that powers of a and b generate an F_2 in H. This is simply beautiful and everyone should want to do geometric group theory after seeing this proof!

For a CAT(0) group G acting on X, one could try to use the action of G on X. However, this is not a convergence group action. In particular, $\mathbb{Z} \oplus \mathbb{Z}$ acts trivially on $\partial \mathbb{E}^2 \equiv S^1$. The most recent result of interest here is from M. Sageev and D. Wise (2004) for groups acting properly on finite dimensional CAT(0) cube complexes. If such a group G has a bound on the order of finite subgroups (a necessary condition) then any subgroup either contains F_2 or is virtually a finitely generated abelian subgroup.

54. **Roman Sauer and Thomas Schick**

Homotopy invariance of almost flat Betti numbers

Dear Guido, we would like to present a question that belongs to the circle of analytical-topological problems around the Novikov conjecture.

The Novikov conjecture about the homotopy invariance of higher signatures has been a driving force for research in topology for many years. One of the approaches, which gives partial results, uses the signature operator twisted with almost flat bundles.

In this context Hilsum and Skandalis (J. Reine Ang. Math. 423, 1992) prove the following result.

Theorem. Let M_1 and M_2 be closed, oriented Riemannian manifolds, and $f\colon M_1 \to M_2$ a homotopy equivalence. Then there is a constant $c > 0$ with the following property:

Let (E, ∇) be an Euclidean vector bundle over M_2. Let $s_2(E)$ denote the index of the signature operator on M_2 twisted with E, and let $s_1(f^*E)$ denote the index of the signature operator on M_1 twisted with the pullback bundle f^*E.

If $||\nabla^2|| < c$, i.e. if the curvature of the bundle (E, ∇) is sufficiently small (where the norm is the supremum of the operator norm in the unit sphere bundle of $\Lambda^2 TM_2$), then

$$s_1(E) = s_2(f^*E).$$

In other words, the index of the twisted signature operator is a homotopy invariant for twisting bundles with sufficiently small curvature.

Hilsum and Skandalis prove this with a clever deformation argument (with a proof that evidently also covers the flat case). An alternative proof, which reduces the statement to the homotopy invariance for flat twisting bundles of C^*-algebra modules and bases on calculations in the K-theory of C^*-algebras, has been recently worked out by Bernhard Hanke and one of us (Thomas).

The following question is motivated by the theorem above and arose from discussions with Paolo Piazza and Sara Azzali.

Question. Retain the situation of the Theorem stated above. The kernels of the twisted signature operators are graded by the degree of differential forms. By taking their dimensions we obtain the so-called *twisted Betti numbers* of E and f^*E, which we denote by $b_k(E)$ and $b_k(f^*E)$.

Is it true that, for sufficiently small $c > 0$ as in the theorem, $||\nabla^2|| < c$ implies $b_k(E) = b_k(f^*E)$?

Note that this *is* the case for the corresponding Euler characteristic:

$$\sum_k (-1)^k b_k(E) = \sum_k (-1)^k b_k(f^*E)$$

for sufficiently small curvature $||\nabla^2||$, since the Euler characteristic is again an index. The proof for the invariance of the twisted Euler characteristic is actually considerably easier than the one for the invariance of the twisted signature.

We find the question interesting since a positive answer would imply that the η-invariants of twisted signature operators are much better behaved than one could expect *a priori*. This could open up the way to construct new "higher" homotopy invariants of smooth manifolds.

It seems that there is no "standard" approach to answer the question. Analysists tend to think the statement should not be true. However, if we twist with a flat bundle, the answer to our question is *yes* — but requires the insight of the de Rham theorem. Analysis alone can be misleading dealing with such questions, because it suggests too many deformations that do not have a topological meaning.

55. **Benjamin Schmidt**
Blocking light in closed Riemannian manifolds

To what extent does the collision of light determine the global geometry of space? In this note we'll discuss two conjectures, both of which assert that the focusing behavior of light in locally symmetric Riemannian manifolds are unique to these spaces. Throughout, (M, g) denotes a C^∞-smooth, connected, and compact manifold without boundary equipped with a C^∞-smooth Riemannian metric g. Geodesic segments $\gamma \subset M$ are identified with their unit speed paramaterization $\gamma : [0, L_\gamma] \to M$, where L_γ is the length of the segment γ.

Definition 55.1 (Light)**.** Let $X, Y \subset (M, g)$ be two nonempty subsets, and let $G_g(X, Y)$ denote the set of geodesic segments $\gamma \subset M$ with initial point $\gamma(0) \in X$ and terminal point $\gamma(L_\gamma) \in Y$. The *light from X to Y* is the set

$$L_g(X, Y) = \{\gamma \in G_g(X, Y) | \text{interior}(\gamma) \cap (X \cup Y) = \emptyset\}.$$

Definition 55.2 (Blocking Set)**.** Let $X, Y \subset M$ be two nonempty subsets. A subset $B \subset M$ is a *blocking set* for $L_g(X, Y)$ provided that for every $\gamma \in L_g(X, Y)$,

$$\text{interior}(\gamma) \cap B \neq \emptyset.$$

We focus on closed Riemannian manifolds for which the light between pairs of points in M is blocked by a finite set of points. By a theorem of Serre ([Ser]), $G_g(x, y)$ is infinite when $x, y \in M$ are distinct points. However, $L_g(x, y) \subset G_g(x, y)$ may or may not be a infinite subset. This is the case, for example, in a round sphere where all of the infinitely many geodesics between a typical pair of points cover a single periodic geodesic.

Definition 55.3 (Blocking Number)**.** Let $x, y \in M$ be two (possibly not distinct) points in M. The *blocking number* $b_g(x, y)$ for $L_g(x, y)$ is defined as

$$b_g(x, y) = \inf\{n \in \mathbb{N} \cup \{\infty\} | L_g(x, y) \text{is blocked by } n \text{ points}\}.$$

Our starting point is the following surprising theorem from [Gut]:

Theorem 55.4 (Gutkin)**.** *Let (M, g) be a closed flat Riemannian manifold. Then there exist $n \in \mathbb{N}$ depending only on the dimension of M such that $b_g \leq n$ as a function on $M \times M$.*

We believe the following is true:

Conjecture 55.5. *Let (M, g) be a closed Riemannian manifold. If there exists $n \in \mathbb{N}$ such that $b_g \leq n$ then g is a flat metric.*

This conjecture is true for Riemannian metrics of nonpositive sectional curvatures as shown independently by Burns/Gutkin and Lafont/Schmidt ([Bur/Gut], [Laf/Sch]). The focusing of light is also interesting in the context of the compact type locally symmetric spaces. In [Gut/Sch], Gutkin and Schroeder establish the following:

Theorem 55.6 (Gutkin/Schroeder)**.** *Let (M, g) be a closed locally symmetric space of compact type with $\mathbb{R}$-rank $k \geq 1$. Then $b_g(x, y) \leq 2^k$ for almost all $(x, y) \in M \times M$.*

We refer the reader to [Gut/Sch] for a more precise formulation and discussion of this result. Presently, we'll restrict attention to the compact rank one symmetric spaces or CROSSes. The CROSSes are classified and consist of the round spheres and the various projective spaces. The CROSSes all satisfy the following blocking property:

Definition 55.7 (Cross Blocking)**.** A closed Riemannian manifold (M, g) has property CB if

$$0 < d(x, y) < \mathrm{Diam}(M, g) \implies b_g(x, y) \leq 2.$$

Round spheres additionally satisfy the following blocking property, a blocking interpretation of antipodal points:

Definition 55.8 (Sphere Blocking)**.** A closed Riemannian manifold (M, g) has property SB if $b_g(x, x) = 1$ for every $x \in M$.

We believe the following is true:

Conjecture 55.9. *A closed Riemannian manifold (M, g) has property CB if and only if (M, g) is isometric to a compact rank one symmetric space. In particular, (M, g) has properties CB and SB if and only if (M, g) is isometric to a round sphere.*

In [Laf/Sch], special cases of this conjecture are confirmed under various additional hypotheses.

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56. **Warren Sinnott**

Power series that generate class numbers

Let k be a CM field, k^+ its totally real subfield, so that k is totally imaginary and $[k : k^+] = 2$. Let p be a prime, and let K be the basic $\mathbb{Z}_p$-extension of K: then $K \subset k(\mu_{p^\infty})$, and k has a unique extension k_n in K of degree p^n over k. Let h_n^* denote the relative class number of k_n/k_n^+. Then Iwasawa showed that there are integers $\mu \geq 0, \lambda \geq 0$ and ν such that

$$\mathrm{ord}_p(h_n^*) = \mu p^n + \lambda n + \nu \tag{1}$$

for n greater than or equal to some integer n_0. One way to show this (not Iwasawa's original method, which gives more general results) is to use Hecke's analytic class number formula and the theory of p-adic L-functions: these results imply that there is a power series $F(T) \in \mathbb{Z}_p[[T-1]]$ such that

$$h_n^* = h_{n_0}^* \prod_{\substack{\zeta^{p^n}=1 \\ \zeta^{p^{n_0}} \neq 1}} F(\zeta) \text{ for } n \geq n_0. \tag{2}$$

The Weierstrass Preparation Theorem implies that we may write $F(T) = p^\mu Q(T)u(T)$, where $\mu \geq 0$, $Q(T)$ is a monic polynomial of degree λ congruent to $(T-1)^\lambda$ mod p, and $u(T)$ is a unit in $\mathbb{Z}_p[[T-1]]$. From this one can see that (2) $\implies$ (1).

But (2) contains much more information than (1), since it gives a formula for the whole relative class number. My questions (basically just questions about formal power series) are:

Question 56.1. *What does (2) tell us about class numbers? i.e. what constraints are imposed on the sequence $\{h_n^*\}$ by the formula (2)?*

For example, (2) has the following curious consequence: let $(h_n^*)'$ denote the "prime-to-p" part of h_n^*. Then (2) implies that

$$\lim_{n\to\infty} (h_n^*)' \text{ exists in } \mathbb{Z}_p^\times. \tag{3}$$

H. Kisilevsky[69] pointed out that (as with (1)) the limit (3) exists for the prime-to-p part (in fact for the ℓ-primary part for any $\ell \neq p$) of the class numbers of *any* $\mathbb{Z}_p$-extension.

Conversely, we can ask:

Question 56.2. *What does (2) tell us about $F(T)$?*

For example, if $a \in \mathbb{Z}_p^\times$ then $F(T^a)$ gives the same sequence h_n^*, so $F(T)$ not completely determined by (2). How much information about $F(T)$ is contained in (2)?

[69] see Pacific J. Math. 1997, Special Issue, *(Olga Taussky-Todd: in memoriam)*, 225–229.

The latter question is interesting since the "Main Conjecture" of Iwasawa theory (proved in the 1980s by Wiles) identifies $F(T)$ — up to a unit in $\mathbb{Z}_p[[T-1]]$ — with a characteristic polynomial defined from the action of $\mathrm{Gal}(K/k)(\simeq \mathbb{Z}_p)$ on the p-primary part of the ideal class group of K.

57. **Ron Solomon and Radu Stancu**
Conjectures on Finite p-Local Groups

Dear Guido,

Congratulations and warmest wishes for the post-retirement continuation of a glorious career! We know you have been interested in localization in topology and group theory. A fundamental conjecture in this context is Alperin's Weight Conjecture. Here is a formulation with a topological flavor.

Definition 57.1. Let G be a finite group. A p-chain C of G is a strictly increasing chain

$$C : P_0 < P_1 < \cdots < P_n$$

of p-subgroups of G. We let C_i denote the initial subchain terminating at P_i. The chain C is radical if $P_0 = O_p(G)$ and, for each i, $P_i = O_p(N_G(C_i))$.

We denote by $\mathcal{R}(G)$ the set of all radical p-chains of G, and by $\mathcal{R}(G)/G$ its orbit space.

Definition 57.2. Fix a prime p. Let ϕ be an ordinary irreducible character of the finite group H. The defect $d(\phi)$ of ϕ is the largest non-negative integer d such that p^d divides $\frac{|H|}{\phi(1)}$.

Definition 57.3. Let G be a finite group and B a p-block of G. For any p-chain C of G we denote by $k(C, B, d)$ the number of characters $\phi \in Irr(N_G(C))$ having defect $d(\phi) = d$ and belonging to a p-block $B(\phi)$ of $N_G(C)$ such that the induced p-block $B(\phi)^G$ is B.

Conjecture 57.4. *Let p be a prime, G a finite group with $O_p(G) = 1$, and B a p-block of G which is not of defect 0. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} k(C, B, d) = 0.$$

Next, here is a statement about finite groups. In some sense it is not a conjecture because Ron is fairly certain it can be proved easily as a corollary of the Classification of the Finite Simple Groups. The conjecture is that there is an elementary proof, perhaps following from further advances in the p-modular representation theory of finite groups.

Definition 57.5. Let p be a prime and P a p-group. Then $\Omega_1(P)$ is the subgroup of P generated by all elements of order p.

Conjecture 57.6. *Let p be a prime and let G be a finite group having an abelian Sylow p-subgroup A. Suppose that B is a strongly closed subgroup of A with respect to G, i.e. if $x \in B$ and $g \in G$ with $x^g \in A$, then $x^g \in B$. Then there exists a normal subgroup N of G having* *a Sylow p-subgroup* B^* *such that* $\Omega_1(B) = \Omega_1(B^*)$. *In particular, if* $A = \Omega_1(A)$ *and* B *is strongly closed in* A *with respect to* G, *then* $B \in \mathrm{Syl}_p(N)$ *for some normal subgroup* N *of* G.

Finally, you are probably familiar with the recent work of Broto, Levi, and Oliver on the theory of so-called p-local groups. The underlying structure of a p-local groups are the fusion systems on a finite p-group P. Here is the definition of a fusion system on P. First let us start with a more general definition

Definition 57.7. A *category* $\mathcal{F}$ *on a finite* p*-group* P is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is the set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of injective group homomorphisms from Q to R, with the following properties:
(a) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$.
(b) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ the induced isomorphism $Q \simeq \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.
(c) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

And now the definition of a fusion system:

Definition 57.8. A *fusion system* $\mathcal{F}$ on a finite p-group P is a category on P satisfying the following properties:
(1) $\mathrm{Hom}_P(Q, R) \subset \mathrm{Hom}_{\mathcal{F}}(Q, R)$ for all $Q, R \leq P$.
(2) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.
(3) Every $\phi : Q \to P$ such that $|N_P(\phi(Q))|$ is maximal in the $\mathcal{F}$-isomorphism class of Q, extends to $\bar{\phi} : N_\phi \to P$ where

$$N_\phi = \{x \in N_P(Q) \,|\, \exists\, y \in N_P(\phi(Q)),\, \phi({}^x u) = {}^y\phi(u) \forall u \in Q\}.$$

Axiom (2) is saying that P is a 'Sylow p-subgroup' of $\mathcal{F}$ and the extension in Axiom (3) is equivalent to saying that any p-subgroup can be embedded by conjugation in a Sylow p-subgroup. In particular, if G is a finite group, p is a prime divisor of $|G|$, and $P \in \mathrm{Syl}_p(G)$, then the morphisms given by conjugation by elements of G between the subgroups of P determine a fusion system $\mathcal{F}_G(P)$ on P.

We say that a fusion system is *exotic* if it does not arise in this way. There is some dispute about the correct definition of a normal subobject in this theory. Here is Markus Linckelmann's definition of a normal subsystem. First let's introduce the notion of *strongly* $\mathcal{F}$*-closed* subgroups.

Definition 57.9. Let $\mathcal{F}$ be a fusion system on a finite p-group P and Q a subgroup of P. We say that Q is *strongly* $\mathcal{F}$*-closed* if for any subgroup R of Q and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ we have $\phi(R) \leq Q$.

And now the notion of normal fusion subsystem.

Definition 57.10. Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on a subgroup P' of P. We say that $\mathcal{F}'$

is *normal* in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\phi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have

$$\phi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \phi^{-1} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R), \phi(R'))\,.$$

Here is a conjecture on normal fusion systems:

Conjecture 57.11. *Let $\mathcal{F}$ be a fusion system on P and P' a strongly $\mathcal{F}$-closed subgroup of P. Then there exists a normal fusion subsystem $\mathcal{F}'$ of $\mathcal{F}$ on P'.*

A *simple fusion* system is a fusion system that has no non-trivial normal subsystems. In particular, if G is a finite simple group which does not have a proper strongly p-embedded subgroup, then $\mathcal{F}_G(P)$ is a simple fusion system.

When p is odd, it seems to be fairly easy to construct examples of exotic simple p-local groups. On the other hand when $p = 2$, the only known exotic examples live in a single infinite family, $\mathcal{F}_{Sol}(q)$, which may be regarded as the analogue of finite Chevalley groups with respect to the exotic 2-compact group of Dwyer and Wilkerson.

Conjecture 57.12. *The family $\mathcal{F}_{Sol}(q)$ contains all of the exotic simple 2-local groups.*

This conjecture is hard to believe. On the other hand, it has thus far been impossible to dream up other examples. It is conceivable that it could be proved by a lengthy and elaborate analysis in the vein of the traditional 2-local analysis of finite simple groups used in the proof of the Classification Theorem. It would be much more interesting if homotopy-theoretic tools could be brought to bear to prove this result. That would really give hope for new and exciting applications of topology to finite group theory.

Another source of examples for fusion systems are the fusion systems coming from p-blocks of group algebras, given by the conjugations between the Brauer pairs in a maximal Brauer pair of the p-block. Such examples are called Brauer categories. Here's a natural question one can ask.

Question 57.13. *Are there Brauer categories which are exotic fusion systems?*

It is pretty hard to check that a given fusion system is not a Brauer category. Up to now, the only known way to do it is by reduction to Brauer categories of quasisimple groups and then by using the classification of finite simple groups. There are examples of embedded fusion systems where the minimal and the maximal ones come from finite groups and the intermediate ones are exotic. It is not yet known whether these exotic fusion systems are Brauer categories.

58. **Olympia Talelli**

On algebraic characterizations for the finiteness of the dimension of $\underline{E}G$.

In [K-M][70] the following theorem was proved:

Theorem 58.1. *If G is an HF-group of type FP_∞ then G admits a finite dimensional model for $\underline{E}G$.*

The class HF was introduced by P. H. Kropholler in [K][71] and it is defined as the smallest class of groups containing the class of finite groups, with the property: if a group G admits a finite dimensional contractible G-CW-complex with all cell stabilizers in HF then G is in HF.

This theorem, especially its proof, was the motivation for defining groups of type Φ in [T] [72] and propose those as the ones which admit a finite dimensional model for $\underline{E}$G.

Definition 58.2. [T]: A group G is said to be of type Φ if it has the property that for every ZG-module M, $\text{projdim}_{ZG}M < \infty$ if and only if $\text{projdim}_{ZH}M < \infty$ for every finite subfroup of H of G.

Conjecture 58.3. *[T] The following statements are equivalent for a group G:*

(1) *G admits a finite dimensional model for $\underline{E}G$.*
(2) *G admits a finite dimensional contractible G-CW-complex with finite cell stabilizers.*
(3) *G is of type Φ.*
(4) *spliZG$< \infty$.*
(5) *silpZG$< \infty$.*
(6) *findimZG$< \infty$.*

The algebraic invariants spliZG and silpZG were defined in [G-G][73] silpZG is the supremum of the injective lengths of the projective ZG-modules and spliZG is the supremum of the projective lengths of the injective ZG-modules. It was shown in [G-G] that silpZG$\leq$spliZG, and that if spliZG$< \infty$ then spliZG=silpZG. The finitistic dimension of ZG, findimZG, is the supremum of the projective dimensions of the ZG-modules of finite projective dimension.

Now (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). It follows from Theorem 58.1 that (6) $\Rightarrow$ (1) if G is an HF-group with a bound on the orders of

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[71][K] P. H. Kropholler On groups of type FP_∞, J. Pure Appl. Algebra 90 (1993), 55-67.

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the finite subgroups. Moreover in [T] it is shown that (6) $\Rightarrow$ (1) if G is a torsion-free locally soluble group.

59. **Alain Valette**
Short exact sequences and a-T-menability

A locally compact, σ-compact group is *a-T-menable*, or has *the Haagerup property*, if it admits a metrically proper isometric action on a Hilbert space. The class of a-T-menable groups is a huge class, containing amenable groups, free groups, surface groups, Coxeter groups, and much more...(see CCJJV [74] for more information on that class). The interest of this class stems from a remarkable result by N. Higson and G. Kasparov [75], that a-T-menable groups satisfy the strongest possible form of the Baum-Connes conjecture, namely the Baum-Connes conjecture with coefficients.

In presence of an interesting class of groups, it is a natural question to ask whether it is stable under short exact sequences. For a-T-menability, this is well-known *not* to be the case: e.g. $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ are a-T-menable, but the semi-direct product $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is not, because of the relative property (T) with respect to the normal subgroup.

Question 59.1. *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of locally compact groups, with N and Q a-T-menable. Under which conditions is G a-T-menable?*

For example, this is known to be the case if Q is amenable, as shown in CCJJV. Let us single out the case of central extensions in Question 59.1:

Conjecture 59.2. *Let $1 \to Z \to G \to Q \to 1$ be a central extension. If Q is a-T-menable, then so is G.*

Some evidence for that conjecture appears in CCJJV (in particular the case of the universal cover of $SU(n, 1)$).

When H, Q are (non-trivial) countable groups, recall that the *wreath product* $H \wr Q$ is the semi-direct product $N \rtimes Q$, where $N = \oplus_Q H$ is a direct sum of copies of H indexed by Q, and Q acts on N by shifting indices.

Conjecture 59.3. *Assume that H and Q are a-T-menable. Then so is $H \wr Q$.*

Evidence for this conjecture comes from a result of M. Neuhauser [76]: if H, Q are a-T-menable, then $H \wr Q$ has no infinite subgroup with the relative property (T). Admittedly, this evidence is limited in view of a recent construction by Y. de Cornulier [77]: there exists countable

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[77] J. Lie Theory 16, 67-82, 2006.

groups (actually S-arithmetic lattices) which are *not* a-T-menable, and do not contain any infinite subgroup with the relative property (T). As a particular case of Conjecture 59.3, we single out what seems to be the first case to look at:

Conjecture 59.4. *Let $\mathbb{F}_2$ denote the free group on 2 generators, and let H be a (non-trivial) finite group. Then $H \wr \mathbb{F}_2$ is a-T-menable.*

Dear Guido, my best wishes for a happy and active retirement!

60. **Kalathoor Varadarajan**

A realization problem

I consider myself singularly fortunate in knowing Guido personally as well as mathematically. He belongs to the rare group of people who are simultaneously outstanding as mathematicians and perfect gentlemen. I wish him a long, happy and mathematically productive retired life.

In his groundbreaking papers[78], C.T.C. Wall associated with each (always assumed 0-connected) finitely dominated space X an element $w(X)$ in $\tilde{K_0}(Z(\pi))$ where $\pi = \pi_1(X)$ and proved that X is of the homotopy type of a finite CW-complex if and only if $w(X) = 0$. Also $w(X)$ is an invariant of the homotopy type of X. In subsequent literature $w(X)$ is referred to as the finiteness obstruction (alternatively as the Wall obstruction) of X. Another major result proved by Wall asserts that given any finitely presented group π and any element x in $\tilde{K_0}(Z(\pi))$, there exists a finitely dominated CW-complex X with $\pi_1(X)$ isomorphic to π and $w(X) = x$ in $\tilde{K_0}(Z(\pi))$. Using Dock Sang Rim's result[79] that $\tilde{K_0}(Z(\pi_p))$ for any prime p is isomorphic to the ideal class group $Cl(Z[\omega])$, where π_p denotes a cyclic group of order p and $\omega = exp(2\pi i/p)$ and the fact that $Cl(Z[\omega])$ is not zero when $p = 23$, Wall shows that there exist finitely dominated CW-complexes which are not of the homotopy type of a finite CW-complex. This settled a famous problem of J.H.C. Whitehead [80] in the negative.

Guido is the first person who started studying the Wall obstruction of finitely dominated nilpotent spaces [81]. In his 1976 work he proved that $w(X) = 0$ for any finitely dominated nilpotent space with $\pi_1(X)$ infinite. In his 1975 work he showed that if X is a finitely dominated nilpotent space with $\pi_1(X)$ finite cyclic, then $w(X)$ has to satisfy certain restrictions. Inspired by his results, I extended his 1975 results to finitely dominated nilpotent spaces with finite abelian fundamental groups. My result[82] appeared in 1978. For any nilpotent group π, let $\overline{Z(\pi)}$ denote a maximal order in $Q(\pi)$ containing $Z(\pi)$ and $D(Z(\pi))$ denote the kernel of

$$j_* : \tilde{K_0}(Z(\pi)) \to \tilde{K_0}(\overline{Z(\pi)}).$$

[78] *Finiteness conditions for CW-complexes*, Ann. of Math.**81** (1965), 57-69, and *Finiteness conditions for CW-complexes ll*, Proc. Royal Soc. London Ser. A**295**(1966),129-139.

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[81] G. Mislin: *Wall obstruction for nilpotent spaces*, Topology**14**(1975) 311-317. and *Finitely dominated nilpotent spaces*, Ann. of Math.**103** (1976) 547-556.

[82] *Finiteness obstruction for nilpotent spaces*, J. Pure and Appl. Alg.**12** (1978), 137-146.

In the joint paper[83] in 1979, Guido and myself showed that for any finitely dominated nilpotent space X with a finite (necessarily nilpotent) fundamental group π , the Wall obstruction $w(X)$ satisfies the restriction that $w(X)$ is in $D(Z(\pi))$. This considerably strengthened the result in 1978.

As stated earlier in this article, for any finitely presented group π and any element x in $\tilde{K}_0(Z(\pi))$, there exists a finitely dominated CW-complex X with $\pi_1(X) = \pi$ and $w(X) = x$ (Wall's work in 1965, 1966). This suggests the following

Conjecture 60.1. *Let π be a finite nilpotent group and x any element in $D(Z(\pi))$. Then there exists a finitely dominated nilpotent space X with $\pi_1(X) = \pi$ and $w(X) = x$.*

In this article, I have concentrated on just one aspect of Guido's work. His work is very profound and has influenced the development of Topology in many ways.

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61. **Clarence Wilkerson**

Finiteness properties of the monoid of self-homotopy equivalences of a simply connected finite CW complex

Roughly speaking, the monoid $\mathrm{Aut}(X)$ plays a role in the homotopy category for a simply connected CW complex X that the group $\mathrm{Isom}(M)$ of isometries plays in differential geometry for a compact Riemannian manifold$M$. The latter is a compact Lie group and thus possesses many finiteness properties. The first analogous properties for $\mathrm{Aut}(X)$ are due to Sullivan and Wilkerson (70's) and Lannes-Schwartz-Zarati (80's):

1) $\pi_0(\mathrm{Aut}(X)$ is commensurable to an arithmetic group.
2) $H^*(\mathrm{Aut}(X), \mathbb{F}_p)$ is a locally finite module over the Steenrod algebra.

These are weak analogues of $\pi_0(G)$ is finite and G is a finite CW complex. It's therefore natural to ask what properties the classifying space $\mathrm{BAut}(X)$ might have in common with BG, for G a compact Lie group. In light of the arithmetic nature of π_0, it suffices to study this question for those maps homotopic to the identity, $\mathrm{Aut}Z_1(X)$. The first example is encouraging:

(Milgram and F. Cohen): Let $BSO(n) \to \mathrm{BAut}_1(S^{n-1})$ be induced from the forgetful map. The induced map on mod 2 cohomology is an isomorphism modulo nilpotent elements.

Dwyer-Wilkerson translated this using the T-functor:

Up to FHE, the only actions of $Z/2Z$ on S^{n-1} are linear actions.

The translation is as follows. Dwyer and Kan have shown that actions of a group G on X correspond to the Borel fibrations $X \to EG \times_G X \to BG$. These, up to FHE, are classified by maps of BG to $\mathrm{BAut}(X)$, which in the case $G = Z/2Z$ can be calculated using T and the information provided by Milgram-Cohen.

However, efforts to find similar phenomena in general failed for several years. Finally, Jeff Smith and Clarence Wilkerson produced a decisive example:

There exists a s.c. finite CW complex X with infinitely many distinct (up to FHE) $Z/2Z$ actions.

The X was constructed as a suspension with infinitely many non-homotopy equivalent desuspensions. The example has a corollary:

There exists a s.c. finite CW complex X such that $H^*(\mathrm{BAut}_1(X), \mathbb{F}_2)$ is not finite modulo nilpotents, decomposables, and the action of the Steenrod algebra.

This seems to eliminate the possibility in general of nice finiteness properties for $\mathrm{BAut}_1(X)$.

However, recently in work of Dwyer, Smith, and Wilkerson, an attractive candidate has emerged. Recall that a group action $G \times X \to X$ is deemed effective if only the identity in G moves no point of X. That is, for any cyclic subgroup C of G, the fixed point set $X^C \neq X$.

One can define a homotopical version of this for a p-group G by requiring that fot $Z/pZ \to G$, the induced fibration pulled back from the Borel fibration over BG to BZ/pZ not be trivial. (This is equivalent to the homotopy fixed point set $X^{hZ/pZ} \neq X$).

Conjecture: If X is a s.c. finite CW complex, then there exists $N_X \geq 0$ such that if E is an elementary abelian p-group and $\mathrm{rank}(E) > N_X$, then E has no homotopocally effective action on X.

We believe that work of Mann (60's) with Smith theory can be translated and applied to this in the case that X is a mod p Poincaré duality complex.

62. **Kevin Wortman**

Finiteness properties of function-field-arithmetic groups

Let K be a global function field, and S a finite nonempty set of pairwise inequivalent valuations on K. We let $\mathcal{O}_S \leq K$ be the corresponding ring of S-integers, and we let $\mathbf{G}$ be a simple K-group. For any $v \in S$, we let K_v be the completion of K with respect to v.

The finiteness properties of function-field-arithmetic groups such as $\mathbf{G}(\mathcal{O}_S)$ have been of interest for at least the past 47 years. Some of that interest has fallen on which of these groups are of type FP_m for a given m. For example, for which m is it true that $\mathbf{SL_n}(\mathbb{F}_q[t])$ is of type FP_m where $\mathbb{F}_q[t]$ denotes a polynomial ring with one variable and coefficients in a finite field with q elements.

We recall that a group Γ is *of type FP_m* if there exists a partial projective resolution

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

of finitely generated $\mathbb{Z}\Gamma$ modules, where the action of $\mathbb{Z}\Gamma$ on $\mathbb{Z}$ is trivial.

All of the evidence thus far indicates the existence of a solution for the following

Conjecture. *With $\mathbf{G}(\mathcal{O}_S)$ as above and with $k = \sum_{v \in S} \mathrm{rank}_{K_v}(\mathbf{G})$, the arithmetic group $\mathbf{G}(\mathcal{O}_S)$ is of type FP_m if and only if either $\mathrm{rank}_K(\mathbf{G}) = 0$ or $k > m$.*

For example, according to the above conjecture $\mathbf{SL_n}(\mathbb{F}_q[t])$ should be of type $FP_{(n-2)}$ but not of type $FP_{(n-1)}$. In fact this was shown independently by Abels and Abramenko for large values of q.

Plenty of other evidence exists to support the conjecture in general including a proof of the "only if" implication; see the papers of Abels, Abramenko, Behr, Bux-Wortman, Hurrelbrink, Keller, Kneser, Lubotzky, McHardy, Nagao, O'Meara, Rehmann-Soulé, Serre, Splitthoff, and Stuhler [Abl], [Abr 1], [Abr 2], [Abr 3], [Be 1], [Be 2], [Be 3], [Be 4], [Bu-Wo], [Hu], [Ke], [Lu], [McH], [Na], [OM], [R-S], [Se 1], [Se 2], [Spl], and [St], [St 2].

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Editor's comment

Dear Guido,

It has been an honor and a lot of fun to edit this gift for you. I am really grateful to the contributors and even more to Gwynyth, Henry, Ian, Mike and Tadeusz for their help, for the enthusiastic support this idea received and for all the efforts put at such short notice. It was really cool (but not at all a surprise) to see that so many people are that fond of you and your work.

Most likely, very important people are missing as we forgot a few, some had aggressive spam filters that directed our mass mailings to the trash and finally others were already gone on holiday: some lack of organization made us start the project kind of late (less than two months ago)!

Somehow I feel that the present volume is a beginning rather than a finished thing, let's see...

Indira