# ON THE BOUSFIELD CLASSES OF $H_\infty$-RING SPECTRA

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ABSTRACT. We prove that any $K(n)$-acyclic, $H_\infty$-ring spectrum is $K(n+1)$-acyclic, affirming an old conjecture of Mark Hovey.

CONTENTS



Throughout this paper, all spectra will be $p$-local for a fixed prime $p$.

## 1. INTRODUCTION

A bedrock result of chromatic homotopy theory is that any $K(n)$-acyclic, finite spectrum is $K(n-1)$-acyclic. Our goal here is to prove that $H_\infty$-ring spectra enjoy the opposite phenomenon:

**Theorem 1.1.** *Suppose $R$ is a $K(n)$-acyclic, $H_\infty$-ring spectrum. Then $R$ is $K(n+1)$-acyclic.*

**Corollary 1.1.1.** *Suppose $R$ is a complex-orientable, $H_\infty$-ring spectrum that kills a finite complex. Then $R$ has the Bousfield class of $E(n)$ for some $n$.*

These results settle 'Miscellaneous Problem 2' from Mark Hovey's 1999 list of unsolved problems in algebraic topology [Hov99].

We will prove Theorem 1.1 for $n > 0$. The theorem is already known when $n = 0$, where it is a consequence of an old conjecture due to J.P. May:

**May Nilpotence Conjecture [MNN15, Theorem 2.1].** *If $R$ is an $H_\infty$-ring spectrum, and $R \otimes \mathbb{Q} \simeq 0$, then $R$ is $K(n)$-acyclic for all $n > 0$.*

The first written proof of May's conjecture is due to Mathew, Naumann, and Noel [MNN15], and these authors found spectacular applications in joint work with Clausen [NN16].

Let $E$ denote the height $n+1$ Morava $E$-theory with $\pi_0 E \cong \mathbb{Z}_p[[u_1, u_2, ..., u_n]]$. Standard techniques, which we review in Section 2, reduce Theorem 1.1 to the following Theorem 1.2:

**Theorem 1.2.** *Suppose $R$ is a $K(n+1)$-local, $H_\infty$-$E$-algebra such that, in $\pi_0 R$, some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1})$. Then 1 is in the ideal $(p, u_1, ..., u_{n-1})$.*

Our proof of Theorem 1.2 is by infinite descent: we use power operations to show that, if some power of $u_n$ lies in $(p, u_1, ..., u_{n-1})$, then so must some lower power. This is analogous to the technique featured in [MNN15].

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## 2. Useful reductions

In this section we reduce Theorem 1.1 to Theorem 1.2. Since the May Nilpotence Conjecture is proved [MNN15], we need only prove Theorem 1.1 when $n > 0$.

Fix such an integer $n > 0$ for the remainder of the paper. Recall that, for us, $E$ denotes a height $n+1$ variant of Morava $E$-theory with $\pi_0 E \cong \mathbb{Z}_p[[u_1, u_2, ..., u_n]]$. For more details on $E$, see Section 3.

**Lemma 2.1.** *Suppose $R$ is a spectrum. Then $R$ is $K(n+1)$-acyclic if and only if $R \wedge E$ is.*

*Proof.* This is Proposition 3.4 of [HS99]. The argument is that $K(n+1)$ is a field spectrum, and so $K(n+1) \wedge E$ splits as a wedge of suspensions of $K(n+1)$. It follows that $K(n+1) \wedge E \wedge R$ is a wedge of suspensions of $K(n+1) \wedge R$. □

By an $H_\infty$-$E$-algebra we simply mean an $H_\infty$-ring spectrum $R$ equipped with a map of $H_\infty$-rings $E \to R$. A small piece of this structure is a ring map $\pi_0(E) \to \pi_0(R)$, which allows us to speak of $u_1, u_2, ..., u_n \in \pi_0(R)$.

**Lemma 2.2.** *Suppose $R$ is a $K(n)$-acyclic, $H_\infty$-$E$-algebra. Then, in $\pi_0(R)$, some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1})$.*

*Proof.* Let $\mathbb{S}/I$ denote a type $n$ Moore spectrum $\mathbb{S}/(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$, as in [HS99, §4]. The spectrum $X = R \wedge \mathbb{S}/I$ is $K(n)$-acyclic by assumption, but also $K(j)$-acyclic for $j < n$. Since $R$ is $L_{n+1}$-local, $X$ is $L_{n+1}$-local, but $L_n X \simeq 0$. By [HS99, 7.10], $L_n^f X \simeq 0$. Also by [HS99, 7.10],

$$L_n^f(R \wedge \mathbb{S}/I) \simeq R \wedge T(\mathbb{S}/I),$$

where $T(\mathbb{S}/I)$ is the telescope of a $v_n$-self map on $\mathbb{S}/I$.

On $\pi_0$, the map $R \wedge \mathbb{S}/I \to R \wedge T(\mathbb{S}/I)$ inverts $u_n$. Since the image of this map is null, it follows that some power of $u_n$ is 0 in $\pi_0(R \wedge \mathbb{S}/I)$.

To finish the proof, I will show that any element in the kernel of $\pi_0 R \to \pi_0(R \wedge \mathbb{S}/I)$ is a member of the ideal $(p, u_1, ..., u_{n-1})$. Indeed, we can decompose this map as a composition

$$\pi_0(R) \to \pi_0(R \wedge \mathbb{S}/p^{i_0}) \to \pi_0\left(R \wedge \mathbb{S}/(p^{i_0}, v_1^{i_1})\right) \to \cdots \to \pi_0(R \wedge \mathbb{S}/I).$$

The kernel of the map that kills $v_k^{i_k}$ consists of elements that are multiples of $u_k^{i_k}$, and the result follows. □

For the moment assume Theorem 1.2, which the rest of the paper is devoted to proving. We will deduce Theorem 1.1 from this assumption.

*Proof of Theorem 1.1.* If $R$ is any $K(n)$-acyclic, $H_\infty$-ring spectrum, then $R \wedge E$ will be a $K(n)$-acyclic, $H_\infty$-$E$-algebra. By Lemma 2.2, some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1}) \subseteq \pi_0(R \wedge E)$. The same fact must be true in $\pi_0 L_{K(n+1)}(R \wedge E)$. By Theorem 1.2, 1 is in the ideal

$(p, u_1, ..., u_{n-1}) \subseteq \pi_0(L_{K(n+1)}(R \wedge E))$. It follows that, upon smashing $L_{K(n+1)}(R \wedge E)$ with any type $(n+1)$-Moore spectrum $M$, one obtains 0. In particular, $L_{K(n+1)}(R \wedge E)$ is acyclic with respect to the telescope of $M$, and hence $K(n+1)$-acyclic. This implies that $R \wedge E$ is $K(n+1)$-acyclic. By Lemma 2.1, $R$ is itself $K(n+1)$-acyclic. □

**Corollary 1.1.1.** *Let $R$ be a complex-orientable, $H_\infty$-ring spectrum that kills a finite complex. Then $R$ has the Bousfield class of $E(n)$ for some $n$.*

*Proof.* This follows immediately from [Hov95, 1.11], which states that $R$ has the Bousfield class of some wedge of Morava $K$-theories. □

## 3. Power operations for $H_\infty$-$E$-algebras

Recall that, given any height $n+1$ formal group $\mathbb{G}_0$ over $\mathrm{Spec}(\mathbb{F}_p)$, there is a universal deformation $\mathbb{G}_E$ defined over $\mathrm{Spf}\,\mathbb{Z}_p[[u_1, u_2, ..., u_n]]$. By work of Goerss and Hopkins [GH04, GH05], there is an associated $\mathbb{E}_\infty$-ring spectrum $E$, a height $n+1$ Morava $E$-theory. The coefficient ring $E_0 = \pi_0 E \cong \mathbb{Z}_p[[u_1, u_2, ..., u_n]]$.

Let $BC_p$ denote the classifying space of the cyclic group with $p$-elements. As noted in [HKR00, §5],

$$E^0(BC_p) \cong E_0[[a]]/[p](a).$$

There is a stable transfer map $\Sigma^\infty_+ BC_p \to \Sigma^\infty_+ Be \simeq \mathbb{S}$. This yields a map $E_0 \to E^0(BC_p)$, the image of which generates an ideal $\mathrm{tr} \subset E^0(BC_p)$. A simple calculation with a Gysin sequence [HKR00, 6.15] shows that $\mathrm{tr} = \left(\frac{[p](a)}{a}\right)$.

The **total power operation** is a ring homomorphism

$$P : E_0 \to E^0(BC_p)/\,\mathrm{tr}\,.$$

In [AHS04, §3], the power operation is described in terms of the moduli problem associated to the ring

$$D = E^0(BC_p)/\mathrm{tr} \cong E_0[[a]]/\left(\frac{[p](a)}{a}\right).$$

To summarize their work, the $E_0$-algebra morphism $E_0 \to D$ specifies a formal group $\mathbb{G}_{\mathrm{source}}$ over $\mathrm{Spf}(D)$. There is an isogeny of formal groups $\mathbb{G}_{\mathrm{source}} \to \mathbb{G}_{\mathrm{target}}$ over $\mathrm{Spf}(D)$, and this latter formal group is specified by the ring homomorphism $P : E_0 \to D$. The interested reader may consult [AHS04] or [Str97] to learn more.

Now, suppose that $x$ is an element of $E_0^\vee(BC_p) = \pi_0(L_{K(n+1)}E \wedge \Sigma^\infty_+ BC_p)$. For each element $\alpha \in E^0(BC_p)$, we obtain a diagram

$$\begin{array}{ccccc} & & E \wedge \Sigma^\infty_+ BC_p & \xrightarrow{1 \wedge \alpha} & E \wedge E \\ & & \downarrow & & \downarrow m \\ \mathbb{S} & \xrightarrow{x} & L_{K(n+1)}(E \wedge \Sigma^\infty_+ BC_p) & \dashrightarrow & E, \end{array}$$

giving an element in $E_0$. Assembling this construction over all $\alpha$ gives an $E_0$-module map

$$\phi_x : E^0(BC_p) \to E_0,$$

which is the cap product with the class $x$.

In the case that the transfer ideal is in the kernel of $\phi_x$, we obtain an additive operation

$$\widetilde{\phi}_x : E_0 \xrightarrow{P} E^0(BC_p)/\,\mathrm{tr} \xrightarrow{\phi_x} E_0.$$

Suppose now that $R$ is a homotopy commutative $E$-algebra, with associated homomorphisms $\iota : E_0 \to \pi_0 R$ and $\tau : E^0(BC_p) \to R^0(BC_p)$. If $\alpha \in E^0(BC_p)$ is such that $\tau(\alpha) = 0$, then in the diagram

$$\begin{CD} E \wedge \Sigma^\infty_+ BC_p @>{1 \wedge \alpha}>> E \wedge E @>{1 \wedge \iota}>> E \wedge R \\ @VVV @VV{m}V @VVV \\ L_{K(n+1)}(E \wedge \Sigma^\infty_+ BC_p) @>>> E @>{\iota}>> R, \end{CD}$$

the composite $E \wedge \Sigma^\infty_+ BC_p \to R$ is null. If $R$ is furthermore $K(n+1)$-local, the map

$$L_{K(n+1)}(E \wedge \Sigma^\infty_+ BC_p) \longrightarrow R$$

must also be null. When $R$ is a $K(n+1)$-local, $H_\infty$-$E$-algebra, there is a commutative diagram of ring homomorphisms

$$\begin{CD} \pi_0 R @>>> R^0(BC_p)/\operatorname{tr} \\ @A{\iota}AA @AAA \\ \pi_0 E @>{P}>> E^0(BC_p)/\operatorname{tr}. \end{CD}$$

The combined structure ensures that, if $\iota(\beta) = 0$ for some $\beta \in E_0$, then $\iota(\widetilde{\phi}_x(\beta)) = 0$ as well.

## 4. An explicit formula for a reduced power operation

Here we remark that, with careful choice of coordinate, one can explicitly describe the total power operation $P$ (modulo certain ideals). We follow [Str97, §15] to select a height $n+1$ formal group law over $\mathbb{F}_p$ and a coordinate on the resultant $\mathbb{G}_E$. The multiplication on $\mathbb{G}_E$ is then presented by a formal group law $F(x, y) \in E_0[[x, y]]$ with properties outlined in the following proposition:

**Proposition 4.1.** [Str97, 15.6] *For any integer $m > 0$, let $C_{p^m}$ denote the polynomial in $\mathbb{Z}[x, y]$ defined by*

$$C_{p^m}(x, y) = \frac{x^{p^m} + y^{p^m} - (x + y)^{p^m}}{p}.$$

*Then,*

(1) *For any $0 < k \leq n$,*

$$F(x, y) \equiv x + y + u_k C_{p^k}(x, y) \text{ modulo } (u_1, u_2, ..., u_{k-1}) + (x, y)^{p^k+1}.$$

(2) *$F(x, y) \equiv x + y + C_{p^{n+1}}(x, y)$ modulo $(u_1, u_2, ..., u_n) + (x, y)^{p^{n+1}+1}$.*

**Corollary 4.1.1.** *For any integer $i$, we use $[i]_F(x)$ to denote the $i$-series of $x$. For $i \geq 0$, $1 \leq k \leq n$, let $\gamma_{i,k}$ denote $\frac{i - i^{p^k}}{p}$. Then,*

$$[i]_F(x) \equiv ix + u_k \gamma_{i,k} x^{p^k} \text{ modulo } (p, u_1, u_2, ..., u_{k-1}, x^{p^k+1}).$$

*In particular, $[p]_F(x) \equiv u_k x^{p^k}$. Furthermore,*

$$[i]_F(x) \equiv ix + \gamma_{i,k} x^{p^{n+1}} \text{ modulo } (p, u_1, u_2, ..., u_n, x^{p^{n+1}+1}),$$

*and so, modulo this ideal, $[p]_F(x) \equiv x^{p^{n+1}}$.*

*Proof.* See also [Rez98, 5.7]. This is a simple induction on $i$, the statement being true when $i = 0$. For larger $i$, setting $u_{n+1} = 1$,

$$\begin{aligned}
[i]_F(x) &= F([i-1]_F(x), x) \\
&\equiv [i-1]_F(x) + x + u_k C_{p^k}(x, [i-1]_F(x)) \\
&\equiv (i-1)x + u_k\gamma_{i-1,k}x^{p^k} + x + u_k C_{p^k}(x, (i-1)x + u_k\gamma_{i-1,k}x^{p^k}) \\
&\equiv (i-1)x + u_k\gamma_{i-1,k}x^{p^k} + x + x^{p^k}u_k\left(\frac{(i-1)^{p^k} + 1 - i^{p^k}}{p}\right) \\
&= ix + u_k x^{p^k}\left(\frac{p\gamma_{i-1,k} + (i-1)^{p^k} + 1 - i^{p^k}}{p}\right) \\
&= ix + u_k x^{p^k}\left(\frac{(i-1) - (i-1)^{p^k} + (i-1)^{p^k} + 1 - i^{p^k}}{p}\right) \\
&= ix + u_k x^{p^k}\left(\frac{i - i^{p^k}}{p}\right) \\
&= ix + u_k\gamma_{i,k}x^{p^k},
\end{aligned}$$

as desired. □

Recall that the total power operation is a ring map

$$P : E_0 \to D \cong E_0[[a]]/([p](a)/a).$$

The ring homomorphism $P$ classifies a formal group law $F'$ on $D$. The natural $E_0$-algebra map $E_0 \to D$ classifies a second formal group law on $D$, which, by abuse of notation, we denote $F$.

**Lemma 4.2.** *There is an equality of elements in $D[[x]]$,*

$$\prod_{k=0}^{p-1}([p]_F(x) -_F [k]_F a) = [p]_{F'}\left(\prod_{k=0}^{p-1}(x -_F [k]_F a)\right)$$

*Proof.* In the language of Section 3, we have a diagram of formal groups over $\mathrm{Spf}(D)$

$$\begin{array}{ccc}
\mathbb{G}_{\mathrm{source}} & \longrightarrow & \mathbb{G}_{\mathrm{target}} \\
{\scriptstyle p}\uparrow & & {\scriptstyle p}\uparrow \\
\mathbb{G}_{\mathrm{source}} & \longrightarrow & \mathbb{G}_{\mathrm{target}}
\end{array}$$

Applying global sections, we obtain a commuting diagram of $E_0$-algebra homomorphisms

$$\begin{array}{ccc}
D[[y]] & \longrightarrow & D[[x]] \\
{\scriptstyle y \mapsto [p]_F(y)}\uparrow & & {\scriptstyle x \mapsto [p]_{F'}(x)}\uparrow \\
D[[y]] & \longrightarrow & D[[x]]
\end{array}$$

By [Str97, 7.13], both horizontal arrows send $y$ to $\prod_{k=0}^{p-1}(x -_F [k]_F a)$. □

**Remark 4.3.** As elements of $D$,

$$\prod_{i=1}^{p-1}([-i]_F a) = \prod_{i=1}^{p-1}([i]_F a).$$

We denote their common value by $\Psi$.

**Proposition 4.4.** *For* $0 < k \leq n$,

$$P(u_k)\Psi^{p^k} \equiv -u_k\Psi \text{ modulo } (p, P(u_1), P(u_2), ..., P(u_{k-1}), u_1, u_2, ..., u_{k-1}).$$

*Proof.* Corollary 4.1.1 implies both of the following equations:

$$[p]_{F'}(x) \equiv P(u_k)x^{p^k} \text{ modulo } (p, P(u_1), ..., P(u_{k-1}), x^{p^k+1}), \text{ and}$$
$$[p]_F(x) \equiv u_k x^{p^k} \text{ modulo } (p, u_1, ..., u_{k-1}, x^{p^k+1}).$$

In particular, both of the equations hold modulo $(p, u_1, ..., u_{k-1}, P(u_1), ..., P(u_{k-1}), x^{p^k+1})$, which is also where we perform the following calculations:

$$\begin{aligned} [p]_{F'}\left(\prod_{i=0}^{p-1}(x -_F [i]_F a)\right) &= [p]_{F'}\left(x \cdot \prod_{i=1}^{p-1}(x -_F [i]_F a)\right)^{p^k} \\ &\equiv P(u_k)x^{p^k}\left(\prod_{i=1}^{p-1} -_F [i]_F a\right)^{p^k} \\ &\equiv P(u_k)x^{p^k}\Psi^{p^k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \prod_{i=0}^{p-1}([p]_F(x) -_F [i]_F a) &\equiv \prod_{i=0}^{p-1}(u_k x^{p^k} -_F [i]_F a) \\ &\equiv u_k x^{p^k}\prod_{i=1}^{p-1}(u_k x^{p^k} -_F [i]_F a) \\ &\equiv u_k x^{p^k}\prod_{i=1}^{p-1}(-_F [i]_F a) \\ &\equiv u_k x^{p^k}\Psi \end{aligned}$$

The result follows by Lemma 4.2. □

In the remainder of this section, we attempt to reduce the complexity of the total power operation $P : E_0 \to D$ by modding out both the domain and codomain by $(p, u_1, u_2, ..., u_{n-1})$. It is not possible to do this directly because $P$ is not an $E_0$-algebra map, and indeed we will need to mod out more of the codomain than just $(p, u_1, ..., u_{n-1})$.

**Proposition 4.5.** *In the ring* $E_0[[a]]/(p, u_1, u_2, ..., u_{n-1}) \cong \mathbb{F}_p[[u_n, a]]$*, the element* $[p]_F(a)$ *is a product*

$$[p]_F(a) = Ua^{p^n}g(a),$$

*where*

*(1) $U$ is a unit in $\mathbb{F}_p[[u_n, a]]$,*
*(2) $g(a)$ is a monic polynomial in $(\mathbb{F}_p[[u_n]])[a]$ of degree $p^{n+1} - p^n$,*
*(3) $g(a) \equiv a^{p^{n+1}-p^n}$ modulo $u_n$, and*
*(4) The constant term of $g(a)$ is divisible by $u_n$ but not $u_n^2$.*

*Proof.* By Corollary 4.1.1, $[p]_F(a) \equiv u_n a^{p^n}$ modulo $a^{p^n+1}$. This means that we may factor

$$[p]_F(a) = a^{p^n}(u_n + aq(a))$$

for some power series $q(a) \in \mathbb{F}_p[[u_n, a]]$. Corollary 4.1.1 also states that $[p]_F(a) \equiv a^{p^{n+1}}$ modulo $(u_n, a^{p^{n+1}+1})$, and so the Weierstrass preparation theorem [HKR00, 5.1] implies

$$u_n + aq(a) = Ug(a)$$

for some unit $U$ and some monic polynomial $g(a)$ of degree $a^{p^{n+1}-p^n}$. Modding out both sides by $a$, we learn that the constant term of $g(a)$ is a unit times $u_n$. Modding out both sides by $u_n$, we arrive at an equation $\overline{U}\overline{g(a)} = a^{p^{n+1}-p^n} + \mathcal{O}(a^{p^{n+1}-p^n+1})$, where the right-hand-side has no terms of degree less than $p^{n+1} - p^n$. By looking at each coefficient of $\overline{g(a)}$ in turn, starting with the constant coefficient, we learn that $\overline{g(a)} = a^{p^{n+1}-p^n}$. □

**Corollary 4.5.1.** *The polynomial $g(a)$ is irreducible, and $\mathbb{F}_p[[u_n]][a]/g(a)$ is a DVR valued by powers of its maximal ideal $\mathfrak{m} = (a)$. The element $u_n$ is in $\mathfrak{m}^{p^{n+1}-p^n}$ but no higher power of $\mathfrak{m}$. The element $\Psi$ is not $0$ inside $\mathbb{F}_p[[u_n]][a]/g(a)$.*

*Proof.* The ring $\mathbb{F}_p[[u_n]][a]$ is a UFD, and so Eisenstein's criterion applies to show that $g(a)$ is irreducible. It follows that the quotient $\mathbb{F}_p[[u_n]][a]/g(a)$ is a local domain. When we further mod out by $a$, we get $\mathbb{F}_p[[u_n]]/\overline{g(a)} \cong \mathbb{F}_p$, since $\overline{g(a)}$ is a unit times $u_n$. Thus, $\mathbb{F}_p[[u_n]][a]/g(a)$ is a DVR with maximal ideal generated by $a$. We have that

$$u_n = (\text{some unit})a^{p^{n+1}-p^n} + (\text{terms of strictly higher valuation than } u_n),$$

and so $u_n$ must have valuation $p^{n+1} - p^n$.

To see that $\Psi$ is not zero, recall that

$$\Psi = \prod_{k=1}^{p-1} [k]_F(a),$$

and so can only be $0$ if one of its factors is $0$. However, for each $1 \leq k < p$, $[k]_F(a) = ka + ...$ has valuation 1. □

By Proposition 4.5, we may compose with a quotient homomorphism to obtain a reduced power operation

$$N : E_0 \xrightarrow{P} D \to \mathbb{F}_p[[u_n]][a]/g(a).$$

**Proposition 4.6.** *For $1 \leq i \leq n-1$, $N(u_i) = 0$. Also, $N(p) = 0$.*

*Proof.* Since $N$ is a ring homomorphism, $N(p) = p$. That $N(p) = 0$ follows, since $p = 0$ in $\mathbb{F}_p[[u_n]][a]/g(a)$. The rest we prove by induction on $i$, assuming that $N(u_1), ..., N(u_{i-1})$ are all zero. Since

$$P(u_i)\Psi^{p^i} \equiv u_i\Psi \text{ modulo } (p, P(u_1), P(u_2), ..., P(u_{i-1}), u_1, u_2, ..., u_{i-1}),$$

we may conclude that

$$N(u_i)\Psi^{p^i} = 0.$$

By Corollary 4.5.1, $N(u_i) = 0$. ☐

**Corollary 4.6.1.** *The ring homomorphism $N : E_0 \to \mathbb{F}_p[[u_n]][a]/g(a)$ factors through a ring homomorphism*

$$\overline{N} : E_0/(p, u_1, u_2, ..., u_{n-1}) \cong \mathbb{F}_p[[u_n]] \to \mathbb{F}_p[[u_n]][a]/g(a)$$

**Proposition 4.7.** $\overline{N}(u_n)\Psi^{p^n-1} = u_n$

*Proof.* We have that

$$P(u_n)\Psi^{p^n} \equiv u_n\Psi \text{ modulo } (p, P(u_1), P(u_2), ..., P(u_{n-1}), u_1, u_2, ..., u_{n-1}).$$

The result follows from Corollary 4.5.1. ☐

## 5. A proof of Theorem 1.2

In the previous section, we learned that the total power operation $P : E_0 \to D$ induces a ring homomorphism $\overline{N} : \mathbb{F}_p[[u_n]] \to \mathbb{F}_p[[u_n]][a]/g(a)$ such that $\overline{N}(u_n)\Psi^{p^n-1} = u_n$.

By Corollary 4.5.1, the ring $\mathbb{F}_p[[u_n]][a]/g(a)$ is a valuation ring. We define the **weight** $\mathrm{wt}(f)$ of $f \in \mathbb{F}_p[[u_n]][a]/g(a)$ such that $\mathrm{wt}(u_n) = 1$ and $\mathrm{wt}(a) = \frac{1}{p^{n+1}-p^n}$. In other words, $\mathrm{wt}(f)$ is just a rescaling of the natural valuation of $f$ by powers of the maximal ideal $\mathfrak{m} = (a)$. We also use $\mathrm{wt}(f)$ to refer to the $u_n$-valuation of any $f \in \mathbb{F}_p[[u_n]]$.

**Proposition 5.1.** *$wt(\Psi) = \frac{p-1}{p^{n+1}-p^n}$.*

*Proof.* We have that $[i]_F(a) = ia + \mathcal{O}(a^2)$, and so for $0 < i < p$ this has weight $\frac{1}{p^{n+1}-p^n}$. By definition, $\Psi$ is the product of all of these elements and the result follows. ☐

**Proposition 5.2.** *$\overline{N}(u_n)$ has weight $\frac{p-1}{p^{n+1}-p^n}$*

*Proof.* We have that

$$1 = \mathrm{wt}(u_n) = \mathrm{wt}(\overline{N}(u_n)\Psi^{p^n-1}) = \mathrm{wt}(\overline{N}(u_n)) + \frac{(p^n-1)(p-1)}{p^{n+1}-p^n},$$

so

$$\mathrm{wt}(\overline{N}(u_n)) = \frac{p^{n+1}-p^n-(p^n-1)(p-1)}{p^{n+1}-p^n} = \frac{p-1}{p^{n+1}-p^n}.$$

☐

**Corollary 5.2.1.** *For any non-zero power series $z \in \mathbb{F}_p[[u_n]]$ of weight at least 1, the weight of $\overline{N}(z)$ is less than the weight of $z$.*

*Proof.* This follows from the facts that $\mathrm{wt}(u_n^t) = t\,\mathrm{wt}(u_n)$ and $\mathrm{wt}(f_1 + f_2) = \mathrm{wt}(f_1) + \mathrm{wt}(f_2)$ whenever $\mathrm{wt}(f_1) \neq \mathrm{wt}(f_2)$. ☐

Recall from the end of Section 3 that, for every element $x \in E_0^\vee(BC_p) = \pi_0 L_{K(n+1)}(E \wedge \Sigma^\infty_+ BC_p)$, there is an $E_0$-module homomorphism $\phi_x : E^0(BC_p) \to E_0$. We can tensor over $E_0$ with the module $E_0/(p, u_1, u_2, ..., u_{p-1})$ to obtain a module homomorphism

$$\overline{\phi}_x : \mathbb{F}_p[[u_n, a]]/([p](a)) \to \mathbb{F}_p[[u_n]].$$

**Proposition 5.3.** *For any non-zero $z$ in $\mathbb{F}_p[[u_n]]$ of weight at least 1, there exists an $x \in E_0^\vee(BC_p)$ such that:*

- *The $E_0$-module map $\overline{\phi}_x : \mathbb{F}_p[[u_n, a]]/([p](a)) \to \mathbb{F}_p[[u_n]]$ kills $g(a)$.*

- *The resultant additive operation*

$$\mathbb{F}_p[[u_n]] \xrightarrow{\overline{N}} \mathbb{F}_p[[u_n]][a]/g(a) \xrightarrow{\overline{\phi}_x} \mathbb{F}_p[[u_n]]$$

*sends $z$ to a power series of strictly smaller weight.*

*Proof.* The operation $x \mapsto \phi_x$ gives a map $E_0^\vee(BC_p) \to \mathrm{Hom}_{E_0-\mathrm{modules}}(E^0(BC_p), E)$. By, e.g. [Str98, §3], this map is in fact bijective. Now, $\overline{N}(z) \in \mathbb{F}_p[[u_n]][a]/g(a)$, has a unique representative polynomial $f(a) \in \mathbb{F}_p[[u_n]][a]$ of degree $< p^{n+1} - p^n$. By Corollary 5.2.1, there is some $i < p^{n+1} - p^n$ such that the coefficient of $a^i$ in $f$ (an element $q \in \mathbb{F}_p[[u_n]]$) has weight less than $\mathrm{wt}(z)$. I claim that there is a choice of $x$ which will induce an additive operation sending $z$ to $q$, finishing the proof. Indeed, Proposition 4.5 implies that there is a unique $x \in \mathrm{Hom}_{E_0-\mathrm{modules}}(E^0(BC_p), E)$ sending $a^i$ to 1, sending all other $a^j$ for $0 \leq j < p^{n+1} - p^n$ to 0, and killing $g(a), ag(a), a^2g(a), \cdots$. □

**Theorem 1.2.** *Suppose $R$ is a $K(n+1)$-local, $H_\infty$-$E$-algebra such that, in $\pi_0 R$, some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1})$. Then 1 is in the ideal $(p, u_1, ..., u_{n-1})$.*

*Proof.* The natural $E_0$-algebra morphism $\iota : E_0 \to \pi_0 R$ yields an $E_0$-algebra morphism

$$\overline{\iota} : \mathbb{F}_p[[u_n]] \to \pi_0 R/(p, u_1, u_2, ..., u_{n-1}).$$

By assumption, there is some non-zero $z \in \mathbb{F}_p[[u_n]]$ that is sent to 0 by $\overline{\iota}$. Choose such a $z$ of minimal weight $\geq 1$. The previous proposition provides an additive operation $E_0 \to E_0$ that sends $z$ to a power series $\hat{z}$ of smaller weight. As explained at the end of Section 3, the $H_\infty$-structure ensures $\overline{\iota}(\hat{z}) = 0$. By the minimality of $\mathrm{wt}(z)$, it must be that $\mathrm{wt}(\hat{z}) = 0$, so $\hat{z}$ is just a unit in $\mathbb{F}_p$. □

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