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Redshift and multiplication for truncated Brown–Peterson spectra

By Jeremy Hahn and Dylan Wilson

Abstract

We equip $\mathrm{BP}\langle n\rangle$ with an $\mathbb{E}_3$-BP-algebra structure for each prime p and height n. The algebraic K-theory of this ring is of chromatic height exactly $n+1$, and the map $\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)} \to \mathrm{L}^f_{n+1}\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$ has bounded above fiber.

Contents

1. Introduction

Our main aim here is to prove the following:

Theorem. *For each prime p and height n, there exists an $\mathbb{E}_3$-BP-algebra structure on $\mathrm{BP}\langle n\rangle$. The algebraic K-theory of the p-completion of this ring has finitely presented cohomology over the mod p Steenrod algebra, and it is of fp-type $n+1$ after p-completion.*

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The principal connective theories in the chromatic approach to stable homotopy theory are thus more structured than previously known, and they satisfy higher height analogs of the Lichtenbaum–Quillen conjecture. The $\mathbb{E}_3$ forms of $\mathrm{BP}\langle n\rangle$ constructed here give the first known examples, for $n > 1$, of chromatic height n theories with algebraic K-theory provably of height $n+1$.

The redshift philosophy. In his 1974 ICM address, Quillen [Qui75] stated as a "hope" the now proven Lichtenbaum–Quillen conjecture [Voe03], [Voe11]. His hope was that the algebraic K-theory of regular noetherian rings could be well approximated by étale cohomology, at least in large degrees. Ten years later, Waldhausen [Wal84] investigated interactions between his K-theory of spaces and the chromatic filtration. He observed that, in the presence of a descent theorem of Thomason [Tho82], the Lichtenbaum–Quillen conjecture could be restated in terms of localization at complex K-theory. Let L_1^f denote the localization that annihilates those finite spectra with vanishing p-adic complex K-theory; for suitable rings R, the Lichtenbaum–Quillen conjecture is equivalent to the statement that

$$\pi_*\mathrm{K}(R)_{(p)} \to \pi_* L_1^f \mathrm{K}(R)_{(p)}$$

is an isomorphism for $* \gg 0$.

Algebraic K-theory is defined not only on rings, but (crucially for applications to smooth manifold theory) on ring spectra. One of the deepest computations of the algebraic K-theory of ring spectra to date is by Ausoni and Rognes [AR02], who for primes $p \geq 5$ computed the mod (p, v_1) K-theory of the p-completed Adams summand $\ell_p^\wedge$. Their computations imply that

$$\mathrm{K}(\ell_p^\wedge)_{(p)} \to L_2^f \mathrm{K}(\ell_p^\wedge)_{(p)}$$

is a π_*-isomorphism for $* \gg 0$. Here L_2^f is the next localization in a hierarchy of chromatic localizations L_n^f for each $n \geq 0$ (at an implicit prime p). This of course suggests a higher height analog of the Lichtenbaum–Quillen conjecture. In the Oberwolfach lecture [Rog00], Rognes laid out a far-reaching vision of how this higher height analog might go, which is now known as the *chromatic redshift* philosophy. The name *redshift* refers to the hypothesis that algebraic K-theory should raise the chromatic height of ring spectra by exactly 1.

To give a more precise statement, we will need the notion of fp-type, due to Mahowald–Rezk [MR99]: A p-complete, bounded below spectrum X is of fp-type n if the thick subcategory of p-local finite complexes F such that $|\pi_*(F \otimes X)| < \infty$ is generated by a type $(n+1)$ complex (i.e., a complex with a v_{n+1} self-map).

With this definition, Ausoni–Rognes conjecture that

Conjecture. For suitable $\mathbb{E}_1$-rings R of fp-type n, $\mathrm{K}(R)_p^\wedge$ is of fp-type $n+1$.

As we review below (see Theorem 3.1.3), this statement also implies that $\mathrm{K}(R) \to L_{n+1}^{f}\mathrm{K}(R)$ is a p-local equivalence in large degrees, so we can think of it as a higher height analog of the Lichtenbaum–Quillen conjecture.

In the years since the Ausoni–Rognes computations, redshift has been verified for additional height 1 ring spectra, including $\mathrm{ku}_p^\wedge$, $\mathrm{KU}_p^\wedge$, and ku/p at primes $p \geq 5$ [BM08], [Aus10], [AR12a], and evidence for redshift has accumulated in general [BDR04], [Rog14], [Wes17], [Vee18], [AK21], [AKQ21a], [CSY21]. Recent conceptual advances show that the algebraic K-theories of many height n rings are of height *at most* $n+1$ [LMMT22], [CMNN20]. Here, we give the first arbitrary height examples of ring spectra for which redshift provably occurs.

Main Results. The truncated Brown–Peterson spectra, $\mathrm{BP}\langle n\rangle$, are among the simplest and most important cohomology theories in algebraic topology. There is one such spectrum for every prime p and height $n \geq 0$, though we will follow tradition by localizing at the prime and omitting it from notation.[1] The height 1 spectrum $\mathrm{BP}\langle 1\rangle$ is the Adams summand ℓ, while $\mathrm{BP}\langle 2\rangle$ is a summand of either topological modular forms (at $p \geq 5$), or topological modular forms with level structure (at $p = 2, 3$).

Both ℓ and tmf are extraordinarily structured: they are $\mathbb{E}_\infty$-ring spectra, inducing power operations on the cohomology of spaces. Our first main result, proven in Section 2, is a construction of part of this structure at an arbitrary height n. To make sense of the statement, we remind the reader that BP admits the structure of an $\mathbb{E}_4$-ring by [BM13].

Theorem A (Multiplication). *For an appropriate choice of indecomposable generators*

$$v_{n+1}, v_{n+2}, \ldots \in \pi_*\mathrm{BP},$$

the quotient map

$$\mathrm{BP} \to \mathrm{BP}/(v_{n+1}, \ldots) = \mathrm{BP}\langle n\rangle$$

is the unit of an $\mathbb{E}_3$-BP-*algebra structure on* $\mathrm{BP}\langle n\rangle$.

Our second main theorem establishes the above conjecture for $R = \mathrm{BP}\langle n\rangle_p^\wedge$.

Theorem B (Redshift). *Let* $\mathrm{BP}\langle n\rangle$ *denote any* $\mathbb{E}_3$-BP-*algebra such that the unit* $\mathrm{BP} \to \mathrm{BP}\langle n\rangle$ *is obtained by modding out a sequence of indecomposable generators* $v_{n+1}, v_{n+2}, \ldots$ *. Then* $\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)_p^\wedge$ *is of* fp-*type* $n+1$.

[1] At each prime p and height $n \geq 0$, $\mathrm{BP}\langle n\rangle$ is conjectured to be unique as a p-local spectrum. For $n > 1$, uniqueness is only proved up to p-completion, by work of Angeltveit and Lind [AL17].

Corollary. For $\mathrm{BP}\langle n\rangle$ any $\mathbb{E}_3$-BP-algebra as above, both maps

$$\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)_{(p)} \to L_{n+1}^f \mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)_{(p)},$$
$$\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)} \to L_{n+1}^f \mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$$

induce isomorphisms on π_* for $* \gg 0$.

To prove Theorem B by trace methods, the critical thing to show is that $\pi_*(V \otimes \mathrm{TC}(\mathrm{BP}\langle n\rangle))$ is bounded above for some type $(n+2)$ complex V. We recall [NS18] that the p-completed topological cyclic homology of $\mathrm{BP}\langle n\rangle$ can be computed as the fiber

$$\mathrm{TC}(\mathrm{BP}\langle n\rangle) \simeq \mathrm{fib}\left(\left(\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1}\right)_p^\wedge \xrightarrow{\varphi^{hS^1}-\mathrm{can}} \left(\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tS^1}\right)_p^\wedge\right),$$

where the map

$$\varphi : \mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$$

is the cyclotomic Frobenius. (See Section 3.2 for our conventions on cyclotomic spectra.)

One would like to argue that $(\varphi^{hS^1} - \mathrm{can})$ is an equivalence in large degrees after tensoring with a type $(n+2)$ complex. We will deduce this from the following two theorems:

Theorem C (Segal conjecture). *Let F be any type $n+1$ complex. Then the cyclotomic Frobenius $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$ induces an isomorphism*

$$F_*\mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong F_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p})$$

in all sufficiently large degrees $* \gg 0$.

Theorem D (Canonical vanishing). *Let F be any type $n+2$ complex. There exists an integer $d \geq 0$ (depending on F) such that, for all $0 \leq k \leq \infty$, the composite*

$$\tau_{\geq d}(F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}}) \to F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}} \xrightarrow{\mathrm{can}} F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$$

is nullhomotopic.

We note that the first theorem involves only the cyclotomic Frobenius map, and the second theorem only the canonical map. We use different techniques to analyze each one.

In order to prove Theorem C, we use a filtration on $\mathrm{BP}\langle n\rangle$ to reduce the statement to a graded version of the Segal conjecture for polynomial algebras over $\mathbb{F}_p$, which we then prove directly. This is done in Section 4.

To prove Theorem D, we investigate the S^1-spectrum $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ of Hochschild homology *relative* to MU. This spectrum is much simpler to understand because of the following analog of Bökstedt's periodicity theorem:

Theorem E (Polynomial THH). *The ring* $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_*$ *is polynomial over* $\pi_*\mathrm{BP}\langle n\rangle$ *on even-degree generators, one of which can be chosen to be the double-suspension class* $\sigma^2 v_{n+1}$. (*For the definition of double-suspension classes, see* A.2.4.)

We may take advantage of the circle action on THH to shift the class $\sigma^2 v_{n+1}$ down to a class detecting v_{n+1}. More precisely, we prove

Theorem F (Detection). *There is an isomorphism of* $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$-*algebras*

$$\pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}) \cong (\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))[\![t]\!],$$

where $|t| = -2$. *This isomorphism can be chosen such that, under the unit map*

$$\pi_*(\mathrm{MU}_{(p)}^{hS^1}) \to \pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}),$$

the canonical complex orientation maps to t *and* v_{n+1} *is sent to* $t(\sigma^2 v_{n+1})$.

Theorem F already implies the following weak form of redshift:

Corollary (Corollary 5.0.2). $L_{K(n+1)}\mathrm{K}(\mathrm{BP}\langle n\rangle)$ is non-zero.

Finally, in order to prove Theorem D, we must descend information along the S^1-equivariant map

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}).$$

In Section 6, we study this descent spectral sequence after tensoring with a type $n+1$ complex. Using this information we are able to understand enough about the homotopy and Tate fixed point spectral sequences associated to $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ to prove a weak form of the Canonical Vanishing Theorem. As explained to us by an anonymous referee, and proven in Section 3, this weak form of canonical vanishing together with the Segal conjecture is enough to prove the strong form of canonical vanishing as well as the main theorem. In fact, we establish the following result, which is equivalent to the combination of Theorems C and D and also directly implies Theorem B.

Theorem G (Bounded TR). *For any type* $n+2$ *complex* F, *the spectrum* $F \otimes \mathrm{TR}(\mathrm{BP}\langle n\rangle)$ *is bounded.*

For a review of the functor TR, see Section 3.2.

Remarks on the Multiplication Theorem. As pointed out by Morava, $\mathrm{BP}\langle n\rangle$ may be equipped with different homotopy ring structures [Mor89]. Our redshift arguments only apply to forms of $\mathrm{BP}\langle n\rangle$ that are $\mathbb{E}_3$-BP-algebras, which we guarantee to exist by Theorem A. To check whether previously studied forms of $\mathrm{BP}\langle n\rangle$ admit such structure, there is a convenient criterion due to Basterra and Mandell:

Remark 1.0.1. Suppose that a form of $\mathrm{BP}\langle n\rangle$ is equipped with an $\mathbb{E}_4$-algebra structure. Then there are no obstructions to producing an $\mathbb{E}_4$-ring map from BP to $\mathrm{BP}\langle n\rangle$ [BM13, Cor. 4.4 and Lemma 5.1]. Any such $\mathbb{E}_4$-map is the unit of an $\mathbb{E}_3$-BP-algebra structure, allowing us to apply Theorem B.

Example 1.0.2. At $p = 2$, connective complex K-theory is an $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 1\rangle$, and it follows that $\mathrm{K}(\mathrm{ku}_2^\wedge)_2^\wedge$ is of fp-type 2. Even the non-vanishing of $L_{K(2)}\,\mathrm{K}(\mathrm{ku})$ was previously known only for $p \geq 5$ [AR02].

Similarly, we can deduce at $p = 2$ that $\mathrm{K}(\mathrm{tmf}_1(3)_2^\wedge)_2^\wedge$ is of fp-type 3, since $\mathrm{tmf}_1(3)$ is the Lawson–Naumann $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 2\rangle$ [LN12]. Applying algebraic K-theory to the E_∞-ring map $\mathrm{tmf} \to \mathrm{tmf}_1(3)$, we conclude that $L_{K(3)}\,\mathrm{K}(\mathrm{tmf}) \neq 0$.

Remark 1.0.3. Our methods may help to prove that the algebraic K-theories of many other height n rings are not $\mathrm{K}(n+1)$-acyclic, especially when combined with the descent and purity results of [CMNN20], [LMMT22]. For example, at the prime 3 these results imply that the non-vanishing of $L_{K(2)}\,\mathrm{K}(\mathrm{ku})$ is equivalent to the non-vanishing of $L_{K(2)}\,\mathrm{K}(\mathrm{ko})$ (cf. [Aus05]), and the latter follows from the fact that 3-localized ko is an $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 1\rangle$.

To give context to Theorem A, the question of whether $\mathrm{BP}\langle n\rangle$ can be made $\mathbb{E}_\infty$ was once a major open problem in algebraic topology [May75]. In breakthrough work, Tyler Lawson [Law18] and Andrew Senger [Sen22] showed this to be impossible whenever $n \geq 4$.

While the non-existence of structure is of great theoretical interest, it is the presence of structure that powers additional computations. For example, in this work we use the $\mathbb{E}_3$-algebra structure guaranteed by Theorem A in order to prove the Polynomial THH Theorem (2.5.3), which is the key computational input to many of the remaining results of the paper. Our proof of Theorem A relies on a number of ideas that we have not discussed so far; see Section 2.1 for an outline of the proof of Theorem A.

Remark 1.0.4. Prior to our work, other authors had succeeded in equipping $\mathrm{BP}\langle n\rangle$ with additional structure. Notably, Baker and Jeanneret produced $\mathbb{E}_1$-ring structures [BJ02] (cf. [Laz01], [Ang08]), and Richter produced Robinson $(2p-1)$-stage structures on related Johnson–Wilson theories [Ric06]. Lawson and Naumann equipped $\mathrm{BP}\langle 2\rangle$ with $\mathbb{E}_\infty$-structure at the prime 2 [LN12], and Hill and Lawson produced an $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 2\rangle$ at $p = 3$ [HL10].

Remark 1.0.5. Basterra and Mandell proved that BP admits a unique $\mathbb{E}_4$-algebra structure, a fact that is necessary to make sense of $\mathbb{E}_3$-BP-algebras [BM13]. They also show that BP is an $\mathbb{E}_4$-algebra retract of $\mathrm{MU}_{(p)}$, so a p-local $\mathbb{E}_3$-MU-algebra inherits an $\mathbb{E}_3$-BP-algebra structure. Our proof of Theorem A most naturally produces an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n\rangle$. In fact, if one also formulates Theorem B in terms of $\mathbb{E}_3$-MU-algebra structures, then none of the statements or proofs in this paper rely on [BM13].

Remark 1.0.6. It is not surprising that $\mathbb{E}_3$-algebra structure on $\mathrm{BP}\langle n\rangle$ is useful in the proof of redshift. As far back as 2000, Ausoni and Rognes observed that redshift could be proved whenever $\mathrm{BP}\langle n\rangle$ is $\mathbb{E}_\infty$ and the Smith–Toda complex $V(n)$ exists as a homotopy ring spectrum [Rog00]. Unfortunately, both of these hypotheses are known to generically fail [Nav10], [Law18].

Open Questions. Our work leaves open many natural questions, chief of which is to determine the homotopy type of $\mathrm{K}(\mathrm{BP}\langle n\rangle)$. Since we show this homotopy type to be closely related to its localization $L_{n+1}^f\mathrm{K}(\mathrm{BP}\langle n\rangle)$, one might hope to assemble an understanding via chromatic fracture squares (cf. [AR12b]). We would also like to highlight the following:

Question 1.0.7. For what ring spectra R, other than $R = \mathrm{BP}\langle n\rangle$, is it possible to prove a version of the Segal conjecture?

Question 1.0.8. For what ring spectra R, other than $R = \mathrm{BP}\langle n\rangle$, is it possible to prove a version of the Canonical Vanishing Theorem?

While variants of the Segal conjecture have received much study (see Section 4 for some history), the Canonical Vanishing result does not seem as widely analyzed. It seems plausible that a ring R might satisfy Canonical Vanishing but not the Segal conjecture, or vice versa.

Question 1.0.9. What ring spectra R, other than $R = \mathrm{BP}\langle n\rangle$, satisfy redshift, or various less precise forms of the Lichtenbaum–Quillen conjecture?

For an arbitrary $\mathrm{BP}\langle n\rangle$-algebra R satisfying the Segal conjecture, Akhil Mathew has deduced (given our work here) various Lichtenbaum–Quillen statements. He has graciously allowed us to reproduce his results at the end of Section 3.3.

Remark 1.0.10. Redshift for $\mathbb{E}_1$-rings that are far from complex oriented remains mysterious. For some intriguing results in this direction, see the work of Angelini-Knoll and Quigley [AKQ21a] on the family of spectra $y(n)$.

One would also like to make many of the above results *effective*, rather than asserting an isomorphism in degrees above an unspecified dimension. Especially the following question is interesting, since it does not depend on a choice of a finite complex:

Question 1.0.11. In precisely what range of degrees is the map

$$\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)} \to L_{n+1}^f\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$$

a π_*-isomorphism?

Theorem A proves that *some* form of $\mathrm{BP}\langle n\rangle$ admits an $\mathbb{E}_3$-BP-algebra structure, and it remains an interesting open question to determine exactly which forms admit such structure.

Question 1.0.12. Which forms of $\mathrm{BP}\langle n\rangle$ admit an $\mathbb{E}_3$-BP-algebra structure? Which of these can be built by the procedure in Section 2?

The subtleties behind Question 1.0.12 are indicated by work of Strickland [Str99, Rem. 6.5], who observed at $p = 2$ that neither the Hazewinkel or Araki generators may be used as generators in Theorem A.

Remark 1.0.13. We suspect that our $\mathbb{E}_3$-algebra structure will be of use in additional computations. For example, Ausoni and Richter give an elegant formula for the THH of a height 2 Johnson–Wilson theory, under the assumption that the theory can be made $\mathbb{E}_3$ [AR20]. Our result does not directly feed into their work, for the simple reason that they use a form of $\mathrm{BP}\langle 2\rangle[v_2^{-1}]$ specified by the Honda formal group. It seems unlikely that their theorem relies essentially on this choice.

Remark 1.0.14. By imitating our construction of an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n\rangle$, we suspect one could produce an $\mathbb{E}_{2\sigma+1}$-$\mathrm{MU}_{\mathbb{R}}$-algebra structure on $\mathrm{BP}\langle n\rangle_{\mathbb{R}}$. As a result, the fixed points $\mathrm{BP}\langle n\rangle_{\mathbb{R}}^{C_2}$ would acquire an $\mathbb{E}_1$-ring structure. At the moment, these fixed points are not even known to be homotopy associative [KLW18].

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Conventions and notation.

- We work in the setting of ∞-categories as used in [Lur17]. We will say *category* and *groupoid* instead of ∞-category and ∞-groupoid. We will denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the mapping space between objects $X, Y \in \mathcal{C}$.
- We let Sp denote the category of spectra.
- If $\mathcal{C}$ is enriched over a category $\mathcal{A}$, we denote by $\mathrm{map}_{\mathcal{C}}(X, Y)$ the morphism object associated to a pair of objects $X, Y \in \mathcal{C}$. For example, if $\mathcal{C}$ is stable, then it is canonically enriched over Sp, and we use $\mathrm{map}_{\mathcal{C}}(X, Y)$ to denote the internal mapping spectrum. In cases below where $\mathcal{C}$ is stable and canonically enriched over Mod_R for some $\mathbb{E}_\infty$-ring, no confusion should arise because the underlying spectrum of the morphism object in R-modules will agree with the morphism object in spectra.
- We do not distinguish between a space and its corresponding groupoid; in particular, we will speak about functors $X \to \mathcal{C}$, where X is a space and $\mathcal{C}$ is a category.
- If M is a (discrete) module over a (discrete) ring R with elements $x, y \in M$, then we write $x \doteq y$ to mean that $x = \lambda y$ where λ is a unit in R.
- If $\mathcal{C}$ is a category and G is a (possibly topological) group, then the category of *objects of* $\mathcal{C}$ *with G-action* is the functor category $\mathsf{Fun}(\mathrm{B}G, \mathcal{C})$. When $\mathcal{C} = \mathsf{Sp}$, we will sometimes refer to these objects as G-spectra. The theory of "genuine G-spectra" is not used in this paper, so there should be no confusion.
- Our conventions on grading spectral sequences associated to towers differs from the usual one, since we prefer to begin our spectral sequences at the second page. See Section C.1.
- If $\mathcal{C}$ is a stable category equipped with a t-structure, we say that an object $X \in \mathcal{C}$ is *bounded above* if $X = \tau_{\leq d} X$ for some $d \in \mathbb{Z}$. We say that X is *bounded below* if $X = \tau_{\geq d} X$ for some $d \in \mathbb{Z}$. We say that a map $f : X \to Y$ is *truncated* if the fiber of f is bounded above.
- If A is an $\mathbb{E}_1$-R-algebra where R is an $\mathbb{E}_\infty$-ring, we denote by $\mathrm{THH}(A/R)$ the R-module $A \otimes_{A \otimes A^{\mathrm{op}}} A$.
- Our conventions on cyclotomic spectra differ somewhat from those in [NS18] since we are only interested in constructions with p-complete spectra. See Section 3.2 for a discussion.

2. The Multiplication Theorem

We begin by giving a more precise formulation of Theorem A. Recall that there is a canonical inclusion [Qui69],

$$\mathrm{BP}_* \to (\mathrm{MU}_{(p)})_*,$$

classifying the p-typification of the universal formal group law. We will write $\{x_i\}_{i\geq 0}$ for a choice of indecomposable polynomial generators of $(\mathrm{MU}_{(p)})_*$, with $|x_i|=2i$, such that the classes $\{x_{p^j-1}\}_{j\geq 1}$ form polynomial generators for BP_* over $\mathbb{Z}_{(p)}$. It will be convenient, at times, to change these generators, so we do not fix such a choice. We write v_j for the generator x_{p^j-1}. By convention we agree that $v_0 = p$. The following definition may be compared with [LN12, 3.1, 3.2].

Definition 2.0.1. Let $1 \leq k \leq \infty$ and $n \geq 0$. Let B be a p-local $\mathbb{E}_k$-MU-algebra. We say that B is an $\mathbb{E}_k$-MU-*algebra form of* $\mathrm{BP}\langle n\rangle$ if the composite

$$\mathbb{Z}_{(p)}[v_1, \ldots, v_n] \subseteq \mathrm{BP}_* \subseteq (\mathrm{MU}_{(p)})_* \to B_*$$

is an isomorphism. By convention, we consider $\mathbb{F}_p$ as the unique $\mathbb{E}_k$-MU-algebra form of $\mathrm{BP}\langle -1\rangle$.

Remark 2.0.2. The subring

$$\mathbb{Z}_{(p)}[v_1, \ldots, v_n] \subseteq \mathrm{BP}_*$$

is equal to the subring generated by all elements of degree at most $2p^n - 2$, and hence is independent of our choice of polynomial generators. It follows that the definition of an $\mathbb{E}_k$-MU-algebra form of $\mathrm{BP}\langle n\rangle$ also does not depend on this choice.

Example 2.0.3. For any $k \geq 1$, there is a unique $\mathbb{E}_k$-MU-algebra form of $\mathrm{BP}\langle 0\rangle$, which is the p-local integers $\mathbb{Z}_{(p)}$. The Adams summand ℓ of $ku_{(p)}$ can be equipped with an $\mathbb{E}_\infty$-MU-algebra structure, which makes ℓ into an $\mathbb{E}_\infty$-MU-algebra form of $\mathrm{BP}\langle 1\rangle$.

We will now relate the notion of a form of $\mathrm{BP}\langle n\rangle$ to the quotients in Theorem A.

Notation 2.0.4. Let $J \subseteq \mathbb{Z}_{\geq 0}$ be an indexing set, and $\{z_j\}_{j\in J}$ a sequence of elements in $\pi_*\mathrm{MU}_{(p)}$. Define

$$\mathrm{MU}_{(p)}/(z_j : j \in J) := \operatorname*{colim}_m \bigotimes_{j\in J, j\leq m} \mathrm{MU}_{(p)}/z_j,$$

where the tensor product is taken over $\mathrm{MU}_{(p)}$, and $\mathrm{MU}_{(p)}/z_j$ is defined by the cofiber sequence

$$\Sigma^{|z_j|}\mathrm{MU}_{(p)} \xrightarrow{z_j} \mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}/z_j.$$

LEMMA 2.0.5. *If B is an $\mathbb{E}_1$-MU-algebra form of $\mathrm{BP}\langle n\rangle$, then there is a choice of indecomposable generators $x_j \in \pi_{2j}\mathrm{MU}_{(p)}$, $j \geq 1$, and there is an extension of the unit map $\iota : \mathrm{MU}_{(p)} \to B$ to an equivalence of MU-modules*

$$\big(\mathrm{MU}_{(p)}/(x_j : j \neq p^i - 1, 1 \leq i \leq n)\big) \simeq B.$$

Proof. Let x'_j be any choice of indecomposable generators such that

$$(x'_{p-1}, x'_{p^2-1}, \dots, x'_{p^n-1}) = (v_1, \dots, v_n).$$

By definition, if $j \neq p^i - 1$ for $1 \leq i \leq n$, then $\iota(x'_j) = f_j(v_1, \dots, v_n)$ for some polynomial f_j with coefficients in $\mathbb{Z}_{(p)}$. Define

$$x_j := \begin{cases} x'_j & j = p^i - 1, 1 \leq i \leq n, \\ x'_j - f_j(x_{p-1}, \dots, x_{p^n-1}) & \text{else.} \end{cases}$$

Then $\{x_j\}$ gives a new set of indecomposable generators for $\pi_* \mathrm{MU}_{(p)}$ with the property that $\iota(x_j) = 0$ when $j \neq p^i - 1$ for some $1 \leq i \leq n$. We may then construct maps

$$\bigotimes_{j \leq m, j \neq p^i - 1, 1 \leq i \leq n} \mathrm{MU}_{(p)}/x_j \to \bigotimes_{j \leq m, j \neq p^i - 1, 1 \leq i \leq n} B \to B,$$

where the second map is the multiplication on B. Passing to the colimit gives the desired equivalence

$$\mathrm{MU}_{(p)}/(x_j : j \neq p^i - 1, 1 \leq i \leq n) \to B. \qquad \square$$

It follows from the above lemma and Remark 1.0.5 that Theorem A is a consequence of the following theorem, which will be the main result of this section:

THEOREM 2.0.6. *For all $n \geq -1$, there exists an $\mathbb{E}_3$-MU-algebra form of* $\mathrm{BP}\langle n \rangle$.

Remark 2.0.7. There are only a few results from Section 2 that will be needed later in the paper. In addition to Theorem 2.0.6, the reader interested in redshift need only understand Proposition 2.5.3 and Theorem 2.5.4.

2.1. *Outline of the proof.* For ease of exposition, we will not take care in this outline to distinguish between different forms of $\mathrm{BP}\langle n \rangle$. Our proof of Theorem 2.0.6 proceeds by induction on n: assuming that $\mathrm{BP}\langle n \rangle$ is an $\mathbb{E}_3$-MU-algebra, we will construct $\mathrm{BP}\langle n+1 \rangle$ as an $\mathbb{E}_3$-MU-algebra.

Consider the tower of MU-modules

$$\mathrm{BP}\langle n+1 \rangle \to \cdots \to \mathrm{BP}\langle n+1 \rangle/(v_{n+1}^k) \to \cdots \to \mathrm{BP}\langle n \rangle.$$

By our inductive hypothesis, the base of the tower, $\mathrm{BP}\langle n \rangle$, has been refined to an $\mathbb{E}_3$-MU-algebra. One possible way to proceed would be to inductively equip each $\mathrm{BP}\langle n+1 \rangle/(v_{n+1}^k)$ with an $\mathbb{E}_3$-MU-algebra structure. Unfortunately, this would involve understanding the $\mathbb{E}_3$-MU-algebra cotangent complex of $\mathrm{BP}\langle n+1 \rangle/(v_{n+1}^k)$, which becomes increasingly difficult to control as k grows.

Instead, we will make a stronger inductive hypothesis. As we review in Section 2.6, there are $\mathbb{E}_\infty$-MU-algebras $\mathrm{MU}[y]/(y^k)$ refining the truncated polynomial algebras $\mathrm{MU}_*[y]/(y^k)$, where $|y| = 2p^{n+1} - 2$. We will induct on k to build $\mathrm{BP}\langle n+1\rangle/(v_{n+1}^k)$ as an $\mathbb{E}_3$-$\mathrm{MU}[y]/(y^k)$-algebra. Taking a limit, we produce $\mathrm{BP}\langle n+1\rangle$ as an $\mathbb{E}_3$-$\mathrm{MU}[y]$-algebra, where y acts by v_{n+1}.

In Sections 2.2 and 2.3 we review some background in deformation theory. In Sections 2.4 and 2.5 we make the key non-formal computations of enveloping algebras and cotangent complexes, which ultimately rest on Steinberger's computation of the action of Dyer–Lashof operations on the dual Steenrod algebra and on Kochman's computation of the action of Dyer–Lashof operations on the homology of BU. Finally, in Section 2.6, we put the pieces together and prove Theorem 2.0.6.

Remark 2.1.1. Our original argument, appearing in a preprint version of this paper, relied on the theory of centers and some manipulations with Koszul duality. We are extremely grateful to the first referee for explaining how our two uses of Koszul duality "cancel each other out," suggesting the more intuitive argument sketched above.

Remark 2.1.2. Our argument constructs $\mathrm{BP}\langle n+1\rangle$ as an $\mathbb{E}_3$-$\mathrm{MU}[y]$-algebra, where y acts by v_{n+1}, but we remember $\mathrm{BP}\langle n+1\rangle$ only as an $\mathbb{E}_3$-MU-algebra when constructing $\mathrm{BP}\langle n+2\rangle$. One might wonder whether, with more care, it is possible to construct $\mathrm{BP}\langle n\rangle$ as an $\mathbb{E}_3$-$\mathrm{MU}[y_0, y_1, \ldots, y_n]$-algebra, where y_i acts as v_i. In fact, this is not possible, even when $n = 1$. If ℓ were an $\mathbb{E}_3$-$\mathrm{MU}[y_0]$-algebra, tensoring over $\mathrm{MU}[y_0]$ with the augmentation $\mathrm{MU}[y_0] \to \mathrm{MU}$ would construct $\ell/p = k(1)$ as an $\mathbb{E}_3$-algebra. However, any $\mathbb{E}_2$-algebra with $p = 0$ in its homotopy groups must be an $\mathbb{F}_p$-module [MNN15, Th. 4.18].

2.2. *Background*: *Operadic modules and enveloping algebras.* Fix an $\mathbb{E}_{n+1}$-algebra k, and let $\mathcal{C} = \mathsf{LMod}_k$. If $A \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is an $\mathbb{E}_n$-algebra, then we can define an $\mathbb{E}_n$-monoidal category, $\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$, of $\mathbb{E}_n$-A-modules ([Lur17, 3.3.3.9]). The relevance of this category in our case is the equivalence of $\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$ ([Lur17, 7.3.4.14]) with the tangent category $\mathsf{Sp}(\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})_{/A})$ controlling deformations of A (see Recollection 2.3.1).

It follows from [Lur17, 7.1.2.1] that we have an equivalence

$$\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{LMod}_{\mathcal{U}^{(n)}(A)},$$

where $\mathcal{U}^{(n)}(A)$, the $\mathbb{E}_n$-k-enveloping algebra of A, is the endomorphism algebra spectrum of the free $\mathbb{E}_n$-A-module on k.

Remark 2.2.1. It follows from [Lur17, 4.8.5.11] that the assignment $B \mapsto \mathcal{U}^{(n-j)}(B)$ is a lax $\mathbb{E}_j$-monoidal functor of B. In particular, if A is an $\mathbb{E}_n$-algebra in $\mathcal{C}$, then $\mathcal{U}^{(n-j)}(A)$ has a canonical $\mathbb{E}_{j+1}$-algebra structure.

We will need the following standard fact:

Proposition 2.2.2. *There is a canonical equivalence of algebras*

$$\mathcal{U}^{(n)}(A) \simeq A \otimes_{\mathcal{U}^{(n-1)}(A)} A^{\mathrm{op}},$$

where A^{op} denotes A regarded as an $\mathbb{E}_1$-$\mathcal{U}^{(n-1)}(A)$-algebra with its opposite multiplication.

Proof. The enveloping algebra is obtained by taking the endomorphism algebra of a free object. So it suffices, by [Lur17, 4.8.5.11, 4.8.5.16], to provide an equivalence

$$\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{LMod}_A(\mathcal{C}) \otimes_{\mathsf{Mod}_A^{\mathbb{E}_{n-1}}(\mathcal{C})} \mathsf{RMod}_A(\mathcal{C}).$$

By [Lur17, 4.8.4.6,4.3.2.7], we may identify the right-hand side as a category of bimodules

$$\mathsf{LMod}_A(\mathcal{C}) \otimes_{\mathsf{Mod}_A^{\mathbb{E}_{n-1}}(\mathcal{C})} \mathsf{RMod}_A(\mathcal{C}) \simeq \mathsf{BMod}_A(\mathsf{Mod}_A^{\mathbb{E}_{n-1}}(\mathcal{C})).$$

The result now follows from [HNP19, 1.0.4] by taking tangent categories at A of the equivalence ([Lur17, 5.1.2.2]):

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C})). \qquad \square$$

Remark 2.2.3. One can use this result and induction on n to prove that there is an equivalence

$$\mathcal{U}^{(n)}(A) \simeq \int_{\mathbb{R}^n - \{0\}} A$$

of the enveloping algebra with the factorization homology of A over $\mathbb{R}^n - \{0\}$.

2.3. *Background: Deformation theory.* In this section we will review the obstruction theory for deforming an algebra over a square-zero extension. Throughout, if $f : S \to S'$ is a map of $\mathbb{E}_\infty$-rings, we will denote by f^* the induced functor $S' \otimes_S (-)$ and by f_* the restriction of scalars along f, to emphasize the dependence on f.

In this section we will be using the cotangent complex formalism as in [Lur17, §7.3], which we briefly review now.

Recollection 2.3.1. If $\mathcal{C}$ is a presentable category, then there is a cocartesian fibration $T_{\mathcal{C}} \to \mathcal{C}$ whose fiber over $A \in \mathcal{C}$ is given by the stabilization $\mathsf{Sp}(\mathcal{C}_{/A})$. If $M \in \mathsf{Sp}(\mathcal{C}_{/A})$, then we will denote by $A \oplus M$ the image of M under the functor $\Omega^\infty : \mathsf{Sp}(\mathcal{C}_{/A}) \to \mathcal{C}_{/A}$, and we will refer to this object as the *trivial square-zero extension* of A by M. The *cotangent complex of* A, denoted $\mathbf{L}_A$, is the image of id_A under the adjoint $\Sigma^\infty_+ : \mathcal{C}_{/A} \to \mathsf{Sp}(\mathcal{C}_{/A})$, so that

$$\mathrm{Map}_{\mathsf{Sp}(\mathcal{C}_{/A})}(\mathbf{L}_A, M) \simeq \mathrm{Map}_{\mathcal{C}_{/A}}(A, A \oplus M).$$

Given $\eta : \mathbf{L}_A \to M$ we will refer to the adjoint map $d_\eta : A \to A \oplus M$ as the *derivation classified by* η. Given such a derivation, we may form the pullback

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[d, "d_0"] \\ A \arrow[r, "d_\eta"'] & A \oplus M, \end{tikzcd}$$

where d_0 is classified by the zero map $0 : \mathbf{L}_A \to M$. We refer to A^η as the *square-zero extension classified by* η.

In the special case $\mathcal{C} = \mathsf{Alg}_{\mathbb{E}_m}(\mathcal{D})$, where $\mathcal{D}$ is a stable and presentably $\mathbb{E}_m$-monoidal category, there is a canonical equivalence [Lur17, Th. 7.3.4.13] $\mathsf{Sp}(\mathsf{Alg}_{\mathbb{E}_m}(\mathcal{D})_{/A}) \simeq \mathsf{Mod}_A^{\mathbb{E}_m}(\mathcal{D})$ with the category of $\mathbb{E}_m$-A-modules. (When $m = \infty$, this is equivalent to the ordinary category of A-modules.) We denote $\mathbf{L}_A$ by $\mathbf{L}_A^{\mathbb{E}_m}$ and further decorate it as $\mathbf{L}_{A/R}^{\mathbb{E}_m}$ in the setting where $\mathcal{D} = \mathsf{Mod}_R$ for an $\mathbb{E}_\infty$-ring R.

We now turn to the problem of classifying algebras over square-zero extensions. Let R be a connective $\mathbb{E}_\infty$-ring and I a connective R-module. Let $\eta : \mathbf{L}_R \to \Sigma I$ be a map of R-modules from the $\mathbb{E}_\infty$-cotangent complex of R to ΣI, and denote by R^η the corresponding square-zero extension. By definition, this sits in a pullback diagram

$$\begin{tikzcd} R^\eta \arrow[r] \arrow[d] & R \arrow[d, "d_0"] \\ R \arrow[r, "d"'] & R \oplus \Sigma I, \end{tikzcd}$$

where d is adjoint to η and d_0 is the trivial derivation.

Recollection 2.3.2. If S is a connective $\mathbb{E}_\infty$-ring, we will denote by $\mathsf{Mod}_S^{\mathrm{cn}}$ the category of connective S-modules. By [Lur18, Th. 16.2.0.2], the pullback diagram above induces a symmetric monoidal equivalence

$$\mathsf{Mod}_{R^\eta}^{\mathrm{cn}} \xrightarrow{\simeq} \mathsf{Mod}_R^{\mathrm{cn}} \times_{\mathsf{Mod}_{R\oplus\Sigma I}^{\mathrm{cn}}} \mathsf{Mod}_R^{\mathrm{cn}},$$

and hence an equivalence upon taking categories of $\mathbb{E}_m$-algebras

$$\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}_{R^\eta}^{\mathrm{cn}}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}_R^{\mathrm{cn}}) \times_{\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}_{R\oplus\Sigma I}^{\mathrm{cn}})} \mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}_R^{\mathrm{cn}}).$$

Denoting an element in the target by $(A, B, \alpha : d^*A \simeq d_0^*B)$, the inverse to this equivalence is implemented by the functor $(A, B, \alpha) \mapsto A \times_{d^*A} B$.

LEMMA 2.3.3. *Suppose $\eta = 0$ classifies the trivial derivation, so that $R^\eta = R \oplus I$ and $d = d_0$. Then $\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}_{R^\eta}^{\mathrm{cn}})$ is equivalent to the category of pairs $(A, \rho : \mathbf{L}_{A/R}^{\mathbb{E}_m} \to A \otimes_R \Sigma I)$ where A is a connective $\mathbb{E}_m$-R-algebra and ρ is a map of $\mathbb{E}_m$-A-modules. Under this equivalence, the $\mathbb{E}_m$-R^η-algebra corresponding to (A, ρ) has underlying $\mathbb{E}_m$-R-algebra given by the square-zero extension classified by ρ.*

Proof. By Recollection 2.3.2, the category of connective $\mathbb{E}_m$-R^η-algebras is equivalent to the category of triples $(A, B, \alpha : A\otimes_R(R\oplus\Sigma I) \simeq B\otimes_R(R\oplus\Sigma I))$, where A and B are connective $\mathbb{E}_m$-R-algebras and α is an equivalence. Since d_0 is a section of the projection $p : R \oplus \Sigma I \to R$, we have a canonical equivalence $p^* d_0^* A = A$. It follows that this category of triples is equivalent to the subcategory where $A = B$, so we are left with understanding automorphisms of the $\mathbb{E}_m$-$(R \oplus \Sigma I)$-algebra $A \otimes_R (R \oplus \Sigma I) = d_0^* A$. By adjunction, this is equivalent to understanding the space of $\mathbb{E}_m$-R-algebra sections of the projection

$$(d_0)_* d_0^* A = A \otimes_R (R \oplus \Sigma I) \to (d_0)_* p_* p^* d_0^* A = A.$$

If A is an $\mathbb{E}_m$-R-algebra, then $A \otimes_R (R \oplus \Sigma I)$ is canonically a spectrum object in $\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}_R)_{/A}$ with deloopings given by $A\otimes_R(R\oplus\Sigma^{j+1}I)$. In other words, $A\otimes_R(R\oplus\Sigma I)$ is a trivial square-zero extension of A by $A\otimes_R\Sigma I$. The result now follows by the universal property of the cotangent complex. $\square$

Construction 2.3.4. Returning to the case of a general square-zero extension R^η, suppose that A is an $\mathbb{E}_m$-R-algebra. Then d^*A is an $\mathbb{E}_m$-$(R \oplus \Sigma I)$-algebra. Moreover, since d is a section of the projection map, base changing d^*A along $R \oplus \Sigma I \to R$ recovers A. By the previous lemma (applied to the trivial square-zero extension $R \oplus \Sigma I$ rather than $R \oplus I$), the $\mathbb{E}_m$-$(R \oplus \Sigma I)$-algebra d^*A is determined by a pair $(A, o(A) : \mathbf{L}^{\mathbb{E}_m}_{A/R} \to A \otimes_R \Sigma^2 I)$. We refer to $o(A)$ as the *obstruction class for deforming* A. Though it is not indicated in the notation, this class also depends on η.

Definition 2.3.5. Let R and R^η be as above, and let A be a connective $\mathbb{E}_m$-R-algebra. Define the *category of lifts of* A by the pullback

$$\begin{tikzcd}
\mathsf{Lifts}(A) \arrow[r] \arrow[d] & \{A\} \arrow[d] \\
\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}^{\mathrm{cn}}_{R^\eta}) \arrow[r] & \mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}^{\mathrm{cn}}_{R}).
\end{tikzcd}$$

PROPOSITION 2.3.6. *The category* $\mathsf{Lifts}(A)$ *is a groupoid equivalent to the space of nullhomotopies of the* $\mathbb{E}_m$-A-*module map* $o(A) : \mathbf{L}^{\mathbb{E}_m}_{A/R} \to A \otimes_R \Sigma^2 I$. *In particular*,

(i) *there exists an* $\mathbb{E}_m$-R^η-*algebra* $\widetilde{A}$ *such that* $\widetilde{A} \otimes_{R^\eta} R \simeq A$ *if and only if* $o(A)$ *is nullhomotopic*;

(ii) *if* $o(A)$ *is nullhomotopic then the space* $\mathsf{Lifts}(A)$ *is equivalent to*

$$\mathrm{Map}_{\mathsf{Mod}^{\mathbb{E}_m}_A(\mathsf{Mod}_R)}(\mathbf{L}^{\mathbb{E}_m}_{A/R}, A \otimes_R \Sigma I).$$

Proof. By Recollection 2.3.2 and the definition of the category of lifts, each square in the rectangle

$$\begin{tikzcd}
\mathsf{Lifts}(A) \arrow[r] \arrow[d] & \{A\} \arrow[d] \\
\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}^{\mathrm{cn}}_{R^\eta}) \arrow[r] \arrow[d] & \mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}^{\mathrm{cn}}_{R}) \arrow[d] \\
\mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}^{\mathrm{cn}}_{R}) \arrow[r] & \mathsf{Alg}_{\mathbb{E}_m}(\mathsf{Mod}^{\mathrm{cn}}_{R\oplus\Sigma I})
\end{tikzcd}$$

is a pullback. Thus $\mathsf{Lifts}(A)$ is equivalent to the category of pairs $(B, d^*B \simeq d_0^*A)$, where B is a connective $\mathbb{E}_m$-R-algebra, d is the derivation classified by η, and d_0 is the trivial derivation. Let $p : R \oplus \Sigma I \to R$ be the projection so that $p \circ d = p \circ d_0 = \mathrm{id}_R$. By Lemma 2.3.3, an equivalence $d^*B \simeq d_0^*A$ corresponds to an equivalence $B = p^*d^*B \simeq p^*d_0^*A = A$ together with a homotopy between the two resulting maps $\mathbf{L}^{\mathbb{E}_m}_{A/R} \to A \otimes_R \Sigma I$. Again, we may restrict to the equivalent subcategory with $A = B$, and we have arrived at an equivalence between $\mathsf{Lifts}(A)$ and the category of homotopies between $o(A)$, which yields d^*A, and the zero map, which yields d_0^*A. This completes the proof. $\square$

2.4. *Grounding the induction.* The purpose of this section is to compute the higher MU-enveloping algebras of $\mathbb{F}_p$. This will allow us to resolve extension problems when computing the $\mathbb{E}_3$-MU-cotangent complex of $\mathrm{BP}\langle n\rangle$ during the inductive step.

In the course of this computation we will make use of the Kudo-Araki-Dyer–Lashof operations [BMMS86], which are natural maps of spectra (see, e.g., [GL20]) for any $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra, A,

$$\begin{cases} Q^i : A \to \Sigma^{-2i(p-1)}A & p > 2, \\ Q^i : A \to \Sigma^{-i}A & p = 2. \end{cases}$$

We will also use the suspension operation σ discussed in Section A.

LEMMA 2.4.1. *The* $\mathbb{E}_1$-MU-*enveloping algebra of* $\mathbb{F}_p$ *has homotopy given by*

$$\pi_*\mathcal{U}^{(1)}_{\mathrm{MU}}(\mathbb{F}_p) \simeq \Lambda(\sigma v_i : i \geq 0) \otimes_{\mathbb{F}_p} \Lambda(\sigma x_j : j \neq p^k - 1).$$

When regarded as an $\mathbb{E}_\infty$-$\mathbb{F}_p$-*algebra via the map*[2] $\mathrm{id}_{\mathbb{F}_p} \otimes 1 : \mathbb{F}_p \to \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p$, *we have the identities*

$$Q^{p^i}\sigma v_i \doteq \sigma v_{i+1}, \qquad p > 2,$$
$$Q^{2^{i+1}}\sigma v_i \doteq \sigma v_{i+1} \qquad p = 2,$$

[2] We make this choice for definiteness, but it does not have a significant effect on our computations. Indeed, once $k \geq 2$, there is a canonical $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebra structure on $\mathcal{U}^{(k)}_{\mathrm{MU}}(\mathbb{F}_p) = \int_{\mathbb{R}^k - \{0\}} \mathbb{F}_p$ up to equivalence, since $\mathbb{R}^k - \{0\}$ is connected.

$$Q^{j+1}\sigma x_j \doteq \sigma x_{jp+p-1}, \text{mod decomposables} \qquad p>2,$$
$$Q^{2j+2}\sigma x_j \doteq \sigma x_{jp+p-1}, \text{mod decomposables} \qquad p=2.$$

Proof. The algebra structure follows from [Ang08, Prop. 3.6]. To compute the action of the operations we use the two $\mathbb{E}_\infty$-maps:

$$\mathbb{F}_p \otimes \mathbb{F}_p \xrightarrow{f} \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p \xrightarrow{g} \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{MU}} \mathbb{F}_p$$

We have $f(\overline{\tau}_i) \doteq \sigma v_i$ (independently of our choice of v_i) and $g(\sigma x_j) = \sigma b_j$, where b_j is the Hurewicz image of x_j (for $j \neq p^k - 1$). The first identity now follows from Steinberger's calculation [BMMS86, III.2] that

$$Q^{p^i}\overline{\tau}_i = \overline{\tau}_{i+1},$$

(and the analogous result at $p=2$). For the second identity, first recall that the Thom isomorphism is an equivalence

$$\mathbb{F}_p \otimes \mathrm{MU} \simeq \mathbb{F}_p \otimes \mathrm{BU}_+$$

of $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, and hence we have an equivalence

$$\mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{MU}} \mathbb{F}_p \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{BU}_+} \mathbb{F}_p \simeq \mathbb{F}_p \otimes \mathrm{B}^2\mathrm{U}_+.$$

of $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras.

Since Σ^∞_+ is symmetric monoidal, the canonical map $\Sigma^\infty_+(\Omega X) \to \Omega\Sigma^\infty_+ X$ is a map of non-unital $\mathbb{E}_\infty$-algebras for any $\mathbb{E}_\infty$-space X. In particular, taking $X = \mathrm{B}^2 U$, we see that the homology suspension $H_*(\mathrm{BU};\mathbb{F}_p) \to H_{*+1}(\mathrm{B}^2U;\mathbb{F}_p)$ preserves Dyer–Lashof operations. The result now follows from Kochman's computation [Koc73, Th. 6] of the action of Dyer–Lashof operations on $H_*(\mathrm{BU})$. $\square$

Lemma 2.4.1 implies that the Künneth spectral sequence for $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p)$ collapses at the E^2 term, which is a divided power algebra:

$$E^2 = E^\infty = \Gamma\{\sigma^2 v_i, \sigma^2 x_j : i \geq 0, j \neq p^k - 1\} \Rightarrow \pi_*(\mathbb{F}_p \otimes_{\mathcal{U}^{(1)}_{\mathrm{MU}}(\mathbb{F}_p)} \mathbb{F}_p) = \pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p).$$

Here we recall that, in the bar complex computing $\mathrm{Tor}^{\Lambda(z)}(\mathbb{F}_p, \mathbb{F}_p)$, the class $\gamma_{p^i}(\sigma z)$, where $i \geq 0$, is represented by the element

$$z \otimes z \otimes \cdots \otimes z \in \Lambda(z)^{\otimes p^i}.$$

Proposition 2.4.2. *There are non-trivial multiplicative extensions in the bar spectral sequence for $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p)$ as follows:*

(i) *If $w_{0,i} \in \pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p)$ is detected by the divided power $\gamma_{p^i}(\sigma^2 v_0)$, then $w_{0,i}^{p^j}$ is detected by $\gamma_{p^i}(\sigma^2 v_j)$, up to a unit.*

(ii) *If $y_{j,i} \in \pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p)$ is detected by the divided power $\gamma_{p^i}(\sigma^2 x_j)$, then $y_{j,i}^p$ is detected by $\gamma_{p^i}(\sigma^2 x_{jp+p-1})$, up to a unit.*

In particular, the homotopy groups of $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p)$ *are polynomial on even dimensional classes, one of which can be chosen to be* $\sigma^2 v_0$.

Proof. This follows from the computation of power operations in the previous lemma by applying [BM13, Th. 3.6] to the standard representatives of divided powers in the bar complex. ☐

Remark 2.4.3. From the equivalence $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p) \simeq \mathrm{THH}(\mathbb{F}_p/\mathrm{MU})$, we deduce the crucial fact that the homotopy groups of $\mathrm{THH}(\mathbb{F}_p/\mathrm{MU})$ are polynomial. An anonymous referee points out that one may also deduce this from Bökstedt's perioidicity theorem, as we now sketch. We have $\mathrm{THH}(\mathbb{F}_p/\mathrm{MU}) \simeq \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}$. By Bökstedt's theorem, $\pi_*\mathrm{THH}(\mathbb{F}_p)$ is polynomial, and one is reduced to proving that $\mathbb{F}_p \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}$ has polynomial homotopy. By a Thom spectrum argument [BCS10], this spectrum is equivalent to $\mathbb{F}_p \otimes \mathrm{BSU}_+$, which is known to have polynomial homotopy groups.

PROPOSITION 2.4.4. *The* $\mathbb{E}_3$-MU-*enveloping algebra of* $\mathbb{F}_p$ *has homotopy given by an exterior algebra on odd dimensional generators, one of which can be chosen to be* $\sigma^3 v_0$.

Proof. The proof is immediate from [Ang08, Prop. 3.6]. ☐

The spectral sequence

$$\mathrm{Ext}_{\pi_*\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$$

then immediately collapses with no possible $\mathbb{F}_p$-algebra extensions, and so proves

THEOREM 2.4.5. *The spectrum* $\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p)$ *has homotopy given by a polynomial algebra on even degree generators.*

2.5. *Computation at the inductive step.* In this section, we will assume that we have constructed an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n\rangle$, and simply denote it by $\mathrm{BP}\langle n\rangle$. We will also choose our polynomial generators for $\pi_*(\mathrm{MU}_{(p)})$ in such a way that $\ker((\mathrm{MU}_*)_{(p)} \to \mathrm{BP}\langle n\rangle_*)$ is generated by the v_i with $i \geq n+1$ and by the x_j with $j \neq p^k - 1$.

Remark 2.5.1. If R is a p-local $\mathbb{E}_k$-MU-algebra, then, with notation as in Section A, $\int_M R$, for any non-empty k-manifold M, can be computed in $\mathrm{MU}_{(p)}$-modules instead of MU-modules. We may therefore make sense of the suspension operations from Section A for elements in $\pi_*(\mathrm{cofib}(\mathrm{MU}_{(p)} \to R))$, rather than just elements of $\pi_*(\mathrm{cofib}(\mathrm{MU} \to R))$.

LEMMA 2.5.2. *The* $\mathbb{E}_1$-MU-*enveloping algebra of* $\mathrm{BP}\langle n\rangle$ *has homotopy given, as a* $\mathrm{BP}\langle n\rangle_*$-*algebra, by*

$$\pi_*\mathcal{U}^{(1)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle) \simeq \Lambda_{\mathrm{BP}\langle n\rangle_*}(\sigma v_i : i \geq n+1) \otimes_{\mathrm{BP}\langle n\rangle_*} \Lambda_{\mathrm{BP}\langle n\rangle_*}(\sigma x_j : j \neq p^k - 1).$$

Proof. The proof is immediate from [Ang08, Prop. 3.6]. □

It follows that the bar spectral sequence for $\pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ collapses to a divided power algebra on even classes:

$$E^2 = E^\infty = \Gamma_{\mathrm{BP}\langle n\rangle_*}(\sigma^2 v_i : i \geq n+1) \otimes_{\mathrm{BP}\langle n\rangle_*} \Gamma_{\mathrm{BP}\langle n\rangle_*}(\sigma^2 x_j : j \neq p^k - 1).$$

PROPOSITION 2.5.3. *For $i \geq 0$, choose any lift $w_{n+1,i}$ of the divided power class $\gamma_{p^i}(\sigma^2 v_{n+1})$. For $j \not\equiv -1 \bmod p$ and $i \geq 0$, choose any lift $y_{j,i}$ of $\gamma_{p^i}(\sigma^2 x_j)$. The $\mathbb{E}_2$-MU-enveloping algebra of $\mathrm{BP}\langle n\rangle$ has homotopy given, as a $\mathrm{BP}\langle n\rangle_*$-algebra, by*

$$\begin{aligned}&\pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)\\ &\simeq \mathrm{BP}\langle n\rangle_*[w_{n+1,i} : i \geq 0] \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{BP}\langle n\rangle_*[y_{j,i} : j \not\equiv -1 \bmod p, i \geq 0, j \geq 1].\end{aligned}$$

Moreover, we may choose $w_{n+1,0} = \sigma^2 v_{n+1}$.

Proof. The enveloping algebra $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ is an $\mathbb{E}_2$-algebra (see Remark 2.2.1) and, in particular, its homotopy groups are a graded commutative algebra. Thus, our choice of elements $w_{n+1,i}$ and $y_{j,i}$ extends to a map

$$\begin{aligned}&f : \mathrm{BP}\langle n\rangle_*[w_{n+1,i} : i \geq 0] \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{BP}\langle n\rangle_*[y_{j,i} : j \not\equiv -1 \bmod p, i \geq 0, j \geq 1]\\ &\to \pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle),\end{aligned}$$

which we would like to be an isomorphism. From the bar spectral sequence we already know that $\pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ is a connective, free $\mathrm{BP}\langle n\rangle_*$-module with finitely many generators in each degree. It suffices from this and a dimension count to prove that f is injective modulo $(p, v_1, \ldots, v_n)$.

But now observe that the map

$$\pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)/(p, v_1, \ldots, v_n) \longrightarrow \pi_*\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathbb{F}_p)$$

is injective by our previous calculation of the target and naturality of the bar spectral sequence, since it is so on the E^∞-page of the bar spectral sequence. The result now follows by Proposition 2.4.2. □

Since $\mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$ coincides with $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ as an $\mathbb{E}_1$-algebra, this is also the computation of Hochschild homology given in the introduction:

THEOREM 2.5.4 (Polynomial THH). *There is an isomorphism of $\mathrm{BP}\langle n\rangle_*$-algebras*

$$\begin{aligned}&\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_*\\ &\simeq \mathrm{BP}\langle n\rangle_*[w_{n+1,i} : i \geq 0] \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{BP}\langle n\rangle_*[y_{j,i} : j \not\equiv -1 \bmod p, i \geq 0, j \geq 1].\end{aligned}$$

Moreover, we may take $w_{n+1,0} = \sigma^2 v_{n+1}$.

Again, it follows from [Ang08, Prop. 3.6] that the $\mathbb{E}_3$-MU-enveloping algebra has homotopy given by an exterior algebra, and hence that the spectral sequence

$$\mathrm{Ext}_{\pi_*\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle_*, \mathrm{BP}\langle n\rangle_*) \Rightarrow \pi_*\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)$$

collapses at the E_2 page. This proves

THEOREM 2.5.5. *The spectrum* $\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)$ *has homotopy groups isomorphic to a polynomial algebra over* $\mathbb{Z}_{(p)}[v_1, \dots, v_n]$ *on even degree generators. In particular, the homotopy groups are concentrated in even degrees.*

For the purposes of our obstruction theory argument, we will require the following closely related statement:

PROPOSITION 2.5.6. *Let* $\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}$ *denote the* $\mathbb{E}_3$*-MU-algebra cotangent complex of* $\mathrm{BP}\langle n\rangle$*. Let* I *denote the fiber* $\mathrm{fib}(\mathrm{MU_{(p)}} \to \mathrm{BP\langle n\rangle})$.

(i) *The groups* $\pi_{-2k}\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}, \mathrm{BP}\langle n\rangle)$ *vanish for* $k \geq 0$.

(ii) *Let* $\delta v_{n+1} \in \pi_*\mathrm{map}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)$ *denote the* $\mathrm{BP}\langle n\rangle_*$*-linear dual of the element* σv_{n+1} *with respect to the standard monomial basis of*

$$\pi_*(\mathrm{BP}\langle n\rangle \otimes_{\mathrm{MU}} \mathrm{BP}\langle n\rangle) \simeq \Lambda(\sigma v_i : i \geq n+1) \otimes_{\mathbb{F}_p} \Lambda(\sigma x_j : j \neq p^k - 1).$$

Identifying $\pi_*\mathrm{map}_{\mathrm{MU}}(\Sigma I, \mathrm{BP}\langle n\rangle)$ *with the* $\mathrm{BP}\langle n\rangle_*$*-module summand of* $\pi_*\mathrm{map}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle, \mathrm{BP}\langle n\rangle)$ *complementary to the unit, the class* δv_{n+1} *lies in the image of the forgetful map*

$$\pi_*\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}, \mathrm{BP}\langle n\rangle) \longrightarrow \pi_*\mathrm{map}_{\mathrm{MU}}(\Sigma I, \mathrm{BP}\langle n\rangle).$$

Proof. By [Lur17, Th. 7.3.5.1], we have a cofiber sequence of $\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$-modules

$$\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle) \to \mathrm{BP}\langle n\rangle \to \Sigma^3\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}.$$

Claim (i) then follows by applying the functor $\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(-, \mathrm{BP}\langle n\rangle)$ and the previous theorem. The same reasoning also shows that the spectral sequence

$$\mathrm{Ext}_{\pi_*\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\pi_*\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}, \pi_*\mathrm{BP}\langle n\rangle) \Rightarrow \pi_*\mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}, \mathrm{BP}\langle n\rangle)$$

collapses at the E_2-page. It will therefore suffice to show that δv_{n+1} lies in the image of the following map (which arises from the forgetful functor from $\mathbb{E}_3$-algebras to $\mathbb{E}_0$-algebras):

$$\mathrm{Hom}_{\pi_*\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\pi_*\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}, \pi_*\mathrm{BP}\langle n\rangle) \to \mathrm{Hom}_{\mathrm{MU}_*}(\pi_*\Sigma I, \pi_*\mathrm{BP}\langle n\rangle).$$

So we must study the MU_*-module map

$$\pi_*\Sigma I \to \pi_* \mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}.$$

The homotopy groups of the source are given by the suspension of $I_* = \ker(\mathrm{MU}_* \to \mathrm{BP}\langle n\rangle_*)$ while the homotopy groups of the target are given by $\Sigma^4 J_*$, where $J_* = \ker(\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)_* \to \mathrm{BP}\langle n\rangle_*)$. Under these identifications, Lemma A.3.2 implies that $x_j \mapsto \sigma^3 x_j$ for those x_j lying in I_*. By our computation of $\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)_*$, the subset of *non-zero* classes of the form $\{\sigma^3 x_j\}$ spans an MU_*-module summand of $\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)_*$. Among these is the non-zero class $\sigma^3 v_{n+1}$, and the result follows. □

2.6. *Proof of Theorem* A. We will now prove Theorem 2.0.6, and hence Theorem A. We will deduce the theorem as a consequence of a more precise assertion. In order to state it we will need to recall a construction.

Construction 2.6.1. Recall that we have an $\mathbb{E}_\infty$-map of spaces

$$J_{\mathbb{C}} : \mathrm{BU} \times \mathbb{Z} \to \mathrm{Pic}(\mathsf{Sp})$$

where the target denotes the Picard space of the category of spectra. Left Kan extension then yields a symmetric monoidal functor

$$\begin{tikzcd} \mathrm{BU}\times\mathbb{Z} \arrow[r] \arrow[d] & \mathrm{Pic}(\mathsf{Sp}) \arrow[r] & \mathsf{Sp}. \\ \mathbb{Z} \arrow[urr, dashed, "\mathrm{MU}[z^{\pm1}]"'] & & \end{tikzcd}$$

We interpret $\mathrm{MU}[z^{\pm1}]$ as a graded $\mathbb{E}_\infty$-ring. Here the notation is justified by the fact that the homotopy groups of the underlying spectrum (i.e., the direct sum of the graded components) are given by $\mathrm{MU}_*[z^{\pm1}]$, where z is a class in degree 2. For any $j \in \mathbb{Z}$, we may then construct a (non-negatively) graded $\mathbb{E}_\infty$-ring

$$\mathrm{MU}[y] : \mathbb{Z}_{\geq 0} \to \mathbb{Z} \xrightarrow{\cdot j} \mathbb{Z} \xrightarrow{\mathrm{MU}[z^{\pm1}]} \mathsf{Sp}.$$

Here, the homotopy groups of the underlying spectrum of $\mathrm{MU}[y]$ are given by $\mathrm{MU}_*[y]$ where $|y| = 2j$. By Lemma B.0.6, we may write $\mathrm{MU}[y]$ as the limit of a tower of $\mathbb{E}_\infty$-MU-algebra square-zero extensions

$$\mathrm{MU}[y] \to \cdots \to \mathrm{MU}[y]/(y^k) \to \mathrm{MU}[y]/(y^{k-1}) \to \cdots \to \mathrm{MU}.$$

When $j > 0$, this is also a limit diagram of underlying $\mathbb{E}_\infty$-MU-algebras and square-zero extensions thereof. In our work below, we regard $\mathrm{MU}[y]$ and $\mathrm{MU}[y]/(y^k)$ as *ungraded* $\mathbb{E}_\infty$-MU-algebras.

Proposition 2.6.2. *Fix $n \geq -1$, and let B be any $\mathbb{E}_3$-MU-algebra form of* $\mathrm{BP}\langle n\rangle$. *Then there exists a sequence of maps*

$$\cdots \to B_k \to B_{k-1} \to \cdots B_1 = B$$

where

(a) B_k *is given the structure of an* $\mathbb{E}_3$-$\mathrm{MU}[y]/(y^k)$-*algebra, where* $|y| = 2p^{n+1}-2$.
(b) *each map* $B_k \to B_{k-1}$ *is given the structure of a map of* $\mathbb{E}_3$-$\mathrm{MU}[y]/(y^k)$-*algebras*,

and such that the following properties are satisfied:

(i) *Each map* $B_k \to B_{k-1}$ *induces an equivalence of* $\mathbb{E}_3$-$\mathrm{MU}[y]/(y^k)$-*algebras*

$$\mathrm{MU}[y]/(y^{k-1}) \otimes_{\mathrm{MU}[y]/(y^k)} B_k \xrightarrow{\simeq} B_{k-1}.$$

(ii) *The map*

$$B = B_1 \to \mathrm{cofib}(B_2 \to B_1) \simeq \Sigma^{|y|+1} B$$

is detected by δv_{n+1} *in*

$$E_2 = E_\infty = \mathrm{Ext}^*_{\mathrm{MU}_*}(B_*, B_*) \Rightarrow \pi_* \mathrm{map}_{\mathrm{MU}}(B, B).$$

Proof. First we prove that we can build B_2 satisfying (ii). Observe that $\mathrm{MU}[y]/(y^2)$ is a trivial square-zero extension of MU by $\Sigma^{|y|}\mathrm{MU}$. According to Lemma 2.3.3, it suffices to supply a map

$$\mathbf{L}^{\mathbb{E}_3}_{B/\mathrm{MU}} \to \Sigma^{|y|+1}\mathrm{BP}\langle n\rangle$$

in $\mathsf{Mod}^{\mathbb{E}_3}_B(\mathsf{Mod}_{\mathrm{MU}})$ whose image under the forgetful map

$$\pi_* \mathrm{map}_{\mathcal{U}^{(3)}_{\mathrm{MU}}(\mathrm{BP}\langle n\rangle)}(\mathbf{L}^{\mathbb{E}_3}_{\mathrm{BP}\langle n\rangle/\mathrm{MU}}, \mathrm{BP}\langle n\rangle)$$
$$\longrightarrow \pi_* \mathrm{map}_{\mathrm{MU}}(\mathrm{fib}(\mathrm{MU}_{(p)} \to \mathrm{BP}\langle n\rangle), \mathrm{BP}\langle n\rangle)$$

detects δv_{n+1}. But this is precisely the content of Proposition 2.5.6(ii).

Suppose by induction that we have constructed the algebras B_j for $j \leq k$ as in (a) and (b), satisfying (i) and (ii). By Proposition 2.3.6, the obstruction to building B_{k+1} is a map

$$o(B_k) : \mathbf{L}^{\mathbb{E}_3}_{B_k/(\mathrm{MU}[y]/(y^k))} \to B_k \otimes_{\mathrm{MU}[y]/(y^k)} \Sigma^{(k+1)|y|+2}\mathrm{MU}$$

in $\mathsf{Mod}^{\mathbb{E}_3}_{B_k}(\mathsf{Mod}_{\mathrm{MU}[y]/(y^k)})$. Base change along the augmentation $\mathrm{MU}[y]/(y^k) \to \mathrm{MU}$ gives rise to a functor

$$\varepsilon^* : \mathsf{Mod}^{\mathbb{E}_3}_{B_k}(\mathsf{Mod}_{\mathrm{MU}[y]/(y^k)}) \longrightarrow \mathsf{Mod}^{\mathbb{E}_3}_B(\mathsf{Mod}_{\mathrm{MU}}),$$

where we have used (i) to identify $\varepsilon^* B_k \simeq B$. So the obstruction $o(B_k)$ is adjoint to a map

$$\mathbf{L}^{\mathbb{E}_3}_{B/\mathrm{MU}} \to \Sigma^{(k+1)|y|+2} B$$

in $\mathsf{Mod}^{\mathbb{E}_3}_B(\mathsf{Mod}_{\mathrm{MU}})$. By Proposition 2.5.6(i), any such map is nullhomotopic. This completes the proof. $\square$

Proof of Theorem 2.0.6. We prove the claim by induction on $n \geq 1$, the cases $n = -1, 0$ being trivial. By induction, suppose there exists an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n\rangle$, say B, and construct a tower as in the previous proposition.

Let $\widetilde{B} := \lim B_k$ be the $\mathbb{E}_3$-$\mathrm{MU}[y]$-algebra at the limit of the tower. We claim that $\widetilde{B}$ is an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n+1\rangle$.

By (i), the associated graded tower is a sum of shifts of B, and we see that

$$\mathrm{gr}(\pi_*\widetilde{B}) \simeq B_*[y].$$

By (ii), the exact sequence of MU_*-modules

$$0 \to \Sigma^{|y|} B_* \xrightarrow{\cdot y} \pi_* B_2 \to B_* \to 0$$

corresponds to the class δv_{n+1}, so that y acts by v_{n+1} on B_2. Combining these observations we see that the composite

$$\mathbb{Z}_{(p)}[v_1, \dots, v_{n+1}] \to (\mathrm{MU}_*)_{(p)} \to \widetilde{B}_*$$

is an isomorphism, which is what we wanted to show. $\square$

3. Unraveling Lichtenbaum–Quillen

Having constructed an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n\rangle$, we aim in the remainder of the paper to study its p-localized algebraic K-theory spectrum $\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$. The philosophy of chromatic homotopy theory, together with the vanishing results of [CMNN20], suggests that we should study $\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$ by computing its chromatic localizations $L_{T(i)}\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$ for $0 \leq i \leq n+1$, which assemble into the smashing localization $L^f_{n+1}\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$. One wants to know whether the localization $L^f_{n+1}\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$ faithfully reproduces $\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$. This is far from assured: for example, Quillen proved that $\mathrm{K}(\mathbb{F}_p)_{(p)} \cong \mathbb{Z}_{(p)}$, but $L^f_0\mathbb{Z}_{(p)} = \mathbb{Q}$.

It turns out, however, that the difference between $\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$ and its L^f_{n+1} localization is entirely concentrated in low degrees. In short, in the remainder of the paper we aim to prove that the localization map $\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)} \to L^f_{n+1}\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}$ is truncated.

In the case $n = 0$, this becomes the classical Lichtenbaum–Quillen conjecture for $\mathbb{Z}_{(p)}$, which is a celebrated theorem of Voevodsky and Rost [Voe03], [Voe11]. Our goal is to reduce the general case to the Voevodsky–Rost theorem. We accomplish this first by leaning on the Dundas–Goodwillie–McCarthy theorem [DGM13, Th. 0.0.2], which relates K to the more computable (p-completed) TC invariant that we review in Section 3.2. The purpose of this section is to discuss a general strategy for proving, for any connective $\mathbb{E}_1$-ring spectrum R, that

$$\mathrm{TC}(R) \to L^f_{n+1}\mathrm{TC}(R)$$

is truncated. Future sections of the paper then implement the strategy in the case of $R = \mathrm{BP}\langle n\rangle$.

In broad outline, our strategy proceeds as follows. First, following Ausoni and Rognes, we apply work of Mahowald and Rezk to reduce to proving that $\pi_*(F \otimes \mathrm{TC}(R))$ is finite for some type $n+2$ complex F [MR99]. As we will review, $\mathrm{TC}(R)$ is constructed from the simpler invariant $\mathrm{THH}(R)$ together with two pieces of structure, a *cyclotomic Frobenius* and a *circle action*. One of the main observations of this paper is that, while the definition of TC mixes these structures by taking S^1 fixed points of the Frobenius, it actually suffices to study the two structures independently.

We reduce the problem of checking that $\pi_*(F \otimes \mathrm{TC}(R))$ is bounded to checking that $\mathrm{THH}(R)$ satisfies the *Segal conjecture*, which is purely about the Frobenius, and that $\mathrm{THH}(R)$ satisfies *canonical vanishing*, which is purely about the S^1 action.

These two properties of $\mathrm{THH}(R)$ together imply that $F \otimes \mathrm{TR}(R)$ is bounded above, and we thank the third referee for pointing out that they are in fact *equivalent* to the statement that $F \otimes \mathrm{TR}(R)$ is bounded above. After checking that the homotopy groups $\pi_i R$ are finitely generated, the statement that $F \otimes \mathrm{TR}(R)$ is bounded above implies that $\pi_*(F \otimes \mathrm{TC}(R))$ is finite.

The remainder of this section fixes conventions and makes precise the reductions to canonical vanishing and the Segal conjecture. Section 4 verifies the Segal conjecture for $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$, while Section 6 verifies canonical vanishing via entirely different means.

3.1. *The work of Mahowald–Rezk.*

Definition 3.1.1. A p-complete, bounded below spectrum X is said to be *fp* if $H^*(X;\mathbb{F}_p)$ is finitely presented over the mod p Steenrod algebra.

THEOREM 3.1.2 (Mahowald–Rezk). *Suppose that X is an fp spectrum. Then there exists a non-zero p-local finite complex F such that $\pi_*(X \otimes F)$ is finite. In other words, $X \otimes F$ has finitely many non-zero homotopy groups, and $\pi_i(X \otimes F)$ is finite for each i.*

On the other hand, suppose Y is any bounded below p-complete spectrum. If there exists a non-zero p-local finite complex F such that $\pi_*(Y \otimes F)$ is finite, then Y is an fp spectrum.

Proof. The proof follows from [MR99, Prop. 3.2]. □

The collection of F such that $\pi_*(X \otimes F)$ is finite is obviously a thick subcategory. One says that an fp spectrum X is *of fp-type n* if $\pi_*(X \otimes F)$ is infinite when F has type n, but finite when F has type $n+1$. Our key interest in fp spectra comes from the following result of Mahowald and Rezk:

THEOREM 3.1.3 (Mahowald–Rezk). *If Y is a spectrum of fp-type n, then the localization map $Y \to L_n^f Y$ is an equivalence on homotopy in large degrees.*

Proof. Mahowald and Rezk [MR99, Th. 8.2(2)] prove that the fiber C of the map $Y \to L_n^f Y$ has Brown–Comenetz dual IC that is bounded below. It follows that C is bounded above (since for any abelian group N, if $\mathrm{Hom}(N, \mathbb{Q}/\mathbb{Z}) = 0$, then $N = 0$), whence the claim. $\square$

3.2. *Background and conventions on cyclotomic spectra.*

Definition 3.2.1. A *p-typical cyclotomic spectrum* X is a p-complete object $X \in \mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp})$ equipped with an S^1-equivariant map $\varphi : X \to X^{tC_p}$, where the action on the target is via the equivalence $S^1 \cong S^1/C_p$. If X is a bounded below p-typical cyclotomic spectrum, so that $(X^{tC_p})^{hC_{p^k}} \simeq X^{tC_{p^{k+1}}}$ by [NS18, II.4.1], then we define invariants

$$\mathrm{TR}^j(X) := X \times_{X^{tC_p}} X^{hC_p} \times_{X^{tC_{p^2}}} X^{hC_{p^2}} \times_{X^{tC_{p^3}}} \cdots \times_{X^{tC_{p^j}}} X^{hC_{p^j}}$$

using the maps $\varphi^{hC_{p^k}}$ and the canonical maps from homotopy fixed points to the Tate fixed points. We define $\mathrm{TR}(X) = \lim_j \mathrm{TR}^j(X)$ where the connecting maps

$$R : \mathrm{TR}^j(X) \to \mathrm{TR}^{j-1}(X)$$

are projection away from the last factor. Observe that each object $\mathrm{TR}^j(X)$ and the limit $\mathrm{TR}(X)$ has a residual S^1-action.

Remark 3.2.2. This is slightly different than the notion of a "p-cyclotomic spectrum" considered in [NS18]. However, when restricting attention to bounded below and p-complete objects, as we do here, the two notions coincide (see [NS18, Rem. II.1.3]). The definition above is the same as in [AN21] except that we have added the hypothesis that X be p-complete.

Definition 3.2.3. If X is a bounded below, p-typical cyclotomic spectrum, then we define

$$\mathrm{TC}(X) := \mathrm{fib}(\varphi^{hS^1} - \mathrm{can} : X^{hS^1} \to X^{tS^1}).$$

Remark 3.2.4. There are maps $F : \mathrm{TR}^n(X) \to \mathrm{TR}^{n-1}(X)$ corresponding to projecting away from the first factor and then using the inclusion of each C_{p^k} homotopy fixed points into the $C_{p^{k-1}}$ homotopy fixed points. These assemble to a map $F : \mathrm{TR}(X) \to \mathrm{TR}(X)$, and the original definition of (p-adic) topological cyclic homology was as the fiber:

$$\mathrm{fib}(1 - F : \mathrm{TR}(X) \to \mathrm{TR}(X)).$$

It is shown in [NS18, Th. II.4.10] that this agrees with the definition above.

Remark 3.2.5. Antieau–Nikolaus construct [AN21, Ex. 3.4] an S^1-equivariant map $V : \mathrm{TR}(X)_{hC_p} \to \mathrm{TR}(X)$ that fits into a cofiber sequence

$$\mathrm{TR}(X)_{hC_p} \xrightarrow{V} \mathrm{TR}(X) \to X$$

of S^1-spectra. Thus, one can recover both $\mathrm{TC}(X)$ and X from knowledge of $\mathrm{TR}(X)$.

Definition 3.2.6. Suppose that A is a connective $\mathbb{E}_1$-ring spectrum. Then $\mathrm{THH}(A)^\wedge_p$ is a bounded below p-typical cyclotomic spectrum. We will in this circumstance abbreviate $\mathrm{TR}^j(\mathrm{THH}(A)^\wedge_p)$ by $\mathrm{TR}^j(A)$, and similarly for $\mathrm{TR}(\mathrm{THH}(A)^\wedge_p)$ and $\mathrm{TC}(\mathrm{THH}(A)^\wedge_p)$.

3.3. *Bounds on* TR *and related conditions on cyclotomic spectra.*

Definition 3.3.1. Let X be a bounded below p-typical cyclotomic spectrum. We will be interested in the following conditions on X, which may or may not hold:

- $\boxed{\text{Bounded TR}}$ The spectrum $\mathrm{TR}(X)$ is bounded.
- $\boxed{\text{Segal Conjecture}}$ The Frobenius $\varphi : X \to X^{tC_p}$ is truncated.
- $\boxed{\text{Canonical Vanishing}}$ There is an integer $d \geq 0$ such that the composite
$$\tau_{\geq d}(X^{hC_{p^k}}) \to X^{hC_{p^k}} \xrightarrow{\mathrm{can}} X^{tC_{p^k}}$$
is nullhomotopic for all $0 \leq k \leq \infty$.
- $\boxed{\text{Weak Canonical Vanishing}}$ There is an integer $d \geq 0$ such that, for $* \geq d$, the map
$$\pi_*(\mathrm{can}) : \pi_* X^{hC_{p^k}} \to \pi_* X^{tC_{p^k}}$$
is zero for all $0 \leq k \leq \infty$.
- $\boxed{\text{Tate Nilpotence}}$ X^{tC_p} lies in the thick tensor ideal of $\mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp})$ generated by $\mathbb{D}S^1_+$, the Spanier–Whitehead dual of S^1_+.
- $\boxed{\mathbb{F}_p \text{ Nilpotence}}$ $\mathrm{TR}(X) \in \mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp})$ lies in the thick tensor ideal generated by $\mathbb{F}_p$, where $\mathbb{F}_p$ is considered to have trivial S^1 action.
- $\boxed{\text{Finiteness}}$ For each $i \in \mathbb{Z}$ and $0 \leq k \leq \infty$, the groups $\pi_i X^{hC_{p^k}}$ and $\pi_i X^{tC_{p^k}}$ are finite, and hence so too are the groups $\pi_i \mathrm{TC}(X)$.

As suggested by its name, the $\boxed{\text{Segal Conjecture}}$ condition holds particular historical significance, some of which we recall in Section 4. It turns out that there are many non-trivial implications between the conditions, summarized by the following theorem:

Theorem 3.3.2. *Let X be a bounded below, p-power torsion p-typical cyclotomic spectrum. That is, we assume there is some $N \geq 0$ for which*

$p^N : X \to X$ is nullhomotopic as a map of p-typical cyclotomic spectra. Then the following implications hold:

(a) (*Antieau–Nikolaus*) $\boxed{\text{Bounded TR}} \Rightarrow \boxed{\text{Segal Conjecture}}$.

(b) $\boxed{\text{Bounded TR}} \Rightarrow \boxed{\mathbb{F}_p \text{ Nilpotence}}$.

(c) (*Mathew*) $\boxed{\mathbb{F}_p \text{ Nilpotence}} \Rightarrow \boxed{\text{Tate Nilpotence}}$.

(d) *If each homotopy group $\pi_i X$ is finite, then* $\boxed{\text{Bounded TR}} \Rightarrow \boxed{\text{Finiteness}}$.

(e) $\boxed{\text{Segal Conjecture}} + \boxed{\text{Tate Nilpotence}} \Rightarrow \boxed{\text{Canonical Vanishing}}$.

(f) $\boxed{\text{Segal Conjecture}} + \boxed{\text{Weak Canonical Vanishing}} \Rightarrow \boxed{\text{Bounded TR}}$.

Remark 3.3.3. We thank the anonymous referee for suggesting both the formulations and proofs of several of the statements in the above Theorem 3.3.2. Of the statements, (a) appeared in previous work of Antieau and Nikolaus [AN21, Prop. 2.25], and (c) was communicated to us by Akhil Mathew. We thank both the referee and Mathew for suggesting that we present their work within this paper.

We postpone the proof of Theorem 3.3.2 to Section 3.5. Let us now describe how we apply it. The main theorem of the remainder of the paper, stated as Theorem G in the introduction, is the following:

Theorem 3.3.4. *Let* $\mathrm{BP}\langle n\rangle$ *denote any* $\mathbb{E}_3$-MU-*algebra form of* $\mathrm{BP}\langle n\rangle$, *and suppose that F is a type $n+2$ complex. Then* $F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ *satisfies* $\boxed{\text{Bounded TR}}$.

By the thick subcategory theorem of Hopkins and Smith [HS98], Theorem 3.3.4 holds for an arbitrary type $n+2$ complex F if and only if it holds for *some* type $n+2$ complex F. Thus, given Theorem 3.3.2(d,f), we can prove Theorem 3.3.4 by checking the following two results independently:

Theorem 3.3.5 (see Theorem 4.0.1). *For all type $n+2$ complexes F,* $F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ *satisfies the Segal conjecture.*

Theorem 3.3.6 (see Theorem 6.3.1). *For some type $n+2$ complex F,* $F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$ *satisfies weak canonical vanishing.*

While it is convenient for our proof of Theorem 3.3.6 that we pick a particularly nice F, it follows from Theorem 3.3.2(a,b) that it holds for all choices of F. As Akhil Mathew explained to us, we may also use Theorem 3.3.2 to deduce results about general $\mathbb{E}_1$-$\mathrm{BP}\langle n\rangle$-algebras:

Proposition 3.3.7 (Mathew). *Suppose that A is a connective* $\mathbb{E}_1$-$\mathrm{BP}\langle n\rangle$-*algebra and that F is a type $n+2$ complex. Then, if* $F \otimes \mathrm{THH}(A)$ *satisfies the Segal conjecture,* $F \otimes \mathrm{TR}(A)$ *is bounded.*

Proof. It suffices to show that $F \otimes \mathrm{TR}(A)$ satisfies $\boxed{\mathbb{F}_p \text{ Nilpotence}}$. However, $F \otimes \mathrm{TR}(A)$ is a retract of $F \otimes \mathrm{TR}(\mathrm{BP}\langle n \rangle) \otimes \mathrm{TR}(A)$, so it suffices to check that $F \otimes \mathrm{TR}(\mathrm{BP}\langle n \rangle)$ satisfies $\boxed{\mathbb{F}_p \text{ Nilpotence}}$. This follows from Theorem 3.3.4 and Theorem 3.3.2(b). □

3.4. *Lichtenbaum–Quillen and bounded* TR.

THEOREM 3.4.1. *Let A be a connective $\mathbb{E}_1$-ring and F a type $n+2$ complex. Suppose that*

(i) $F \otimes \mathrm{TR}(A)$ *is bounded*;

(ii) $\pi_i(A^\wedge_p)$ *is a finitely generated* $\mathbb{Z}_p$*-module for all* i.

Then $\mathrm{TC}(A)$ *is an fp-spectrum of fp-type at most* $n+1$. *In particular, this implies that*

$$\mathrm{TC}(A) \to L^f_{n+1}\mathrm{TC}(A)$$

is truncated.

Proof. Since TC is calculated as the fiber of a self-map of TR, we know by assumption (i) that $\pi_*(F \otimes \mathrm{TC}(A))$ is bounded. It remains only to check that each $\pi_i(F \otimes \mathrm{TC}(A))$ is finite. By Theorem 3.3.2(e), it suffices to show that each homotopy group $\pi_i(F \otimes \mathrm{THH}(A))$ is finite. Recall that $\mathrm{THH}(A)$ can be computed as the geometric realization of the cyclic bar construction $\bullet \mapsto A^{\otimes \bullet+1}$. Since A is connective, it therefore suffices to prove $\pi_{i-k}(F \otimes A^{\otimes k+1})$ is finite for each i and k. The p-completion of the tensor product $A^{\otimes k+1}$ will have finitely generated homotopy groups, by connectivity and (ii). Since F is not type 0, the result follows. □

THEOREM 3.4.2. *Let A be a connective $\mathbb{E}_1$-ring and F a type $n+2$ complex. Suppose that*

(i) $F \otimes \mathrm{TR}(A)$ *is bounded*;

(ii) $\pi_i(A^\wedge_p)$ *is a finitely generated* $\mathbb{Z}_p$*-module for all* i;

(iii) $F \otimes \mathrm{TR}\,(\pi_0 A)$ *is bounded/*

Then,

(a) *if* $\pi_*(F \otimes \mathrm{K}(\pi_0 A))$ *is finite, then* $\mathrm{K}(A)^\wedge_p$ *is an fp-spectrum of fp-type at most* $n+1$;

(b) *if the map* $\mathrm{K}(\pi_0 A)_{(p)} \to L^f_{n+1}\mathrm{K}(\pi_0 A)_{(p)}$ *is truncated, then the map*

$$\mathrm{K}(A)_{(p)} \to L^f_{n+1}\mathrm{K}(A)_{(p)}$$

is truncated.

Remark 3.4.3. The condition in (a) of the above theorem is that $\mathrm{K}(\pi_0 A)^\wedge_p$ is fp of fp type at most $n+1$. Mitchell's theorem [Mit90] ensures that, if $\mathrm{K}(\pi_0 A)^\wedge_p$ is fp, then it will be of fp type at most 1. Similarly, Mitchell's theorem implies that the spectrum $L^f_{n+1}\mathrm{K}(\pi_0 A)$ appearing in (b) is equivalent to $L^f_1\mathrm{K}(\pi_0 A)$.

Proof. By p-completing the pullback square in the Dundas–Goodwillie–McCarthy theorem [DGM13, Ch. VII, Th. 0.0.2], we obtain a pullback square

$$\begin{tikzcd} \mathrm{K}(A)^\wedge_p \arrow[r] \arrow[d] & \mathrm{TC}(A) \arrow[d] \\ \mathrm{K}(\pi_0 A)^\wedge_p \arrow[r] & \mathrm{TC}(\pi_0(A)). \end{tikzcd}$$

Here we note that the symbol TC above agrees, by our conventions, with the p-completion of what the authors of [DGM13] denote by TC.

The assumption that each homotopy group $\pi_i A^\wedge_p$ is finitely generated ensures additionally that $(\pi_0 A)^\wedge_p$, by which we mean the p-completion of the Eilenberg–MacLane spectrum $\pi_0 A$, also has finitely generated homotopy groups. By Theorem 3.4.1, we know that $\mathrm{TC}(\pi_0 A)$ and $\mathrm{TC}(A)$ are fp-spectra of type at most $n+1$. We then observe that the condition of being an fp-spectrum of type at most $n+1$ is preserved under fiber sequences and finite coproducts, proving (a).

Similarly, to prove (b) we observe that the collection of spectra X such that

$$X_{(p)} \to L^f_{n+1} X_{(p)}$$

is truncated is also closed under fiber sequences and finite coproducts. This class of spectra includes all rational spectra. The claim (b) now follows from Theorems 3.1.3 and 3.4.1, the Dundas–Goodwillie–McCarthy square above, and the arithmetic pullback square

$$\begin{tikzcd} X_{(p)} \arrow[r] \arrow[d] & X^\wedge_p \arrow[d] \\ X_{(p)}[p^{-1}] \arrow[r] & X^\wedge_p[p^{-1}]. \end{tikzcd}$$ □

As a corollary of these results, we deduce the main theorems of the introduction.

COROLLARY 3.4.4. *Let A denote any $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n\rangle$. Then*

- $\mathrm{TC}(A)$ *is fp of fp-type at most $n+1$, as is* $\mathrm{K}(A^\wedge_p)^\wedge_p$;
- *the map*
$$\mathrm{K}(A)_{(p)} \to L^f_{n+1}\mathrm{K}(A)_{(p)}$$
is an equivalence in large degrees.

Proof. We observe that the $\mathbb{Z}_p$-module

$$\pi_*(\mathrm{BP}\langle n\rangle^\wedge_p) \cong \mathbb{Z}_p[v_1, v_2, \ldots, v_n]$$

is finitely generated in each degree. If we let F denote any type $n+2$ complex, our main Theorem 3.3.4 states that $F \otimes \mathrm{TR}(A)$ is bounded. We also observe that $F \otimes \mathrm{TR}(\pi_0 A) \simeq F \otimes \mathrm{TR}(\mathbb{Z}_{(p)})$ is bounded, for example by our work here and the fact that $\mathbb{Z}_{(p)}$ is an $\mathbb{E}_\infty$-MU-algebra form of $\mathrm{BP}\langle 0\rangle$ (cf. [BM94] and [Rog99] for more explicit proofs of this fact). It remains to check only that

$$\mathrm{K}(\mathbb{Z}_{(p)})_{(p)} \to L_{n+1}^f \mathrm{K}(\mathbb{Z}_{(p)})_{(p)} \simeq L_1^f \mathrm{K}(\mathbb{Z}_{(p)})_{(p)}$$

is an equivalence in large degrees. But this is exactly Waldhausen's reformulation [Wal84, §4] of the Lichtenbaum–Quillen conjecture for $\mathbb{Z}_{(p)}$, which is proven by the celebrated work of Voevodsky and Rost [Voe03], [Voe11]. $\square$

Remark 3.4.5. In fact, in the notation of the above corollary, $\mathrm{TC}(A)$ is of fp-type exactly $n+1$, as follows from Corollary 5.0.2.

3.5. *Proof of Theorem* 3.3.2. In this section we supply the proof of Theorem 3.3.2. The statements and proofs will rely on the following notion of nilpotence, which goes back at least to Bousfield [Bou79]. For an excellent survey of modern uses of this notion, we recommend [Mat18].

Definition 3.5.1. Let $\mathcal{C}$ be a stable, symmetric monoidal category, and let A be an $\mathbb{E}_1$-algebra object. We say that $M \in \mathcal{C}$ is *A-nilpotent* if M lies in the thick tensor subcategory generated by A. Equivalently, we can ask that M lies in the thick subcategory generated by those objects of $\mathcal{C}$ that admit the structure of a left A-module.

Definition 3.5.2. Let G be a compact Lie group. We say that a (Borel) G-spectrum Y is *G-nilpotent* if it is $\mathbb{D}G_+$-nilpotent, where $\mathbb{D}G_+$ denotes the Spanier–Whitehead dual of G_+.

LEMMA 3.5.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor between stable, symmetric monoidal categories, and let $A \in \mathsf{Alg}(\mathcal{C})$ and $B \in \mathsf{Alg}(\mathcal{D})$ be algebra objects. If $F(A)$ is B-nilpotent, then $F(M)$ is B-nilpotent for any A-nilpotent object M.*

Proof. The subcategory $\mathcal{E} \subseteq \mathcal{C}$ of objects M such that $F(M)$ is B-nilpotent is thick so we need only show that it contains all A-modules. If M is an A-module, then $F(M)$ is an $F(A)$-module and hence a retract of $F(A)\otimes F(M)$. But $F(A)$ is B-nilpotent and hence so is $F(A)\otimes F(M)$ and the retract $F(M)$. $\square$

LEMMA 3.5.4. *If $Y \in \mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp})$ is $\mathbb{F}_p$-nilpotent, where $\mathbb{F}_p$ is given the trivial action, then Y_{hC_p}, Y^{hC_p}, and Y^{tC_p} are also $\mathbb{F}_p$-nilpotent, where we give each the residual $S^1/C_p \simeq S^1$ action.*

Proof. From the cofiber sequence $Y_{hC_p} \to Y^{hC_p} \to Y^{tC_p}$ it is enough to prove the claim for Y^{hC_p} and Y^{tC_p}. By Lemma 3.5.3 it is enough to check that $\mathbb{F}_p^{hC_p}$ and $\mathbb{F}_p^{tC_p}$ are $\mathbb{F}_p$-nilpotent. Both are modules over $\mathbb{F}_p^{hC_p}$ and hence also, by restriction, modules over $\mathbb{F}_p$, hence $\mathbb{F}_p$-nilpotent. $\square$

LEMMA 3.5.5. *If* $Y \in \mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp})$ *is* $\mathbb{F}_p$*-nilpotent*, Y^{tC_p} *is* S^1*-nilpotent.*

Proof. First we claim that $(\mathbb{F}_p^{hC_p})^{tS^1} = 0$. One can check this by direct calculation, or else argue as follows. Since $\mathbb{F}_p^{hC_p}$ is a module over $\mathbb{F}_p$, we have that $(\mathbb{F}_p^{hC_p})^{tS^1}$ is p-complete and $(\mathbb{F}_p^{hC_p})^{tS^1}/p = (\mathbb{F}_p^{hC_p})^{tC_p}$, by [NS18, Lemma IV.4.12]. But $(\mathbb{F}_p^{hC_p})^{tC_p} = 0$ by [NS18, Lemma I.2.2], whence the claim. It now follows from [MNN17, Th. 4.19] that $\mathbb{F}_p^{hC_p}$ is S^1-nilpotent. The lemma now follows from Lemma 3.5.3. □

LEMMA 3.5.6. *If* $Y \in \mathsf{Fun}(\mathrm{B}G, \mathsf{Sp})$ *is* G*-nilpotent, there is a* $d \geq 0$ *such that, for all integers* n*, the map* $\tau_{\geq d+n}Y \to \tau_{\geq n}Y$ *factors through a* G*-nilpotent spectrum.*

Proof. Choose an N so that the map $Y \to \mathrm{map}(\mathrm{sk}_N(EG)_+, Y)$ has a retract r. Let d be the dimension of the finite complex $\mathrm{sk}_N(EG)_+$. Then, for all $n \in \mathbb{Z}$, the spectrum $\mathrm{map}(\mathrm{sk}_N(EG)_+, \tau_{\geq d+n}Y)$ is n-connective, so the composite

$$\mathrm{map}(\mathrm{sk}_N(EG)_+, \tau_{\geq d+n}Y) \to \mathrm{map}(\mathrm{sk}_N(EG)_+, Y) \xrightarrow{r} Y$$

factors through $\tau_{\geq n}Y$. The map $\tau_{\geq d+n}Y \to \tau_{\geq n}Y$ then factors through the diagonal

$$\tau_{\geq d+n}Y \to \mathrm{map}(\mathrm{sk}_N(EG)_+, \tau_{\geq d+n}Y),$$

the target of which is G-nilpotent. □

LEMMA 3.5.7. *Let* X *be an* S^1*-spectrum, and suppose we have a map of* S^1*-spectra* $f : X \to Y$*, where* Y *is* S^1*-nilpotent, which induces an equivalence* $\tau_{\geq m}X \simeq \tau_{\geq m}Y$ *for some* $m \geq 0$*. Then there is a* $d \geq 0$ *such that the map* $\tau_{\geq d}X \to X$ *factors through an* S^1*-nilpotent spectrum.*

Proof. By the previous lemma there is a $d' \geq 0$ such that $\tau_{\geq d'+n}Y \to \tau_{\geq n}Y$ factors through an S^1-nilpotent spectrum for all integers n. Set $d = d' + m$. Then $\tau_{\geq d}X \simeq \tau_{\geq d'+m}Y \to \tau_{\geq m}Y \simeq \tau_{\geq m}X$ factors through an S^1-nilpotent spectrum and hence so does the composite $\tau_{\geq d}X \to \tau_{\geq m}X \to X$. □

LEMMA 3.5.8. *Let* X *and* d *be as in Lemma 3.5.7. Then, for all* $0 \leq k \leq \infty$,

(i) $(\tau_{\geq d}X)^{tC_{p^k}} \to X^{tC_{p^k}}$ *is nullhomotopic*;

(ii) $\tau_{\geq d}(X^{hC_{p^k}}) \to X^{tC_{p^k}}$ *is nullhomotopic*;

(iii) *the map* $X^{tC_{p^k}} \to (\tau_{<d}X)^{tC_{p^k}}$ *has a retract.*

Proof. The Tate construction $(-)^{tC_{p^k}}$ annihilates all S^1-nilpotent spectra, so (i) is immediate from the previous lemma. The map in (ii) factors as

$$\tau_{\geq d}(X^{hC_{p^k}}) \to (\tau_{\geq d}X)^{hC_{p^k}} \to (\tau_{\geq d}X)^{tC_{p^k}} \to X^{tC_{p^k}}$$

and so is nullhomotopic by (i). The claim (iii) follows from (i) and the cofiber sequence

$$(\tau_{\geq d}X)^{tC_{p^k}} \to X^{tC_{p^k}} \to (\tau_{<d}X)^{tC_{p^k}}. \qquad \square$$

Proof of Theorem 3.3.2. Claim (a) is [AN21, Prop. 2.25].

We now prove (b). By assumption, there is some N for which $p^N : X \to X$ is nullhomotopic as a map in CycSp_p. It follows that p^N annihilates each homotopy group $\pi_i\mathrm{TR}(X)$. It follows that each object $\pi_i\mathrm{TR}(X)$ admits the structure of a $\mathbb{Z}/p^N$-module in S^1-spectra, and hence that each object $\pi_i\mathrm{TR}(X)$ is $\mathbb{F}_p$-nilpotent. If $\mathrm{TR}(X)$ is bounded, then we conclude that $\mathrm{TR}(X)$ is also $\mathbb{F}_p$-nilpotent, as an S^1-spectrum.

Now we prove (c). Assume that $\mathrm{TR}(X)$ is $\mathbb{F}_p$-nilpotent. By Lemma 3.5.4, we deduce that $\mathrm{TR}(X)_{hC_p}$ is $\mathbb{F}_p$-nilpotent as well and, from the S^1-equivariant cofiber sequence

$$\mathrm{TR}(X)_{hC_p} \xrightarrow{V} \mathrm{TR}(X) \to X,$$

we deduce that X is $\mathbb{F}_p$-nilpotent. By Lemma 3.5.5, we deduce that X^{tC_p} is S^1-nilpotent, which completes the proof of (c).

Claim (e) is immediate from Lemma 3.5.8(ii).

For claim (d), first observe that (a), (b), (c), and Lemma 3.5.8(iii) imply that $X^{tC_{p^k}}$ is a retract of $(\tau_{<d}X)^{tC_{p^k}}$. The finiteness assumption on X ensures that the homotopy groups of $(\tau_{<d}X)^{tC_{p^k}}$ are finite and hence so are the homotopy groups of $X^{tC_{p^k}}$. The homotopy groups of X being finite also implies that the homotopy groups of $X_{hC_{p^k}}$ are finite, since X was assumed bounded below. The claim (d) now follows from the cofiber sequence $X_{hC_{p^k}} \to X^{hC_{p^k}} \to X^{tC_{p^k}}$.

We are left with establishing the claim (f), for which we argue as in [Mat21]. Recall that we have pullback squares

$$\begin{tikzcd}
\mathrm{TR}^k(X) \arrow[r, "R"] \arrow[d] & \mathrm{TR}^{k-1}(X) \arrow[d] \\
X^{hC_{p^k}} \arrow[r, "\mathrm{can}"'] & X^{tC_{p^k}}.
\end{tikzcd}$$

The right vertical map is an equivalence in large degrees, independent of k, by Tsalidis's theorem [NS18, II.4.9] and the assumption that $\boxed{\text{Segal Conjecture}}$ holds. The bottom horizontal map is zero in in large degrees by the assumption that $\boxed{\text{Weak Canonical Vanishing}}$ holds. It follows that the top horizontal map is zero in large degrees, and hence that the limit $\mathrm{TR}(X)$ is bounded above. Since X was assumed bounded below, $\mathrm{TR}(X)$ is bounded below, and hence $\mathrm{TR}(X)$ is bounded. $\square$

4. The Segal conjecture

We fix throughout this section an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n\rangle$. Our purpose is to prove the Segal conjecture (Theorem C), which we restate here for convenience.

THEOREM 4.0.1. *Let F denote any type $n+1$ finite complex. Then the cyclotomic Frobenius* $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$ *induces an isomorphism*

$$F_*\mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong F_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p})$$

in all sufficiently large degrees $* \gg 0$.

Remark 4.0.2. This theorem implies the corresponding statement for F a type $n+2$ complex, which is all that is used in deducing the Lichtenbaum–Quillen statements as in the previous section.

The idea of the proof is to use (the décalage of) the Adams filtration on $\mathrm{BP}\langle n\rangle$ to reduce the claim to a much simpler one about graded polynomial algebras over $\mathbb{F}_p$.

Remark 4.0.3. There are several antecedents to the Segal conjecture. First, the classical Segal conjecture for the group C_p states that the map

$$S^0 = \mathrm{THH}(S^0) \to \mathrm{THH}(S^0)^{tC_p} = (S^0)^{tC_p}$$

is p-completion; this is a theorem of Lin [Lin80] (at $p = 2$) and Gunawardena [AGM85] (for p odd). For various classes of ordinary commutative rings R, the map

$$\varphi : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

is a p-adic equivalence in large degrees: this is the case for DVRs of mixed characteristic with perfect residue field in odd characteristic [HM03], [HM04] for smooth algebras in positive characteristic [Hes18, Prop. 6.6], and for p-torsionfree excellent noetherian rings R with R/p finitely generated over its pth powers [Mat21, Cor. 5.3].

When $R = \ell$ is the Adams summand, it is proved in [AR02, Th. 5.5] for $p \geq 5$ that

$$\varphi : \mathrm{THH}(\ell)/(p, v_1) \to \mathrm{THH}(\ell)^{tC_p}/(p, v_1)$$

is an equivalence in degrees larger than $2(p-1)$ (cf. [LN05]). When $R = \mathrm{MU}$, Lunøe-Nielsen and Rognes show [LNR11] that

$$\varphi : \mathrm{THH}(\mathrm{MU}) \to \mathrm{THH}(\mathrm{MU})^{tC_p}$$

is a p-adic equivalence. In another direction, Angelini-Knoll and Quigley [AKQ21b] have shown that φ is an equivalence for Ravenel's $X(n)$ spectra.

4.1. *Polynomial rings over the sphere.* First we record some facts about polynomial rings over the sphere spectrum, starting with their construction.

Construction 4.1.1. For $r, w \in \mathbb{Z}$, we will denote by $S^{2r}(w)$ the graded spectrum that is S^{2r} in weight w and zero elsewhere. Recall (see, e.g., [Lur15, 3.4.1,3.4.2]) that there is a graded $\mathbb{E}_2$-ring $S^0[y_{-2,-1}]$ equipped with a class $y_{-2,-1} : S^{-2}(-1) \to S^0[y_{-2,-1}]$ that exhibits the target as the free graded $\mathbb{E}_1$-algebra on $S^{-2}(-1)$. This graded $\mathbb{E}_2$-ring corresponds to an $\mathbb{E}_2$-monoidal functor

$$S^0[y_{-2,-1}] : \mathbb{Z}^{\mathrm{ds}} \to \mathsf{Sp}$$

that factors through the subcategory $\mathrm{Pic}(\mathsf{Sp})$ of invertible spectra. When $r \leq 0$ we define the graded $\mathbb{E}_2$-ring $S^0[y_{2r,-1}]$ as the composite

$$\mathbb{Z}^{\mathrm{ds}} \xrightarrow{\cdot r} \mathbb{Z}^{\mathrm{ds}} \xrightarrow{S^0[y_{-2,-1}]} \mathrm{Pic}(\mathsf{Sp}) \to \mathsf{Sp}.$$

When $r \geq 0$ we define the graded $\mathbb{E}_2$-ring $S^0[y_{2r,-1}]$ as the composite

$$\mathbb{Z}^{\mathrm{ds}} \xrightarrow{\cdot(-r)} \mathbb{Z}^{\mathrm{ds}} \xrightarrow{S^0[y_{-2,-1}]} \mathrm{Pic}(\mathsf{Sp}) \xrightarrow{D} \mathrm{Pic}(\mathsf{Sp}) \to \mathsf{Sp},$$

where D denotes the duality functor. Finally, we define $S^0[y_{2r,w}]$ for arbitrary $r, w \in \mathbb{Z}$ by left Kan extending $S^0[y_{2r,-1}]$ along the map $(-w) : \mathbb{Z}^{\mathrm{ds}} \to \mathbb{Z}^{\mathrm{ds}}$. Thus, for each $r, w \in \mathbb{Z}$, we have constructed a graded $\mathbb{E}_2$-ring $S^0[a]$ equipped with a class $a : S^{2r}(w) \to S^0[a]$ which exhibits the target as the free graded $\mathbb{E}_1$-ring on $S^{2r}(w)$.

Next we establish an important finiteness property for $\mathrm{THH}(S^0[a])$. But first we recall a definition.

Definition 4.1.2. If G is a (topological) group, we will say that a spectrum with G-action, $X \in \mathsf{Fun}(\mathrm{B}G, \mathsf{Sp})$, is *finite* if it lies in the thick subcategory generated by the objects G/H_+, where $H \subseteq G$ is a closed subgroup and G/H_+ denotes $\Sigma^\infty_+(G/H)$.

Lemma 4.1.3. *The graded $\mathbb{E}_1$-ring map*

$$S^0[a] \to \mathrm{THH}(S^0[a])$$

induces on $\mathbb{F}_p$-homology the ring map

$$\mathbb{F}_p[a] \to \mathbb{F}_p[a] \otimes \Lambda_{\mathbb{F}_p}(\sigma a).$$

Here, the weights of a and σa are both w. Furthermore, if $w \neq 0$, then, as a graded C_p-spectrum, $\mathrm{THH}(S^0[a])$ is pointwise finite. That is, at each weight j, the C_p-spectrum $\mathrm{THH}(S^0[a])_j$ lies in the thick subcategory generated by S^0 and C_{p+}.

Proof. We have that $\mathbb{F}_p \otimes \mathrm{THH}(-) = \mathrm{THH}(\mathbb{F}_p \otimes (-)/\mathbb{F}_p)$, so the induced map on homology is given by

$$\mathbb{F}_p[a] \to \mathrm{THH}(\mathbb{F}_p[a]/\mathbb{F}_p)_*.$$

This map depends only on the $\mathbb{E}_1$-algebra structure of $\mathbb{F}_p[a]$, which is free, so this is equivalent to the classical calculation of $\mathrm{THH}(\mathbb{F}_p[a]/\mathbb{F}_p)_*$ (i.e., ordinary Hochschild homology over $\mathbb{F}_p$).

We now show that $\mathrm{THH}(S^0[a])$ is pointwise finite as a graded C_p-spectrum. This statement only depends on the graded $\mathbb{E}_1$-algebra structure on $S^0[a]$, which is free. The Hochschild homology of free $\mathbb{E}_1$-algebras is well known (see, e.g., the argument in [Mat21, Th. 3.8], which applies verbatim to the graded case), and in this case specializes to

$$\mathrm{THH}(S^0[a]) \simeq \bigoplus_{k\geq 0} \mathrm{Ind}_{C_k}^{S^1}(S^{2rk}(wk)).$$

Here $\mathrm{Ind}_{C_k}^{S^1}$ denotes the induction functor, given by left Kan extension along the functor $\mathrm{B}C_k \to \mathrm{B}S^1$, and C_k acts by permuting the factors in $(S^{2r}(w))^{\otimes k} = S^{2rk}(wk)$. Observe that, since $w \neq 0$, there is at most one non-zero summand in each fixed weight. To complete the proof we need to show that each summand is finite as a C_p-spectrum.

The property of finiteness is always preserved by induction. In this case, the restriction functor $\mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp}) \to \mathsf{Fun}(\mathrm{B}C_p, \mathsf{Sp})$ also preserves finiteness. Indeed, when $k = mp$, the object S^1/C_{k+} is equivalent to $S^1_+ = \Sigma(C_p/C_p)_+$ as a C_p-spectrum, and when k is coprime to p, then we have a cofiber sequence

$$C_{p+} \to S^1/C_{k+} \to \Sigma C_{p+}.$$

So it suffices to show that $(S^{2r})^{\otimes k}$ is finite as a C_k-spectrum. After possibly dualizing we may assume that $r \geq 0$, and then this is the suspension spectrum of the one-point compactification of $2r$ copies of the regular representation of C_k, which admits a finite C_p-CW-structure. □

We now prove the Segal conjecture for these graded polynomial rings over the sphere. For the statement, recall that the cyclotomic Frobenius on filtered objects multiplies filtrations by p; we review the formalism for this using the functor L_p in Section C.5.

Proposition 4.1.4. *Suppose $w \neq 0$. Then the cyclotomic Frobenius*

$$L_p\mathrm{THH}(S^0[a]) \to \mathrm{THH}(S^0[a])^{tC_p}$$

witnesses the target as the p-completion of the source.

Proof. As in the previous proposition, we may compute

$$\mathrm{THH}(S^0[a]) \simeq \bigoplus_{k\geq 0} \mathrm{Ind}_{C_k}^{S^1}(S^{2rk}(wk)).$$

Since there is at most one non-zero summand in each fixed weight, taking the Tate fixed points (in the category of graded spectra) commutes with this sum, so that we have

$$\mathrm{THH}(S^0[a])^{tC_p} \simeq \bigoplus_{k\geq 0} (\mathrm{Ind}_{C_k}^{S^1}(S^{2rk}(wk)))^{tC_p}.$$

When k is not divisible by p, the restriction of $\mathrm{Ind}_{C_k}^{S^1}(S^{2rk})$ to a C_p-spectrum lies in the thick subcategory generated by C_{p+}, and so is annihilated by $(-)^{tC_p}$. When $k = mp$ is divisible by p, then the restriction of $\mathrm{Ind}_{C_k}^{S^1}(S^{2rk})$ is equivalent to $S^1_+ \otimes (S^{2rm})^{\otimes p}$, where C_p acts trivially on the first term and by cyclic permutations on the second. Thus

$$\mathrm{THH}(S^0[a])^{tC_p} \simeq \bigoplus_{m\geq 0} ((S^{2rm})^{\otimes p}(wmp))^{tC_p} \oplus \Sigma((S^{2rm})^{\otimes p}(wmp))^{tC_p}.$$

To compute what the cyclotomic Frobenius does, recall that, directly from the construction of the cyclotomic Frobenius, we have a commutative diagram for any graded $\mathbb{E}_1$-ring A:

$$\begin{tikzcd} L_pA^{\otimes m} \arrow[r] \arrow[d] & (A^{\otimes mp})^{tC_p} \arrow[d] \\ L_p\mathrm{THH}(A) \arrow[r] & \mathrm{THH}(A)^{tC_p}. \end{tikzcd}$$

The bottom arrow is $S^1 \cong S^1/C_p$-equivariant, so we may induce up the targets of the vertical maps to get a diagram

$$\begin{tikzcd} S^1_+ \otimes L_pA^{\otimes m} \arrow[r] \arrow[d] & S^1_+ \otimes (A^{\otimes mp})^{tC_p} \arrow[d] \\ L_p\mathrm{THH}(A) \arrow[r] & \mathrm{THH}(A)^{tC_p}. \end{tikzcd}$$

If we now take $A = S^0[a]$ and restrict to the summand corresponding to a^m, then we learn that the cyclotomic Frobenius map in weight mp is given by tensoring the Tate diagonal

$$S^{2rm} \to ((S^{2rm})^{\otimes p})^{tC_p}$$

with S^1_+. The Tate diagonal here witnesses the target as the p-completion of the source by the classical Segal conjecture. $\square$

4.2. *The Segal conjecture for polynomial* $\mathbb{F}_p$*-algebras.* In this section we consider a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra R, with homotopy groups a polynomial ring

$$\pi_*(R) \cong \mathbb{F}_p[a_1, a_2, \ldots, a_n].$$

Each a_i will have non-negative even degree $|a_i|$ and positive weight $\mathrm{wt}(a_i)$, though we suppress the weights from the notation. In fact, there is a unique ring R with the above description:

Proposition 4.2.1. *As a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra, the ring R above must be equivalent to*

$$\mathbb{F}_p \otimes S^0[a_1] \otimes S^0[a_2] \otimes \cdots \otimes S^0[a_n],$$

where $S^0[a_i]$ is the ring constructed in Construction 4.1.1 with $2r = |a_i|$ and $w = \mathrm{wt}(a_i)$.

Proof. Let us denote $\mathbb{F}_p \otimes S^0[a_1] \otimes S^0[a_2] \otimes \cdots \otimes S^0[a_n]$ by A. We first claim that A has a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra cell structure with cells in even degrees. Indeed, this algebra is canonically augmented over $\mathbb{F}_p$, so we may apply [GKRW21, Th. 11.21,Th. 13.7] which, together, show that, if

$$\mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_A \mathbb{F}_p} \mathbb{F}_p$$

has homotopy groups in even degrees, then A has a minimal cell structure as a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra with cells in even degrees.[3] But the Künneth spectral sequence computing these homotopy groups collapses at the E_2-page as a divided power algebra on even degree classes, so the claim follows.

There is then no obstruction to constructing an $\mathbb{E}_2$-map $A \to R$ sending a_i to a_i, since the homotopy groups of R are concentrated in even degrees. The result follows. □

Our main theorem about this $\mathbb{E}_2$-$\mathbb{F}_p$-algebra R is as follows:

Proposition 4.2.2. *The cyclotomic Frobenius*

$$L_p\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

induces on homotopy groups the ring map

$$\begin{aligned}&\mathbb{F}_p[x, a_1, a_2, \ldots, a_n] \otimes \Lambda(\sigma a_1, \sigma a_2, \cdots, \sigma a_n)\\ &\to \mathbb{F}_p[x^{\pm 1}, a_1, a_2, \ldots, a_n] \otimes \Lambda(\sigma a_1, \sigma a_2, \cdots, \sigma a_n)\end{aligned}$$

that inverts x. Here, x is in degree 2 and weight 0. The degree of σa_i is one more than the degree of a_i, and the weight of σa_i is the same as the weight of a_i.

A version of the above is well known in the case that all a_i are in degree 0 and weight zero, so R is a classical commutative ring. (See, e.g., [Hes18, 6.6] for a much stronger result.) Our main observation is that the result extends to the case where not all a_i are in degree 0, in which case R is not discrete. Since an exterior algebra on classes of degree $|a_i| + 1$ has no homotopy above degree $n + \sum |a_i|$, we obtain the following result:

[3] Here it is important that we are considering $\mathbb{E}_2$-algebras: the iterated bar construction is related to the $\mathbb{E}_k$-cotangent complex up to a shift by k. Since $k = 2$, the property of being concentrated in even degrees is insensitive to this shift.

Corollary 4.2.3 (Segal conjecture for graded polynomial $\mathbb{F}_p$-algebras). *The map*

$$\pi_*(\mathrm{THH}(R)/(a_1,\dots,a_n)) \to \pi_*(\mathrm{THH}(R)^{tC_p}/(a_1,\dots,a_n))$$

is an equivalence in degrees $* > n + \sum_{i=1}^n |a_i|$.

Proof of Propostion 4.2.2. For convenience, we will omit the grading shear, L_p, from the notation throughout.

By Proposition 4.2.1, we may assume that R is a tensor product of graded $\mathbb{E}_2$-rings

$$R \simeq \mathbb{F}_p \otimes S^0[a_1] \otimes \cdots \otimes S^0[a_n].$$

Since THH is symmetric monoidal as a functor to cyclotomic spectra [NS18, p.341], we may compute

$$\mathrm{THH}(R) \simeq \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{THH}(S^0[a_1]) \otimes \cdots \otimes \mathrm{THH}(S^0[a_n])$$

as a C_p-equivariant $\mathbb{E}_1$-ring spectrum. We next compute the cyclotomic Frobenius on each component of the above tensor product. It follows from Bökstedt's unpublished computation of $\mathrm{THH}(\mathbb{F}_p)$ (see [NS18, §IV.4] for a modern reference) that the map

$$\varphi : \mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$$

induces, on homotopy groups, the map

$$\mathbb{F}_p[x] \to \mathbb{F}_p[x^{\pm 1}],$$

which inverts x. Here $x = \sigma^2 v_0$ is in degree 2 and weight zero. We have already seen (Proposition 4.1.4) that each map

$$\varphi : \mathrm{THH}(S^0[a_i]) \to \mathrm{THH}(S^0[a_i])^{tC_p}$$

is an equivalence after p-completion. It follows that the map

$$\mathrm{THH}(\mathbb{F}_p) \otimes \bigotimes_i \mathrm{THH}(S^0[a_i]) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p} \otimes \bigotimes_i \mathrm{THH}(S^0[a_i])^{tC_p}$$

has the desired effect on homotopy groups. To finish the proof we need to show that the lax monoidal structure map

$$\mathrm{THH}(\mathbb{F}_p)^{tC_p} \otimes \bigotimes_i \mathrm{THH}(S^0[a_i])^{tC_p} \to \left(\mathrm{THH}(\mathbb{F}_p) \otimes \bigotimes_i \mathrm{THH}(S^0[a_1])\right)^{tC_p}$$

is an equivalence. By Lemma 4.1.3 it suffices to prove the following more general statement: if X and Y are non-negatively graded C_p-spectra, and, for each weight j the C_p-spectrum Y_j lies in the thick subcategory generated by C_{p+} and S^0, then the map

$$\alpha : X^{tC_p} \otimes Y^{tC_p} \to (X \otimes Y)^{tC_p}$$

is an equivalence after p-completion.

The Tate construction on graded C_p-spectra is computed pointwise, so we need to prove that

$$\bigoplus_{i+j=w} X_i^{tC_p} \otimes Y_j^{tC_p} \to \Big(\bigoplus X_i \otimes Y_j\Big)^{tC_p}$$

is a p-adic equivalence. Since X and Y are non-negatively graded, these sums are finite. We are therefore reduced to proving the analogous ungraded statement: that α is a p-adic equivalence, where X and Y are (ungraded) C_p-spectra and where we assume that Y belongs to the thick subcategory generated by S^0 and C_{p+}. Since the Tate construction is exact, the category of C_p-spectra Y for which α an equivalence after p-completion is a thick subcategory. So we need only prove the claim for $Y = S^0$ and $Y = C_{p+}$. When $Y = C_{p+}$, both sides vanish. When $Y = S^0$, we may identify Y^{tC_p} with the p-complete sphere, $(S^0)^\wedge_p$, by the Segal conjecture, so this map becomes the canonical one

$$X^{tC_p} \otimes (S^0)^\wedge_p \to X^{tC_p},$$

which is indeed an equivalence after p-completion. □

4.3. *The Segal conjecture for* $\mathrm{BP}\langle n\rangle$. The key to the proof of Theorem C is the following:

THEOREM 4.3.1. *The map of* BP-*algebras*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle)/(p, v_1, v_2, \dots, v_n) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(p, v_1, v_2, \dots, v_n)$$

is an equivalence in large degrees. Here we regard $p, v_1, \dots, v_n$ as elements in the homotopy of the right-hand side via the ring map φ.

Before proving it, we need to recall a few things about the Adams spectral sequence for $\mathrm{BP}\langle n\rangle$.

Recollection 4.3.2. Recall the descent tower $\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)$ discussed in Section C. We claim that the associated graded object has homotopy groups given by

$$\pi_*(\mathrm{gr}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}\mathrm{BP}\langle n\rangle)) \simeq \mathbb{F}_p[v_0, v_1, \dots, v_n],$$

where each v_i lies in weight $2p^i - 1$. (Recall that the *weight* of a class in $E_2^{s,t}$ is t; see Convention C.1.1.) Indeed, from the definition of the descent tower, these homotopy groups agree with

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, \mathrm{H}_*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p)).$$

Recall that the homology of $\mathrm{BP}\langle n\rangle$ as a comodule is coextended from the quotient Hopf algebra $\Lambda(\overline{\tau_0}, \dots, \overline{\tau_n})$ (where we write τ_j for ζ_{j+1} at the prime 2):[4]

$$\mathrm{H}_*(\mathrm{BP}\langle n\rangle; \mathbb{F}_p) = \mathcal{A}_* \square_{\Lambda(\overline{\tau_0}, \dots, \overline{\tau_n})} \mathbb{F}_p.$$

[4] Here we use the convention of [Mil58] for the definition of ζ_j.

(This goes back to the construction of $\mathrm{BP}\langle n\rangle$; see [Wil75, Prop. 1.7].) By the change of rings isomorphism [Rav86, A1.3.13], we have

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_p, \mathrm{H}_*(\mathrm{BP}\langle n\rangle;\mathbb{F}_p)) \cong \mathrm{Ext}^{*,*}_{\Lambda(\overline{\tau_0},\dots,\overline{\tau_n})}(\mathbb{F}_p,\mathbb{F}_p) \cong \mathbb{F}_p[v_0,\dots,v_n],$$

where the v_i are represented by $[\overline{\tau_i}]$ in the cobar complex. The classes are so-named because they detect the corresponding classes in $\pi_*\mathrm{BP}\langle n\rangle$, and v_0 detects p. For $i>0$, we denote by $\tilde{v}_i$ chosen lifts of each v_i to elements in $\pi_*\mathrm{desc}^{\geq 2p^i-1}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)$, and we observe that v_0 admits a unique lift to an element in $\pi_*\mathrm{desc}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)$, which we denote by the same symbol.

Proof of Theorem 4.3.1. For convenience, in this proof we will suppress the functor L_p from the notation when discussing the cyclotomic Frobenius for filtered and graded objects.

First observe that we may reformulate this claim as saying that the map

$$\mathbb{F}_p\otimes_{\mathrm{BP}\langle n\rangle}\varphi : \mathbb{F}_p\otimes_{\mathrm{BP}\langle n\rangle}\mathrm{THH}(\mathrm{BP}\langle n\rangle)\to \mathbb{F}_p\otimes_{\mathrm{BP}\langle n\rangle}\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$$

is an equivalence in large degrees, since

$$\mathrm{BP}\langle n\rangle/(p,v_1,\dots,v_n)\simeq \mathbb{F}_p.$$

To define this map, we are using that $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ is an $\mathbb{E}_1$-$\mathrm{BP}\langle n\rangle$-algebra, and the map φ is an $\mathbb{E}_2$-algebra map, and hence φ in particular has the structure of a map of modules over $\mathrm{BP}\langle n\rangle$ (where the module structure on the target is defined using the map φ).

The $\mathbb{E}_2$-algebra $\mathrm{BP}\langle n\rangle$ refines to a filtered $\mathbb{E}_2$-algebra $\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)$, and $\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathbb{F}_p)=\mathbb{F}_p$ is a module over this algebra, where the right-hand side is the tower with 0 in positive filtration and $\mathbb{F}_p$ in non-positive filtration. Moreover, THH inherits a filtration, and so we can ask whether the map

$$\begin{aligned}&\mathbb{F}_p\otimes_{\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)}\mathrm{THH}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle))\\ &\to \mathbb{F}_p\otimes_{\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)}\mathrm{THH}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle))^{tC_p}\end{aligned}$$

is an equivalence in large degrees on homotopy groups.

We would like to reduce this to a claim on the associated graded, but in order to do so we need to know that the towers on both sides are conditionally convergent. By Proposition C.5.4, the towers $\mathrm{THH}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}\mathrm{BP}\langle n\rangle)$ and $\mathrm{THH}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}\mathrm{BP}\langle n\rangle)^{tC_p}$ are conditionally convergent, after v_0-completion. Using the notation in Recollection 4.3.2, it suffices by a thick subcategory argument to prove that

$$\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)/(v_0,\tilde{v_1},\dots,\tilde{v_n})\simeq\mathbb{F}_p$$

as filtered modules over $\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)$. Again, since $\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)$ is conditionally convergent after v_0-completion, it suffices to check this equivalence upon taking the associated graded, where it is clear from Recollection 4.3.2.

We are now reduced to checking that the associated graded of the map

$$\mathbb{F}_p \otimes_{\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)} \mathrm{THH}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle))$$
$$\to \mathbb{F}_p \otimes_{\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle)} \mathrm{THH}(\mathrm{desc}^{\geq\bullet}_{\mathbb{F}_p}(\mathrm{BP}\langle n\rangle))^{tC_p}$$

is an equivalence in high enough degrees.

Upon taking associated graded, we may, by Recollection 4.3.2, identify this map with

$$\mathrm{THH}(\mathbb{F}_p[v_0, v_1, \dots, v_n])/(v_0, \dots, v_n) \to \mathrm{THH}(\mathbb{F}_p[v_0, v_1, \dots, v_n])^{tC_p}/(v_0, \dots, v_n),$$

and it follows from Corollary 4.2.3 that this map is an equivalence in large degrees. This completes the proof. $\square$

From a thick subcategory argument in BP-modules, we then learn the following:

Corollary 4.3.3. *For any positive integers* $i_0, i_1, \dots, i_n$, *the map of* BP-*algebras*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle)/(p^{i_0}, v_1^{i_1}, v_2^{i_2}, \dots, v_n^{i_n}) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(p^{i_0}, v_1^{i_1}, v_2^{i_2}, \dots, v_n^{i_n})$$

is an equivalence in large degrees.

In particular, if we let S/I *denote a generalized Moore spectrum of the form* $S^0/(p^{i_0}, v_1^{i_1}, \dots, v_n^{i_n})$, *then*

$$(S/I)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to (S/I)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$$

is an equivalence in large degrees.

The Segal conjecture (Theorem C) now follows by a thick subcategory argument in spectra, since any S/I generates the thick subcategory of type $n+1$ spectra.

5. The Detection Theorem

Throughout this section, we will use $\mathrm{BP}\langle n\rangle$ to denote a fixed $\mathbb{E}_3$-MU-algebra form $\mathrm{BP}\langle n\rangle$. By $v_{n+1} \in \pi_{2p^{n+1}-2}\mathrm{MU}_{(p)}$ we will refer to a specific indecomposable generator, with

- trivial mod p Hurewicz image, and
- the key property that the unit map $\mathrm{MU}_{(p)} \to \mathrm{BP}\langle n\rangle$ sends v_{n+1} to 0 in homotopy.

This last assumption ensures that v_{n+1} admits a unique lift to an element in the homotopy of the fiber of the unit map $\mathrm{MU}_{(p)} \to \mathrm{BP}\langle n\rangle$. Our main aim will be to prove Theorem F from the introduction, which we restate for convenience:

THEOREM 5.0.1 (Detection). *There is an isomorphism of* $\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$-*algebras*

$$\pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}) \cong (\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))\,[\![t]\!],$$

where $|t| = -2$. *This isomorphism can be chosen such that, under the unit map*

$$\pi_*(\mathrm{MU}_{(p)}^{hS^1}) \to \pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}),$$

the canonical complex orientation maps to t *and* v_{n+1} *is sent to* $t(\sigma^2 v_{n+1})$.

Before turning to the proof, we observe that the Detection Theorem implies a weak form of redshift.

COROLLARY 5.0.2. *For each* $0 \leq m \leq n+1$, $L_{K(m)}\mathrm{K}(\mathrm{BP}\langle n\rangle) \neq 0$. *In particular*, $L_{K(n+1)}\mathrm{K}(\mathrm{BP}\langle n\rangle) \neq 0$.

Proof. By [BGT14], the cyclotomic trace map

$$\mathrm{K}(-) \to \mathrm{TC}(-)$$

is a lax symmetric monoidal natural transformation. It follows that the trace $\mathrm{K}(\mathrm{BP}\langle n\rangle) \to \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ is a map of $\mathbb{E}_2$-rings. Recall that there is a canonical map $\mathrm{TC}(-) \to \mathrm{THH}(-)^{hS^1}$, to negative cyclic homology. Thus we have a sequence of $\mathbb{E}_2$-ring maps,

$$\mathrm{K}(\mathrm{BP}\langle n\rangle) \to \mathrm{TC}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1} \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1},$$

and hence an $\mathbb{E}_2$-ring map

$$L_{K(m)}\mathrm{K}(\mathrm{BP}\langle n\rangle) \to L_{K(m)}\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$$

for each height $m \leq n+1$. If the source of this map were zero, then the target would be zero as well, since this is a map of rings. The relative negative cyclic homology $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$ has the structure of an MU-module. It follows from [Hov95, Th. 1.9] and [Hov97, Th. 1.5.4] that

$$L_{K(m)}\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1} = (\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1})[v_m^{-1}]^{\wedge}_{(p,v_1,\ldots,v_{m-1})}.$$

By Theorems 5.0.1 and 2.5.4, this completion and localization can be computed algebraically, and the result is non-zero. $\square$

Remark 5.0.3. In the statement and proof of the theorem we have used that the S^1-action on $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ is compatible with the algebra structure. One way to see this is to use the generality in which THH is defined.

Recall that for any symmetric monoidal category $\mathcal{C}$ with tensor product compatible with sifted colimits, Hochschild homology gives a functor

$$\mathrm{HH}_{\mathcal{C}} : \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C}) \to \mathsf{Fun}(\mathrm{B}S^1, \mathcal{C}).$$

For a reference, one could observe that the construction of THH with its circle action in [NS18, §III.2] works just the same for $\mathcal{C}$ in place of Sp. Alternatively, one can use the identification of THH with factorization homology over S^1, which is defined in this generality ([Lur17, §5.5.2], [AF15]). Now apply this in the case $\mathcal{C} = \mathsf{Alg}_{\mathbb{E}_2}(\mathsf{Mod}_{\mathrm{MU}})$ to see that $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ has a canonical enhancement to an object in $\mathsf{Fun}(\mathrm{B}S^1, \mathsf{Alg}_{\mathbb{E}_2}(\mathsf{Mod}_{\mathrm{MU}}))$.

Let us now proceed with the proof of Theorem 5.0.1. First we compute $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$. Recall that we computed the homotopy groups of $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ in Theorem 2.5.4. An immediate consequence of that calculation is the following proposition:

PROPOSITION 5.0.4. *The homotopy fixed point spectral sequence for*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$$

collapses at the E_2-page, with

$$E_\infty = \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_*[t],$$

where $t \in H^2(\mathbb{C}P^\infty)$ is the standard generator.

Proof. The homotopy fixed point spectral sequence computing

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$$

is concentrated in even degrees by Theorem 2.5.4, and hence collapses as indicated. □

As we will shortly explain, the remainder of the argument for Theorem 5.0.1 is a formal consequence of the relationship between the suspension map σ^2 and the circle action. We explore this relationship in Section A.

Proof of 5.0.1. The image of the canonical complex orientation under the unit map

$$\pi_*\mathrm{MU}_{(p)}^{hS^1} \to \pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$$

will be detected by t in the homotopy fixed point spectral sequence. We recall that $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ is a polynomial $\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$-algebra generated by classes $w_{n+1,i}$ (for $i \geq 0$) and $y_{j,i}$ (for $i \geq 0$, $j \geq 1$, and $j \not\equiv -1$ modulo p). Furthermore, we may set $w_{n+1,0}$ equal to $\sigma^2 v_{n+1}$.

Since polynomial algebras are free commutative algebras, an isomorphism

$$\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1} \cong (\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))\,[\![t]\!]$$

is determined by a choice of elements $\widetilde{w_{n+1,i}}, \widetilde{y_{j,i}} \in \pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$ detecting the similarly named classes in the homotopy fixed point spectral sequence.

If we make any such choice, then Lemma A.4.1 ensures that $t\widetilde{w_{n+1,0}}$ will be v_{n+1} modulo t^2, say $t\widetilde{w_{n+1,0}} = v_{n+1} + t^2 y$. We may then replace $\widetilde{w_{n+1,0}}$ by $\widetilde{w_{n+1,0}} - ty$, which also lifts the class $w_{n+1,0}$, and so guarantee that $t\widetilde{w_{n+1,0}} = v_{n+1}$. □

Remark 5.0.5. Rognes has sketched an alternative proof that v_{n+1} is detected in the homotopy of $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$. The strategy is to consider the exact sequence in mod p homology:

$$H_*(\lim_{\mathbb{C}P^1} \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})) \to H_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})) \xrightarrow{B} H_{*+1}(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})).$$

One computes that $B\overline{\tau}_{n+1}$ is non-zero and hence does not lie in the kernel, i.e., does not arise in the first term. It follows from an Adams spectral sequence argument that v_{n+1} must be detected in $\pi_* \lim_{\mathbb{C}P^1} \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$.

6. **Canonical vanishing**

Fix an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n\rangle$. In this section, we will study the canonical map

$$\mathrm{can} : \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1} \longrightarrow \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tS^1}.$$

Our goal will be to establish Theorem D, which by the results of Section 3 can be reduced to weak canonical vanishing (Theorem 6.3.1).

To be specific, we will choose a convenient type $n+1$ complex M, with v_{n+1} self map v, and consider the map

$$1 \otimes \mathrm{can} : M/v \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1} \to M/v \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tS^1},$$

where 1 is the identity map of the type $n+2$ complex M/v. We will prove that the $\pi_*(1 \otimes \mathrm{can})$ map is zero for all sufficiently large degrees $*$.

As a prototype for the result and its proof, consider the case $n = -1$, where the statement is that

$$\mathrm{can}/p : \mathrm{THH}(\mathbb{F}_p)^{hS^1}/p \longrightarrow \mathrm{THH}(\mathbb{F}_p)^{tS^1}/p$$

induces the zero map on homotopy groups in large degrees.

We may compute $\mathrm{THH}(\mathbb{F}_p)^{hS^1}$ via the homotopy fixed point spectral sequence

$$E_2 = \mathbb{F}_p[\sigma^2 v_0, t] \implies \pi_*\mathrm{THH}(\mathbb{F}_p)^{hS^1}.$$

Here, $\sigma^2 v_0$ is in homotopy dimension 2 and filtration 0, while t is in homotopy dimension -2 and filtration 2.

We may also understand $\mathrm{THH}(\mathbb{F}_p)^{tS^1}$ via the Tate fixed point spectral sequence, with E_2 page $\mathbb{F}_p[\sigma^2 v_0, t^{\pm 1}]$. The canonical map is compatible with homotopy and Tate fixed point spectral sequences, and at the level of E_2-pages it is approximated by the map

$$\mathbb{F}_p[\sigma^2 v_0, t] \longrightarrow \mathbb{F}_p[\sigma^2 v_0, t^{\pm 1}]$$

that inverts t.

The element $p \in \pi_* \mathrm{THH}(\mathbb{F}_p)^{hS^1}$ is detected by $t\sigma^2 v_0$, which lives in filtration 2 in both spectral sequences. By killing a filtration 2 lift of p, we build a map between a (modified) homotopy fixed point spectral sequence converging to $\pi_*\left(\mathrm{THH}(\mathbb{F}_p)^{hS^1}/p\right)$ and a (modified) Tate fixed point spectral sequence converging to $\pi_*\left(\mathrm{THH}(\mathbb{F}_p)^{tS^1}/p\right)$. At the level of E_2 pages, the mod p canonical map is approximated by

$$E_2 = \mathbb{F}_p[\sigma^2 v_0, t]/(t\sigma^2 v_0) \to E_2 = \mathbb{F}_p[t^{\pm 1}].$$

This map of E_2 pages is trivial in positive homotopy dimension, and we would like to conclude that the mod p canonical map is zero in positive degrees. We might be worried about filtration jumps, but in fact this is no issue. The source spectral sequence is concentrated in non-negative filtration, while the target spectral sequence, in positive homotopy dimension, is concentrated in negative filtration.

Our strategy for proving Theorem 6.3.1 is to mimic the above argument at a general height. The main challenge in carrying this out (especially in the absence of Smith–Toda complexes) is to find and name an appropriate class in the homotopy fixed point spectral sequence for $M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1}$ that detects the v_{n+1} self map v. We address this issue by descending information from $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$, which we understand well thanks to the previous section.

6.1. *Descent.* We will need to know that $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ is well approximated by $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ in a way made precise in the below proposition. We will use notation as in Section C.2 (also note Remark C.2.2).

PROPOSITION 6.1.1. *For any type $(n+1)$-complex F, the spectral sequence computing $\pi_*(F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))$ by descent along the map*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$$

collapses at a finite page with a horizontal vanishing line. In particular, if F is equipped with a homotopy ring structure, then the kernel of the map

$$\pi_*(F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)) \to \pi_*(F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))$$

is nilpotent.

Remark 6.1.2. It is possible to obtain much stronger results about horizontal vanishing lines in these and related descent spectral sequences by further developing the methods used below. We hope to return to this in future work.

In the proof we will use Hochschild homology with coefficients in a bimodule, which we now recall.

Definition 6.1.3. If M is a bimodule over an $\mathbb{E}_1$-algebra A in $\mathcal{C}$, then we define

$$\mathrm{HH}(A;M) := M \otimes_{A\otimes A^{\mathrm{op}}} A.$$

Remark 6.1.4. (Compare [AHL10, §2].) If A admits the structure of an $\mathbb{E}_2$-algebra, so that $\mathrm{HH}(A)$ has the structure of a module over A, and M is a right A-module viewed as an A-bimodule by restriction along $A \otimes A^{\mathrm{op}} \to A$, then we have a canonical equivalence

$$\mathrm{HH}(A;M) = M \otimes_{A\otimes A^{\mathrm{op}}} A \simeq M \otimes_A (A \otimes_{A\otimes A^{\mathrm{op}}} A) \simeq M \otimes_A \mathrm{HH}(A).$$

If, moreover, the bimodule structure on M arises from an $\mathbb{E}_1$-algebra map $A \to M$, then we have an equivalence

$$\mathrm{HH}(A;M) = M \otimes_{A\otimes A^{\mathrm{op}}} A \simeq M \otimes_{M\otimes A^{\mathrm{op}}} M \otimes A^{\mathrm{op}} \otimes_{A\otimes A^{\mathrm{op}}} A \simeq M \otimes_{M\otimes A^{\mathrm{op}}} M.$$

Construction 6.1.5. By the previous remark we have an equivalence

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbb{F}_p) \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p\otimes \mathrm{BP}\langle n\rangle} \mathbb{F}_p.$$

Recall that

$$\pi_*(\mathbb{F}_p \otimes \mathrm{BP}\langle n\rangle) \simeq \Lambda(\overline{\tau}_i : i \geq n+1) \otimes_{\mathbb{F}_p} \mathbb{F}_p[t_1,\dots,t_{n+1}],$$

where the t_i come from the homology of BP. Thus we have well-defined elements $\sigma\overline{\tau}_{n+1}$ and $\sigma t_1,\dots,\sigma t_{n+1}$ in $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbb{F}_p)$. We will write $\sigma\overline{\tau}_{n+1}$ as $\sigma^2 v_{n+1}$ since this is its image inside $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU};\mathbb{F}_p)$.

Proposition 6.1.6. *The descent spectral sequence for*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbb{F}_p) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU};\mathbb{F}_p)$$

collapses at the E_2-page as

$$E_2 = \mathbb{F}_2[\sigma^2 v_{n+1}] \otimes \Lambda(\sigma t_1,\dots,\sigma t_{n+1}).$$

Here $\sigma^2 v_{n+1}$ has filtration 0 and homotopy dimension $2p^{n+1}$, and each σt_i has filtration 1 and homotopy dimension $2p^i - 1$.

Proof of Proposition 6.1.1 from Proposition 6.1.6. By a thick subcategory argument (using [HPS99]) it suffices to establish the claim for a generalized Moore complex $F = S^0/(p^{i_0},\dots,v_n^{i_n})$. Observe that

$$F \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle) \simeq (F \otimes \mathrm{BP}\langle n\rangle) \otimes_{\mathrm{BP}\langle n\rangle} \mathrm{THH}(\mathrm{BP}\langle n\rangle).$$

The $\mathrm{BP}\langle n\rangle$-module $F \otimes \mathrm{BP}\langle n\rangle$ lies in the thick subcategory generated by the $\mathrm{BP}\langle n\rangle$-module $\mathbb{F}_p$, so we are reduced to the statement in Proposition 6.1.6. □

Proof of Proposition 6.1.6. For this proof, we will abbreviate

$$A := \mathrm{THH}(\mathrm{BP}\langle n\rangle; \mathbb{F}_p),\ B := \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}; \mathbb{F}_p).$$

It follows from [AR05, Th. 5.12] that

$$\pi_* A = \mathbb{F}_p[\sigma^2 v_{n+1}] \otimes \Lambda(\sigma t_1, \ldots, \sigma t_{n+1}).$$

We will see shortly that $\Sigma := \pi_*(B \otimes_A B)$ is flat over $\pi_* B$. The proposition will follow if we can show that $\mathrm{Ext}^*_\Sigma(\pi_* B, \pi_* B)$ already has the correct size. We will prove this by constructing a further spectral sequence computing $\mathrm{Ext}^*_\Sigma(\pi_* B, \pi_* B)$ whose E_2-term has the same size as $\pi_* A$.

Regard A and B as filtered algebras via the Whitehead filtrations $\{\tau_{\geq j} A\}$ and $\{\tau_{\geq j} B\}$, so that the map $A \to B$ is a map of filtered algebras. We may then regard the cosimplicial object

$$[n] \mapsto B^{\otimes_A (n+1)}$$

as a cosimplicial filtered spectrum. The associated graded cosimplicial object is then given by

$$[n] \mapsto (\pi_* B)^{\otimes^{\mathbb{L}}_{\pi_* A}(n+1)}$$

(where we have used $\otimes^{\mathbb{L}}$ to remind the reader that the tensor products are derived). Since $\pi_* B$ is concentrated in even degrees, the exterior classes must vanish under the map $\pi_* A \to \pi_* B$. It follows that

$$\overline{\Sigma} := \pi_* \left(\pi_* B \otimes^{\mathbb{L}}_{\pi_* A} \pi_* B\right) \simeq \pi_* B \otimes_{\mathbb{F}_p} P \otimes_{\mathbb{F}_p} \Gamma,$$

where $\Gamma = \Gamma\{\sigma^2 t_1, \ldots, \sigma^2 t_{n+1}\}$ is a divided power algebra on the indicated generators and P is a polynomial algebra on even degree classes.

Since $\overline{\Sigma}$ is concentrated in even degrees, we learn that each of the spectral sequences

$$\pi_* \left((\pi_* B)^{\otimes^{\mathbb{L}}_{\pi_* A}(n+1)}\right) \Rightarrow \pi_*(B^{\otimes_A (n+1)})$$

collapses at the E_2-page. In other words, we have a filtration on each group $\pi_*(B^{\otimes_A(n+1)})$ whose associated graded is given by $\pi_* \left((\pi_* B)^{\otimes^{\mathbb{L}}_{\pi_* A}(n+1)}\right)$. This, in particular, implies that Σ is flat over $\pi_* B$ as we claimed earlier.

Using this filtration on homotopy groups, we may then extract a spectral sequence ([Rav86, Th. A.1.3.9]):

$$\mathrm{Ext}^*_{\overline{\Sigma}}(\pi_* B, \pi_* B) \Rightarrow \mathrm{Ext}^*_\Sigma(\pi_* B, \pi_* B).$$

It will now suffice to prove that $\mathrm{Ext}^*_{\overline{\Sigma}}(\pi_* B, \pi_* B)$ has the same size as $\pi_* A$. The map $\pi_* A \to \pi_* B$ can be written as a tensor product (over $\mathbb{F}_p$) of the three maps

$$\mathbb{F}_p[\sigma^2 v_{n+1}] \xrightarrow{\mathrm{id}} \mathbb{F}_p[\sigma^2 v_{n+1}],\ \ \mathbb{F}_p \to P,\ \ \Lambda(\sigma t_1, \ldots, \sigma t_{n+1}) \to \mathbb{F}_p.$$

The descent Hopf algebroid for the first map is just the pair

$$(\mathbb{F}_p[\sigma^2 v_{n+1}], \mathbb{F}_p[\sigma^2 v_{n+1}]),$$

which has cohomology $\mathbb{F}_p[\sigma^2 v_{n+1}]$, concentrated in cohomological dimension zero. The descent Hopf algebroid for the second map is $(P, P \otimes_{\mathbb{F}_p} P)$, which has cohomology $\mathbb{F}_p$ concentrated in cohomological dimension zero. The descent Hopf algebroid for the last map is the divided power Hopf algebra $(\mathbb{F}_p, \Gamma\{\sigma^2 t_1, \dots, \sigma^2 t_{n+1}\})$.

It follows that we may compute our Ext as

$$\begin{aligned} \mathrm{Ext}^*_{\overline{\Sigma}}(\pi_* B, \pi_* B) &\simeq \mathbb{F}_p[\sigma^2 v_{n+1}] \otimes \mathrm{Ext}^*_{\Gamma\{\sigma^2 t_1, \dots, \sigma^2 t_{n+1}\}}(\mathbb{F}_p, \mathbb{F}_p) \\ &\simeq \mathbb{F}_p[\sigma^2 v_{n+1}] \otimes \Lambda(\sigma t_1, \dots, \sigma t_{n+1}), \end{aligned}$$

which completes the proof. □

6.2. *Recollection on Hopkins–Smith.* It will be convenient for our argument to use a type $n+1$ complex with a v_{n+1}-element that has as high an Adams filtration as possible. We do not know whether it is possible to do this and also equip our complex with a homotopy commutative ring structure, but the below proposition will suffice for our purposes.

Proposition 6.2.1. *There is a finite p-local $\mathbb{E}_1$-ring spectrum M with the following properties*:

(i) *M admits a non-nilpotent v_{n+1}-element, $v \in \pi_* M$.*
(ii) *The element v is central.*
(iii) *$\mathrm{BP}\langle n\rangle \otimes M$ splits, as a $\mathrm{BP}\langle n\rangle$-module, as a direct sum of suspensions of $\mathbb{F}_p$.*
(iv) *Let $\mathrm{fil}(v)$ denote the Adams filtration of v, and $|v|$ the dimension. Then* $\dfrac{|v|}{\mathrm{fil}(v)} = 2p^{n+1} - 2.$
(v) *The map $\mathrm{MU}_*(M) \to \mathrm{BP}\langle n\rangle_*(M)$ is surjective.*

Proof. We may take $M = \mathrm{End}(X)$, where X is the type $(n+1)$ spectrum constructed by Jeff Smith in [Rav92, §6.4]. The claims (i), (ii), and (iv) are shown in the course of the proof of [HS98, Th. 4.12]. Since the Margolis homology of $H^*(X; \mathbb{F}_p)$ vanishes with respect to each Q_i with $i \le n$, [MW81, Prop. 2.7] shows that $H^*(X; \mathbb{F}_p)$ is a finitely generated free module over $\Lambda(Q_0, \dots, Q_n)$, so the same is true of M. Choosing a basis of $H^*(M; \mathbb{F}_p)$ as a $\Lambda(Q_0, \dots, Q_n)$-module gives a map $M \to V$ into a direct sum of suspensions of $\mathbb{F}_p$. After extending scalars of the source to $\mathrm{BP}\langle n\rangle$ this becomes an equivalence on $\mathbb{F}_p$-cohomology, and hence an equivalence, proving (iii).

We now turn to the proof of (v). It will suffice to prove the statement for BP in place of MU. We will use descent along $\mathrm{BP} \to \mathrm{BP}\langle n\rangle$ to study the

BP-modules $\mathrm{BP}\otimes M$ and $\mathrm{BP}/(p,\ldots,v_n)\otimes M$. We have a map between descent spectral sequences:

$$\begin{tikzcd}
\mathrm{Ext}_{\Lambda_{\mathrm{BP}\langle n\rangle_*}(\sigma v_{n+1},..)}(\mathrm{BP}\langle n\rangle_*, \mathrm{BP}\langle n\rangle_*(M)) \arrow[r, Rightarrow] \arrow[d] & \mathrm{BP}_*(M) \arrow[d] \\
\mathrm{Ext}_{\Lambda_{\mathrm{BP}\langle n\rangle_*}(\sigma v_{n+1},..)}(\mathrm{BP}\langle n\rangle, H_*(M;\mathbb{F}_p)) \arrow[r, Rightarrow] & (\mathrm{BP}/(p,\ldots,v_n))_*(M).
\end{tikzcd}$$

Here we have used that $\mathrm{BP}\langle n\rangle \otimes_{\mathrm{BP}} \mathrm{BP}/(p,\ldots,v_n)\simeq \mathbb{F}_p$.

Since $\mathrm{BP}\langle n\rangle\otimes M$ and $\mathbb{F}_p\otimes M$ are $\mathbb{F}_p$-modules, by (iii), we may rewrite the map of Ext groups as

$$\mathrm{Ext}_{\Lambda_{\mathbb{F}_p}(\overline{\tau}_{n+1},\ldots)}(\mathbb{F}_p, \mathrm{BP}\langle n\rangle_*(M)) \to \mathrm{Ext}_{\Lambda_{\mathbb{F}_p}(\overline{\tau}_{n+1},\ldots)}(\mathbb{F}_p, H_*(M;\mathbb{F}_p)).$$

As argued above, the map $\mathrm{BP}\langle n\rangle\otimes M\to \mathbb{F}_p\otimes M$ has a retract, so the map $\mathrm{BP}\langle n\rangle_*(M)\to H_*(M;\mathbb{F}_p)$ is injective. Finally, $H_*(M;\mathbb{F}_p)$ is a trivial comodule over the Hopf algebra $\Lambda(\overline{\tau}_{n+1},\ldots)\simeq\Lambda(\sigma v_{n+1},\ldots)$, and hence so too is $\mathrm{BP}\langle n\rangle_*(M)$. It follows that the map on E_2-terms above is an injection, and that every class in $\mathrm{BP}\langle n\rangle_*(M)$ has a representative on the 0-line of the spectral sequence computing $\mathrm{BP}_*(M)$. It remains to show that these representative classes survive to the E_∞-page. By the above injectivity, it will suffice to prove that the spectral sequence

$$\mathrm{Ext}_{\Lambda_{\mathbb{F}_p}(\overline{\tau}_{n+1},\ldots)}(\mathbb{F}_p, H_*(M;\mathbb{F}_p)) \Longrightarrow (\mathrm{BP}/(p,\ldots,v_n))_*(M)$$

collapses at the E_2-page.

Observe that the property of a descent spectral sequence collapsing at the E_2-page is closed under direct sums, suspensions, and retracts. Since the descent spectral sequence for $\mathrm{BP}/(p,\ldots,v_n)$ collapses at the E_2-page, it will suffice to prove that $\mathrm{BP}/(p,\ldots,v_n)\otimes M$ is a direct summand of a finite direct sum of suspensions of $\mathrm{BP}/(p,\ldots,v_n)$.

Recall [Rav92, Lemma 6.2.6, Th. C.3.2] that Smith's complex X is obtained as a summand of a tensor power of a finite complex Y with cells in dimensions 2 through $2p^{n+1}$. It follows that $H_*(Y;\mathbb{F}_p)$ is a trivial comodule over $\Lambda(\overline{\tau}_{n+1},\ldots)$ and that, for dimension reasons, the Adams spectral sequence

$$\mathrm{Ext}_{\Lambda(\overline{\tau}_{n+1},\ldots)}(\mathbb{F}_p, H_*(Y;\mathbb{F}_p))\simeq H_*(Y;\mathbb{F}_p)[v_{n+1},\ldots]\Rightarrow(\mathrm{BP}/(p,\ldots,v_n))_*(Y)$$

collapses at the E_2-page. The E_2-page is a finite free module over $\mathbb{F}_p[v_{n+1},\ldots]$. Using any homotopy ring structure on $\mathrm{BP}/(p,\ldots,v_n)$ as a BP-module, we may then lift a basis to construct an equivalence between $\mathrm{BP}/(p,\ldots,v_n)\otimes Y$ and a finite direct sum of suspensions of $\mathrm{BP}/(p,\ldots,v_n)$.

Similarly, using any homotopy ring structure on $\mathrm{BP}/(p,\ldots,v_n)$, we deduce that both

$$\mathrm{BP}/(p,\ldots,v_n)\otimes(Y^{\otimes j}),\ \text{and}\ \mathrm{BP}/(p,\ldots,v_n)\otimes((\mathbb{D}Y)^{\otimes j})$$

are equivalent to finite direct sums of suspensions of $\mathrm{BP}/(p,\dots,v_n)$, as BP-modules. So we conclude that $\mathrm{BP}/(p,\dots,v_n)\otimes M$ is a summand of a finite direct sum of suspensions of $\mathrm{BP}/(p,\dots,v_n)$. This completes the proof. $\square$

We will, several times, use the following elementary lemma found in [HS98], which we recall for the reader's convenience.

LEMMA 6.2.2 ([HS98, Lemma 3.4]). *Suppose that x and y are commuting elements of a $\mathbb{Z}_{(p)}$-algebra. If $x-y$ is both torsion and nilpotent, then for $N\gg 0$,*

$$x^{p^N}=y^{p^N}.$$

Proof. Expand $(y+(x-y))^{p^N}$ and use that $p^k(x-y)=0$ for some k. $\square$

From Proposition 6.2.1(iii), we see that $\mathrm{THH}(\mathrm{BP}\langle n\rangle;\mathbb{F}_p)$ is a summand of

$$M\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)\simeq (M\otimes \mathrm{BP}\langle n\rangle)\otimes_{\mathrm{BP}\langle n\rangle}\mathrm{THH}(\mathrm{BP}\langle n\rangle),$$

arising from the unit map. In particular, there is a class that lifts $\sigma^2 v_{n+1}$ from $M\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$. We will need the following result ensuring the uniqueness and centrality of such lifts, up to taking large powers.

LEMMA 6.2.3. *If $x\in\pi_*(M\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle))$ is a lift of a power of $\sigma^2v_{n+1}\in\pi_*(M\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))$, then there is some $k\geq 0$ for which x^{p^k} is central. Moreover, if y is another such lift, then there are $j,j'\geq 0$ such that $x^{p^j}=y^{p^{j'}}$ and both elements are central.*

Proof. Let $F=\mathrm{End}(M)$, and denote by L_x and R_x the elements in homotopy corresponding to left and right multiplication by x, respectively. These elements commute, and their difference is nilpotent by Proposition 6.1.6. It follows from Lemma 6.2.2 that $L_x^{p^k}=R_x^{p^k}$ for some $k\geq 0$, and hence that x^{p^k} is central. For the second claim, first replace x and y by x^{p^k} and $y^{p^{k'}}$ so that x^{p^k} is central and both elements map to the same power of σ^2v_{n+1} inside $\pi_*(M\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))$. Then x and y are commuting elements and $x-y$ maps to zero in $\pi_*(M\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))$. By Proposition 6.1.6, $x-y$ is nilpotent and again Lemma 6.2.2 implies that $x^{p^j}=y^{p^j}$ for some $j\geq 0$. This completes the proof. $\square$

6.3. *Proof of canonical vanishing.*

THEOREM 6.3.1. *There are a v_{n+1}-element $v\in\pi_*M$ and an integer $d\geq 0$ such that, for all $0\leq k\leq\infty$, the map*

$$\pi_*(M/v\otimes \mathrm{can}):\pi_*(M/v\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}})\to\pi_*(M/v\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}})$$

is zero when $*\geq d$.

Remark 6.3.2. In the proof of the theorem and the lemmas below we will make use of the homotopy fixed point spectral sequence. If a group G acts on a spectrum X, then we will take the homotopy fixed point spectral sequence computing $\pi_*(X^{hG})$ to be the one associated to the tower $\{(\tau_{\geq j}X)^{hG}\}$ according to our conventions in Section C. However, it will be convenient to know that, for fixed s, an element $x \in \pi_*(X^{hG})$ is detected by a class in $E_2^{s',t}$ for some $s' > s$ if and only if x vanishes when restricted to $\mathrm{map}(\mathrm{sk}_s(EG)_+, X)^{hG}$. This follows from [GM95, Th. B.8].

LEMMA 6.3.3. *Let $v \in \pi_* M$ be the v_{n+1}-element from Proposition 6.2.1. Then v is detected in the homotopy fixed point spectral sequence for $M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1}$ in filtration at least $\frac{|v|}{p^{n+1}-1}$.*

Proof. Set $m = \frac{|v|}{2p^{n+1}-2}$. We need to prove that the image of v vanishes inside

$$Y := \lim_{\mathbb{C}P^{m-1}} M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle).$$

Since

$$M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle) \simeq (M \otimes \mathrm{BP}\langle n\rangle) \otimes_{\mathrm{BP}\langle n\rangle} \mathrm{THH}(\mathrm{BP}\langle n\rangle)$$

is a direct sum of shifts of $\mathbb{F}_p$, the skeletal filtration on $\mathbb{C}P^{m-1}$ gives rise to an Adams resolution of Y of length $m-1$. The claim now follows from Proposition 6.2.1(iv). ☐

LEMMA 6.3.4. *The homotopy fixed point spectral sequence converging to $\pi_*(\mathrm{M} \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1})$ collapses at the E_2-page.*

Proof. The E_2-page can be described as

$$\mathrm{BP}\langle n\rangle_*(M) \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_*[t]$$

By Proposition 6.2.1(v), the images of the equivariant maps

$$\mathrm{MU} \otimes M \to \mathrm{M} \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}),$$
$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}) \to \mathrm{M} \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$$

induce maps of spectral sequences whose images generate the E_2-page of the target as a ring. Every element in the homotopy fixed point spectral sequence for both $\mathrm{MU}^{hS^1} \otimes M$ and $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ is a permanent cycle, so the claim follows. ☐

LEMMA 6.3.5. *There is an element $z \in \pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))$ with the following properties*:

(i) *z is central*;
(ii) *z maps to a power of $\sigma^2 v_{n+1}$ inside $\pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))$*;

(iii) *for some* $m > 0$, $t^m z$, *in the* E_2*-term of the homotopy fixed point spectral sequence, detects the image of a central* v_{n+1}*-element from* $\pi_*(M)$ *inside* $\pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1})$;

(iv) $\pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))$ *is a finitely generated* $\mathbb{Z}_{(p)}[z]$*-module.*

Proof. First observe that each of these properties is preserved after replacing z by any power of itself, so we may do this at any time in the argument.

By Proposition 6.1.6 and Proposition 6.2.1(iii), we may choose $z \in \pi_*(M\otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))$ that lifts $\sigma^2 v_{n+1}$ and for which $\pi_*(M\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle))$ is a finitely generated $\mathbb{Z}_{(p)}[z]$-module. By Lemma 6.2.3, after replacing z by a power, we may assume that z is central as well. So we have chosen a z that satisfies (i), (ii), and (iv).

Let

$$f : M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle) \to M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$$

be the canonical map. Let us denote by $\{E'_r\}$ the homotopy fixed point spectral sequence computing $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hS^1}$ and by $\{E''_r\}$ the homotopy fixed point spectral sequence computing $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$. We will denote by $E_r(f) : E'_r \to E''_r$ the map induced by f.

By Theorem 5.0.1, we know that v_{n+1} is detected in E''_2 by $t(\sigma^2 v_{n+1})$. Let v denote a central v_{n+1}-element in $\pi_*(M)$, projected to $\pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))$. By the definition of a v_{n+1}-element, there is an $m > 0$ such that $f(v) = v_{n+1}^m$ modulo the ideal $(p, \ldots, v_n)$. Property (iii) in Proposition 6.2.1 guarantees that $M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ has $(p, \ldots, v_n) = 0$, and hence that v is detected by $t^m(\sigma^2 v_{n+1})^m$ in $E''_2 = E''_\infty$.

It follows that v cannot be detected in E'_2 in filtration higher than $2m$. By Lemma 6.3.3, v must be detected in E'_2 by a class in filtration at least $2m$. Say that v is detected by $t^m z'$, where $z' \in \pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$.

Then, since $E_2(f)(t^m z') = t^m(\sigma^2 v_{n+1})^m$, and $E''_2 = E''_\infty$, we must have that $f(z') = (\sigma^2 v_{n+1})^m$.

After replacing z and v by suitable powers, the result now follows from Lemma 6.2.3 applied to the elements z and z'. □

Remark 6.3.6. At height one and primes $p \geq 5$ a version of Lemma 6.3.5(iii) was obtained by Ausoni–Rognes in [AR02, Prop. 4.8].

Proof of Theorem 6.3.1. Fix v, z, and m as in the previous lemma. Let

$$X = \{\tau_{\geq j}(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))\}$$

denote the filtered spectrum corresponding to taking connective covers of $M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)$. We can choose a lift $\tilde{v} \in \pi_*(X^{\geq 2mp^{n+1}})^{hS^1}$ of v and form the cofibers in filtered spectra:

$$Y := X^{hC_{p^k}}/\tilde{v},\ Z := X^{tC_{p^k}}/\tilde{v}.$$

The filtered spectra Y and Z give spectral sequences converging to $M/v \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}}$ and $M/v \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$ respectively, and the canonical map $Y \to Z$ converges to the canonical map between these two spectra. Observe that the E_2-page of the spectral sequence for Y is concentrated in non-negative filtration (with our grading conventions). It will therefore suffice to prove that the E_2-page of the spectral for Z is eventually concentrated in negative filtration (uniformly in k). From the cofiber sequence (with suspensions and grading shifts omitted)

$$\mathrm{gr}(X)^{tC_{p^k}} \xrightarrow{t^m z} \mathrm{gr}(X)^{tC_{p^k}} \to \mathrm{gr}(Z),$$

it is enough to show that multiplication by z on $\pi_*(\mathrm{gr}(X)^{tC_{p^k}})$ is eventually an isomorphism in non-negative filtration. By [NS18, Lemma IV.4.12], we have $\mathrm{gr}(X)^{tC_{p^k}} = \mathrm{gr}(X)^{tS^1}/p^k$. So it suffices to prove that multiplication by z is eventually an isomorphism in non-negative filtration for the group

$$\pi_*(M \otimes \mathrm{THH}(\mathrm{BP}\langle n\rangle))[t^{\pm 1}].$$

But, more generally, if L is any finitely generated $\mathbb{Z}_{(p)}[z]$-module, then the analogous claim is true for $L[t^{\pm 1}]$. □

Appendix A. Suspension maps

Suppose R is an augmented (discrete) algebra over a field k with augmentation ideal I. Then there is a homomorphism of abelian groups

$$\sigma : I \to \mathrm{Tor}_1^R(k,k),$$

where σx is represented by the class $[x]$ in the bar complex. At various points in the paper we use a generalization of this construction to the spectrum level. Specifically, it is used in Section 2 to provide canonical lifts of elements in Künneth spectral sequences and, more crucially, in Section 5 in order to prove the Detection Theorem (Theorem 5.0.1). We make no claim of originality for the material in this appendix, though we were not able to find the Detection Lemma (Lemma A.4.1) in the literature.

Convention A.0.1. Throughout this section $\mathcal{C}$ will denote a stable, presentably symmetric monoidal category with unit object $\mathbf{1}$.

A.1. *Construction of suspension maps.* For the purposes of functoriality, it is convenient to construct our suspension maps in the setting of factorization homology. Let $\mathsf{Mfld}_n^{\mathrm{fr}}$ denote the category of framed n-manifolds as constructed in [AF15], equipped with its symmetric monoidal structure under disjoint unions. Let $\mathsf{Disk}_n^{\mathrm{fr}}$ be the full subcategory spanned by n-manifolds equivalent to disjoint unions of copies of $\mathbb{R}^n$. This category is equivalent to the symmetric monoidal envelope of the $\mathbb{E}_n$-operad. Factorization homology is then given by a functor

$$\int : \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{Fun}^{\otimes}(\mathsf{Disk}_n^{\mathrm{fr}}, \mathcal{C}) \longrightarrow \mathsf{Fun}^{\otimes}(\mathsf{Mfld}_n^{\mathrm{fr}}, \mathcal{C})$$

that is left adjoint to restriction. Here $\mathsf{Fun}^{\otimes}(-,-)$ denotes the category of symmetric monoidal functors.

Construction A.1.1 (Unreduced suspension). Since factorization homology is functorial on $\mathsf{Mfld}_n^{\mathrm{fr}}$, we always have an (unpointed) map of spaces:

$$\mathrm{Map}_{\mathsf{Mfld}_n^{\mathrm{fr}}}(N, M) \to \mathrm{Map}_{\mathcal{C}}\left(\int_N A, \int_M A\right).$$

If we set $N = \mathbb{R}^n$, then $\int_N A = A$, and the above is adjoint to a map

$$s^M : \mathrm{Map}_{\mathsf{Mfld}_n^{\mathrm{fr}}}(\mathbb{R}^n, M)_+ \otimes A = M_+ \otimes A \to \int_M A$$

that is functorial in M and A. Here we have used $X_+ \otimes (-)$ to denote the tensoring of $\mathcal{C}$ over the category of unpointed spaces.

We observe that, when $A = \mathbf{1}$, this map is canonically identified with the collapse

$$M_+ \otimes \mathbf{1} \to \mathbf{1} \simeq \int_M \mathbf{1}.$$

Construction A.1.2 (Suspension). Let M be a framed n-manifold equipped with a basepoint. From the previous construction, we have a functorial diagram

$$\begin{tikzcd} M_+ \otimes \mathbf{1} \arrow[r, "s^M"] \arrow[d] & \mathbf{1} \arrow[d] \\ M_+ \otimes A \arrow[r, "s^M"'] & \int_M A. \end{tikzcd}$$

The choice of basepoint provides a splitting of the top map and hence a commutative square (functorial in A and basepoint preserving maps in M):

$$\begin{tikzcd} M \otimes \mathbf{1} \arrow[r] \arrow[d] & 0 \arrow[d] \\ M \otimes A \arrow[r] & \int_M A. \end{tikzcd}$$

Thus we get a map from the pushout of the diagram with the lower right vertex deleted:

$$\sigma^M : M \otimes (A/\mathbf{1}) \to \int_M A,$$

where $A/\mathbf{1}$ denotes the cofiber of the unit map for A.

Remark A.1.3. If one instead used the transposed diagram

$$\begin{tikzcd} M \otimes \mathbf{1} \arrow[r] \arrow[d] & M \otimes A \arrow[d] \\ 0 \arrow[r] & \int_M A, \end{tikzcd}$$

then this would alter the definition of σ^M by -1. Since neither choice seems canonical, and we will often find ourselves rotating distinguishing triangles in the arguments below, we will mostly make and prove statements about σ^M only up to a factor of ± 1. For convenience and for the purposes of this paper, we will, in Section A.4, fix a choice so as to make a certain equation true on the nose rather than up to a factor of ± 1.

A.2. *Examples of suspension maps.*

Example A.2.1 (Dimension 0). If A is an $\mathbb{E}_0$-algebra, then we denote σ^{S^0} by σ, which is a map

$$\sigma : (A/\mathbf{1}) \to \int_{S^0} A = A \otimes A.$$

From the construction, σ comes as the induced map from the square

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d] & 0 \arrow[d] \\ A \arrow[r, "\mathrm{id}\otimes 1 - 1\otimes \mathrm{id}"'] & A \otimes A. \end{tikzcd}$$

Equivalently, we can describe this map (up to sign) as arising from the large square in the diagram

$$\begin{tikzcd} \Sigma^{-1}(A/\mathbf{1}) \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[d] \\ & \mathbf{1} \arrow[r] \arrow[d] & A \arrow[d, "1\otimes \mathrm{id}"] \\ 0 \arrow[r] & A \arrow[r, "\mathrm{id}\otimes 1"'] & A \otimes A. \end{tikzcd}$$

Variant A.2.2. Recall that the category $\mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})$ carries an action of C_2 given informally by sending an $\mathbb{E}_1$-algebra B to the algebra B^{op} equipped with the opposite multiplication.

Let $R \in \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C})^{hC_2}$ be an object in the fixed points so that, in particular, R comes equipped with an equivalence $\tau : R \simeq R^{\mathrm{op}}$. This induces an equivalence

$$(-)^\tau : \mathsf{LMod}_R(\mathcal{C}), \to \mathsf{RMod}_R(\mathcal{C})$$

which is the identity on underlying objects. Now let k be a left R-module equipped with a map $\mathbf{1} \to k$ in $\mathcal{C}$. We can extend this to a left R-module map $1_k : R \to k$ and to a right R-module map $1_k^\tau : R \to k^\tau$. Since τ is an equivalence, there is a canonical identification between the fibers

$$\mathrm{fib}(1_k) \simeq \mathrm{fib}(1_k^\tau),$$

and we denote either by I.

This is enough to make sense of the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
I \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[d] \\
& R \arrow[r, "1_k"] \arrow[d, "1_k^\tau"'] & k \arrow[d, "1_k^\tau \otimes \mathrm{id}"] \\
0 \arrow[r] & k \arrow[r, "\mathrm{id}\otimes 1_k"'] & k^\tau \otimes_R k.
\end{tikzcd}$$

Thus we may extend the definition of σ in this case to

$$\sigma : \Sigma I \to k^\tau \otimes_R k.$$

Lemma A.2.3 (Compatibility with Künneth spectral sequence). *Take $\mathcal{C} =$ Sp and adopt notation as in Variant A.2.2. Let $i : I \to R$ denote the fiber of $R \to k$. Suppose that the map*

$$1_k : \pi_* R \to \pi_* k$$

is surjective. Then, for any $x \in \pi_* I$, $\sigma(x) \in \pi_{*+1}(k^\tau \otimes_R k)$ is detected in the Künneth spectral sequence in filtration 1 by the class

$$[1 \otimes i(x) \otimes 1] \in \mathrm{Tor}_1^{\pi_* R}(\pi_* k, \pi_* k),$$

up to sign.

Proof. First we claim that the composite

$$\Sigma I \xrightarrow{\sigma} k^\tau \otimes_R k \to k^\tau \otimes_R \Sigma I$$

is homotopic, up to sign, to the map $1_k^\tau \otimes \mathrm{id}$. Indeed, consider the following diagram:

$$\begin{tikzcd}
& I \arrow[r] \arrow[d] \arrow[dl] & 0 \arrow[d] \arrow[ddr] & \\
& R \arrow[r] \arrow[d] & k \arrow[d] & \\
0 \arrow[r] & k \arrow[r] \arrow[d] & k \amalg_R k \arrow[r] \arrow[d] & k^\tau \otimes_R k \arrow[d] \\
& 0 \arrow[r] & \Sigma I \arrow[r] & k^\tau \otimes_R \Sigma I.
\end{tikzcd}$$

The vertices of the large trapezoid form the commutative square

$$\begin{tikzcd} I \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & k^\tau \otimes_R k \end{tikzcd}$$

used to define σ, and hence the induced map on the pushout of $(0 \leftarrow I \rightarrow 0)$ gives the desired factorization.

It follows from naturality of the Künneth spectral sequence that the map $R \otimes_R I = \Sigma I \to k^\tau \otimes_R I$ induces, on E^2-terms, the projection map

$$\pi_* I \to \mathrm{Tor}_0^{\pi_* R}(\pi_* k, \pi_* I) = \pi_* I/(\pi_* I)^2.$$

Finally, recall the construction of the Künneth spectral sequence for $M \otimes_R N$ proceeds by lifting a (graded) free $\pi_* R$-resolution of N to a filtration by left R-modules [Til18], and that this construction is natural in the resolution. Since

$$0 \to \pi_* I \to \pi_* R \to \pi_* k \to 0$$

is exact, we may choose a resolution C_* of $\pi_* k$ that begins with $C_0 = R$ and with the property that

$$\pi_* I \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$$

is a resolution of $\pi_* I$. Considering $\pi_* R$ as a complex concentrated in degree zero, the quotient map $C_* \to (C_*/R)$ can then be lifted to a map of filtered objects and then we may apply $k^\tau \otimes_R (-)$ to this map. This gives a map of spectral sequences that, on the E^2-page, gives the boundary map $\partial : \mathrm{Tor}_1^{\pi_* R}(\pi_* k, \pi_* k) \to \Sigma \mathrm{Tor}_0^{\pi_* R}(\pi_* k, \pi_* I)$, which is an isomorphism. The result follows. □

Example A.2.4 (Dimension 1). The circle acts on itself by framed maps, where we use the Lie group framing, and hence the map to Hochschild homology

$$s^{S^1} : S^1_+ \otimes A \to \int_{S^1} A = \mathrm{HH}(A)$$

is circle equivariant. Since the source of s^{S^1} is induced, the map must be induced from its restriction along the identity; i.e., s^{S^1} is adjoint to the non-equivariant map $A \to \mathrm{HH}(A)$ corresponding to the identity element in S^1. We abbreviate the reduced suspension map by $\sigma^2 : \Sigma(A/\mathbf{1}) \to \mathrm{HH}(A)$.

A.3. *Relationship with the cotangent complex.* Let A be an $\mathbb{E}_n$-algebra in $\mathcal{C}$. We will abbreviate by $\mathbf{L}_A^{(n)}$ the $\mathbb{E}_n$-algebra cotangent complex of A, which is an $\mathbb{E}_n$-A-module. Recall [Lur17, Th. 7.3.5.1] that we have a functorial cofiber sequence of $\mathbb{E}_n$-A-modules:

$$\mathcal{U}^{(n)}(A) \to A \to \Sigma^n \mathbf{L}_A^{(n)}.$$

Example A.3.1. When $n = 0$, this is the cofiber sequence

$$\mathbf{1} \to A \to (A/\mathbf{1}).$$

If $k \leq n$, we may apply $A \otimes_{\mathcal{U}^{(k-1)}} (-)$ to the cofiber sequence computing the $\mathbb{E}_{k-1}$-cotangent complex and get a cofiber sequence in $\mathcal{C}$:

$$A \to A \otimes_{\mathcal{U}^{(k-1)}(A)} A \simeq \mathcal{U}^{(k)}(A) \to A \otimes_{\mathcal{U}^{(k-1)}(A)} \Sigma^{k-1}\mathbf{L}_A^{(k-1)}.$$

This gives a functorial splitting of objects in $\mathcal{C}$,

$$A \oplus \Sigma^{k-1}\mathbf{L}_A^{(k)} \simeq_{\mathcal{C}} \mathcal{U}^{(k)}(A),$$

and an identification

$$\mathbf{L}_A^{(k)} \simeq_{\mathcal{C}} A \otimes_{\mathcal{U}^{(k-1)}(A)} \mathbf{L}_A^{(k-1)}.$$

(Here we have places a subscript on the equivalence to emphasize that this equivalence is not one of $\mathbb{E}_k$-A-modules).

LEMMA A.3.2. *For $k \leq n$, the following diagram commutes* (*up to sign*):

$$\begin{tikzcd} \Sigma^{k-1}A/\mathbf{1} \arrow[r, equal] \arrow[d, "\sigma^k"'] & \Sigma^{k-1}\mathbf{L}_A^{(0)} \arrow[d] \\ \mathcal{U}^{(k)}(A) \arrow[r] & \Sigma^{k-1}\mathbf{L}_A^{(k)}. \end{tikzcd}$$

Proof. We prove this by induction on k, the base case being trivial. For the inductive step, observe that, by functoriality of of σ^M in M, we have a diagram

$$\begin{tikzcd} 0 \arrow[r, equal] \arrow[d] & (\mathbb{R}^k) \otimes (A/\mathbf{1}) \arrow[d] & \Sigma^{k-1}(A/\mathbf{1}) \arrow[l] \arrow[r] \arrow[d, "\sigma^k"'] & (\mathbb{R}^k) \otimes (A/\mathbf{1}) \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\ A \arrow[r, equal] & \int_{\mathbb{R}^k} A & \int_{\mathbb{R}^k - \{0\}} A \arrow[l] \arrow[r] & \int_{\mathbb{R}^k} A \arrow[r, equal] & A. \end{tikzcd}$$

The induced map on the pushout is, on the one hand, given by σ^{k+1} and, on the other hand, by the inductive hypothesis, given by the composite

$$\Sigma^k(A/\mathbf{1}) \xrightarrow{\sigma^k} \Sigma\mathcal{U}^{(k)}(A) \to \Sigma^k\mathbf{L}_A^{(k)} \xrightarrow{\sigma} \mathcal{U}^{k+1}(A),$$

where σ is constructed as in Variant A.2.2.[5] On the other hand, as explained in the beginning of the proof of Lemma A.2.3, the composite

$$\Sigma^k\mathbf{L}_A^{(k)} \xrightarrow{\sigma} \mathcal{U}^{(k+1)}(A) \to \Sigma^k\mathbf{L}_A^{(k+1)} \simeq A \otimes_{\mathcal{U}^{(k)}(A)} \Sigma^k\mathbf{L}_A^{(k)}$$

is given by the map $1 \otimes \mathrm{id}$. This completes the proof. $\square$

[5]Notice that, in this inductive step, $k < n$, so $\mathcal{U}^{(k)}(A)$ is at least an $\mathbb{E}_2$-algebra, and hence the involution τ is trivial.

A.4. *Undoing suspension in Hochschild homology.* If $X \in \mathsf{Fun}(\mathrm{B}S^1, \mathcal{C})$ is an object equipped with an S^1-action, then we may compute the limit over $\mathbb{C}P^1 \subseteq \mathrm{B}S^1$ using the fiber sequence

$$\lim_{\mathbb{C}P^1} X \to X \to \Sigma^{-1} X,$$

where $X \to \Sigma^{-1}X$ is adjoint to the (reduced) action map $\Sigma X \to S^1_+ \otimes X \to X$. We denote the connecting homomorphism by

$$t : \Sigma^{-2} X \to \lim_{\mathbb{C}P^1} X.$$

Our goal in this section is to prove the following lemma, which allows us to use the circle action on Hochschild homology to "undo" the suspension.

LEMMA A.4.1 (Detection Lemma). *There is a functorial diagram*[6]

$$\begin{tikzcd}
\Sigma^{-1}(A/\mathbf{1}) \arrow[r] \arrow[d, "\sigma^2"'] & \mathbf{1} \arrow[d] \\
\Sigma^{-2}\mathrm{HH}(A) \arrow[r, "t"'] & \lim_{\mathbb{C}P^1} \mathrm{HH}(A).
\end{tikzcd}$$

Here, the map $\mathbf{1} \to \lim_{\mathbb{C}P^1} \mathrm{HH}(A)$ *arises from the* S^1*-equivariant map* $\mathbf{1} \to \mathrm{HH}(A)$ *where* $\mathbf{1}$ *has the trivial action.*

Proof. We have a diagram, functorial in A,

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d] & 0 \arrow[dd] \\
A \arrow[d] & \\
\mathrm{HH}(A) \arrow[r] & \Sigma^{-1}\mathrm{HH}(A).
\end{tikzcd}$$

We can left Kan extend to a diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[drr] & 0 \arrow[drrr] \arrow[dd] & & & \\
& & A \arrow[rr] \arrow[dd] & & \overline{A} \arrow[dd] \\
& 0 \arrow[drrr] & & & \\
& & \mathrm{HH}(A) \arrow[rr] & & \Sigma^{-1}\mathrm{HH}(A).
\end{tikzcd}$$

[6]Recall that the definition of the suspension map requires a choice, and that altering this choice multiplies the map by (-1). For the purposes of this paper, we will fix this choice so that the diagram in this lemma commutes.

The map $\overline{A} \to \Sigma^{-1}\mathrm{HH}(A)$ is adjoint to $\sigma : \Sigma\overline{A} \to \mathrm{HH}(A)$, since this latter map was constructed as the induced map on the pushout of the diagram

$$\begin{tikzcd}
\Sigma\mathbf{1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Sigma A \arrow[r] & \mathrm{HH}(A).
\end{tikzcd}$$

Now we may further right Kan extend to a diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[drr] \arrow[dd] & 0 \arrow[drr] \arrow[dd] & & \\
& & A \arrow[r] \arrow[dd] & \overline{A} \arrow[dd] \\
\lim_{\mathbb{C}P^1} \mathrm{HH}(A) \arrow[r] \arrow[drr] & 0 \arrow[drr] & & \\
& & \mathrm{HH}(A) \arrow[r] & \Sigma^{-1}\mathrm{HH}(A).
\end{tikzcd}$$

Here, the map $\mathbf{1} \to \lim_{\mathbb{C}P^1} \mathrm{HH}(A)$ arises from the canonical trivialization of the S^1-action $S^1_+ \otimes \mathbf{1} \to \mathbf{1} \simeq \mathrm{HH}(\mathbf{1})$.

We may view the above cube as a map of fiber sequences

$$\begin{tikzcd}
\mathbf{1} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \overline{A} \arrow[d] \\
\lim_{\mathbb{C}P^1} \mathrm{HH}(A) \arrow[r] & \mathrm{HH}(A) \arrow[r] & \Sigma^{-1}\mathrm{HH}(A),
\end{tikzcd}$$

which then yields the desired diagram:

$$\begin{tikzcd}
\Sigma^{-1}\overline{A} \arrow[r] \arrow[d, "\sigma^2"'] & \mathbf{1} \arrow[d] \\
\Sigma^{-2}\mathrm{HH}(A) \arrow[r, "t"'] & \lim_{\mathbb{C}P^1} \mathrm{HH}(A).
\end{tikzcd}$$

□

Appendix B. Recollections on graded objects

In this section we briefly review some facts about graded rings used in the body of the paper.

Definition B.0.1. Let k be an $\mathbb{E}_\infty$-ring. The category of graded k-modules is defined by

$$\mathsf{grMod}_k := \mathsf{Fun}(\mathbb{Z}^{\mathrm{ds}}, \mathsf{Mod}_k),$$

where $\mathbb{Z}^{\mathrm{ds}}$ denotes the integers viewed as a 0-category. The category grMod_k is a presentably symmetric monoidal category under Day convolution. If M is a graded k-module, we will denote by M_i its values at i, and by $M(n)$ the precomposition with addition by $-n$ (so that $M(n)_i = M_{i-n}$). We will refer to the grading as the *weight* throughout.

We will need various (co)connectivity conditions and finiteness conditions.

Definition B.0.2.

- Let A be an augmented, graded $\mathbb{E}_n$-k-algebra. We say that A is *weight-connected* (resp. *weight-coconnected*) if the fiber of the augmentation $A \to k$ is concentrated in positive grading (resp. negative grading). We denote the corresponding categories with superscripts wt-cn and wt-ccn, respectively.
- We denote by $\mathsf{grMod}_k^{\mathrm{wt}\geq n}$ (resp. $\mathsf{grMod}_k^{\mathrm{wt}\leq n}$) the full subcategory of graded k-modules concentrated in weights at least n (resp. at most n). We will write $M \geq n$ (resp. $M \leq n$) to indicate that M belongs to this subcategory.

Remark B.0.3. Observe that the map $-1 : \mathbb{Z}^{\mathrm{ds}} \to \mathbb{Z}^{\mathrm{ds}}$ is a symmetric monoidal equivalence, and hence induces a symmetric monoidal equivalence on the category of graded k-modules, algebras, etc. It follows that any result about weight-connected algebras, or modules of weight bounded below by n, has a counterpart for weight-coconnected algebras or modules of weight bounded above by $-n$.

Lemma B.0.4. *Let A be an augmented, graded k-algebra, and denote by $\overline{A}$ the fiber of the augmentation. Let $M \in \mathsf{LMod}_A$ and $N \in \mathsf{RMod}_A$. Then there is a filtration on $M \otimes_A N$,*

$$M \otimes N = F_0 \to F_1 \to \cdots \to \operatorname{colim} F_i = M \otimes_A N,$$

such that

$$\mathrm{gr}_i(M \otimes_A N) \simeq \Sigma^i M \otimes \overline{A}^{\otimes i} \otimes N.$$

Proof. The relative tensor product is computed by the geometric realization of the standard simplicial object with nth term $M \otimes A^{\otimes n} \otimes N$ ([Lur17, 4.4.2.8]). Hence, by the Dold–Kan correspondence ([Lur17, 1.2.4.1]), it is also computed as the colimit of a filtered object with associated graded corresponding to the normalized complex (which can be computed in the homotopy category), as indicated. □

Corollary B.0.5. *Let A be a weight connected algebra $L, N \in \mathsf{LMod}_A$ and $M \in \mathsf{RMod}_A$. If $M \geq \alpha$ and $N \geq \beta$, then $M \otimes_A N \geq \alpha+\beta$ and $(M \otimes_A N)_{\alpha+\beta} = M_\alpha \otimes N_\beta$.*

We will now study a natural filtration on the category of graded modules over a weight-connected algebra. If A is weight-connected, we denote by

$$\mathsf{LMod}_A^{\mathrm{wt}\geq j}, \mathsf{LMod}_A^{\mathrm{wt}\leq j} \subseteq \mathsf{LMod}_A$$

the full subcategories spanned by those modules that are concentrated in weights at least j and at most j, respectively.

Lemma B.0.6. *Let A be a weight-connected $\mathbb{E}_n$-k-algebra for some $1 \leq n \leq \infty$.*

(i) *The inclusion $\mathsf{LMod}_A^{\mathrm{wt}\geq j} \to \mathsf{LMod}_A$ admits a right adjoint, $(-)_{\geq j}$, computed as*

$$(M_{\geq j})_i = \begin{cases} M_i & i \geq j, \\ 0 & \textit{else.} \end{cases}$$

(ii) *The inclusion $\mathsf{LMod}_A^{\mathrm{wt}\leq j} \to \mathsf{LMod}_A$ admits a left adjoint, $(-)_{\leq j}$, computed as $M \mapsto M/M_{\geq j+1}$. In particular,*

$$(M_{\leq j})_i = \begin{cases} M_i & i \leq j, \\ 0 & \textit{else.} \end{cases}$$

(iii) *The subcategory $\mathsf{LMod}_A^{\mathrm{wt}\geq 0}$ inherits an $\mathbb{E}_{n-1}$-monoidal structure.*

(iv) *The localizations $(-)_{\leq m}$ are compatible with the $\mathbb{E}_{n-1}$-monoidal structure on $\mathsf{LMod}_A^{\mathrm{wt}\geq 0}$.*

(v) *The tower*

$$A \to \cdots \to A_{\leq m} \to A_{\leq m-1} \to \cdots \to k$$

of $\mathbb{E}_n$-k-algebras is a tower of square-zero extensions; i.e., we have pullback diagrams of $\mathbb{E}_n$-k-algebras:

$$\begin{tikzcd} A_{\leq m} \arrow[r] \arrow[d] & k \arrow[d] \\ A_{\leq m-1} \arrow[r] & k \oplus \Sigma A_m(m). \end{tikzcd}$$

Proof. The existence of these adjoints is immediate since the inclusions preserve all limits and colimits. To compute $M_{\geq j}$, observe that, for $i \geq j$, the A-module $A(i)$ is in weights at least j, since A is weight-connected. The homogeneous component $(M_{\geq j})_i$ is computed as the spectrum of maps of A-modules from $A(i)$ to $M_{\geq j}$ which, by the adjunction, is the same as the spectrum of maps from $A(i)$ to M, which is M_i. This proves (i).

Claim (ii) follows formally from the observation that, if $M \geq m+1$ and $N \leq m$, then every map $M \to N$ is zero.

Claim (iii) follows, using [Lur17, 2.2.1.1], from the fact that $\mathsf{LMod}_A^{\mathrm{wt}\geq 0}$ contains the unit and is closed under tensor products, by Corollary B.0.5.

For claim (iv), we must show that if $M \to M'$ is an equivalence in weights at most m, then so is $Z \otimes_A M \to Z \otimes_A M'$ and $M \otimes_A Z \to M \otimes_A Z'$ for $Z \geq 0$. Let F be the fiber of $M \to M'$ so that $F \geq m+1$. Then the result follows from Corollary B.0.5 applied to $Z \otimes_A F$ and $F \otimes_A Z$.

For the final claim, we need to produce a derivation, i.e., a map,

$$\mathbf{L}^{(n)}_{A_{\leq m-1}} \to \Sigma A_m(m)$$

refining the map $A_{\leq m-1} \to \mathrm{cofib}(A_{\leq m} \to A_{\leq m-1}) = A_m(m)$. Here $\mathbf{L}^{(n)}$ denotes the $\mathbb{E}_n$-cotangent complex. We will produce this refinement as a composite

$$\mathbf{L}^{(n)}_{A_{\leq m-1}} \to \mathbf{L}^{(n)}_{A_{\leq m-1}/A_{\leq m}} \to \Sigma A_m(m),$$

where the first map is the canonical one to the relative cotangent complex and the second is projection onto the first non-zero weight. By [Lur17, 7.5.3.1] applied to the $\mathbb{E}_n$-monoidal category $\mathsf{Mod}^{\mathbb{E}_n}_{A_{\leq m}}$ of $\mathbb{E}_n$-$A_{\leq m}$-modules, we can compute the relative cotangent complex using the cofiber sequence

$$\mathcal{U}^{(n)}_{A_{\leq m}}(A_{\leq m-1}) \to A_{\leq m-1} \to \Sigma^n \mathbf{L}^{(n)}_{A_{\leq m-1}/A_{\leq m}}$$

of $\mathbb{E}_n$-$A_{\leq m-1}$-modules. Using the recursive construction of the enveloping algebra, we are reduced to proving the following claim:

($*$) If $A \to B$ is a map of weight-connected $\mathbb{E}_1$-algebras with cofiber $C \geq j$, denote by C' the cofiber of $B \otimes_A B \to B$. Then $C' \geq j$ and $C'_j = \Sigma C_j$.

To prove ($*$), observe that the multiplication map admits a section so that $C' \simeq \Sigma B \otimes_A C$. The result now follows from Corollary B.0.5. □

Appendix C. Spectral sequences

In the body of the paper, we use various spectral sequences and maps of spectral sequences obtained by applying certain functors and natural transformations to towers. The purpose of this appendix is to check that these maneuvers produce convergent spectral sequences under certain conditions satisfied in the cases of interest.

Convention C.0.1. Throughout this section, $\mathcal{C}$ will denote a presentably symmetric monoidal stable category with a t-structure. We will assume that $\mathcal{C}$ satisfies the following properties (all of which are satisfied, for example, by modules over a connective $\mathbb{E}_\infty$-ring, equipped with an action of a group):

(i) The t-structure is compatible with filtered colimits; i.e., $\mathcal{C}_{\leq 0}$ is closed under filtered colimits.
(ii) The t-structure is left and right complete, which in this case is equivalent to saying that

$$\operatorname*{colim}_{n \to -\infty} \tau_{\leq n} X = 0 = \lim_{n \to \infty} \tau_{\geq n} X.$$

(iii) The t-structure is compatible with the symmetric monoidal structure; i.e., $\mathbf{1} \in \mathcal{C}_{\geq 0}$ and $X \otimes Y \in \mathcal{C}_{\geq n+m}$ whenever $X \in \mathcal{C}_{\geq n}$ and $Y \in \mathcal{C}_{\geq m}$.

C.1. *Towers and convergence.*

Convention C.1.1. Given a tower $\{X^{\geq s}\} \in \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$, we index the associated spectral sequence so that

$$E_2^{s,t} = \pi_{t-s}\mathrm{gr}^t X = \pi_{t-s}(\mathrm{cofib}(X^{\geq t+1} \to X^{\geq t})).$$

We write $X^{-\infty} := \operatorname{colim} X^{\geq s}$.

Warning C.1.2. There is not a typo here: we mean $\pi_{t-s}\mathrm{gr}^t$ and not $\pi_{t-s}\mathrm{gr}^s$. The latter would have differentials as in the E_1-term of a spectral sequence, whereas the former will behave as an E_2-term.

Definition C.1.3. Suppose that $\{X^{\geq s}\}$ is a tower with associated spectral sequence $\{E_r^{s,t}\}$. We say that E_r *converges conditionally* to $\pi_* X^{-\infty}$ if $\lim X^{\geq s} = 0$. We say that E_r *converges strongly* if the associated filtration $F^s(\pi_{t-s}X^{-\infty}) := \mathrm{im}(\pi_{t-s}X^{\geq 2t-s} \to \pi_{t-s}X^{-\infty})$ satisfies

$$\operatorname*{holim}_s F^s \pi_n X^{-\infty} = 0.$$

We will content ourselves below with establishing general conditions under which conditional convergence holds. In the body of the paper, when we claim that some spectral sequence actually converges strongly, it is because it also satisfies the conditions of Boardman's theorem [Boa99, Th. 7.1] for spectral sequences with entering differentials:

THEOREM C.1.4 (Boardman). *Suppose that E_r converges conditionally and that, for each fixed (s,t), there are only finitely many non-trivial differentials entering with target in the (s,t) spot. Thus, we eventually have $E_r^{s,t} \supseteq E_{r+1}^{s,t}$. Suppose further that $\lim^1_r E_r^{s,t} = 0$ for each (s,t). Then E_r converges strongly to $\pi_* X$.*

C.2. *Descent towers.* Let B be a connective, commutative algebra object in $\mathcal{C}$. Then we may form the descent tower functor (see, e.g., [BHS20, §B-C])

$$\mathrm{desc}_B : \mathcal{C} \to \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}),$$

which is lax symmetric monoidal and is specified by

$$\mathrm{desc}_B^{\geq j}(X) := \lim_{\Delta}(\tau_{\geq j}(X \otimes B^{\otimes \bullet+1})).$$

When X is bounded below, this yields a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(\pi_t(X) \to \pi_t(X \otimes B) \to \cdots) \Rightarrow \pi_{t-s}X_B^{\wedge},$$

where $X_B^{\wedge} = \lim(X \otimes B^{\otimes \bullet})$. When $\pi_*(B \otimes B)$ is flat over $\pi_* B$, we can further identify the E_2-term with Ext in the category of comodules over the Hopf algebroid $(\pi_* B, \pi_*(B \otimes B))$.

Remark C.2.1. When $\mathcal{C} = \mathsf{Mod}_A$ and B is regarded as a commutative algebra object through a map $A \to B$ of commutative algebras, we will often refer to the above procedure as "descent along the map $A \to B$."

Remark C.2.2. It is possible to make this construction with much weaker hypotheses (at the cost of losing some multiplicative structure). For example, the cobar construction and construction of the descent tower makes sense when $\mathcal{C} = \mathsf{Mod}_A$ for a connective $\mathbb{E}_2$-ring A and B a connective $\mathbb{E}_1$-A-algebra.

Remark C.2.3. The tower $\mathrm{desc}_B^{\geq *}(X)$ is not the usual Adams tower, but rather its *décalage* (compare [Del71, 1.3.3, 1.3.4] and [Hed20, §II]), which is why its associated graded has homotopy groups corresponding to the E_2-page of the Adams spectral sequence rather than the E_1-page.

Warning C.2.4. The descent tower shears the filtration in the Adams spectral sequence. If we fix $t - s = n$, then contributions to Adams filtration s come from $\mathrm{desc}^{\geq s+n}$. So, for example, a *horizontal* vanishing line on, say, the E_2-term of the Adams spectral sequence would correspond to behavior in the descent filtration that is more like a vanishing line of slope 1. Of course, if one is only interested in a finite range of values of n, there is no difference.

This story is especially well behaved when $\mathrm{fib}(\mathbf{1} \to B)$ is 1-connective.

PROPOSITION C.2.5. *Suppose that $I = \mathrm{fib}(\mathbf{1} \to B)$ lies in $\tau_{\geq 1}\mathcal{C}$. Then, for any d-connective object X, the descent tower has the following properties*:

(a) *The natural map $X \to X_B^\wedge$ is an equivalence.*
(b) *$E_2^{s,t}$ vanishes when $2s - t \geq d$.*
(c) *$\pi_n \mathrm{desc}_B^{\geq j}(X) = 0$ whenever $j \geq d + 2n$.*
(d) *For each k, there exists an N such that $\mathrm{desc}_B^{\geq j}(X)$ is k-connective for $j \geq N$.*

Proof. Since $\lim_j \mathrm{desc}_B^{\geq j}(X) = 0$, we can study the vanishing of the homotopy groups of each $\mathrm{desc}_B^{\geq j}(X)$ by establishing a vanishing range in the associated graded. Thus (b) $\Rightarrow$ (c) $\Rightarrow$ (d), so we need only establish (a) and (b). But these claims can be proven using the usual construction of the descent spectral sequence, via the tower $\{\mathrm{Tot}^{\leq s}(B^{\bullet+1} \otimes X)\}$, where the result is clear. $\square$

C.3. *Classical Adams spectral sequence.* The classical Adams spectral sequence, given by descent along $S^0 \to \mathbb{F}_p$, has slightly more involved convergence issues since the fiber of the unit map $S^0 \to \mathbb{F}_p$ is not 1-connective. We review the classical approach to getting around this issue and leverage this to understand the convergence behavior of the Tate fixed point spectral sequence below.

Throughout this section $\mathrm{desc}(-) = \mathrm{desc}_{\mathbb{F}_p}(-)$.

Construction C.3.1. Since $\mathrm{desc}(-)$ is lax symmetric monoidal, every descent tower is a module over $\mathrm{desc}(S^0)$. Recall that the element $p \in \pi_0(S^0)$ is detected in Adams filtration 1, and hence lifts to an element $v_0 \in \pi_0\mathrm{desc}^{\geq 1}(S^0)$. Thus, given any spectrum X, we have a natural map

$$v_0 : \mathrm{desc}(X)(1) \to \mathrm{desc}(X),$$

where, for a filtered spectrum Y, $Y(j)$ refers to the filtered spectrum with $Y(j)^{\geq s} = Y^{\geq s-j}$.

Remark C.3.2. The composite of the shift operator with v_0 is multiplication by p. It follows that v_0 induces multiplication by p on both $\mathrm{colim}\, X$ and $\lim X$.

Remark C.3.3. There is a canonical identification $\mathrm{desc}(X)/v_0 \simeq \mathrm{desc}(X/p)$. However, when $k \geq 2$, $\mathrm{desc}(X)/v_0^k$ and $\mathrm{desc}(X/p^k)$ differ. The former tower has E_2-term computed by the homotopy groups of an object of the derived category of $\mathcal{A}_*$-comodules that does not lie in the heart.

Proposition C.3.4. *Let X be d-connective. Then $\mathrm{desc}(X)/v_0^m$ has the property that, for each k, there is an N such that, for all $j \geq N$, $\mathrm{desc}^{\geq j}(X)/v_0^m$ is k-connective. Moreover, each term $\mathrm{desc}^{\geq j}(X)/v_0^m$ is d-connective.*

Proof. The conclusion about the tower is stable under extensions, so we are reduced to the case when $m = 1$ and $\mathrm{desc}(X)/v_0 = \mathrm{desc}(X/p)$. Since the tower is conditionally convergent, it suffices to establish a vanishing line on the E_2-page, and to show this is concentrated in stems starting in dimension d. The E_2-page is computed by $\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(H_*(X) \otimes \Lambda(\tau_0))$, which classically has the desired vanishing line. (See [Ada66, Th. 2.1] at the prime 2, and [Liu63, Prop. 2] at odd primes.) □

C.4. *Fixed point spectral sequences.* Given a tower X in the category of spectra with an action of a group G, we can take homotopy fixed points, orbits, or Tate fixed points levelwise and produce a new tower. In this section we establish some criteria for the conditional convergence of this tower.

Proposition C.4.1. *Suppose $X \in \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Fun}(\mathrm{BG}, \mathcal{C}))$ is conditionally convergent (i.e., $\lim X = 0$). Then so is X^{hG}.*

Proof. Limits commute with limits. □

The analogous result for Tate fixed points requires a proposition. We are grateful to the referee for pointing out the following result, which simplifies our earlier treatment of convergence in this section and the next.

Proposition C.4.2. *Let Y be a filtered G-spectrum that is uniformly bounded below. Then*

$$(\lim Y)^{tG} \to \lim(Y^{tG})$$

is an equivalence.

Proof. Without loss of generality we may assume that Y is uniformly 0-connective, and hence that $\lim Y$ is (-1)-connective. If Z is a G-spectrum, then Z_{hG} is computed as $\operatorname{colim}_{\Delta^{\mathrm{op}}}(G_+^{\otimes\bullet}\otimes Z)$. Denote by $\mathrm{sk}_r(Z_{hG})$ the colimit over $\Delta^{\mathrm{op}}_{\leq r}$. If Z is (-1)-connective, then the cofiber $(Z_{hG})/(\mathrm{sk}_r Z_{hG})$ is $(r-1)$-connective. We learn that, in the diagram

$$\begin{tikzcd} \mathrm{sk}_r(\lim Y)_{hG} \arrow[r] \arrow[d] & \lim \mathrm{sk}_r Y_{hG} \arrow[d] \\ (\lim Y)_{hG} \arrow[r] & \lim(Y_{hG}), \end{tikzcd}$$

the vertical arrows are an equivalence in a range increasing with r, and the upper horizontal arrow is always an equivalence since $\lim(-)$ commutes with finite colimits. It follows that

$$(\lim Y)_{hG} \to \lim(Y_{hG})$$

is an equivalence, and hence so is $(\lim Y)^{tG} \to \lim(Y^{tG})$. $\square$

COROLLARY C.4.3. *If Y is a uniformly bounded below, conditionally convergent tower, then Y^{tG} is also conditionally convergent.*

C.5. *Hochschild homology of filtered rings.* If A is a filtered $\mathbb{E}_1$-ring, one can construct a corresponding filtration of $\mathrm{THH}(A)$ and spectral sequence (see [AKS18]). We will need to understand how this spectral sequence interacts with the Tate-valued Frobenius, and for this we need a construction of $\mathrm{THH}(A)$ as a filtered cyclotomic object. We refer the reader to [AMMN21, §A] for details and review the relevant definitions here.

Definition C.5.1. Let $L_p : \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Sp}) \to \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Sp})$ denote left Kan extension along multiplication by p.

PROPOSITION C.5.2 ([AMMN21, §A]). *Let A be a filtered or graded $\mathbb{E}_1$-ring. Then $\mathrm{THH}(A)$ admits a natural L_p-twisted diagonal, that is, an S^1-equivariant map*

$$\varphi : L_p\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}.$$

In the filtered case, this map is compatible with passage to the associated graded and, in both cases, the map is compatible with forgetting to underlying objects.

Remark C.5.3. Since L_p is adjoint to restriction along multiplication by p, the Frobenius gives S^1-equivariant maps

$$\varphi : \mathrm{THH}(A)^{\geq j} \to (\mathrm{THH}(A)^{tC_p})^{\geq jp}$$

for all j, and similarly for the graded case.

In particular, this produces maps of spectral sequences (which shear the gradings). We will be using these spectral sequences in the case when we are filtering A by its descent tower for $S^0 \to \mathbb{F}_p$. The following proposition guarantees convergence (after p-completion) when A is connective.

Proposition C.5.4. *Let A be a connective $\mathbb{E}_1$-ring. Then, for each $1 \leq m \leq \infty$, the tower* $\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}(A))^{tC_{p^m}}$ *converges conditionally to* $\mathrm{THH}(A)^{tC_{p^m}}$. *The tower* $\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}(A))^{\wedge}_{v_0}$ *converges conditionally to* $\mathrm{THH}(A)^{\wedge}_p$.

Proof. Since $\mathrm{THH}(\mathrm{desc}(A))$ is uniformly bounded below, we have

$$\lim \mathrm{THH}(\mathrm{desc}(A))^{tC_{p^m}} = (\lim \mathrm{THH}(\mathrm{desc}(A)))^{tC_{p^m}}$$

by virtue of Proposition C.4.2.

On bounded below C_{p^k}-spectra Z we have that $Z^{tC_{p^k}} = (Z^{\wedge}_p)^{tC_{p^k}}$ (see [NS18, Lemma I.2.9, Lemma II.4.9]). So to prove conditional convergence it will suffice to show that

$$(\lim \mathrm{THH}(\mathrm{desc}(A)))^{\wedge}_p = 0.$$

We have

$$(\lim \mathrm{THH}(\mathrm{desc}(A)))/p = \lim(\mathrm{THH}(\mathrm{desc}(A))/v_0),$$

so it suffices to show that this vanishes. This, in turn, will prove the second claim that $\mathrm{THH}(\mathrm{desc}(A))^{\wedge}_{v_0}$ converges to $\mathrm{THH}(A)^{\wedge}_p$.

We recall that $\mathrm{THH}(\mathrm{desc}(A))$ is defined as the geometric realization of a simplicial object with terms $\mathrm{desc}(A)^{\otimes \bullet+1}$, and therefore $\mathrm{THH}(\mathrm{desc}A)/v_0$ is computed as the geometric realization of a simplicial object with terms $\mathrm{desc}(A)/v_0 \otimes \mathrm{desc}(A)^{\otimes \bullet}$. Observe that, if $Z_\bullet$ is any simplicial spectrum with each Z_i connective, then, by the Dold–Kan correspondence, $\mathrm{sk}_r|Z_\bullet|/\mathrm{sk}_{r-1}|Z_\bullet|$ is a summand of $\Sigma^r Z_r$ and hence must be r-connective. Thus, to check that $\lim \mathrm{THH}(\mathrm{desc}(A))/v_0 = 0$ we need only check that $\lim \mathrm{sk}_r \mathrm{THH}(\mathrm{desc}(A))/v_0 = 0$ for all r. Since this skeleton is a finite colimit, we are reduced to checking that $\lim(\mathrm{desc}(A)/v_0 \otimes \mathrm{desc}(A)^{\otimes n}) = 0$ for all n. In fact, the terms in this tower become increasingly connective by Proposition C.3.4, so the result is proved. ☐

References

[Ada66] J. F. Adams, A periodicity theorem in homological algebra, *Proc. Cambridge Philos. Soc.* **62** (1966), 365–377. MR 0194486. Zbl 0163.01602. https://doi.org/10.1017/s0305004100039955.

[AGM85] J. F. Adams, J. H. Gunawardena, and H. Miller, The Segal conjecture for elementary abelian p-groups, *Topology* **24** no. 4 (1985), 435–460. MR 0816524. Zbl 0611.55010. https://doi.org/10.1016/0040-9383(85)90014-X.

[AK21] G. Angelini-Knoll, Detecting β elements in iterated algebraic K-theory, 2021. arXiv 1810.10088.

[AKQ21a] G. Angelini-Knoll and J. D. Quigley, Chromatic complexity of the algebraic K-theory of $y(n)$, 2021. arXiv 1908.09164.

[AKQ21b] G. Angelini-Knoll and J. D. Quigley, The Segal conjecture for topological Hochschild homology of Ravenel spectra, *J. Homotopy Relat. Struct.* **16** no. 1 (2021), 41–60. MR 4225506. Zbl 1467.55007. https://doi.org/10.1007/s40062-021-00275-7.

[AKS18] G. Angelini-Knoll and A. Salch, A May-type spectral sequence for higher topological Hochschild homology, *Algebr. Geom. Topol.* **18** no. 5 (2018), 2593–2660. MR 3848395. Zbl 1410.18016. https://doi.org/10.2140/agt.2018.18.2593.

[Ang08] V. Angeltveit, Topological Hochschild homology and cohomology of A_∞ ring spectra, *Geom. Topol.* **12** no. 2 (2008), 987–1032. MR 2403804. Zbl 1149.55006. https://doi.org/10.2140/gt.2008.12.987.

[AHL10] V. Angeltveit, M. A. Hill, and T. Lawson, Topological Hochschild homology of ℓ and k_o, *Amer. J. Math.* **132** no. 2 (2010), 297–330. MR 2654776. Zbl 1271.55009. https://doi.org/10.1353/ajm.0.0105.

[AL17] V. Angeltveit and J. A. Lind, Uniqueness of $BP\langle n\rangle$, *J. Homotopy Relat. Struct.* **12** no. 1 (2017), 17–30. MR 3613020. Zbl 1379.55007. https://doi.org/10.1007/s40062-015-0120-0.

[AR05] V. Angeltveit and J. Rognes, Hopf algebra structure on topological Hochschild homology, *Algebr. Geom. Topol.* **5** (2005), 1223–1290. MR 2171809. Zbl 1087.55009. https://doi.org/10.2140/agt.2005.5.1223.

[AMMN21] B. Antieau, A. Mathew, M. Morrow, and T. Nikolaus, On the Beilinson fiber square, 2021. arXiv 2003.12541.

[AN21] B. Antieau and T. Nikolaus, Cartier modules and cyclotomic spectra, *J. Amer. Math. Soc.* **34** no. 1 (2021), 1–78. MR 4188814. Zbl 1467.14058. https://doi.org/10.1090/jams/951.

[Aus05] C. Ausoni, Topological Hochschild homology of connective complex K-theory, *Amer. J. Math.* **127** no. 6 (2005), 1261–1313. MR 2183525. Zbl 1107.55006. https://doi.org/10.1353/ajm.2005.0036.

[Aus10] C. Ausoni, On the algebraic K-theory of the complex K-theory spectrum, *Invent. Math.* **180** no. 3 (2010), 611–668. MR 2609252. Zbl 1204.19002. https://doi.org/10.1007/s00222-010-0239-x.

[AR20] C. Ausoni and B. Richter, Towards topological Hochschild homology of Johnson-Wilson spectra, *Algebr. Geom. Topol.* **20** no. 1 (2020), 375–393. MR 4071375. Zbl 1437.55009. https://doi.org/10.2140/agt.2020.20.375.

[AR02] C. Ausoni and J. Rognes, Algebraic K-theory of topological K-theory, *Acta Math.* **188** no. 1 (2002), 1–39. MR 1947457. Zbl 1019.18008. https://doi.org/10.1007/BF02392794.

[AR12a] C. Ausoni and J. Rognes, Algebraic K-theory of the first Morava K-theory, *J. Eur. Math. Soc. (JEMS)* **14** no. 4 (2012), 1041–1079. MR 2928844. Zbl 1253.19001. https://doi.org/10.4171/JEMS/326.

[AR12b] C. Ausoni and J. Rognes, Rational algebraic K-theory of topological K-theory, *Geom. Topol.* **16** no. 4 (2012), 2037–2065. MR 2975299. Zbl 1260.19004. https://doi.org/10.2140/gt.2012.16.2037.

[AF15] D. Ayala and J. Francis, Factorization homology of topological manifolds, *J. Topol.* **8** no. 4 (2015), 1045–1084. MR 3431668. Zbl 1350.55009. https://doi.org/10.1112/jtopol/jtv028.

[BDR04] N. A. Baas, B. I. Dundas, and J. Rognes, Two-vector bundles and forms of elliptic cohomology, in *Topology, Geometry and Quantum Field Theory*, *London Math. Soc. Lecture Note Ser.* **308**, Cambridge Univ. Press, Cambridge, 2004, pp. 18–45. MR 2079370. Zbl 1106.55004. https://doi.org/10.1017/CBO9780511526398.005.

[BJ02] A. Baker and A. Jeanneret, Brave new Hopf algebroids and extensions of MU-algebras, *Homology Homotopy Appl.* **4** no. 1 (2002), 163–173. MR 1937961. Zbl 1380.55009. https://doi.org/10.4310/hha.2002.v4.n1.a9.

[BM13] M. Basterra and M. A. Mandell, The multiplication on BP, *J. Topol.* **6** no. 2 (2013), 285–310. MR 3065177. Zbl 1317.55005. https://doi.org/10.1112/jtopol/jts032.

[BCS10] A. J. Blumberg, R. L. Cohen, and C. Schlichtkrull, Topological Hochschild homology of Thom spectra and the free loop space, *Geom. Topol.* **14** no. 2 (2010), 1165–1242. MR 2651551. Zbl 1219.19006. https://doi.org/10.2140/gt.2010.14.1165.

[BGT14] A. J. Blumberg, D. Gepner, and G. Tabuada, Uniqueness of the multiplicative cyclotomic trace, *Adv. Math.* **260** (2014), 191–232. MR 3209352. Zbl 1297.19002. https://doi.org/10.1016/j.aim.2014.02.004.

[BM08] A. J. Blumberg and M. A. Mandell, The localization sequence for the algebraic K-theory of topological K-theory, *Acta Math.* **200** no. 2 (2008), 155–179. MR 2413133. Zbl 1149.18008. https://doi.org/10.1007/s11511-008-0025-4.

[Boa99] J. M. Boardman, Conditionally convergent spectral sequences, in *Homotopy Invariant Algebraic Structures* (Baltimore, MD, 1998), *Contemp. Math.* **239**, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84. MR 1718076. Zbl 0947.55020. https://doi.org/10.1090/conm/239/03597.

[BM94] M. Bökstedt and I. Madsen, Topological cyclic homology of the integers, in K-*Theory* (Strasbourg, 1992), *Astérisque* **226**, 1994, pp. 57–143. MR 1317117. Zbl 0816.19001. Available at http://www.numdam.org/item/AST_1994__226__57_0/.

[Bou79] A. K. Bousfield, The localization of spectra with respect to homology, *Topology* **18** no. 4 (1979), 257–281. MR 0551009. Zbl 0417.55007. https://doi.org/10.1016/0040-9383(79)90018-1.

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, H_∞ *Ring Spectra and their Applications*, *Lecture Notes in Math.* **1176**, Springer-Verlag, Berlin, 1986. MR 0836132. Zbl 0585.55016. https://doi.org/10.1007/BFb0075405.

[BHS20] R. Burklund, J. Hahn, and A. Senger, Galois reconstruction of Artin-Tate $\mathbb{R}$-motivic spectra, 2020. arXiv 2010.10325.

[CSY21] S. Carmeli, T. M. Schlank, and L. Yanovski, Ambidexterity and height, *Adv. Math.* **385** (2021), Paper No. 107763, 90. MR 4246977. Zbl 1466.18014. Available at https://doi.org/10.1016/j.aim.2021.107763.

[CMNN20] D. Clausen, A. Mathew, N. Naumann, and J. Noel, Descent and vanishing in chromatic algebraic K-theory via group actions, 2020. arXiv 2011.08233.

[Del71] P. Deligne, Théorie de Hodge. II, *Inst. Hautes Études Sci. Publ. Math.* no. 40 (1971), 5–57. MR 0498551. Zbl 0219.14007. Available at http://www.numdam.org/item?id=PMIHES_1971__40__5_0.

[DGM13] B. I. Dundas, T. G. Goodwillie, and R. McCarthy, *The Local Structure of Algebraic* K*-Theory*, *Algebra Appl.* **18**, Springer-Verlag London, Ltd., London, 2013. MR 3013261. Zbl 1272.55002. https://doi.org/10.1007/978-1-4471/4393-2.

[GKRW21] S. Galatius, A. Kupers, and O. Randal-Williams, Cellular E_k-algebras, 2021. arXiv 1805.07184.

[GL20] S. Glasman and T. Lawson, Stable power operations, 2020. arXiv 2002.02035.

[GM95] J. P. C. Greenlees and J. P. May, Generalized Tate cohomology, *Mem. Amer. Math. Soc.* **113** no. 543 (1995), viii+178. MR 1230773. Zbl 0876.55003. https://doi.org/10.1090/memo/0543.

[HNP19] Y. Harpaz, J. Nuiten, and M. Prasma, Tangent categories of algebras over operads, *Israel J. Math.* **234** no. 2 (2019), 691–742. MR 4040842. Zbl 1435.55010. https://doi.org/10.1007/s11856-019-1933-z.

[Hed20] A. P. Hedenlund, Multiplicative Tate spectral sequences, 2020, Thesis (Ph.D.)–Univ. of Oslo; available on John Rognes' website.

[Hes18] L. Hesselholt, Topological Hochschild homology and the Hasse-Weil zeta function, in *An Alpine Bouquet of Algebraic Topology*, *Contemp. Math.* **708**, Amer. Math. Soc., Providence, RI, 2018, pp. 157–180. MR 3807755. Zbl 1402.19002. https://doi.org/10.1090/conm/708/14264.

[HM03] L. Hesselholt and I. Madsen, On the K-theory of local fields, *Ann. of Math.* (2) **158** no. 1 (2003), 1–113. MR 1998478. Zbl 1033.19002. https://doi.org/10.4007/annals.2003.158.1.

[HM04] L. Hesselholt and I. Madsen, On the de Rham-Witt complex in mixed characteristic, *Ann. Sci. École Norm. Sup.* (4) **37** no. 1 (2004), 1–43. MR 2050204. Zbl 1062.19003. https://doi.org/10.1016/j.ansens.2003.06.001.

[HL10] M. Hill and T. Lawson, Automorphic forms and cohomology theories on Shimura curves of small discriminant, *Adv. Math.* **225** no. 2 (2010), 1013–1045. MR 2671186. Zbl 1220.55006. https://doi.org/10.1016/j.aim.2010.03.009.

[HPS99] M. J. Hopkins, J. H. Palmieri, and J. H. Smith, Vanishing lines in generalized Adams spectral sequences are generic, *Geom. Topol.* **3** (1999), 155–165. MR 1697180. Zbl 0920.55020. https://doi.org/10.2140/gt.1999.3.155.

[HS98] M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. II, *Ann. of Math.* (2) **148** no. 1 (1998), 1–49. MR 1652975. Zbl 0924.55010. https://doi.org/10.2307/120991.

[Hov95] M. Hovey, Bousfield localization functors and Hopkins' chromatic splitting conjecture, in *The Čech Centennial* (Boston, MA, 1993), *Contemp. Math.* **181**, Amer. Math. Soc., Providence, RI, 1995, pp. 225–250. MR 1320994. Zbl 0830.55004. https://doi.org/10.1090/conm/181/02036.

[Hov97] M. A. Hovey, v_n-elements in ring spectra and applications to bordism theory, *Duke Math. J.* **88** no. 2 (1997), 327–356. MR 1455523. Zbl 0880.55006. https://doi.org/10.1215/S0012-7094-97-08813-X.

[KLW18] N. Kitchloo, V. Lorman, and W. S. Wilson, Multiplicative structure on real Johnson-Wilson theory, in *New Directions in Homotopy Theory*, *Contemp. Math.* **707**, Amer. Math. Soc., Providence, RI, 2018, pp. 31–44. MR 3807740. Zbl 1404.55015. https://doi.org/10.1090/conm/707/14252.

[Koc73] S. O. Kochman, Homology of the classical groups over the Dyer-Lashof algebra, *Trans. Amer. Math. Soc.* **185** (1973), 83–136. MR 0331386. Zbl 0271.57013. https://doi.org/10.2307/1996429.

[LMMT22] M. Land, A. Mathew, L. Meier, and G. Tamme, Purity in chromatically localized algebraic K-theory, 2022. arXiv 2001.10425.

[Law18] T. Lawson, Secondary power operations and the Brown-Peterson spectrum at the prime 2, *Ann. of Math.* (2) **188** no. 2 (2018), 513–576. MR 3862946. Zbl 1431.55011. https://doi.org/10.4007/annals.2018.188.2.3.

[LN12] T. Lawson and N. Naumann, Commutativity conditions for truncated Brown-Peterson spectra of height 2, *J. Topol.* **5** no. 1 (2012), 137–168. MR 2897051. Zbl 1280.55007. https://doi.org/10.1112/jtopol/jtr030.

[Laz01] A. Lazarev, Homotopy theory of A_∞ ring spectra and applications to MU-modules, *K-Theory* **24** no. 3 (2001), 243–281. MR 1876800. Zbl 1008.55007. https://doi.org/10.1023/A:1013394125552.

[Lin80] W. H. Lin, On conjectures of Mahowald, Segal and Sullivan, *Math. Proc. Cambridge Philos. Soc.* **87** no. 3 (1980), 449–458. MR 0556925. Zbl 0455.55007. https://doi.org/10.1017/S0305004100056887.

[Liu63] A. Liulevicius, Zeroes of the cohomology of the Steenrod algebra, *Proc. Amer. Math. Soc.* **14** (1963), 972–976. MR 0157383. Zbl 0148.17102. https://doi.org/10.2307/2035037.

[LN05] S. Lunøe-Nielsen, The Segal conjecture for topological Hochschild homology of commutative S-algebras, 2005. Available at http://urn.nb.no/URN:NBN:no-98057.

[LNR11] S. Lunøe-Nielsen and J. Rognes, The Segal conjecture for topological Hochschild homology of complex cobordism, *J. Topol.* **4** no. 3 (2011), 591–622. MR 2832570. Zbl 1229.55006. https://doi.org/10.1112/jtopol/jtr015.

[Lur15] J. Lurie, Rotation invariance in algebraic K-theory, 2015, available on author's webpage.

[Lur17] J. Lurie, Higher algebra, 2017, available on author's webpage.

[Lur18] J. Lurie, Spectral algebraic geometry, 2018, available on author's webpage.

[MR99] M. Mahowald and C. Rezk, Brown-Comenetz duality and the Adams spectral sequence, *Amer. J. Math.* **121** no. 6 (1999), 1153–1177. MR 1719751. Zbl 0942.55012. https://doi.org/10.1353/ajm.1999.0043.

[Mat18] A. Mathew, Examples of descent up to nilpotence, in *Geometric and Topological Aspects of the Representation Theory of Finite Groups*, *Springer Proc. Math. Stat.* **242**, Springer, Cham, 2018, pp. 269–311. MR 3901164. https://doi.org/10.1007/978-3-319-94033-5_11.

[Mat21] A. Mathew, On $K(1)$-local TR, *Compos. Math.* **157** no. 5 (2021), 1079–1119. MR 4256236. Zbl 1471.19002. https://doi.org/10.1112/S0010437X21007144.

[MNN15] A. Mathew, N. Naumann, and J. Noel, On a nilpotence conjecture of J. P. May, *J. Topol.* **8** no. 4 (2015), 917–932. MR 3431664. Zbl 1335.55009. https://doi.org/10.1112/jtopol/jtv021.

[MNN17] A. Mathew, N. Naumann, and J. Noel, Nilpotence and descent in equivariant stable homotopy theory, *Adv. Math.* **305** (2017), 994–1084. MR 3570153. Zbl 1420.55024. https://doi.org/10.1016/j.aim.2016.09.027.

[May75] J. P. May, Problems in infinite loop space theory, in *Conference on Homotopy Theory* (*Evanston, Ill.*, 1974), *Notas Mat. Simpos.* **1**, Soc. Mat. Mexicana, México, 1975, pp. 111–125. MR 0761724. Zbl 0332.55007.

[MW81] H. Miller and C. Wilkerson, Vanishing lines for modules over the Steenrod algebra, *J. Pure Appl. Algebra* **22** no. 3 (1981), 293–307. MR 0629336. Zbl 0469.55012. https://doi.org/10.1016/0022-4049(81)90104-3.

[Mil58] J. Milnor, The Steenrod algebra and its dual, *Ann. of Math.* (2) **67** (1958), 150–171. MR 0099653. Zbl 0080.38003. https://doi.org/10.2307/1969932.

[Mit90] S. A. Mitchell, The Morava K-theory of algebraic K-theory spectra, *K-Theory* **3** no. 6 (1990), 607–626. MR 1071898. Zbl 0709.55011. https://doi.org/10.1007/BF01054453.

[Mor89] J. Morava, Forms of K-theory, *Math. Z.* **201** no. 3 (1989), 401–428. MR 0999737. Zbl 0709.55003. https://doi.org/10.1007/BF01214905.

[Nav10] L. S. Nave, The Smith-Toda complex $V((p+1)/2)$ does not exist, *Ann. of Math.* (2) **171** no. 1 (2010), 491–509. MR 2630045. Zbl 1194.55017. https://doi.org/10.4007/annals.2010.171.491.

[NS18] T. Nikolaus and P. Scholze, On topological cyclic homology, *Acta Math.* **221** no. 2 (2018), 203–409. MR 3904731. Zbl 1457.19007. https://doi.org/10.4310/ACTA.2018.v221.n2.a1.

[Qui69] D. Quillen, On the formal group laws of unoriented and complex cobordism theory, *Bull. Amer. Math. Soc.* **75** (1969), 1293–1298. MR 0253350. Zbl 0199.26705. https://doi.org/10.1090/S0002-9904-1969-12401-8.

[Qui75] D. Quillen, Higher algebraic K-theory, in *Proceedings of the International Congress of Mathematicians, Vol.* 1 (Vancouver, B. C., 1974), 1975, pp. 171–176. MR 0422392. Zbl 0359.18014.

[Rav86] D. C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, *Pure Appl. Math.* **121**, Academic Press, Inc., Orlando, FL, 1986. MR 0860042. Zbl 0608.55001.

[Rav92] D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, *Annals of Mathematics Studies* **128**, Princeton Univ. Press, Princeton, NJ, 1992, Appendix C by Jeff Smith. MR 1192553. Zbl 0774.55001. https://doi.org/10.1515/9781400882489.

[Ric06] B. Richter, A lower bound for coherences on the Brown-Peterson spectrum, *Algebr. Geom. Topol.* **6** (2006), 287–308. MR 2199461. Zbl 1095.55005. https://doi.org/10.2140/agt.2006.6.287.

[Rog99] J. Rognes, Algebraic K-theory of the two-adic integers, *J. Pure Appl. Algebra* **134** no. 3 (1999), 287–326. MR 1663391. Zbl 0929.19004. https://doi.org/10.1016/S0022-4049(97)00156-4.

[Rog00] J. Rognes, Algebraic K-theory of finitely presented ring spectra, 2000, Available on author's webpage.

[Rog14] J. Rognes, Chromatic redshift, 2014. arXiv 1403.4838.

[Sen22] A. Senger, The Brown-Peterson spectrum is not $E_{2(p^2+2)}$ at odd primes, 2022. arXiv 1710.09822.

[Str99] N. P. Strickland, Products on MU-modules, *Trans. Amer. Math. Soc.* **351** no. 7 (1999), 2569–2606. MR 1641115. Zbl 0924.55005. https://doi.org/10.1090/S0002-9947-99-02436-8.

[Tho82] R. W. Thomason, The Lichtenbaum-Quillen conjecture for $K/\ell_*[\beta^{-1}]$, in *Current Trends in Algebraic Topology, Part* 1 (*London, Ont.*, 1981), *CMS Conf. Proc.* **2**, Amer. Math. Soc., Providence, R.I., 1982, pp. 117–139. MR 0686114. Zbl 0563.14009.

[Til18] S. Tilson, Power operations in the Künneth spectral sequence and commutative $H\mathbb{F}_p$-algebras, 2018. arXiv 1602.06736.

[Vee18] T. Veen, Detecting periodic elements in higher topological Hochschild homology, *Geom. Topol.* **22** no. 2 (2018), 693–756. MR 3748678. Zbl 1384.55007. https://doi.org/10.2140/gt.2018.22.693.

[Voe03] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2$-coefficients, *Publ. Math. Inst. Hautes Études Sci.* no. 98 (2003), 59–104. MR 2031199. Zbl 1057.14028. https://doi.org/10.1007/s10240-003-0010-6.

[Voe11] V. Voevodsky, On motivic cohomology with $\mathbf{Z}/l$-coefficients, *Ann. of Math.* (2) **174** no. 1 (2011), 401–438. MR 2811603. Zbl 1236.14026. https://doi.org/10.4007/annals.2011.174.1.11.

[Wal84] F. Waldhausen, Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy, in *Algebraic Topology* (Aarhus, 1982), *Lecture Notes in Math.* **1051**, Springer, Berlin, 1984, pp. 173–195. MR 0764579. Zbl 0562.55002. https://doi.org/10.1007/BFb0075567.

[Wes17] C. Westerland, A higher chromatic analogue of the image of J, *Geom. Topol.* **21** no. 2 (2017), 1033–1093. MR 3626597. Zbl 1371.19003. https://doi.org/10.2140/gt.2017.21.1033.

[Wil75] W. S. Wilson, The Ω-spectrum for Brown-Peterson cohomology. II, *Amer. J. Math.* **97** (1975), 101–123. MR 0383390. Zbl 0303.55003. https://doi.org/10.2307/2373662.

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Massachusetts Institute of Technology, Cambridge, MA
E-mail: jhahn01@mit.edu

Harvard University, Cambridge, MA
Current address: West Virginia University, Morgantown, WV
E-mail: dylan.wilson2@mail.wvu.edu