On model structure for coreflective subcategories of a model category

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1 Introduction

Let $C$ be a coreflective subcategory of a cofibrantly generated model category $D$. In this paper we show that under suitable conditions $C$ admits a cofibrantly generated model structure which is left Quillen adjunct to the model structure on $D$. As an application, we prove that well-known convenient categories of topological spaces, such as $k$-spaces, compactly generated spaces, and $\Delta$-generated spaces [3] (called numerically generated in [12]) admit a finitely generated model structure which is Quillen equivalent to the standard model structure on the category $\text{Top}$ of topological spaces.

2 Coreflective subcategories of a model category

Let $D$ be a cofibrantly generated model category [7, 2.1.17] with generating cofibrations $I$, generating trivial cofibrations $J$ and the class of weak equivalences $W_D$. If the domains and codomains of $I$ and $J$ are finite relative to $I$-cell [7, 2.1.4], then $D$ is said to be finitely generated.

Recall that a subcategory $C$ of $D$ is said to be coreflective if the inclusion functor $i: C \to D$ has a right adjoint $G: D \to C$, so that there is a natural isomorphism $\varphi: \text{Hom}_D(X, Y) \to \text{Hom}_C(X, GY)$. The counit of this adjunction $\varepsilon: GY \to Y$ ($Y \in D$) is called the coreflection arrow.

Theorem 2.1. Let $C$ be a coreflective subcategory of a cofibrantly generated model category $D$ which is complete and cocomplete. Suppose that the unit of the adjunction $\eta: X \to GX$ is a natural isomorphism, and that the classes $I$ and $J$ of cofibrations and trivial cofibrations in $D$ are contained in $C$. Then $C$ has a cofibrantly generated model structure with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W_C$ as the class of weak equivalences, where $W_C$ is the class of all weak equivalences contained in $C$. If $D$ is finitely generated, then so is $C$. Moreover, the adjunction $(i, G, \varphi): C \to D$ is a Quillen adjunction in the sense of [7, 1.3.1].

Proof. It suffices to show that $C$ satisfies the six conditions of [7, 2.1.19] with respect to $I$, $J$ and $W_C$. Clearly, the first condition holds because $W_C$ satisfies the two out of three property and is closed under retracts. To see that the
second and the third conditions hold, let $I_C$-cell and $J_C$-cell be the collections of relative $I$-cell and $J$-cell complexes contained in $C$, respectively. Since $I_C$-cell and $J_C$-cell are subcollections of the collections of relative $I$-cell and $J$-cell complexes in $D$, respectively, the domains of $I$ and $J$ are small relative to $I_C$-cell and $J_C$-cell, respectively. The rest of the conditions are verified as follows.

Let $f: X \to Y$ be a map in $C$. Since $\eta: X \to GX$ is isomorphic for $X \in D$, $f$ is $I$-injective in $C$ if and only if it is $I$-injective in $D$. Similarly, $f$ is $J$-injective in $C$ if and only if it is $J$-injective in $D$. Let $f$ be an $I$-cofibration in $D$. Then it has the left lifting property with respect to all $I$-injective maps in $C$. Hence $f$ is an $I$-cofibration in $C$. Conversely, let $f$ be an $I$-cofibration in $C$. Suppose we are given a commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow f & & \downarrow p \\
Y & \longrightarrow & B
\end{array}
$$

where $p$ is $I$-injective in $D$. Then there is a relative $I$-cell complex $g: X \to Z$ [7, 2.1.9] such that $f$ is a retract of $g$ by [7, 2.1.15]. Since $g$ is an $I$-cofibration in $D$, there is a lift $Z \to A$ of $g$ with respect to $p$. Then the composite $Y \to Z \to A$ is a lift of $f$ with respect to $p$. Therefore $f$ is an $I$-cofibration in $D$. Similarly, $f$ is a $J$-cofibration in $C$ if and only if it is a $J$-cofibration in $D$. Thus we have the desired inclusions

- $J_C$-cell $\subseteq W_C \cap I_C$-cof,
- $I_C$-inj $\subseteq W_C \cap J_C$-inj, and
- either $W_C \cap I_C$-cof $\subseteq J_C$-cof or $W_C \cap J_C$-inj $\subseteq I_C$-inj.

Here $I_C$-inj and $I_C$-cof denote, respectively, the classes of $I$-injective maps and $I$-cofibrations in $C$, and similarly for $J_C$-inj and $J_C$-cof. Therefore $C$ is a cofibrantly generated model category by [7, 2.1.19].

It is clear, by the definition, that $C$ is finitely generated if so is $C$.

Finally, to prove that $(i, G, \varphi)$ is a Quillen adjunction, it suffices to show that $G: D \to C$ is a right Quillen functor, or equivalently, $G$ preserves $J$-injective maps in $D$ by [7, 1.3.4] and [7, 2.1.17]. Let $p: X \to Y$ be a $J$-injective map in $D$. Suppose there is a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & GX \\
\downarrow f & & \downarrow Gp \\
B & \longrightarrow & GY
\end{array}
$$

where $f \in J$. Then we have a commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & GX & \longrightarrow & X \\
\downarrow f & & \downarrow p & & \downarrow \varepsilon \\
B & \longrightarrow & GY & \longrightarrow & Y.
\end{array}
$$
Since $p$ is $J$-injective in $D$, there is a lift $h: B \to X$ of $f$. Thus we have a lift $Gh \circ \eta: B \cong GB \to GX$ of $f$ with respect to $Gp$. Therefore $Gp: GX \to GY$ is $J$-injective in $C$. Similarly, we can show that $G$ preserves $I$-injective maps in $C$, and so $G$ preserves trivial fibrations in $C$. Hence $(i, G, \varphi)$ is a Quillen adjunction.

We turn to the case of pointed categories [7, p.4]. Let $D^*$ be the pointed category associated with $D$, and let $U: D^* \to D$ be the forgetful functor. We denote by $I_+$ and $J_+$ the classes of those maps $f: X \to Y$ in $D^*$ such that $Uf: UX \to UY$ belongs to $I$ and $J$, respectively. Then we have the following. (Compare [7, 1.1.8], [7, 1.3.5], and [7, 2.1.21].)

**Theorem 2.2.** Let $D$ be a cofibrantly (resp. finitely) generated model category, and let $C$ be a coreflective subcategory satisfying the conditions of Theorem 2.1. Then the pointed category $C^*$ has a cofibrantly (resp. finitely) generated model structure, with generating cofibrations $I_+$ and generating trivial cofibrations $J_+$, such that the induced adjunction $(i_*, G_*, \varphi_*): C^* \to D^*$ is a Quillen adjunction.

We also have the following Proposition.

**Proposition 2.3.** Suppose $C$ and $D$ satisfy the conditions of Theorem 2.1. Suppose, further, that the coreflection arrow $\epsilon: GY \to Y$ is a weak equivalence for any fibrant object $Y$ in $D$. Then the adjunctions $(i, G, \varphi): C \to D$ and $(i_*, G_*, \varphi_*): C^* \to D^*$ are Quillen equivalences.

**Proof.** Let $X$ be a cofibrant object in $C$ and $Y$ a fibrant object in $D$. Let $f: X \to Y$ be a map in $D$. Then we have $\varphi f = Gf \circ \eta: X \cong GX \to GY$. Since $f$ coincides with the composite $X \xrightarrow{\varphi} GY \xrightarrow{\epsilon} Y$ and $\epsilon$ is a weak equivalence in $D$, $\varphi f$ is a weak equivalence in $C$ if and only if $f$ is a weak equivalence in $D$. It follows by [7, 1.3.17] that the induced adjunction $(i_*, G_*, \varphi_*)$ is a Quillen equivalence. □

### 3 On a model structure of the category $NG$

In [12] we introduced the notion of numerically generated spaces which turns out to be the same notion as $\Delta$-generated spaces introduced by Jeff Smith (cf. [3]). Let $X$ be a topological space. A subset $U$ of $X$ is numerically open if for every continuous map $P: V \to X$, where $V$ is an open subset of Euclidean space, $P^{-1}(U)$ is open in $V$. Similarly, $U$ is numerically closed if for every such map $P$, $P^{-1}(U)$ is closed in $V$. A space $X$ is called a numerically generated space if every numerically open subset is open in $X$.

Let $NG$ denote the full subcategory of $Top$ consisting of numerically generated spaces. Then the category $NG$ is cartesian closed [12, 4.6]. To any $X$ we can associate the numerically generated space topology, denoted $\nu X$, by letting $U$ open in $\nu X$ if and only if $U$ is numerically open in $X$. Therefore we have a functor $\nu: Top \to NG$ which takes $X$ to $\nu X$. Clearly, the identity map $\nu X \to X$ is continuous. By the results of [7, §3] the following holds.
Proposition 3.1. The functor $\nu: \text{Top} \to \text{NG}$ is a right adjoint to the inclusion functor $i: \text{NG} \to \text{Top}$, so that $\text{NG}$ is a coreflective subcategory of $\text{Top}$.

A continuous map $f: X \to Y$ between topological spaces is called a weak homotopy equivalence in $\text{Top}$ if it induces an isomorphism of homotopy groups

$$f_*: \pi_n(X,x) \to \pi_n(Y,f(x))$$

for all $n > 0$ and $x \in X$. Let $I$ be the set of boundary inclusions $S^{n-1} \to D^n$, $n \geq 0$, $J$ the set of inclusions $D^n \times \{0\} \to D^n \times I$, and $W_{\text{Top}}$ the class of weak homotopy equivalences. The standard model structure on $\text{Top}$ can be described as follows.

Theorem 3.2 ([7, 2.4.19]). There is a finitely generated model structure on $\text{Top}$ with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W_{\text{Top}}$ as the class of weak equivalences.

The category $\text{NG}$ is complete and cocomplete by [12, 3.4]. A space $X$ is numerically generated if and only if $\nu X = X$ holds. Thus the unit of the adjunction $\eta: X \to \nu X$ is a natural homeomorphism. Moreover, since CW-complexes are numerically generated spaces by [12, 4.4], the classes $I$ and $J$ are contained in $\text{NG}$. Let $W_{\text{NG}}$ be the class of maps $f: X \to Y$ in $\text{NG}$ which is a weak equivalence in $\text{Top}$. Since the coreflection arrow $\nu Y \to Y$, given by the identity of $Y \in \text{Top}$, is a weak equivalence (cf. [12, 5.4]), we have the following by Theorem 2.1 and Proposition 2.3.

Theorem 3.3. The category $\text{NG}$ has a finitely generated model structure with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W_{\text{NG}}$ as the class of weak equivalences. Moreover the adjunction $(i, \nu, \varphi): \text{NG} \to \text{Top}$ is a Quillen equivalence.

We turn to the case of pointed spaces. Let $\text{Top}_*$ be the category of pointed topological spaces. By [7, 2.4.20], there is a finitely generated model structure on the category $\text{Top}_*$, with generating cofibrations $I_+$ and generating trivial cofibrations $J_+$. Then we have the following by Theorem 2.2 and Proposition 2.3.

Corollary 3.4. There is a finitely generated model structure on the category $\text{NG}_*$ of pointed numerically generated spaces, with generating cofibrations $I_+$ and generating trivial cofibrations $J_+$. Moreover, the inclusion functor $i_*: \text{NG}_* \to \text{Top}_*$ is a Quillen equivalence.

Remark. (1) The argument of Theorem 3.3 can be applied to the subcategories $\mathbf{K}$ of $k$-spaces and $\mathbf{T}$ of compactly generated spaces. Similarly, the argument of Corollary 3.4 can be applied to the pointed categories $\mathbf{K}_*$ and $\mathbf{T}_*$. Compare [2.4.28], [2.4.25], [2.4.26] of [7].

(2) Let $\text{Diff}$ be the category of diffeological spaces (cf. [8]). In [12] we introduced a pair of functors $T: \text{Diff} \to \text{Top}$ and $D: \text{Diff} \to \text{Top}$, where $T$ is a left adjoint to $D$, and showed that the composite $TD$ coincides with...
Thus $\nu : \text{Top} \to \text{NG}$. It is natural to ask whether $\text{Diff}$ has a model category structure with respect to which the pair $(T, D)$ gives a Quillen adjunction between $\text{Top}$ and $\text{Diff}$.

Let $I$ be the unit interval, and let $\lambda : \mathbb{R} \to I$ be the smashing function, that is, a smooth function such that $\lambda(t) = 0$ for $t \leq 0$ while $\lambda(t) = 1$ for $t \geq 1$. Let $\tilde{I}$ denote the unit interval equipped with the quotient diffeology $\lambda_*(D\mathbb{R})$, where $D\mathbb{R}$ is the standard diffeology of $\mathbb{R}$. In [5] we introduce a finitely generated model category structure on $\text{Diff}$ with the boundary inclusions $\partial\tilde{I}^{n-1} \to \tilde{I}^n$ as generating cofibrations, and with the inclusions $\partial\tilde{I}^{n-1} \times I \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times I$ as generating trivial cofibrations. Its class of weak equivalences consists of those smooth maps $f : X \to Y$ inducing an isomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for every $n \geq 0$ and $x_0 \in X$. Here, the homotopy set $\pi_n(X, x_0)$ is defined to be the set of smooth homotopy classes of smooth maps $(\tilde{I}^n, \partial\tilde{I}^n) \to (X, x_0)$.

It is expected that with respect to the model structure on $\text{Diff}$ described above, the pair $(T, D)$ induces a Quillen adjunction between $\text{Top}$ and $\text{Diff}$.

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References


