Oberwolfach Seminar
Topological Cyclic Homology and Arithmetic

Topological Cyclic Homology

Lars Hesselholt
Nagoya University and University of Copenhagen

Based on notes by
Lucy Yang

Lecture 1

Higher algebra: The basic difference between algebra and higher algebra is that the role of sets in the former are replaced by anima in the latter. Properties in the former tend to become structure in the latter, e.g. equality becomes a path/homotopy between two things, which needs to be specified. The free commutative monoid is $\mathbb{N}$ and (the free $\mathbb{E}_\infty$-monoid is) $\mathrm{Fin}^{\simeq}$, respectively. The free commutative group on one generator is $\mathbb{Z}$ and $\mathbb{S}$, respectively. Moreover, the stable ∞-categories $D(\mathbb{Z})$ and $D(\mathbb{S}) \simeq \mathrm{Sp}$ have "Postnikov" t-structures and the heart in both is the abelian category of abelian groups, but $D(\mathbb{S})$ is much better. Sometimes we can descend $\mathbb{E}_\infty$-algebras in $D(\mathbb{Z})$ to $\mathbb{E}_\infty$-algebras in $D(\mathbb{S})$. For example, for the ring of Witt vectors $W(k)$ over a perfect field of characteristic $p > 0$, there exists $\mathbb{S}_{W(k)}$ such that $\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} \mathbb{Z} \simeq W(k)$ with compatible Frobenii. There also exists $\mathbb{S}_{W(k)}[z]$ such that $\mathbb{S}_{W(k)}[z] \otimes_{\mathbb{S}} \mathbb{Z} \simeq W(k)[z]$ with compatible Frobenii. This follows from [6, Theorem 5.2.5].

Given $k \to R$ in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$, we can form

$$\mathrm{THH}(R/k) \simeq R^{\otimes_k S^1},$$

where more generally we can replace S^1 by any anima.

Remark. This has an action of the circle group.

How do we understand the homotopy groups of this? Nonequivariantly, we can write the circle as $S^1 = \Delta^1 \cup_{\partial \Delta^1} \Delta^1$. Since the functor $R^{\otimes_k (-)}$ is a left adjoint, and hence, preserves colimits, we have

$$\mathrm{THH}(R/k) \simeq R^{\otimes_k \Delta^1} \otimes_{R^{\otimes_k \partial\Delta^1}} R^{\otimes_k \Delta^1} \simeq R \otimes_{R \otimes_k R} R$$

We work in a world, where everything is derived, so we do not write $\otimes^{\mathbb{L}}$.

Example. For $\mathbb{Z} \to \mathbb{F}_p$, we have

$$\mathrm{THH}(\mathbb{F}_p/\mathbb{Z}) \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p} \mathbb{F}_p$$

Recall that $\mathbb{F}_p \simeq \Lambda_{\mathbb{Z}}\{x\}$ where $|x| = 1$, $dx = p$. Hence $\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \Lambda_{\mathbb{F}_p}\{x\}$. As a $\Lambda_{\mathbb{F}_p}\{x\}$-module, we have $\mathbb{F}_p \simeq \Lambda_{\mathbb{F}_p}\{x\}\langle u \rangle$ divided power algebra, where $|u| = 2$ and $d(u^{[i]}) = u^{[i-1]}x$. (We write $u^{[i]} = \gamma_i(y)$ for the ith divided power of u.) Therefore, we have

$$\mathrm{THH}(\mathbb{F}_p/\mathbb{Z}) \simeq \mathbb{F}_p\langle u \rangle \qquad |u| = 2$$

Exercise. Calculate $\mathrm{THH}_*(R/R[z])$, where $R/\mathbb{Z}$ and $|z| = 0$, $z \mapsto 0$.

Theorem (Bökstedt periodicity)**.**

$$\mathrm{THH}_*(\mathbb{F}_p) := \mathrm{THH}_*(\mathbb{F}_p/\mathbb{S}) = \mathbb{F}_p[u], \qquad |u| = 2.$$

Philosophically, the denominators $n!$ in $\mathrm{THH}_*(\mathbb{F}_p/\mathbb{Z})$ appear because we have identified the $n!$ ways of counting to n to the same entity. In $\mathrm{THH}_*(\mathbb{F}_p/\mathbb{S})$, we have not, so the denominators disappear!

Recall that if $\mathcal{O}_K$ is a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field $k = \mathcal{O}_K/\mathfrak{m}_K$, then

$$\begin{tikzcd} W(k) \arrow[r, "\exists!"] \arrow[d] & \mathcal{O}_K \arrow[d] \\ k \arrow[r, equal] & k \end{tikzcd} \tag{1}$$

We now consider the map

$$W(k)[z] \overset{\theta}{\twoheadrightarrow} \mathcal{O}_K$$
$$z \mapsto \pi_K$$

where $p\mathcal{O}_K = \mathfrak{m}_K^{e_K}$. The kernel of θ is generated by an Eisenstein polynomial $E_K(z)$.[1] In [3], it is proved that the base-change map

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}) \to \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)})$$

is a p-completion. The induced map on $\pi_1(-)$ is the base-change map

$$\Omega^1_{\mathcal{O}_K/\mathbb{Z}} \to \Omega^1_{\mathcal{O}_K/W(k)},$$

[1] It will be important later to normalize $E_K(z)$ to have constant term equal to p.

which is a surjection onto a $W(k)$-module of finite length with kernel a K-vector space of (uncountable) dimension $[K : \mathbb{Q}]$.

Will understand $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)})$, which is complicated, via descent along

$$\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}) \to \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]),$$

where we view $\mathcal{O}_K$ an $\mathbb{S}_{W(k)}[z]$-algebra via $z \mapsto \pi_K$.

Now let's look at the square

$$\begin{tikzcd} \mathbb{S}_{W(k)}[z] \arrow[r, "z\mapsto 0"] \arrow[d, "z\mapsto\pi_K"] & \mathbb{S}_{W(k)} \arrow[d] \\ \mathcal{O}_K \arrow[r] & k \end{tikzcd} \tag{2}$$

which is a pushout in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$.

Remark. We view Eilenberg–MacLane construction as a forgetful functor from $\mathbb{Z}$-modules to $\mathbb{S}$-modules, so, as usual with forgetful functors, we won't write it.

Exercise. Argue that

$$\begin{aligned}\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \otimes_{\mathcal{O}_K} k &\simeq \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) \otimes_{\mathbb{S}_{W(k)}[z]} \mathbb{S}_{W(k)} \\ &\simeq \mathrm{THH}(\mathcal{O}_K \otimes_{\mathbb{S}_{W(k)}[z]} \mathbb{S}_{W(k)}/\mathbb{S}_{W(k)}) \\ &\simeq \mathrm{THH}(k/\mathbb{S}_{W(k)}).\end{aligned}$$

The second equivalence is an exercise in the definition of THH.

The exercise together with Bökstedt periodicity show that the π_K-Bockstein spectral sequence

$$E^1_{i,j} = \pi_{i+j}(\mathrm{gr}^{-i}_{\pi_K} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])) \Rightarrow \mathrm{THH}_{i+j}(\mathcal{O}_K/\mathbb{S}_{W(k)}[z])$$

collapses, and that

$$\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_{W(k)}[z]) = \mathcal{O}_K[u]$$

with $|u| = 2$. Indeed, the E^1-term is concentrated in even total degrees.

Let $\mathbb{T}$ be the circle group and $C_p \subset \mathbb{T}$ the subgroup of prime order p. Nikolaus and Scholze [8] define a p-cyclotomic spectrum to be a pair (X, φ) of a spectrum with $\mathbb{T}$-action X and a map of spectra with $\mathbb{T}$ action

$$X \xrightarrow{\varphi} X^{tC_p}.$$

Here, on the right-hand side, $\mathbb{T}$ acts via the pth root $\mathbb{T} \to \mathbb{T}/C_p$. The map φ is called the cyclotomic Frobenius.

We wish to construct a p-cyclotomic structure on $\mathrm{THH}(R/k)$. So we follow the construction by Nikolaus and Scholze from [8, Section 4] in the case $k = \mathbb{S}$, which we recall. The inclusion $\{1\} \to \mathbb{T}$ induces a map of $\mathbb{E}_\infty$-algebras in spectra

$$R \simeq R^{\otimes\{1\}} \longrightarrow \mathrm{THH}(R/\mathbb{S}) \simeq R^{\otimes\mathbb{T}}$$

which is initial among maps of $\mathbb{E}_\infty$-algebras in spectra from R to the underlying $\mathbb{E}_\infty$-algebra in spectra of an $\mathbb{E}_\infty$-algebra in spectra with $\mathbb{T}$-action. Similarly, the inclusion $\{1\} \to C_p$ induces a map of $\mathbb{E}_\infty$-algebras in spectra

$$R \simeq R^{\otimes\{1\}} \longrightarrow R^{\otimes C_p}$$

which is initial among maps of $\mathbb{E}_\infty$-algebras in spectra from R to the underlying $\mathbb{E}_\infty$-algebra in spectra of an $\mathbb{E}_\infty$-algebra in spectra with C_p-action. The universal property of the latter map gives a unique[2] map

$$R^{\otimes C_p} \longrightarrow \mathrm{THH}(R/\mathbb{S})$$

of $\mathbb{E}_\infty$-algebras in spectra with C_p-action, which, in turn, induces a map

$$(R^{\otimes C_p})^{tC_p} \longrightarrow \mathrm{THH}(R/\mathbb{S})^{tC_p}$$

of $\mathbb{E}_\infty$-algebras in spectra. Precomposing this map by the Tate diagonal[3]

$$R \xrightarrow{\Delta_R} (R^{\otimes C_p})^{tC_p},$$

we get a map of $\mathbb{E}_\infty$-algebras in spectra

$$R \longrightarrow \mathrm{THH}(R/\mathbb{S})^{tC_p}.$$

The target has a residual $\mathbb{T}/C_p$-action, which we may consider a $\mathbb{T}$-action via the isomorphism $\mathbb{T} \to \mathbb{T}/C_p$ given by the pth root. So by the universal property of the map $R \to \mathrm{THH}(R/\mathbb{S})$, there is a unique commutative diagram $\mathbb{E}_\infty$-algebras in spectra with $\mathbb{T}$-action

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\Delta_R"] & \mathrm{THH}(R/\mathbb{S}) \arrow[d, dotted, "\exists!\ \varphi"] \\ (R^{\otimes p})^{tC_p} \arrow[r] & \mathrm{THH}(R/\mathbb{S})^{tC_p} \end{tikzcd}$$

and the map φ is the cyclotomic Frobenius on $\mathrm{THH}(R/\mathbb{S})$.

We wish to repeat this construction for a general base $\mathbb{E}_\infty$-ring k. So we consider

[2] By "unique," we mean "unique, up to contractible ambiguity."

[3] Morally, this map takes a to $a\otimes\cdots\otimes a$, but it only exists in higher algebra; not in algebra.

the following diagram:

$$\begin{tikzcd}
k \arrow[r] \arrow[d, "\Delta_k"] & R \simeq R^{\otimes_k \{1\}} \arrow[r] \arrow[d, "\Delta_R"] & \mathrm{THH}(R/k) \simeq R^{\otimes_k \mathbb{T}} \arrow[dd, dotted, "\exists\varphi\,?"] \\
(k^{\otimes C_p})^{tC_p} \arrow[r] \arrow[d] & (R^{\otimes C_p})^{tC_p} \arrow[d] & \\
k^{tC_p} \arrow[r] & (R^{\otimes_k C_p})^{tC_p} \arrow[r] & \mathrm{THH}(R/k)^{tC_p}
\end{tikzcd}$$

The terms and maps in the top and bottom rows have canonical $\mathbb{E}_\infty$-k-algebra structures, but the terms and map in the middle row do not. In particular, the Nikolaus–Scholze Frobenius, which is defined to be the composition[4]

$$k \xrightarrow{\varphi} k^{tC_p}$$

of the left-hand vertical maps, is not k-linear. But it is a map of $\mathbb{E}_\infty$-algebras in spectra, so if we consider the bottom row as a sequence of $\mathbb{E}_\infty$-k-algebras via this map, then the outer square (which lacks an edge) is a diagram of $\mathbb{E}_\infty$-k-algebras. However, to invoke the universal property of the upper horizontal map in this square, we also need the Nikolaus–Scholze Frobenius to be $\mathbb{T}$-equivariant with respect to the trivial action on the domain and the residual $\mathbb{T} \simeq \mathbb{T}/C_p$-action on the target. Here is a more precise statement.

Proposition. *Let k be an $\mathbb{E}_\infty$-algebra in spectra, and let $\varphi\colon k \to k^{tC_p}$ be the Nikolaus–Scholze Frobenius. A diagram in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ of the form*

$$\begin{tikzcd}
 & (k^{tC_p})^{h\mathbb{T}/C_p} \arrow[dr, "\mathrm{fgt}"] & \\
k \arrow[ur, "\tilde{\varphi}"] \arrow[rr, "\varphi"] & & k^{tC_p}
\end{tikzcd} \tag{3}$$

defines a structure of $\mathbb{E}_\infty$-k-algebra in p-cyclotomic spectra on $\mathrm{THH}(R/k)$, for all $\mathbb{E}_\infty$-k-algebras R.

We stress that the resulting cyclotomic Frobenius

$$\mathrm{THH}(R/k) \xrightarrow{\varphi} \mathrm{THH}(R/k)^{tC_p}$$

is linear with respect to the Nikolaus–Scholze Frobenius $\varphi\colon k \to k^{tC_p}$; it is not k-linear with respect to the canonical k-algebra structure on $\mathrm{THH}(R/k)^{tC_p}$.

Exercise. Show that a lift $\tilde{\varphi}$ exists if $k = \mathbb{S}_{W(k)}$ or $k = \mathbb{S}_{W(k)}[z]$, but not if $k = \mathbb{Z}$. Hint: The Tate-orbit lemma shows that $k^{t\mathbb{T}} \to (k^{tC_p})^{h(\mathbb{T}/C_p)}$ is a p-completion.

[4] For instance, if $k = \mathbb{F}_p$, then this map is the total Steenrod power operation.

Lecture 2

We wish to understand the p-cyclotomic spectrum $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W)$, and we will do so, following Liu–Wang [5], by descent along the map

$$k = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W) \xrightarrow{f} R = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W[z])$$

of $\mathbb{E}_\infty$-algebras in p-cyclotomic spectra.

How does descent work?

Let $f\colon k \to R$ be a map of $\mathbb{E}_\infty$-algebras in spectra. We let Δ be the category of non-empty finite ordinals and order-preserving maps and consider the functor

$$\begin{aligned} \Delta &\to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Mod}_k), \\ [n] &\mapsto R^{\otimes_k [n]} \end{aligned}$$

where $[n] = \{0, 1, \dots, n\}$, and the map

$$k \longrightarrow \lim_{\Delta} R^{\otimes_k[-]} \simeq \lim_n \lim_{\Delta^{\leq n}} R^{\otimes_k[-]}. \tag{4}$$

induced by $f\colon k \to R$. The description of the right-hand side as the limit of a "tower" gives rise to a "descent" spectral sequence[5]

$$E^1_{i,j} = \pi_j(R^{\otimes_k[-i]}) \implies \pi_{i+j}(\lim_{\Delta} R^{\otimes_k[-]})$$

with $d^1\colon E^1_{i,j} \to E^1_{i-1,j}$ given by the alternating sum of the coface maps.

Let I be the fiber of $f\colon k \to R$. It is a k-module, and for all $n \geq 0$, there is a fiber sequence of k-modules

$$I^{\otimes_k[n]} \longrightarrow k \longrightarrow \lim_{\Delta^{\leq n}} R^{\otimes_k[-]};$$

for a proof of this, see [7, Proposition 2.14]. Now, if $\pi_i(I) = 0$ for $i \leq 0$, then $\pi_i(I^{\otimes_k[n]}) = 0$ for $i \leq n$, so in this case, the descent spectral sequence converges strongly to $\pi_*(k)$. In particular, the map (4) is an equivalence.

The E^2-term is given by the cohomotopy groups

$$E^2_{i,j} = \pi^{-i}(\pi_j(R^{\otimes_k[-]})).$$

To understand this, we let

$$A = \pi_*(R^{\otimes_k[0]}) \underset{d^1}{\overset{d^0}{\rightrightarrows}} B = \pi_*(R^{\otimes_k[1]})$$

and make the following assumptions:

[5] If $k = \mathbb{S}$, then this is the R-based Adams spectral sequence.

(a) The maps $d^0, d^1 \colon A \to B$ are flat.

(b) The graded rings A and B are concentrated in even degrees.

Assumption (a) implies that the maps $p_i \colon [1] \to [n]$, where $1 \le i \le n$, defined by $p_i(0) = i - 1$ and $p_i(1) = i$ induce an isomorphism

$$B \otimes_A \cdots \otimes_A B \longrightarrow \pi_*(R^{\otimes_k [n]})$$

that to $b_1 \otimes \cdots \otimes b_n$ assigns $p_1(b_1) \cdots p_n(b_n)$. So we get a cocategory object

$$\begin{tikzcd} B \otimes_A B & B \arrow[l, "\psi"'] \arrow[r, shift left] \arrow[r, shift right] & A \arrow[l, shift right, "d^0"'] \arrow[l, shift left, "d^1"] \end{tikzcd}$$

in the symmetric monoidal category of graded-commutative graded rings, and if we also include the automorphism $\chi \colon B \to B$ induced by the unique non-identity bijection of $[1] = \{0, 1\}$, then we get a cogroupoid object. Assumption (b) implies that A and B are commutative rings with a grading, so we can apply Spec and obtain a groupoid object in schemes with $\mathbb{G}_m$-action, [6]

$$\begin{tikzcd} \operatorname{Spec}(B) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B) \arrow[r, "\circ"] & \operatorname{Spec}(B) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \operatorname{Spec}(A), \arrow[l] \end{tikzcd}$$

where the inverse "χ" is omitted. This determines a stack with $\mathbb{G}_m$-action $\tilde{X}$, or equivalently, a stack $X = [\tilde{X}/\mathbb{G}_m]$ together with a line bundle $\mathcal{O}_X(1)$, namely, the line bundle determined by the $\mathbb{G}_m$-torsor $\tilde{X} \to X$. We write $\mathcal{O}_X(m)$ for its m-fold tensor product.

Exercise. Show that the morphism of stacks

$$\operatorname{Spec}(A) \xrightarrow{f} \tilde{X}$$

is faithfully flat, and conclude that it is affine.

The E^2-term of the descent spectral sequence is now canonically identified with the cohomology of the stack with $\mathbb{G}_m$-action $\tilde{X}$ with coefficients in its structure sheaf, or equivalently, with the cohomology of the stack X with coefficients in the tensor powers of the line bundle ω. So it takes the form

$$E^2_{i,j} = H^{-i}(\tilde{X}, \mathcal{O}_{\tilde{X}})_j = H^{-i}(X, \mathcal{O}_X(j/2)) \implies \pi_{i+j}(k)$$

with $d^r \colon E^r_{i,j} \to E^r_{i-r,j+r-1}$, since we use homological Serre grading.

(Aside: In Adams grading, we instead have

$$E^2_{n,s} = H^s(X, \mathcal{O}_X(n+s/2)) \implies \pi_n(k)$$

[6] A grading of A is a ring homomorphism $A \to A \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$.

with $d^r\colon E^r_{n,s} \to E^r_{n-1,r+s}$, and in motivic grading, we have

$$E^2_{i,j} = H^{j-i}(X, \mathcal{O}_X(j)) \implies \pi_{i+j}(k)$$

with $d^r\colon E^r_{i,j} \to E^r_{i-r,j+r-1}$.)

The sheaf cohomology groups are given by

$$H^s(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \operatorname{Ext}^s_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}) = H^s \operatorname{Hom}_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{\tilde{X}}, I^\cdot),$$

where $\mathcal{O}_{\tilde{X}} \to I^\cdot$ is a resolution by injective graded $\mathcal{O}_{\tilde{X}}$-modules. However, since the morphism $f\colon \operatorname{Spec}(A) \to \tilde{X}$ is affine, the direct image functor

$$\operatorname{Mod}_A \xrightarrow{f_*} \operatorname{Mod}_{\mathcal{O}_{\tilde{X}}}$$

is exact, and therefore, we may instead use a resolution $\mathcal{O}_{\tilde{X}} \to J^\cdot$ by graded $\mathcal{O}_{\tilde{X}}$-modules in its essential image.

We will now use the descent spectral sequence for

$$k = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W) \xrightarrow{f} R = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W[z])$$

to calculate the homotopy groups of k. (Later, we will do the more difficult calculation, where THH is replaced by TP.) The fiber of f has trivial homotopy groups in degrees ≤ 0, so the descent spectral sequence converges strongly. In the first lecture, we proved that

$$A = \pi_*(R) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) = \mathcal{O}_K[u]$$

with $|u| = 2$, and in the next lecture, we will prove that

$$B = \pi_*(R \otimes_k R) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) = A\langle t\rangle$$

with $|t| = 2$. Here we consider B as an A-algebra via $\eta_L = d^1\colon A \to B$. We will also show that the $\mathcal{O}_K$-algebra map $\eta_R = d^0\colon A \to B$ is given by

$$d^0(u) = u + E'_K(\pi_K)t,$$

and that the cocomposition $\psi\colon B \to B \otimes_A B$ is given by

$$\psi(t) = 1 \otimes t + t \otimes 1.$$

Granting this, we find:

Proposition. *In this situation, the nontrivial stack cohomology groups are*

$$\begin{aligned} &H^0(X, \mathcal{O}_X(0)) \simeq \mathcal{O}_K \\ &H^1(X, \mathcal{O}_X(n)) \simeq \mathcal{O}_K/nE'_K(\pi_K), \qquad n > 0. \end{aligned}$$

Proof. We sketch the proof, but it is a good exercise to work this out in detail. Equivalently, we wish to show that

$$\begin{aligned} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})_0 &\simeq \mathcal{O}_K \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})_{2n} &\simeq \mathcal{O}_K/nE'_K(\pi_K), \qquad n > 0. \end{aligned}$$

To this end, we use the resolution

$$\mathcal{O}_{\tilde{X}} \xrightarrow{\ \eta\ } f_*f^*\mathcal{O}_{\tilde{X}} \xrightarrow{\ D\ } f_*f^*\mathcal{O}_{\tilde{X}},$$

where D is the adjunct of the A-linear map

$$\begin{tikzcd} f^*f_*f^*\mathcal{O}_{\tilde{X}} \arrow[r, "\tilde{D}"] \arrow[d, equal] & f^*\mathcal{O}_{\tilde{X}} \arrow[d, equal] \\ B \arrow[r] & A \end{tikzcd} \tag{5}$$

that to $t^{[i]}$ assigns 1, if $i = 1$, and 0, otherwise. To prove that this sequence is indeed a resolution, one first identifies

$$\mathrm{QCoh}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \simeq \mathrm{Mod}_{(A,B)},$$

where the right-hand side is the category of A-modules with descent data with respect to the cogroupoid (A, B), and explicitly works out the adjunction

$$\begin{tikzcd} \mathrm{Mod}_{(A,B)} \arrow[r, shift left, "f^*"] & \mathrm{Mod}_A . \arrow[l, shift left, "f_*"] \end{tikzcd}$$

Having done so, one can calculate D as a map of A-modules with descent data to see that the sequence is a resolution. Finally, using this adjunction, one calculates that the cohomology groups are as stated. ∎

This gives a new proof of the following result. Krause and Nikolaus [3] have recently given a slightly different but equally leisurely proof thereof.

Corollary (Lindenstrauss–Madsen [4]). *There are isomorphisms*

$$\mathrm{THH}_j(\mathcal{O}_K/\mathbb{S}_W) \simeq \begin{cases} \mathcal{O}_K & \text{if } j = 0, \\ \mathcal{O}_K/nE'_K(\pi_K) & \text{if } j = 2n-1 \text{ with } n > 0, \end{cases}$$

and the remaining groups are zero. The isomorphisms depend on the choice of uniformizer $\pi_K \in \mathcal{O}_K$.

Proof. The descent spectral sequence

$$E^2_{i,j} = H^{-i}(X, \mathcal{O}_X(j/2)) \Longrightarrow \mathrm{THH}_{i+j}(\mathcal{O}_K/\mathbb{S}_W)$$

converges strongly and collapses, since, according to the proposition, all nonzero terms are concentrated on the lines $i = 0$ and $i = -1$. ∎

We finally recall the Nikolaus–Scholze definition of topological cyclic homology from [8, Section II]. Recall from the first lecture that if $k \to R$ is a map of $\mathbb{E}_\infty$-algebras in spectra, then we have the $\mathbb{E}_\infty$-k-algebra with $\mathbb{T}$-action

$$\mathrm{THH}(R/k) = R^{\otimes_k S^1}.$$

We showed that a lift of the Nikolaus–Scholze Frobenius on k to a structure of $\mathbb{E}_\infty$-algebra in p-cyclotomic spectra makes $\mathrm{THH}(R/k)$ an $\mathbb{E}_\infty$-k-algebra in p-cyclotomic spectra. We write

$$\begin{array}{ccc} \mathrm{TC}^-(R/k) & \xrightarrow{\mathrm{can}} & \mathrm{TP}(R/k) \\ \| & & \| \\ \mathrm{THH}(R/k)^{h\mathbb{T}} & \xrightarrow{\mathrm{can}} & \mathrm{THH}(R/k)^{t\mathbb{T}} \end{array} \tag{6}$$

and refer to the homotopy fixed points spectrum and the Tate spectrum as the negative topological cyclic homology of R/k and the periodic topological cyclic homology of R/k, respectively. Assuming that $\mathrm{THH}(R/k)$ is p-complete, the cyclotomic Frobenius induces a different map between these two spectra, which is defined to be the composition

$$\begin{array}{ccccc} \mathrm{THH}(R/k)^{h\mathbb{T}} & \xrightarrow{\varphi^{h\mathbb{T}}} & (\mathrm{THH}(R/k)^{tC_p})^{h\mathbb{T}/C_p} & \xleftarrow{\sim} & \mathrm{THH}(R/k)^{t\mathbb{T}} \\ \| & & & & \| \\ \mathrm{TC}^-(R/k) & & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & & \mathrm{TP}(R/k) \end{array} \tag{7}$$

of the map of homotopy fixed points induced by the cyclotomic Frobenius and the (p-adic) equivalence given by the Tate-orbit lemma [8, Lemma I.2.1]. We also write φ for the composite map and refer to it as the Frobenius map. Finally, the topological cyclic homology of R/k is defined to be the equalizer

$$\mathrm{TC}(R/k) \longrightarrow \mathrm{TC}^-(R/k) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(R/k)$$

of these two maps.

Lecture 3

Recall that $k = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W)$ and $R = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W[z])$. We wish to prove that the structure of the cogroupoid in graded rings

$$\begin{aligned} A &= \pi_*(R) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) \\ B &= \pi_*(R \otimes_k R) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) \end{aligned}$$

is as stated in Lecture 2.[7] So we let $u := d^1u \in B$ and begin by evaluating the associated graded $\mathrm{gr}_u^{\boldsymbol{\cdot}}(B)$ of B for the u-adic filtration. We would like to

[7] In fact, we will not see this until Lecture 4.

understand the cofiber of $u \otimes \mathrm{id}\colon R \otimes_k R \to R \otimes_k R$. This is the same as

$$\begin{aligned} \mathrm{cofib}(R \xrightarrow{u} R) \otimes R &\simeq \mathrm{THH}(\mathcal{O}_K/\mathcal{O}_K) \otimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W)} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W[z]) \\ &\simeq \mathcal{O}_K \otimes_{\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W)} \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W[z]) \\ &\simeq \mathrm{THH}(\mathcal{O}_K/\mathcal{O}_K[z_0]), \end{aligned}$$

and by an earlier exercise, $\mathrm{THH}_*(\mathcal{O}_K/\mathcal{O}_K[z_0]) = \mathcal{O}_K\langle t\rangle$ with $\deg(t) = 2$. Now we run the u-Bockstein spectral sequence. Since everything sits in even total degree, all differentials are zero, so we conclude that

$$\mathrm{gr}_u^{\cdot}(B) = A\langle t\rangle.$$

We will prove below that the generator t can be chosen so that this is true before taking associated graded for the u-adic filtration.

In the last lecture, we used descent along

$$k = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W) \xrightarrow{\ f\ } R = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_{W[z]}) \tag{8}$$

to calculate the homotopy groups of the domain, assuming that the structure of the cogroupoid (A, B) is as stated in the last lecture. Recall that since the fiber I of $f\colon k \to R$ is 1-connective, the canonical map

$$k \longrightarrow \lim_\Delta R^{\otimes_k[-]}$$

is an equivalence. Now, this is a map of ($\mathbb{E}_\infty$-k-algebras in) cyclotomic spectra, and hence, also the induced map of Tate spectra

$$k^{t\mathbb{T}} \longrightarrow (\lim_\Delta R^{\otimes_k[-]})^{t\mathbb{T}} \tag{9}$$

is an equivalence. In order to use descent to understand the homotopy groups of the domain, we also need to know that the canonical map

$$(\lim_\Delta R^{\otimes_k[-]})^{t\mathbb{T}} \longrightarrow \lim_\Delta (R^{\otimes_k[-]})^{t\mathbb{T}} \tag{10}$$

is an equivalence.

Exercise. Show that given $f\colon k \to R$ in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}^{B\mathbb{T}})$ with 1-connective fiber, the map (10) is an equivalence. (Hint: Use the fiber sequence

$$X_{h\mathbb{T}}[1] \xrightarrow{\ \mathrm{Nm}\ } X^{h\mathbb{T}} \xrightarrow{\ \mathrm{can}\ } X^{t\mathbb{T}}$$

together with the fact that $\Delta^{\leq n}$ finite.)

We wish to show that the cosimplicial graded W-algebra

$$[n] \mapsto \pi_*((R^{\otimes_k[n]})^{t\mathbb{T}}) = \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0, \ldots, z_n])$$

is the (co)nerve of a cogroupoid and to understand its structure. In general, for X is a spectrum with $\mathbb{T}$-action, we have the Tate spectral sequence

$$E^2_{*,*} = \pi_*(X)[v^{\pm 1}] \implies \pi_*(X^{t\mathbb{T}}).$$

In the case $X = R$, we have

$$\pi_*(X) = \pi_*(R) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) = \mathcal{O}_K[u]$$

with $\deg(u) = 2$, so the Tate spectral sequence collapses, since the E^2-term is located in even total degrees. The Tate spectral sequence defines a descending filtration of $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0])$ that we call the Nygaard filtration. (Here we will not count the odd rows and columns, which are all zero.) It is a complete and separated filtration. We have

$$\mathrm{gr}^i_N(\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0])) = \mathrm{THH}_{2i}(\mathcal{O}_K/\mathbb{S}_W[z_0]),$$

so we may view the filtered ring $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0])$ as a deformation of the graded ring $\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0])$. The following exercise shows that $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0])$ is an integral domain.

Exercise. If a ring A has a separated descending filtration such that $\mathrm{gr}^\cdot(A)$ is an integral domain, then A is an integral domain.

The composition of the canonical map $W[z_0] = \pi_0(\mathbb{S}_W[z_0]) \to \pi_0(R^{t\mathbb{T}})$ and the edge homomorphism $\pi_0(R^{t\mathbb{T}}) \to \pi_0 R = \mathcal{O}_K$ is equal to the unique W-algebra map $W[z_0] \to \mathcal{O}_K$ that maps $z_0 \mapsto \pi_K$. Since this map annihilates the Eisenstein polynomial $E_K(z_0) \in W[z_0]$, we conclude that

$$E_K(z_0) \in N^{\geq 1}\,\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0])$$

has Nygaard filtration at least 1.

Claim. The canonical map

$$N^{\geq 1} TP_0(\mathcal{O}_K/\mathbb{S}_W[z_0]) \to \mathrm{gr}^1_N\, TP_0(\mathcal{O}_K/\mathbb{S}_W[z_0]) = \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_W[z_0])$$

takes $E_K(z_0)$ to an $\mathcal{O}_K$-module generator of the target.

This is a calculation in Hochschild homology, which we omit. We now define

$$u \in \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_W[z_0])$$

to be the image of $E_K(z_0)$ by this map and conclude from

$$\mathrm{gr}^\cdot_N\,\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0]) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) = \mathcal{O}_K[u]$$

and from the Nygaard filtration being complete and separated that

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0]) = W[z_0]^\wedge_{(E_K(z_0))} = W[[z_0]].$$

Here the right-hand equality holds, since $E_K(z_0)$ is Eisenstein.

We have a map of spectral sequences

$$\begin{tikzcd} E^2 = \pi_*(X)[v^{\pm 1}] \arrow[r, Rightarrow] & \pi_*(X^{t\mathbb{T}}) \\ E^2 = \pi_*(X)[v] \arrow[u] \arrow[r, equal] & \pi_*(X^{h\mathbb{T}}) \arrow[u, "\mathrm{can}"'] \end{tikzcd}$$

and conclude from the above that

$$\begin{aligned} \mathrm{TC}^-_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) &= \pi_*(R^{h\mathbb{T}}) = W[[z_0]][u,v]/(uv - E_K(z_0)) \\ \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) &= \pi_*(R^{t\mathbb{T}}) = W[[z_0]][v^{\pm 1}] \end{aligned}$$

and that

$$\mathrm{TC}^-_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) \xrightarrow{\mathrm{can}} \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0])$$

is the unique $W[z_0]$-algebra homomorphism such that

$$\begin{aligned} \mathrm{can}(u) &= u = E_K(z_0)v^{-1} \\ \mathrm{can}(v) &= v. \end{aligned}$$

The Frobenius map

$$\mathrm{TC}^-_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) \xrightarrow{\varphi} \mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0])$$

is not a $W[z_0]$-algebra homomorphism. Instead, it is a ring homomorphism that is linear with respect to the Frobenius on $W[z_0]$. It is given by

$$\begin{aligned} \varphi(u) &= v^{-1} \\ \varphi(v) &= \varphi(E_K(z_0)) \cdot v. \end{aligned}$$

We note that the Frobenius does NOT preserve the Nygaard filtration.

Remark. It is easy to see that $\varphi(u) = \text{unit} \cdot v^{-1}$. To see that u and v can be chosen such that the unit is 1, we must normalize $E_K(z)$ such that $E_K(0) = p$ and use that there is a $\mathbb{T}$-equivariant map

$$\mathbb{Z} \longrightarrow \mathbb{Z}_p \simeq K(\mathbb{F}_p)^\wedge_p \longrightarrow \mathrm{THH}(k/\mathbb{S}_S[z_0])$$

compatible with the Frobenius. Here the $\mathbb{T}$-action and the Frobenius are both (necessarily) trivial on $\mathbb{Z}$.

We now consider

$$R \otimes_k R = \mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]),$$

where $\mathbb{S}_W[z_0, z_1] \to \mathcal{O}_K$ maps $z_0, z_1 \mapsto \pi_K$. We have calculated that

$$\mathrm{gr}^{\boldsymbol{\cdot}}_u \, \pi_*(R \otimes_k R) = \mathcal{O}_K[u]\langle t \rangle$$

with $\deg(u) = \deg(t) = 2$. This shows that $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$ is an integral domain. (Apply exercise twice.) Moreover, it is complete and separated with respect to the Nygaard filtration. The following observation is key.

Lemma. *Suppose that $x \in \mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$ has Nygaard filtration $\geq n$. Then $\varphi(x)$ is divisible by $\varphi(E_K(z_0))^n$.*

Proof. We write $x = v^n y$ with $v = d^1(v)$ for some y and calculate

$$\varphi(x) = \varphi(v^n y) = \varphi(v)^n \varphi(y) = \varphi(E_K(z_0))^n v^n \varphi(y).$$

This proves the lemma $\square$

We apply the lemma to $z_1 - z_0$. Since $\theta(z_1 - z_0) = \pi_K - \pi_K = 0$, we have

$$z_1 - z_0 \in N^{\geq 1}\,\mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]).$$

Therefore, there exists an element $h \in \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$ such that

$$\varphi(z_1 - z_0) = \varphi(E_K(z_0))h,$$

and since $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$ is an integral domain, this element h is unique.

The Frobenius $\varphi\colon W[z_0, z_1] \to W[z_0, z_1]$ is a Frobenius lift in the sense that there exists an element $\delta(x) \in W[z_0, z_1]$ such that

$$\varphi(x) = x^p + p\delta(x).$$

Moreover, since p is a non-zero-divisor in $W[z_0, z_1]$, the element $\delta(x)$ is uniquely determined by x, and the map $\delta\colon W[z_0, z_1] \to W[z_0, z_1]$ is an example of a δ-ring structure. The map δ is neither additive nor multiplicative, but rather satisfies the axioms necessary to make φ a ring homomorphism. In p-torsion free rings, a Frobenius lift φ and a δ-ring structure determine each other uniquely, but in rings with p-torsion, a δ-ring structure is the better notion.

The free δ-ring over $W[z_0, z_1]$ on a generator T is the ring

$$W[z_0, z_1]\{T\} = W[z_0, z_1][T, \delta(T), \delta(\delta(T)), \dots]$$

with the Frobenius φ that extends the Frobenius φ on $W[z_0, z_1]$ and maps

$$\varphi(\delta^n(T)) = \delta^n(T)^p + p\delta^{n+1}(T).$$

The ring $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$ is p-torsion free, but it is not clear that the map

$$\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) = \mathrm{TC}_0^-(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) \xrightarrow{\varphi} \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$$

is a Frobenius lift. That this nevertheless is the case is a consequence of the following more precise result.

Theorem (Liu–Wang [5]). *There is a (unique) map of $W[z_0, z_1]$-algebras*

$$W[z_0, z_1]\{T\} \xrightarrow{f} \mathrm{TP}_0(\mathcal{O}_K/S_W[z_0, z_1])$$

that is compatible with φ and maps T to

$$h = \varphi(z_1 - z_0)/\varphi(E_K(z_0)).$$

Moreover, the image of f is dense with respect to the topology given by the ideals

$$J_n = \textstyle\sum_{i+j=n} p^i N^{\geq j}\, \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) \subset \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]).$$

The "J-topology" determined by the ideals J_n has the advantage (compared to the one determined by the Nygaard filtration) that the Frobenius map φ is continuous with respect to this topology. We will not (need to) determine the kernel of the map f in the theorem. However, the element

$$\varphi(z_1 - z_0) - \varphi(E_K(z_0)) \cdot T$$

lies is this kernel, and hence, so does its image by δ^n for all $n \geq 0$.

Proof. (Sketch) The proof that the map f exists is a consequence of the following very clever definition/calculation: Liu and Wang set

$$f^{(0)} = z_1 - z_0$$

and show, by induction on $m \geq 0$, that the elements

$$f^{(m+1)} = \frac{(f^{(m)})^p - E_K(z_0)^{p^{m+1}}(-\delta)^m(h)}{p}$$

exist (meaning that the division by p is possible) and satisfy

$$f^{(m)} \in N^{\geq p^m}\, \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]).$$

To prove that its image is dense in the stated topology, it suffices to show that it is dense in the topology determined by the Nygaard filtration. To this end, we define $u, t \in \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$ to be the images by the map

$$\mathrm{gr}^1_N\, \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) \longrightarrow \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$$

of $E_K(z_0))$ and $z_1 - z_0$, respectively, and show that the map

$$\mathrm{gr}^{\cdot}_{E_K(z_0)}\, \mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1]) \longrightarrow \mathrm{gr}^{\cdot}_u\, \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])$$

is an isomorphism. Now, we have identified the target ring with $\mathcal{O}_K[u]\langle t\rangle$, and, by the definition of $f^{(m)}$, we have

$$\text{image of } f^{(m+1)} = \frac{1}{(p-1)!} \cdot (\text{image of } f^{(m)})^{[p]}.$$

So, up to a (known) unit, the image of $f^{(m)}$ is equal to $t^{[p^m]}$, which shows that the map in question is an isomorphism. □

Lecture 4

Last time, we proved that

$$\mathrm{gr}_u^{\bullet}\,\mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]) = \mathcal{O}_K[u]\langle t\rangle$$

with u and t defined to be the images of $E_K(z_0)$ and $z_1 - z_0$ by the map

$$N^{\geq 1}\,\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]) \longrightarrow \mathrm{THH}_2(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]),$$

respectively. Let us now prove that this is true before taking associated graded with respect to the u-filtration. To this end, we use the theorem from last time that the Frobenius φ makes $\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1])$ a δ-ring.

Lemma. *For all $m \geq 1$,*

$$\delta^m(h) \in N^{\geq 1}\,\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]).$$

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]) \arrow[r, "\theta"] \arrow[d, "s^0"] & \mathrm{THH}_0(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]) \arrow[d, "s^0"] \\
\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0]) \arrow[r, "\theta"] & \mathrm{THH}_0(\mathcal{O}_K/\mathbb{S}_W[z_0])
\end{tikzcd}$$

and wish to show that the top horizontal map annihilates $\delta^m(h)$. Here, we recall,

$$h = \varphi(z_1 - z_0)/\varphi(E_K(z_0)).$$

Now, the right-hand vertical map s^0 is an isomorphism (the common ring is isomorphic to $\mathcal{O}_K$), so we may instead show that the left-hand vertical map s^0 annihilates $\delta^m(h)$. This is true for $m = 0$, since s^0 is a ring homomorphism, which maps both z_0 and z_1 to z_0, and therefore, it is true for all $m \geq 0$, since s^0 is compatible with φ, and therefore, is a map of δ-rings. $\square$

Corollary. *There is a $\mathbb{Z}_p$-algebra map*

$$\mathbb{Z}_p\langle x\rangle \to \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1])$$

that to x assigns t.

Proof. The lemma shows that, in

$$\mathrm{gr}^{\bullet}\,\mathrm{TP}_0(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]) = \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]),$$

we have

$$\text{class of } f^{(m+1)} = \text{class of } \frac{(f^{(m)})^p}{p} = \text{class of } \frac{1}{(p-1)!}\cdot (f^{(m)})^{[p]}.$$

Since $(p-1)!$ is a unit in $\mathbb{Z}_p$, we conclude that the element

$$t = \text{class of } f^{(0)}$$

admits divided powers. $\square$

Corollary. *As graded $\mathcal{O}_K$-algebras,*

$$\begin{array}{ccc} \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0,z_1]) & = & \mathcal{O}_K[u]\langle t\rangle \\ {\scriptstyle d^0}\uparrow\;\uparrow{\scriptstyle d^1} & & {\scriptstyle d^0}\uparrow\;\uparrow{\scriptstyle d^1} \\ \mathrm{THH}_*(\mathcal{O}_K/\mathbb{S}_W[z_0]) & = & \mathcal{O}_K[u], \end{array}$$

where $d^1(u) = u$ and $d^0(u) = u + E_K'(\pi_K)\cdot t$.

Proof. The top identity follows from the previous corollary and from the fact, which we proved earlier, that it holds after taking associated graded with respect to the u-adic filtration. So only the formula for $d^0(u)$ needs proof. We have

$$\begin{aligned} d^0(u) &= \text{class of } E_K(z_1) \\ &= \text{class of } E_K(z_0) + \frac{E_K(z_1)-E_K(z_0)}{z_1-z_0}\cdot(z_1-z_0) \\ &= u + E_K'(\pi_K)\cdot t \end{aligned}$$

as desired. $\square$

Remark. Note that we used TP_0 to understand THH_*!

This completes the calculation of $\mathrm{THH}(\mathcal{O}_K/\mathbb{S}_W)$ that we started in Lecture 2.

We now wish to understand the descent spectral sequence[8]

$$E^1_{i,j}(\mathrm{TP}) = \mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_W[z_0,\dots,z_{-i}]) \implies \mathrm{TP}_{i+j}(\mathcal{O}_K/\mathbb{S}_W).$$

The Nygaard filtration gives a filtration of the cochain complexes

$$C^\cdot(j/2) := (\mathrm{TP}_j(\mathcal{O}_K/\mathbb{S}_W[z]^{\otimes_{\mathbb{S}_W}[-]}), d = \textstyle\sum(-1)^s d^s), \tag{11}$$

and the (purely algebraic) spectral sequence associated with these filtered cochain complexes takes the form

$$E_1 = E^2(\mathrm{THH})[v^{\pm 1}] \implies E^2(\mathrm{TP})$$

We saw earlier that $E^2_{i,j}(\mathrm{THH}) = 0$, unless $i=0$ or $i=-1$, so we conclude:

Corollary. *The groups $E^2_{i,j}(\mathrm{TP})$ vanish, unless $i=0$ or $i=-1$.*

[8] More precisely, this spectral sequence is obtained by applying $(-)^{t\mathbb{T}}$ to the cosimplicial cyclotomic spectrum that gives rise to the THH descent spectral sequence.

Hence, the calculation of $\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W)$ is now reduced to the purely algebraic problem of calculating the cohomology

$$E^2_{i,j}(\mathrm{TP}) = H^{-i}(\mathcal{X}, \mathcal{O}_X(j/2))$$

of the stack[9] $\mathcal{X} = [\tilde{\mathcal{X}}/\mathbb{G}_m]$, where

$$\begin{tikzcd}
\operatorname{Spec}(\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0, z_1])) \arrow[r, "d^1"] \arrow[d, "d^0"] & \operatorname{Spec}(\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0])) \arrow[d, "f"] \\
\operatorname{Spec}(\mathrm{TP}_*(\mathcal{O}_K/\mathbb{S}_W[z_0])) \arrow[r, "f"] & \tilde{\mathcal{X}}.
\end{tikzcd}$$

That doesn't mean it's easy. In fact, this problem is too difficult (currently), so we reduce modulo p. At the end of the day, we wish to show that

$$\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_W) \longrightarrow L_1\,\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_W)$$

is n-truncated for some small n (such as $n = 1$), and this is equivalent to showing that the multiplication by $v_1 \in \pi_{2p-3}(\mathbb{S}_W)$ map

$$\mathrm{TC}_j(\mathcal{O}_K/\mathbb{S}_W, \mathbb{Z}/p) \xrightarrow{v_1} \mathrm{TC}_{j+2(p-1)}(\mathcal{O}_K/\mathbb{S}_W, \mathbb{Z}/p)$$

is an isomorphism for $j \geq n$. So we consider (11) with $\mathbb{Z}/p$-coefficients. We can do this one j at a time, and we will only consider $j = 0$ here. (For $j \neq 0$, we must understand the Breuil–Kisin twist.) The algebraic spectral sequence becomes

$$E_1 = E^2(\mathrm{THH}) \implies E^2(\mathrm{TP}_0).$$

Before reducing modulo p, THH looks quite reasonable. However, reduction modulo p does not interact well with the Nygaard filtration. But we are helped by the fact that we are doing algebra, as opposed to higher algebra: We define a refined Nygaard filtration by

$$\tilde{N}^{\geq de_k + r}\,\mathrm{TP}_0 = z_0^r N^{\geq d}\,\mathrm{TP}_0,$$

where $d \geq 0$ and $0 \leq r < e_k$, and consider the algebraic spectral sequence obtained from this refined filtration of $C^{\cdot}(0)/p$ instead.

Exercise (Addendum to earlier exercise). Show that if $e = e_K > 1$, then

$$E_1 = H^*(\mathrm{gr}^{\cdot}_{\tilde{N}}\, C^{\cdot}(0)) = k[z_0] \otimes_k \Lambda_k\{t\}$$

with $z_0 \in E_1^{1,-1}$ and $t \in E_1^{e,1-e}$.

[9] We expect that $\mathcal{X}$ is the prismatization of $\operatorname{Spf}(\mathcal{O}_K)$.

Differentials go $d_r\colon E_r^{i,j} \to E_r^{i+r,j-(r-1)}$, so

$$d_{p-1}(z_0) \doteq t$$

is the first possible non-zero differential. (Here we write "$\doteq$" to indicate equality up to a unit.) But, modulo p, $E_K'(\pi_K) \doteq \pi_K^{e-1}$, so

$$d(z_0) = d^0(z_0) - d^1(z_0) = u - E_K'(\pi_K) \cdot t - u \doteq \pi_K^{e-1} \cdot t,$$

which shows that this differential is nonzero. We find that

$$E_e = k[z_0^p] \otimes_k \Lambda_k\{z_0^{p-1}t\},$$

and $E_{(p+1)e-1} = E_e$ for degree reasons. The next possible nonzero differential is

$$d_{(p+1)e-1}(z_0^p) \doteq z_0^{pe} z_0^{p-1} t.$$

To prove that this is indeed nonzero, Liu–Wang prove the congruence

$$d(z_0^p) = z_1^p - z_0^p \equiv z_0^{pe} \cdot z_0^{p-1}(z_1 - z_0) \mod (\tilde{N}^{\geq 2pe}, p),$$

up to an explicit unit. So we calculate that

$$E_{(p+1)e} = k[z_0^{p^2}] \otimes_k \Lambda\{z_0^{pe-p^2-1}t\} \oplus k\{z_0^{pi} \cdot z_0^{p-1}t \mid 0 \leq i < e\},$$

where $k\{\cdots\}$ indicates the k-vector space spanned by $(\cdots)$ and not the free δ-ring. Proceeding in this manner, Liu–Wang determine the differential structure of the algebraic spectral sequences defined by the refined Nygaard filtration completely. We note the similarity with [2, Theorem 5.5.1].

Theorem (Liu–Wang [5])**.** *Assuming that p is odd and that $e_K > 1$ (or that $p = 2$ and $e_K > 2$), the nonzero differentials in the spectral sequence*

$$E_1 = k[z_0] \otimes_k \Lambda_k\{t\} \cdot v^{-a} \implies H^{\cdot}(C^{\cdot}(a))$$

are multiplicatively generated from

$$d_{\frac{p^{v+1}-1}{p-1}e-1}(z_0^r v^{-a}) \doteq z_0^{pe\frac{p^v-1}{p-1}+r-1} t v^{-a},$$

where $v = v_p(r - \frac{pe}{p-1}a)$

The calculation of TC^- is similar, and for TC, we consider the bicomplex

$$\mathrm{Tot}(E^1(\mathrm{TC}^-) \xrightarrow{\varphi-\mathrm{can}} E^1(\mathrm{TP})) = E^1(\mathrm{TC})$$

with the total filtration obtained from the refined Nygaard filtrations of $E^1(\mathrm{TC}^-)$ and $E^1(\mathrm{TP})$. We have normalized $E_K(z)$ such that $E_K(0) = p$, so we have

$$E_K(z) \equiv \alpha \cdot z^e$$

modulo p, where α is a unit in W that depends on the structure of field K. The unit α enters into the calculation of φ, and hence, $E^2(\mathrm{TC})$. For example, if K contains the pth roots of unity, then Lubin–Tate theory shows that K contains a $(p-1)$th root of α. More precisely, a choice of a primitive pth root of unity ζ determines a $(p-1)$th root of α; see [2, Lemma 5.2.1]. In this case, the calculation shows that

$$\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_W) \longrightarrow L_1 \,\mathrm{TC}(\mathcal{O}_K/\mathbb{S}_W)$$

is 1-truncated. In general, the degree of the extension $K(\mu_p)/K$ divides $p-1$, and hence, is a unit modulo p. So one concludes that the map above is 1-truncated for all K, at least if p is odd. This, in turn, implies that

$$K(K, \mathbb{Z}_p) \longrightarrow L_1 K(K, \mathbb{Z}_p)$$

is (-1)-truncated. Finally, Thomason's hyperdescent theorem makes it possible to understand the right-hand side in terms of Iwasawa theory. This is explained well in the paper [1] by Dwyer–Mitchell.

References

[1] William Dwyer and Stephen Mitchell. On the K-theory spectrum of a ring of algebraic integers. *K-Theory*, 14(3):201–263, 1998.

[2] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math.*, 158:1–113, 2003.

[3] Achim Krause and Thomas Nikolaus. Bökstedt periodicity and quotients of DVRs. arXiv1907.03477.

[4] Ayelet Lindenstrauss and Ib Madsen. Topological Hochschild homology of number rings. *Trans. Amer. Math. Soc.*, 352:2179–2004, 2000.

[5] Ruochuan Liu and Guozhen Wang. Topological cyclic homology of local fields. arXiv:2012.15014.

[6] Jacob Lurie. Ellictic cohomology II: Orientations. available at `math.ias.edu/~lurie`, April 2018.

[7] Akhil Mathew, Niko Naumann, and Justin Noel. Nilpotence and descent in equivariant stable homotopy theory. *Advances in Mathematics*, 305:994–1084, 2017.

[8] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Mathematica*, 221(2):203–409, 2018.