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Topological cyclic homology

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ABSTRACT: This survey of topological cyclic homology is a chapter in the Handbook on Homotopy Theory. We give a brief introduction to topological cyclic homology and the cyclotomic trace map following Nikolaus-Scholze, followed by a proof of Bökstedt periodicity that closely resembles Bökstedt's original unpublished proof. We explain the extension of Bökstedt periodicity by Bhatt-Morrow-Scholze from perfect fields to perfectoid rings and use this to give a purely p-adic proof of Bott periodicity. Finally, we evaluate the cofiber of the assembly map in p-adic topological cyclic homology for the cyclic group of order p and a perfectoid ring of coefficients.

Topological cyclic homology is a manifestation of Waldhausen's vision that the cyclic theory of Connes and Tsygan should be developed with the initial ring $\mathbb{S}$ of higher algebra as base. In his philosophy, such a theory should be meaningful integrally as opposed to rationally. Bökstedt realized this vision for Hochschild homology [9], and he made the fundamental calculation that

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) = \mathbb{F}_p[x]$$

is a polynomial algebra on a generator x in degree two [10]. By comparison,

$$\mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p\langle x\rangle$$

is the divided power algebra,[1] so Bökstedt's periodicity theorem indeed shows that by replacing the base $\mathbb{Z}$ by the base $\mathbb{S}$, denominators disappear. In fact, the base-change map $\mathrm{HH}_*(\mathbb{F}_p/\mathbb{S}) \to \mathrm{HH}_*(\mathbb{F}_p/\mathbb{Z})$ can be identified with the edge homomorphism of a spectral sequence

$$E^2_{i,j} = \mathrm{HH}_i(\mathbb{F}_p/\pi_*(\mathbb{S}))_j \Rightarrow \mathrm{HH}_{i+j}(\mathbb{F}_p/\mathbb{S}),$$

so apparently the stable homotopy groups of spheres have exactly the right size to eliminate the denominators in the divided power algebra.

The appropriate definition of cyclic homology relative to $\mathbb{S}$ was given by Bökstedt–Hsiang–Madsen [11]. It involves a new ingredient not present in the Connes–Tsygan cyclic theory: a Frobenius map. The nature of this Frobenius map is now much better understood by the work of Nikolaus–Scholze [32]. As

[1] The divided power algebra $\mathbb{F}_p\langle x\rangle$ has generators $x^{[i]}$ with $i \in \mathbb{N}$ subject to the relations that $x^{[i]} \cdot x^{[j]} = \binom{i+j}{i} \cdot x^{[i+j]}$ for all $i, j \in \mathbb{N}$ and $x^{[0]} = 1$. So $x^i = (x^{[1]})^i = i! \cdot x^{[i]}$.

in the Connes–Tsygan theory, the circle group $\mathbb{T}$ acts on topological Hochschild homology, and negative topological cyclic homology and periodic topological cyclic homology are defined to be the homotopy fixed points and the Tate construction, respectively, of this action:

$$\mathrm{TC}^-(A) = \mathrm{THH}(A)^{h\mathbb{T}} \qquad \text{and} \qquad \mathrm{TP}(A) = \mathrm{THH}(A)^{t\mathbb{T}}.$$

There is always a canonical map from the homotopy fixed points to the Tate construction, but, after p-completion, the pth Frobenius gives rise to another such map and topological cyclic homology is the equalizer

$$\mathrm{TC}(A) \longrightarrow \mathrm{TC}^-(A) \underset{\mathrm{can}}{\overset{(\varphi_p)}{\rightrightarrows}} \mathrm{TP}(A)^\wedge = \textstyle\prod_p \mathrm{TP}(A)^\wedge_p.$$

Here "$(-)^\wedge$" and "$(-)^\wedge_p$" indicates profinite and p-adic completion.

Topological cyclic homology receives a map from algebraic K-theory, called the cyclotomic trace map. Roughly speaking, this map records traces of powers of matrices and may be viewed as a denominator-free version of the Chern character. There are two theorems that concern the behavior of this map applied to cubical diagrams of connective $\mathbb{E}_1$-algebras in spectra. If A is such an n-cube, then the theorems give conditions for the $(n+1)$-cube

$$K(A) \longrightarrow \mathrm{TC}(A)$$

to be cartesian. For $n = 1$, the Dundas–Goodwillie–McCarthy theorem [16] states that this is so provided $\pi_0(A_0) \to \pi_0(A_1)$ is a surjection with nilpotent kernel. And for $n = 2$, the Land–Tamme theorem [26], which strengthens theorems of Cortiñas [15] and Geisser–Hesselholt [21], states that this is so provided A is cartesian and $\pi_0(A_1 \otimes_{A_0} A_2) \to \pi_0(A_{12})$ an isomorphism. For $n \geq 3$, it is an open question to find conditions on A that make $K(A) \to \mathrm{TC}(A)$ cartesian. It is also not clear that the conditions for $n = 1$ and $n = 2$ are optimal. Indeed, a theorem of Clausen–Mathew–Morrow [14] states that for every commutative ring R and ideal $I \subset R$ with (R, I) henselian, the square

$$\begin{tikzcd}
K(R) \arrow[r] \arrow[d] & \mathrm{TC}(R) \arrow[d] \\
K(R/I) \arrow[r] & \mathrm{TC}(R/I)
\end{tikzcd}$$

becomes cartesian after profinite completion. So in this case, the conclusion of the Dundas–Goodwillie–McCarthy theorem holds under a much weaker assumption on the 1-cube $R \to R/I$. The Clausen–Mathew–Morrow theorem may be seen as a p-adic analogue of Gabber rigidity [20]. Indeed, for prime numbers ℓ that are invertible in R/I, the left-hand terms in the square above vanish after ℓ-adic completion, so the Clausen–Mathew–Morrow recovers and extends the Gabber rigidity theorem.

Absolute comparison theorems between K-theory and topological cyclic homology begin with the calculation that for R a perfect commutative $\mathbb{F}_p$-algebra, the cyclotomic trace map induces an equivalence

$$K(R)_p^\wedge \longrightarrow \tau_{\geq 0}\,\mathrm{TC}(R)_p^\wedge.$$

The Clausen–Mathew–Morrow theorem then implies that the same is true for every commutative ring R such that (R, pR) is henselian and such that the Frobenius $\varphi\colon R/p \to R/p$ is surjective. Indeed, in this case, $(R/p)^{\mathrm{red}}$ is perfect and $(R/p, \mathrm{nil}(R/p))$ is henselian; see [14, Corollary 6.9]. In particular, this is true for all semiperfectoid rings R.[2]

The starting point for the calculation of topological cyclic homology and its variants is the Bökstedt periodicity theorem, which we mentioned above. Since Bökstedt's paper [10] has never appeared, we take the opportunity to give his proof in Section 1.2 below. The full scope of this theorem was realized only recently by Bhatt–Morrow–Scholze [4], who proved that Bökstedt periodicity holds for every perfectoid ring. More precisely, their result, which we explain in Section 1.3 below states that if R is perfectoid, then[3]

$$\mathrm{THH}_*(R, \mathbb{Z}_p) = R[x]$$

on a polynomial generator of degree 2. Therefore, as is familiar from complex orientable cohomology theories, the Tate spectral sequence

$$E^2_{i,j} = \hat{H}^{-i}(B\mathbb{T}, \mathrm{THH}_j(R, \mathbb{Z}_p)) \Rightarrow \mathrm{TP}_{i+j}(R, \mathbb{Z}_p)$$

collapses, since all non-zero elements are concentrated in even total degree. However, since the $\mathbb{T}$-action on $\mathrm{THH}(R, \mathbb{Z}_p)$ is non-trivial, the ring homomorphism given by the edge homomorphism of the spectral sequence,

$$\mathrm{TP}_0(R, \mathbb{Z}_p) \xrightarrow{\ \theta\ } \mathrm{THH}_0(R, \mathbb{Z}_p),$$

does not admit a section, and therefore, we cannot identify the domain with a power series algebra over R. Instead, this ring homomorphism is canonically identified with the universal p-complete pro-infinitesimal thickening

$$A = A_{\mathrm{inf}}(R) \xrightarrow{\ \theta\ } R$$

introduced by Fontaine [19], which we recall in Section 1.3 below. In addition, Bhatt–Scholze [6] show that, rather than a formal group over R, there is a canonical p-typical λ-ring structure

$$A \xrightarrow{\ \lambda\ } W(A),$$

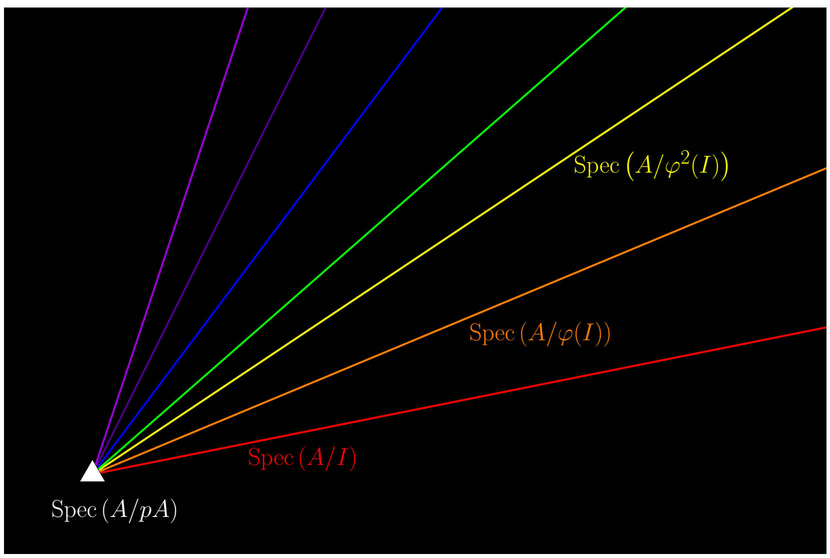
[2] A commutative $\mathbb{Z}_p$-algebra R is perfectoid (resp. semiperfectoid), for example, if there exists a non-zero-divisor $\pi \in R$ with $p \in \pi^p R$ such that the π-adic topology on R is complete and separated and such that the Frobenius $\varphi\colon R/\pi \to R/\pi^p$ a bijection (resp. surjection).

[3] We write $\pi_*(X, \mathbb{Z}_p)$ for the homotopy groups of the p-completion of a spectrum X, and we write $\mathrm{THH}_*(R, \mathbb{Z}_p)$ instead of $\pi_*(\mathrm{THH}(R), \mathbb{Z}_p)$. For p-completion, see Bousfield [12].

the associated Adams operation $\varphi\colon A \to A$ of which is the composition of the inverse of the canonical map $\mathrm{can}\colon \mathrm{TC}_0^-(R,\mathbb{Z}_p) \to \mathrm{TP}_0(R,\mathbb{Z}_p)$ and the Frobenius map $\varphi\colon \mathrm{TC}_0^-(R,\mathbb{Z}_p) \to \mathrm{TP}_0(R,\mathbb{Z}_p)$. The kernel $I \subset A$ of the edge homomorphism $\theta\colon A \to R$ is a principal ideal, and Bhatt–Scholze show that the pair $((A,\lambda), I)$ is a prism in the sense that

$$p \in I + \varphi(I)A,$$

and that this prism is perfect in the sense that $\varphi\colon A \to A$ is an isomorphism. We remark that if $\xi \in I$ is a generator, then, equivalently, the prism condition means that the intersection of the divisors "$\xi = 0$" and "$\varphi(\xi) = 0$" is contained in the special fiber "$p = 0$," whence the name. We thank Riccardo Pengo for the following figure, which illustrates $\mathrm{Spec}(A)$.



Bhatt–Morrow–Scholze further show that, given the choice of ξ, one can choose $u \in \mathrm{TC}_2^-(R,\mathbb{Z}_p)$, $v \in \mathrm{TC}_{-2}^-(R,\mathbb{Z}_p)$, and $\sigma \in \mathrm{TP}_2(R,\mathbb{Z}_p)$ in such a way that

$$\begin{aligned}\mathrm{TC}_*^-(R,\mathbb{Z}_p) &= A[u,v]/(uv-\xi),\\ \mathrm{TP}_*(R,\mathbb{Z}_p) &= A[\sigma^{\pm 1}],\end{aligned}$$

and $\varphi(u) = \alpha \cdot \sigma$, $\varphi(v) = \alpha^{-1}\varphi(\xi)\cdot\xi^{-1}$, $\mathrm{can}(u) = \xi\cdot\sigma$, and $\mathrm{can}(v) = \sigma^{-1}$ with $\alpha \in A$ a unit. In these formulas, the unit α can be eliminated, if one is willing to replace the generator $\xi \in I$ by the generator $\varphi^{-1}(\alpha)\cdot\xi \in I$. We use these results for $R = \mathcal{O}_C$, where C is a complete algebraically closed p-adic field, to give a purely p-adic proof of Bott periodicity. In particular, Bökstedt periodicity implies Bott periodicity, but not vice versa.

The Nikolaus–Scholze approach to topological cyclic homology is also very useful for calculations. To wit, Speirs has much simplified the calculation of the topological cyclic homology of truncated polynomial algebras over a perfect $\mathbb{F}_p$-algebra [34], and we have evaluated the topological cyclic homology of planar cuspical curves over a perfect $\mathbb{F}_p$-algebra [23]. Here, we illustrate this approach in Section 1.4, where we identify the cofiber of the assembly map

$$\mathrm{TC}(R) \otimes BC_{p+} \longrightarrow \mathrm{TC}(R[C_p])$$

for R perfectoid in terms of an analogue of the affine deformation to the normal cone along $I \subset A = A_{\mathrm{inf}}(R)$ with p as the parameter.

Finally, we mention that Bhatt–Morrow–Scholze [5] have constructed weight[4] filtrations of topological cyclic homology and its variants such that, on jth graded pieces, the equalizer of p-completed spectra

$$\mathrm{TC}(S,\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(S,\mathbb{Z}_p) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}(S,\mathbb{Z}_p)$$

gives rise to an equalizer

$$\mathbb{Z}_p(j)[2j] \longrightarrow \mathrm{Fil}^j\, \widehat{\mathbb{\Delta}}_S\{j\}[2j] \underset{\text{“incl”}}{\overset{\text{“}\frac{\varphi}{\xi^j}\text{”}}{\rightrightarrows}} \widehat{\mathbb{\Delta}}_S\{j\}[2j]$$

Here S is any commutative ring, $\widehat{\mathbb{\Delta}}_S = \widehat{\mathbb{\Delta}}_S\{0\}$ is an $\mathbb{E}_\infty$-algebra in the derived ∞-category of S-modules, $\widehat{\mathbb{\Delta}}_S\{j\}$ is an invertible $\widehat{\mathbb{\Delta}}_S$-module, and $\mathrm{Fil}^{\boldsymbol{\cdot}}\, \widehat{\mathbb{\Delta}}_S\{j\}$ is the derived complete descending “Nygaard” filtration thereof. The equalizer $\mathbb{Z}_p(j)$ is a version of syntomic cohomology that works correctly for all weights j, as opposed to only for $j < p-1$. The Bhatt–Morrow–Scholze filtration of $\mathrm{TP}(S,\mathbb{Z}_p)$ gives rise to an Atiyah–Hirzebruch type spectral sequence

$$E^2_{i,j} = H^{j-i}(\mathrm{Spec}(S), \widehat{\mathbb{\Delta}}_S\{j\}) \Rightarrow \mathrm{TP}_{i+j}(S,\mathbb{Z}_p),$$

and similarly for $\mathrm{TC}(S,\mathbb{Z}_p)$ and $\mathrm{TC}^-(S,\mathbb{Z}_p)$. If R is perfectoid, then

$$H^i(\mathrm{Spec}(R), \widehat{\mathbb{\Delta}}_R\{j\}) \simeq \begin{cases} A_{\mathrm{inf}}(R) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, \end{cases}$$

for all integers j, and methods for evaluating these “prismatic” cohomology groups are currently being developed by Bhatt–Scholze [6].

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[4] If S is a smooth $\mathbb{F}_p$-algebra, then, on the jth graded piece of the Bhatt–Morrow–Scholze filtration on $\mathrm{TP}_*(S,\mathbb{Z}_p)[1/p]$, the geometric Frobenius Fr_p acts with pure weight j in the sense that $\mathrm{Fr}_p^* = p^j \varphi_p$, where φ_p is the cyclotomic Frobenius.

1.1 Topological Hochschild homology

We sketch the definition of topological Hochschild homology, topological cyclic homology, and the cyclotomic trace map from algebraic K-theory following Nikolaus–Scholze [32] and Nikolaus [31].

1.1.1 Definition

If R is an $\mathbb{E}_\infty$-algebra in spectra, then we define $\mathrm{THH}(R)$ to be the colimit of the diagram $\mathbb{T} \to \mathrm{Alg}_{\mathbb{E}_\infty}(\mathsf{Sp})$ that is constant with value R, and we write

$$\mathrm{THH}(R) = R^{\otimes \mathbb{T}}$$

to indicate this colimit. Here $\mathbb{T}$ is the circle group. The action of $\mathbb{T}$ on itself by left translation induces a $\mathbb{T}$-action on the $\mathbb{E}_\infty$-algebra $\mathrm{THH}(R)$. In addition, the map of $\mathbb{E}_\infty$-algebras $R \to \mathrm{THH}(R)$ induced by the structure map of the colimit exhibits $\mathrm{THH}(R)$ as the initial $\mathbb{E}_\infty$-algebra with $\mathbb{T}$-action under R.

We let p be a prime number, and let $C_p \subset \mathbb{T}$ be the subgroup of order p. The *Tate diagonal* is a natural map of $\mathbb{E}_\infty$-algebras in spectra

$$R \xrightarrow{\Delta_p} (R^{\otimes C_p})^{tC_p}.$$

Heuristically, this map takes a to the equivalence class of $a \otimes \cdots \otimes a$, but it exist only in higher algebra.[5] Moreover, the map $R \to \mathrm{THH}(R)$ of $\mathbb{E}_\infty$-algebras in spectra extends uniquely to a map $R^{\otimes C_p} \to \mathrm{THH}(R)$ of $\mathbb{E}_\infty$-algebras in spectra with C_p-action, which, in turn, induces a map of Tate spectra

$$(R^{\otimes C_p})^{tC_p} \longrightarrow \mathrm{THH}(R)^{tC_p}.$$

This map also is a map of $\mathbb{E}_\infty$-algebras, and its target carries a residual action of $\mathbb{T}/C_p$, which we identify with $\mathbb{T}$ via the isomorphism given by the pth root. Hence, by the universal property of $R \to \mathrm{THH}(R)$, there is a unique map φ_p of $\mathbb{E}_\infty$-algebras in spectra with $\mathbb{T}$-action which makes the diagram

$$\begin{tikzcd} R \arrow[r, "\Delta_p"] \arrow[d] & (R^{\otimes C_p})^{tC_p} \arrow[d] \\ \mathrm{THH}(R) \arrow[r, "\varphi_p"] & \mathrm{THH}(R)^{tC_p} \end{tikzcd}$$

in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$ commute (in the ∞-categorical sense). The map φ_p is called the pth cyclotomic Frobenius, and the family of maps $(\varphi_p)_{p \in \mathbb{P}}$ indexed by the set $\mathbb{P}$ of prime numbers makes $\mathrm{THH}(R)$ a *cyclotomic* spectrum in the following sense.

[5] In fact, if k is a commutative ring, then the space of natural transformations between the corresponding endofunctors on $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{D}(k))$ is empty.

Definition 1.1.1 (Nikolaus–Scholze)**.** A cyclotomic spectrum is a pair of a spectrum with $\mathbb{T}$-action X and a family $(\varphi_p)_{p\in\mathbb{P}}$ of $\mathbb{T}$-equivariant maps

$$X \xrightarrow{\varphi_p} X^{tC_p}.$$

The ∞-category of cyclotomic spectra is the pullback of simplicial sets

$$\begin{tikzcd}
\mathrm{CycSp} \arrow[r] \arrow[d] & \prod_{p\in\mathbb{P}} \mathrm{Fun}(\Delta^1, \mathrm{Sp}^{B\mathbb{T}}) \arrow[d, "\prod(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\mathrm{Sp}^{B\mathbb{T}} \arrow[r, "(\mathrm{id},(-)^{tC_p})_{p\in\mathbb{P}}"] & \prod_{p\in\mathbb{P}} \mathrm{Sp}^{B\mathbb{T}} \times \mathrm{Sp}^{B\mathbb{T}}.
\end{tikzcd}$$

We remark that, in contrast to the earlier notions of cyclotomic spectra in Hesselholt–Madsen [22] and Blumberg–Mandell [8], the Nikolaus–Scholze definition does not require equivariant homotopy theory.

It is shown in [32] that CycSp is a presentable stable ∞-category, and that it canonically extends to a symmetric monoidal ∞-category

$$\mathrm{CycSp}^{\otimes} \longrightarrow \mathrm{Fin}_*$$

with underlying ∞-category CycSp. Now, the construction of THH given above produces a lax symmetric monoidal functor

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp}^{\otimes}) \xrightarrow{\mathrm{THH}} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{CycSp}^{\otimes}).$$

Topological Hochschild homology may be defined, more generally, for (small) stable ∞-categories $\mathcal{C}$. If R is an $\mathbb{E}_\infty$-algebra in spectra and $\mathcal{C} = \mathrm{Perf}_R$ is the stable ∞-category of perfect R-modules, there is a canonical equivalence

$$\mathrm{THH}(R) \simeq \mathrm{THH}(\mathrm{Perf}_R)$$

of cyclotomic spectra. The basic idea is to define the underlying spectrum with $\mathbb{T}$-action $\mathrm{THH}(\mathcal{C})$ to be the geometric realization of the cyclic spectrum that, in simplicial degree n, is given by

$$\mathrm{THH}(\mathcal{C})_n = \mathrm{colim}\Big(\bigotimes_{0\leq i\leq n} \mathrm{map}_{\mathcal{C}}(x_i, x_{i+1})\Big),$$

where the colimit ranges over the space of $(n+1)$-tuples in the groupoid core $\mathcal{C}^{\simeq}$ of the ∞-category $\mathcal{C}$, $\mathrm{map}_{\mathcal{C}}$ denotes the mapping spectrum in $\mathcal{C}$, and the index i is taken modulo $n+1$. We indicate the steps necessary to make sense of this definition, see [31] and the forthcoming paper [30] for details.

First, to make sense of the colimit above, one must construct a functor

$$(\mathcal{C}^{\simeq})^{n+1} \longrightarrow \mathrm{Sp}$$

that to $(x_0, \dots, x_n)$ assigns $\bigotimes_{0 \leq i \leq n} \mathrm{map}_{\mathcal{C}}(x_i, x_{i+1})$. This can be achieved by a combination of the tensor product functor, the mapping spectrum functor $\mathrm{map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$, and the canonical equivalence $\mathcal{C}^{\simeq} \simeq (\mathcal{C}^{\mathrm{op}})^{\simeq}$. Second, one must lift the assignment $n \mapsto \mathrm{THH}(\mathcal{C})_n$ to a functor $\Lambda^{\mathrm{op}} \to \mathrm{Sp}$ from Connes' cyclic category such that the face and degeneracy maps are given by composing adjacent morphisms and by inserting identities, respectively, while the cyclic operator is given by cyclic permutation of the tensor factors. As explained in [32, Appendix B], for every cyclic spectrum $\Lambda^{\mathrm{op}} \to \mathrm{Sp}$, the geometric realization of the simplicial spectrum $\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}} \to \mathrm{Sp}$ carries a natural $\mathbb{T}$-action. Finally, one must construct the cyclotomic Frobenius maps

$$\mathrm{THH}(\mathcal{C}) \xrightarrow{\varphi_p} \mathrm{THH}(\mathcal{C})^{tC_p}.$$

These are defined following [32, Section III.2] as the Tate-diagonal applied levelwise followed by the canonical colimit-Tate interchange map.

1.1.2 Topological cyclic homology and the trace

Taking R to be the sphere spectrum $\mathbb{S}$, we obtain an $\mathbb{E}_\infty$-algebra in cyclotomic spectra $\mathrm{THH}(\mathbb{S})$, which we denote by $\mathbb{S}^{\mathrm{triv}}$. Its underlying spectrum is $\mathbb{S}$, and its cyclotomic Frobenius map φ_p can be identified with a canonical $\mathbb{T}$-equivariant refinement of the composition

$$\mathbb{S} \xrightarrow{\mathrm{triv}} \mathbb{S}^{hC_p} \xrightarrow{\mathrm{can}} \mathbb{S}^{tC_p}$$

of the map triv induced from the projection $BC_p \to \mathrm{pt}$ and the canonical map. Since the ∞-category CycSp is stable, we have associated with every pair of objects $X, Y \in \mathrm{CycSp}$ a mapping spectrum $\mathrm{map}_{\mathrm{CycSp}}(X, Y)$, which depends functorially on X and Y.

Definition 1.1.2. The topological cyclic homology of a cyclotomic spectrum X is the mapping spectrum $\mathrm{TC}(X) = \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X)$.

If $X = \mathrm{THH}(R)$ with R an $\mathbb{E}_\infty$-algebra in spectra, then we abbreviate and write $\mathrm{TC}(R)$ instead of $\mathrm{TC}(X)$. Similarly, if $X = \mathrm{THH}(\mathcal{C})$ with $\mathcal{C}$ a stable ∞-category, then we write $\mathrm{TC}(\mathcal{C})$ instead of $\mathrm{TC}(X)$.

This definition of topological cyclic homology as given above is abstractly elegant and useful, but for concrete calculations, a more concrete formula is necessary. Therefore, we unpack Definition 1.1.2. We assume that X is bounded below and write

$$\mathrm{TC}^{-}(X) \xrightarrow{\mathrm{can}} \mathrm{TP}(X)$$

for the canonical map $X^{h\mathbb{T}} \to X^{t\mathbb{T}}$. We call the domain and target of this map the negative topological cyclic homology and the periodic topological cyclic homology of X, respectively. Since the cyclotomic Frobenius maps

$$X \xrightarrow{\varphi_p} X^{tC_p}$$

are $\mathbb{T}$-equivariant, they give rise to a map of $\mathbb{T}$-homotopy fixed points spectra

$$X^{h\mathbb{T}} \xrightarrow{(\varphi_p^{h\mathbb{T}})} \prod_{p\in\mathbb{P}} (X^{tC_p})^{h\mathbb{T}}.$$

Moreover, there is a canonical map

$$\prod_{p\in\mathbb{P}} (X^{tC_p})^{h\mathbb{T}} \longleftarrow X^{t\mathbb{T}},$$

which becomes an equivalence after profinite completion, since X is bounded below, by the Tate-orbit lemma [32, Lemma II.4.2]. Hence, we get a map

$$\mathrm{TC}^-(X) \xrightarrow{\varphi} \mathrm{TP}(X)^\wedge,$$

where "$(-)^\wedge$" indicates profinite completion. There is also a canonical map

$$\mathrm{TC}^-(X) \xrightarrow{\mathrm{can}} \mathrm{TP}(X)^\wedge$$

given by the composition of the canonical from the homotopy fixed point spectrum to the Tate construction followed by the completion map. This gives the following description of $\mathrm{TC}(X)$.

Proposition 1.1.3. *For bounded below cyclotomic spectra X, there is natural equalizer diagram*

$$\mathrm{TC}(X) \longrightarrow \mathrm{TC}^-(X) \overset{\varphi}{\underset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(X)^\wedge.$$

We now explain the definition of the cyclotomic trace map from K-theory to topological cyclic homology. Let $\mathrm{Cat}_\infty^{\mathrm{stab}}$ be the ∞-category of small, stable ∞-categories and exact functors. The ∞-category of noncommutative motives (or a slight variant thereof) of Blumberg–Gepner–Tabuada [7] is defined to be the initial (large) ∞-category with a functor

$$z : \mathrm{Cat}_\infty^{\mathrm{stab}} \longrightarrow \mathrm{NMot}$$

such that the following hold:

(1) (Stability) The ∞-category NMot is stable.

(2) (Localization) For every Verdier sequence[6] $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ in $\mathrm{Cat}_\infty^{\mathrm{stab}}$, the image sequence $z(\mathcal{C}) \to z(\mathcal{D}) \to z(\mathcal{D}/\mathcal{C})$ in NMot is a fiber sequence.

(3) (Morita invariance) For every map $\mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{\mathrm{stab}}$ that becomes an equivalence after idempotent completion, the image map $z(\mathcal{C}) \to z(\mathcal{D})$ in NMot is an equivalence.

[6] This means that $\mathcal{C} \to \mathcal{D} \to \mathcal{C}/\mathcal{D}$ is both a fiber sequence and cofiber sequence in $\mathrm{Cat}_\infty^{\mathrm{stab}}$. In particular, $\mathcal{D} \to \mathcal{C}$ is fully faithful and its image in $\mathcal{D}$ is closed under retracts.

The main theorem of op. cit. states[7] that for every (small) stable ∞-category $\mathcal{C}$, there is a canonical equivalence

$$K(\mathcal{C}) \simeq \mathrm{map}_{\mathrm{NMot}}(z(\mathrm{Perf}_{\mathbb{S}}), z(\mathcal{C}))$$

between its nonconnective algebraic K-theory spectrum and the indicated mapping spectrum in NMot. In general, one may view the mapping spectra in NMot as bivariant versions of nonconnective algebraic K-theory. Accordingly, the mapping spectra in CycSp are bivariant versions of TC. As we outlined in the previous section, topological Hochschild homology is a functor

$$\mathrm{Cat}_{\infty}^{\mathrm{stab}} \xrightarrow{\mathrm{THH}} \mathrm{CycSp},$$

and one can show that it satisfies the properties (1)–(3) above. There is a very elegant proof of (2) and (3) based on work of Keller, Blumberg–Mandell, and Kaledin that uses the trace property of THH, see the forthcoming paper [30] for a summary. Accordingly, the functor THH admits a unique factorization

$$\mathrm{Cat}_{\infty}^{\mathrm{stab}} \xrightarrow{z} \mathrm{NMot} \xrightarrow{\mathrm{tr}} \mathrm{CycSp}$$

with tr exact. In particular, for every stable ∞-category $\mathcal{C}$, we have an induced map of mapping spectra

$$\mathrm{map}_{\mathrm{NMot}}(z(\mathrm{Perf}_{\mathbb{S}}), z(\mathcal{C})) \xrightarrow{\mathrm{tr}} \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, \mathrm{THH}(\mathcal{C})).$$

This map, by definition, is the cyclotomic trace map, which we write

$$K(\mathcal{C}) \xrightarrow{\mathrm{tr}} \mathrm{TC}(\mathcal{C}).$$

More concretely, on connective covers, considered here as $\mathbb{E}_{\infty}$-groups in spaces, the cyclotomic trace map is given by the composition

$$\begin{aligned} \Omega^{\infty} K(\mathcal{C}) \simeq \Omega(\mathrm{colim}_{\Delta^{\mathrm{op}}}(S\mathcal{C}^{\mathrm{idem}})^{\simeq}) &\to \Omega(\mathrm{colim}_{\Delta^{\mathrm{op}}} \Omega^{\infty}\,\mathrm{TC}(S\mathcal{C}^{\mathrm{idem}})) \\ &\to \Omega^{\infty}\Omega(\mathrm{colim}_{\Delta^{\mathrm{op}}}\mathrm{TC}(S\mathcal{C}^{\mathrm{idem}})) \simeq \Omega^{\infty}\,\mathrm{TC}(\mathcal{C}), \end{aligned}$$

where $S(-)$ and $(-)^{\mathrm{idem}}$ indicate Waldhausen's construction and idempotent completion, respectively, where the second map is induced from the map

$$\mathcal{D}^{\simeq} \simeq \mathrm{Map}_{\mathrm{Cat}_{\infty}^{\mathrm{stab}}}(\mathrm{Perf}_{\mathbb{S}}, \mathcal{D}) \xrightarrow{\mathrm{THH}} \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, \mathrm{THH}(\mathcal{D})) \simeq \Omega^{\infty}\,\mathrm{TC}(\mathcal{D}),$$

and where last equivalence follows from TC satisfying (2) and (3) above.

[7] In fact, we do not require $z\colon \mathrm{Cat}_{\infty}^{\mathrm{stab}} \to \mathrm{NMot}$ to preserve filtered colimits, as do [7].

1.1.3 Connes' operator

The symmetric monoidal ∞-category of spectra with $\mathbb{T}$-action is canonically equivalent to the symmetric monoidal ∞-category of modules over the group algebra $\mathbb{S}[\mathbb{T}]$. The latter is an $\mathbb{E}_\infty$-algebra in spectra, and

$$\pi_*(\mathbb{S}[\mathbb{T}]) = (\pi_*\mathbb{S})[d]/(d^2 - \eta d),$$

where d has degree 1 and is obtained from a choice of generator of $\pi_1(\mathbb{T})$ by translating it to the basepoint in the group ring. The relation $d^2 = \eta d$ is a consequence of the fact that, stably, the multiplication map $\mu\colon \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ splits off the Hopf map $\eta \in \pi_1(\mathbb{S})$. From this calculation we conclude that a $\mathbb{T}$-action on an $\mathbb{E}_\infty$-algebra in spectra T gives rise to a graded derivation

$$\pi_j(T) \xrightarrow{d} \pi_{j+1}(T),$$

and this is Connes' operator. The operator d is not quite a differential, since we have $d \circ d = d \circ \eta = \eta \circ d$.

The $\mathbb{E}_\infty$-algebra structure on T gives rise to power operations in homology. In singular homology with $\mathbb{F}_2$-coefficients, there are power operations

$$\pi_j(\mathbb{F}_2 \otimes T) \xrightarrow{Q^i} \pi_{i+j}(\mathbb{F}_2 \otimes T)$$

for all integers i introduced by Araki–Kudo [25], and in singular homology with $\mathbb{F}_p$-coefficients, where p is odd, there are similar power operations

$$\pi_j(\mathbb{F}_p \otimes T) \xrightarrow{Q^i} \pi_{2i(p-1)+j}(\mathbb{F}_p \otimes T)$$

for all integers i defined by Dyer–Lashof [17]. The power operations are natural with respect to maps of $\mathbb{E}_\infty$-rings, but it is not immediately clear that they are compatible with Connes' operator, too. We give a proof that this is the nevertheless the case, following Angeltveit–Rognes [1, Proposition 5.9] and the very nice exposition of Höning [24].

Proposition 1.1.4. *If T is an $\mathbb{E}_\infty$-ring with $\mathbb{T}$-action, then*

$$Q^i \circ d = d \circ Q^i$$

for all integers i.

Proof. The adjunct $\tilde{\mu}\colon T \to \operatorname{map}(\mathbb{T}_+, T)$ of the map $\mu\colon T \otimes \mathbb{T}_+ \to T$ induced by the $\mathbb{T}$-action on T is a map of $\mathbb{E}_\infty$-rings, as is the canonical equivalence $T \otimes \operatorname{map}(\mathbb{T}_+, \mathbb{S}) \to \operatorname{map}(\mathbb{T}_+, T)$. Composing the former map with an inverse of the latter map, we obtain a map of $\mathbb{E}_\infty$-rings

$$T \xrightarrow{\tilde{d}} T \otimes \operatorname{map}(\mathbb{T}_+, \mathbb{S}),$$

and hence, the induced map on homology preserves power operations. We identify the homology of the target via the isomorphism

$$\pi_*(\mathbb{F}_p \otimes T) \otimes_{\pi_*(\mathbb{F}_p)} \pi_*(\mathbb{F}_p \otimes \mathrm{map}(\mathbb{T}_+, \mathbb{S})) \longrightarrow \pi_*(\mathbb{F}_p \otimes T \otimes \mathrm{map}(\mathbb{T}_+, \mathbb{S}))$$

and the direct sum decomposition of $\mathrm{map}(\mathbb{T}_+, \mathbb{S})$ induced by the direct sum decomposition above, and under this idenfication, we have

$$\tilde{d}(a) = a \otimes 1 + da \otimes [S^{-1}].$$

Now, the Cartan formula for power operations shows that

$$Q^i(\tilde{d}(a)) = Q^i(a \otimes 1 + da \otimes [S^{-1}]) = Q^i(a) \otimes 1 + Q^i(da) \otimes [S^{-1}],$$

since $Q^0([S^{-1}]) = [S^{-1}]$ and $Q^i([S^{-1}]) = 0$ for $i \neq 0$. But we also have

$$Q^i(\tilde{d}(a)) = \tilde{d}(Q^i(a)) = Q^i(a) \otimes 1 + d(Q^i(a)) \otimes [S^{-1}],$$

so we conclude that $Q^i \circ d = d \circ Q^i$ as stated. □

We finally discuss the HKR-filtration. If k is a commutative ring and A a simplicial commutative k-algebra, then the Hochschild spectrum

$$\mathrm{HH}(A/k) = A^{\otimes_k \mathbb{T}}$$

has a complete and $\mathbb{T}$-equivariant descending filtration

$$\cdots \subseteq \mathrm{Fil}^n \mathrm{HH}(A/k) \subseteq \cdots \subseteq \mathrm{Fil}^1 \mathrm{HH}(A/k) \subseteq \mathrm{Fil}^0 \mathrm{HH}(A/k) \simeq \mathrm{HH}(A/k)$$

defined as follows. If A/k is smooth and discrete, then

$$\mathrm{Fil}^n \mathrm{HH}(A/k) = \tau_{\geq n} \mathrm{HH}(A/k),$$

and in general, the filtration is obtained from this special case by left Kan extension. The filtration quotients are identified as follows. If A/k is discrete, then $\mathrm{HH}(A/k)$ may be represented by a simplicial commutative A-algebra, and hence, its homotopy groups $\mathrm{HH}_*(A/k)$ form a strictly[8]anticommutative graded A-algebra. Moreover, Connes' operator gives rise to a differential on $\mathrm{HH}_*(A/k)$, which raises degrees by one and is a graded k-linear derivation. By definition, the de Rham-complex $\Omega^*_{A/k}$ is the universal example of this algebraic structure, and therefore, we have a canonical map

$$\Omega^*_{A/k} \longrightarrow \mathrm{HH}_*(A/k),$$

which, by the Hochschild–Kostant–Rosenberg theorem, is an isomorphism, if A/k smooth. By the definition of the cotangent complex, this shows that

$$\mathrm{gr}^j \mathrm{HH}(A/k) \simeq (\Lambda^j_A L_{A/k})[j]$$

with trivial $\mathbb{T}$-action. Here Λ^j_A indicates the non-abelian derived functor of the jth exterior power over A.

[8] Here "strictly" indicates that elements of odd degree square to zero. This follows from [13, Théorème 4] by considering the universal case of Eilenberg–MacLane spaces.

1.2 Bökstedt periodicity

Bökstedt periodicity is the fundamental result that $\mathrm{THH}_*(\mathbb{F}_p)$ is a polynomial algebra over $\mathbb{F}_p$ on a generator in degree two. We present a proof, which is close to Bökstedt's original proof in the unpublished manuscript [10]. The skeleton filtration of the standard simplicial model for the circle induces a filtration of the topological Hochschild spectrum. For every homology theory, this gives rise to a spectral sequence, called the Bökstedt spectral sequence, that converges to the homology of the topological Hochschild spectrum. It is a spectral sequence of Hopf algebras in the symmetric monoidal category of quasi-coherent sheaves on the stack defined by the homology theory in question, and to handle this rich algebraic structure, we find it useful to introduce the geometric language of Berthelot and Grothendieck [3, II.1].

1.2.1 The Adams spectral sequence

If $f\colon A \to B$ is a map of anticommutative graded rings, then extension of scalars along f and restriction of scalars along f define adjoint functors

$$\mathrm{Mod}_A \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}_B$$

between the respective categories of graded modules. Moreover, the extension of scalars functor f^* is symmetric monoidal, while the restriction of scalars functor f_* is lax symmetric monoidal with respect to the tensor product of graded modules.

We let k be an $\mathbb{E}_\infty$-ring and form the cosimplicial $\mathbb{E}_\infty$-ring

$$\Delta \xrightarrow{k^{\otimes[-]}} \mathrm{Alg}_{\mathbb{E}_\infty}(\mathsf{Sp}).$$

Here, as usual, $[n] \in \Delta$ denotes the finite ordinal $\{0, 1, \dots, n\}$, so $k^{\otimes[n]}$ is an $(n+1)$-fold tensor product. We will assume that the map

$$A = \pi_*(k^{\otimes[0]}) \xrightarrow{d^1} \pi_*(k^{\otimes[1]}) = B$$

is flat, so that $d^0, d^2\colon k^{\otimes[1]} \to k^{\otimes[2]}$ induce an isomorphism of graded rings

$$B \otimes_A B = \pi_*(k^{\otimes[1]}) \otimes_{\pi_*(k^{\otimes[0]})} \pi_*(k^{\otimes[1]}) \xrightarrow{d^2+d^0} \pi_*(k^{\otimes[2]}).$$

The map $d^1\colon k^{\otimes[1]} \to k^{\otimes[2]}$ now gives rise to a map of graded rings

$$B \xrightarrow{d^1} B \otimes_A B$$

and the sextuple

$$(A, B,\ B \xrightarrow{s^0} A\ ,\ A \underset{d^1}{\overset{d^0}{\rightrightarrows}} B\ ,\ B \xrightarrow{d^1} B \otimes_A B\ )$$

forms a cocategory object in the category of graded rings with the cocartesian symmetric monoidal structure. Here the maps $s^0\colon A \to B$, $d^0, d^1\colon A \to B$, and $d^1\colon B \to B \otimes_A B$ are the opposites of the unit map, the source and target maps, and the composition map. Likewise, the septuple, where we also include the map $\chi\colon B \to B$ induced by the unique non-identity automorphism of the set $[1] = \{0, 1\}$, forms a cogroupoid in this symmetric monoidal category. We will abbreviate and simply write (A, B) for this cogroupoid object.

In general, given a cogroupoid object (A, B) in graded rings, we define an (A, B)-module[9] to be a pair (M, ϵ) of an A-module M and a B-linear map

$$d^{1*}(M) \xrightarrow{\epsilon} d^{0*}(M)$$

that makes the following diagrams, in which the equality signs indicate the unique isomorphisms, commute.

$$\begin{tikzcd}
s^{0*}d^{1*}(M) \arrow[rr, "s^{0*}(\epsilon)"] \arrow[dr, equal] & & s^{0*}d^{0*}(M) \arrow[dl, equal] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
& d^{1*}d^{1*}(M) \arrow[rr, "d^{1*}(\epsilon)"] \arrow[dl, equal] & & d^{1*}d^{0*}(M) \arrow[dr, equal] & \\
(B \otimes d^1)^* d^{1*}(M) \arrow[dr, "(B\otimes d^1)^*(\epsilon)"'] & & & & (d^0 \otimes B)^* d^{0*}(M) \\
& (B \otimes d^1)^* d^{0*}(M) \arrow[rr, equal] & & (d^0 \otimes B)^* d^{1*}(M) \arrow[ur, "(d^0\otimes B)^*(\epsilon)"'] &
\end{tikzcd}$$

We say that ϵ is a stratification of the A-module M relative to the cogroupoid (A, B). The map ϵ, we remark, is necessarily an isomorphism. We define a map of (A, B)-modules $f\colon (M_0, \epsilon_0) \to (M_1, \epsilon_1)$ to be a map of A-modules $f\colon M_0 \to M_1$ that makes the diagram of B-modules

$$\begin{tikzcd}
d^{1*}(M_0) \arrow[r, "\epsilon_0"] \arrow[d, "d^{0*}(f)"] & d^{0*}(M_0) \arrow[d, "d^{1*}(f)"] \\
d^{0*}(M_1) \arrow[r, "\epsilon_1"] & d^{1*}(M_1)
\end{tikzcd}$$

[9] The cogroupoid (A, B) defines a stack $\mathcal{X}$, and the categories of (A, B)-modules and quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules are equivalent. For this reason, we prefer to say (A, B)-module instead of (A, B)-comodule, as is more common in the homotopy theory literature.

commute. In this case, we also say that the A-linear map $f\colon M_0 \to M_1$ is horizontal with respect ϵ. The category $\mathrm{Mod}_{(A,B)}$ of (A,B)-modules admits a symmetric monoidal structure with the monoidal product defined by

$$(M_1, \epsilon_1) \otimes_{(A,B)} (M_2, \epsilon_2) = (M_1 \otimes_A M_2, \epsilon_{12}),$$

where ϵ_{12} is the unique map that makes the diagram

$$\begin{tikzcd}
d^{1*}(M_1) \otimes_B d^{1*}(M_2) \arrow[r, equal] \arrow[d, "\epsilon_1 \otimes \epsilon_2"] & d^{1*}(M_1 \otimes_A M_2) \arrow[d, "\epsilon_{12}"] \\
d^{0*}(M_1) \otimes_B d^{0*}(M_2) \arrow[r, equal] & d^{0*}(M_1 \otimes_A M_2)
\end{tikzcd}$$

commute. The unit for the monoidal product is given by the A-module A with its unique structure of (A,B)-module, where $\epsilon\colon d^{1*}(A) \to d^{0*}(A)$ is the unique B-linear map that makes the diagram

$$\begin{tikzcd}
s^{0*}d^{1*}(A) \arrow[rr, "s^{0*}(\epsilon)"] \arrow[dr, equal] & & s^{0*}d^{0*}(A) \arrow[dl, equal] \\
& A &
\end{tikzcd}$$

commute.

We again let (A,B) be the cogroupoid associated with the $\mathbb{E}_\infty$-ring k. For every spectrum X, we consider the cosimplicial spectrum $k^{\otimes[-]} \otimes X$. The homotopy groups $\pi_*(k^{\otimes[0]} \otimes X)$ and $\pi_*(k^{\otimes[1]} \otimes X)$ form a left A-module and a left B-module, respectively. Moreover, we have A-linear maps

$$\pi_*(k^{\otimes[0]} \otimes X) \xrightarrow{d^i} d^i_*(\pi_*(k^{\otimes[1]} \otimes X))$$

induced by $d^i\colon k^{\otimes[0]} \otimes X \to k^{\otimes[1]} \otimes X$, and their adjunct maps

$$d^{i*}(\pi_*(k^{\otimes[0]} \otimes X)) \xrightarrow{\widetilde{d^i}} \pi_*(k^{\otimes[1]} \otimes X)$$

are B-linear isomorphisms. We now define the (A,B)-module associated with the spectrum X to be the pair (M,ϵ) with

$$M = \pi_*(k^{\otimes[0]} \otimes X)$$

and with ϵ the unique map that makes the following diagram commute.

$$\begin{tikzcd}
d^{1*}(M) \arrow[rr, "\epsilon"] \arrow[dr, "\widetilde{d^1}"'] & & d^{0*}(M) \arrow[dl, "\widetilde{d^0}"] \\
& \pi_*(k^{\otimes[1]} \otimes X) &
\end{tikzcd}$$

We often abbreviate and write $\pi_*(k \otimes X)$ for the (A,B)-module (M,ϵ).

The skeleton filtration of the cosimplicial spectrum $k^{\otimes[-]} \otimes X$ gives rise to the conditionally convergent k-based Adams spectral sequence

$$E^2_{i,j} = \operatorname{Ext}^{-i}_{(\pi_*(k),\pi_*(k\otimes k))}(\pi_*(k), \pi_*(k \otimes X))_j \Rightarrow \pi_{i+j}(\lim_\Delta k^{\otimes[-]} \otimes X),$$

where the Ext-groups are calculated in the abelian category of modules over the cogroupoid $(\pi_*(k), \pi_*(k \otimes k))$. An $\mathbb{E}_\infty$-algebra structure on X gives rise to a commutative monoid structure on $\pi_*(k \otimes X)$ in the symmetric monoidal category of $(\pi_*(k), \pi_*(k \otimes k))$-modules and makes the spectral sequence one of bigraded rings.

If X is a k-module, then the augmented cosimplicial spectrum

$$X \xrightarrow{d^0} k^{\otimes[-]} \otimes X$$

acquires a nullhomotopy. Therefore, the spectral sequence collapses and its edge homomorphism becomes an isomorphism

$$\pi_j(X) \longrightarrow \operatorname{Hom}_{(\pi_*(k),\pi_*(k\otimes k))}(\pi_*(k), \pi_*(k \otimes X))_j.$$

This identifies $\pi_j(X)$ with the subgroup of elements $x \in \pi_j(k \otimes X)$ that are horizontal[10] with respect to the stratification relative to $(\pi_*(k), \pi_*(k \otimes k))$ in the sense that $\epsilon(1 \otimes_{A,d^1} x) = 1 \otimes_{A,d^0} x$.

1.2.2 The Bökstedt spectral sequence

In general, if R is an $\mathbb{E}_\infty$-ring, then

$$\operatorname{THH}(R) \simeq R^{\otimes \operatorname{colim}_{\Delta^{\mathrm{op}}} \Delta^1[-]/\partial\Delta^1[-]} \simeq \operatorname{colim}_{\Delta^{\mathrm{op}}} R^{\otimes \Delta^1[-]/\partial\Delta^1[-]}.$$

A priori, the right-hand term is the colimit in $\operatorname{Alg}_{\mathbb{E}_\infty}(\mathsf{Sp})$, but since the index category Δ^{op} is sifted [28, Lemma 5.5.8.4], the colimit agrees with the one in Sp. The increasing filtration of $\Delta^1[-]/\partial\Delta^1[-]$ by the skeleta induces an increasing filtration of $\operatorname{THH}(R)$,

$$\operatorname{Fil}_0 \operatorname{THH}(R) \to \operatorname{Fil}_1 \operatorname{THH}(R) \to \cdots \to \operatorname{Fil}_n \operatorname{THH}(R) \to \cdots.$$

We let k be an $\mathbb{E}_\infty$-ring and let (A, B) be the associated cogroupoid in graded rings, where $A = \pi_*(k)$ and $B = \pi_*(k \otimes k)$. We also let C be the commutative monoid $\pi_*(k \otimes R)$ in the symmetric monoidal category of (A, B)-modules. Here we assume that $d^0\colon A \to B$ and $\eta\colon A \to C$ are flat. The filtration above gives rise to a spectral sequence

$$E^2_{i,j} = \operatorname{HH}_i(C/A)_j \Rightarrow \pi_{i+j}(k \otimes \operatorname{THH}(R)),$$

[10] In comodule nomenclature, horizontal elements are called primitive elements.

called the Bökstedt spectral sequence. It is a spectral sequence of C-algebras in the symmetric monoidal category of (A, B)-modules, and Connes' operator on $\pi_*(k \otimes \mathrm{THH}(R))$ induces a map

$$E^r_{i,j} \xrightarrow{d} E^r_{i+1,j}$$

of spectral sequences, which, on the E^2-term, is equal to Connes' operator

$$\mathrm{HH}_i(C/A)_j \xrightarrow{d} \mathrm{HH}_{i+1}(C/A)_j.$$

In particular, if $y \in E^2_{i,j}$ and $dy \in E^2_{i+1,j}$ both survive the spectral sequence, and if y represents a homology class $\tilde{y} \in \pi_{i+j}(k \otimes \mathrm{THH}(R))$, then dy represents the homology class $d\tilde{y} \in \pi_{i+j+1}(k \otimes \mathrm{THH}(R))$.

Theorem 1.2.1 (Bökstedt)**.** *The canonical map of graded $\mathbb{F}_p$-algebras*

$$\mathrm{Sym}_{\mathbb{F}_p}(\mathrm{THH}_2(\mathbb{F}_p)) \longrightarrow \mathrm{THH}_*(\mathbb{F}_p)$$

is an isomorphism, and $\mathrm{THH}_2(\mathbb{F}_p)$ is a 1-dimensional $\mathbb{F}_p$-vector space.

Proof. We let $k = R = \mathbb{F}_p$ and continue to write $A = \pi_*(k)$, $B = \pi_*(k \otimes k)$, and $C = \pi_*(k \otimes R)$. We apply the Bökstedt spectral sequence to show that, as a C-algebra in the symmetric monoidal category of (A, B)-modules,

$$\pi_*(k \otimes \mathrm{THH}(R)) = C[x]$$

on a horizontal generator x of degree 2, and use the Adams spectral sequence to conclude that $\mathrm{THH}_*(R) = R[x]$, as desired. Along the way, we will use the fact, observed by Angeltveit–Rognes [1], that the maps

$$S^1 \underset{\phi}{\overset{\psi}{\rightleftarrows}} S^1 \vee S^1 \qquad \{\infty\} \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} S^1 \qquad S^1 \xrightarrow{\chi} S^1$$

where ψ and ϕ are the pinch and fold maps, η and ε the unique maps, and χ the flip map, give $\pi_*(k \otimes \mathrm{THH}(R))$ the structure of a C-Hopf algebra in the symmetric monoidal category of (A, B)-modules, assuming that the unit map is flat. Moreover, the Bökstedt spectral sequence is a spectral sequence of C-Hopf algebras, provided that the unit map $C \to E^r$ is flat for all $r \geq 2$. We remark that the requirement that the comultiplication on $E = E^r$ be a map of C-modules in the symmetric monoidal category of (A, B)-modules is equivalent to the requirement that the diagram

$$\begin{tikzcd}
d^{1*}(E) \arrow[r, "d^{1*}(\psi)"] \arrow[d, "\epsilon_E"] & d^{1*}(E \otimes_C E) \arrow[r, equal] & d^{1*}(E) \otimes_{d^{1*}(C)} d^{1*}(E) \arrow[d, "\epsilon_E \otimes_{\epsilon_C} \epsilon_E"] \\
d^{0*}(E) \arrow[r, "d^{0*}(\psi)"] & d^{0*}(E \otimes_C E) \arrow[r, equal] & d^{0*}(E) \otimes_{d^{0*}(C)} d^{0*}(E)
\end{tikzcd}$$

commutes.

To begin, we recall from Milnor [29] that, as a graded A-algebra,

$$C = \begin{cases} A[\,\bar{\xi}_i \mid i \geq 1\,] & \text{for } p = 2, \\ A[\,\bar{\xi}_i \mid i \geq 1\,] \otimes_A \Lambda_A\{\bar{\tau}_i \mid i \geq 0\} & \text{for } p \text{ odd,} \end{cases}$$

where $\bar{\tau}_i = \chi(\tau_i)$ and $\bar{\xi}_i = \chi(\xi_i)$ are the images by the antipode of Milnor's generators τ_i and ξ_i. Here, for $p = 2$, $\deg(\bar{\xi}_i) = p^i - 1$, while for p odd, $\deg(\bar{\xi}_i) = 2(p^i - 1)$ and $\deg(\bar{\tau}_i) = 2p^i - 1$. The stratification

$$d^{1*}(C) \xrightarrow{\epsilon} d^{0*}(C)$$

is given by

$$\epsilon(1 \otimes_{A,d^1} \bar{\xi}_i) = \sum \bar{\xi}_s \otimes_{A,d^0} \bar{\xi}_t^{p^s}$$
$$\epsilon(1 \otimes_{A,d^1} \bar{\tau}_i) = 1 \otimes_{A,d^0} \bar{\tau}_i + \sum \bar{\tau}_s \otimes_{A,d^0} \bar{\xi}_t^{p^s}$$

with the sums indexed by $s, t \geq 0$ with $s + t = i$. Moreover, we recall from Steinberger [35] that the power operations on $C = \pi_*(k \otimes R)$ satisfy

$$Q^{p^i}(\bar{\xi}_i) = \bar{\xi}_{i+1},$$
$$Q^{p^i}(\bar{\tau}_i) = \bar{\tau}_{i+1}.$$

A very nice brief account of this calculation is given in [38].

We first consider $p = 2$. The E^2-term of the Bökstedt spectral sequence, as a C-Hopf algebra in (A, B)-modules, takes the form

$$E^2 = \Lambda_C\{d\bar{\xi}_i \mid i \geq 1\}$$

with $\deg(\bar{\xi}_i) = (0, 2^i - 1)$ and $\deg(d\bar{\xi}_i) = (1, 2^i - 1)$, and all differentials in the spectral sequence vanish. Indeed, they are C-linear derivations and, for degree reasons, the algebra generators $d\bar{\xi}_i$ cannot support non-zero differentials. We define x to be the image of $\bar{\xi}_1$ by the composite map

$$\pi_1(k \otimes R) \xrightarrow{\eta} \pi_1(k \otimes \mathrm{THH}(R)) \xrightarrow{d} \pi_2(k \otimes \mathrm{THH}(R)),$$

and proceed to show, by induction on $i \geq 0$, that the homology class x^{2^i} is represented by the element $d\bar{\xi}_{i+1}$ in the spectral sequence. The case $i = 0$ follows from what was said above, so we assume the statement has been proved for $i = r - 1$ and prove it for $i = r$. We have

$$x^{2^r} = (x^{2^{r-1}})^2 = Q^{2^r}(x^{2^{r-1}}),$$

which, by induction, is represented by $Q^{2^r}(d\bar{\xi}_r)$. But Proposition 1.1.4 and Steinberger's calculation show that

$$Q^{2^r}(d\bar{\xi}_r) = d(Q^{2^r}(\bar{\xi}_r)) = d\bar{\xi}_{r+1},$$

so we conclude that x^{2^r} is represented by $d\bar{\xi}_{r+1}$. Hence, as a graded C-algebra,

$$\pi_*(k \otimes \mathrm{THH}(R)) = C[x].$$

Finally, we calculate that

$$\begin{aligned}
\epsilon(1 \otimes_{A,d^1} x) &= \epsilon((\mathrm{id} \otimes_{A,d^1} d)(\eta \otimes_{A,d^1} \eta)(1 \otimes_{A,d^1} \bar{\xi}_1)) \\
&= (\mathrm{id} \otimes_{A,d^0} d)(\eta \otimes_{A,d^0} \eta)(\epsilon(1 \otimes_{A,d^1} \bar{\xi}_1)) \\
&= (\mathrm{id} \otimes_{A,d^0} d)(\eta \otimes_{A,d^0} \eta)(\bar{\xi}_1 \otimes_{A,d^0} 1 + 1 \otimes_{A,d^0} \bar{\xi}_1) \\
&= 1 \otimes_{A,d^0} x,
\end{aligned}$$

which shows that the element x is horizontal with respect to the stratification of $\pi_*(k \otimes \mathrm{THH}(R))$ relative to (A, B).

We next let p be odd. As a C-Hopf algebra,

$$E^2 = \Lambda_C\{d\bar{\xi}_i \mid i \geq 1\} \otimes_C \Gamma_C\{d\bar{\tau}_i \mid i \geq 0\}$$

with $\deg(\bar{\xi}_i) = (0, 2p^i - 2)$, $\deg(d\bar{\xi}_i) = (1, 2p^i - 2)$, $\deg(\bar{\tau}_i) = (0, 2p^i - 1)$, and $\deg(d\bar{\tau}_i) = (1, 2p^i - 1)$, and with the coproduct given by

$$\begin{aligned}
\psi(d\bar{\xi}_i) &= 1 \otimes d\bar{\xi}_i + d\bar{\xi}_i \otimes 1, \\
\psi((d\bar{\tau}_i)^{[r]}) &= \textstyle\sum (d\bar{\tau}_i)^{[s]} \otimes (d\bar{\tau}_i)^{[t]},
\end{aligned}$$

where the sum ranges over $s, t \geq 0$ with $s + t = r$. Here $(-)^{[r]}$ indicates the rth divided power. We define x to be the image of $\bar{\tau}_0$ by the composite map

$$\pi_1(k \otimes R) \xrightarrow{\eta} \pi_1(k \otimes \mathrm{THH}(R)) \xrightarrow{d} \pi_2(k \otimes \mathrm{THH}(R))$$

and see, as in the case $p = 2$, that the homology class x^{p^i} is represented by the element $d\bar{\tau}_i$. This is also shows that for $i \geq 1$, the element

$$d\bar{\xi}_i = d(\beta(\bar{\tau}_i)) = \beta(d\bar{\tau}_i)$$

represents the homology class $\beta(x^{p^i}) = p^i x^{p^i - 1}\beta(x)$, which is zero. Hence, this element is annihilated by some differential. We claim that for all $i, s \geq 0$,

$$d^{p-1}((d\bar{\tau}_i)^{[p+s]}) = a_i \cdot d\bar{\xi}_{i+1} \cdot (d\bar{\tau}_i)^{[s]}$$

with $a_i \in k^*$ a unit that depends on i but not on s. Grating this, we find as in the case $p = 2$ that, as a C-algebra,

$$\pi_*(k \otimes \mathrm{THH}(R)) = C[x]$$

with x horizontal of degree 2, which proves the theorem.

To prove the claim, we note that a shortest possible non-zero differential between elements of lowest possible total degree factors as a composition

$$E^r_{i,j} \xrightarrow{\pi} QE^r_{i,j} \xrightarrow{\bar{d}^r} PE^r_{i-r,j+r-1} \xrightarrow{\iota} E^r_{i-r,j+r-1},$$

where π is the quotient by algebra decomposables and ι is the inclusion of the coalgebra primitives. We further observe that $QE^2_{i,j}$ is zero, unless i is a power of p, and that $PE^2_{i,j}$ is zero, unless $i = 1$. Hence, the shortest possible non-zero differential of lowest possible total degree is

$$d^{p-1}((d\bar{\tau}_0)^{[p]}) = a_0 \cdot d\bar{\xi}_1$$

with $a_0 \in k$. In particular, we have $E^{p-1} = E^2$. If $a_0 = 0$, then $d\bar{\xi}_1$ survives the spectral sequence, so $a_0 \in k^*$. This proves the claim for $i = s = 0$.

We proceed by nested induction on $i, s \geq 0$ to prove the claim in general. We first note that if, for a fixed $i \geq 0$, the claim holds for $s = 0$, then it holds for all $s \geq 0$. For let $s \geq 1$ and assume, inductively, that the claim holds for all smaller values of s. One calculates that the difference

$$d^{p-1}((d\bar{\tau}_i)^{[p+s]}) - a_i \cdot d\bar{\xi}_{i+1} \cdot (d\bar{\tau}_i)^{[s]}$$

is a coalgebra primitive element, which shows that it is zero, since all non-zero coalgebra primitives in $E^{p-1} = E^2$ have filtration $i = 1$.

It remains to prove that the claim holds for all $i \geq 0$ and $s = 0$. We have already proved the case $i = 0$, so we let $j \geq 1$ and assume, inductively, that the claim has been proved for all $i < j$ and all $s \geq 0$. The inductive assumption implies that E^p is a subquotient of the C-subalgebra

$$D = \Lambda_C\{d\bar{\xi}_i \mid i \geq j+1\} \otimes_C \Gamma_C\{d\bar{\tau}_i \mid i \geq j\} \otimes_C C[d\bar{\tau}_i \mid i < j]/((d\bar{\tau}_i)^p \mid i < j)$$

of $E^{p-1} = E^2$. Now, since $\pi_*(k \otimes \mathrm{THH}(R))$ is an augmented C-algebra, all elements of filtration 0 survive the spectral sequence. Hence, if $x \in E^r$ with $r \geq p$ supports a non-zero differential, then x has filtration at least $p+1$. But all algebra generators in D of filtration at least $p+1$ have total degree at least $2p^{j+2}$, so either $d^{p-1}((d\bar{\tau}_j)^{[p]})$ is non-zero, or else all elements in D of total degree at most $2p^{j+2} - 2$ survive the spectral sequence. Since we know that the element $d\bar{\xi}_{j+1} \in D$ of total degree $2p^{j+1} - 1$ does not survive the spectral sequence, we conclude that the former is the case. We must show that

$$d^{p-1}((d\bar{\tau}_j)^{[p]}) = a_j \cdot d\bar{\xi}_{j+1}$$

with $a_j \in k^*$, and to this end, we use the fact that E^{p-1} is a C-Hopf algebra in the symmetric monoidal category of (A, B)-modules. We have

$$\begin{aligned}
\epsilon(1 \otimes_{A,d^1} d\bar{\xi}_i) &= \epsilon((\mathrm{id} \otimes_{A,d^1} d)(1 \otimes_{A,d^1} \bar{\xi}_i)) = (\mathrm{id} \otimes_{A,d^0} d)(\epsilon(1 \otimes_{A,d^1} \bar{\xi}_i)) \\
&= (\mathrm{id} \otimes_{A,d^0} d)(\textstyle\sum \bar{\xi}_s \otimes_{A,d^0} \bar{\xi}_t^{p^s}) = 1 \otimes_{A,d^0} d\bar{\xi}_i, \\
\epsilon(1 \otimes_{A,d^1} d\bar{\tau}_i) &= \epsilon((\mathrm{id} \otimes_{A,d^1} d)(1 \otimes_{A,d^1} \bar{\tau}_i)) = (\mathrm{id} \otimes_{A,d^0} d)(\epsilon(1 \otimes_{A,d^1} \bar{\tau}_i)) \\
&= (\mathrm{id} \otimes_{A,d^0} d)(1 \otimes_{A,d^0} \bar{\tau}_i + \textstyle\sum \bar{\tau}_s \otimes_{A,d^0} \bar{\xi}_t^{p^s}) = 1 \otimes_{A,d^0} d\bar{\tau}_i,
\end{aligned}$$

where the sums range over $s, t \geq 0$ with $s + t = i$. Hence, the sub-k-vector space of E^{p-1} that consists of the horizontal elements of bidegree $(1, 2p^{j+1} - 2)$

is spanned by $d\bar{\xi}_{j+1}$. Therefore, it suffices to show that $(d\bar{\tau}_j)^{[p]}$, and hence, $d^{p-1}((d\bar{\tau}_j)^{[p]})$ is horizontal. We have already proved that $d\bar{\tau}_j$ is horizontal, and using the fact that E^{p-1} is a C-algebra in the symmetric monoidal category of (A, B)-modules, we conclude that $(d\bar{\tau}_j)^s$, and therefore, $(d\bar{\tau}_j)^{[s]}$ is horizontal for all $0 \leq s < p$. Finally, we make use of the fact that E^{p-1} is a C-coalgebra in the symmetric monoidal category of (A, B)-modules. Since

$$\psi((d\bar{\tau}_j)^{[p]}) = \sum (d\bar{\tau}_j)^{[s]} \otimes (d\bar{\tau}_j)^{[t]}$$

with the sum indexed by $s, t \geq 0$ with $s + t = p$, and since we have already proved that $(d\bar{\tau}_j)^{[s]}$ with $0 \leq s < p$ are horizontal, we find that $(d\bar{\tau}_j)^{[p]}$ is horizontal. This completes the proof. □

We finally recall the following analogue of the Segal conjecture. This is a key result for understanding topological cyclic homology and its variants.

Addendum 1.2.2. *The Frobenius induces an equivalence*

$$\mathrm{THH}(\mathbb{F}_p) \xrightarrow{\varphi} \tau_{\geq 0} \mathrm{THH}(\mathbb{F}_p)^{tC_p}.$$

Proof. See [32, Section IV.4]. □

1.3 Perfectoid rings

Perfectoid rings are to topological Hochschild homology what separably closed fields are to K-theory: they annihilate Kähler differentials. In this section, we present the proof by Bhatt–Morrow–Scholze that Bökstedt periodicity holds for every perfectoid ring R. As a consequence, the Tate spectral sequence

$$E^2_{i,j} = \hat{H}^{-i}(B\mathbb{T}, \mathrm{THH}_j(R, \mathbb{Z}_p)) \Rightarrow \mathrm{TP}_{i+j}(R, \mathbb{Z}_p)$$

collapses and gives the ring $\mathrm{TP}_0(R, \mathbb{Z}_p)$ a complete and separated descending filtration, the graded pieces of which are free R-modules of rank 1. The ring $\mathrm{TP}_0(R, \mathbb{Z}_p)$, however, is not a power series ring over R. Instead, it agrees, up to unique isomorphism over R, with Fontaine's p-adic period ring $A_{\mathrm{inf}}(R)$, the definition of which we recall below. Finally, we use these results to prove that Bökstedt periodicity implies Bott periodicity.

1.3.1 Perfectoid rings

A $\mathbb{Z}_p$-algebra R is perfectoid, for example, if there exists a non-zero-divisor $\pi \in R$ with $p \in \pi^p R$ such that R is complete and separated with respect to the π-adic topology and such that the Frobenius $\varphi\colon R/\pi \to R/\pi^p$ is an isomorphism. We will give the general definition, which does not require $\pi \in R$

to be a non-zero-divisor, below. Typically, perfectoid rings are large and highly non-noetherian. Moreover, the ring R/π is typically not a field, but is also a large non-noetherian ring with many nilpotent elements. An example to keep in mind is the valuation ring $\mathcal{O}_C$ in an algebraically closed field $C/\mathbb{Q}_p$ that is complete with respect to a non-archimedean absolute value extending the p-adic absolute value on $\mathbb{Q}_p$; here we can take π to be a pth root of p.

We recall some facts from [4, Section 3]. If a ring S contains an element $\pi \in S$ such that $p \in \pi S$ and such that π-adic topology on S is complete and separated, then the canonical projections

$$\begin{tikzcd}
\lim_{n,F} W(S) \arrow[r] \arrow[d] & \lim_{n,F} W(S/p) \arrow[r] \arrow[d] & \lim_{n,F} W(S/\pi) \arrow[d] \\
\lim_{n,F} W_n(S) \arrow[r] & \lim_{n,F} W_n(S/p) \arrow[r] & \lim_{n,F} W_n(S/\pi)
\end{tikzcd}$$

all are isomorphisms. Here the limits range over non-negative integers n with the respective Witt vector Frobenius maps as the structure maps. Moreover, since the Witt vector Frobenius for $\mathbb{F}_p$-algebras agrees with the map of rings of Witt vectors induced by the Frobenius, we have a canonical map

$$W(\lim_{n,\varphi} S/p) \longrightarrow \lim_{n,F} W(S/p),$$

and this map, too, is an isomorphism, since the Witt vector functor preserves limits. The perfect $\mathbb{F}_p$-algebra

$$S^\flat = \lim_{n,\varphi} S/p = \lim(S/p \xleftarrow{\varphi} S/p \xleftarrow{\varphi} \cdots)$$

is called the tilt of S, and its ring of Witt vectors

$$A_{\mathrm{inf}}(S) = W(S^\flat)$$

is called Fontaine's ring of p-adic periods. The Frobenius automorphism φ of $S^\flat$ induces the automorphism $W(\varphi)$ of $A_{\mathrm{inf}}(S)$, which, by abuse of notation, we also write φ and call the Frobenius.

We again consider the diagram of isomorphisms at the beginning of the section. By composing the isomorphisms in the diagram with the projection onto $W_n(S)$ in the lower left-hand term of the diagram, we obtain a ring homomorphism $\widetilde{\theta}_n \colon A_{\mathrm{inf}}(S) \to W_n(S)$, and we define

$$A_{\mathrm{inf}}(S) \xrightarrow{\theta_n} W_n(S)$$

to be $\theta_n = \widetilde{\theta}_n \circ \varphi^n$. It is clear from the definition that the diagrams

$$\begin{tikzcd}
A_{\mathrm{inf}}(S) \arrow[r, "\theta_n"] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\varphi"] & W_n(S) \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
A_{\mathrm{inf}}(S) \arrow[r, "\theta_{n-1}"] & W_{n-1}(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
A_{\mathrm{inf}}(S) \arrow[r, "\widetilde{\theta}_n"] \arrow[d, shift right, "\varphi^{-1}"'] \arrow[d, shift left, "\mathrm{id}"] & W_n(S) \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
A_{\mathrm{inf}}(S) \arrow[r, "\widetilde{\theta}_{n-1}"] & W_{n-1}(S)
\end{tikzcd}$$

commute. The map $\theta = \theta_1 \colon A_{\inf}(S) \to S = W_1(S)$ is Fontaine's map from [19], which we now describe more explicitly. There is a well-defined map

$$S^\flat \xrightarrow{(-)^\#} S$$

that to $a = (x_0, x_1, \dots) \in \lim_{n,\varphi} S/p = S^\flat$ assigns $a^\# = \lim_{n\to\infty} y_n^{p^n}$ for any choice of lifts $y_n \in S$ of $x_n \in S/p$. It is multiplicative, but it is not additive unless S is an $\mathbb{F}_p$-algebra. Using this map, we have

$$\theta(\textstyle\sum_{i\geq 0}[a_i]\, p^i) = \textstyle\sum_{i\geq 0} a_i^\# p^i,$$

where $[-]\colon S^\flat \to W(S^\flat)$ is the Teichmüller representative. We can now state the general definition of a perfectoid ring that is used in [4], [5], and [6].

Definition 1.3.1. A $\mathbb{Z}_p$-algebra R is perfectoid if there exists $\pi \in R$ such that $p \in \pi^p R$, such that the π-adic topology on R is complete and separated, such that the Frobenius $\varphi\colon R/p \to R/p$ is surjective, and such that the kernel of $\theta\colon A_{\inf}(R) \to R$ is a principal ideal.

The ideal $I = \ker(\theta) \subset A = W(R^\flat)$ is typically not fixed by the Frobenius on A, but it always satisfies the prism property that

$$p \in I + \varphi(I)A.$$

If an ideal $J \subset A$ satisfies the prism property, then the quotient A/J is an untilt of $R^\flat$ in the sense that it is perfectoid and that its tilt is $R^\flat$. In fact, every untilt of $R^\flat$ arises as A/J for some ideal $J \subset A$ that satisfies the prism property. The set of such ideals is typically large, but it has a very interesting p-adic geometry. Indeed, for $R = \mathcal{O}_C$, there is a canonical one-to-one correspondence between orbits under the Frobenius of such ideals and closed points of the Fargues–Fontaine curve FF_C [18]. Among all ideals $J \subset A$ that satisfy the prism property, the ideal $J = pA$ is the only one for which the untilt $A/J = R^\flat$ is of characteristic p; all other untilts A/J are of mixed characteristic $(0,p)$. One can show that every untilt A/J is a reduced ring and that a generator ξ of the ideal $J \subset A$ necessarily is a non-zero-divisor. Hence, such a generator is well-defined, up to a unit in A. An untilt A/J may have p-torsion, but if an element is annihilated by some power of p, then it is in fact annihilated by p. We refer to [6] for proofs of these statements.

Bhatt–Morrow–Scholze prove in [5, Theorem 6.1] that Bökstedt periodicity for $R = \mathbb{F}_p$ implies the analogous result for R any perfectoid ring.

Theorem 1.3.2 (Bhatt–Morrow–Scholze)**.** *If R is a perfectoid ring, then the canonical map is an isomorphism of graded rings*

$$\mathrm{Sym}_R(\mathrm{THH}_2(R,\mathbb{Z}_p)) \longrightarrow \mathrm{THH}_*(R,\mathbb{Z}_p),$$

and $\mathrm{THH}_2(R,\mathbb{Z}_p)$ is a free R-module of rank 1.

Proof. We follow the proof in loc. cit. We first claim that the canonical map

$$\Gamma_R(\mathrm{HH}_2(R/\mathbb{Z},\mathbb{Z}_p)) \longrightarrow \mathrm{HH}_*(R/\mathbb{Z},\mathbb{Z}_p)$$

is an isomorphism and that the R-module $\mathrm{HH}_2(R/\mathbb{Z},\mathbb{Z}_p)$ is free of rank 1. To prove this, we first notice that the base-change map

$$\mathrm{HH}(R/\mathbb{Z}) \longrightarrow \mathrm{HH}(R/A_{\mathrm{inf}}(R))$$

is a p-adic equivalence. Indeed, we always have

$$\mathrm{HH}(R/A_{\mathrm{inf}}(R)) \simeq \mathrm{HH}(R/\mathbb{Z}_p) \otimes_{\mathrm{HH}(A_{\mathrm{inf}}(R)/\mathbb{Z}_p)} A_{\mathrm{inf}}(R),$$

and $\mathrm{HH}(A_{\mathrm{inf}}(R)/\mathbb{Z}) \to A_{\mathrm{inf}}(R)$ is a p-adic equivalence, because

$$\mathrm{HH}(A_{\mathrm{inf}}(R)/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \mathrm{HH}(A_{\mathrm{inf}}(R) \otimes_{\mathbb{Z}} \mathbb{F}_p/\mathbb{F}_p) \simeq \mathrm{HH}(R^\flat/\mathbb{F}_p) \simeq R^\flat.$$

The last equivalence holds since $R^\flat$ is perfect. Now, we write $R \simeq \Lambda_{A_{\mathrm{inf}}(R)}\{y\}$ with $dy = \xi$ to see that $R \otimes_{A_{\mathrm{inf}}(R)} R \simeq \Lambda_R\{y\}$ with $dy = 0$, and similarly, we write $R \simeq \Lambda_R\{y\} \otimes \Gamma_R\{x\}$ with $dx^{[i]} = x^{[i-1]}y$ to see that

$$\mathrm{HH}(R/A_{\mathrm{inf}}(R)) \simeq R \otimes_{R \otimes_{A_{\mathrm{inf}}(R)} R} R \simeq \Gamma_R\{x\},$$

which proves the claim.

It follows, in particular, that the R-module

$$\mathrm{THH}(R,\mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \simeq \mathrm{HH}(R/\mathbb{Z},\mathbb{Z}_p)$$

is pseudocoherent in the sense that it can be represented by a chain complex of finitely generated free R-modules that is bounded below. Since $\mathrm{THH}(\mathbb{Z})$ has finitely generated homotopy groups, we conclude, inductively, that

$$\mathrm{THH}(R,\mathbb{Z}_p) \otimes_{\mathrm{THH}(\mathbb{Z})} \tau_{\leq n} \mathrm{THH}(\mathbb{Z})$$

is a pseudocoherent R-module for all $n \geq 0$. Therefore, also $\mathrm{THH}(R,\mathbb{Z}_p)$ is a pseudocoherent R-module.

Next, we claim that any ring homomorphism $R \to R'$ between perfectoid rings induces an equivalence

$$\mathrm{THH}(R,\mathbb{Z}_p) \otimes_R R' \longrightarrow \mathrm{THH}(R',\mathbb{Z}_p).$$

Indeed, it suffices to prove that the claim holds after extension of scalars along the canonical map $\mathrm{THH}(\mathbb{Z}) \to \mathbb{Z}$. This reduces us to proving that

$$\mathrm{HH}(R,\mathbb{Z}_p) \otimes_R R' \longrightarrow \mathrm{HH}(R',\mathbb{Z}_p)$$

is an equivalence, which follows from the first claim.

We now prove that the map in the statement is an isomorphism. The case of $R = \mathbb{F}_p$ is Theorem 1.2.1, and the case of a perfect $\mathbb{F}_p$-algebra follows from the base-change formula that we just proved. In the general case, we show, inductively, that the map is an isomorphism in degree $i \geq 0$. So we assume that the map is an isomorphism in degrees $< i$ and prove that it is an isomorphism in degree i. By induction, the R-module $\tau_{<i} \operatorname{THH}(R, \mathbb{Z}_p)$ is perfect, and hence, the R-module $\tau_{\geq i} \operatorname{THH}(R, \mathbb{Z}_p)$ is pseudocoherent. It follows that the R-module $\operatorname{THH}_i(R, \mathbb{Z}_p)$ is finitely generated. Since R is perfectoid, the composition

$$R \longrightarrow R/p \longrightarrow \bar{R} = \operatorname{colim}_{n,\varphi} R/p = R/\sqrt{pR}$$

of the canonical projection and the canonical map from the initial term in the colimit is surjective. Since $\bar{R}$ is a perfect $\mathbb{F}_p$-algebra, the base-change formula and the inductive hypothesis show that, in the diagram

$$\begin{tikzcd}
\operatorname{Sym}_R(\operatorname{THH}_2(R, \mathbb{Z}_p)) \otimes_R \bar{R} \arrow[r] \arrow[d] & \operatorname{THH}_*(R, \mathbb{Z}_p) \otimes_R \bar{R} \arrow[d] \\
\operatorname{Sym}_{\bar{R}}(\operatorname{THH}_2(\bar{R}, \mathbb{Z}_p)) \arrow[r] & \operatorname{THH}_*(\bar{R}, \mathbb{Z}_p),
\end{tikzcd}$$

the vertical maps are isomorphisms in degrees $\leq i$, and we have already seen that the lower horizontal map is an isomorphism. Hence, the upper horizontal map is an isomorphism in degrees $\leq i$. Since the kernel of the map $R \to \bar{R}$ contains the Jacobson radical, Nakayama's lemma shows that the map in the statement of the theorem is surjective in degrees $\leq i$. To prove that it is also injective, we consider the diagram

$$\begin{tikzcd}
\operatorname{Sym}_R(\operatorname{THH}_2(R, \mathbb{Z}_p)) \arrow[r] \arrow[d] & \operatorname{THH}_*(R, \mathbb{Z}_p) \arrow[d] \\
\prod_{\mathfrak{p}} \operatorname{Sym}_R(\operatorname{THH}_2(R, \mathbb{Z}_p)) \otimes_R R_{\mathfrak{p}} \arrow[r] & \prod_{\mathfrak{p}} \operatorname{THH}_*(R, \mathbb{Z}_p) \otimes_R R_{\mathfrak{p}}
\end{tikzcd}$$

where the products range over the minimal primes $\mathfrak{p} \subset R$. Since R is reduced, the left-hand vertical map is injective and the local rings $R_{\mathfrak{p}}$ are fields. Hence, it suffices to prove that for every minimal prime $\mathfrak{p} \subset R$, the map

$$\operatorname{Sym}_R(\operatorname{THH}_2(R, \mathbb{Z}_p)) \otimes_R R_{\mathfrak{p}} \longrightarrow \operatorname{THH}_*(R, \mathbb{Z}_p) \otimes_R R_{\mathfrak{p}}$$

is injective in degrees $\leq i$. To this end, we write $\operatorname{Spec}(R)$ as the union of the closed subscheme $\operatorname{Spec}(\bar{R})$ and its open complement $\operatorname{Spec}(R[1/p])$. If $\mathfrak{p}$ belongs to $\operatorname{Spec}(\bar{R})$, then the map in question is an isomorphism in degrees $\leq i$ by what was proved above. Similarly, if $\mathfrak{p}$ belongs to $\operatorname{Spec}(R[1/p])$, then the map in a question is an isomorphism in all degrees, since the map

$$\operatorname{Sym}_R(\operatorname{THH}_2(R, \mathbb{Z}_p)) \otimes_R R[1/p] \longrightarrow \operatorname{THH}_*(R, \mathbb{Z}_p) \otimes_R R[1/p]$$

is so, by the claim at the beginning of the proof. This completes the proof. $\square$

Addendum 1.3.3. *If R is a perfectoid ring, then*

$$\mathrm{THH}(R,\mathbb{Z}_p) \xrightarrow{\varphi} \tau_{\geq 0}\,\mathrm{THH}(R,\mathbb{Z}_p)^{tC_p}$$

is an equivalence.

Proof. See [5, Proposition 6.2]. □

We show that Fontaine's map $\theta\colon A_{\mathrm{inf}}(R) \to R$ is the universal p-complete pro-infinitesimal thickening following [19, Théorème 1.2.1]. We remark that, in loc. cit., Fontaine defines $A_{\mathrm{inf}}(R)$ to be the $\ker(\theta)$-adic completion of $W(R^\flat)$. We include a proof here that this is not necessary in that the $\ker(\theta)$-adic topology on $W(R^\flat)$ is already complete and separated.

Proposition 1.3.4 (Fontaine)**.** *If R is a perfectoid ring, then the map*

$$A_{\mathrm{inf}}(R) \xrightarrow{\theta} R$$

is initial among ring homomorphisms $\theta_D\colon D \to R$ such that D is complete and separated in both the p-adic topology and the $\ker(\theta_D)$-adic topology.

Proof. We first show that $A = A_{\mathrm{inf}}(R)$ is complete and separated in both the p-adic topology and the $\ker(\theta)$-adic topology. Since $R^\flat$ is a perfect $\mathbb{F}_p$-algebra, we have $p^nA = V^n(A) \subset A$, so the p-adic topology on $A = W(R^\flat)$ is complete and separated. Moreover, since $p \in A$ is a non-zero-divisor, this is equivalent to A being derived p-complete. As we recalled above, a generator $\xi \in \ker(\theta)$ is necessariy a non-zero-divisor. Therefore, the $\ker(\theta)$-adic topology on A is complete and separated if and only if A is derived ξ-adically complete, and since A is derived p-adically complete, this, in turn, is equivalent to A/p being derived ξ-adically complete. Now, we have

$$\begin{tikzcd}
A/p \arrow[r] \arrow[d, equal] & A/(p,\xi^n) \arrow[r, "\varphi^{-n}"] \arrow[d, "\varphi"] & A/(p,\xi) \arrow[r, "\theta"] \arrow[d, "\varphi"] & R/p \arrow[d, "\varphi"] \\
A/p \arrow[r] & A/(p,\xi^{n-1}) \arrow[r, "\varphi^{-(n-1)}"] & A/(p,\xi) \arrow[r, "\theta"] & R/p
\end{tikzcd}$$

with the middle and right-hand maps bijective and with the remaining maps surjective. Taking derived limits, we obtain

$$A/p \longrightarrow (A/p)^\wedge_\xi \longrightarrow (A/\xi)^\flat \longrightarrow R^\flat$$

with the middle and right-hand maps equivalences. The composite map takes the class of $\sum [a_i]p^i$ to a_0, and therefore, it, too, is an equivalence. This proves that the left-hand map is an equivalence, as desired.

Let $\theta_D\colon D \to R$ be as in the statement. We wish to prove that there is a unique ring homomorphism $f\colon A \to D$ such that $\theta = \theta_D \circ f$. Since A and

D are derived p-complete and $L_{(A/p)/\mathbb{F}_p} \simeq 0$, this is equivalent to showing that there is a unique ring homomorphism $\bar{f}\colon A/p \to D/p$ with the property that $\bar{\theta} = \bar{\theta}_D \circ \bar{f}$, where $\bar{\theta}\colon A/p \to R/p$ and $\bar{\theta}_D\colon D/p \to R/p$ are induced by $\theta\colon A \to R$ and $\theta_D\colon D \to R$, respectively. Identifying A/p with $R^\flat$, we wish to show that there is a unique ring homomorphism $\bar{f}\colon R^\flat \to D/p$ such that $a^{\#} + pR = \bar{\theta}_D(\bar{f}(a))$ for all $a \in R^\flat$. Since the $\ker(\theta_D)$-adic topology on D is complete and separated, so is the $\ker(\bar{\theta}_D)$-adic topology on D/p. It follows that for $a = (x_0, x_1, \dots) \in R^\flat$, the limit

$$\bar{f}(a) = \lim_{n\to\infty} \widetilde{x}_n^{p^n} \in D/p,$$

where we choose $\widetilde{x}_n \in D/p$ with $\bar{\theta}_D(\widetilde{x}_n) = x_n \in R/p$, exists and is independent of the choices made. This defines a map $\bar{f}\colon R^\flat \to D/p$, and the uniqueness of the limit implies that it is a ring homomorphism. It satisfies $\bar{\theta} = \bar{\theta}_D \circ \bar{f}$ by construction, and it is unique with this property, since the $\ker(\bar{\theta}_D)$-adic topology on D/p is separated. $\square$

We identify the diagram of p-adic homotopy groups

$$\mathrm{TC}^-_*(R,\mathbb{Z}_p) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}_*(R,\mathbb{Z}_p)$$

for R perfectoid. By Bökstedt periodicity, the spectral sequences

$$\begin{aligned} E^2_{i,j} &= H^{-i}(B\mathbb{T}, \mathrm{THH}_j(R,\mathbb{Z}_p)) \Rightarrow \mathrm{TC}^-_{i+j}(R,\mathbb{Z}_p) \\ E^2_{i,j} &= \hat{H}^{-i}(B\mathbb{T}, \mathrm{THH}_j(R,\mathbb{Z}_p)) \Rightarrow \mathrm{TP}_{i+j}(R,\mathbb{Z}_p) \end{aligned}$$

collapse. It follows that the respective edge homomorphisms in total degree 0 satisfy the hypotheses of Proposition 1.3.4, and therefore, there exists a unique ring homomorphism making the diagram

$$\begin{tikzcd} A_{\inf}(R) \arrow[r] \arrow[d, "\theta"] & \mathrm{TC}^-_0(R,\mathbb{Z}_p) \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{edge}"] & \mathrm{TP}_0(R,\mathbb{Z}_p) \arrow[d, "\mathrm{edge}"] \\ R \arrow[r, equal] & \mathrm{THH}_0(R,\mathbb{Z}_p) \arrow[r, equal] & \mathrm{THH}_0(R,\mathbb{Z}_p) \end{tikzcd}$$

commute. We view $\mathrm{TC}^-_*(R,\mathbb{Z}_p)$ and $\mathrm{TP}_*(R,\mathbb{Z}_p)$ as graded $A_{\inf}(R)$-algebras via the top horizontal maps in the diagram. Bhatt–Morrow–Scholze make the following calculation in [5, Propositions 6.2 and 6.3].

Theorem 1.3.5 (Bhatt–Morrow–Scholze)**.** *Let R be a perfectoid ring, and let ξ be a generator of the kernel of Fontaine's map $\theta\colon A_{\inf}(R) \to R$.*

(1) *There exists $\sigma \in \mathrm{TP}_2(R,\mathbb{Z}_p)$ such that*

$$\mathrm{TP}_*(R,\mathbb{Z}_p) = A_{\inf}(R)[\sigma^{\pm 1}].$$

(2) *There exists* $u \in \mathrm{TC}_2^-(R,\mathbb{Z}_p)$ *and* $v \in \mathrm{TC}_{-2}^-(R,\mathbb{Z}_p)$ *such that*

$$\mathrm{TC}_*^-(R,\mathbb{Z}_p) = A_{\mathrm{inf}}(R)[u,v]/(uv-\xi).$$

(3) *The graded ring homomorphisms*

$$\mathrm{TC}_*^-(R,\mathbb{Z}_p) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} \mathrm{TP}_*(R,\mathbb{Z}_p)$$

are φ*-linear and* $A_{\mathrm{inf}}(R)$*-linear, respectively, and* u*,* v*, and* σ *can be chosen in such a way that* $\varphi(u) = \alpha\cdot\sigma$*,* $\varphi(v) = \alpha^{-1}\varphi(\xi)\cdot\sigma^{-1}$*,* $\mathrm{can}(u) = \xi\cdot\sigma$*, and* $\mathrm{can}(v) = \sigma^{-1}$*, where* $\alpha \in A_{\mathrm{inf}}(R)$ *is a unit.*[11]

Proof. The canonical map $A_{\mathrm{inf}}(R) \to \mathrm{TP}_0(R,\mathbb{Z}_p)$ is a map of filtered rings, where the domain and target are given the ξ-adic filtration and the filtration induced by the Tate spectral sequence, respectively. Since both filtrations are complete and separated, the map is an isomorphism if and only if the induced maps of filtration quotients are isomorphisms. These, in turn, are R-linear maps between free R-modules of rank 1, and to prove that they are isomorphisms, it suffices to consider the case $R = \mathbb{F}_p$.

The canonical map $\mathrm{can}\colon \mathrm{TC}_*^-(R,\mathbb{Z}_p) \to \mathrm{TP}_*(R,\mathbb{Z}_p)$ induces the map of spectral sequences that, on E^2-terms, is given by the localization map

$$R[t,x] \longrightarrow R[t^{\pm1},x],$$

where $t \in E^2_{-2,0}$ and $x \in E^2_{0,2}$ are any R-module generators. It follows that $\mathrm{TP}_*(R,\mathbb{Z}_p)$ is 2-periodic and concentrated in even degrees, so (1) holds for any $\sigma \in \mathrm{TP}_2(R,\mathbb{Z}_p)$ that is an $A_{\mathrm{inf}}(R)$-module generator, or equivalently, is represented in the spectral sequence by an R-module generator of $E^2_{2,0}$. We now fix a choice of $\sigma \in \mathrm{TP}_2(R,\mathbb{Z}_p)$, and let $t \in E^2_{-2,0}$ and $x \in E^2_{0,2}$ be the unique elements that represent $\sigma^{-1} \in \mathrm{TP}_{-2}(R,\mathbb{Z}_p)$ and $\xi\sigma \in \mathrm{TP}_2(R,\mathbb{Z}_p)$, respectively. The latter classes are the images by the canonical map of unique classes $v \in \mathrm{TC}_{-2}^-(R,\mathbb{Z}_p)$ and $u \in \mathrm{TC}_2^-(R,\mathbb{Z}_p)$, and $uv = \xi$. This proves (2) and the part of (3) that concerns the canonical map.

It remains to identify $\varphi\colon \mathrm{TC}_*^-(R,\mathbb{Z}_p) \to \mathrm{TP}_*(R,\mathbb{Z}_p)$. In degree zero, we have fixed identifications of domain and target with $A_{\mathrm{inf}}(R)$, and we first prove that, with respect to these identifications, the map in question is given by the Frobenius $\varphi\colon A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(R)$. To this end, we consider the diagram

$$\begin{array}{ccccc}
\mathrm{THH}(R)^{h\mathbb{T}} & \xrightarrow{\varphi^{h\mathbb{T}}} & (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}} & \longrightarrow & (R^{tC_p})^{h\mathbb{T}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{THH}(R) & \xrightarrow{\varphi} & \mathrm{THH}(R)^{tC_p} & \longrightarrow & R^{tC_p},
\end{array}$$

[11] If one is willing to replace ξ by $\varphi^{-1}(\alpha)\xi$, then the unit α can be eliminated.

where, on the right, we view R as an $\mathbb{E}_\infty$-ring with trivial $\mathbb{T}$-action, and where the right-hand horizontal maps both are induced by the unique extension $\mathrm{THH}(R) \to R$ of the identity map of R to a map of $\mathbb{E}_\infty$-rings with $\mathbb{T}$-action. Applying $\pi_0(-, \mathbb{Z}_p)$, we obtain the diagram of rings

$$\begin{tikzcd} A_{\mathrm{inf}}(R) \arrow[r, "\pi_0(\varphi^{h\mathbb{T}})"] \arrow[d, "\theta"] & A_{\mathrm{inf}}(R) \arrow[r, "\theta"] \arrow[d] & R \arrow[d, "\mathrm{pr}"] \\ R \arrow[r, "\pi_0(\varphi)"] & R/p \arrow[r, equal] & R/p, \end{tikzcd}$$

where we identify the upper right-hand horizontal map by applying naturality of the edge homomorphism of the Tate spectral sequence to $\mathrm{THH}(R) \to R$. Now, it follows from the proof of Proposition 1.3.4 that the map $\pi_0(\varphi^{h\mathbb{T}})$ is uniquely determined by the map $\pi_0(\varphi)$. Moreover, the latter map is identified in [32, Corollary IV.2.4] to be the map that takes x to the class of x^p. We conclude that $\pi_0(\varphi^{h\mathbb{T}})$ is equal to the Frobenius $\varphi\colon A_{\mathrm{inf}}(R) \to A_{\mathrm{inf}}(R)$, since the latter makes the left-hand square commute. Finally, since

$$\mathrm{TC}_2^-(R, \mathbb{Z}_p) \xrightarrow{\varphi} \mathrm{TP}_2(R, \mathbb{Z}_p)$$

is an isomorphism, we have $\varphi(u) = \alpha \cdot \sigma$ with $\alpha \in A_{\mathrm{inf}}(R)$ a unit, and the relation $uv = \xi$ implies that $\varphi(v) = \alpha^{-1}\varphi(\xi) \cdot \sigma^{-1}$. $\square$

1.3.2 Bott periodicity

We fix a field C that contains $\mathbb{Q}_p$ and that is both algebraically closed and complete with respect to a non-archimedean absolute value that extends the p-adic absolute value on $\mathbb{Q}_p$. The valuation ring

$$\mathcal{O}_C = \{x \in C \mid |x| \leq 1\} \subset C$$

is a perfectoid ring, and we proceed to evaluate its topological cyclic homology. We explain that this calculation, which uses Bökstedt periodicity, gives a purely p-adic proof of Bott periodicity.

The calculation uses a particular choice of generator ξ of the kernel of the map $\theta\colon A_{\mathrm{inf}}(\mathcal{O}_C) \to \mathcal{O}_C$, which we define first. We fix a generator

$$\epsilon \in \mathbb{Z}_p(1) = T_p(C^*) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C^*)$$

of the p-primary Tate module of C^*. It determines and is determined by the sequence $(1, \zeta_p, \zeta_{p^2}, \dots)$ of compatible primitive p-power roots of unity in C, where $\zeta_{p^n} = \epsilon({}^1\!/_{p^n} + \mathbb{Z}_p) \in C$. By abuse of notation, we also write

$$\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^\flat}.$$

We now define the elements $\mu, \xi \in A_{\inf}(\mathcal{O}_C)$ by

$$\begin{aligned} \mu &= [\epsilon] - 1 \\ \xi &= \mu/\varphi^{-1}(\mu) = ([\epsilon]-1)/([\epsilon^{1/p}]-1). \end{aligned}$$

The element ξ lies in the kernel of $\theta \colon A_{\inf}(\mathcal{O}_C) \to \mathcal{O}_C$, since

$$\theta(\xi) = \theta(\textstyle\sum_{0 \leq k < p} [\epsilon^{1/p}]^k) = \sum_{0 \leq k < p} \zeta_p^k = 0,$$

and it is a generator. More generally, the element

$$\xi_r = \mu/\varphi^{-r}(\mu) = ([\epsilon]-1)/([\epsilon^{1/p^r}]-1)$$

generates the kernel of $\theta_r \colon A_{\inf}(\mathcal{O}_C) \to W_r(\mathcal{O}_C)$, and the element μ generates the kernel of the induced map[12]

$$A_{\inf}(\mathcal{O}_C) \xrightarrow{(\theta_r)} \lim_{r,R} W_r(\mathcal{O}_C) = W(\mathcal{O}_C).$$

Theorem 1.3.6. *Let C be a field extension of $\mathbb{Q}_p$ that is both algebraically closed and complete with respect to a non-archimedean absolute value that extends the p-adic absolute value on $\mathbb{Q}_p$, and let $\mathcal{O}_C \subset C$ be the valuation ring. The canonical map of graded rings*

$$\mathrm{Sym}_{\mathbb{Z}_p}(\mathrm{TC}_2(\mathcal{O}_C, \mathbb{Z}_p)) \longrightarrow \mathrm{TC}_*(\mathcal{O}_C, \mathbb{Z}_p)$$

is an isomorphism, and $\mathrm{TC}_2(\mathcal{O}_C, \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module of rank 1. Moreover, the map $\mathrm{TC}_{2m}(\mathcal{O}_C, \mathbb{Z}_p) \to \mathrm{TC}^{-}_{2m}(\mathcal{O}_C, \mathbb{Z}_p)$ takes a $\mathbb{Z}_p$-module generator of the domain to $\varphi^{-1}(\mu)^m$ times an $A_{\inf}(\mathcal{O}_C)$-module generator of the target.

Proof. We let $\epsilon \in T_p(C^*)$ and $\xi, \mu \in A_{\inf}(\mathcal{O}_C)$ be as above. According to Theorem 1.3.5, the even (resp. odd) p-adic homology groups of $\mathrm{TC}(\mathcal{O}_C)$ are given by the kernel (resp. cokernel) of the $\mathbb{Z}_p$-linear map

$$A_{\inf}(\mathcal{O}_C)[u,v]/(uv - \xi) \xrightarrow{\varphi - \mathrm{can}} A_{\inf}(\mathcal{O}_C)[\sigma^{\pm 1}]$$

given by

$$\begin{aligned} (\varphi - \mathrm{can})(a \cdot u^m) &= (\alpha^m \varphi(a) - \xi^m a) \cdot \sigma^m \\ (\varphi - \mathrm{can})(a \cdot v^m) &= (\alpha^{-m} \varphi(\xi)^m \varphi(a) - a) \cdot \sigma^{-m}, \end{aligned}$$

where $m \geq 0$ is an integer, and where $\alpha \in A_{\inf}(\mathcal{O}_C)$ is a fixed unit. We need only consider the top formula, since we know, for general reasons, that the p-adic homotopy groups of $\mathrm{TC}(\mathcal{O}_C)$ are concentrated in degrees ≥ -1.

We first prove that an element $a \cdot u^m$ in the image of the map

$$\mathrm{TC}_{2m}(\mathcal{O}_C, \mathbb{Z}_p) \longrightarrow \mathrm{TC}^{-}_{2m}(\mathcal{O}_C, \mathbb{Z}_p)$$

[12] Its cokernel $R^1 \lim_r \ker(\theta_r)$ is a huge $A_{\inf}(\mathcal{O}_C)$-module that is almost zero.

is of the form $a = \varphi^{-1}(\mu)^m b$, for some $b \in A_{\inf}(\mathcal{O}_C)$. The element b is uniquely determined by a, since $\mathcal{O}_C$, and hence, $A_{\inf}(\mathcal{O}_C)$ is an integral domain. This image consists of the elements $a \cdot u^m$, where $a \in A_{\inf}(\mathcal{O}_C)$ satisfies

$$\alpha^m \varphi(a) = \xi^m a.$$

Rewriting this equation in the form

$$a = \varphi^{-1}(\alpha)^{-m} \varphi^{-1}(\xi)^m \varphi^{-1}(a),$$

we find inductively that for all $r \geq 1$,

$$a = \varphi^{-1}(\alpha_r)^{-m} \varphi^{-1}(\xi_r)^m \varphi^{-r}(a),$$

where $\alpha_r = \prod_{0 \leq i < r} \varphi^{-i}(\alpha)$ and $\xi_r = \prod_{0 \leq i < r} \varphi^{-i}(\xi)$. Therefore, we have

$$\varphi(a) \in \bigcap\nolimits_{r \geq 1} \xi_r^m A_{\inf}(\mathcal{O}_C) = \mu^m A_{\inf}(\mathcal{O}_C),$$

as desired. Moreover, the element $a \cdot u^m = \varphi^{-1}(\mu)^m b \cdot u^m$ belongs to the image of the map above if and only if $b \in A_{\inf}(\mathcal{O}_C)$ solves the equation

$$\alpha^m \varphi(b) = b.$$

Indeed, the elements $\mu, \xi \in A_{\inf}(\mathcal{O}_C)$ satisfy $\mu = \xi \varphi^{-1}(\mu)$.

To complete the proof, it suffices to show that the canonical map

$$\mathrm{Sym}_{\mathbb{Z}/p}(\mathrm{TC}_2(\mathcal{O}_C, \mathbb{Z}/p)) \longrightarrow \mathrm{TC}_*(\mathcal{O}_C, \mathbb{Z}/p)$$

is an isomorphism, and that the $\mathbb{Z}/p$-vector space $\mathrm{TC}_2(\mathcal{O}_C, \mathbb{Z}/p)$ has dimension 1. First, to show that $\mathrm{TC}_{2m-1}(\mathcal{O}_C, \mathbb{Z}/p)$ is zero, we must show that for every $c \in \mathcal{O}_{C^\flat}$, there exists $a \in \mathcal{O}_{C^\flat}$ such that

$$\alpha^m a^p - \xi^m a = c.$$

The ring $\mathcal{O}_{C^\flat}$ is complete with respect to the non-archimedean absolute value given by $|x|_{C^\flat} = |x^\#|_C$, and its quotient field $C^\flat$ is algebraically closed. Hence, there exists $a \in C^\flat$ that solves the equation in question, and we must show that $|a|_{C^\flat} \leq 1$. If $|a| = |a|_{C^\flat} > 1$, then

$$|\alpha^m a^p| = |a|^p > |a| \geq |\xi|^m |a| = |\xi^m a|,$$

and since $|\ \ |$ is non-archimedean, we conclude that

$$|\alpha^m a^p - \xi^m a| = |\alpha^m a^p| > 1 \geq |c|.$$

Hence, every solution $a \in C^\flat$ to the equation in question is in $\mathcal{O}_{C^\flat}$. This shows that the group $\mathrm{TC}_{2m-1}(\mathcal{O}_C, \mathbb{Z}/p)$ is zero.

Finally, we determine the $\mathbb{Z}/p$-vector space $\mathrm{TC}_{2m}(\mathcal{O}_C, \mathbb{Z}/p)$, which we have identified with the subspace of $\mathrm{TC}^-_{2m}(\mathcal{O}_C, \mathbb{Z}/p)$ that consists of the elements of the form $b\varphi^{-1}(\mu)^m \cdot u^m$, where $b \in \mathcal{O}_{C^\flat}$ satisfies the equation

$$\alpha^m b^p = b.$$

Since $C^\flat$ is algebraically closed, there are p solutions to this equations, namely, 0 and the $(p-1)$th roots of α^{-m}, all of which are units in $\mathcal{O}_{C^\flat}$. This shows that $\mathrm{TC}_{2m}(\mathcal{O}_C, \mathbb{Z}/p)$ is a $\mathbb{Z}/p$-vector space of dimension 1, for all $m \geq 0$. It remains only to notice that if b_1 and b_2 satisfy $\alpha^{m_1} b_1^p = b_1$ and $\alpha^{m_2} b_2^p = b_2$, respectively, then $b = b_1 b_2$ satisfies $\alpha^{m_1+m_2} b^p = b$, which shows that the map in the statement is an isomorphism. ☐

We thank Antieau–Mathew–Morrow for sharing the elegant proof of the following statement with us.

Lemma 1.3.7. *With notation as in Theorem 1.3.6, the map*

$$K(\mathcal{O}_C, \mathbb{Z}_p) \xrightarrow{j^*} K(C, \mathbb{Z}_p)$$

is an equivalence.

Proof. For every ring R, the category of coherent R-modules is abelian, and we define $K'(R)$ to be the algebraic K-theory of this abelian category. If R is coherent as an R-module, then the category of coherent R-modules contains the category of finite projective R-modules as a full exact subcategory. So in this situation the canonical inclusion induces a map of K-theory spectra

$$K(R) \longrightarrow K'(R).$$

If, in addition, the ring R is of finite global dimension, then the resolution theorem [33, Theorem 3] shows that this map is an equivalence. In particular, this is so, if R is a valuation ring. For every finitely generated ideal in R is principal, so R is coherent, and it follows form [2] that R is of finite global dimension.

We now choose any pseudouniformizer $\pi \in \mathcal{O}_C$ and apply the localization theorem [33, Theorem 5] to the abelian category of coherent $\mathcal{O}_C$-modules and the full abelian subcategory of coherent $\mathcal{O}_C/\pi$-modules. The localization sequence then takes the form

$$K'(\mathcal{O}_C/\pi) \xrightarrow{i_*} K(\mathcal{O}_C) \xrightarrow{j^*} K(C),$$

since $\mathcal{O}_C$ and C both are valuation rings. In a similar manner, we obtain, for any pseudouniformizer $\pi^\flat \in \mathcal{O}_{C^\flat}$, the localization sequence

$$K'(\mathcal{O}_{C^\flat}/\pi^\flat) \xrightarrow{i_*} K(\mathcal{O}_{C^\flat}) \xrightarrow{j^*} K(C^\flat).$$

We may choose π and $\pi^\flat$ such that $\mathcal{O}_C/\pi$ and $\mathcal{O}_{C^\flat}/\pi^\flat$ are isomorphic rings, so we conclude that the lemma is equivalent to the statement that

$$K(\mathcal{O}_{C^\flat}, \mathbb{Z}_p) \xrightarrow{j^*} K(C^\flat, \mathbb{Z}_p)$$

is an equivalence. But $\mathcal{O}_{C^\flat}$ and $C^\flat$ are both perfect local $\mathbb{F}_p$-algebras, so the domain and target are both equivalent to $\mathbb{Z}_p$. Finally, the map in question is a map of $\mathbb{E}_\infty$-algebras in spectra, so it is necessarily an equivalence. □

Corollary 1.3.8 (Bott periodicity)**.** *The canonical map of graded rings*

$$\mathrm{Sym}_{\mathbb{Z}}(K_2^{\mathrm{top}}(\mathbb{C})) \longrightarrow K_*^{\mathrm{top}}(\mathbb{C})$$

is an isomorphism, and $K_2^{\mathrm{top}}(\mathbb{C})$ is a free abelian group of rank 1*.*

Proof. The homotopy groups of $K^{\mathrm{top}}(\mathbb{C})$ are finitely generated[13], so it suffices to prove the analogous statements for the p-adic homotopy groups, for all prime numbers p. We fix p, let C be as in Theorem 1.3.6, and choose a ring homomorphism $f\colon \mathbb{C} \to C$. By Suslin [36, 37], the canonical maps

$$K^{\mathrm{top}}(\mathbb{C}) \longleftarrow K(\mathbb{C}) \xrightarrow{f^*} K(C)$$

become weak equivalences upon p-completion. Moreover, by Lemma 1.3.7, the map $j^*\colon K(\mathcal{O}_C) \to K(C)$ becomes a weak equivalence after p-completion. The ring $\mathcal{O}_C$ is a henselian local ring with algebraically closed residue field k of characteristic p. Therefore, by Clausen–Mathew–Morrow [14], the cyclotomic trace map induces an equivalence

$$K(\mathcal{O}_C, \mathbb{Z}_p) \xrightarrow{\mathrm{tr}} \mathrm{TC}(\mathcal{O}_C, \mathbb{Z}_p),$$

so the statement follows from Theorem 1.3.6. □

1.4 Group rings

Let G be a discrete group. We would like to understand the topological cyclic homology of the group ring $R[G]$, where R is a ring or, more generally, a connective $\mathbb{E}_1$-algebra in spectra. Since the assignment $G \mapsto \mathrm{TC}(R[G])$ is functorial in the 2-category of groups,[14] we get an "assembly" map

$$\mathrm{TC}(R) \otimes BG_+ \longrightarrow \mathrm{TC}(R[G]),$$

[13] This follows by a Serre class argument from the fact that the homology groups of the underlying space are finitely generated.

[14] By the latter we mean the full subcategory of the ∞-category of spaces consisting of spaces of the form BG. Concretely objects are groups, morphisms are group homomorphisms and 2-morphisms are conjugations.

and what we will actually do here is to consider the cofiber of this map. By [27, Theorem 1.2], topological cyclic homology for a given group G can be assembled from the cyclic subgroups of G. We will focus on the case $G = C_p$ of a cyclic group of prime order p, but the methods that we present here can be generalized to the case of cyclic p-groups. We will be interested in the p-adic homotopy type of the cofiber. To this end, we remark that the p-completion of $\mathrm{THH}(R)$, which we denote by $\mathrm{THH}(R, \mathbb{Z}_p)$ as before, inherits a cyclotomic structure. For R connective, we have

$$\mathrm{TC}(R, \mathbb{Z}_p) \simeq \mathrm{TC}(\mathrm{THH}(R, \mathbb{Z}_p)).$$

The formula we give involves the non-trivial extension of groups

$$C_p \longrightarrow \mathbb{T}_p \longrightarrow \mathbb{T}.$$

The middle group $\mathbb{T}_p$ is a circle, but the right-hand map is a p-fold cover, and by restriction along this map, a spectrum with $\mathbb{T}$-action X gives rise to a spectrum with $\mathbb{T}_p$-action, which we also write X.

Theorem 1.4.1. *For a connective $\mathbb{E}_1$-algebra R in spectra, there is a natural cofiber sequence of spectra*

$$\mathrm{TC}(R, \mathbb{Z}_p) \otimes BC_{p+} \longrightarrow \mathrm{TC}(R[C_p], \mathbb{Z}_p) \longrightarrow \mathrm{THH}(R, \mathbb{Z}_p)_{h\mathbb{T}_p}[1] \otimes C_p,$$

where C_p is considered as a pointed set with basepoint 1.

We note that the right-hand term in the sequence above is non-canonically equivalent to a $(p-1)$-fold sum of copies of $\mathrm{THH}(R, \mathbb{Z}_p)_{h\mathbb{T}_p}[1]$. We do not determine the boundary map in the sequence. The proof of Theorem 1.4.1 requires some preparation and preliminary results. First, we recall that for a spherical group ring $\mathbb{S}[G]$, there is a natural equivalence

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \mathbb{S} \otimes LBG_+,$$

where $LBG = \mathrm{Map}(\mathbb{T}, BG)$ is the free loop space. Moreover, the equivalence is $\mathbb{T}$-equivariant for the $\mathbb{T}$-action on LBG induced from the action of $\mathbb{T}$ on itself by multiplication. Hence, for general R, we have

$$\mathrm{THH}(R[G]) \simeq \mathrm{THH}(R) \otimes LBG_+,$$

where $\mathbb{T}$ acts diagonally on the right-hand side. Since G is discrete, we further have a $\mathbb{T}$-equivariant decomposition of spaces

$$LBG \simeq \coprod BC_G(x),$$

where x ranges over a set of representatives of the conjugacy classes of elements of G, and where $C_G(x) \subset G$ is the centralizer of $x \in G$. The action by $\mathbb{T} \simeq B\mathbb{Z}$ on $BC_G(x)$ is given by the map $B\mathbb{Z} \times BC_G(x) \to BC_G(x)$ induced by the group homomorphism $\mathbb{Z} \times C_G(x) \to C_G(x)$ that to (n, g) assigns $x^n g$. Specializing to the case $G = C_p$, we get the following description.

Lemma 1.4.2. *There is a natural $\mathbb{T}$-equivariant cofiber sequence of spaces*

$$BC_p^{\mathrm{triv}} \to LBC_p \to (BC_p^{\mathrm{res}})_+ \otimes C_p,$$

where BC_p^{triv} has the trivial $\mathbb{T}$-action and $BC_p^{\mathrm{res}} \simeq (\mathrm{pt})_{hC_p}$ has the residual action by $\mathbb{T} = \mathbb{T}_p/C_p$. [15]

As a consequence, we obtain for every $\mathbb{E}_1$-algebra in spectra R, a cofiber sequence of spectra with $\mathbb{T}$-action

$$\mathrm{THH}(R) \otimes BC_{p+}^{\mathrm{triv}} \longrightarrow \mathrm{THH}(R[G]) \longrightarrow (\mathrm{THH}(R) \otimes BC_{p+}^{\mathrm{res}}) \otimes C_p,$$

where the left-hand map is the assembly map. To determine the cyclotomic structure on the terms of this sequence, we prove the following result.

Lemma 1.4.3. *Let X be a spectrum with $\mathbb{T}$-action that is bounded below, and let $\mathbb{T}$ act diagonally on $X \otimes BC_{p+}^{\mathrm{res}}$. Then $(X \otimes BC_{p+}^{\mathrm{res}})^{tC_p} \simeq 0$.*

Proof. We write $X \simeq \lim_n \tau_{\leq n} X$ as the limit of its Postnikov tower. The spectra $\tau_{\leq n} X$ inherit a $\mathbb{T}$-action, and the equivalence is $\mathbb{T}$-equivariant. The map induced by the canonical projections,

$$X \otimes BC_{p+}^{\mathrm{res}} \longrightarrow \lim_n(\tau_{\leq n} X \otimes BC_{p+}),$$

is an equivalence, since the connectivity of the fibers tend to infinity with n, and therefore, also the map

$$(X \otimes BC_{p+}^{\mathrm{res}})^{tC_p} \longrightarrow \lim_n((\tau_{\leq n} X \otimes BC_{p+})^{tC_p})$$

is an equivalence. Indeed, the analogous statements for homotopy fixed points and homotopy orbits is respectively clear and a consequence of the fact that the connectivity of the fibers tend to infinity with n.

Since X is bounded below, we may assume that X is concentrated in a single degree with necessarily trivial $\mathbb{T}$-action. As spectra with $\mathbb{T}$-action,

$$X \otimes BC_{p+}^{\mathrm{res}} \simeq X_{hC_p},$$

where the right-hand side has the residual $\mathbb{T}$-action. But $(X_{hC_p})^{tC_p} \simeq 0$ by the Tate orbit lemma [32, Lemma I.2.1], so the lemma follows. $\square$

Corollary 1.4.4. *Let X be a spectrum with $\mathbb{T}$-action that is bounded below and p-complete. The spectrum $X \otimes BC_{p+}^{\mathrm{res}}$ with the diagonal $\mathbb{T}$-action admits a unique cyclotomic structure, and, with respect to this structure,*

$$\mathrm{TC}(X \otimes BC_{p+}^{\mathrm{res}}) \simeq X_{h\mathbb{T}_p}[1].$$

[15] This comes from the fact that pt carries a (necessarily trivial) $\mathbb{T}_p$-action. In a point set model, it can be described as $E\mathbb{T}_p/C_p$.

Proof. A cyclotomic structure on a spectrum with $\mathbb{T}$-action Y consists of a family of $\mathbb{T}$-equivariant maps $\varphi_\ell \colon Y \to Y^{tC_\ell}$, one for every prime number ℓ, including p. In the case of $Y = X \otimes BC_p^{\mathrm{res}}$, the target of this map is contractible for all ℓ. Indeed, for $\ell \neq p$, this follows from Y being p-complete, and for $\ell = p$, it follows from Lemma 1.4.3. Hence, there is a unique such family of maps. In order to evaluate TC, we first note that Lemma 1.4.3 also implies that

$$\mathrm{TP}(X \otimes BC_{p+}^{\mathrm{res}}) \simeq \mathrm{TP}(X \otimes BC_{p+}^{\mathrm{res}})^{\wedge} \simeq 0.$$

Accordingly,

$$\mathrm{TC}(X \otimes BC_{p+}^{\mathrm{res}}) \simeq \mathrm{TC}^{-}(X \otimes BC_{p+}^{\mathrm{res}}) = (X \otimes BC_{p+}^{\mathrm{res}})^{h\mathbb{T}},$$

and by the vanishing of $(-)^{t\mathbb{T}}$, this is further equivalent to

$$(X \otimes BC_{p+}^{\mathrm{res}})_{h\mathbb{T}}[1] \simeq (X_{hC_p})_{h\mathbb{T}}[1] \simeq X_{h\mathbb{T}_p}[1],$$

where, in the middle term, $C_p \subset \mathbb{T}_p$ acts trivially on X. $\square$

Proof of Theorem 1.4.1. We have proved that there is a cofiber sequence

$$\mathrm{THH}(R) \otimes BC_{p+}^{\mathrm{triv}} \longrightarrow \mathrm{THH}(R[C_p]) \longrightarrow \mathrm{THH}(R) \otimes BC_{p+}^{\mathrm{res}} \otimes C_p$$

of spectra with $\mathbb{T}$-action in which the left-hand map is the assembly map. We get an induced fiber sequence with p-adic coefficients. Since the forgetful functor from cyclotomic spectra to spectra with $\mathbb{T}$-action creates colimits, this map is also the assembly map in the ∞-category of cyclotomic spectra, and by Corollary 1.4.4, the induced cyclotomic structure on cofiber necessarily is the unique cyclotomic structure, for which

$$\mathrm{TC}(\mathrm{THH}(R, \mathbb{Z}_p) \otimes BC_{p+}^{\mathrm{res}} \otimes C_p) \simeq \mathrm{THH}(R, \mathbb{Z}_p)_{h\mathbb{T}_p}[1] \otimes C_p.$$

Finally, by the universal property of the colimit, there is a canonical map

$$\mathrm{TC}(R, \mathbb{Z}_p) \otimes BC_p \longrightarrow \mathrm{TC}(\mathrm{THH}(R, \mathbb{Z}_p) \otimes BC_p^{\mathrm{triv}}),$$

and by [14, Theorem 2.7], this map is an equivalence. $\square$

Example 1.4.5. For $R = \mathbb{S}$, the sequence in Theorem 1.4.1 becomes

$$\mathrm{TC}(\mathbb{S}, \mathbb{Z}_p) \otimes BC_{p+} \longrightarrow \mathrm{TC}(\mathbb{S}[C_p], \mathbb{Z}_p) \longrightarrow (\mathbb{S}_p \otimes B\mathbb{T}_{p+})[1] \otimes C_p,$$

where $B\mathbb{T}_p \simeq \mathbb{P}^\infty(\mathbb{C})$. One can also give a formula for $\mathrm{TC}(\mathbb{S}[C_p], \mathbb{Z}_p)$ in this case, but this is more complicated than the formula for the cofiber of the assembly map.

Finally, we evaluate the homotopy groups of $\mathrm{THH}(R,\mathbb{Z}_p)_{h\mathbb{T}_p}$ in the case, where R is a p-torsion free perfectoid ring. We consider the diagram

$$\begin{tikzcd}
\pi_0(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}}) \arrow[r] \arrow[d] & \pi_0(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}_p}) \arrow[d] \\
\pi_0(\mathrm{THH}(R,\mathbb{Q}_p)^{h\mathbb{T}}) \arrow[r] & \pi_0(\mathrm{THH}(R,\mathbb{Q}_p)^{h\mathbb{T}_p}),
\end{tikzcd}$$

where the horizontal maps are given by restriction along $\mathbb{T}_p \to \mathbb{T}$, and where the vertical maps are given by change-of-coefficients. The respective homotopy fixed point spectral sequences endow each of the four rings with a descending filtration, which we refer to as the Nygaard filtration, and they are all complete and separated in the topology. We have identified the top left-hand ring with Fontaine's ring $A = A_{\mathrm{inf}}(R)$. The lower horizontal map is an isomorphism, and the common ring is identified with $A[1/p]^\wedge$, where "$(-)^\wedge$" indicates Nygaard completion. We further have compatible edge homomorphisms

$$\begin{tikzcd}
\pi_0(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}}) \arrow[r] \arrow[d, "\theta"] & \pi_0(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}_p}) \arrow[d, "\theta_p"] \\
\mathrm{THH}_0(R,\mathbb{Z}_p) \arrow[r, equal] & \mathrm{THH}_0(R,\mathbb{Z}_p),
\end{tikzcd}$$

whose kernels $I \subset A$ and $I_p \subset A_p$ are principal ideals, and we can choose generators ξ and ξ_p such that the top horizontal map takes ξ to $p\xi_p$. This identifies the top right-hand ring A_p with the subring

$$A_p = (\textstyle\sum_{n\geq 0} p^{-n} I^n)^\wedge \subset A[1/p]^\wedge$$

given by the Nygaard completion of the Rees construction.

Proposition 1.4.6. *Let R be a p-torsion free perfectoid ring. The map*

$$\pi_*(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}_p}) \xrightarrow{\mathrm{can}} \pi_*(\mathrm{THH}(R,\mathbb{Z}_p)^{t\mathbb{T}_p})$$

is given by the localization of graded A_p-algebras

$$A_p[u,v_p]/(uv_p - \xi_p) \longrightarrow A_p[u,v_p^{\pm 1}]/(uv_p - \xi_p) = A_p[v_p^{\pm 1}],$$

where u and v_p are homogeneous elements of degree 2 and -2, and where ξ_p is a generator of the kernel I_p of the edge homomorphism $\theta_p \colon A_p \to R$.

Proof. The proof is analogous to the proof of Theorem 1.3.5. □

Corollary 1.4.7. *Let R be a p-torsion free perfectoid ring. For $m \geq 0$,*

$$\pi_{2m}(\mathrm{THH}(R,\mathbb{Z}_p)_{h\mathbb{T}_p}) = A_p/I_p^{m+1} \cdot v_p^{-(m+1)},$$

and the remaining homotopy groups are zero.

Remark 1.4.8. It is interesting to compare the calculation above to the case, where R is a perfect $\mathbb{F}_p$-algebra. In this case, the map

$$\pi_0(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}}) \longrightarrow \pi_0(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}_p})$$

takes $uv - p = 0$ to $(uv_p - 1)p = 0$, and since $1 - uv_p$ is a unit, we conclude that, in the target ring, $p = 0$. Hence, this ring is a power series ring $R[[y]]$ on a generator y that is represented by $t_p x$ in the spectral sequence

$$E^2 = R[t_p, x] \Rightarrow \pi_*(\mathrm{THH}(R,\mathbb{Z}_p)^{h\mathbb{T}_p}).$$

Hence, for $m \geq 0$, we have

$$\pi_{2m}(\mathrm{THH}(R,\mathbb{Z}_p)_{h\mathbb{T}_p}) = R[[y]]/y^{m+1} \cdot v_p^{-(m+1)}$$

and the remaining homotopy groups are zero.

We also consider the case of the semi-direct product $G = \mathrm{Aut}(C_p) \ltimes C_p$. The conjugacy classes of elements in G are represented by the elements $(\mathrm{id}, 0)$, $(\mathrm{id}, 1)$, and $(\alpha, 0)$ with $\alpha \in \mathrm{Aut}(C_p) \smallsetminus \{\mathrm{id}\}$, the centralizers of which are G, C_p, and $\mathrm{Aut}(C_p)$, respectively. Hence, for every $\mathbb{E}_1$-algebra in spectra R, there is a cofiber sequence of spectra with $\mathbb{T}$-action

$$\begin{aligned} &\mathrm{THH}(R) \otimes BG_+^{\mathrm{triv}} \longrightarrow \mathrm{THH}(R[G]) \\ &\longrightarrow \mathrm{THH}(R) \otimes BC_{p+}^{\mathrm{res}} \oplus \mathrm{THH}(R) \otimes B\,\mathrm{Aut}(C_p)_+ \otimes \mathrm{Aut}(C_p), \end{aligned}$$

where the last tensor factor $\mathrm{Aut}(C_p)$ is viewed as a pointed space with id as basepoint. Moreover, as cyclotomic spectra, the right-hand summand splits off. Therefore, after p-completion, we arrive at the following statement.

Theorem 1.4.9. *Let $G = \mathrm{Aut}(C_p) \ltimes C_p$, and let R be a connective $\mathbb{E}_1$-ring. There is a canonical fiber sequence of spectra*

$$\begin{aligned} &\mathrm{TC}(R,\mathbb{Z}_p) \otimes BG_+ \longrightarrow \mathrm{TC}(R[G],\mathbb{Z}_p) \\ &\qquad\qquad \longrightarrow \mathrm{THH}(R,\mathbb{Z}_p)_{h\mathbb{T}_p}[1] \oplus \mathrm{TC}(R,\mathbb{Z}_p) \otimes \mathrm{Aut}(C_p), \end{aligned}$$

and moreover, the summand $\mathrm{TC}(R,\mathbb{Z}_p) \otimes \mathrm{Aut}(C_p)$ *splits off* $\mathrm{TC}(R[G],\mathbb{Z}_p)$.

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