2020.11.10

Condensed Sets

Idea: In terms of phenomena

Cond. sets <~~~> Top spaces

but categorically

Cond. sets <~~~> Sets

so easier to add-on algebraic structure.

Basic building blocks: CHaus (compact Hausdorff spaces). Glue these together along certain relations. Formally, modulo set-theoretic issues, will define a notion of covering on CHaus, giving a site, and define

$$\mathrm{CondSets} = \mathrm{Shv}(\mathrm{CHaus}).$$

Def For $Y \in \mathrm{CHaus}$, $\mathrm{Cov}(Y)$ consists of finite families $(X_i \to Y)_{i \in I}$ of continuous maps $X_i \to Y$ s.t.

$$\coprod_{i \in I} X_i \twoheadrightarrow Y.$$

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Why this definition? Always two things to consider:

1) Strong enough so that you can work locally — this buys you something.

2) Not too strong so that invariant you care about vanish.

Consider 2) first:

Ex If T is a top. space, then

$$X \longmapsto \mathcal{F}(X) = \mathrm{Cont}(X, T)$$

is a sheaf for this topology. Indeed, set-theoretically, the sheaf condition expresses that given $Y \twoheadrightarrow X$ in CHaus, maps $X \to T$ are in 1-1 correspondence with maps $Y \to T$ that are constant along the fibers of $Y \twoheadrightarrow X$. Must show that $Y \to T$ is continuous if and only if the induced map $X \to T$ is continuous. The "only if" direction is clear; for "if" direction use that a map of cpt. Hausdorff spaces maps closed subsets to closed subsets. //

Ex (Will be discussed tomorrow)

$$X \longmapsto R\Gamma(X, M)$$

satisfies descent for this Grothendieck top. on CHaus. //

So this Grothendieck topology is suitable both for point-set topology and for algebraic top.

Next consider 1):

Gleason: The category CHaus has enough projectives. Here T is proj. if for all $Y \twoheadrightarrow X$ and $T \to X$, a lift

$$\begin{tikzcd} & Y \arrow[d, two heads] \\ T \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$$

exists. "Enough" means that

for all $X \in \mathrm{CHaus}$, there exists $T \twoheadrightarrow X$ with T proj.

Pf If S is a set viewed as a discrete top. space, then its Stone-Čech compactification is the limit

$$\beta(S) = \lim_{\substack{S \twoheadrightarrow S_i \\ S_i \text{ finite}}} S_i$$

The functor β is left adjoint to the forgetful functor

$$\begin{tikzcd} \mathrm{Sets} \arrow[r, shift left, "\beta"] & \mathrm{CHaus} \arrow[l, shift left, "\mathrm{fgt}"] \end{tikzcd}.$$

So $\beta(S)$ is projective (by axiom of choice):

$$\begin{tikzcd} & Y \arrow[d, two heads, dashed] \\ \beta(S) \arrow[ur, dashed, "\exists"] \arrow[r] & X \end{tikzcd}.$$

Now, if $S \to X$ has dense image, then $\beta(S) \twoheadrightarrow X$, since its image is both dense and closed. //

A sheaf on CHaus is uniquely determined by its restriction to the full subcategory spanned by the $\beta(S)$'s:

$$\begin{tikzcd}
\beta(S') \arrow[r, two heads] & \beta(S) \times_X \beta(S) \arrow[d, shift left] \arrow[d, shift right] \\
 & \beta(S) \arrow[d, two heads] \\
 & X
\end{tikzcd}$$

closed subset of $\beta(S) \times \beta(S)$

Warning: A product of proj. is not proj., and neither is a closed subspace of proj.

So $\mathcal{F}(X)$ is the equalizer

$$\mathcal{F}(X) \to \mathcal{F}(\beta(S)) \rightrightarrows \mathcal{F}(\beta(S')).$$

Moreover, on the $\beta(S)$'s, the Grothendieck topology is much simpler: The sheaf cond. is:

(i) $\mathcal{F}(\emptyset) \simeq *$

(ii) $\mathcal{F}(\beta(S) \amalg \beta(S')) \xrightarrow{\sim} \mathcal{F}(\beta(S)) \times \mathcal{F}(\beta(S')).$

The fact that the $\beta(S)$'s give enough proj. implies that every proj. in CHaus is a retract of a $\beta(S)$. The projective objects in CHaus are called "extremally disconnected cpt. Hausdorff spaces." They can be characterized point-set topologically by the following property: The closure of every open subset is an open subset!

Recap: Strict inclusions

$$\text{CHaus} \supset \text{profinite sets} \supset \beta(S)\text{'s}$$

CHaus	profinite sets	$\beta(S)$'s
↑	↑	↑
have all limits	o-dim'l	good for arguments

Grothendieck top. on CHaus restricts to both subcat. and these three sites all define the same topos.

Want to define

$$\mathrm{Cond}(\mathrm{Sets}) \simeq \mathrm{Shv}\left(\begin{matrix}\text{profinite}\\ \text{sets}\end{matrix}\right),$$

but this is not allowed, since profinite sets do not form a small category. So Hom-sets would be large. First bound cardinality.

Def A cardinal κ is a strong limit cardinal if

$$\lambda < \kappa \implies 2^\lambda < \kappa.$$

Easy to make examples

beth hirachy: For any ordinal α,

$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$

and for α limit ordinal,

$$\beth_\alpha = \bigcup_{\beta<\alpha} \beth_\beta.$$

The latter is a strong limit cardinal. For all practical purposes, can use $\kappa = \beth_\alpha$.

<u>Def</u> Category of κ-cond. sets :

$$\mathrm{Shv}\big(\kappa\text{-small } \overset{\beta(S)'s}{\underset{\text{CHaus}}{\text{profinite sets}}}\big).$$

<u>Rmk</u> Barwick–Haine use κ strongly inaccessible = strong limit + regular. Cannot be proved to exist in ZFC.

<u>Def</u> The category of condensed sets is the union

$$\mathrm{Cond(Sets)} = \bigcup_{\kappa} \kappa\text{-}\mathrm{Cond(Sets)}.$$

Explain union: If $\kappa < \kappa'$, then

$$\{\kappa\text{-small } \beta(S)'s\} \hookrightarrow \{\kappa'\text{-small } \beta(S)'s\}$$

gives a pullback functor

$$\kappa\text{-}\mathrm{Cond(Sets)} \longrightarrow \kappa'\text{-}\mathrm{Cond(Sets)}$$

$$X \longmapsto \Big(T' \longmapsto \operatorname*{colim}_{\substack{T' \to T \\ T\ \kappa\text{-small}}} X(T)\Big)^{\mathrm{shv}}$$

Sheafification is not needed if we use $\beta(S)$'s, but (possibly) if we use CHaus or prof. sets.

<u>Rmk</u> Alternative definition:

$$\mathrm{Cond}(\mathrm{Sets}) \subset \mathcal{P}(\beta(S)'s)$$

is the full subcat. spanned by all small <u>sifted</u> <u>colimits</u> of $\beta(S)$'s. Upshot is that Cond. sets satisfy all of Giraud's axioms (HTT, Chap. 6), except that it is only class-presentable and not presentable. So

1) Finite limits and arbitrary colimits interact as in Sets. (Topos)

2) Arbitrary limits and sifted colimits interact as in Sets. (Algebraic theory)

The property 2) comes from from the fact that Cond(Sets) has enough compact projectives.

Cond(Ab) is an abelian category with enough compact projectives.

M cpt. proj. $\Longleftrightarrow$ $\mathrm{Hom}(M, -)$ pres. all lim/colim.