FREENESS AND EQUIVARIANT STABLE HOMOTOPY

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Abstract. We introduce a notion of freeness for RO-graded equivariant generalized homology theories, considering spaces or spectra E such that the R-homology of E splits as a wedge of the R-homology of induced virtual representation spheres. The full subcategory of these spectra is closed under all of the basic equivariant operations, and this greatly simplifies computation. Many examples of spectra and homology theories are included along the way.

We refine this to a collection of spectra analogous to the pure and isotropic spectra considered by Hill–Hopkins–Ravenel. For these spectra, the RO-graded Bredon homology is extremely easy to compute, and if these spaces have additional structure, then this can also be easily determined. In particular, the homology of a space with this property naturally has the structure of a co-Tambara functor (and compatibly with any additional product structure). We work this out in the example of $BU_{\mathbb{R}}$ and coinduced versions of this.

We finish by describing a readily computable bar and twisted bar spectra sequence, giving Bredon homology for various E_∞ pushouts, and we apply this to describe the homology of $BBU_{\mathbb{R}}$.

1. Introduction

Equivariant cohomology theories are often viewed as very difficult to compute. In full generality, this is often true, as many computations which non-equivariantly were completely in the 1950s and 1960s are still out of reach. Addtionally, the kinds of cellular decompositions which arise most naturally geometrically are often not adapted to easy computation, further compounding the problem. Many computations in the literature require significant amounts of hard work, even for ordinary (Bredon) homology (see, for example, the recent papers of Dugger on equivariant Grassmanians [4] and Hazel on C_2-surfaces [8]).

In this paper, we build on a class of spectra introduced by Ferland–Lewis [5], focusing on a certain subcategory of spaces and spectra for which essentially all of these problems go away. Given a commutative ring-valued equivariant cohomology theory R, we say that a spectrum E is R-free if the R homology of E splits as a wedge of the R-homology of induced representation spheres. These spectra contains many of the geometrically meaningful spaces and spectra. Delightfully, these R-free spectra are closed under most of the usual operations in equivariant homotopy, including restriction, induction, and the symmetric monoidal product. If R is represented by an equivariant commutative ring spectrum, then the class of R-free spectra is also closed under the norm maps. In the classical, Bredon case, this means that the cohomology is easy to describe with almost arbitrary coefficients, and most excitingly, it means we can describe a full co-algebra (in fact, co-Tambara functor)

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structure on the homology of these spaces and on the cohomology of equivariant commutative monoid objects.

After describing a host of examples, we restrict focus to a class of spectra for which everything is described by the underlying homology. The slice filtration of [10] gives a version of the Postnikov tower where we use various representation spheres instead of ordinary spheres. In the nicest cases, such that those built out of the norms of the Fujii–Landweber spectrum of Real bordism $\mathrm{MU}_{\mathbb{R}}$, the slice associated graded is a wedge of regular representation spheres smashed with computationally tractible Eilenberg–Mac Lane spectra [6, 20] (see also [15]). For these kinds of spectra, we consider free spectra where the induced spheres are also in regular representation dimensions. These assumptions allow us to reduce any computational question to a question about the non-equivariant homotopy, tying things to classically studied objects. We demonstrate the efficiency of this by giving the full Tambara and co-Tambara functor structures on the homology of $BU_{\mathbb{R}}$ and by describing the action of the Dyer–Lashof algebra on the mod 2-homology of $BU_{\mathbb{R}}$.

We close with applications to the bar/Rothenberg–Steenrod and Eilenberg–Moore spectral sequences. When the spaces in question are R-free, the E_2-terms of the usual spectral sequences have the expected form, and we use this to compute the homology of $BBU_{\mathbb{R}}$ and of the coinduced space $\mathrm{Map}^{C_2}(G, BBU_{\mathbb{R}})$ for all finite G. As an aside, we also mention the sign-twisted analogues of these classical spectral sequences when $G = C_2$, giving ways to compute the homology of the signed bar construction or the cohomology of the twisted homotopy pullback and signed loop spaces.

Throughout the paper, our emphasis is on the conceptual understanding of the objects and on explicit examples. We include many examples of spaces and spectra of interest, showing how they fit into this framework, working to demystify equivariant computations.

2. RO-graded homology

2.1. **Gradings and ring structures.** Many of the spaces which arise naturally geometrically can be built not out of representation cells but rather out of more general cells of the form
$$G_+ \underset{H}{\wedge} D(V),$$
where V is a [virtual] H-representation. Algebraic constructions like the norm automatically build in this more general kind of $\mathrm{RO}(G)$-grading, considering instead objects graded on pairs consisting of a subgroup H and a virtual representation of H. A more coordinate free version is given by considering Thom spectra of virtual bundles over finite G-sets. A particular model of this is the restriction of work of Angeltveit–Bohmann to incomplete Tambara functors or Mackey functors [1].

Definition 2.1 ([12, Definition 2.7])**.** If T is a finite G-set and V is an equivariant virtual bundle over T, then let $M(V)$ be the Thom spectrum of V and
$$\underline{\pi_{\star}}(E)\big(T, V\big) = \big[M(V), E\big]^G.$$

Remark 2.2. If T is a transitive G-set, then a choice of point $t \in T$ gives an equivariant equivalence
$$T \cong G/\mathrm{Stab}(t),$$

and restriction to t gives an equivalence of categories between $\mathrm{Stab}(t)$-equivariant virtual representations and virtual equivariant vector bundles over T.

Notation 2.3. In the case $T = G/H$, so V gives a virtual H-representation V_H, let

$$E^H_{V_H}(S^0) = \underline{\pi}_\star E(T, V).$$

These abelian groups assemble into a kind of Mackey functor, twisted by these bundles. This generalizes the earlier work of Ferland–Lewis [5].

Proposition 2.4. *If $f\colon S \to T$ is a map of finite G-sets and if $V \to T$ is a virtual equivariant bundle, then f induces a transfer map*

$$\pi_{\underline{\star}}(E)\big(S, f^*V\big) \xrightarrow{T_f} \pi_{\underline{\star}}(E)\big(T, V\big)$$

and a restriction map

$$\pi_{\underline{\star}}(E)\big(T, V\big) \xrightarrow{R_f} \pi_{\underline{\star}}(E)\big(S, f^*V\big).$$

Remark 2.5. There is a slight subtlety here with the Mackey double coset formula: there can be signs introduced which reflect the degree of the map on the underlying representation sphere. See, for example, [10, Lemma 7.20].

Smashing together maps gives us the usual external product.

Definition 2.6. If $x \in \pi_{\underline{\star}}(E)(T, V)$ and $y \in \pi_{\underline{\star}}(E')(S, W)$, then we have an external product

$$x \wedge y \in \pi_{\underline{\star}}(E \wedge E')(T \times S, V \times W)$$

given by the smash product of representing maps.

Since this pairing is the one arising from the pairing of homotopy classes of functions in G-spectra, it has the usual properties.

Proposition 2.7. *The external product is linear in both factors and satisfies the Frobenius relation.*

The multiplication in the RO-graded context can be a little more confusing, since elements are attached to virtual representations for different groups. To effectively compare them, the elements must first be restricted to a maximal common subgroup. Again, we have many ways to represent this. Conceptually, the RO-graded group actually remembers more information, including not only the elements but also the various Weyl transfers. Thinking in this way, the RO-graded products will not only record the product we would expect but also include any of the pairwise products of restrictions to conjugate subgroups.

If $T = S$, then we have a canonical pullback diagram

$$\begin{tikzcd} V \oplus W \arrow[r] \arrow[d] & V \times W \arrow[d] \\ T \arrow[r, "\Delta_T"'] & T \times T. \end{tikzcd}$$

Composing the external product with the restriction along the diagonal Δ_T gives the usual product structure on the $RO(T)$-graded homotopy of the "restriction to T" of a ring spectrum R.

At the other extreme, we if $T = G/H$ and $S = G/K$, then the product can be a little more confusing. The classes x and y are maps

$$G_+ \underset{H}{\wedge} S^V \xrightarrow{x} E \quad \& \quad G_+ \underset{K}{\wedge} S^W \xrightarrow{y} E',$$

and smashing them together gives the map

$$\big(G_+ \underset{H}{\wedge} S^V\big) \wedge \big(G_+ \underset{K}{\wedge} S^W\big) \xrightarrow{x\wedge y} E \wedge E'.$$

The source is naturally the Thom spectrum of a virtual bundle on

$$G/H \times G/K \cong G \underset{H}{\times} i_H^* G/K \cong \coprod_{HgK \in H\backslash G/K} G/(H \cap gKg^{-1}),$$

and the bundle over the summand associated to g is

$$i^*_{H\cap gKg^{-1}} V \oplus i^*_{H\cap gKg^{-1}} c_g^* W.$$

The corresponding map on this summand is

$$res^H_{H\cap gKg^{-1}}(x) \wedge res^{gKg^{-1}}_{H\cap gKg^{-1}}(c_g^* y).$$

Corollary 2.8. *If E has a multication in the homotopy category, then the composition with the multiplication in E makes $\underline{\pi}_\star(E)$ into an* RO*-graded Green functor.*

In fact, there is a good G-symmetric monoidal category of $\underline{R_\star}$-modules for any equivariant commutative ring spectrum R. This has been developed in forthcoming joint work with Angeltveit–Bohmann. We will make use of this structure somewhat heavily in what follows. However, the only cases in which we will consider it are ones for which the structure is immediate from the definition of the objects, so there should be no confusion.

2.2. R-free spectra modules.

2.2.1. *Free and projective.* Although our desired applications will be to ordinary Bredon homology with various coefficients, it will be helpful to being considering any generalized equivariant homology theory R represented by an E_∞-ring spectrum R. Equivariantly, this is weaker than being a commutative monoid in any of the good point-set models of spectra, but this is sufficient to have a good, symmetric monoidal category of modules over R [3].

It greatly simplifies much of the notation (and of our discussion of a basis) to ourselves to evaluate our homotopy Mackey functors on infinite G-sets and virtual representations on these.

Notation 2.9. If T is a discrete G-set and V is a virtual bundle over T, then let

$$\underline{R_\star}(E)(T,V) = \varprojlim \underline{R_\star}(E)(S, i_S^* V),$$

where S ranges over all finite subsets of T.

Since Thom spectra of disjoint unions of spaces is the coproduct of the associated Thom spectra, we natural have

$$\underline{R_\star}(E)(T,V) \cong \big[M(V), R\wedge E\big]^G.$$

Definition 2.10. A G-spectrum E has **free R-homology** or "is R-free" if there is a G-set T_E and a virtual vector bundle V_E over T_E such that we have an equivalence of R-modules
$$R \wedge E \simeq R \wedge M(V_E).$$
The full subcategory of Sp^G spanned by the spectra with free R-homology will be denoted
$$\mathcal{S}p^G_{R,fr}.$$
It has **projective R-homology** if $R \wedge E$ is a retract of an R-module of the form $R \wedge M(V)$ for some virtual vector bundle V over a G-set. The full subcategory of Sp^G spanned by the spectra with projective R-homology will be denoted
$$\mathcal{S}p^G_{R,pr}.$$

Remark 2.11. The use of "free" here is to bring to mind a free module. In the homotopy category of R-modules, the R-module $R \wedge (G_+ \underset{H}{\wedge} S^V)$ corepresents the functor
$$E \mapsto \pi^H_V(E),$$
on the category of R-modules, and hence maps out of it correspond to certain elements in this $\underline{RO}$-graded Mackey functor.

Definition 2.12. If E has free R-homology, then a **basis** for the R-homology of E is an element
$$\vec{x} \in R_{\underline{\star}}(E)(T_E, V_E)$$
for a G-set T_E and a virtual bundle V_E over T_E, such that the induced map
$$R \wedge M(V_E) \xrightarrow{R\wedge\vec{x}} R \wedge E$$
is an equivalence.

We can restate the definition of a basis using an orbit decomposition of T. A choice of points in each orbit for T gives an equivariant isomorphism
$$T \cong \coprod_{t\in T/G} G/H_t,$$
and if we let $V_{E,t}$ be the restriction of V_E to the orbit G/H_t, then
$$R_{\underline{\star}}(E)(T,V) \cong \prod_{t\in T/G} R^{H_t}_{V_{E,t}}(E).$$
A basis then is a collection of elements
$$x_t\colon G_+ \underset{H_t}{\wedge} S^{V_{E,t}} \to R \wedge E$$
such that the induced map
$$R \wedge \left(\bigvee_{t\in T/G} G_+ \underset{H_t}{\wedge} S^{V_{E,t}} \right) \to R \wedge E$$
is an equivalence. We will use both formulations.

Just as for vector spaces, a basis is a choice of additional data which aids in explicit computation. In particular, describing product structures is greatly simplified with a basis.

It is always helpful to always keep in mind the example of Bredon homology with coefficients in a commutative Green functor $\underline{R}$. The Eilenberg–Mac Lane spectrum associated to a commutative Green functor is always E_∞, so we can apply this general formalism.

Example 1. *If $G = \{e\}$, then a basis for the homology of E with coefficients in $\underline{R} = R$ (an ordinary commutative ring) is the same as a basis for the graded R-module $H_*(E;R)$.*

Example 2. *Kronholm showed that if X is a Rep(C_2)-complex (meaning a C_2-complex formed by attaching disks in representations along their boundaries), then X has free $H\underline{\mathbb{F}}_2$-homology* [18] *(and with no summands induced up from the trivial group).*

Example 3. *C. May's decomposition theorem for the [co]homology of a finite C_2-CW complex says that for any finite C_2-complex X, we have a splitting*

$$H\underline{\mathbb{F}}_2 \wedge X \simeq H\underline{\mathbb{F}}_2 \wedge \Big(\bigvee C_{2+} \underset{H}{\wedge} S^V \vee \bigvee S(n_i\sigma)_+\Big)$$

where the second sum, $n_i \geqslant 2$ and σ is the sign representation [23]*. Thus C_2 spaces have free $H\underline{\mathbb{F}}_2$-homology if and only if this second sum vanishes.*

Example 4. *Hazel's computation of the Bredon homology of C_2-surfaces shows that every connected C_2-surface for which the action is not free has free $H\underline{\mathbb{F}}_2$-homology* [8, Theorem 6.6].

Example 5. *Ricka extended the Hu–Kriz computation of the dual Steenrod algebra for $\underline{\mathbb{F}}_2$, and showed that $H\underline{\mathbb{F}}_2$ has free $H\underline{\mathbb{F}}_2$-homology* [28], [15].

Notation 2.13. Let $\_r$ stand for either "fr" or "pr".

2.2.2. *Closure under sums.*

Proposition 2.14. *The adjoint pair $G_+ \underset{H}{\wedge} (\text{-}) \dashv i_H^*$ descends to an adjoint pair*

$$G_+ \underset{H}{\wedge} (\text{-})\colon \mathcal{S}p^H_{i_H^* R,\_r} \rightleftarrows \mathcal{S}p^G_{R,\_r} \colon i_H^*.$$

A basis for one gives the other via restriction or induction.

Proof. Since these are full subcategories, and since retracts are preserved by any functor, it suffices to show that restriction and induction preserve R-free spectra. For restriction, we just use the restriction of the Thom spectrum. For induction, we note that inducing up a Thom spectrum is again a Thom spectrum of the desired form. □

Proposition 2.15. *The category $\mathcal{S}p^G_{R,\_r}$ is a closed under arbitrary coproducts.*
A basis for the wedge is the sum of the bases.

Proof. The smash product distributes over wedges, and the wedge of Thom spectra of virtual bundles over G-sets is again a Thom spectrum of a virtual bundle over a G-set. □

The free and projectives also work well with base-change.

Proposition 2.16. *A map $f\colon R \to R'$ of E_∞-ring spectra induces a map*

$$f_*\colon \mathcal{S}p^G_{R,\_r} \to \mathcal{S}p^G_{R',\_r}.$$

A basis $\vec{x}$ for E over R gives a basis $f_*(\vec{x})$ be composing with f.

Proof. This follows from base-changing the equivalence $R \wedge E \simeq R \wedge M(V_E)$ along the map $R \to R'$. □

2.2.3. *Closure under products.* The categories of frees and projectives are also closed under the [twisted] smash products on G-spectra, being closed under the norms which R has.

Proposition 2.17. *The category $\mathcal{S}p^G_{R,\_r}$ is a symmetric monoidal subcategory of $\mathcal{S}p^G$ for the smash product.*

Moreover, a basis for E and E' gives a basis for $E \wedge E'$ by boxing them together.

Proof. Again, it suffices to show this for free spectra. If V and V' are virtual vector bundles on T and T' respectively, then we have a natural equivalence

$$M(V) \wedge M(V') \simeq M(V \times V'),$$

where the latter is just the Thom spectrum of product of V and V' over $T \times T'$. The result follows from recalling that the functor $R \wedge (\text{-})$ is a strong symmetric monoidal functor from G-spectra to R-modules. $\square$

This gives us a kind of weak Künneth theorem.

Theorem 2.18. *If $E \in \mathcal{S}p^G_{R,\_r}$, then for any R-module M, then $M \wedge E$ is a summand of the R-module $M \wedge M(V_E)$ for some virtual vector bundle V_E, and hence the multiplication gives a natural isomorphism*

$$R_{\underline{\star}}(E) \underset{R_{\underline{\star}}}{\Box} \underline{\pi}_{\underline{\star}}(M) \to \underline{\pi}_{\underline{\star}}(E \wedge M).$$

Proof. Again, it suffice to show for E R-free. By assumption, there is a splitting of R-modules

$$R \wedge M(V_E) \simeq R \wedge E.$$

This gives an equivalence of R-modules

$$E \wedge M \simeq (R \wedge E) \underset{R}{\wedge} M \simeq \big(R \wedge M(V_E)\big) \underset{R}{\wedge} M \simeq M(V_E) \wedge M.$$

Since the smash product distributes over the wedge, the latter spectrum is a wedge of R-modules of the form

$$(G_+ \underset{H}{\wedge} S^V) \wedge M.$$

The result follows by the definition of the representables. $\square$

Corollary 2.19. *If $E, E' \in \mathcal{S}p^G_{R,\_r}$, then*

$$R_{\underline{\star}}(E \wedge E') \cong R_{\underline{\star}}(E) \underset{R_{\underline{\star}}}{\Box} R_{\underline{\star}}(E').$$

For the norms, we recall some properties of the norm and these relatively simple Thom spectra.

Notation 2.20. If T is an H-set and $V \to T$ is a virtual vector bundle, then let

$$\mathrm{Map}^H(G, V) \to \mathrm{Map}^H(G, T)$$

be the coinduced vector bundle over $\mathrm{Map}^H(G, T)$.

Proposition 2.21. *For any virtual vector bundle V, we have*

$$M\big(\mathrm{Map}^H(G, V)\big) \simeq N_H^G M(V).$$

Proof. All of the functors considered commute with filtered colimits, so it suffices to consider the case that T is finite. This is then the distributive law for norms, together with the formula for $T = H/H$:

$$M\big(\mathrm{Map}^H(G,V)\big) = S^{\uparrow_H^G V} \simeq N_H^G S^V \cong N_H^G M(V). \qquad \square$$

The norm is also a strong symmetric monoidal functor, and it induces a map

$$N_H^G \colon R\text{-}\mathcal{M}od \to N_H^G R\text{-}\mathcal{M}od.$$

Proposition 2.22. *The norm induces a functor*

$$N_H^G \colon \mathcal{S}p^H_{R,\text{-}r} \to \mathcal{S}p^G_{N_H^G R,\text{-}r}.$$

Proof. It suffices to show this for E having free R-homology. In this case, we simply apply the norm to the equivalence

$$R \wedge E \simeq R \wedge M(V_E)$$

for some virtual vector bundle V_E and use Proposition 2.17. $\square$

If R is an E_∞-ring spectrum that has an E_∞-map

$$N_H^G i_H^* R \to R,$$

then we have a relative norm map on R-modules given by

$$M \mapsto R \underset{N_H^G i_H^* R}{\wedge} N_H^G M.$$

The usual case is when R is an equivariant commutative ring spectrum (i.e. a G-E_∞-ring spectrum), but this has also been worked out for algebras over linear isometries operads [3].

Proposition 2.23. *Let R be an E_∞-ring spectrum that has an E_∞-map*

$$N_H^G i_H^* R \to R,$$

then N_H^G induces a functor

$$\mathcal{S}p^H_{i_H^* R,\text{-}r} \to \mathcal{S}p^G_{R,\text{-}r}.$$

The norm of a basis for E gives one for the norm.

In the Bredon case, if $\underline{R}$ has the structure of a Tambara functor [31], then Ullman has shown that $H\underline{R}$ has the structure of a G-E_∞ ring spectrum [32]. This gives us many examples for Bredon homology. In particular, the absolute norms (i.e. the norms from the trivial group) of an ordinary commutative ring are always Tambara functors.

Generalizing the C_2-equivariant examples of [11], we get that absolute norms are free for a host of Green functors.

Example 6. *Let k be a field and let $\underline{R}$ be a Green functor under $N_e^G k$. Then for any spectrum E, $N_e^G E$ has free $H\underline{R}$-homology.*

The more general integral story also follows.

Example 7. *If E is an ordinary, non-equivariant spectrum such that $H_*(E;\mathbb{Z})$ is free, then $N_e^G E$ has free $\underline{A}$-homology, and hence free Bredon homology for any $\underline{M}$.*

We also have several chromatic examples.

Example 8. *Since*

$$\mathrm{MU} \wedge \mathrm{MU} \simeq \mathrm{MU} \wedge BU_+ \simeq \mathrm{MU}[b_1, \dots],$$

where $|b_i| = 2i$, the spectrum MU *and the space BU have free* MU*-homology. This implies that the same is true for the norms: $N_e^G \mathrm{MU}$ and $N_e^G \Sigma_+^\infty BU$ have free $N_e^G \mathrm{MU}$-homology.*

We have identical statements for $\mathbb{C}P^n$ for all $n \leqslant \infty$ and the spaces $BU(n)$.

Using the orientations given by the norm of MU, we produce a host of other interesting examples.

Example 9. *Let R be an E_∞ G-spectrum that admits an E_∞ norm map*

$$N_e^G i_e^* R \to R,$$

and assume that $i_e^* R$ can be given a commutative complex orientation. Then for any spectrum E such that $\mathrm{MU}_* E$ is a free MU_*-module, $N_e^G(E)$ has free R-homology.

In particular, the spaces and spectra considered in Example 8 have free MU_G and KU_G-homology.

Example 10. *If E is any finite type, bounded below spectrum with free integral homology, then $N_e^G E$ has free KU_G and MU_G homology.*

There is a norm in R-homology, specified by the norms in Mackey functors (or equivalently in spectra), and the following holds by definition.

Corollary 2.24. *If E has free $i_H^* R$-homology, then we have a natural isomorphism*

$$R_{\underline{\star}}(N_H^G E) \cong N_H^G\big(i_H^* R_{\underline{\star}}(E)\big).$$

As a specific example, this gives us the equivariant homology of the topological Singer construction [22].

Example 11. *Let $k = \mathbb{F}_p$, let $G = C_p$, and let $\underline{\mathbb{F}}_p$ be the constant Green functor $\mathbb{F}_p$. This is a Tambara functor, so for any spectrum E, we have a natural isomorphism*

$$H\big(N_e^{C_p}(E); \underline{\mathbb{F}}_p\big) \cong N_e^{C_p}\big(H_*(E; \mathbb{F}_p)\big).$$

In particular, for $p = 2$, the homology of $N_e^{C_2} H\mathbb{F}_2$ is free.

Unpacking this a little more, a basis is given by the monomial basis in

$$\Big(\mathbb{F}_p[\xi_1, \dots] \otimes E(\tau_0, \dots)\Big)^{\otimes p}.$$

The group C_p acts on this by permuting the factors, so associated to every monic monomial, f, there is a stabilizer subgroup H_f. This is the subgroup associated to f. The degree of f is given by

$$||f|| = \frac{|f|}{|H_f|} \rho_{H_f},$$

where $|f|$ is the ordinary, underlying degree induced by the degrees in the dual Steenrod algebra.

These freeness results can also give us interesting information about non-free spectra. Snaith showed that we have an equivalence of E_∞-ring spectra

$$KU \simeq \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}],$$

where β is the map on $\Sigma_+^\infty \mathbb{C}P^\infty$ induced by the inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$.

The norm functor commutes with filtered colimits, so this gives us an equivariant version of Snaith's theorem.

Theorem 2.25. *For any finite group G, we have an equivalence of G-E_∞-ring spectra*
$$N_e^G KU \simeq \Sigma^\infty_+ \operatorname{Map}(G, \mathbb{C}P^\infty)[N(\beta)^{-1}],$$
where
$$N(\beta)\colon S^{2\rho_G} \to \Sigma^\infty_+ \operatorname{Map}(G, \mathbb{C}P^\infty)$$
is induced by the norm.

Corollary 2.26. *Let R be a G-E_∞-ring spectrum such that $\mathbb{C}P^\infty$ has free $i_e^* R$-homology. Then we have an isomorphism*
$$\underline{R_\star} N_e^G KU \simeq \varinjlim \Sigma^{-2n\rho_G} N_e^G\big(i_e^* R_*(\mathbb{C}P^\infty)\big).$$
In particular, this is always a flat $\underline{R_\star}$-module.

Example 12. *Because $H\underline{A}$ is a G-E_∞-commutative ring spectrum and $\mathbb{C}P^\infty$ has free $H\mathbb{Z}$-homology, we have*
$$\underline{H_\star}\big(\operatorname{Map}(G, \mathbb{C}P^\infty); \underline{A}\big) \cong N_e^G H_*(\mathbb{C}P^\infty; \mathbb{Z}) \cong N_e^G\big(\Gamma(x)\big),$$
where $|x| = 2$. The Bott element we invert is the norm of x, and we deduce
$$\underline{H_\star} N_e^G KU \simeq N_e^G \mathbb{Q}[x^{\pm 1}].$$

Since KU_G is a $N_e^G KU$-algebra, we also deduce that $H\underline{A} \wedge KU_G$ is rational.

2.2.4. *Duality.* We also have a weak Universal Coefficients Theorem, provided our spectrum is small.

Definition 2.27. Let $E \in \mathcal{S}p^G_{R,\text{-}r}$, and let V_E be the associated virtual bundle such that $R \wedge E \simeq R \wedge M(V_E)$. We say E is **finite type** if for each $k \leqslant j \in \mathbb{Z}$, only finitely many orbits of T_E contribute to $\underline{\pi}_\ell(R \wedge M(V_E))$ for $k \leqslant \ell \leqslant j$.

Clearly, if the set T_E can be chosen to be finite, then it is finite type. This more general condition is analogous to only having finitely many cells in each degree.

Theorem 2.28. *If $E \in \mathcal{S}p^G_{R,\text{-}r}$ is a finite complex, then $D(X)$ is also in $\mathcal{S}p^G_{R,\text{-}r}$.*

More generally, if $E \in \mathcal{S}p^G_{R,\text{-}r}$ is finite type, then for any R-module M, we have a weak equivalence of R-modules
$$F(E, M) \simeq M \wedge M(-V_E).$$

We have a universal coefficients isomorphism that computes the M-cohomology of E out of the R-homology of E:
$$\underline{M^{-\star}}(E) \cong \operatorname{Hom}_{\underline{R_\star}}\big(\underline{R_\star}(E), \underline{M_\star}\big).$$

Proof. If E is a finite complex, then
$$D(E) \wedge R \simeq F(E, R),$$
and the first will follow from the second.

Since M is an R-module, we have an equivalence
$$F(E, M) \simeq F_R(R \wedge E, M).$$
A basis for the R-homology of E gives an equivalence
$$F_R(R \wedge E, M) \simeq F_R(R \wedge M(V_E), M),$$

and this is equivalent to $F\big(M(V_E), M\big)$. Since maps out of a wedge is the product, we first check the case of an orbit. The result is then the classical Wirthmüller isomorphism:

$$F(G_+ \underset{H}{\wedge} S^V, M) \simeq G_+ \underset{H}{\wedge} S^{-V} \wedge M.$$

Finally, the finite type condition ensures that the natural map from the wedge to the product is in fact an equivalence.

The second part follows from this by taking homotopy and observing the result for orbits. □

A surprising final feature of the universal coefficients theorem is that we can also describe the cohomology of the norms of R-free spectra.

Proposition 2.29. *Let R be an E_∞-ring spectrum that has an E_∞-map*

$$N_H^G i_H^* R \to R.$$

If an H-spectrum E has free $i_H^* R$-homology with a finite basis, then the function spectrum $F(N_H^G E, R)$ is equivalent to a free R-module, and the basis is the dual to the one for $N_H^G E$.

In particular, analyzing the Thom spectrum for the functional dual, we have that for E as in the proposition, the R-cohomology of $N_H^G E$ can be described as the norm of the $i_H^* R$-cohomology of E.

2.2.5. *Pullbacks.* Projectivity is also preserved by restricting along quotient maps (also called "pulling back").

Notation 2.30. If N is a normal subgroup of G and $q\colon G \to Q = G/N$, then let

$$q^* \colon \mathcal{S}p^Q \to \mathcal{S}p^G$$

be the inclusion of Q-spectra into G-spectra.

Proposition 2.31. *The functor q^* induces a functor*

$$q^* \colon \mathcal{S}p^Q_{R,-r} \to \mathcal{S}p^G_{q^* R,-r}.$$

If $E \in \mathcal{S}p^Q$ has a basis for R, then $q^* E$ has a basis for $q^* R$.

Proof. Again, it suffices to check on the full subcategory of R-free spectra, and since q^* is strong symmetric monoidal, it suffices to show on the associated Thom spectra. By construction,

$$q^* M(V_E) \simeq M(q^* V_E),$$

where $q^* V_E$ is just V_E viewed as a G-virtual bundle. □

Remark 2.32. The fixed points functors do not preserve projective objects, as the tom Dieck splitting shows. However, the canonical map

$$q^*(R^G) \to R$$

gives us a map

$$q^* \colon \mathcal{S}p_{R^G,-r} \to \mathcal{S}p^G_{R,-r}.$$

Example 13. *If E is an ordinary, non-equivariant spectrum such that $H_*(E;\mathbb{Z})$ is free in each degree, then q^*E has free Bredon homology for any coefficients. For any G,*

$$\underline{\pi}_0 q^* H\mathbb{Z} = \underline{A},$$

and the negative homotopy groups are all zero. The zeroth Postnikov section then gives us an E_∞-map

$$q^* H\mathbb{Z} \to H\underline{A}.$$

The result then follows from Proposition 2.31, Proposition 2.16, and Theorem 2.18.

Example 14. *If E is an ordinary, non-equivariant spectrum such that $H_*(E;\mathbb{F}_p)$ is free in each degree, then for any G and for any Green functor $\underline{R}$ in which $p\cdot 1 = 0 \in \underline{R}(G/G)$, q^*E has free $\underline{R}$-homology. This is because the pullback of $H\mathbb{F}_p$ has $\underline{\pi}_0 = \underline{A}/p$, the initial example of such a Green functor.*

In particular, for any G and for any $\underline{R}$ of this form, this applies to

$$E = \Sigma^\infty K(\mathbb{F}_p, m),$$

the pullback of which is the suspension spectrum of the Eilenberg–Mac Lane space for the constant coefficient system $\underline{\mathbb{F}}_p$.

2.3. **Freeness and spaces.** Our primary interest in these freeness results comes from the connection between the [twisted] smash products in spectra and [twisted] Cartesian products in spaces.

Proposition 2.33. *If X is an H-space, then we have a natural equivalence*

$$N_H^G \Sigma_+^\infty X \simeq \Sigma_+^\infty \operatorname{Map}_H(G, X).$$

If X and Y are G-spaces, then

$$\Sigma_+^\infty (X \times Y) \simeq \Sigma_+^\infty X \wedge \Sigma_+^\infty Y.$$

We can assemble all of our results so far into a summary theorem.

Theorem 2.34. *Let X be an K-space such that X has free $i_K^* R$-homology. Then we have a natural isomorphism*

$$N_K^G R_{\underline{\star}}(X) \cong R_{\underline{\star}}\big(\operatorname{Map}_H(G,X); \underline{R}\big),$$

and moreover, this is free on the basis $N_H^G \vec{x}$, where $\vec{x}$ is a basis for the homology of X.

If X and Y are G-spaces that have R-free homology, then

$$R_{\underline{\star}}(X \times Y) \cong R_{\underline{\star}}(X) \underset{R_{\underline{\star}}}{\Box} R_{\underline{\star}}(Y),$$

with a basis given by the product of the bases.

Example 15. *In general, coinduction preserves Eilenberg–Mac Lane spaces: if $\underline{M}$ is an H-Mackey functor, then we have an equivalence*

$$\operatorname{Map}^H\big(G, K(\underline{M}, n)\big) \simeq K\big(\uparrow_H^G \underline{M}, n\big).$$

(More generally, the G-space $\operatorname{Map}^H\big(K(\underline{M}, V)\big)$ represents the functor

$$X \mapsto H^V(i_H^* X; \underline{M}),$$

so these in general are kinds of Eilenberg–Mac Lane spaces.)

When $H = \{e\}$, this allows us to determine the homology of Eilenberg–Mac Lane space attached to any induced Mackey functor with coefficients in a $N_e^G k$-algebra, for k a field. In particular, we have

$$H_{\underline{\star}}\Big(K\big(\uparrow_e^{C_p}\mathbb{F}_p, n\big); \underline{\mathbb{F}}_p\Big) \cong N_e^{C_p}\Big(H_*\big(K(\mathbb{F}_p, n); \mathbb{F}_p\big)\Big),$$

and the latter was determined by Cartan and Serre.

This is closely connected to some additional structure that is often difficult to access. Equivariant spaces are canonically G-cocommutative comonoids. In addition to the coproduct

$$X \to X \times X,$$

they have conorm maps

$$X \xrightarrow{\Delta^{G/H}} \mathrm{Map}(G/H, X) \cong \mathrm{Map}^H(G, X).$$

In fact, the contravariant Yoneda functor gives for any X a functor

$$よ_X \colon \mathrm{Map}(\text{-}, X) \colon (\mathcal{F}in^G)^{op} \to \mathcal{T}op^G,$$

and on passage to fixed points, these maps are exactly giving the usual coefficient system of fixed points for any G-space.

Definition 2.35. If $f \colon S \to T$ is a map of finite G-sets, then let

$$\psi_f \colon R_{\underline{\star}}\big(\mathrm{Map}(T, X)\big) \to R_{\underline{\star}}\big(\mathrm{Map}(S, X)\big)$$

be the "conorm" map associated to f. When $f = \nabla_S \colon S \amalg S \to S$ is the fold map, we call this the "coproduct".

In general, this is difficult to work with, since we need not have a good [twisted] Künneth theorem. In the case we are considering, however, we do!

Theorem 2.36. *Let R be an equivariant commutative ring spectrum, and let X be a space that has free R-homology. Then $R_{\underline{\star}}(X)$ has a comultiplication map*

$$R_{\underline{\star}}(X) \to R_{\underline{\star}}(X) \underset{R_{\underline{\star}}}{\square} R_{\underline{\star}}(X)$$

making it a "co-Green functor". Moreover, we have for any map of finite G-sets $f \colon S \to T$ a conorm map

$$N^T R_{\underline{\star}}(X) \to N^S R_{\underline{\star}}(X)$$

which is a map of co-Green functors.

Proof. Since X has R-free homology, so do all of its restrictions, and hence so do all of the spaces $\mathrm{Map}(T, X)$ for any finite G-set T. The comultiplication and conorm maps then follow immediately from our earlier analysis of the homology of the spaces involved.

That the conorm maps are maps of coGreen functors follows from naturality. □

Rephrased, a space with free R-homology gives a strong G-symmetric monoidal functor

$$\mathcal{S}et^{G,op} \to R_{\underline{\star}}\text{-}\mathcal{M}od,$$

where the G-symmetric monoidal structure on $\mathcal{S}et^{G,op}$ is the dual to the co-Cartesian one.

All told, this gives $\underline{R}_\star(X)$ naturally the structure of a co-Tambara functor. The exact axiomatic treatment of the norm maps is dual to that of an ordinary G-commutative monoid [9]. Via the Universal Coefficients theorem for $\underline{R}$-free spaces, this structure is exactly the structure which gives rise to the usual Tambara functor structure on the $\underline{R}$-cohomology of a G-space.

Remark 2.37. The usual formulation of a Tambara functor describes norm maps $n_H^K\colon \underline{R}(G/H) \to \underline{R}(G/K)$. These connect, via work of Mazur and Hoyer ([13], [14]) to G-commutative monoids in Mackey functors via canonical set maps

$$\underline{R}(G/H) \cong i_K^*\underline{R}(K/H) \to (N^{K/H} i_K^*\underline{R})(K/K).$$

The maps go the wrong way to be able to interpret a co-Tambara functor easily in the more traditional way.

2.3.1. *Coalgebras.* There are also many examples of spectra for which we have similar kinds of comultiplications. These are ubiquitous amongst spectra which have free homology over themselves. This was shown and used by Hu–Kriz; it works very generally.

Proposition 2.38 ([15])**.** *If R is an E_∞ ring spectrum such that has free R-homology, then the pair*

$$(R_{\underline{\star}}, R_{\underline{\star}}R)$$

forms a Hopf algebroid, and moreover, the R-homology of any space or spectrum is a comodule over this.

If R is actually a G-E_∞-ring spectrum, then we have the much more structure. The R-homology of itself inherits the structure of a Tambara functor, and again, all of the maps are maps of Tambara functors. We include one interesting example.

Proposition 2.39. *If R is a C_2-E_∞-ring spectrum, then $H_{\underline{\star}}(R;\underline{\mathbb{F}}_2)$ is a comodule Tambara functor over the equivariant dual Steenrod algebra: the comodule structure map is a map of Green functors and commutes with the norms:*

$$\begin{tikzcd}
N_e^{C_2} H_*(i_e^* R;\mathbb{F}_2) \arrow[r, "N_e^{C_2}\psi"] \arrow[d, "N"'] & N_e^{C_2}\mathcal{A}_* \underset{H_{\underline{\star}}}{\square} N_e^{C_2} H_*(i_e^* R;\mathbb{F}_2) \arrow[d, "N\square N"] \\
H_{\underline{\star}}(R;\underline{\mathbb{F}}_2) \arrow[r, "\psi"'] & \mathcal{A}_{\underline{\star}} \underset{H_{\underline{\star}}}{\square} H_{\underline{\star}}(R;\underline{\mathbb{F}}_2).
\end{tikzcd}$$

Proof. The coaction map is the map induced by the unit:

$$H\underline{\mathbb{F}}_2 \wedge R \cong H\underline{F}_2 \wedge S^0 \wedge R \to H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2 \wedge R,$$

and hence is a map of C_2-E_∞-ring spectra. In particular, it commutes with all of the structure arising from the multiplication. $\square$

This means that the coaction on the spectrum $N_e^{C_2} H\mathbb{F}_2$ should be completely determined by the coaction on $H\mathbb{F}_2$, allowing us to analyze the homotopy groups of this spectrum by a Hu–Kriz style Adams spectral sequence [15].

Aside: what are G-Hopf algebras? We pause here to sketch the analogue of Hopf algebras, explaining what structure we see. We focus on the Bredon case with coefficients in a Tambara functor $\underline{R}$ for simplicity. The dual, cohomology statements to the comuliplications and conorms do not need freeness; the Bredon cohomology of a space with coefficients in a Tambara functor is always a Tambara functor. This is because we always have a map

$$N^T H^{\underline{\star}}(X;\underline{R}) \to H^{\underline{\star}}\big(\operatorname{Map}(T,X);\underline{R}\big),$$

and this gives us the canonical G-commutative monoid (i.e. Tambara) structure on the cohomology of a G-space. In our free case, this map is an isomorphism.

Proposition 2.40. *If X has free $\underline{R}$-homology and is finite type, then we have a natural isomorphism*

$$N^T H^{\underline{\star}}(X;\underline{R}) \xrightarrow{\cong} H^{\underline{\star}}\big(\operatorname{Map}(T,X);\underline{R}\big).$$

It is helpful to think of what the products and norms in the cohomology of X imply for the representable functor

$$よ_X : \underline{B} \mapsto \mathcal{T}amb^G_{\underline{R}_\star}\big(H^{\underline{\star}}(X;\underline{R}),\underline{B}\big).$$

In general, any representable functor like this is just set-valued. The external norm maps are maps of Tambara functors, and the norm functor is left adjoint to the forgetful functor

$$i_H^* \colon \mathcal{T}amb^G_{\underline{R}_\star} \to \mathcal{T}amb^H_{\underline{R}_\star}.$$

The Tambara structure maps then give us the usual lift:

$$\begin{tikzcd} & \mathcal{C}oeff \arrow[d, "ev_{G/G}"] \\ \mathcal{T}amb^G_{\underline{R}_\star} \arrow[ur, dashed, "よ_X"] \arrow[r, "よ_X"'] & \mathcal{S}et \end{tikzcd}$$

If X is a G-commutative monoid in the homotopy category, then we have added compatibility with the twisted coproducts, since the [twisted] coproducts are natural in maps of spaces. By construction, we have a Mackey functor object in the homotopy category of spaces.

Proposition 2.41. *If X is a G-commutative monoid in the homotopy category of spaces and if X has free $\underline{R}$-homology and is finite type, then the functor $よ_X$ naturally lifts to a functor valued in semi-Mackey functors:*

$$\begin{tikzcd} & \mathcal{M}ackey \arrow[d, "U"] \\ & \mathcal{C}oeff \arrow[d, "ev_{G/G}"] \\ \mathcal{T}amb^G_{\underline{R}_\star} \arrow[uur, dashed, "よ_X"] \arrow[ur] \arrow[r, "よ_X"'] & \mathcal{S}et \end{tikzcd}$$

We can summarize connecting this to actual spaces of maps. First recall that if Y is a G-commutative monoid in the homotopy category of spaces, then the set valued functor

$$X \mapsto [X,Y]$$

naturally extends to a semi-Mackey functor valued functor.

Theorem 2.42. *If Y is a G-commutative monoid in the homotopy category of spaces, and if Y is $\underline{R}$-free and finite type, then the $\underline{R}$-homology functor extends to a map of semi-Mackey functors*

$$\underline{[X,Y]} \to \underline{\mathcal{T}amb}^G_{\underline{R}_\star}\big(H^{\star}(Y;\underline{R}), H^{\star}(X;\underline{R})\big).$$

Remark 2.43. Examples of Y for which this holds are $\mathbb{C}P^\infty$ with the standard C_2-structure or the spaces $BU_{\mathbb{R}}$. In all cases, we take $\underline{R} = \underline{\mathbb{Z}}$ to be the constant Mackey functor $\mathbb{Z}$. We will return to a more general version of this in work with Meier.

3. An even nicer class of spectra

3.1. **Homological purity.** We single out a class of spectra for which computations are strikingly simple, being completely determined by the homology of the underlying spectrum.

Definition 3.1. A **regular slice sphere** is a G-spectrum of the form

$$G_+ \underset{H}{\wedge} S^{k\rho_H},$$

for some integer k. The dimension of such a regular slice sphere is $k|H|$.

In [10], a spectrum E was called "pure" if the slice associated graded of E is a wedge of regular slice spheres smashed with $\underline{\mathbb{Z}}$. We build on that here.

Definition 3.2. A G-spectrum E is **homologically pure** if we have an equivalence of $\underline{\mathbb{Z}}$-modules

$$\underline{\mathbb{Z}} \wedge E \simeq \underline{\mathbb{Z}} \wedge \bigvee_{(k,H)\in\mathcal{I}_E} G_+ \underset{H}{\wedge} S^{k\rho_H}.$$

A homologically pure G-spectrum E is **isotropic** if there are no summands with a trivial stabilizer.

Remark 3.3. A slightly restricted form of this definition was independently given by Pitsch–Ricka–Scherer in their analysis of conjugation spaces [25]. The choice name and reason for the name are the same as the one here: analogy with [10].

Since the zero-slice of the zero sphere is $\underline{\mathbb{Z}}$, any zero-slice is a module over $\underline{\mathbb{Z}}$. This shows that we could have instead used arbitrary zero-slices.

Proposition 3.4. *A G-spectrum E is homologically pure if and only if for every zero-slice $\underline{M}$, we have an equivalence of $\underline{\mathbb{Z}}$-modules*

$$H\underline{M} \wedge E \simeq H\underline{M} \wedge \bigvee_{(k,H)\in\mathcal{I}_E} G_+ \underset{H}{\wedge} S^{k\rho_H}.$$

Notation 3.5. If we have a decomposition like that of homological purity or isotropic homological purity only for particular Green zero slices $\underline{R}$, then we will say that E has [isotropic] homological purity for $\underline{R}$.

The induced regular representations are much nicer representations than we might have initially expected. These are closed under restrictions, conjugations, and inductions. This gives the following.

Proposition 3.6. *If E is a homologically pure H-spectrum, then*

(1) *$G_+ \underset{H}{\wedge} E$ is homologically pure,*

(2) *if $K \subset H$, then $i_K^* E$ is homologically pure, and*
(3) *$N_H^G E$ is homologically pure.*

3.1.1. *Homology.* The main benefit of this definition is from the defining property of zero-slices: all restriction maps are injections, and hence statements can usually be checked at the level of underlying homology.

Notation 3.7. Given an indexing set $\mathcal{I}_E$ for a homologically pure E, for each integer n, let

$$\mathcal{I}_E^n = \big\{(k,H) \in \mathcal{I}_E \mid k|H| = n\big\}.$$

Proposition 3.8. *Let E be homologically pure and let $\underline{M}$ be a zero-slice. For any subgroup K and for any integer k, we have*

$$\underline{H}_{k\rho_K}(E;\underline{M}) \cong \bigoplus_{(j,J)\in\mathcal{I}_E^{k|K|}} \underline{M}_{i_K^* G/J}.$$

We also have

$$\underline{H}_{k\rho_K-1}(E;\underline{M}) \cong \bigoplus_{(j,J)\in\mathcal{I}_E^{k|K|-1}} \bigoplus_{\substack{g\in K\backslash G/J \\ K\cap gJg^{-1}=\{e\}}} \underline{M}_G.$$

In particular, all restriction maps are injections.

Proof. By assumption, $H\underline{M} \wedge E$ is a wedge of regular slices, and hence we have

$$\underline{\big[G_+ \underset{K}{\wedge} S^{k\rho_K}, H\underline{M} \wedge E\big]} \cong \uparrow_K^G \underline{\big[S^{k\rho_K}, i_K^*(H\underline{M} \wedge E)\big]} \cong$$

$$\bigoplus_{(j,J)\in\mathcal{I}_E^{k|K|}} \uparrow_K^G \underline{\big[S^{k\rho_K}, i_K^*(G_+ \underset{J}{\wedge} S^{j\rho_J} \wedge H\underline{M})\big]} \cong \bigoplus_{(j,J)\in\mathcal{I}_E^{k|K|}} \uparrow_K^G \underline{\big[S^0, i_K^*(G/J_+ \wedge H\underline{M})\big]}$$

The result follows by the definition of $\underline{M}_{i_K^* G/J}$.

For the case of a $k\rho_K - 1$, the argument is identical until the last step. Here, we have a direct sum

$$\bigoplus \uparrow_K^G \underline{\pi}_{-1}\big(S^{-k\rho_K} \wedge i_K^*(G_+ \underset{J}{\wedge} S^{j\rho_J} \wedge H\underline{M})\big)$$

The double coset decomposition of G as a (K,J)-biset allows us to rewrite each summand:

$$\underline{\pi}_{-1}\big(S^{-k\rho_K} \wedge i_K^*(G_+ \underset{J}{\wedge} S^{j\rho_J} \wedge H\underline{M})\big) \cong \bigoplus_{g\in K\backslash G/J} \uparrow_{K\cap gJg^{-1}}^G \underline{\pi}_{-1}\big(S^{(n-m)\rho_{K\cap gJg^{-1}}} \wedge H\underline{M}\big),$$

where

$$n = j[J : J \cap g^{-1}Kg] \text{ and } m = k[K : K \cap gJg^{-1}].$$

The only regular representation sphere that has a non-trivial homology in degree -1 is the one for the trivial group in degree -1, which gives the second part. $\square$

Corollary 3.9. *If $G = C_{p^n}$ and E is homologically pure and isotropic, then the homology groups in dimensions of the form $(i\rho_H - 1)$ vanish.*

Definition 3.10. A homologically pure G-spectrum E is **generalized isotropic** if there is no pair

$$(k,K) \in \mathcal{I}_E^{k|K|} \text{ and } (h,H) \in \mathcal{I}_E^{k|K|-1}$$

such that $G/K \times G/H$ contains a free summand.

This generalized isotropic condition allows us to have other ways to check homological purity.

Theorem 3.11. *Let E be a G spectrum that admits a filtration such that $gr(E)$ is homologically pure and generalized isotropic. Then E is homologically pure and generalized isotropic.*

Proof. The filtration on E gives a spectral sequence with E_1-term

$$\pi_{\underline{\star}}\big(gr(E) \wedge H\underline{\mathbb{Z}}\big).$$

By assumption, this is a free $H\underline{\mathbb{Z}}_{\underline{\star}}$-module, and the generators are in dimensions $k\rho_K$ for $(k, K) \in \mathcal{I}_{gr(E)}$. The generalized isotropic condition guarantees that these classes are permanent cycles, since there are no possible targets for the differentials on the generators by Proposition 3.8. Thus $E_1 = E_\infty$, and since this is a free module, there are no possible extensions. □

3.1.2. *Cohomology.* We can make similar statements about the cohomology.

Proposition 3.12. *If E is homologically pure and $|\mathcal{I}_E^n| < \infty$ for all n, then for any zero slice $\underline{M}$, we have an equivalence of $H\underline{\mathbb{Z}}$-modules*

$$F(E, H\underline{M}) \simeq H\underline{M} \wedge \bigvee_{(k,H)\in\mathcal{I}_E} G_+ \underset{H}{\wedge} S^{-k\rho_H}.$$

Proof. Since zero-slices are $H\underline{\mathbb{Z}}$-modules, we have an equivalance of $H\underline{\mathbb{Z}}$-modules

$$F(E, H\underline{M}) \simeq F_{H\underline{\mathbb{Z}}}(H\underline{\mathbb{Z}} \wedge E, H\underline{M}).$$

The homological purity of E gives an equivalence of $H\underline{\mathbb{Z}}$-modules

$$H\underline{\mathbb{Z}} \wedge E \simeq H\underline{\mathbb{Z}} \wedge \bigvee_{(k,H)\in\mathcal{I}_E} G_+ \underset{H}{\wedge} S^{k\rho_H},$$

and hence we have

$$F(E, H\underline{M}) \simeq \prod_n \prod_{(k,H)\in\mathcal{I}_E^n} G_+ \underset{H}{\wedge} S^{-k\rho_H} \wedge H\underline{M}.$$

Since $\mathcal{I}_E^n$ is finite, the inner products are the same as wedges. Since for all (k, H), the homotopy Mackey functors of $G_+ \underset{H}{\wedge} S^{-k\rho_H} \wedge H\underline{M}$ are zero outside of a finite range (depending only on k and H), the outer product is also equivalent to the wedge. □

Example 16. *An theorem of Pitsch–Ricka–Schrerer shows that any conjugation space of Hausman–Holm–Puppe* [7] *are "mod 2" homologically pure and isotropic* [25]*. This gives a large class of examples.*

3.2. **Consequences in computations.** The condition of homological purity gives surprising computational control.

3.2.1. *Green functor structure.*

Theorem 3.13. *Let E be a homologically pure spectrum, and assume that E comes equipped with a [commutative, associative] multiplication in the homotopy category. Then for any commutative Green functor $\underline{R}$ which is a zero-slice, the multiplication on*

$$\underline{H}_{\underline{\star}}(E; \underline{R})$$

is completely determined by the restrictions to

$$H_*(i_e^* E; \underline{R}(G)).$$

Proof. The homological purity of E guarantees that the homology and cohomology are free modules over the RO-graded homology of a point. In particular, the ring structure is completely determined by the products of basis vectors. These occur in dimensions of the form $k\rho_H$ for various k and H. If $x \in H_{k\rho_H}(E;\underline{\mathbb{Z}})$ and $y \in H_{\ell\rho_J}(E;\underline{\mathbb{Z}})$, then the product of x and y is represented by a map out of

$$(G_+ \underset{H}{\wedge} S^{k\rho_H}) \wedge (G_+ \underset{J}{\wedge} S^{\ell\rho_J}).$$

This is a wedge of spaces of the form

$$G_+ \underset{K}{\wedge} S^{m\rho_K},$$

where K ranges over all subgroups of the form $H \cap gJg^{-1}$ and where

$$m\rho_K = i_K^*(k\rho_H) + i_K^* c_g^*(\ell\rho_J).$$

In particular, this is a wedge of regular slice spheres, again, and hence the product takes values in a zero slice by Proposition 3.8s. Since all restriction maps are injections here, the result follows. □

Corollary 3.14. *If E is a homologically pure spectrum, then the* RO*-graded ring structure on the cohomology of E with coefficients in any commutative Green zero slice is functorially determined by the underlying cohomology ring.*

3.2.2. *Tambara functor structure.* If, moreover, E is a G-E_∞-ring spectrum, then we also have good control over norms.

Theorem 3.15. *If E is a G-commutative monoid in the homotopy category and if X is a homologically pure spectrum, then for any Tambara zero-slice $\underline{R}$, we have that the norms in*

$$H_{\underline{\star}}(E;\underline{R})$$

are determined by the formula

$$i_e^* N_H^G(x) = \prod_{\gamma \in G/H} \gamma(i_e^* x).$$

Proof. The proof is the same as for the products. Here we use that the collection of regular representations is a sub-semi-Mackey functor of the representation ring. □

Remark 3.16. Tambara functors which are also zero-slices were independently studied by Nakaoka, who called these "MRC" Tambara functors, in his study of localizations of Tambara functors [24].

3.2.3. *CoTambara structure.* Again, all of the desired structure can be read out of the underlying homology. The conorm maps are detected as twisted coproducts. The proofs are identical.

Theorem 3.17. *Let E be a homologically pure spectrum, and assume that E comes equipped with a co-[commutative, associative] comultiplication in the homotopy category. Then for any commutative Green functor $\underline{R}$ which is a zero-slice, the comultiplication on*

$$H_{\underline{\star}}(E;\underline{R})$$

is completely determined by

$$H_*(i_e^* E;\underline{R}(G)).$$

Theorem 3.18. *If E is a G-co-commutative comonoid in the homotopy category and if X is a homologically pure spectrum, then for any Tambara zero-slice $\underline{R}$, we have that the conorms in*

$$H_{\underline{\star}}(E;\underline{R})$$

are determined by the formula

$$i_e^* N_H^G(x) = \Big(\big(\bigotimes_{\gamma \in G/H} \gamma \big) \circ \psi \Big) (i_e^* x).$$

3.2.4. *Dyer–Lashof operations.* Finally, we restrict to C_2. None of the arguments here are that specific to C_2; the only issue is in defining the appropriate Dyer–Lashof operations. For groups which contain C_2, norm arguments provide analogous classes, but we have no idea in general. We recall Wilson's $RO(C_2)$-graded stable operations.

Theorem 3.19 ([2, §3], [33])**.** *For each $i \geqslant 0$ and for each $\epsilon = 0, 1$, we have Dyer–Lashof operations*

$$Q^{i\rho_2 - \epsilon} \colon H_\star(\text{-};\underline{\mathbb{F}}_2) \to H_{\star + i\rho_2 - \epsilon}(\text{-};\underline{\mathbb{F}}_2).$$

When $\star = i\rho_2$, $Q^{i\rho_2}$ is the square.

In this case, homological purity says that the underlying structure describes everything.

Theorem 3.20. *If E is a homologically pure C_2-E_∞-ring spectrum, then we have*

$$Q^{i\rho_2 - \epsilon} \colon H_\star(E;\underline{\mathbb{F}}_2) \to H_{\star + i\rho_2 - \epsilon}(E;\underline{\mathbb{F}}_2)$$

is determined by the restrictions $i_e^* Q^{i\rho_2 - \epsilon}$. The "odd" operations $Q^{i\rho_2 - 1}$ can only land in cells induced from the trivial group.

Proof. This again follows immediately from the assumption of homological purity. □

3.3. **Example: the homology of $BU_{\mathbb{R}}$.**

3.3.1. *The ring structure.* We begin with the computation of the homology of $BU_{\mathbb{R}}$ with coefficients in $\underline{\mathbb{Z}}$. We give a slightly different proof than that of [16] and [26], using instead our formulae above. This line of argument was undoubtedly known by Araki and Landweber.

Theorem 3.21. *There are classes*

$$\bar{a}_i \in H_{i\rho_2}(BU_{\mathbb{R}};\underline{\mathbb{Z}})$$

such that the induced map on A_∞-rings

$$\underline{\mathbb{Z}} \wedge \mathbb{S}^0[\bar{a}_1, \bar{a}_2, \dots] \to \underline{\mathbb{Z}} \wedge BU_{\mathbb{R}}$$

is an equivalence of C_2-equivariant associative algebras, and hence the C_2-space $BU_{\mathbb{R}}$ is homologically pure and isotropic.

Proof. Araki lifted the classical, non-equivariant description of $\mathrm{MU}_* \mathrm{MU}$, showing

$$\mathrm{MU}_{\mathbb{R}} \wedge \mathrm{MU}_{\mathbb{R}} \simeq \mathrm{MU}_{\mathbb{R}}[\bar{a}_1, \dots],$$

and in particular, this is free with a basis in regular representation dimensions. The Thom isomorphism shows

$$\mathrm{MU}_{\mathbb{R}} \wedge BU_{\mathbb{R}+} \simeq \mathrm{MU}_{\mathbb{R}} \wedge \mathrm{MU}_{\mathbb{R}}$$

as C_2-E_∞-rings. Since $H\underline{\mathbb{Z}}$ is a commutative ring spectrum under $\mathrm{MU}_{\mathbb{R}}$, the result follows by base-change. □

Corollary 3.22. *For any finite group G which contains C_2, the coinduced G-space $\mathrm{Map}^{C_2}(G, BU_{\mathbb{R}})$ is homologically pure and isotropic with basis given by the norm of the monomial basis.*

Remark 3.23. The classical Schubert cell analysis works equally well here, and the underlying argument is essential the same as that of [7].

Notation 3.24. Let
$$H_{\underline{\star}}^{\underline{\mathbb{Z}}} = \underline{\pi}_\star H\underline{\mathbb{Z}}.$$

Corollary 3.25. *We have an isomorphism of $RO(C_2)$-graded Green functors*
$$H_{\underline{\star}}(BU_{\mathbb{R}};\underline{\mathbb{Z}}) \cong H_{\underline{\star}}^{\underline{\mathbb{Z}}}[\bar{a}_1, \dots],$$
where $|\bar{a}_i| = i\rho_2$.

We can also deduce the norms, coproducts, and conorms.

Proposition 3.26. *The norms are given by*
$$N_e^{C_2}(a_i) = (-1)^i \bar{a}_i^2.$$

Finally, the co-Tambara structure is lifting the usual dual polynomial structure. Since the space $BU_{\mathbb{R}}$ is finite type, we can equivalently describe the cohomology ring and the norms there.

Proposition 3.27 ([16])**.** *The cohomology ring of $BU_{\mathbb{R}}$ is*
$$H^{\underline{\star}}(BU_{\mathbb{R}};\underline{\mathbb{Z}}) \cong \underline{\mathbb{Z}}[\bar{c}_1, \dots].$$
Moreover, the inclusions of equivariant maximal tori into the $U_{\mathbb{R}}(n)$ identify these Chern classes with the usual symmetric functions in the Chern roots.

Proof. Only the second part requires proof, since $BU_{\mathbb{R}}$ is homologically pure, isotropic, and of finite type. The same is true for the space $(\mathbb{C}P^\infty)^{\times n}$. The induced map on cohomology is the determined by the underlying homology, and we reduce to the classical case. □

Proposition 3.28. *The norms of the Chern classes are also the squares:*
$$N_e^{C_2}(c_i) = (-1)^i \bar{c}_i^2.$$

Finally, using Theorem 3.20, we deduce the action of Wilson's Dyer–Lashof operations.

Theorem 3.29. *The Dyer–Lashof operations $Q^{i\rho_2}$ on $H_\star(BU_{\mathbb{R}};\underline{\mathbb{F}}_2)$ act as*
$$Q^{i\rho_2}(\bar{a}_j) = \binom{n}{r-n-1}\bar{a}_{i+j} \text{ mod decomposables.}$$
The Dyer–Lashof operations $Q^{i\rho_2-1}$ are identically zero.

Proof. Theorem 3.20 implies that these operations are completely determined by the underlying action. The ordinary Dyer–Lashof action on the homology of BU was determined by Kochman [17, 19]. □

As an aside, this also gives the Dyer–Lashof action on the space BO by applying geometric fixed points.

Corollary 3.30 ([17, Theorem 36]). *In*

$$H_\star(BO;\mathbb{F}_2)\cong\mathbb{F}_2[e_1,\dots],$$

we have for all $r\geqslant 0$ and $n\geqslant 1$,

$$Q^r(e_n)=\binom{n}{r-n-1}e_{n+r}\text{ mod decomposables.}$$

4. Bar and twisted bar spectral sequences

For R-free spectra, we have readily computable equivariant versions of the classical Rothenberg–Steenrod and Eilenberg–Moore spectral sequences. For $G=C_2$, we also have twisted versions of these where the group acts also on the homotopy pullback diagram. We explain how these work here, giving an example for the bar spectral sequence.

4.1. **Bar and Rothenberg–Steenrod.** Let A be an associative monoid in G-spaces. Let X be a right A-space and let Y be a left A-space. In this case, the derived balanced product can be computed via the bar construction:

$$X\underset{A}{\times}Y=B(X,A,Y),$$

where $B(X,A,Y)$ is the geometric realization of the simplicial complex

$$k\mapsto B_k(X,A,Y)=X\times A^{\times k}\times Y,$$

and where as usual, the structure maps are the actions or product in A.

If A and either X or Y are R-free, then we have a bar spectral sequence computing the R-homology of this.

Theorem 4.1. *If A and either X or Y are R-free, then we have an Adams-graded spectral sequence*

$$E_2^{s,\underline{\star}}=\operatorname{Tor}_{-s}^{R_{\underline{\star}}(A)}\big(R_{\underline{\star}}(X),R_{\underline{\star}}(Y)\big)\Rightarrow R_{\underline{\star}-s}(X\underset{A}{\times}Y).$$

Proof. Our assumptions guarantee that for each k, the R-homology of $B_k(X,A,Y)$ is given by

$$R_{\underline{\star}}\big(B_k(X,A,Y)\big)\cong R_{\underline{\star}}(X)\underset{R_{\underline{\star}}}{\square}R_{\underline{\star}}(A)^{\square k}\underset{R_{\underline{\star}}}{\square}R_{\underline{\star}}(Y),$$

and the maps are the standard resolution computing Tor. □

Remark 4.2. Lewis–Mandell give an $RO(G)$-graded version of the Künneth spectral sequence which gives the exact same result, since our bar complex becomes the relative smash product upon taking Σ^∞_+. The resulting spectral sequence is the same [21], since it is built the same way.

Applying cohomology instead to the bar construction when $X=Y=*$ gives the Rothenberg–Steenrod spectral sequence [29]. Our assumptions allow this to be determined as well.

Theorem 4.3. *If A is R-free and A is finite type, then we have a spectral sequence*

$$E_2^{*,\underline{\star}}=\operatorname{Ext}^s_{R^{\underline{\star}}(A)}\big(R^{\underline{\star}},R^{\underline{\star}}\big)\Rightarrow R^{\underline{\star}-s}(BA).$$

4.1.1. *Example:* $BBU_{\mathbb{R}}$. Since $BU_{\mathbb{R}}$ is $H\underline{\mathbb{Z}}$-free, we can run the bar spectral sequence to compute the homology of $BBU_{\mathbb{R}}$.

Proposition 4.4 ([21])**.** *There is an Adams-style spectral sequence with*

$$\underline{E}^2_{s,\star} = \underline{\mathrm{Tor}}^{-s}_{H_{\underline{\star}}BU_{\mathbb{R}}}(H^{\underline{\mathbb{Z}}}_{\underline{\star}}, H^{\underline{\mathbb{Z}}}_{\underline{\star}}) \cong E_{H_{\underline{\star}}}\big(\bar{y}_1, \dots\big) \Rightarrow \underline{\pi}_{\star-s}\big(H\underline{\mathbb{Z}} \wedge BBU_{\mathbb{R}}\big),$$

where $\bar{y}_i$ is the element in Tor^1 *represented by $\bar{a}_i$ and has bidegree $(-1, i\rho_2)$.*

Since all of the algebra generators are in filtration (-1), this spectral sequence collapses at E_2. This is a free $H^{\underline{\mathbb{Z}}}_{\underline{\star}}$-module, hence there are no additive extensions. There are, however, multiplicative extensions.

Theorem 4.5. *As an $RO(C_2)$-graded Green functor,*

$$H_{\underline{\star}}(BBU_{\mathbb{R}}; \underline{\mathbb{Z}}) \cong H_{\underline{\star}}[\bar{y}_1, \bar{y}_2, \dots]/\bar{y}_i^2 = a_\sigma \bar{y}_{2i+1},$$

where $\bar{y}_i$ is a fixed element of degree $i\rho_2 + 1$.

Proof. The Dyer–Lashof operations commute with the homology suspension, and since this factors through the indecomposables, our earlier analysis gives on-the-nose identifications of the Dyer–Lashof actions.

Wilson has shown that for a class in degree $(n\rho_2 + 1)$, the square is stable and can be written as

$$(\text{-})^2 = a_\sigma Q^{(n+1)\rho_2}.$$

In particular, the squares are given by

$$\bar{y}_n^2 = a_\sigma Q^{(n+1)\rho_2}\bar{y}_n = a_\sigma\big[Q^{(n+1)\rho_2}\bar{a}_n\big] = a_\sigma[\bar{a}_{2n+1}] = a_\sigma \bar{y}_{2n+1}.$$

□

Remark 4.6. The geometric fixed points of this are again polynomial, and we recover the result of Kochman [17].

Since the homology of $BBU_{\mathbb{R}}$ is free, we can easily determine the homology of the coinduced $BBU_{\mathbb{R}}$.

Theorem 4.7. *For any $G = C_{2^n}$, we have*

$$H_{\underline{\star}}\big(\mathrm{Map}_{C_2}(G, BBU_{\mathbb{R}}); \underline{\mathbb{Z}}\big) \cong N_{C_2}^G\big(H_{\underline{\star}}(BBU_{\mathbb{R}}; \underline{\mathbb{Z}})\big).$$

4.1.2. *Twisted bar spectral sequence.* In C_2-equivariant homotopy, we have an additional version of the E_1-operad: the E_σ-operad. Algebras for this have no multiplication on their fixed points, but they do have a transfer map and an underlying multiplication. A summary can be found in [11].

If A is an E_σ-algebra, then we can form a kind of balanced product

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & X \underset{A}{\times} X, \end{array}$$

where C_2 acts on the whole diagram by swapping the two copies of X. This amounts to the data of a space X acted on by the associative monoid $i_e^* A$. The E_σ-structure on A means that the group action gives an isomorphism $i_e^* A \cong i_e^* A^{op}$, and hence the action on X also canonically gives a right action. The twisted balanced product swaps the two factors of X and also then necessarily changes these left and right actions.

Definition 4.8. If A is an E_σ-algebra and X is an i_e^*A-module, then let

$$B^\sigma(A;X) = B\big(A, \mathrm{Map}(C_2,A), \mathrm{Map}(C_2,X)\big),$$

where the action of $\mathrm{Map}(C_2,A)$ on A is via the E_σ-structure.

Perhaps the most interest case is when X is a point. In this case, the balance product is a model for the signed classifying space.

Theorem 4.9. *If A has R-free homology, then we have a spectral sequence which Adams indexed has the form*

$$E_2^{s,\underline{\star}} = \mathrm{Tor}_{-s}^{N_e^{C_2}\left(i_e^* R_*(i_e^*A)\right)}\Big(R_{\underline{\star}}\big(\mathrm{Map}(C_2,X)\big), R_{\underline{\star}}(A)\Big) \Rightarrow R_{\underline{\star}-s}\big(B^\sigma(A;X)\big).$$

If X also has R-free homology, then the action of $N_e^{C_2}\big(i_e^*R_*(i_e^*A)\big)$ on

$$R_{\underline{\star}}\big(\mathrm{Map}(C_2,X)\big) \cong N_e^{C_2}\big(i_e^*R_*(X)\big)$$

is the one induced by functoriality.

4.2. **Eilenberg–Moore.** Following Rector, we build a geometric model of the Eilenberg–Moore spectral sequence [27, 30]. Just as non-equivariantly, a space X together with a map to B can be viewed as a B-comodule (and in fact, we have much more structure equivariantly coming from the twisted diagonals). This allows us to form the cobar cosimplicial complex as a model for the homotopy pullback.

If $X \to B$ and $B \leftarrow Y$ are maps of G-spaces, then a model for the homotopy pullback is given by

$$X \underset{B}{\times}^h Y \simeq coB(X,B,Y),$$

where $coB(X,B,Y)$ is the totalization of the cosimplicial complex

$$k \mapsto X \times B^{\times k} \times Y,$$

and where the structure maps are the diagonal of B or the respective coaction maps. If B and either X or Y are R-free and finite type, then we have a spectral sequence computing cohomology. In general, convergence of this spectral sequence is very delicate, just as classically. For this reason, we state the result only for Bredon homology with coefficients in a Green functor.

Theorem 4.10. *If B and either X or Y has $\underline{R}$-free homology, then we have a spectral sequence*

$$E_2 = \mathrm{Tor}_{-s}^{H^{\underline{\star}}(B;\underline{R})}\big(H^{\underline{\star}}(X;\underline{R}), H^{\underline{\star}}(Y;\underline{R})\big) \Rightarrow H^{\underline{\star}+s}(X \underset{B}{\times}^h Y; \underline{R}).$$

4.2.1. *Twisted Eilenberg–Moore.* Dual to the twisted pushout, we have a twisted homotopy pullback.

Definition 4.11. If $f\colon X \to i_e^*B$, then let $X \underset{B}{\overset{\leftrightarrow}{\times}} X$ be the defined by the homotopy pullback

$$\begin{tikzcd} X \underset{B}{\overset{\leftrightarrow}{\times}} X \arrow[r] \arrow[d] & \mathrm{Map}(C_2,X) \arrow[d, "\mathrm{Map}(C_2,f)"] \\ B \arrow[r, "\Delta"'] & \mathrm{Map}(C_2,B). \end{tikzcd}$$

Explicitly, a point in the homotopy pullback is given by a triple:

$$\big((x_0, x_1), b, (\gamma_0, \gamma_1)\big) \in \mathrm{Map}(C_2, X) \times B \times \mathrm{Map}(C_2, B)^I.$$

This can be viewed as a pair of paths: one from b to $f(x_0)$ and one from gb to $gf(x_1)$. The group action swaps the points and the paths. A fixed point is then one for which $b = gb$, $x_0 = x_1$, and $g\gamma_0 = \gamma_1$. Put another way, this is an equivariant map

$$\gamma \colon [-1, 1] \to B,$$

where $\gamma(-1) = f(x_0)$, and where $[-1, 1] \subset \mathbb{R}_-$ is the balanced interval in the sign representation. In other words, this is the signed version of the homotopy pullback.

Remark 4.12. If X is a point, then this gives us the space of signed loops into B.

If $X = B^{C_2}$, then the homotopy pullback gives the space of paths connecting points in B^{C_2} but possibly passing through B.

This pullback gives a cobar complex and hence an Eilenberg–Moore spectral sequence via Theorem 4.10.

Theorem 4.13. *If B has $\underline{R}$-free homology, and if $\underline{R}$ is a Tambara functor then we have a spectral sequence*

$$E_2 = \mathrm{Tor}_{-s}^{N_e^{C_2} H^*(i_e^* B; \underline{R}(C_2))}\Big(H^{\underline{\star}}\big(\mathrm{Map}(C_2, X); \underline{R}\big), H^{\underline{\star}}(B; \underline{R})\Big) \Rightarrow H^{\underline{\star}+s}(X \overset{\leftrightarrow}{\underset{B}{\times}} X; \underline{R}).$$

Moreover, if X also has $\underline{R}$-free homology, then the action on

$$H^{\underline{\star}}\big(\mathrm{Map}(C_2, X); \underline{R}\big) \cong N_e^{C_2} H^*(X; \underline{R}(C_2))$$

is induced by the non-equivariant one.

We believe that these spectral sequences will be useful in computing the cohomology of equivariant Eilenberg–Mac Lane spaces.

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