# A NOTE ON THE DIVISIBILITY OF THE WHITEHEAD SQUARE

HARUO MINAMI

Abstract. We show that if we suppose $n \geq 4$ and $\pi^S_{2n-1}$ has no 2-torsion, then the Whitehead squares of the identity maps of $S^{2n+1}$ and $S^{4n+3}$ are divisible by 2. By applying the result of G. Wang and Z. Xu on $\pi^S_{61}$, we find that the Kervaire invariant one elements in dimensions 62 and 126 exist.

arXiv:2106.14604v1 [math.AT] 14 Apr 2021

## 1. Introduction

Let $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$ denote the Whitehead square of $\iota_n \in \pi_n(S^n)$ where $\iota_n$ is the homotopy class of the identity map of $S^n$. For $n$ odd $\neq 1, 3, 7$, it is well known that $[\iota_n, \iota_n]$ generates a subgroup of order 2 [2]. Furthermore, when $n$ is not of the form $2^r - 1$, this subgroup splits off as a direct summand [3]. Let $n_k = 2^k - 1$ and write $w_k$ for $[\iota_{2n_k+1}, \iota_{2n_k+1}] \in \pi_{4n_k+1}(S^{2n_k+1})$. In this note we consider the divisibility of $w_k$ by 2. But, since $w_1 = 0$ and $w_2 = 0$ [7], we assume here that $k \geq 3$. The main result is then the following

Theorem. *Suppose $\pi^S_{2n_k-1}$ has no 2-torsion. Then $w_k$ and $w_{k+1}$ are divisible by 2.*

From [7] and [8] we know that ${}_2\pi^S_{13} = 0$ and ${}_2\pi^S_{61} = 0$ where the subscript 2 represents the 2-primary part. By applying these two results to the theorem we obtain

Corollary. *$w_3$, $w_4$, $w_5$ and $w_6$ are divisible by 2.*

Because the Kervaire invariant one element $\theta_k \in \pi^S_{2n_k}$ exists if and only if $w_k \in 2\pi_{4n_k+1}(S^{2n_k+1})$ [1], this corollary together with the fact that $w_1 = 0$ and $w_2 = 0$ shows that there exist $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and $\theta_6$. This settles the problem of determining whether or not there exists $\theta_6$ which has been perhaps left unsolved. Based on the result of [4] that $\theta_k$ does not exist for $k \geq 7$, it can be therefore concluded that the only $\theta_k$ which exist are these six ones.

In order to prove the theorem we use an expression for $w_k$ by means of the characteristic map of a principal bundle over a sphere. Let $T_{n+1} : S^{n-1} \to SO(n)$ denote the characteristic map of the canonical principal $SO(n)$-bundle $SO(n+1) \to S^n$ and let $J$ be the $J$-homomorphism $\pi_{n-1}(SO(n)) \to \pi_{2n-1}(S^n)$. Then from [5, p. 521] we know that, when $n$ is odd $\geq 9$, $[\iota_n, \iota_n]$ can be written $[\iota_n, \iota_n] = J([T_{n+1}])$, so that

$$w_k = J([T_{n_k+2}]) \quad (k \geq 3)$$

(the bracket [ ] denotes the homotopy class).

2010 *Mathematics Subject Classification.* 55Q15, 55Q50.

Let $\mathbb{R}^n$ be euclidean space: $x = (x_1, \cdots, x_n)$. Let $S^{n-1} \subset \mathbb{R}^n$ be the unit sphere with base point $x_0 = (0, \cdots, 0, 1)$. According to [6], $T_{n+1}$ is then given by

$$T_{n+1}(x) = \big(\delta_{ij} - 2x_i x_j\big) \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \quad (1 \leq i, j \leq n)$$

where $I_{n-1}$ is the identity matrix of dimension $n-1$. Then obviously $T_{n+1}(\pm x_0) = I_n$. Let $\Sigma^n = \mathbb{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^n$ with $\infty$ as base-point. Applying the Hopf construction to $T_{n+1}$ we obtain a map

$$\tau_n : \Sigma^n \wedge S^{n-1} \to \Sigma^n.$$

By virtue of $T_{n+1}(\pm x_0) = I_n$ it follows that

$$\tau_n \mid \Sigma^n \wedge S^0 \simeq c_\infty$$

where $S^0 = \{x_0, -x_0\}$ and $c_\infty$ denotes the constant map at $\infty$. Since $(\Sigma^n, \infty) \simeq (S^n, x_0)$, ii is clear that $[\tau_n] = J([T_{n+1}]) \in \pi_{2n-1}(S^n)$ and so $w_k = [\tau_{n_k+1}]$ for $k \geq 3$.

For $1 \leq s \leq n-1$, if we set $x' = (x_1, \cdots, x_s)$ and $x'' = (x_{s+1}, \cdots, x_n)$, then the map $(x', x'') \to (-x', x'')$ of $\mathbb{R}^n$ defines involutions on $\Sigma^n$ and $S^{n-1}$, denoted by $\bar{a}_{s,n-s}$ and $a_{s,n-s}$, respectively. Let $I_{s,n-s} = (-I_s) \times I_{n-s}$ be the diagonal matrix whose first $s$ diagonal elements equal to $-1$ and the remaining $n-s$ diagonal elements equal to 1. Then we find that

$$T_{n+1}(a_{s,n-s}(x)) = I_{s,n-s} T_{n+1}(x) I_{s,n-s} \quad (x \in S^{n-1}).$$

This gives $\tau_n(\bar{a}_{s,n-s}(v) \wedge a_{s,n-s}(x)) = \bar{a}_{s,n-s}(\tau_n(v \wedge x))$ where $v \in \Sigma^n$, $x \in S^{n-1}$, i.e.

$$\tau_n \circ (\bar{a}_{s,n-s} \wedge a_{s,n-s}) = \bar{a}_{s,n-s} \circ \tau_n \qquad (0 \leq s \leq n-1).$$

If $s$ can be written as $s = 2j + r$ $(j, r \geq 0)$, then considering the above formula with $I_{s,n-s} T_{n+1}(x) I_{s,n-s}$ replaced by

$$(-I_r \times \nu(t) \times I_{n-s}) T_{n+1}(x) (-I_r \times \nu(t)^{-1} \times I_{n-s})$$

where $\nu(t)$ is a path in $SO(2j)$ from $-I_{2j}$ to $I_{2j}$, we obtain

$$\tau_n \circ (\bar{a}_{r,n-r} \wedge a_{s,n-s}) \simeq \bar{a}_{r,n-r} \circ \tau_n \qquad (0 \leq s \leq n-1).$$

This is a homotopy relative to $\Sigma^n \wedge S^0$, so that it maintains the relation $\tau_n \mid \Sigma^n \wedge S^0 \simeq c_\infty$. In particular, when $s$ is even, it becomes a homotopy

$$\tau_n \circ (1 \wedge a_{s,n-s}) \simeq \tau_n \qquad (0 \leq s \leq n-1)$$

where 1 denotes the identity map of $\Sigma^n$. This exhibits a certain symmetry property of $\tau_n$ about $x_n$-axis. In the case when $s$ is odd, considering its suspension $S\tau_n$ instead of $\tau_n$ we have

$$S\tau_n \circ (1 \wedge a_{s,n-s}) \simeq S\tau_n \qquad (0 \leq s \leq n-1)$$

where 1 is the identity map of $\Sigma^{n+1}$ (here $\Sigma^{n+1}$ is identified with the suspension $S\Sigma^n$ of $\Sigma^n$ in the usual way).

For $1 \leq s \leq n-1$, let $\mathbb{R}^{n-s} \subset \mathbb{R}^n$ be the subspace defined by $x_1 = \cdots = x_s = 0$ and let $(\Sigma^{n-s}, S^{n-s-1})$ be the pair defined for it above. Then according to the definition of $\tau_n$ we see that the restriction of $\tau_n$ to $\Sigma^n \wedge S^{n-s-1}$ can be written as the composition

$$\Sigma^n \wedge S^{n-s-1} \xrightarrow{p \wedge 1} \Sigma^{n-s} \wedge S^{n-s-1} \xrightarrow{\tau_{n-s}} \Sigma^{n-s} \xrightarrow{i} \Sigma^n$$

where $p$ and $i$ denote the maps induced by the canonical projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-s}$ and inclusion of $\mathbb{R}^{n-s}$ into $\mathbb{R}^n$, respectively and 1 denotes the identity map of $S^{n-s-1}$. This means that the restriction of $\tau_n$ to $\Sigma^n \wedge S^{n-s-1}$ coincides with the $s$-fold suspension of $\tau_{n-s}$, i.e. $\tau_n \mid \Sigma^n \wedge S^{n-s-1} = S^s\tau_{n-s}$. But, by abuse of notation, we use the notation $\tau_n \mid \Sigma^n \wedge S^{n-s-1}$ to denote the composite of the first two maps, i.e. we write

$$\tau_n \mid \Sigma^n \wedge S^{n-s-1} = \tau_{n-s} \circ (p \wedge 1).$$

Given $f, g : \Sigma^n \wedge S^{n-1} \to \Sigma^n$, the sum $f + g$ is given by the composition

$$\Sigma^n \wedge S^{n-1} \xrightarrow{\Delta} (\Sigma^n \wedge S^{n-1}) \vee (\Sigma^n \wedge S^{n-1}) \xrightarrow{f \vee g} \Sigma^n \vee \Sigma^n \xrightarrow{\mu} \Sigma^n$$

where $\Delta$ is the inclusion induced by the diagonal map of $S^{n-1}$ or $\Sigma^n$ as necessary and $\mu$ the folding map.

Let $D^{n-1}_{\pm} = S^{n-1} \cap \{x \mid \pm x_1 \geq 0\}$. Then $S^{n-1} = D^{n-1}_{+} \cup D^{n-1}_{-}$ and $S^{n-2} = D^{n-1}_{+} \cap D^{n-1}_{-}$. Put $S^{n-1}_{\pm} = D^{n-1}_{\pm}/S^{n-2}$. Then $S^{n-1}/S^{n-2}$ becomes homeomorphic to $S^{n-1}_{+} \vee S^{n-1}_{-}$. Denote by $\pi : \Sigma^n \wedge S^{n-1} \to (\Sigma^n \wedge S^{n-1}_{+}) \vee (\Sigma^n \wedge S^{n-1}_{-})$ the composition of the quotient map $\Sigma^n \wedge S^{n-1} \to \Sigma^n \wedge S^{n-1}/\Sigma^n \wedge S^{n-2}$ and the homeomorphism to $(\Sigma^n \wedge S^{n-1}_{+}) \vee (\Sigma^n \wedge S^{n-1}_{-})$. Let $\pi_{\pm} : S^{n-1} \to S^{n-1}_{\pm}$ denote the collapsing maps. We use also the same symbols $\pi_{\pm}$ to denote the $m$-fold suspension $S^m\pi_{\pm}$ $(m \geq n)$.

The proof of the theorem proceeds in four steps. First, we consider the decomposition of $S\tau_n$ into two homotopic maps.

LEMMA 1. *Suppose $S\tau_n \mid \Sigma^{n+1} \wedge S^{n-2} \simeq c_\infty$ where $S^{n-2}$ denotes the equator of $S^{n-1}$ defined by $x_1 = 0$. Then there exist maps $f_{\pm} : \Sigma^{n+1} \wedge S^{n-1}_{\pm} \to \Sigma^{n+1}$ such that $S\tau_n \simeq f_+ \circ S\pi_+ - f_- \circ S\pi_-$, $f_- \circ S\pi_- \simeq -f_+ \circ S\pi_+$, so that $S\tau_n \simeq 2f_+ \circ S\pi_+$.*

*Proof.* Because of the assumption, $S\tau_n : \Sigma^{n+1} \wedge S^{n-1} \to \Sigma^{n+1}$ can be factorized through the quotient $\Sigma^{n+1} \wedge S^{n-1}/\Sigma^{n+1} \wedge S^{n-2}$ into the composition

$$\Sigma^{n+1} \wedge S^{n-1} \xrightarrow{S\pi} (\Sigma^{n+1} \wedge S^{n-1}_{+}) \vee (\Sigma^{n+1} \wedge S^{n-1}_{-}) \xrightarrow{f_+ \vee f_-} \Sigma^{n+1} \vee \Sigma^{n+1} \xrightarrow{\mu} \Sigma^{n+1}$$

where $f_{\pm} : \Sigma^{n+1} \wedge S^{n-1}_{\pm} \to \Sigma^{n+1}$. This shows due to the definition of the sum of maps that

$$S\tau_n \simeq f_+ \circ \pi_+ - f_- \circ \pi_-.$$

Consider the homeomorphism $1 \wedge a_{1,n-1}$ of $\Sigma^{n+1} \wedge S^{n-1}$ onto itself given above. Then, since its restriction to $\Sigma^{n+1} \wedge S^{n-2}$ is the identity map, it induces homeomorphisms $(1 \wedge a_{1,n-1})_{\pm} : \Sigma^{n+1} \wedge S^{n-1}_{\pm} \to \Sigma^{n+1} \wedge S^{n-1}_{\mp}$ (the double signs as usual are to be taken in the same order). Using the formula $S\tau_n \circ (1 \wedge a_{1,n-1}) \simeq S\tau_n$ we can see that $f_{\pm}$ satisfy

$$f_- \simeq f_+ \circ (1 \wedge a_{1,n-1})_-,$$

so that $f_- \circ \pi_- \simeq f_+ \circ (1 \wedge a_{1,n-1})_- \circ \pi_-$. Since clearly $(1 \wedge a_{1,n-1})_- \circ \pi_- \simeq \pi_+ \circ (1 \wedge a_{1,n-1})$ and $\pi_+ \circ (1 \wedge a_{1,n-1}) \simeq -\pi_+$, we have $(1 \wedge a_{1,n-1})_- \circ \pi_- \simeq -\pi_+$ and so $f_+ \circ (1 \wedge a_{1,n-1})_- \circ \pi_- \simeq -f_+ \circ \pi_+$. Substituting the relation obtained above in this formula we get $f_- \circ \pi_- \simeq -f_+ \circ \pi_+$, which completes the proof.

## 2. Decompositions of $\tau_{2n+1}$ and $\tau_{2n+2}$

From now, let $n = n_k/2$ and assume that the assumption of the theorem is fulfilled, i.e., ${}_2\pi^S_{2n-1} = 0$. Also we work modulo odd torsion since we consider the 2-primary homotopy decomposition of maps.

From the fact that the suspension homomorphism $E : \pi_{4n-1}(S^{2n}) \to \pi_{4n}(S^{2n+1})$ of the *EHP* sequence is a surjection with kernel $\mathbb{Z}$, generated by $[\iota_{2n}, \iota_{2n}]$, we see that it induces an isomorphism ${}_2\pi_{4n-2}(S^{2n}) \cong {}_2\pi^S_{2n-1}$ between their 2-primary parts. Hence from the assumption above we have

$$(^*) \qquad {}_2\pi_{4n-1}(S^{2n}) = 0.$$

In the above, when we write $\mathbb{R}^n$ as $\mathbb{R}^n_\alpha$ by attaching a suffix $\alpha$, to denote its associated spaces and maps given above we use here the notations $\Sigma^n_\alpha$, $S^{n-1}_\alpha$, $(D^{n-1}_\alpha)_\pm$, $(S^{n-1}_\alpha)_\pm$, $(\tau_n)_\alpha$, $(\pi_\alpha)_\pm$ with adding the suffix $\alpha$.

Lemma 2. *Under the assumption of Theorem, there exist maps $g_\pm : \Sigma^{2n+q} \wedge S^{2n+q-1}_\pm \to \Sigma^{2n+q}$ for $q = 1, 2$ such that $\tau_{2n+q} \simeq g_+ \circ \pi_+ - g_- \circ \pi_-$, $g_- \circ \pi_- \simeq -g_+ \circ \pi_+$, so that $\tau_{2n+q} \simeq 2g_+ \circ \pi_+$.*

*Proof of Case q=1.* We first consider the suspension $S\tau_{2n} : \Sigma^{2n+1} \wedge S^{2n-1} \to \Sigma^{2n+1}$ of $\tau_{2n}$. Then $S\tau_{2n} \mid \Sigma^{2n+1} \wedge S^{2n-2}$ represents a map from $\Sigma^{2n} \wedge S^{2n-1}$ to $\Sigma^{2n}$, so by $(^*)$ we have

$$(1) \qquad S\tau_{2n} \mid \Sigma^{2n+1} \wedge S^{2n-2} \simeq c_\infty.$$

This shows that the null homotopy condition of Lemma 1 with $n$ replaced by $2n$ is satisfied and therefore we see that there exists a decomposition of $S\tau_{2n}$ such that

$$(2) \qquad S\tau_{2n} \simeq f_+ \circ \pi_+ - f_- \circ \pi_-, \quad f_- \circ \pi_- \simeq -f_+ \circ \pi_+,$$

where $f_\pm : \Sigma^{2n+1} \wedge S^{2n-1}_\pm \to \Sigma^{2n+1}$.

Using polar coordinates for the first two variables $x_1, x_2$ we express $x \in \mathbb{R}^{2n+1}$ as

$$x_t = (r\cos t, r\sin t, x_3, \cdots, x_{2n+1}) \qquad (0 \le t < \pi,\ r \in \mathbb{R}).$$

For any fixed $t$, let $\mathbb{R}^{2n}_t \subset \mathbb{R}^{2n+1}$ denote the $2n$-dimensional subspace generated by the $x_t$. Regard $\mathbb{R}^{2n}$ as $\mathbb{R}^{2n}_t$ with $t = 0$ and put

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & I_{2n-1} \end{pmatrix}.$$

Then the map $x \to xM(t)^T$ gives a linear isomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}_t$ (the subscript $T$ denotes the transpose). This induces homeomorphisms $\bar{e}_t : \Sigma^{2n+1} \to S\Sigma^{2n}_t$, $e_t : S^{2n-1} \to S^{2n-1}_t$ and $e_{t\pm} : S^{2n-1}_\pm \to (S^{2n-1}_t)_\pm$. Here $(S^{2n-1}_t)_\pm = (D^{2n-1}_t)_\pm / S^{2n-2}$ where $(D^{2n-1}_t)_\pm = S^{2n-1}_t \cap \{x_t \mid \pm r \ge 0\}$ and $S^{2n-2}$ is the unit sphere in $\mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}_t$ defined by $r = 0$. Then clearly $\pi_{t\pm} \circ (\bar{e}_t \wedge e_t) = (\bar{e}_t \wedge e_{t\pm}) \circ \pi_\pm$. Let $(\tau_{2n})_t = \tau_{2n+1} \mid \Sigma^{2n}_t \wedge S^{2n-1}_t$ and $f_{t\pm} : S\Sigma^{2n}_t \wedge (S_t^{2n-1})_\pm \to S\Sigma^{2n}_t$ be the maps defined by the formula $f_{t\pm} \circ (\bar{e}_t \wedge e_{t\pm}) = \bar{e}_t \circ f_\pm$ where $f_\pm$ are as in (2). Then we have

$$S(\tau_{2n})_t \circ (\bar{e}_t \wedge e_t) = \bar{e}_t \circ S\tau_{2n}, \quad (f_{t\pm} \circ \pi_{t\pm}) \circ (\bar{e}_t \wedge e_{t\pm}) = \bar{e}_t \circ (f_\pm \circ \pi_\pm),$$

respectively. Substituting $S\tau_{2n}$ and $f_\pm \circ \pi_\pm$ derived from these equalities by composing $\bar{e}_t^{-1}$ in (2) we get

$$S(\tau_{2n})_t \simeq f_{t+} \circ \pi_{t+} - f_{t-} \circ \pi_{t-}, \quad f_{t-} \circ \pi_{t-} \simeq -f_{t+} \circ \pi_{t+} \qquad (0 \le t < \pi). \tag{3}$$

Similarly, through $S(\tau_{2n})_t \circ (\bar{e}_t \wedge e_t) = \bar{e}_t \circ S\tau_{2n}$, the formula (1) can be rewritten as

$$S(\tau_{2n})_t \mid S\Sigma_t^{2n} \wedge S^{2n-2} \simeq c_\infty.$$

Also we inherit $S(\tau_{2n})_t \circ (1 \wedge a_{1,2n-1}) \simeq S(\tau_{2n})_t$ from $S\tau_{2n} \circ (1 \wedge a_{1,2n-1}) \simeq S\tau_{2n}$. Considering (3) together with these two formulas we can determine the behavior of maps $f_{t\pm}$ when, for any fixed $t$, rotating along the region of $t$ by $\pi$ degrees.

According to the definition of the sum of maps we have a homotopy

$$f_{t+\pi+} \circ \pi_{t+\pi+} \simeq -f_{t+} \circ \pi_{t+}$$

On the other hand, according to the definition of $f_{t-}$, we have

$$f_{t+\pi+} \circ \pi_{t+\pi+} \simeq -f_{t-} \circ \pi_{t-},$$

so that it follows that

$$f_{t+} \circ \pi_{t+} \simeq f_{t-} \circ \pi_{t-}.$$

By applying this to the former formula of (3) we get $S(\tau_{2n})_t \mid S\Sigma_t^{2n} \wedge S_t^{2n-1} \simeq c_\infty$. So identifying $S\Sigma_t^{2n}$ with $\Sigma^{2n+1}$ in the usual way we obtain

$$\tau_{2n+1} \mid \Sigma^{2n+1} \wedge S_t^{2n-1} \simeq c_\infty \qquad (0 \le t < \pi). \tag{4}$$

Take $t = \pi/2$. Then $S^{2n}/S_t^{2n-1} \approx S_+^{2n} \vee S_-^{2n}$ where $S_\pm^{2n} = S^{2n} \cap \{x_t \mid \pm x_1 \ge 0\}$. Thereby the null homotopy of (4), as in the case of $S\tau_{2n}$, yields a factorization of $\tau_{2n+1}$ into the composition

$$\Sigma^{2n+1} \wedge S^{2n} \xrightarrow{\pi} (\Sigma^{2n+1} \wedge S_+^{2n}) \vee (\Sigma^{2n+1} \wedge S_-^{2n}) \xrightarrow{g_+ \vee g_-} \Sigma^{2n+1} \vee \Sigma^{2n+1} \xrightarrow{\mu} \Sigma^{2n+1},$$

so that a decomposition $\tau_{2n+1} \simeq g_+ \circ \pi_+ - g_- \circ \pi_-$ where $g_\pm : \Sigma^{2n+1} \wedge S_\pm^{2n} \to \Sigma^{2n+1}$. But in fact, $g_\pm$ can be obtained by unifying $f_{t\pm}$ according to the former formula of (3) under the null homotopy of (4), respectively, and therefore we have $g_- \circ \pi_- \simeq -g_+ \circ \pi_+$ from the latter of (3).

*Proof of Case q=2.* In order to use the results in the proof of the case $q = 1$, we adopt spherical polar coordinate representation for the first three variables $x_1, x_2, x_3$ and express $x \in \mathbb{R}^{2n+2}$ as

$$x_{\theta\, t} = (r\cos\theta, r\sin\theta\cos t, r\sin\theta\sin t, x_4, \cdots, x_{2n+2}) \quad (0 \le \theta \le \pi,\ 0 \le t < \pi,\ r \in \mathbb{R}).$$

For a fixed $0 < \theta < \pi$, let $\mathbb{R}_\theta^{2n+1} \subset \mathbb{R}^{2n+2}$ denote the $(2n+1)$-dimensional subspace generated by the $x_{\theta\, t}$. In addition, fix $t$, then these elements define the $2n$-dimensional subspace $\mathbb{R}_{\theta\, t}^{2n} \subset \mathbb{R}_\theta^{2n+1}$. For any fixed $t$, let $\mathbb{R}_t^{2n} \subset \mathbb{R}^{2n+2}$ denote the subspace generated by the $x_{\theta\, t}$ with $\theta = \pi/2$ and put

$$M(\theta\, t) = \begin{pmatrix} \sin\theta & \cos\theta\cos t & \cos\theta\sin t & 0 \\ -\cos\theta\cos t & \sin\theta & 0 & 0 \\ -\cos\theta\sin t & 0 & \sin\theta & 0 \\ 0 & 0 & 0 & I_{2n-1} \end{pmatrix} \qquad (0 < \theta < \pi).$$

Then the map $x \to xM(\theta\, t)^T$ gives a linear isomorphism $\mathbb{R}^{2n}_t \to \mathbb{R}^{2n}_{\theta\, t}$. Denote by $\bar{e}_{\theta\, t} : S\Sigma^{2n}_t \to S\Sigma^{2n}_{\theta\, t}$, $e_{\theta\, t} : S^{2n-1}_t \to S^{2n-1}_{\theta\, t}$ and $(b_{\theta\, t})_\pm : (S^{2n-1}_t)_\pm \to (S^{2n-1}_{\theta\, t})_\pm$ the homeomorphisms induced by this isomorphism. Here $(S^{2n-1}_{\theta\, t})_\pm = (D^{2n-1}_{\theta\, t})_\pm / S^{2n-2}_{\theta\, t}$ where $(D^{2n-1}_{\theta\, t})_\pm = S^{2n-1}_{\theta\, t} \cap \{x_{\theta\, t} \mid \pm r \geq 0\}$ and $S^{2n-2}_{\theta\, t} = S^{2n-1}_{\theta\, t} \cap \{x_{\theta\, t} \mid r = 0\}$. Then putting $(\tau_{2n})_{\theta\, t} = \tau_{2n+2} \mid \Sigma^{2n}_{\theta\, t} \wedge S^{2n-1}_{\theta\, t}$ we have

$$S(\tau_{2n})_{\theta\, t} \circ (\bar{e}_{\theta\, t} \wedge e_{\theta\, t}) = \bar{e}_{\theta\, t} \circ S(\tau_{2n})_t.$$

We also have

$$((f_{\theta\, t})_\pm \circ (\pi_{\theta\, t})_\pm) \circ (\bar{b}_{\theta\, t} \wedge b_{\theta\, t}) = \bar{b}_{\theta\, t} \circ (f_{t\pm} \circ \pi_{t\pm})$$

where $(f_{\theta\, t})_\pm : S\Sigma^{2n}_{\theta\, t} \wedge (S^{2n-1}_{\theta\, t})_\pm \to S\Sigma^{2n}_{\theta\, t}$ denote the maps defined by the formula $(f_{\theta\, t})_\pm \circ (\bar{e}_{\theta\, t} \wedge (e_{\theta\, t})_\pm) = \bar{e}_{\theta\, t} \circ f_{t\pm}$. Substituting $S(\tau_{2n})_t$ and $f_{t\pm} \circ \pi_{t\pm}$ derived from these equalities by composing $\bar{b}^{-1}_{\theta\, t}$ in (3) we obtain

(5) $\quad S(\tau_{2n})_{\theta\, t} \simeq (f_{\theta\, t})_+ \circ (\pi_{\theta\, t})_+ - (f_{\theta\, t})_- \circ (\pi_{\theta\, t})_-, \quad (f_{\theta\, t})_+ \circ (\pi_{\theta\, t})_+ \simeq -(f_{\theta\, t})_-. \circ (\pi_{\theta\, t})_-$

where $0 < \theta < \pi$.

Similarly, inheriting (4) through $S(\tau_{2n})_{\theta\, t} \circ (\bar{e}_{\theta\, t} \wedge e_{\theta\, t}) = \bar{e}_{\theta\, t} \circ S(\tau_{2n})_t$ we have

$$S(\tau_{2n})_{\theta\, t} \mid \Sigma^{2n+1}_\theta \wedge S^{2n-1}_{\theta\, t} \simeq c_\infty \qquad (0 < \theta < \pi).$$

We look at (4) from another angle. Replacing formally the domain $\Sigma^{2n+1} \wedge S^{2n-1}_t$ of the null homotopy of $S(\tau_{2n})_t$ given there by $\Sigma^{2n+1} \wedge S^{2n-1}$ where $S^{2n-1} = S^{2n+1} \cap \{x_{\theta\, t} \mid \theta = 0\}$, we have a null homotopy of $S(\tau_{2n})_{\theta\, t}$ over $\Sigma^{2n+1} \wedge S^{2n-1}$. Combining this with the null homotopy obtained above we can form a homotopy

$$\tau_{2n+2} \mid \Sigma^{2n+2} \wedge S^{2n} \simeq c_\infty \tag{6}$$

where $S^{2n} = S^{2n+1} \cap \{x_{\theta\, t} \mid t = \pi/2\}$. In the case here, differently from the case $q = 1$, imposing the coordinate condition $\pm x_1 \geq 0$ on $x_2$, we let $D^{2n+1}_\pm = S^{2n+1} \cap \{x_{\theta\, t} \mid \pm x_2 \geq 0\}$ and put $S^{2n+1}_\pm = D^{2n+1}_\pm / S^{2n}$. Then according to the former formula of (5) under the null homotopy of (6), in the same way as in the case of $\tau_{2n+1}$ we obtain a decomposition $\tau_{2n+2} \simeq g_+ \circ \pi_+ - g_- \circ \pi_-$ of $\tau_{2n+2}$ where $g_\pm : \Sigma^{2n+2} \wedge S^{2n+1}_\pm \to \Sigma^{2n+2}$. Then $g_+ \circ \pi_+ \simeq -g_- \circ \pi_-$ follows from the latter formula of (5) as in the case of $q = 1$ above. This completes the proof of the lemma.

## 3. Proof of Theorem

The theorem follows immediately from Lemma 2, Case $q = 1$, and the following

Lemma 3. *Under the assumption of Theorem, there exist maps* $g_\pm : \Sigma^{4n+3} \wedge S^{4n+2}_\pm \to \Sigma^{4n+3}$ *such that* $\tau_{4n+3} \simeq g_+ \circ \pi_+ - g_- \circ \pi_-$, $g_+ \circ \pi_+ \simeq -g_- \circ \pi_-$, *so that* $\tau_{4n+3} \simeq 2g_+ \circ \pi_+$.

*Proof.* Put $m = 2n + 1$ and let $\phi = (\phi_1, \cdots, \phi_m)$ where $0 \leq \phi_i < \pi$ $(1 \leq i \leq m)$. We express $x \in \mathbb{R}^{2m+1}$ as

$$x_\phi = (x_1 \cos\phi_1, \cdots, x_m \cos\phi_m, x_1 \sin\phi_1, \cdots, x_m \sin\phi_m, x_{2m+1})$$

where $x_1, \cdots, x_m, x_{2m+1} \in \mathbb{R}$. For a fixed $\phi$, let $\mathbb{R}^{m+1}_\phi \subset \mathbb{R}^{2m+1}$ be the $(m+1)$-dimensional subspace generated by the $x_\phi$. Let

$$c(\phi) = \mathrm{diag}(\cos\phi_1, \cos\phi_2, \cdots, \cos\phi_m), \quad s(\phi) = \mathrm{diag}(\sin\phi_1, \sin\phi_2, \cdots, \sin\phi_m)$$

be the diagonal matrices whose $(i,i)$-entries are $\cos\phi_i$ and $\sin\phi_i$, respectively, and set

$$M(\phi)=\begin{pmatrix} c(\phi) & -s(\phi) & 0\\ s(\phi) & c(\phi) & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We regard $\mathbb{R}^{m+1}_\phi$ with $\phi=(0,\cdots,0)$ as $\mathbb{R}^{m+1}$, so $x_\phi=(x_1,\cdots,x_m,0,\cdots,0,x_{2m+1})\in\mathbb{R}^{m+1}_\phi$ can be taken to be equal to $x=(x_1,\cdots,x_m,x_{2m+1})\in\mathbb{R}^{m+1}$. Here the first three variables $x_1,x_2,x_3$ are supposed to have the same expression in polar form as those of $x_{\theta\, t}$ above. Then, for any fixed $\phi$, the map $x\to xM(\phi)^T$ defines a linear isomorphism $\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}_\phi$. Let $\bar{e}_\phi:\Sigma^{m+1}\to\Sigma^{m+1}_\phi$ and $e_\phi:S^m\to S^m_\phi$ denote the homeomorphisms induced by this isomorphism. If we put $(\tau_{m+1})_\phi=\tau_{2m+1}\mid\Sigma^{m+1}_\phi\wedge S^m_\phi$, then we have

$$(\tau_{m+1})_\phi\circ(\bar{e}_\phi\wedge e_\phi)=\bar{e}_\phi\circ\tau_{m+1}.$$

Similarly as in the case $q=2$ above, taking $(D^m_\phi)_\pm=S^m_\phi\cap\{x_\phi\mid\pm x_2\geq 0\}$ and $S^{m-1}_\phi=S^m_\phi\cap\{x_\phi\mid x_2=0\}$ we put $(S^m_\phi)_\pm=(D^m_\phi)_\pm/S^{m-1}_\phi$. Let $e_{\phi_\pm}:S^m_\pm\to(S^m_\phi)_\pm$ be the homeomorphisms induced by $e_\phi$ and we define the maps $f_{\phi_\pm}:\Sigma^{m+1}_\phi\wedge(S^m_\phi)_\pm\to\Sigma^{m+1}_\phi$ by the formula $f_{\phi_\pm}\circ(\bar{e}_\phi\wedge e_{\phi_\pm})=\bar{e}_\phi\circ g_\pm$ where $g_\pm:\Sigma^{m+1}\wedge S^m_\pm\to\Sigma^{m+1}$ are as the maps occurring in the decomposition of $\tau_{m+1}$ above. Then clearly

$$(f_{\phi_\pm}\circ\pi_{\phi_\pm})\circ(\bar{e}_\phi\wedge e_{\phi_\pm})=\bar{e}_\phi\circ(g_\pm\circ\pi_\pm).$$

Substituting $\tau_{m+1}$ and $g_\pm\circ\pi_\pm$ derived from the above two equalities by composing $\bar{e}_\phi^{-1}$ in the decomposition formula of $\tau_{m+1}$ obtained in Lemma **??**, Case $q=2$, we have

$$(\tau_{m+1})_\phi\simeq f_{\phi_+}\circ\pi_{\phi_+}-f_{\phi_-}\circ\pi_{\phi_-},\quad f_{\phi_+}\circ\pi_{\phi_+}\simeq-f_{\phi_-}\circ\pi_{\phi_-}. \tag{7}$$

In addition, in a similar way we have the null homotopy

$$(\tau_{m+1})_\phi\mid\Sigma^{2m+1}\wedge S^{m-1}_\phi\simeq c_\infty \tag{8}$$

which inherits the null homotopy of $\tau_{m+1}\mid\Sigma^{m+1}\wedge S^{m-1}$ used for proving the case $q=2$ above through the equality relating $(\tau_{m+1})_\phi$ and $\tau_{m+1}$ given above. By using a similar argument to the case $q=1$, i.e. to the proof of $\tau_m\mid\Sigma^m\wedge S^{m-2}_t\simeq c_\infty$, together with (7) and (8) we can determine the behavior of $f_{\phi_\pm}$ in rotating along the regions of $\phi_i$ by $\pi$ degrees.

Consider the composite $\alpha=a_{m,1}\circ a_{1,m}\circ a_{2,m-1}:S^m\to S^m$ which maps $x_i$ to itself if $i=2,m+1$ and to $-x_i$ otherwise where $a_{s,m-s}$ are as above and $x=(x_1,x_2,\cdots,x_m,x_{2m+1})\in S^m$. If we write $\alpha_\phi$ for $e_\phi\circ\alpha\circ{e_\phi}^{-1}$, then we have

$$(\tau_{m+1})_\phi\circ(1\wedge\alpha_\phi)\simeq(\tau_{m+1})_\phi$$

where 1 denotes the identity on $\Sigma^{m+1}_\phi$. Since a number of $x_i$ whose sign is converted by $\alpha$ in reverse is $m-1=2n$ and so is even, this implies that the null homotopy (8) of $\tau_{2m+1}$ over $\Sigma^{2m+1}\wedge S^{m-1}_\phi$ can be extended over the subspace of $\Sigma^{2m+1}\wedge S^{2m}$ consisting of the elements of $\Sigma^{2m+1}\wedge S^{m-1}_\phi$ when $\phi_1,\phi_3,\cdots,\phi_m$ vary over the full range and only the second variable $\phi_2$ remains fixed.

Under this null homotopy, rotating $\phi_2$ by $\pi$ we have

$$(\tau_{m+1})_{(\phi_1,\phi_2+\pi,\phi_3,\cdots,\phi_m)}\simeq-(\tau_{m+1})_\phi.$$

If we take $\phi_2 = 0$ and set $S^{2m-1} = S^{2m} \cap \{x_\phi \mid \phi_2 = 0\}$, then considering this null homotopy together with the extension of (8) obtained above we get a homotopy

$$\tau_{2m+1} \mid \Sigma^{2m+1} \wedge S^{2m-1} \simeq c_\infty.$$

Let $S^{2m}_\pm = D^{2m}_\pm / S^{2m-1}$ where $D^{2m}_\pm = S^{2m} \cap \{x_\phi \mid \pm x_2 \geq 0\}$. Then, as in the cases $q = 1, 2$ above, considering (7) under the null homotopy now obtained we see that $f_{\pi\pm}$ $(0 \leq \phi < \pi)$ can be integrated into the maps $g_\pm : \Sigma^{2m+1} \wedge S^{2m}_\pm \to \Sigma^{2m+1}$, respectively, and therefore the desired decomposition formula can be satisfied. This completes the proof of the lemma.

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H. Minami: Professor Emeritus, Nara University of Education

*Email address*: `hminami@camel.plala.or.jp`