J. Math. Kyoto Univ. (JMKYAZ)
45-1 (2005), 39–55

The Steenrod algebra and the automorphism group of additive formal group law

By

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1. Introduction

Let H_*H be the Hopf algebra of stable co-operations of the mod 2 ordinary cohomology theory $H^*(\ )$. The structure of H_*H is well known as follows. First J. P. Serre [7] determined the unstable cohomology of the Eilenberg-MacLane complex $K(n, \mathbb{Z}/2)$. He has shown the stable part of $H^*(K(n, \mathbb{Z}/2))$ is generated by iterated Steenrod operations and computed the rank of $H^i(K(n, \mathbb{Z}/2))$ in terms of excess operations. He assumed the existence of Steenrod squares Sq^i but did not use the Adem relations. Using the Adem relations, we see that the algebra S^* generated by Steenrod squares modulo the Adem relations is isomorphic to H^*H. Moreover Milnor [4] determined the Hopf algebra structure of S_*, the dual Steenrod algebra which is the polynomial algebra $\mathbb{F}_2[\xi_1, \xi_2, \dots]$ with the coproduct $\psi(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i$, and therefore we obtain the Hopf algebra structure of H_*H.

Now we recall strict automorphisms of the additive formal group law. Let G_a be the additive formal group law, and $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ the set of strict automorphisms of G_a over a non-negatively graded commutative $\mathbb{F}_2$-algebra R_*. An element $f(x)$ in $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is written as a formal power series $x + \sum_{i=1}^{\infty} a_i x^{2^i}$, where $a_i \in R_{2^i-1}$. Here $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(-)$ is a functor from the category of graded algebras to the category of sets. A product of $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is defined by the composition of power series, and induces the group structure. Therefore $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(-)$ is a functor to the category of groups, and is represented by the Hopf algebra $A_* = \mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \dots]$ with the coproduct $\psi(\bar{\xi}_n) = \sum_{i=0}^{n} \bar{\xi}_{n-i}^{2^i} \otimes \bar{\xi}_i$. In other words, we have a natural group isomorphism

$$\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R_*) \cong \mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*).$$

Comparing S_* with A_*, we see that $S_* \cong A_*$ as a Hopf algebra.

We recall the Dickson algebra. Let V^n be the $\mathbb{F}_2$-vector space spanned by elements $x_1, \dots, x_n$. In the polynomial ring $\mathbb{F}_2[x_1, \dots, x_n][t]$, consider the

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2000 *Mathematics Subject Classification*(*s*). Primary 55N22, 55S10; Secondary 55P20
Received December 24, 2003
Revised August 11, 2004
*The author was partially supported by JSPS Research Fellowships for Young Scientists

polynomial

$$\prod_{\alpha \in V^n} (t+\alpha) = \sum_{s=0}^{n} q_{n,s} t^{2^s}, \quad \text{with } q_{n,n} = 1.$$

Then $q_{n,s}$ is invariant under the usual action of $GL_n(\mathbb{F}_2)$ and Dickson [2] has shown that

$$\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)} = \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}].$$

Formally putting $\deg x_i = 1$, we have $\deg q_{n,s} = 2^n - 2^s$. Let Σ_{2^n} be the symmetric group of degree 2^n,

$$P_n : H^*(X) \longrightarrow H^{2^n *}(E\Sigma_{2^n} \times_{\Sigma_{2^n}} X^{2^n})$$

the extended power operations of Steenrod [8], and $d_n : B\Sigma_{2^n} \times X \to E\Sigma_{2^n} \times_{\Sigma_{2^n}} X^{2^n}$ the diagonal map. We regard Σ_{2^n} as the group of set automorphisms of E^n, and we obtain the regular embedding $i : E^n \subset \Sigma_{2^n}$ which takes $g \in E^n$ to the permutation induced by $h \mapsto g + h$. Identify V^n by the dual of E^n over $\mathbb{F}_2$. Then we have canonical isomorphisms $H^1(BE^n) \cong V^n$, and $H^*(BE^n) \cong \mathbb{F}_2[x_1, \ldots, x_n]$. Furthermore Mùi [6] has proved $\operatorname{Im} i^* = \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}]$. Now consider the restriction of $d_n^* P_n$

$$H^*(X) \xrightarrow{d_n^* P_n} H^*(B\Sigma_{2^n}) \otimes H^*(X) \xrightarrow{i^* \otimes 1} H^*(BE^n) \otimes H^*(X),$$

which is written by the same symbol $d_n^* P_n$. Actually $\operatorname{Im} d_n^* P_n \subset \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}] \otimes H^*(X)$, and we can define an operation $S_n : H^*(X) \to \mathbb{F}_2[q_{n,0}^{\pm}, \ldots, q_{n,n-1}] \otimes H^*(X)$ by $S_n(x) = q_{n,0}^{-\deg x} d_n^* P_n(x)$. We set $\xi_i[n] = q_{n,i}/q_{n,0}$ and $D[n]_* = \mathbb{F}_2[\xi_1[n], \ldots, \xi_n[n]] \subset \mathbb{F}_2[q_{n,0}^{\pm}, \ldots, q_{n,n-1}]$. Then by [6] we see S_n takes value in $D[n]_* \otimes H^*(X)$.

Now we have four algebras H_*H, S_*, A_* and $D[n]_*$. The purpose of this paper is to give a new proof of theorem of Milnor. In other words, we have showed directly that there exists a Hopf algebra isomorphism

$$\chi_\psi : A_* \longrightarrow H_*H$$

without the usage of S_*. Since the Hopf algebra structure of A_* is easily seen, we can obtain that of H_*H. Hence we have $S_* \cong H_*H$ as a corollary. The key idea to relate those algebras is the notion of unstable multiplicative operations based on a graded ring R_*

$$H^*(X) \longrightarrow H^*(X) \otimes R_*.$$

A multiplicative operation $\omega : H^*(X) \longrightarrow H^*(X) \otimes R_*$ induces the graded algebra homomorphism $\chi_\omega : A_* \to R_*$. Moreover we have the universal multiplicative operation $\psi : H^*(X) \to H^*(X) \otimes H_*H$. Namely, there exists a unique algebra homomorphism $\bar{\omega} : H_*H \to R_*$ which satisfies $(1 \otimes \bar{\omega}) \circ \psi = \omega$

for a multiplicative operation ω. For the above multiplicative operation $S_n : H^*(X) \to H^*(X) \otimes D[n]_*$, we can get the following diagram:

$$\begin{array}{ccc} A_* & \xrightarrow{\chi_\psi} & H_*H \\ & \searrow^{\chi_{S_n}} & \downarrow \bar{S}_n \\ & & D[n]_*. \end{array}$$

Here χ_{S_n} is an isomorphism in low dimensional range for sufficiently large n. Therefore χ_ψ is injective and we see that χ_ψ is an isomorphism by Serre's result [7]. Furthermore we can show that the algebra homomorphism χ_ψ is actually a Hopf algebra homomorphism.

This paper is constructed as follows. We define a multiplicative operation and construct the universal multiplicative operation ψ in Section 2. In Section 3, we recall the definition of the reduced power operation in Steenrod and Epstein [8] and Mùi's results [5] [6], and introduce the multiplicative operation S_n. In Section 4, we construct A_* and $\chi_\omega : A_* \to R_*$ from a multiplicative operation ω over R_*. We prove main theorem (Theorem 4.2). In appendix, we determine a coproduct of certain elements in the algebra $D_{*,*} = \prod_n D[n]_*$, and consider relations to A_* and H_*H.

This paper is written under the supervision of Professor Goro Nishida. I am grateful to him for his extraordinary patience, generous help and inspiring guidance.

2. Multiplicative operation

We assume that X and Y are spaces, and denote by $H^*(X)$ the mod 2 ordinary cohomology of X in this paper.

Definition 2.1. Let R_* be a non-negatively graded commutative algebra over $\mathbb{F}_2$, namely $R_* = 0$ for $* < 0$. We consider a graded module in which the cohomological degree k-part is $\prod_{n \geq 0} H^{k+n}(X) \otimes R_n$. By abuse of notation we denote the graded module by $H^*(X) \otimes R_*$. We call a natural operation $\beta : H^*(X) \to H^*(X) \otimes R_*$ with cohomological degree preserving multiplicative when β satisfies the following conditions:

(i) The diagram

$$\begin{array}{ccccc} H^*(X) \otimes H^*(Y) & & \xrightarrow{\quad\times\quad} & & H^*(X \times Y) \\ \downarrow{\scriptstyle \beta\otimes\beta} & & & & \downarrow{\scriptstyle \beta} \\ \begin{array}{c} H^*(X) \otimes R_* \\ \otimes H^*(Y) \otimes R_* \end{array} & \xrightarrow{1\otimes\mu\otimes 1} & H^*(X) \otimes H^*(Y) \otimes R_* \otimes R_* & \xrightarrow{(\times)\otimes m} & H^*(X \times Y) \otimes R_*. \end{array}$$

is commutative, where $\times$ is the cross product, μ interchanges the first and second factors, and m is the multiplication on R_*.

(ii) $\beta(u) = u \otimes 1$ where u is the generator of $H^1(S^1)$.

Let $\widetilde{H}^*()$ be the reduced cohomology. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{H}^*(X) & \longrightarrow & H^*(X) & \longrightarrow & H^*(pt) & \longrightarrow & 0 \\
 & & & & \beta\big\downarrow & & \big\downarrow\beta & & \\
0 & \longrightarrow & \widetilde{H}^*(X)\otimes R_* & \longrightarrow & H^*(X)\otimes R_* & \longrightarrow & H^*(pt)\otimes R_* & \longrightarrow & 0,
\end{array}$$

with the horizontal sequences exact. Then we can define the reduced operation $\tilde{\beta} : \widetilde{H}^*(X) \to \widetilde{H}^*(X) \otimes R_*$ such that the above diagram is commutative. Obviously $\tilde{\beta}$ is natural and the following diagram is commutative:

(1)

$$\begin{array}{ccccc}
\widetilde{H}^*(X)\otimes\widetilde{H}^*(Y) & & \xrightarrow{\quad\wedge\quad} & & \widetilde{H}^*(X\wedge Y) \\
\tilde{\beta}\big\downarrow & & & & \big\downarrow\tilde{\beta} \\
\begin{array}{l}\widetilde{H}^*(X)\otimes R_* \\ \quad\otimes\widetilde{H}^*(Y)\otimes R_*\end{array} & \xrightarrow{1\otimes\mu\otimes 1} & \widetilde{H}^*(X)\otimes\widetilde{H}^*(Y)\otimes R_*\otimes R_* & \xrightarrow{(\wedge)\otimes m} & \widetilde{H}^*(X\wedge Y)\otimes R_*,
\end{array}$$

where $\wedge$ is the smash product.

Lemma 2.2. *For a multiplicative operation β, $\tilde{\beta}$ is stable. That is, the following diagram is commutative*:

$$\begin{array}{ccc}
\widetilde{H}^n(X) & \xrightarrow{\sigma} & \widetilde{H}^{n+1}(\Sigma X) \\
\tilde{\beta}\big\downarrow & & \big\downarrow\tilde{\beta} \\
[\widetilde{H}^*(X)\otimes R_*]^n & \xrightarrow{\sigma\otimes 1} & [\widetilde{H}^*(\Sigma X)\otimes R_*]^{n+1},
\end{array}$$

where σ is the suspension isomorphism.

Proof. By the commutative diagram (1), we have the following commutative diagram:

$$\begin{array}{ccccc}
\widetilde{H}^*(X)\otimes\widetilde{H}^*(S^1) & & \xrightarrow{\quad\wedge\quad} & & \widetilde{H}^*(X\wedge S^1) \\
\tilde{\beta}\otimes\tilde{\beta}\big\downarrow & & & & \big\downarrow\tilde{\beta} \\
\begin{array}{l}\widetilde{H}^*(X)\otimes R_* \\ \quad\otimes\widetilde{H}^*(S^1)\otimes R_*\end{array} & \xrightarrow{1\otimes\mu\otimes 1} & \widetilde{H}^*(X)\otimes\widetilde{H}^*(S^1)\otimes R_*\otimes R_* & \xrightarrow{\wedge\otimes m} & \widetilde{H}^*(X\wedge S^1)\otimes R_*.
\end{array}$$

For any element x in $\widetilde{H}^*(X)$,

$$\begin{aligned}
\tilde{\beta}(x\wedge u) &= (\wedge\otimes m)\circ(1\otimes\mu\otimes 1)\circ(\tilde{\beta}(x)\otimes\tilde{\beta}(u)) \\
&= (\wedge\otimes m)\circ(1\otimes\mu\otimes 1)(\tilde{\beta}(x)\otimes u\otimes 1) \\
&= \tilde{\beta}(x)\wedge u,
\end{aligned}$$

where $\tilde{\beta}(x)\wedge u = \sum_n (y_n\wedge u)\otimes\alpha_n$ for $\tilde{\beta}(x) = \sum_n y_n\otimes\alpha_n$. This implies that $\tilde{\beta}$ is a stable operation. □

Let H be the mod 2 Eilenberg-MacLane spectrum, and H_*H be $\pi_*(H \wedge H)$. We want to introduce a multiplicative operation $\psi : H^*(X) \to H^*(X) \otimes H_*H$. We define a map

$$\bar{\psi} : H^*(X) = [X,\ H]^* \longrightarrow [X,\ H \wedge H]^*$$

by $\bar{\psi}(f) = i \wedge f \in [S^0 \wedge X,\ H \wedge H]^*$, where $i : S^0 \to H$ is the unit map.

Let κ be the map

$$\kappa : H^*(X) \otimes H_*H \longrightarrow [X,\ H \wedge H]^*$$

induced by $H \wedge (H \wedge H) \xrightarrow{m \wedge 1} H \wedge H$, where m is the multiplication on H.

Lemma 2.3. *κ is an isomorphism.*

Proof. If $X = S^n$, κ is an isomorphism. Therefore if $H^*(X) \otimes H_*H$ is a cohomology, κ is a cohomology operation. Because H_nH is finite dimensional, we have the result. □

Therefore $\kappa^{-1}\bar{\psi} : H^*(X) \to H^*(X) \otimes H_*H$ is well-defined and it is denoted by ψ.

Theorem 2.4. *The operation $\psi : H^*(X) \longrightarrow H^*(X) \otimes H_*H$ is multiplicative.*

Proof. The map $i \wedge 1 : S^0 \wedge H \to H \wedge H$ is a ring spectra map. Therefore $\bar{\psi} : H^*(X) \to (H \wedge H)^*(X)$ preserves the external product. Since the multiplication $m : H \wedge H \to H$ is a ring spectra map, $m \wedge 1 : H \wedge H \wedge H \to H \wedge H$ is so. Therefore we see that $\kappa : H^*(X) \otimes H_*H \to (H \wedge H)^*(X)$ preserves the external product. Hence ψ satisfies Definition 2.1 (i).

Next we prove that ψ satisfies Definition 2.1 (ii). It is enough to prove for $u = \Sigma i$. Since $\Sigma i \wedge i = i \wedge \Sigma i$ in $[S^1,\ \Sigma(H \wedge H)]$, we see

$$\psi(\Sigma i) = \kappa^{-1} \circ \bar{\psi}(\Sigma i) = \kappa^{-1}(i \wedge \Sigma i) = \kappa^{-1}(\Sigma i \wedge i) = u \otimes 1.$$

□

From now on, we assume any graded algebra R_* is of finite type, that is R_n is finite dimensional for each n. We define $\mathrm{Op}(R_*)$ by the set of all multiplicative operations over R_*. This is a covariant functor from the category of graded algebras over $\mathbb{F}_2$ to the category of sets. We now construct a natural transformation

$$\lambda : \mathrm{Op}(R_*) \longrightarrow \mathrm{Hom}_{\mathbb{F}_2}(H_*H, R_*),$$

where $\mathrm{Hom}_{\mathbb{F}_2}(\ ,\ )$ is the set of all graded linear homomorphisms.

Since $H^*(X) \otimes R_*$ is a cohomology theory in the same way as the proof of Lemma 2.3, we denote the spectrum which represents the cohomology $H^*(\ ) \otimes$

R_* by HR_*. Obviously HR_* is a commutative ring spectrum and an H-module spectrum induced by the products

$$H^*(X) \otimes R_* \otimes H^*(Y) \otimes R_* \longrightarrow H^*(X \times Y) \otimes R_*,$$
$$(x \otimes r \otimes y \otimes r' \mapsto (x \times y) \otimes r \cdot r'),$$

and

$$H^*(X) \otimes (H^*(Y) \otimes R_*) \longrightarrow H^*(X \times Y) \otimes R_*,$$
$$(x \otimes y \otimes r \mapsto (x \times y) \otimes r).$$

Under these conditions, we have

$$\bar{\lambda} : (HR_*)^* H \xrightarrow{\cong} \mathrm{Hom}^*_{\mathbb{F}_2}(H_* H, R_*),$$

from [1, III, 13.5]. This map is defined by $H \wedge H \xrightarrow{1 \wedge x} H \wedge HR_* \xrightarrow{\tau} HR_*$, where $x \in [H,\ HR_*]$ and the H-module map $\tau : H \wedge HR_* \to HR_*$. For an element α in $(HR_*)^* H$, we write $\bar{\lambda}(\alpha)$ as $\bar{\alpha}$.

Let $\beta : H^*(X) \to H^*(X) \otimes R_*$ be a multiplicative operation. Since $\tilde{\beta}$ is stable by Lemma 2.2, we can identify β as a stable cohomology operation. Therefore $\mathrm{Op}(R_*)$ is a subset in $(HR_*)^0 H$. β satisfies the following commutative diagram:

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\psi} & H^*(X) \otimes H_* H \\ & \searrow^{\beta} & \downarrow{\scriptstyle 1 \otimes \bar{\beta}} \\ & & H^*(X) \otimes R_* \end{array}$$

from the commutative diagram:

$$\begin{array}{ccc} S^0 \wedge H & \xrightarrow{i \wedge 1} & H \wedge H \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle 1 \wedge x} \\ HR_* & \xleftarrow{\tau} & H \wedge HR_*. \end{array}$$

Here x is a spectra map which represents β. We define λ by the restriction of $\bar{\lambda}$ to $\mathrm{Op}(R_*)$. Since $\mathrm{Op}(R_*) \subset (HR_*)^0 H$, the image of λ is actually included in $\mathrm{Hom}_{\mathbb{F}_2}(H_* H, R_*)$.

Theorem 2.5. *Let* $\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(\ ,\ )$ *be the set of all graded algebra homomorphisms. Then there is an one to one correspondence*

$$\lambda : \mathrm{Op}(R_*) \longrightarrow \mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(H_* H, R_*)$$

which is natural in R_*.

Proof. We now prove $\lambda(\mathrm{Op}(R_*)) \subset \mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(H_*H, R_*)$. It is enough to prove that the following diagram is commutative:

$$
(2) \qquad \begin{array}{ccc}
H \wedge H \wedge H \wedge H & \xrightarrow{(m\wedge m)\circ(1\wedge\mu\wedge 1)} & H \wedge H \\
{\scriptstyle 1\wedge x\wedge 1\wedge x}\downarrow & & \downarrow{\scriptstyle 1\wedge x} \\
H \wedge HR_* \wedge H \wedge HR_* & \xrightarrow{(m\wedge m_{R_*})\circ(1\wedge\mu\wedge 1)} & H \wedge HR_* \\
{\scriptstyle \tau\wedge\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\
HR_* \wedge HR_* & \xrightarrow{m_{R_*}} & HR_*,
\end{array}
$$

where β is a multiplicative operation, x represents β, and m_{R_*} is the multiplication on HR_*. Because β is a multiplicative operation, the following diagram is commutative:

$$
\begin{array}{ccc}
H \wedge H & \xrightarrow{x\wedge x} & HR_* \wedge HR_* \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m_{R_*}} \\
H & \xrightarrow{x} & HR_*.
\end{array}
$$

Therefore the upper square in the diagram (2) is commutative. The lower square in (2) is commutative since HR_* is a commutative ring spectrum and $m_{R_*} : HR_* \wedge HR_* \to HR_*$ is an H-module spectra map.

For any r in $\mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(H_*H, R_*)$, the operation

$$(1 \otimes r) \circ \psi : H^*(X) \longrightarrow H^*(X) \otimes H_*H \longrightarrow H^*(X) \otimes R_*$$

is multiplicative. This shows $\lambda(\mathrm{Op}(R_*)) = \mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(H_*H, R_*)$. □

3. Construction of the reduced power

Let G be a subgroup of the symmetric group Σ_m of degree m. For a space X, G acts on X^m as a permutation. Steenrod defined the extended power operation

$$P_G : H^q(X) \to H^{mq}(E_G(X)),$$

where $E_G(X)$ is defined by $EG \times_G X^m$ [8, VII]. From the diagonal map $d_G : BG \times X \to E_G(X)$, we have the natural map $d_G^* P_G : H^q(X) \to H^{mq}(BG \times X)$.

Let E^n be the elementary abelian 2-group with dimension n and we write $E^n = E_1 \times \cdots \times E_n$, where $E_i = \mathbb{Z}/2$. Then we can identify $\mathrm{Aut}_{\mathrm{Set}}(E^n)$, the set of all permutations of the set E^n, as Σ_{2^n}. Since E^n acts on itself as a vector space, there is the regular embedding $E^n \subset \Sigma_{2^n}$. The wreath product $E_1 \int \cdots \int E_n$ is a 2-sylow subgroup of Σ_{2^n}, and it is denoted by $\Sigma_{2^n,2}$. Obviously, $\Sigma_{2^n,2}$ contains E^n. We define an inclusion $E^{n-1} \subset E^n$ by $E^{n-1} \cong \{0\} \times E_2 \times \cdots \times E_n \subset E^n$. Then it induces the inclusion $\Sigma_{2^{n-1},2} \subset E_1 \int \Sigma_{2^{n-1},2} = \Sigma_{2^n,2}$.

From $i_{G,G'} : G' \subset G$, we have three maps $BG' \to BG$, $E_{G'}(X) \to E_G(X)$, and $BG' \times X \to BG \times X$. They induce $H^*(BG) \to H^*(BG')$, $H^*(E_G(X)) \to$

$H^*(E_{G'}(X))$ and $H^*(BG \times X) \to H^*(BG' \times X)$, which are denoted by the same symbol $i^*_{G,G'}$. Since we see $i^*_{G,\ G'} P_G = P_{G'}$ by [8, VII, 2.5], we obtain

$$\text{(3)} \qquad \begin{aligned} i^*_{\Sigma_{2^n},\ E^n} d^*_{\Sigma_{2^n}} P_{\Sigma_{2^n}} &= i^*_{\Sigma_{2^n,2},\ E^n} d^*_{\Sigma_{2^n,2}} P_{\Sigma_{2^n,2}} \\ &= d^*_n P_n : H^q(X) \to H^{2^n q}(BE^n \times X), \end{aligned}$$

where $d_n = d_{E^n}$ and $P_n = P_{E^n}$. We can identify $P_{E_1} P_{\Sigma_{2^{n-1},2}}$ with $P_{\Sigma_{2^n,2}}$ from the following commutative diagram:

$$\begin{array}{ccc} H^*(X) & \xrightarrow{P_{\Sigma_{2^n,2}}} & H^*(E_{\Sigma_{2^n,2}}(X)) \\ {\scriptstyle P_{\Sigma_{2^{n-1},2}}}\downarrow & & \downarrow\cong \\ H^*(E_{2^{n-1},2}(X)) & \xrightarrow{P_1} & H^*(E_{E_1}(E_{\Sigma_{2^{n-1},2}}(X))) \cong H^*(E_{E_1 \int \Sigma_{2^{n-1},2}}(X)). \end{array}$$

By the naturality of P, the following diagram is commutative:

$$\begin{array}{ccc} H^*(X) & & \\ {\scriptstyle P_{\Sigma_{2^{n-1},2}}}\downarrow & & \\ H^*(E_{\Sigma_{2^{n-1},2}}(X)) & \xrightarrow{d^*_{n-1}} & H^*(BE^{n-1} \times X) \\ {\scriptstyle P_1}\downarrow & & \downarrow{\scriptstyle P_1} \\ H^*(E_{E_1}(E_{\Sigma_{2^{n-1},2}}(X))) & \xrightarrow{E_{E_1}(d_{n-1})^*} & H^*(E_{E_1}(BE^{n-1} \times X)) \\ {\scriptstyle d^*_1}\downarrow & & \downarrow{\scriptstyle d^*_1} \\ H^*(BE_1 \times E_{\Sigma_{2^{n-1},2}}(X)) & \xrightarrow{(1\times d_{n-1})^*} & H^*(BE^n \times X). \end{array}$$

Hence we see the following lemma:

Lemma 3.1 ([8]). *We have*

$$d^*_n P_n = d^*_1 P_1 d^*_{n-1} P_{n-1}.$$

Given the diagonal maps

$$\lambda : EE^n \times_{E^n} (X \times Y)^{2^n} \longrightarrow EE^n \times EE^n \times_{E^n \times E^n} X^{2^n} \times Y^{2^n},$$

and $\quad d' : BE^n \times X \times Y \longrightarrow BE^n \times BE^n \times X \times Y,$

we obtain the following maps:

$$H^*(X) \otimes H^*(Y) \overset{P_n \times P_n}{\longrightarrow} H^*(EE^n \times_{E^n} X^{2^n}) \otimes H^*(EE^n \times_{E^n} Y^{2^n})$$
$$\xrightarrow{\times} H^*(EE^n \times EE^n \times_{E^n \times E^n} X^{2^n} \times Y^{2^n}) \xrightarrow{\lambda^*} H^*(EE^n \times_{E^n} (X \times Y)^{2^n}),$$

and

$$H^*(X) \otimes H^*(Y) \overset{P_n \times P_n}{\longrightarrow} H^*(EE^n \times_{E^n} X^{2^n}) \otimes H^*(EE^n \times_{E^n} Y^{2^n})$$
$$\overset{d^*_n \times d^*_n}{\longrightarrow} H^*(BE^n \times BE^n \times X \times Y) \xrightarrow{d'} H^*(BE^n \times X \times Y).$$

Lemma 3.2. *We have*

$$d_n^*P_n(u\times v) = {d'}^*(d_n^*\times d_n^*)(P_nu\times P_nv) : H^*(X)\otimes H^*(Y) \longrightarrow H^*(BE^n\times X\times Y).$$

Proof. By Steenrod and Epstein [8, VII, Lemma 2.6], we obtain $\lambda^*(P_nu\times P_nv) = P_n(u\times v)$, where $u\in H^*(X)$ and $v\in H^*(Y)$. The commutative diagram

$$\begin{array}{ccc} BE^n\times X\times Y & \xrightarrow{d_n} & EE^n\times_{E^n}(X\times Y)^{2^n} \\ {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle \lambda} \\ BE^n\times BE^n\times X\times Y & \xrightarrow{d_n\times d_n} & EE^n\times EE^n\times_{E^n\times E^n} X^{2^n}\times Y^{2^n} \end{array}$$

induces

$$\begin{array}{ccc} H^*(BE^n\times X\times Y) & \xleftarrow{d_n^*} & H^*(EE^n\times_{E^n}(X\times Y)^{2^n}) \\ {\scriptstyle {d'}^*}\uparrow & & \uparrow{\scriptstyle \lambda^*} \\ H^*(BE^n\times BE^n\times X\times Y) & \xleftarrow{d_n^*\times d_n^*} & H^*(EE^n\times EE^n\times_{E^n\times E^n} X^{2^n}\times Y^{2^n}). \end{array}$$

Therefore

$$d_n^*P_n(u\times v) = d_n^*\lambda^*(P_nu\times P_nv) = {d'}^*(d_n^*\times d_n^*)(P_nu\times P_nv).$$

□

We recall that $H^*(BE^n) = \mathbb{F}_2[x_1,\dots,x_n]$, where each x_i is of degree 1. It is well known by Mùi [5] that

(4) $\mathrm{Im}(i^*_{\Sigma_{2^n},\ E^n}) = \mathbb{F}_2[x_1,\dots,x_n]^{GL_n(\mathbb{F}_2)}$, $\mathrm{Im}(i^*_{\Sigma_{2^n,2},\ E^n}) = \mathbb{F}_2[x_1,\dots,x_n]^{T_n}$,

where T_n the upper triangular subgroup of $GL_n(\mathbb{F}_2)$. We define v_{k+1} by

$$v_{k+1} = \prod\left(\sum_{i=1}^{k}\lambda_i x_i + x_{k+1}\right),$$

and $q_{n,i}$ by

$$\prod_{\alpha\in E^n}(x+\alpha) = \sum_{s=0}^{n} q_{n,s}x^{2^s} \quad \text{with } q_{n,n}=1. \tag{5}$$

Obviously $\deg q_{n,i} = 2^n - 2^i$. Dickson [2] and Mùi [5] have shown

$$\mathbb{F}_2[x_1,\dots,x_n]^{GL_n(\mathbb{F}_2)} \cong \mathbb{F}_2[q_{n,0},q_{n,1},\dots,q_{n,n-1}],$$
$$\mathbb{F}_2[x_1,\dots,x_n]^{T_n} \cong \mathbb{F}_2[v_1,\dots,v_n].$$

Furthermore the following relations between $q_{n,i}$ and v_i are known.

Theorem 3.3 ([5]). *We have*

$$q_{n,j} = q_{n-1,j}v_n + q_{n-1,j-1}^2,$$

where $q_{n,j} = 0$ *for* $j < 0$ *or* $n < j$.

We need the following definition and theorem in [8].

Definition 3.4 ([8] VII 3.2). Suppose $H^*(BE^1) = \mathbb{F}_2[x]$ and $u \in H^q(X)$. Then we can write $d_1^*P_1(u) = \sum_k x^k \times Sq^{q-k}(u)$, where

$$Sq^k : H^q(X) \longrightarrow H^{q+k}(X).$$

Theorem 3.5 ([8] VII 4.3, 4.4, 3.4). *For each* k, Sq^k *is a homomorphism. If* $u \in H^q(X)$, *then* $Sq^k(u) = 0$ *for* $k < 0$, $Sq^0(u) = u$ *and* $Sq^q(u) = u^2$.

We now consider

$$d_1^*P_1 : H^*(BE^n) \to H^*(BE^1 \times BE^n). \tag{6}$$

Obviously $BE^1 \times BE^n = BE^{n+1}$. Since

$$E^n \cong \{0\} \times E_2 \times \cdots \times E_{n+1} \subset E^{n+1} = E_1 \times \cdots \times E_{n+1},$$

we identify BE^1 as BE_1 and BE^n as $B(E_2 \times \cdots \times E_{n+1})$ in (6).

Theorem 3.6 ([6] Theorem 1.5). *Define an element* v'_n *by*

$$v'_n = \prod_{\lambda_i=0,1} \left(\sum_{i=2}^{n} \lambda_i x_i + x_{n+1} \right) \quad in \quad H^*(BE^n) = \mathbb{F}_2[x_2, \ldots, x_{n+1}].$$

Then we have

$$d_1^*P_1(v'_n) = v_{n+1}, \quad in \quad H^*(BE^{n+1}).$$

Especially

$$d_n^*P_n x_{n+1} = v_{n+1},$$

where $H^*(BE_{n+1}) = \mathbb{F}_2[x_{n+1}]$ *and* $d_n^*P_n : H^*(BE_{n+1}) \to H^*(B(E_1 \times \cdots \times E_n) \times BE_{n+1})$.

Proof. By Theorem 3.5, we have $d_1^*P_1(u) = 1 \times u^2 + x_1 \times u$ for $u \in H^1(X)$. By Lemma 3.2, we have

$$\begin{aligned} d_1^*P_1(v'_n) &= \prod_{\lambda_i=0,1} d_1^*P_1 \left(\sum_{i=2}^{n} \lambda_i x_i + x_{n+1} \right) \\ &= \prod_{\lambda_i=0,1} \left(\sum_{i=2}^{n} \lambda_i x_i + x_{n+1} \right) \left(x_1 + \sum_{i=2}^{n} \lambda_i x_i + x_{n+1} \right) \\ &= v_{n+1}. \end{aligned}$$

The second claim is obvious by Lemma 3.1. □

From (3) and (4), the image of $d_n^* P_n$ is included in $\mathbb{F}_2[x_1,\dots,x_n]^{GL_n(\mathbb{F}_2)} \otimes H^*(X)$. For any $u \in H^q(X)$, we can denote $d_n^* P_n u$ by

$$d_n^* P_n u = \sum_{R=(r_0,\dots r_{n-1})} q_{n,0}^{r_0} q_{n,1}^{r_1} \cdots q_{n,n-1}^{r_{n-1}} \otimes \mathcal{D}_R u, \tag{7}$$

where $\mathcal{D}_R : H^q(X) \to H^{2^q-|R|}(X)$ with $|R| = \sum_{s=0}^{n-1} r_s(2^n - 2^s)$.

Lemma 3.7 ([6], Lemma 2.3). *$\mathcal{D}_R u = 0$ if $q < r_0 + r_1 + \cdots + r_{n-1}$.*

Proof. We now prove by the induction on n. In the case of $n = 1$, it is obvious by Definition 3.4 and Theorem 3.5. We assume that the lemma is true for $n = k - 1$. We consider the case of $n = k$. By Lemma 3.1 we have

$$d_k^* P_k(u) = d_1^* P_1 d_{k-1}^* P_{k-1} u = \sum_{i=0}^{2^{k-1}q} x_1^{2^{k-1}q-i} Sq^i(d_{k-1}^* P_{k-1} u).$$

So the degree of $d_k^* P_k(u)$ in x_1 is equal to $2^{k-1}q$. From Theorem 3.3, $\deg_{x_1} q_{k,s} = \deg_{x_k} q_{k,s} = 2^{k-1}$. From the equality (7), we must have

$$2^{k-1}(r_1 + \cdots + r_{k-1}) \le 2^{k-1}q.$$

Therefore the lemma is true. □

Let $P_n = \mathbb{F}_2[x_1,\dots, x_n](= H^*(BE^n))$, $e_n = \prod(\sum_{i=1}^n \lambda_i x_i) \in P_n$, ($\lambda_i = 0$ or 1, $\sum \lambda_i > 0$), and $\Phi_n = P_n[e_n^{-1}]$. Then there exists the natural action of $GL_n(\mathbb{F}_2)$ on P_n and Φ_n. Define $\Delta_n = \Phi_n^{T_n}$ and $\Gamma_n = \Phi_n^{GL_n}$, where Φ_n^K is the subalgebra of the invariants of K in Φ_n for $K = T_n$ or GL_n. We set $w_{k+1} = v_{k+1}/e_k$. It is easily seen that

$$\Delta_n = \mathbb{F}_2[v_1^{\pm 1},\dots, v_n^{\pm 1}] \cong \mathbb{F}_2[w_1^{\pm 1},\dots, w_n^{\pm 1}],\ \Gamma_n = \mathbb{F}_2[q_{n,0}^{\pm 1}, q_{n,1},\dots, q_{n,n-1}].$$

Let $S_n : H^*(X) \to \Phi_n \otimes H^*(X)$ be the map which sends x to $q_{n,0}^{-\deg(x)} d_n^* P_n(x)$. From the definition, S_n preserves cohomological degree. It is the same as the definition of S_n by Lomonaco [3] substantially.

Let $D[n]_*$ be the subalgebra generated by $\xi_1[n], \xi_2[n], \dots, \xi_n[n]$ in Φ_n, where $\xi_i[n] = q_{n,i}/q_{n,0}$. It is easily seen that $\xi_i[n]$ is an element in $D[n]_{2^i-1}$ and $D[n]_* = \mathbb{F}_2[\xi_1[n],\dots,\xi_n[n]]$.

Corollary 3.8. *Suppose $H^*(B\mathbb{Z}/2) = \mathbb{F}_2[x]$. Then we have*

$$S_n(x) = \sum_{s=0}^n \xi_s[n] x^{2^s}.$$

Proof. From the definition of v_n and the equality (5), we have $v_{n+1} = \sum_{s=0}^n q_{n,s} x_{n+1}^{2^s}$. By Theorem 3.6 and the definition of S_n, we have $S_n(x) = \sum_{s=0}^n \xi_s[n] x^{2^s}$. □

Lemma 3.9. $\mathrm{Im}(S_n) \subset D[n]_* \otimes H^*(X)$.

Proof. Trivial by Lemma 3.7. □

We consider the operation $H^*(X) \xrightarrow{S_n} D[n]_* \otimes H^*(X) \to H^*(X) \otimes D[n]_*$, where the second map interchanges the first and second factors, and denote it by the same symbol S_n.

Lemma 3.10. *The cohomology operation S_n is multiplicative. That is, the following diagram is commutative*:

$$\begin{array}{ccccc} H^*(X) \otimes H^*(Y) & & \xrightarrow{\times} & & H^*(X \times Y) \\ \downarrow{\scriptstyle S_n \otimes S_n} & & & & \downarrow{\scriptstyle S_n} \\ \begin{array}{c} H^*(X) \otimes D[n]_* \\ \otimes H^*(Y) \otimes D[n]_* \end{array} & \xrightarrow{1\otimes\mu\otimes 1} & \begin{array}{c} H^*(X) \otimes H^*(Y) \\ \otimes D[n]_* \otimes D[n]_* \end{array} & \xrightarrow{\times\otimes m} & H^*(X \times Y) \otimes D[n]_*. \end{array}$$

Proof. By Lemma 3.2, it is obvious. □

4. The relation between H_*H and $\mathrm{Aut}_{\mathbb{F}_2} G_a$

Let G_a be the additive formal group law and $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ the set of all strict automorphisms of G_a over a graded $\mathbb{F}_2$-algebra R_*. Then $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(-)$ is a functor from the category of graded algebras to the category of sets. An element in $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is a power series $f(x) \in R_*[[x]]$ satisfying the following three conditions: (i) $f(x+y) = f(x) + f(y)$; (ii) the coefficient of x in $f(x)$ is equal to 1; (iii) that of x^k is an element in R_{k-1}. Therefore for $f(x) \in \mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ we have

$$f(x) = x + a_1x^2 + a_2x^4 + \cdots + a_mx^{2^m} + \cdots, \quad \text{where } a_i \in R_{2^i-1}.$$

Let A_* be the graded polynomial algebra generated by $\{\bar{\xi}_1, \ldots, \bar{\xi}_n, \ldots\}$ with $\bar{\xi}_i \in A_{2^i-1}$. Such a power series is represented by a graded $\mathbb{F}_2$-algebra homomorphism

$$\chi : A_* = \mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \ldots] \longrightarrow R_*$$

defined by $\chi(\bar{\xi}_i) = a_i$, and we have the natural isomorphism

$$(8) \qquad \mathrm{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R_*) \cong \mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*), \qquad \chi \mapsto \sum_{i=0}^{\infty} \chi(\xi_i)x^{2^i},$$

where $\xi_0 = 1$. A product of $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is defined by $(g \cdot f)(x) = f(g(x))$. Then $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is a group, and thereby $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(-)$ is a functor to the category of groups. This induces the coproduct map $\Delta : A_* \to A_* \otimes A_*$. It is easy to check $\Delta(\bar{\xi}_n) = \sum_{i=0}^{n} \bar{\xi}_{n-i}^{2^i} \otimes \bar{\xi}_i$.

Consider a multiplicative operation $\beta : H^*(X) \to H^*(X) \otimes R_*$. The classifying space $B\mathbb{Z}/2$ is an H-space and the Hopf algebra $H^*(B\mathbb{Z}/2) \cong \mathbb{F}_2[x]$ is nothing but the additive formal group. We can identify $\beta(x)$ as an element in $R_*[[x]]$ and write it by $f_\beta(x)$.

Lemma 4.1. *$f_\beta(x)$ is an element in* $\mathrm{Aut}_{\mathbb{F}_2}(G_a)(R_*)$.

Proof. The product map $a : B\mathbb{Z}/2 \times B\mathbb{Z}/2 \to B\mathbb{Z}/2$ induces the commutative diagram:

$$\begin{array}{ccc} H^*(B\mathbb{Z}/2)\otimes H^*(B\mathbb{Z}/2) & \xrightarrow{\beta\times\beta} & H^*(B\mathbb{Z}/2)\otimes R_*\otimes H^*(B\mathbb{Z}/2)\otimes R_* \\ \times\downarrow & & \downarrow (\times)\times m\circ(1\times\mu\times 1) \\ H^*(B\mathbb{Z}/2\times B\mathbb{Z}/2) & \xrightarrow{\beta} & H^*(B\mathbb{Z}/2\times B\mathbb{Z}/2)\otimes R_* \\ a^*\uparrow & & \uparrow a^* \\ H^*(B\mathbb{Z}/2) & \xrightarrow{\beta} & H^*(B\mathbb{Z}/2)\otimes R_*. \end{array}$$

Therefore we see

$$\beta(x\times1+1\times x) = a^*\circ\beta(x) = \beta\circ a^*(x) = \beta(x\times1)+\beta(1\times x) = \beta(x)\times1+1\times\beta(x).$$

□

Let $\chi_\beta : A_* \to R_*$ be the algebra homomorphism corresponding to $f_\beta(x)$ in (8). For the multiplicative operations ψ in Section 2 and S_n in Section 3, we obtain the algebra homomorphisms $\chi_\psi : A_* \to H_*H$ and $\chi_{S_n} : A_* \to D[n]_*$.

The map $H\wedge S^0\wedge H \xrightarrow{1\wedge i\wedge 1} H\wedge H\wedge H$ induces

$$\delta : H_*H = [S^0,\ H\wedge H]_* \longrightarrow [S^0,\ H\wedge H\wedge H]_* \cong H_*H\otimes H_*H,$$

and H_*H is a Hopf algebra. Then $H^*(X)$ is an H_*H-comodule with $\psi : H^*(X)\to H^*(X)\otimes H_*H$.

Theorem 4.2. *$\chi_\psi : A_* \longrightarrow H_*H$ is a Hopf algebra isomorphism.*

Proof. From Theorem 2.5, there exists a unique algebra homomorphism $\bar{S}_n : H_*H \to D[n]_*$ with the commutative diagram

$$\begin{array}{ccc} H^*(X) & \xrightarrow{\psi} & H^*(X)\otimes H_*H \\ & \searrow^{S_n} & \downarrow 1\otimes\bar{S}_n \\ & & H^*(X)\otimes D[n]_*. \end{array}$$

It induces the following commutative diagram:

$$\begin{array}{ccc} A_* & \xrightarrow{\chi_\psi} & H_*H \\ & \searrow^{\chi_{S_n}} & \downarrow \bar{S_n} \\ & & D[n]_*. \end{array}$$

From Corollary 3.8, χ_{S_n} is defined by $\chi_{S_n}(\bar{\xi_i}) = \xi_i[n]$. For sufficiently large m, there exists a number n such that $\chi_{S_n} : H_*H \to D[n]_*$ is an isomorphism on

$* \leq m$. Therefore χ_ψ is injective. Serre [7, §18, Théorème 3] has shown that the Poincaré series of H^*H and H_*H is equal to $\prod_{i=1}^{\infty} 1/(1-t^{2^i-1})$, which is the same as that of A_*. Hence χ_ψ is bijective.

Next we prove χ_ψ is a Hopf algebra homomorphism. Since ψ is an H_*H-comodule map, the following operation is multiplicative:

$$(\psi \otimes 1) \circ \psi = (1 \otimes \delta) \circ \psi : H^*(X) \to H^*(X) \otimes H_*H \otimes H_*H.$$

Since $(\psi \otimes 1) \circ \psi$ is two iteration of ψ, we see $\chi_{(\psi\otimes 1)\circ\psi} = (\chi_\psi \otimes \chi_\psi) \circ \Delta$. Moreover we obtain $\chi_{(1\otimes\delta)\circ\psi} = \delta \circ \chi_\psi$. Since $(\psi \otimes 1) \circ \psi = (1 \otimes \delta) \circ \psi$, we have the following commutative diagram:

$$\begin{array}{ccc} A_* & \xrightarrow{\Delta} & A_* \otimes A_* \\ {\scriptstyle \chi_\psi}\downarrow & & \downarrow{\scriptstyle \chi_\psi \otimes \chi_\psi} \\ H_*H & \xrightarrow{\delta} & H_*H \otimes H_*H. \end{array}$$

□

5. Appendix

Let $D_{*,*}$ be the bigraded algebra $\prod_{n\geq 0} D[n]_*$ with $D_{m,n} = D[n]_m$. In this appendix, we define a coproduct of some elements in $D_{*,*}$, and construct algebra homomorphisms $\chi_D : A_* \to D_{*,*}$ and $\bar{S} : H_*H \to D_{*,*}$ which preserve coproducts.

First we study a coproduct of $D[n]_*$. Define an algebra homomorphism $\delta_{m,n} : \Delta_{n+m} \to \Delta_m \otimes \Delta_n$ by

$$\delta_{m,n}(w_i) = \begin{cases} w_i \otimes 1 & \text{if } 0 \leq i \leq m, \\ 1 \otimes w_{i-m} & \text{if } m+1 \leq i \leq n+m. \end{cases}$$

Lemma 5.1. $\delta_{m,n}(\xi_j[n+m]) = \sum_{0\leq j\leq i} \xi_{i-j}^{2^j}[m] \otimes \xi_j[n]$. *Especially* $\delta_{m,n}(D[n+m]_*) \subset D[m]_* \otimes D[n]_*$.

Proof. We prove the lemma by induction on $n+m$. For $n+m=1$, it is trivial. We now assume that the lemma is true for $n+m \leq k$. For $n+m=k+1$, we consider only the map $\delta_{n,k-n+1}$ because the map $\delta_{k+1,0}$ is trivial. From Theorem 3.3 and $q_{n,0} = v_1 \cdots v_n$,

$$\begin{aligned} \xi_j[n] &= q_{n,0}^{-1}(q_{n-1,j}v_n + q_{n-1,j-1}^2) \\ &= \frac{q_{n-1,j}v_n + q_{n-1,j-1}^2}{v_1 v_2 \cdots v_n} \\ &= \xi_j[n-1] + \xi_{j-1}[n-1]^2 w_n^{-1}. \end{aligned}$$

By this equality, we have

$$\begin{aligned} &\delta_{n,k-n+1}(\xi_j[k+1]) \\ &\quad = \delta_{n,k-n+1}(\xi_j[k] + \xi_{j-1}[k]^2 w_{k+1}^{-1}) \\ &\quad = \delta_{n,k-n}(\xi_j[k]) + \delta_{n,k-n}(\xi_{j-1}[k])^2 \delta_{n,k-n+1}(w_{k+1})^{-1}. \end{aligned}$$

By the induction hypothesis, this is equal to

$$\begin{aligned}
&\sum_{0\le i\le j} \xi_{j-i}^{2^i}[n]\otimes\xi_i[k-n] + \sum_{0\le i'\le j-1} \xi_{j-1-i'}^{2^{i'+1}}[n+1]\otimes\xi_{i'}^2[k-n]w_{k-n+1}^{-1}\\
&\quad = \sum_{0\le i\le j} \xi_{j-i}^{2^i}[n]\otimes\xi_i[k-n] + \sum_{0\le i''\le j} \xi_{j-i''}^{2^{i''}}[n]\otimes\xi_{i''-1}^2[k-n]w_{k-n+1}^{-1}\\
&\quad = \sum_{0\le i\le j} \xi_{j-i}^{2^i}[n]\otimes\xi_i[k-n+1].
\end{aligned}$$

Therefore we have the lemma. □

From Lemma 5.1, we have obtained the coproduct $\delta_{m,n} : D[n+m]_* \to D[m]_*\otimes D[n]_*$. Next we investigate the multiplicative operation $S_n : H^*(X) \to H^*(X)\otimes D[n]_*$.

Lemma 5.2. *For $u \in H^q(X)$, we have*

$$d_n^* P_n(u) = \sum_{i_1,i_2,\ldots,i_n} v_1^{c_1} v_2^{c_2}\cdots v_n^{c_n} \times Sq^{i_1}\cdots Sq^{i_n}(u),$$

where $0 \le i_k \le q + \sum_{j=k+1}^n i_j$ and $c_k = q - i_k + \sum_{j=k+1}^n i_j$ for any $1 \le k \le n$.

Proof. We prove by induction on n. For $n = 1$, it is trivial by the definition of $d_1^*P_1$. We now assume that the lemma is true for $n \le k$. For $k+1$, we use the equality $d_{k+1}^*P_{k+1} = d_1^*P_1 d_k^*P_k$ by Lemma 3.1. Then we have

$$\begin{aligned}
&d_1^*P_1 d_k^*P_k(u)\\
&\quad = d_1^*P_1\left(\sum_{i_2,i_3,\ldots,i_{k+1}} v_1^{c_2}v_2^{c_3}\cdots v_k^{c_{k+1}} \times Sq^{i_2}\cdots Sq^{i_{k+1}}(u)\right)\\
&\quad = \sum_{i_2,\ldots,i_{k+1}} d_1^*P_1(v_1')^{c_2}\cdots d_1^*P_1(v_k')^{c_{k+1}} \times d_1^*P_1(Sq^{i_2}\cdots Sq^{i_{k+1}}(u))\\
&\quad = \sum_{i_2,\ldots,i_{k+1}} (v_2)^{c_2}\cdots(v_{k+1})^{c_{k+1}}\left(\sum_{i_1} v_1^{q+i_2+\cdots+i_{k+1}-i_1} \times Sq^{i_1}(Sq^{i_2}\cdots Sq^{i_{k+1}}(u))\right).
\end{aligned}$$

We have the first equality by the induction hypothesis, the second equality by Steenrod and Epstein [8, VII, 2.6] and the naturality of d_1, and the third equality by Theorem 3.6. By $\deg(Sq^{i_2}\cdots Sq^{i_{k+1}}(u))) = q + \sum_{j=2}^{k+1} i_j$, we have $0 \le i_1 \le q + \sum_{j=2}^{k+1} i_j$. □

Corollary 5.3. *For $u \in H^q(X)$, we have*

$$S_n(u) = \sum_{i_1,i_2,\ldots,i_n} Sq^{i_1}Sq^{i_2}\cdots Sq^{i_n}(u) \times w_1^{-i_1}w_2^{-i_2}\cdots w_n^{-i_n},$$

where $0 \le i_k \le q + \sum_{j=k+1}^n i_j$ for any $0 \le k \le n$.

Proof. By the definition of S_n and Lemma 5.2, we see

$$\begin{aligned} S_n(u) &= q_{n,0}^{-q} d_n^* P_n(u) \\ &= (v_1 \cdots v_n)^{-q} \sum_{i_1, i_2, \ldots, i_n} v_1^{c_1} v_2^{c_2} \cdots v_n^{c_n} \times Sq^{i_1} \cdots Sq^{i_n}(u) \\ &= \sum_{i_1, i_2, \ldots, i_n} w_1^{-i_1} \cdots w_n^{-i_n} \times Sq^{i_1} \cdots Sq^{i_n}(u). \end{aligned}$$

□

Here is a theorem which describes a relation between two iteration of S_n and $\delta_{m,n}$.

Theorem 5.4.

$$(S_m \otimes id_{D[n]_*}) \circ S_n = (id_{H^*(X)} \otimes \delta_{m,n}) \circ S_{n+m} : H^*(X) \to H^*(X) \otimes D[m]_* \otimes D[n]_*.$$

Proof. Let u be an element in $H^q(X)$. From the definitions of S_n and $\delta_{m,n}$, and Corollary 5.3, we obtain

$$\begin{aligned} &(S_m \otimes id_{D[n]_*}) \circ S_n(u) \\ &= (S_m \otimes id_{D[n]_*}) \sum_{i_1, i_2, \ldots, i_n} Sq^{i_1} \cdots Sq^{i_n}(u)) \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \\ &= \sum_{i_1, i_2, \ldots, i_n} S_m(Sq^{i_1} \cdots Sq^{i_n}(u)) \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \\ &= \sum_{i_1, i_2, \ldots, i_n} \left[\left(\sum_{j_1, \ldots, j_m} Sq^{j_1} \cdots Sq^{j_m} Sq^{i_1} \cdots Sq^{i_n}(u) \right) \times w_1^{-j_1} \cdots w_m^{-j_m} \right. \\ &\qquad \left. \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \right] \end{aligned}$$

Since

$$S_{n+m}(u) = \sum_{i_1, \ldots, i_{n+m}} Sq^{i_1} \cdots Sq^{i_{n+m}}(u) \times w_1^{-i_1} \cdots w_{m+n}^{-i_{m+n}},$$

$$\delta_{m,n}(w_1^{-i_1} w_2^{-i_2} \cdots w_{n+m}^{-i_{n+m}}) = w_1^{-i_1} \cdots w_m^{-i_m} \otimes w_1^{-i_{m+1}} \cdots w_n^{-i_{m+n}},$$

we have the result. □

In the same way as the proof of Theorem 4.2, we have the following two commutative diagrams:

(9)

$$\begin{array}{ccc} A_* & \xrightarrow{\Delta} & A_* \otimes A_* \\ {\scriptstyle \chi_{S_{n+m}}}\downarrow & & \downarrow{\scriptstyle \chi_{S_m} \otimes \chi_{S_n}} \\ D[n+m]_* & \xrightarrow{\delta_{m,n}} & D[m]_* \otimes D[n]_*, \end{array} \quad \text{and} \quad \begin{array}{ccc} H_*H & \xrightarrow{\delta} & H_*H \otimes H_*H \\ {\scriptstyle \bar{S}_{n+m}}\downarrow & & \downarrow{\scriptstyle \bar{S}_m \otimes \bar{S}_n} \\ D[n+m]_* & \xrightarrow{\delta_{m,n}} & D[m]_* \otimes D[n]_*. \end{array}$$

We define an element ξ_k in $D_{*,*}$ by $\sum_{k\geq 0}\xi_k[n]$, where $\xi_k[n]=0$ for $n<k$. Then we obtain the coproduct $\xi_n \to \sum_{i=0}^{n}\xi_{n-i}^{2^i}\otimes\xi_i$ of ξ_n induced by $\delta_{m,n}$. We define $\chi_S : A_* \to D_{*,*}$ by $\prod_n \chi_{S_n}$, and $\bar{S} : H_*H \to D_{*,*}$ by $\prod_n \bar{S}_n$. Then χ_S and $\bar{S}$ preserve coproducts. Since $\chi_\psi : A_* \to H_*H$ is a Hopf algebra homomorphism, we get the commutative diagram of formal Hopf algebra homomorphisms

$$\begin{array}{ccc} A_* & \overset{\chi_S}{\searrow} & \\ \chi_\psi \big\downarrow & & D_{*,*}. \\ H_*H & \underset{\bar{S}}{\nearrow} & \end{array}$$

Remark. Since $D_{*,*}$ is not actually a Hopf algebra, χ_S and $\bar{S}$ are not Hopf algebra homomorphisms.

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