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Limits of stable homotopy and cohomotopy groups

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1. *Introduction and statement of results*

In this paper we formulate and prove generalizations of a theorem of Lin [**7**]. Let X be a CW complex with base point x_0. Define a free involution T on $S^\infty \times (X \wedge X)$ by $T(w, x\wedge y) = (-w, y \wedge x)$. The quadratic construction on X is the complex

$$S^\infty \underset{T}{\times} X \wedge X / S^\infty \underset{T}{\times} x_0 \wedge x_0.$$

This construction can be applied to spectra. A complete and thorough account will appear in the work on equivariant stable homotopy theory in preparation by L. G. Lewis, J. P. May, J. McLure and M. Steinberger. Some of the results are announced in [**8**].

Let $\mathscr{S}p$ be the homotopy category of CW spectra. Let S be the suspension functor. There is a natural transformation $SD_2 \to D_2 S$ of functors defined on $\mathscr{S}p$. If X is a spectrum and $k \in \mathbb{Z}$ let $P_k(X)$ be the spectrum $S^{-k}D_2 S^k X$. By applying S^{-k-1} to the map $SD_2 S^k X \to D_2 S^{k+1} X$ we get a map $\alpha_k = \alpha_k(X)\colon P_k(X) \to P_{k+1}(X)$. We study the inverse system of spectra

$$P_0(X) \leftarrow P_{-1}(X) \leftarrow P_{-2}(X) \leftarrow \ldots.$$

Let $\hat{\pi}^*$ and $\hat{\pi}_*$ be the cohomology theory and homology theory defined by the 2-adic completion of the sphere spectrum. When we apply $\hat{\pi}^*$ and $\hat{\pi}_*$ to this inverse system of spectra we get a direct system and an inverse system of abelian groups.

Theorem A. *Let X be a finite spectrum, then there are natural isomorphisms*

(i) $\operatorname*{Dirlim}_{k\to\infty} \hat{\pi}^* P_{-k}(X) \cong \hat{\pi}^* X$.

(ii) $\operatorname*{Invlim}_{k\to\infty} \hat{\pi}_* P_{-k}(X) \cong \hat{\pi}_* S^{-1} X$.

When $X = S^0$, this is Lin's theorem [**7**] (see also [**1**]) and this suggests the proof of Theorem A. First show that $\operatorname{Dirlim} \hat{\pi}^* P_{-k}(X)$ is a cohomology theory of X. Next exhibit a natural transformation $\hat{\pi}^* X \to \operatorname{Dirlim} \hat{\pi}^* P_{-k}(X)$, which is an isomorphism when $X = S^0$. Part (i) of Theorem A follows from the Eilenberg–Steenrod uniqueness theorem.

We issue a warning here: we do not require cohomology theories to satisfy Milnor's wedge axiom. This accounts for the finiteness hypothesis in Theorem A. In § 6 we give an example to show that the conclusion of part (i) of Theorem A is false when X is infinite. A complete discussion of this result with the finiteness hypothesis removed will be given in [**11**].

Our second theorem describes a particular natural isomorphism in part (i) of Theorem A. It is, essentially, the total Steenrod operation in cohomotopy. Note that

$P_{-t}(S^t) = S^t D_2 S^0 = S^t(\mathbb{R}P^\infty_+)$, where $+$ means adjoin a disjoint basepoint. There is a canonical map $\mathbb{R}P^\infty_+ \to S^0$; map base point to base point and $\mathbb{R}P^\infty$ to the other point. This gives a map $\rho: P_{-t}(S^t) \to S^t$. For $l \geqslant 0$ define $\gamma_{t,l}: \pi^t X \to \pi^t P_{-t-l}(X)$ by

$$\gamma_{t,l}(f) = \alpha^*(\rho P_{-t}(f)) \quad (f \in \pi^t X).$$

Here $\alpha^*: \pi^t P_{-t}(X) \to \pi^t P_{-t-l}(X)$ is induced by suitable composites of maps $\alpha_k(X)$.

If $l \geqslant 1$, $\gamma_{t,l}$ is a homomorphism and it extends to a homomorphism

$$\gamma_{t,l}: \hat{\pi}^t X \to \hat{\pi}^t P_{-t-l}(X).$$

Pick $l \geqslant 1$ and let Γ be the composite homomorphism

$$\hat{\pi}^t X \to \hat{\pi}^t P_{-t-l}(X) \to \operatorname{Dirlim} \hat{\pi}^t P_{-k}(X).$$

Here the second homomorphism is the usual homomorphism from a term in a direct system to the limit of the system. By construction Γ is independent of $l \geqslant 1$.

THEOREM B. *Let X be a finite spectrum; then Γ is an isomorphism.*

There are obvious analogues of Theorems A and B with 2 replaced by an odd prime p. The odd primary version of Lin's theorem has been proved by Gunawardena [**5**]. The quadratic construction is replaced by the p-adic construction defined using the group $\mathbb{Z}/p$ in place of $\mathbb{Z}/2$. We comment very briefly on the p-primary versions in § 6.

The rest of this paper is set out as follows. In § 2 we discuss the quadratic construction. In §§ 3 and 4 we prove Theorems A and B. In § 5 we discuss the fact that $RH^*(X; \mathbb{F}_2) = \operatorname{Dirlim} H^*(P_{-k}(X); \mathbb{F}_2)$ is the so-called Singer construction on $H^*(X; \mathbb{F}_2)$ [**9**], [**2**]. This observation is due to Haynes Miller. It has no logical role to play in this paper, but it was the starting point for this piece of work. Finally § 6 contains a counter-example and a comment on the odd primary case.

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2. *The quadratic construction*

Let $\mathscr{S}$ be the full subcategory of $\mathscr{S}p$ whose objects are the spectra $S^k X$, where $k \in \mathbb{Z}$ and X is a complex. Finite spectra are in $\mathscr{S}$ so it is sufficient, for our applications, to restrict attention to $\mathscr{S}$; it is also technically convenient. From now on we regard P_k as a functor $\mathscr{S} \to \mathscr{S}p$ unless we explicitly state otherwise. Let $\mathscr{C}$ be the homotopy category of CW complexes. We will not use any special notation to distinguish between a complex and its suspension spectrum. We now list those properties of P_k that we need.

(2·1) *For each complex Z, there is a map*

$$\phi = \phi(Z, X): Z \wedge P_k(X) \to P_k(Z \wedge X).$$

This map ϕ defines a natural transformation of functors $\mathscr{C} \times \mathscr{S} \to \mathscr{S}p$. Further, $\phi(S^0, X) = 1$ and if W and Z are complexes

$$\phi(W \wedge Z, X) = \phi(W, Z \wedge X)(1_W \wedge \phi(Z, X)).$$

(2·2) *There are natural maps $i_k = i_k(X): S^k X \wedge X \to P_k(X)$ and $\tau_k: P_k(X) \to S^k X \wedge X$ with $\tau_k i_k = 1 + (-1)^k T$, where $T: X \wedge X \to X \wedge X$ is the map which switches factors.*

Let Z be a complex; then the following diagrams commute

$$\begin{array}{ccc} S^kZ\wedge X\wedge X & \xrightarrow{1\wedge i_k} & Z\wedge P_k(X) \\ \downarrow{\scriptstyle S^k(\Delta\wedge 1)} & & \downarrow{\scriptstyle \phi} \\ S^kZ\wedge Z\wedge X\wedge X & \xrightarrow[i_k]{} & P_k(Z\wedge X) \end{array} \qquad \begin{array}{ccc} Z\wedge P_k(X) & \xrightarrow{1\wedge \tau_k} & S^kZ\wedge X\wedge X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle S^k(\Delta\wedge 1)} \\ P_k(Z\wedge X) & \xrightarrow[\tau_k]{} & S^kZ\wedge Z\wedge X\wedge X \end{array}$$

Here $\Delta: Z \to Z \wedge Z$ is the diagonal map and we have identified $Z \wedge Z \wedge X \wedge X$ with $Z \wedge X \wedge Z \wedge X$.

Now suppose we are given a cofibration sequence in $\mathscr{S}$

$$A \underset{f}{\to} X \underset{g}{\to} Y \underset{h}{\to} SA.$$

Let $P_k(X; A)$ be the cofibre of $P_k(f)$.

(2·3) *There is a natural cofibration sequence*

$$S^kA \wedge Y \to P_k(X; A) \to P_k(Y) \to S^{k+1}A \wedge Y.$$

Let Z be a complex; then the maps ϕ define a map of cofibrations

$$\begin{array}{ccccccc} S^kZ\wedge A\wedge Y & \longrightarrow & Z\wedge P_k(X;A) & \longrightarrow & Z\wedge P_k(Y) & \longrightarrow & S^{k+1}Z\wedge A\wedge Y \\ \downarrow{\scriptstyle S^k\Delta\wedge 1} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle S^{k+1}\Delta\wedge 1} \\ S^kZ\wedge Z\wedge A\wedge Y & \longrightarrow & P_k(Z\wedge X; Z\wedge A) & \longrightarrow & P_k(Z\wedge Y) & \longrightarrow & S^{k+1}Z\wedge Z\wedge A\wedge Y \end{array}$$

The maps in the cofibration sequence are related to i_k and τ_k as in the following commutative diagrams

$$\begin{array}{ccccccc} P_k(A) & \longrightarrow & P_k(X) & \longrightarrow & P_k(X;A) & \longrightarrow & SP_k(A) \\ \uparrow{\scriptstyle i_k(A)} & & \uparrow{\scriptstyle i_k(X)\circ(f\wedge 1)} & & \uparrow & & \uparrow{\scriptstyle Si_k(A)} \\ S^kA\wedge A & \xrightarrow[S^k(1\wedge f)]{} & S^kA\wedge X & \xrightarrow[S^k(1\wedge g)]{} & S^kA\wedge Y & \xrightarrow[S^k(1\wedge h)]{} & S^{k+1}A\wedge A \end{array}$$

$$\begin{array}{ccc} P_k(Y) & \xrightarrow{\tau_k(Y)} & S^kY\wedge Y \\ & \searrow & \downarrow{\scriptstyle S^k(h\wedge 1)} \\ & & S^{k+1}A\wedge Y \end{array}$$

We now define the map $\alpha_k(X)$ mentioned in the introduction to be $S^{-1}\phi(S^1, X)$:

(2·4) $\qquad P_k(SX) = SP_{k+1}(X), \quad i_k(SX) = Si_{k+1}(X), \quad \tau_k(SX) = S\tau_{k+1}(X).$

(2·5) *Let X be a complex; then the spectrum $P_0(X)$ is naturally equivalent to the suspension spectrum of D_2X.*

Let P_k be the spectrum $P_k(S^0)$. By regarding u as a 1-dimensional cohomology class, the ring $\mathbb{F}_2[u, u^{-1}]$ admits an action of the mod 2 Steenrod algebra A.

(2·6) (i) *$H^*(P_k; \mathbb{F}_2)$ is isomorphic over A to the A-submodule of $\mathbb{F}_2[u, u^{-1}]$ generated by u^l, $l \geqslant k$.*

(ii) *The map $\alpha_k: P_k \to P_{k+1}$ induces the obvious inclusion in* mod 2 *cohomology.*

(iii) *$H^*(P_k; \mathbb{Z})$ has no odd torsion.*

We note now, for future reference, that if I is the unit interval with base point 0, then the map $\phi(I,X)$ defines an equivalence of $SP_k(X)$ with $P_k(CX,X)$, where $CX = I \wedge X$ is the cone on X. This equivalence, the cofibration $X \to CX \to SX$, (2·3) and (2·4) give a cofibration sequence

$$S^kX \wedge X \xrightarrow{i_k} P_k(X) \xrightarrow{\alpha_k} P_{k+1}(X) \xrightarrow{\tau_{k+1}} S^{k+1}X \wedge X.$$

Next, we discuss space-level analogues of these properties when $k \geqslant 0$ and X is a complex. Let $D_2^n(X)$ be $S^n \underset{T}{\times} X \wedge X / S^n \underset{T}{\times} x_0 \wedge x_0$ and let $[w; x, y]$ be the point of $D_2^n(X)$ determined by $(w, x \wedge y) \in S^n \times X \wedge X$. Let ξ_n be the bundle $S^n \underset{T}{\times} (\mathbb{R} \times X \times X) \to S^n \underset{T}{\times} X \times X$. Here T acts on $X \times X$ by permuting factors, and on $\mathbb{R}$ by -1. If $k \geqslant 1$, there is a homeomorphism $S^{n+k} \underset{T}{\times} (X \times X)/S^{k-1} \underset{T}{\times} (X \times X)$ with the Thom complex of $k\xi_n$. In the case where X is a point this is the classical homeomorphism of truncated projective spaces with Thom complexes. The general proof is a straightforward modification of this special case. Note that $\xi_n \oplus 1$ is the bundle $S^n \underset{T}{\times} (\mathbb{R} \times X) \times (\mathbb{R} \times X)$, where T acts on $(\mathbb{R} \times X) \times (\mathbb{R} \times X)$ by permuting factors $\mathbb{R} \times X$. This gives homeomorphisms

$$S^k(D_2^{n+k}(X_+)/D_2^{k-1}(X_+)) \cong D_2^n(S^kX_+) \quad \text{and} \quad S^k(D_2^{n+k}(X)/D_2^{k-1}(X)) \cong D_2^n(S^kX);$$

we take $D_2^{-1}(X)$ to be the base point. So if $k \geqslant 0$ and X is a complex we can define $P_k(X)$ to be the complex $D_2(X)/D_2^{k-1}(X)$.

Let Z be a complex; then define $\phi(Z,X)\colon Z \wedge D_2 X \to D_2(Z \wedge X)$ by the formula $z \wedge [w; x, y] \to [w; z \wedge x, z \wedge y]$. This gives the maps $\phi\colon Z \wedge P_k(X) \to P_k(Z \wedge X)$. The maps i_k ($k \geqslant 0$) and τ_k ($k \geqslant 1$) are defined by the cofibration sequence

$$S^kX \wedge X = D_2^k(X)/D_2^{k-1}(X) \to P_k(X) \to P_{k+1}(X) \to S(D_2^k(X)/D_2^{k-1}(X)) = S^{k+1}X \wedge X.$$

We deduce that $\tau_k i_k = 1 + (-1)^k T$ ($k \geqslant 1$) and the appropriate diagrams involving ϕ, i, τ commute.

Let A be a sub-complex of X and $Y = X/A$. Since

$$S_+^\infty \wedge X \wedge X = S^\infty \times X \wedge X / S^\infty \times x_0 \wedge x_0,$$

there is a cofibration of spaces

$$S_+^\infty \wedge (A \wedge Y \vee Y \wedge A) \subseteq S_+^\infty \wedge (X \wedge X / A \wedge A) \to S_+^\infty \wedge Y \wedge Y.$$

Divide out by the involution to get cofibrations

$$A \wedge Y \simeq S_+^\infty \wedge A \wedge Y \subseteq D_2(X; A) \to D_2 Y,$$

$$S^kA \wedge Y \simeq S_+^\infty \wedge A \wedge Y / S_+^{k-1} \wedge A \wedge Y \to P_k(X; A) \to P_{k+1}(Y).$$

One may now check that the maps in this cofibration sequence are related to i_k and τ_k in the manner described in 2·3.

We now turn to the construction of $P_k\colon \mathscr{S} \to \mathscr{S}p$. The main point is the definition of the quadratic construction on a spectrum, see the forthcoming work of May *et al.* Here we give a brief account of one such definition. This definition has its deficiencies but it is adequate for our present purposes.

Let C_{n+1} be the real Clifford algebra of $\mathbb{R}^{n+1}$. So S^n is contained in the units of C_{n+1} and we choose the base point $w_0 \in S^n$ to be the unit of C_{n+1}. As a real vector space C_{n+1} has dimension 2^{n+1}. If $w \in S^n$, define $r_n(w) \in GL(2^{n+2}, \mathbb{R})$ by the formula

$$r_n(w)(u, v) = \tfrac{1}{2}(w(u-v) + u + v, u + v - w(u-v)),$$

where $(u, v) \in C_{n+1} \oplus C_{n+1} = \mathbb{R}^{2^{n+2}}$; compare ([6], § 3). Note that

$$r_n(-w) = r_n(w)\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where I is the $2^{n+1} \times 2^{n+1}$ unit matrix.

Let X be a complex. We get a homeomorphism

$$h_n: D_2^n(S^{2^{n+1}} \wedge X) \to S^{2^{n+2}} \wedge D_2^n(X),$$

$$h_n[w; u \wedge x, v \wedge y] = r_n(w)(u, v) \wedge [w; x, y], \quad u, v \in C_{n+1} \cup \infty = S^{2^{n+1}}.$$

For $l \in \mathbb{Z}$ define a spectrum $E = D_2 S^l X$ by

$$E_{2^{n+2}} = D_2^n(S^{2^{n+1}+l}X),$$

where $S^{2^{n+1}+l}X$ is the base point if $2^{n+1}+l < 0$. The maps are the composites

$$S^{2^{n+2}} D_2^n(S^{2^{n+1}+l}X) \xrightarrow{h_n^{-1}} D_2^n(S^{2^{n+2}+l}X) \subseteq D_2^{n+1}(S^{2^{n+2}+l}X).$$

Two things are clear from this definition. Firstly, if F is a spectrum we can modify this construction to define $D_2 F$; replace $S^{2^{n+1}+l}X$ by $F_{2^{n+1}}$ and use the map $S^{2^{n+1}} F_{2^{n+1}} \to F_{2^{n+2}}$. Secondly, $D_2(S^l X_+)$ is the Thom spectrum of the bundle $k(\xi_\infty \oplus 1)$ over $S^\infty \underset{T}{\times} (X \times X)$. From the point of view of Thom spectra, it is not obvious that $D_2 S^l X_+$ is functorial for stable maps of X_+.

Using this definition of the quadratic construction on a spectrum, the verification of (2·1)–(2·6) follows a standard pattern. Take the analogous property of $D_2^n(X)$ when X is a complex and check compatibility with the maps used to define P_k. We omit the details. Alternatively refer to [8] and the forthcoming work of May *et al.*

3. *The proof of Theorem A*

Let X be a complex. The maps $\phi(X, S^0)$ give maps $\beta_k = \beta_k(X): X \wedge P_k \to P_k(X)$ (recall $P_k = P_k(S^0)$). The maps β_k define a natural transformation of functors $\mathscr{C} \to \mathscr{S}p$. Recall $\alpha_k(X): P_k(X) \to P_{k+1}(X)$ is $S^{-1}\phi(S^1, X)$ so from (2·1) we get a map β of inverse systems

$$\begin{array}{ccccccc} P_0(X) & \leftarrow & P_{-1}(X) & \leftarrow & P_{-2}(X) & \leftarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ X \wedge P_0 & \leftarrow & X \wedge P_{-1} & \leftarrow & X \wedge P_{-2} & \leftarrow & \dots \end{array}$$

THEOREM 3·1. *Let E^* be a cohomology theory.*

(*a*) $F^*(X) = \operatorname{Dirlim} E^* P_{-k}(X)$ *is a cohomology theory on* $\mathscr{S}$.

(*b*) $\beta^*: F^*(X) \to \operatorname{Dirlim} E^*(X \wedge P_{-k})$ *is an isomorphism when X is a finite complex.*

Let $\mathscr{S}_0$ and $\mathscr{S}p_0$ be the full subcategories of $\mathscr{S}$ and $\mathscr{S}p$ whose objects are of finite type (i.e. finite n-skeleton for each n). If E_* is a homology theory then E_* is *connective* if there is an integer N such that $E_j(S^0) = 0$ for $j < N$. Note that if $E \in \mathscr{S}p_0$ then the homology theory $\hat{E}_*$ represented by the 2-adic completion of E is connective and, for each j, $\hat{E}_j(S^0)$ is a compact Hausdorff topological group.

THEOREM 3·2. *Let E_* be a connective homology theory such that $E_j(S^0)$ is a compact Hausdorff topological group for each j.*

(*a*) $F_*(X) = \operatorname{Invlim} E_* P_{-k}(X)$ *is a homology theory on* $\mathscr{S}_0$.

(*b*) $\beta_*: \operatorname{Invlim} E_*(X \wedge P_{-k}) \to F_*(X)$ *is an isomorphism when X is a finite complex.*

Proof of 3·1. The homotopy invariance of F^* is obvious. The suspension isomorphism is given on the terms of the direct system as follows:

$$E^j P_{-k}(X) \cong E^{j+1} S P_{-k}(X) = E^{j+1} P_{-k-1} SX.$$

Suppose $A \to X \to Y \to SA$ is a cofibration sequence in $\mathscr{S}$. Take the diagram in 2·3 with $Z = S^1$, apply S^{-1}, use 2·4 and apply E^* to get a direct system of exact sequences

$$\begin{array}{ccccccc} & \uparrow & & \uparrow & & \uparrow & \\ \leftarrow & E^j(S^{-k}A \wedge Y) & \leftarrow & E^j P_{-k}(X;A) & \leftarrow & E^j P_{-k}(Y) & \leftarrow \\ & \uparrow & & \uparrow & & \uparrow & \\ \leftarrow & E^j(S^{-k+1}A \wedge Y) & \leftarrow & E^j P_{-k+1}(X;A) & \leftarrow & E^j P_{-k+1}(Y) & \leftarrow \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

Since direct limits preserve exactness we get a long exact sequence of limits. Now $\Delta: S^1 \to S^1 \wedge S^1$ is trivial so $\operatorname{Dirlim} E^j(S^{-k}A \wedge Y) = 0$ since each homomorphism in the system is zero. Therefore we get natural isomorphisms $F^j(Y) \to \operatorname{Dirlim} E^j P_{-k}(X;A)$. The F^* exact sequence of the cofibration $A \to X \to Y$ comes from the E^* exact sequences of the cofibrations $P_{-k}(A) \to P_{-k}(X) \to P_{-k}(X;A)$ by taking limits and using the above isomorphism. This proves 3·1 (a).

Note that $\operatorname{Dirlim} E^*(X \wedge P_{-k})$ is a cohomology theory. Using 2·1 and 2·4,

$$\alpha_{-k-1}(SX) S\beta_{-k}(X) = \beta_{-k}(SX);$$

therefore β^* commutes with the suspension isomorphisms in the two theories and is a natural transformation of cohomology theories defined on $\mathscr{C}$. When $X = S^0$ each β_k is the identity map so that β^* is an isomorphism. Therefore β^* is an isomorphism for each finite complex X. This proves 3·1 (b).

To prove 3·2, copy the argument but add the fact that inverse limits are exact on the category of compact Hausdorff topological groups. The hypotheses of 3·2 ensure we stay in this category.

Proof of part (i) *of Theorem A*. From 2·5 $P_0(S^0) = \mathbb{R}P^\infty_+$ and there is a canonical map $\rho: P_0(S^0) \to S^0$ such that ρi_0 is the identity. Composing with composites of α_j we get maps $\rho_{-k}: P_{-k} \to S^0$ $(k \geq 0)$. The maps $1 \wedge \rho_{-k}$ give a homomorphism

$$\rho^*: \hat{\pi}^* X \to \operatorname{Dirlim} \hat{\pi}^* X \wedge P_{-k}.$$

When $X = S^0$ this homomorphism is an *isomorphism* [**7**, **1**]; Lin's theorem applies because of 2·6. Since ρ^* is a natural transformation of cohomology theories defined on $\mathscr{S}p$ it is an isomorphism for each finite spectrum X.

When X is a finite complex define the natural isomorphism

$$\Phi: \hat{\pi}^* X \to \operatorname{Dirlim} \hat{\pi}^* P_{-k}(X)$$

by $\Phi = (\beta^*)^{-1}\rho^*$. When X is a finite spectrum pick an integer L such that the maps $SX_n \to X_{n+1}$ are equivalences for $n \geq L$. Then define Φ to be the composite

$$\hat{\pi}^* X \cong \hat{\pi}^{*+L} X_L \to \operatorname{Dirlim} \hat{\pi}^{*+L} P_{-k}(X_L) \cong \operatorname{Dirlim} \hat{\pi}^* P_{-k}(X),$$

where the extreme isomorphisms are the suspension isomorphisms in the appropriate cohomology theories.

Proof of part (ii) *of Theorem A*. According to Lin [7] $\operatorname{Invlim} \pi_{-1} P_{-k} \cong \mathbb{Z}_2^\wedge$, where $\mathbb{Z}_2^\wedge$ is the 2-adic integers. Pick a compatible sequence of maps $\psi_{-k}: S^{-1} \to P_{-k}$ such that $\{\psi_{-k}\} \in \operatorname{Invlim} \pi_{-1} P_{-k}$ is non-zero modulo 2. The maps $1 \wedge \psi_{-k}$ define a natural transformation $\psi_*: \hat{\pi}_* S^{-1} X \to \operatorname{Invlim} \hat{\pi}_*(X \wedge P_{-k})$ of homology theories defined on $\mathscr{S}p$.

Lin shows that, when $X = S^0$, this transformation is an isomorphism. Now copy the argument used for part (i).

4. *The proof of Theorem B*

The construction of the total power operation $\gamma_{t,l}\colon \pi^t X \to \pi^t P_{-t-l}(X)$, $X \in \mathscr{S}$, is given in the introduction. We begin with a lemma concerning the functors P_k. Let q_1 and q_2 be the projections of $X \vee Y$ on to X and Y. Define $f\colon P_k(X \vee Y) \to S^k X \wedge Y$ to be the composite

$$P_k(X \vee Y) \xrightarrow{\tau_k} S^k(X \vee Y) \wedge (X \vee Y) \xrightarrow{S^k q_1 \wedge q_2} S^k X \wedge Y.$$

LEMMA 4.1. *The map $P_k(X \vee Y) \to P_k(X) \vee P_k(Y) \vee S^k X \wedge Y$ with components $P_k(q_1)$, $P_k(q_2)$ and f, is an equivalence.*

Proof. Let j_1, j_2 be the inclusions of X and Y in $X \vee Y$. Define $g\colon S^k X \wedge Y \to P_k(X \vee Y)$ to be the composite

$$S^k X \wedge Y \xrightarrow[S^k j_1 \wedge j_2]{} S^k(X \vee Y) \wedge (X \vee Y) \xrightarrow[i_k]{} P^\lambda(X \vee Y).$$

Then the map $P_k(X) \vee P_k(Y) \vee S^k X \wedge Y \to P_k(X \vee Y)$, with components $P_k(i_1), P_k(i_2)$ and g, is inverse to the map in Lemma 4·1.

LEMMA 4·2. *For $x, y \in \pi^t X$, $\gamma_{t,0}(x+y) = \gamma_{t,0}(x) + \gamma_{t,0}(y) + (S^{-t}x \wedge y)\,\tau_{-t}$.*

Proof. Let $\xi\colon X \to X \vee X$ be the map with both components the identity. Then the composite

$$P_{-t}(X) \xrightarrow{P_{-t}(\xi)} P_{-t}(X \vee X) \xrightarrow{f} P_{-t}(X) \vee P_{-t}(X) \vee S^{-t} X \wedge X$$

has as its first two components the identity and its third τ_{-t}. Lemma 4·2 follows easily.

LEMMA 4·3. (*a*) *$\gamma_{t,l}\colon \pi^t X \to \pi^t P_{-t-l}(X)$ is a homomorphism if $l \geqslant 1$.*

(*b*) *The homomorphism $\gamma_{t,l}$, $l \geqslant 1$ extends to a homomorphism $\gamma_{t,l}\colon \hat{\pi}^t X \to \hat{\pi}^t P_{-t-l}(X)$.*

Proof. (*a*) From 2·2 and the definition of α_{-t}, we see that $\tau_{-t}\,\alpha_{-t} = 0$. Now use 4.2.

(*b*) For $n \geqslant k$ define $P_k^n(X)$ by replacing S^∞ by S^{n-k} throughout the definition of $P_k(X)$. Define $\gamma_{t,l}^n\colon \pi^t X \to \pi^t P_{-t-l}^n(X)$ using P_k^n in place of P_k throughout the definition of $\gamma_{t,l}$. If X is a finite spectrum then $P_{-t-l}^n(X)$ is finite so both $\pi^t X$ and $\pi^t P_{-t-l}^n(X)$ are finitely generated. Therefore $\hat{\pi}^t X$ and $\hat{\pi}^t P_{-t-l}^n(X)$ are the 2-adic completions of the cohomotopy groups, and $\gamma_{t,l}^n$ extends to a homomorphism of completions since it is continuous with respect to the 2-adic topology.

For general X use the fact that $\hat{\pi}^t X = \operatorname{Invlim} \hat{\pi}^t Y$, where Y runs through the finite subspectra of X. The spectra $P_k^n(Y)$ form a cofinal subsystem of the finite subspectra of $P_k(X)$, here Y is a finite subspectrum of X, and therefore

$$\hat{\pi}^t P_{-t-l}(X) = \operatorname*{Invlim}_{n,\,Y} \hat{\pi}^t P_{-t-l}^n(Y).$$

The homomorphisms $\gamma_{t,l}^n$ are compatible as n and Y vary and so define a homomorphism of inverse limits. This is the required extension.

Proof of Theorem B. Let $f\colon X \to S^t$ be a map in $\mathscr{S}$. From the definitions we find that $\gamma_{t+1,l}(Sf) = S\gamma_{t,l}(f)$ and therefore Γ commutes with the suspension isomorphisms in the two theories.

Assume X is a finite spectrum and let $\Phi\colon \hat{\pi}^t X \to \operatorname{Dirlim} \hat{\pi}^t P_{-k}(X)$ be the natural isomorphism used in the proof of part (i) of Theorem A. A straightforward check on definitions shows that $\Gamma(1) = \Phi(1)$, where $1 \in \hat{\pi}^0 S^0$ is the unit. Both Φ and Γ are

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natural and commute with suspensions. Therefore Φ and Γ agree for any finite spectrum and so Γ is an isomorphism.

5. *The relation with the work of W. M. Singer*

In this section we examine the groups $F^*X = \operatorname{Dirlim} H^*P_{-k}(X)$, where H^* means mod 2 cohomology. The results 5·1–5·3 below, when $k \geqslant 0$ and X is a complex, all specialize to standard results about the cohomology of the quadratic construction. They extend to the general case, for example using our explicit description of $P_k(X)$. Alternatively consult [8], [10] and the forthcoming work of May *et al.*

Let W be the usual complete resolution of Σ_2 over $\mathbb{F}_2$. That is, for each $k \in \mathbb{Z}$, $W_k = \mathbb{F}_2[\Sigma_2]$ with generator e_k, and $\partial e_k = e_{k-1} + Te_{k-1}$. Here $T \in \Sigma_2$ is the non-trivial element. If $k \leqslant l$, let $W(k,l)$ be the resolution obtained by replacing W_i with zero if $i < k$ or $i > l$. If $n \geqslant k$ write $P_k^n(X)$ for the spectrum obtained by replacing S^∞ with S^{n-k} throughout the definition of $P_k(X)$. Let X be a spectrum, then form the co-chain complex $W^*(k,l) \underset{\Sigma_2}{\otimes} H^*X \otimes H^*X$ with differential

$$\delta(e^r \otimes a \otimes b) = e^{r+1} \otimes (a \otimes b + b \otimes a),$$

where $e^r \in \operatorname{Hom}(W_r, \mathbb{F}_2)$ is dual to e_r.

(5·1) (*a*) $H^*P_k^n(X) = H(W^*(k,n) \underset{\Sigma_2}{\otimes} H^*X \otimes H^*X; \delta)$.

(*b*) $H^*P_k(X) = H(W^*(k,\infty) \underset{\Sigma_2}{\otimes} H^*X \otimes H^*X; \delta)$.

(*c*) $F^*X = H(W^* \underset{\Sigma_2}{\otimes} H^*X \otimes H^*X; \delta)$.

If $q \in \mathbb{Z}$ and $x \in H^rX$ define $Q^qx \in F^{q+r}X$ to be the class of $e^{q-r} \otimes x \otimes x$.

(5·2) (*a*) *Let B be a basis for H^*X. Then a basis for F^*X is*

$$\{Q^qx \mid q \in \mathbb{Z}, x \in B\}.$$

(*b*) *The action of A on F^*X is given by the formula*

$$Sq^tQ^qx = \sum_{i \geqslant 0} \binom{q-i}{t-2i} Q^{t+q-i}Sq^ix.$$

If X is a complex then the map β of inverse systems defined in §3 gives a homomorphism

$$\beta^*: F^*X \to \operatorname{Dirlim} H^*(X \wedge P_k) = H^*X \otimes \mathbb{F}_2[u, u^{-1}].$$

(5·3) $\beta^*Q^qx = \sum_{j \geqslant 0} Sq^jx \otimes u^{q-j}$, compare [4], §27.

Let M be an A-module. In [9] Singer introduces the module

$$RM = \mathbb{F}_2[u, u^{-1}] \otimes M$$

with A-action given by

$$Sq^t(u^q \otimes m) = \sum_{i \geqslant 0} \binom{q-i}{t-2i} u^{t+q-i} \otimes Sq^im.$$

Lemma 5·4. *$F^*X \cong RH^*X$ as A-modules.*

Proof. The isomorphism is given by $Q^qx \to u^q \otimes x$.

Two algebraic properties of RM have analogues in terms of the inverse system of spectra $\{P_k(X)\}$. Singer studies the homomorphisms $f: RM \to M \otimes F_2[u, u^{-1}]$, $\epsilon: RM \to S^{-1}M$ defined by

$$f(u^b \otimes m) = \sum_{j \geqslant 0} Sq^jm \otimes u^{b-j}, \quad \epsilon(u^b \otimes m) = Sq^{b+1}m.$$

Here $Sq^k = 0$ if $k < 0$. Following Miller, Singer shows that if M is bounded above and below then f is an isomorphism.

When we identify F^*X with RH^*X we get $\beta^* = f$ and $\psi^* = \epsilon$, where ψ is the map of direct systems used in the proof of A (ii). The fact that f is an isomorphism when $M = H^*X$ with X a finite spectrum follows from 3·1.

6. *Final observations*

First we show that the conclusion of A (i) is false when $X = P_0$. From 2·5, $P_0(P_0)$ is the suspension spectrum of BD_4^+, where D_4 is the dihedral group with eight elements. Further $P_0 \wedge P_0$ is the suspension spectrum of $B(\mathbb{Z}/2 \times \mathbb{Z}/2)^+$. We use the verification of the Segal conjecture [**2**, **3**, **7**] to compute cohomotopy groups.

For any finite group G write $\hat{A}(G)$ for the completion of the Burnside ring of G at its augmentation ideal. It is straightforward to check that

$$\hat{A}(D_4) = (\mathbb{Z}_2^\wedge)^8, \quad \hat{A}(\mathbb{Z}/2 \times \mathbb{Z}/2) = (\mathbb{Z}_2^\wedge)^5.$$

From [**3**] and [**2**] we get $\hat{\pi}^0(P_0(P_0)) = (\mathbb{Z}_2^\wedge)^8$, $\hat{\pi}^0(P_0 \wedge P_0) = (\mathbb{Z}_2^\wedge)^5$ and $\hat{\pi}^j(P_0 \wedge P_0) = 0$ if $j \geqslant 1$.

Now use the cohomotopy exact sequences of the cofibrations

$$S^{-k}P_0 \wedge P_0 \to P_{-k}(P_0) \to P_{-k+1}(P_0)$$

to deduce that $\operatorname{Dirlim} \hat{\pi}^0 P_{-k}(P_0) = \hat{\pi}^0 P_{-1}(P_0)$ and from the case $k = 1$

$$\operatorname{rank}_{\mathbb{Z}_2^\wedge} \hat{\pi}^0 P_{-1}(P_0) \geqslant 3.$$

However, from [**7**] $\hat{\pi}^0(P_0) = (\mathbb{Z}_2^\wedge)^2$ so that $\hat{\pi}^0 P_0$ cannot be isomorphic to $\operatorname{Dirlim} \hat{\pi}^0 P_{-k}(P_0)$.

Next we comment on the space level analogue of 2·3 when 2 is replaced by an odd prime p. We use the p-adic construction

$$D_p(X) = S^\infty \underset{\mathbb{Z}/p}{\times} X^p / S^\infty \underset{\mathbb{Z}/p}{\times} x_0^p,$$

where $X^p = X \wedge \ldots \wedge X$ with p-factors of X. Suppose $A \subseteq X$ and write $Y = X/A$. Define subspaces $W^i \subseteq X^p$, $0 \leqslant i \leqslant p$ by letting W^i consist of points with at least i components in A, so

$$A^p = W^p \subseteq W^{p-1} \subseteq \ldots \subseteq W^0 = X^p.$$

Note that W^i/W^{i+1} is a wedge of $\binom{p}{i}$ spaces each of which is homeomorphic to

$$A^i \wedge Y^{p-i}.$$

If $p - 1 \geqslant i \geqslant 1$ the action of $\mathbb{Z}/p$ on W^i/W^{i+1} divides these wedge summands into $\frac{1}{p}\binom{p}{i}$ orbits. Therefore $S_+^\infty \underset{\mathbb{Z}/p}{\wedge} W^i/W^{i+1}$ is a wedge of $\frac{1}{p}\binom{p}{i}$ spaces each of which is homeomorphic to $S_+^\infty \wedge A^i \wedge Y^{p-i}$. We now get $p-1$ cofibrations

$$S_+^\infty \underset{\mathbb{Z}/p}{\wedge} W^{i+1}/W^p \to S_+^\infty \underset{\mathbb{Z}/p}{\wedge} W^i/W^p \to S_+^\infty \underset{\mathbb{Z}/p}{\wedge} W^{i+1}/W^i$$

$(p-1 \geqslant i \geqslant 1)$. When $i = 1$, this gives the cofibration

$$S_+^\infty \underset{\mathbb{Z}/p}{\wedge} W^1/W^p \to D_p(X; A) \to D_p(Y).$$

If we replace the single cofibration of 2·3 with $p-1$ cofibrations, the proofs of our results with 2 replaced by an odd prime p are straightforward modifications of the proofs given here.

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