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Traced monoidal categories

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Abstract

Traced monoidal categories are introduced, a structure theorem is proved for them, and an example is provided where the structure theorem has application.

1. *Introduction*

This paper introduces axioms for an abstract trace on a monoidal category. This trace can be interpreted in various contexts where it could alternatively be called contraction, feedback, Markov trace or braid closure. Each full submonoidal category of a tortile (or ribbon) monoidal category admits a canonical trace. We prove the structure theorem that every traced monoidal category arises in this way. Naively speaking, the construction is a glorification of the construction of the integers from the natural numbers. Less naively, the construct provides a left biadjoint to the forgetful 2-functor from the 2-category of tortile monoidal categories to the 2-category of traced monoidal categories, and we prove that the unit for this biadjunction is fully faithful.

It should be kept in mind that the more familiar symmetric monoidal categories [**ML**] are obtained as special cases of balanced monoidal categories by taking the twist isomorphisms $\theta_A : A \to A$ to be identities. In the same way, compact closed categories [**KL**] are special tortile monoidal categories. In the diagrams for these special cases the reader may replace the ribbons by strings and ignore over and under crossings. Giulio Katis pointed out that an early attempt at the construction by the second author was too simplistic to be correct for the non-symmetric case.

We shall describe here, for motivation, the meaning of our trace for linear functions between finite dimensional vector spaces. Consider a linear function $f: V \otimes U \to W \otimes U$ where U, V, W are vector spaces with bases $(u_i), (v_j), (w_k)$. The trace of f with respect to U is the linear function $t: V \to W$ given by

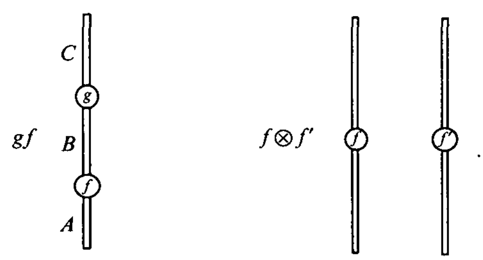
$$t(v_i) = \sum_{j,k} a_{ij}^{kj} w_k \quad \text{where} \quad f(v_i \otimes u_j) = \sum_{k,m} a_{ij}^{km} w_k \otimes u_m.$$

This reduces to the usual trace of $f\colon U \to U$ when V and W are one dimensional. As the symmetric monoidal category $\mathscr{V}ect_{fin}$ of finite-dimensional vector spaces is already compact closed, the main construction of this paper just gives back $\mathscr{V}ect_{fin}$ up to equivalence.

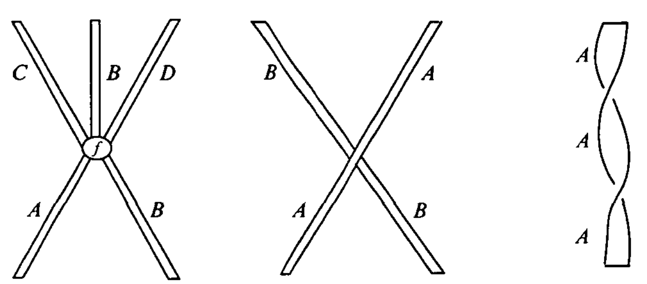
In knot theory, our work relates to that of [Y], [**JS1**, **JS2**], [**FY1**, **FY2**], [**Sm**], [**RT**]. In computer science, the connection between feedback and an iteration operation was pointed out to us by Steve Bloom who suggested an intriguing formula for some examples of traces (see [**BÉ**]); in these cases, the construction coming out of our structure theorem leads to some interesting new compact closed (bi)categories in which the tensor product is direct sum. This notion of trace also appears in the geometry of interaction as the 'execution formula' [**G**].

2. *Abstract trace*

For legibility and without loss of generality, we write as if our monoidal categories were strict. The concept of balanced monoidal category appears in [**JS1**], and the appropriate ribbon diagrams for them are described and justified in [**JS2**]. We remind the reader that our conventions are to compose arrows $f\colon A \to B, g\colon B \to C$ to get $gf\colon A \to C$, we tensor $f\colon A \to B, f'\colon A' \to B'$ to get $f \otimes f'\colon A \otimes A' \to B \otimes B'$, and we depict these respectively as follows.



An arrow $f\colon A \otimes B \to C \otimes B \otimes D$, the braiding $c_{A,B}\colon A \otimes B \to B \otimes A$, and the twist $\theta_A\colon A \to A$ are respectively depicted as follows:



It is the braiding which introduces the third dimension into our diagrams, and the twist which forces the change from strings to ribbons.

Definition. A trace for a balanced monoidal category $\mathscr{V}$ is a natural family of functions

$$\mathrm{Tr}^U_{A,B}\colon \mathscr{V}(A \otimes U, B \otimes U) \to \mathscr{V}(A, B)$$

satisfying three axioms:

vanishing:

$$\mathrm{Tr}^I_{A,B}(f) = f, \quad \mathrm{Tr}^{U \otimes V}_{A,B}(g) = \mathrm{Tr}^U_{A,B}(\mathrm{Tr}^V_{A \otimes U, B \otimes U}(g));$$

superposing:

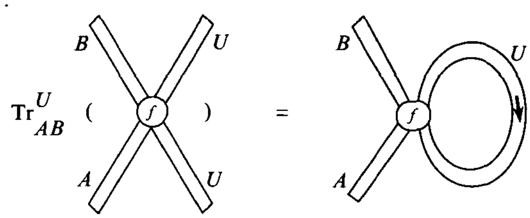
$$\begin{aligned}\mathrm{Tr}^U_{A\otimes C,\,B\otimes D}((1_B \otimes c^{-1}_{D,U}) \circ (f \otimes g) \circ (1_A \otimes c_{C,U}))\\ = \mathrm{Tr}^U_{A,B}(f) \otimes g\\ = \mathrm{Tr}^U_{A\otimes C,\,B\otimes D}((1_B \otimes c_{U,D}) \circ (f \otimes g) \circ (1_A \otimes c^{-1}_{U,C}));\end{aligned}$$

yanking:

$$\mathrm{Tr}^U_{U,U}(c_{U,U} \circ (\theta^{-1}_U \otimes 1_U)) = 1_U = \mathrm{Tr}^U_{U,U}(c^{-1}_{U,U} \circ (\theta_U \otimes 1_U)).$$

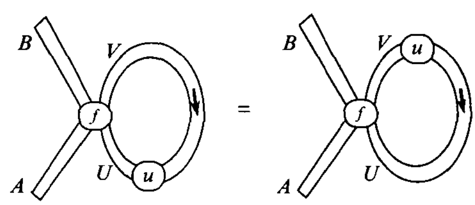
A traced monoidal category is a balanced monoidal category equipped with a trace.

We extend the diagrams for balanced monoidal categories as follows to accommodate trace:

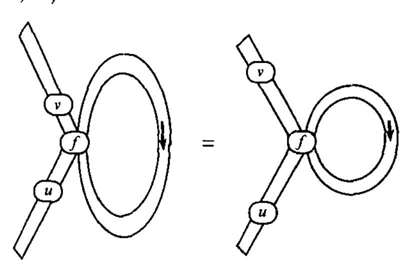


The purpose of the arrow is to remind us of the non-progressive part of the diagram where trace is employed. The axioms on trace can then be illustrated as follows:

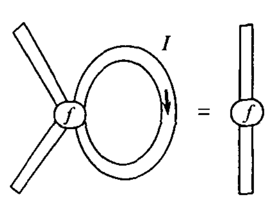
sliding (naturality in U)

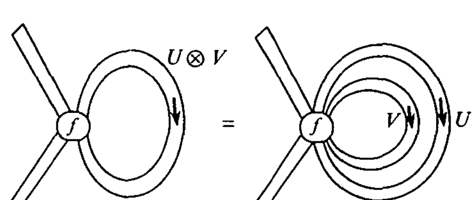


tightening (naturality in A, B)

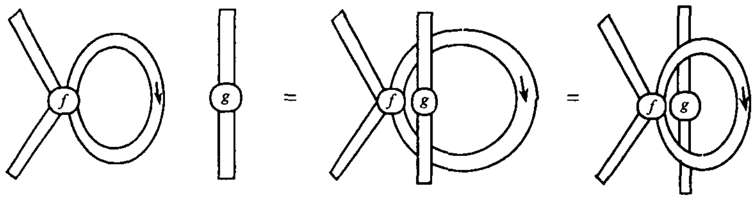


vanishing

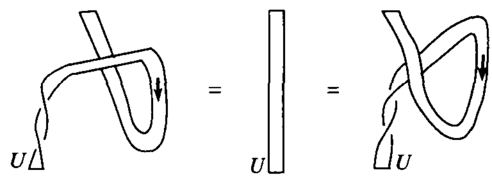




superposing



yanking

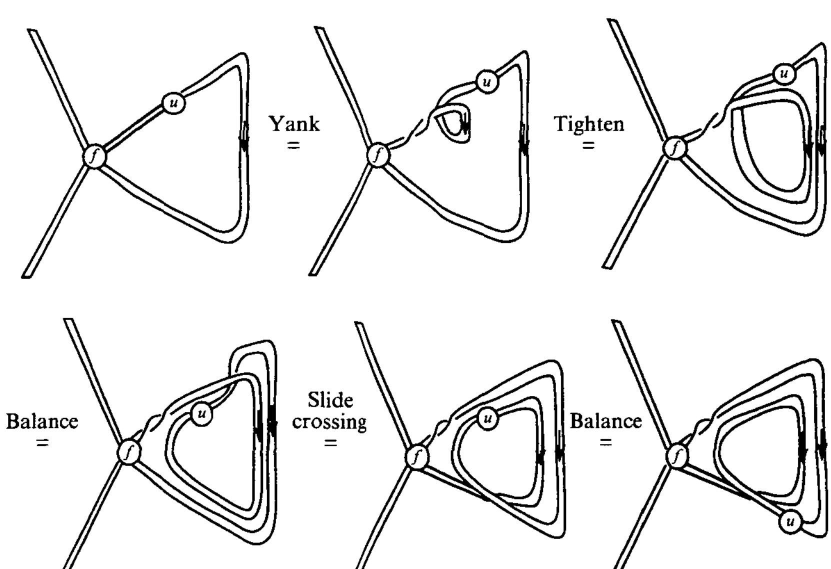


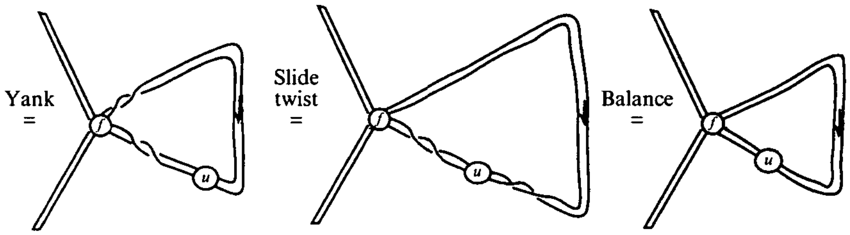
Remark. The reader should be careful in using these diagrams. Diagrams which look justified by three-dimensional reasoning must, at this stage, be deduced from the axioms if they involve trace. Three dimensional reasoning is valid on the progressive parts of the diagrams because of the results of [**JS2**] for balanced monoidal categories. With these provisos, our geometric proofs are completely rigorous. Our next three lemmas will give some exercises in using diagrams and the axioms. Algebraic proofs can be constructed from the geometric ones, but algebraic proofs seem only to obfuscate the intuition.

One of the basic properties of traditional trace is $\mathrm{Tr}(f\,u) = \mathrm{Tr}(u\,f)$ whenever f, u are composable in both orders. This is clearly a special case of sliding. So it might come as a surprise that the sliding axiom can be weakened.

LEMMA 2·1. *Slidings of crossings and twists suffice for all slidings in the presence of the other axioms.*

Proof. Our proof, that slidings of crossings and twists imply all slidings, is diagrammatic.



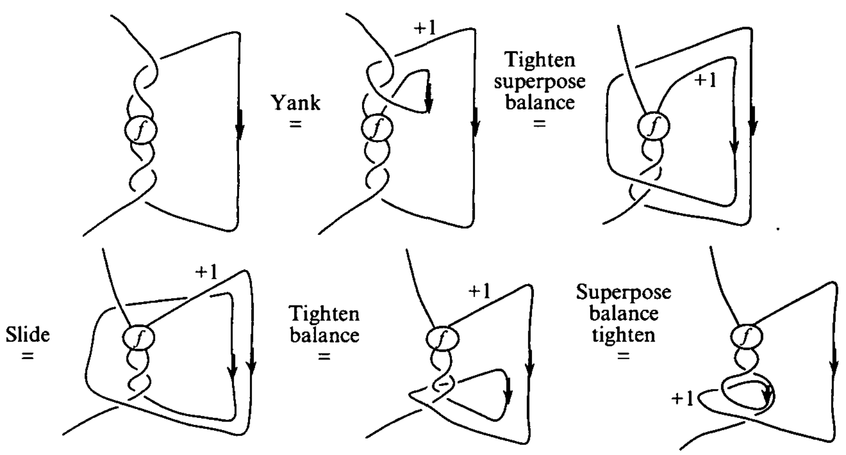


Remark. Instead of ribbons it is permissible to draw strings with integers to record the twists. In fact, the integers can be put anywhere on the string because this is known to be possible in balanced monoidal categories, and can be extended to traced monoidal categories because of sliding.

LEMMA 2·2 (*Flipping*). *The trace* Tr^U *of* $f: A \otimes U \to B \otimes U$ *is equal to the trace of the composite*

$$A \otimes U \xrightarrow{c} U \otimes A \xrightarrow{c} A \otimes U \xrightarrow{f} B \otimes U \xrightarrow{c^{-1}} U \otimes B \xrightarrow{c^{-1}} B \otimes U.$$

Proof. Again we use diagrams beginning with the one for the trace of the above composite.



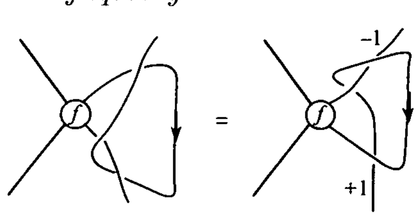
Yanking and an obvious balanced manoeuvre leave us with the trace of f.

LEMMA 2·3 (*Trace swapping*). *For all* $f: A \otimes B \to C \otimes D$, *the following composites have the same trace.*

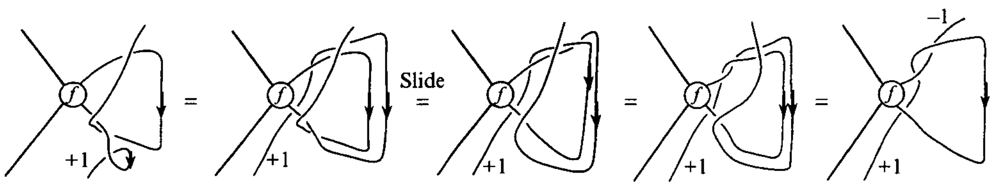
$$A \otimes B \otimes D \xrightarrow{1\otimes c} A \otimes D \otimes B \xrightarrow{1\otimes c} A \otimes B \otimes D \xrightarrow{f\otimes 1} C \otimes D \otimes D \xrightarrow{1\otimes c^{-1}} C \otimes D \otimes D,$$

$$A \otimes B \otimes B \xrightarrow{1\otimes c^{-1}} A \otimes B \otimes B \xrightarrow{1\otimes f} C \otimes D \otimes B \xrightarrow{1\otimes c} C \otimes B \otimes D \xrightarrow{1\otimes c} C \otimes D \otimes B.$$

Pictorially, this is the following equality.



Proof. Starting with the left-hand side by using tightening, we proceed diagrammatically as follows.



∎

Recall that a (strong) monoidal functor $F:\mathscr{V}\to\mathscr{W}$ is a functor equipped with a natural family of isomorphisms $\phi_{A,B}: FA\otimes FB\to F(A\otimes B)$ and an isomorphism $\phi^0: I\to FI$ satisfying the obvious coherence conditions. It is called balanced when it is compatible with the braidings and twists on $\mathscr{V}, \mathscr{W}$. If $\mathscr{V}, \mathscr{W}$ are traced monoidal, we say $F:\mathscr{V}\to\mathscr{W}$ is traced monoidal when it is balanced monoidal and it preserves trace in the following obvious sense:

$$\mathrm{Tr}^{FU}_{FA,FB}(FA\otimes FU\xrightarrow{\phi_{A,U}}F(A\otimes U)\xrightarrow{Ff}F(B\otimes U)\xrightarrow{\phi^{-1}_{B,U}}FB\otimes FU)$$

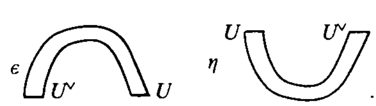
$$= F(\mathrm{Tr}^{U}_{A,B}(f)): FA\longrightarrow FB.$$

We point out the following rather straightforward proposition.

PROPOSITION 2·4. *Suppose $F:\mathscr{V}\to\mathscr{W}$ is a fully faithful, balanced monoidal functor with $\mathscr{W}$ traced monoidal. Then there exists a unique trace on $\mathscr{V}$ for which F is a traced monoidal functor. (This is called the trace on $\mathscr{V}$ transported from $\mathscr{W}$ along F.)*

3. *Canonical trace*

In a tortile monoidal category [**JS1, JS2, Sm**], each object U has a left dual $U^\vee$ with counit $\epsilon: U^\vee\otimes U\to I$ and unit $\eta: I\to U\otimes U^\vee$. The diagrams for these are as follows.

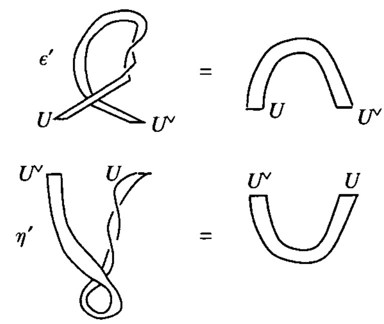


As justified in [**JS4**], the appropriate counit and unit for $U^\vee$ as a right dual of U are

$$\epsilon': U\otimes U^\vee\xrightarrow{c_{U,U^\vee}}U^\vee\otimes U\xrightarrow{1_{U^\vee}\otimes\theta_U}U^\vee\otimes U\xrightarrow{\epsilon}I,$$

$$\eta': I\xrightarrow{\eta}U\otimes U^\vee\xrightarrow{c^{-1}_{U^\vee,U}}U^\vee\otimes U\xrightarrow{1_{U^\vee}\otimes\theta_U^{-1}}U^\vee\otimes U,$$

which are illustrated best by the diagrams on the right-hand sides of the following equalities.



The following definition of trace is essentially classical in the case $A = B = I$ [**SR**, **KL**].

PROPOSITION 3·1. *In any tortile monoidal category, a trace, called the canonical trace, is defined by the formula*

$$\mathrm{Tr}^U_{A,B}(f) = (A \xrightarrow{1\otimes\eta} A \otimes U \otimes U^\vee \xrightarrow{f\otimes 1} B \otimes U \otimes U^\vee \xrightarrow{1\otimes\epsilon'} B).$$

Furthermore, every balanced strong monoidal functor between tortile monoidal categories is traced with respect to the canonical traces.

Proof. The diagrams for tortile monoidal categories will be explained precisely in [**JS5**]. However, for the present purposes, using the above diagrams for the counits and units, we see that the formula for the trace leads to the same diagram for trace as we use in any traced monoidal category. Then the diagrams for the axioms can easily be used to construct a full algebraic proof. The last sentence is clear from the formula for canonical trace. ▮

The goal of the next few sections is to prove a structure theorem for traced monoidal categories by augmenting dual objects. We shall prove that every trace on every balanced monoidal category $\mathscr{V}$ is transported from a canonical trace on a tortile monoidal category $\mathrm{Int}\,\mathscr{V}$.

4. *The monoidal category* Int $\mathscr{V}$

We begin with a traced monoidal category $\mathscr{V}$ and define a category $\mathrm{Int}\,\mathscr{V}$. The objects of $\mathrm{Int}\,\mathscr{V}$ are pairs (X, U) of objects X, U of $\mathscr{V}$. (One should think of the pair (X, U) as a formalization of $X \otimes U^\vee$.) An arrow $f: (X, U) \to (Y, V)$ in $\mathrm{Int}\,\mathscr{V}$ is an arrow $f: X \otimes V \to Y \otimes U$ in $\mathscr{V}$. The composite of $f: (X, U) \to (Y, V)$ and $g: (Y, V) \to (Z, W)$ is the value of the trace function

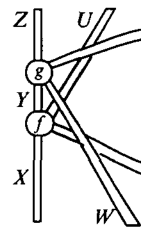
$$\mathrm{Tr}^V_{X\otimes W,\, Z\otimes U}: \mathscr{V}(X \otimes W \otimes V, Z \otimes U \otimes V) \to \mathscr{V}(X \otimes W, Z \otimes U)$$

at the composite

$$X \otimes W \otimes V \xrightarrow{1\otimes c} X \otimes V \otimes W \xrightarrow{f\otimes 1} Y \otimes U \otimes W \xrightarrow{1\otimes c^{-1}} Y \otimes W \otimes U$$

$$\xrightarrow{g\otimes 1} Z \otimes V \otimes U \xrightarrow{1\otimes c} Z \otimes U \otimes V.$$

The composition in $\mathrm{Int}\,\mathscr{V}$ is represented by the following diagram.

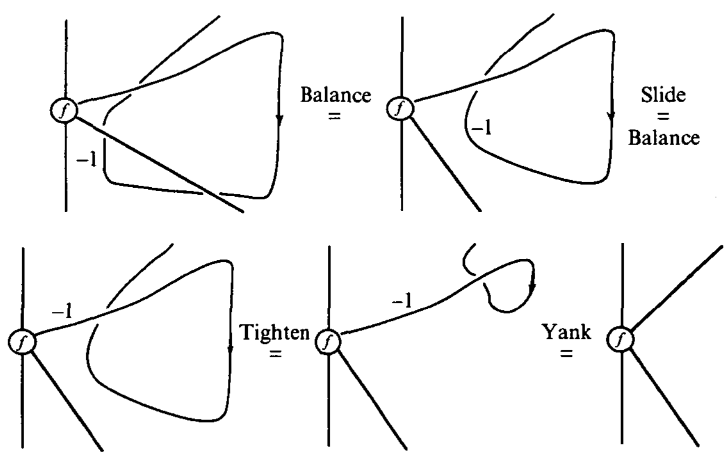


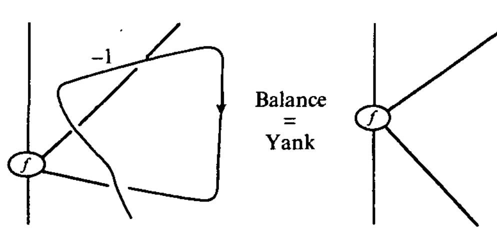
The identity of the object (X, U) in $\mathrm{Int}\,\mathscr{V}$ is $1_X \otimes \theta_U^{-1}: X \otimes U \to X \otimes U$.

Remark. The reader may suspect that there is a degree of choice in the definition of composition in $\mathrm{Int}\,\mathscr{V}$. Indeed, this suspicion will extend to definitions we make later. However, a consistent set of choices is forced on us by the desired universal property of $\mathrm{Int}\,\mathscr{V}$ as a tortile tensor category.

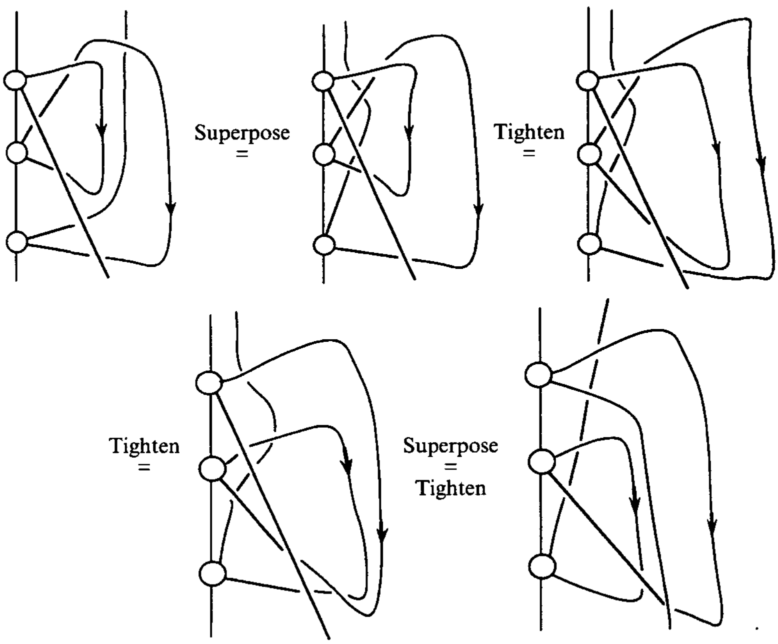
PROPOSITION 4·1. *The above data do define a category* $\mathrm{Int}\,\mathscr{V}$ *and a fully faithful functor* $N: \mathscr{V} \to \mathrm{Int}\,\mathscr{V}$ *is defined by* $N(X) = (X, I), N(f) = f$.

Proof. The following two diagrammatic calculations prove that arrows $1 \otimes \theta^{-1}$ give the identities for composition in $\mathrm{Int}\,\mathscr{V}$.





The proof that composition is associative is as follows:



The fact that N is a fully faithful functor follows from the first vanishing equation. ∎

We now define a tensor product functor $\otimes'\colon \operatorname{Int}\mathscr{V} \times \operatorname{Int}\mathscr{V} \to \operatorname{Int}\mathscr{V}$. The definition on objects is

$$(X, U) \otimes' (X', U') = (X \otimes X', U' \otimes U).$$

For arrows $f\colon (X, U) \to (Y, V), f'\colon (X', U') \to (Y', V')$ in $\operatorname{Int}\mathscr{V}$, the arrow

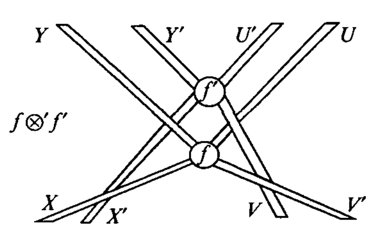
$$f \otimes' f'\colon (X \otimes X', U' \otimes U) \to (Y \otimes Y', V' \otimes V)$$

in $\operatorname{Int}\mathscr{V}$ is defined to be the following composite

$$X \otimes X' \otimes V' \otimes V \xrightarrow{c \otimes c^{-1}} X' \otimes X \otimes V \otimes V' \xrightarrow{1 \otimes f \otimes 1} X' \otimes Y \otimes U \otimes V'$$

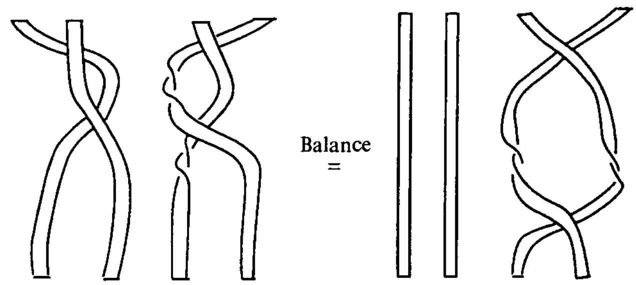
$$\xrightarrow{c^{-1} \otimes c^{-1}} Y \otimes X' \otimes V' \otimes U \xrightarrow{1 \otimes f' \otimes 1} Y \otimes Y' \otimes U' \otimes U,$$

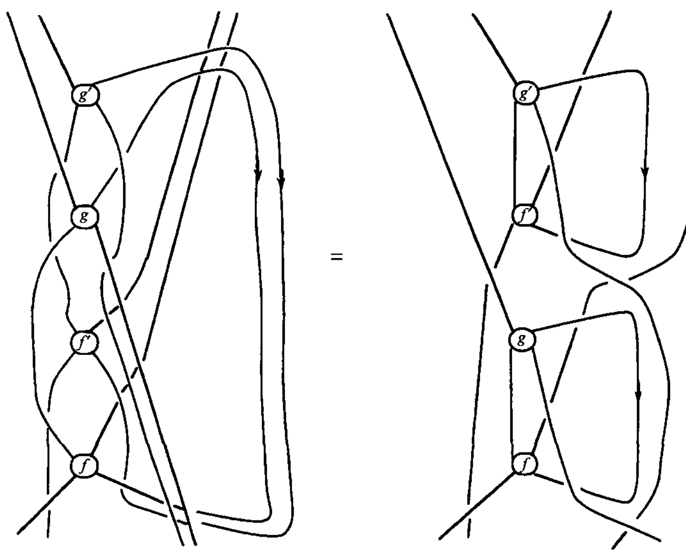
which is diagrammatically represented as below.



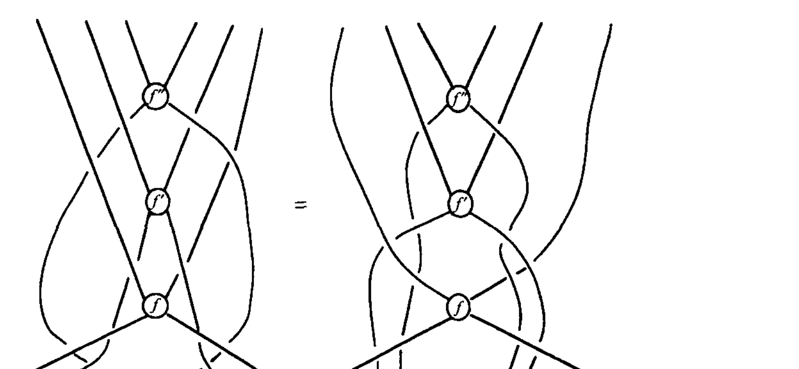
PROPOSITION 4·2. *The above tensor product enriches* $\operatorname{Int}\mathscr{V}$ *with the structure of monoidal category, and the functor* $N\colon \mathscr{V} \to \operatorname{Int}\mathscr{V}$ *is then* (*strong*) *monoidal.*

Proof. The following two diagrammatic equalities (whose verification using the axioms we leave to the reader) mean that $\otimes'$ is a functor.





The calculation that it is associative is the following purely balanced diagrammatic observation.



This completes the proof that Int$\mathscr{V}$ is a monoidal category. The fact that N preserves tensor product and unit is obvious.

5. *The tortile structure on* Int $\mathscr{V}$

For each pair $(X, U), (X', U')$ of objects of Int $\mathscr{V}$, let

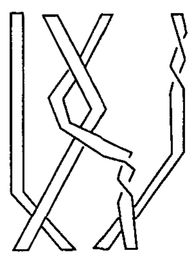
$$c_{(X,U),(X',U')}\colon (X, U) \otimes' (X', U') \to (X', U') \otimes' (X, U)$$

be the arrow in Int $\mathscr{V}$ given by the composite in $\mathscr{V}$ of the following four arrows

$$X \otimes X' \otimes U' \otimes U \xrightarrow{c_{X,X'} \otimes c^{-1}_{U,U'}} X' \otimes X \otimes U \otimes U' \xrightarrow{1 \otimes 1 \otimes \theta^{-1}_U \otimes 1} X' \otimes X \otimes U \otimes U'$$

$$\xrightarrow{1 \otimes c^{-1}_{U,X} \otimes \theta^{-1}_{U'}} X' \otimes U \otimes X \otimes U' \xrightarrow{1 \otimes c^{-1}_{X,U} \otimes 1} X' \otimes X \otimes U \otimes U'.$$

Diagrammatically, this is as follows.



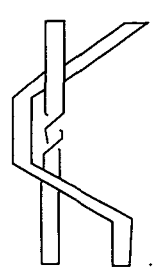
For each object (X, U) of Int $\mathscr{V}$, let

$$\theta_{(X,U)}\colon (X, U) \to (X, U)$$

be the arrow in Int $\mathscr{V}$ given by the composite in $\mathscr{V}$ of the three arrows

$$X \otimes U \xrightarrow{c^{-1}_{U,X}} U \otimes X \xrightarrow{1 \otimes \theta_X} U \otimes X \xrightarrow{c^{-1}_{X,U}} X \otimes U,$$

which is represented by the following diagram.

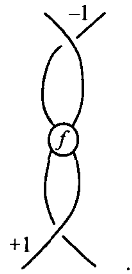


Of course, the dual $(X, U)^\vee$ of (X, U) is (U, X). The counit (or 'exact pairing') is the arrow $\epsilon\colon (U, X) \otimes' (X, U) \to (I, I)$ in Int $\mathscr{V}$ given by the arrow $1_U \otimes \theta_X\colon U \otimes X \to U \otimes X$ in $\mathscr{V}$. The unit $\eta\colon (I, I) \to (X, U) \otimes' (U, X)$ is the arrow $1_X \otimes \theta^{-1}_U\colon X \otimes U \to X \otimes U$ in $\mathscr{V}$.

It seems helpful here to also describe the dual on maps. The dual $f^\vee\colon (Y, V)^\vee \to (X, U)^\vee$ of an arrow $f\colon (X, U) \to (Y, V)$ in Int $\mathscr{V}$ is the following composite in $\mathscr{V}$:

$$V \otimes X \xrightarrow{\theta_V \otimes 1} V \otimes X \xrightarrow{c} X \otimes V \xrightarrow{f} Y \otimes U \xrightarrow{c} U \otimes Y \xrightarrow{1 \otimes \theta^{-1}_Y} U \otimes Y.$$

Or, diagrammatically,

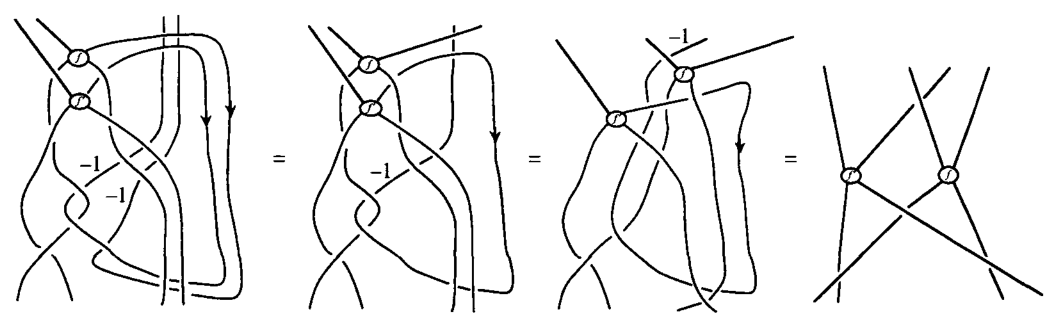


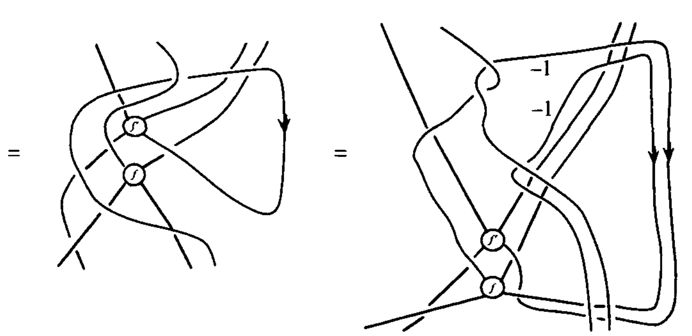
Proposition 5·1. *The arrows* $c_{(X,U),(X',U')}$, $\theta_{(X,U)}$ *enrich the monoidal category* $\mathrm{Int}\,\mathscr{V}$ *with a tortile structure, and the fully faithful monoidal functor* $N\colon \mathscr{V} \to \mathrm{Int}\,\mathscr{V}$ *is then traced.*

Proof. The naturality of c is expressed by the commutativity of the square

$$\begin{tikzcd}
(X,U)\otimes'(X',U') \arrow[r, "c"] \arrow[d, "f\otimes' f'"'] & (X',U')\otimes'(X,U) \arrow[d, "f'\otimes' f"] \\
(Y,V)\otimes'(Y',V') \arrow[r, "c"'] & (Y',V')\otimes'(Y,V)
\end{tikzcd}$$

which is proved diagrammatically as follows.

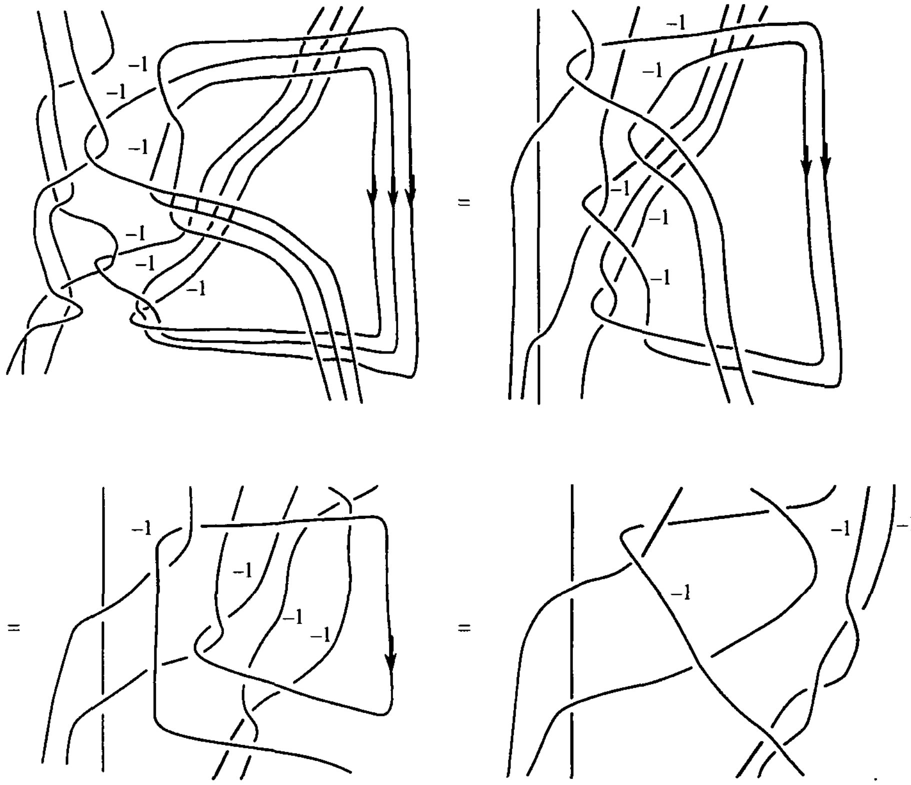




One of the braiding conditions is the commutativity of the triangle

$$\begin{tikzcd}
(X,U)\otimes'(X',U')\otimes'(X'',U'') \arrow[rr, "c"] \arrow[dr, "1\otimes' c"'] & & (X'',U'')\otimes'(X,U)\otimes'(X',U') \\
& (X,U)\otimes'(X'',U'')\otimes'(X',U') \arrow[ur, "c\otimes' 1"'] &
\end{tikzcd}$$

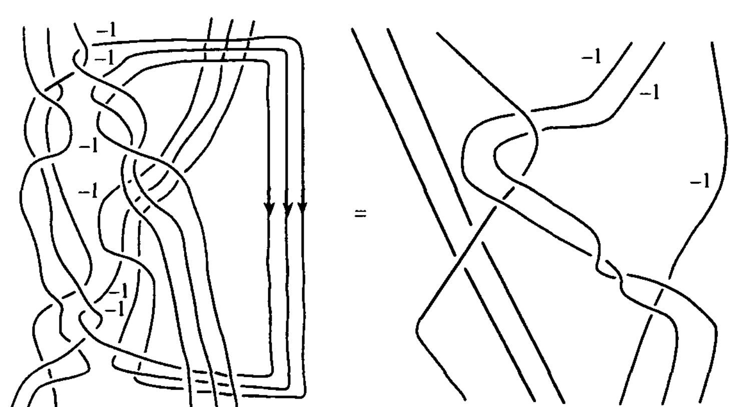
which is proved diagrammatically (with a few steps missing) as follows.



The commutativity of the other braiding triangle

$$\begin{tikzcd}
(X,U)\otimes'(X',U')\otimes'(X'',U'') \arrow[rr, "c"] \arrow[dr, "c\otimes' 1"'] & & (X',U')\otimes'(X'',U'')\otimes'(X,U) \\
& (X',U')\otimes'(X,U)\otimes'(X'',U'') \arrow[ur, "1\otimes' c"'] &
\end{tikzcd}$$

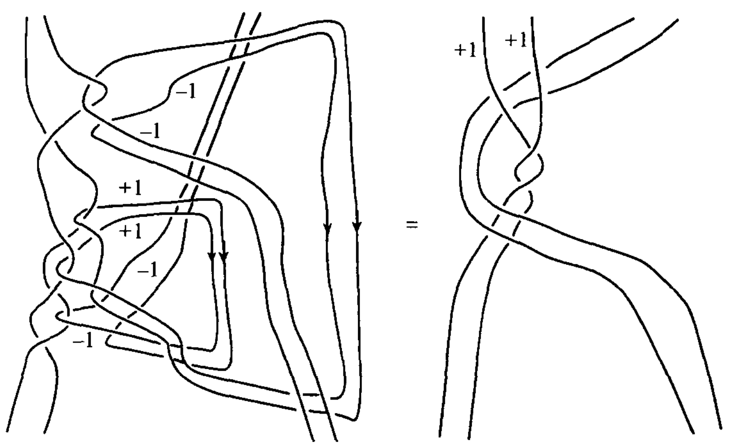
amounts to the following equality of diagrams whose verification we leave to the reader in the interest of saving some space.



The invertibility of c can be proved directly, but, at any rate, it is a consequence of the other axioms on a tortile monoidal category. To complete the proof that Int $\mathscr{V}$ is balanced, we must see that the diagram

$$\begin{tikzcd}
(X,U)\otimes'(X',U') \arrow[r, "c_{(X,U),(X',U')}"] \arrow[d, "\theta_{(X,U)\otimes'(X',U')}"'] & (X',U')\otimes'(X,U) \arrow[d, "\theta_{(X',U')}\otimes'\theta_{(X,U)}"] \\
(X,U)\otimes'(X',U') & (X',U')\otimes'(X,U) \arrow[l, "c_{(X',U'),(X,U)}"']
\end{tikzcd}$$

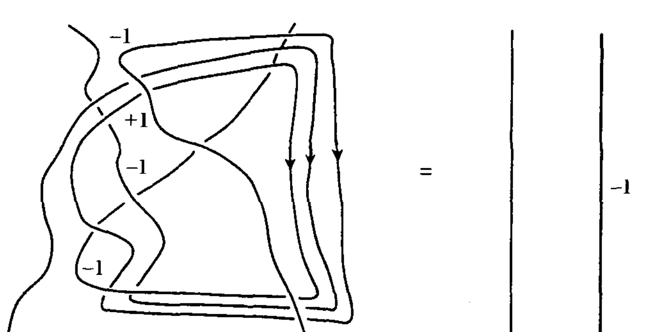
commutes. This amounts to the following equality which, again, we leave to the reader.



We now turn our attention to the duality conditions. Commutativity of the triangle

$$\begin{tikzcd}
(U,X) \arrow[r, "1\otimes'\eta"] \arrow[dr, "1"'] & (U,X)\otimes'(X,U)\otimes'(U,X) \arrow[d, "\epsilon\otimes' 1"] \\
 & (U,X)
\end{tikzcd}$$

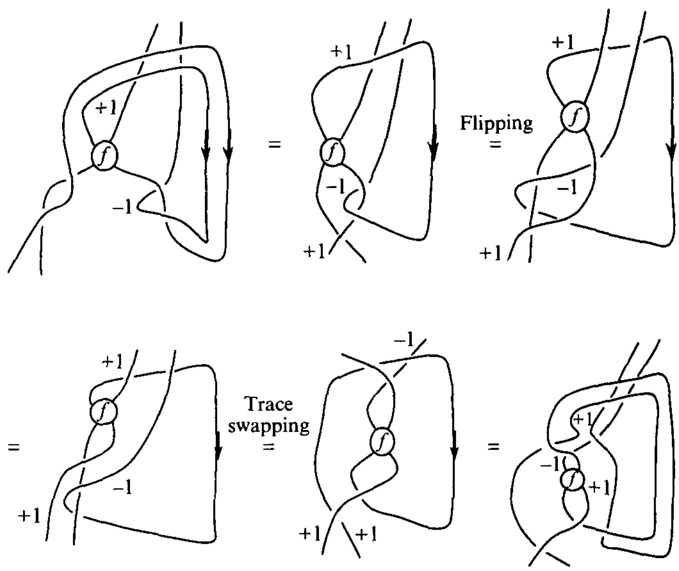
is proved by the following equality, and the other duality triangle is left to the reader.



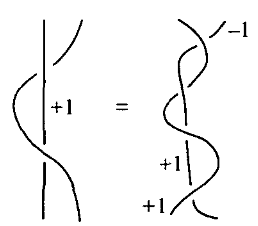
To verify the formula for the dual of a map, we shall prove the commutativity of the square

$$\begin{tikzcd}
(V, Y) \otimes' (X, U) \arrow[r, "f^\vee \otimes' 1"] \arrow[d, "1 \otimes' f"'] & (U, X) \otimes' (X, U) \arrow[d, "\epsilon"] \\
(V, Y) \otimes' (Y, V) \arrow[r, "\epsilon"'] & (I, I)
\end{tikzcd}$$

which comes down to the following diagrammatic calculation where we start with the lower leg of the square. Lemmas 2·2 and 2·3 are used.



With this, we are in a position to verify the remaining non-trivial tortility condition $\theta_{(U,X)} = (\theta_{(X,U)})^\vee$ (the condition $\theta_{(I,I)} = 1$ is trivially true). This is just the following obvious equality.

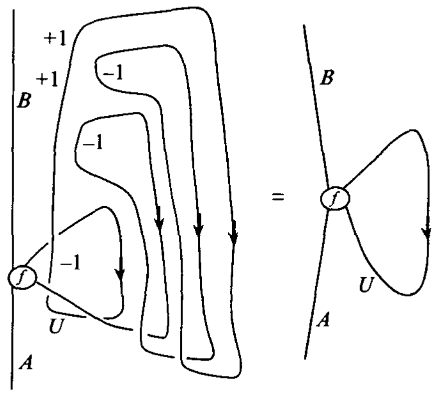


That $N\colon \mathscr{V} \to \operatorname{Int}\mathscr{V}$ preserves the braiding and twist is clear from the definitions of these arrows in $\operatorname{Int}\mathscr{V}$. To see that N preserves trace we must see that, for all arrows $f\colon A \otimes U \to B \otimes U$ in $\mathscr{V}$, the canonical trace of $f\colon (A \otimes U, I) \to (B \otimes U, I)$ in $\operatorname{Int}\mathscr{V}$ is $N(\mathrm{Tr}^U(f))$. That is, we must see that the following composite in $\operatorname{Int}\mathscr{V}$ is equal to $\mathrm{Tr}^U(f)\colon (A, I) \to (B, I)$.

$$(A,I) \xrightarrow{1\otimes'\eta} (A,I) \otimes' (U,I) \otimes' (I,U) \xrightarrow{f\otimes' 1} (B,I) \otimes' (U,I) \otimes' (I,U) \xrightarrow{1\otimes' c}$$

$$(B,I) \otimes' (I,U) \otimes' (U,I) \xrightarrow{1\otimes'\theta} (B,I) \otimes' (I,U) \otimes' (U,I) \xrightarrow{1\otimes'\epsilon} (B,I).$$

This comes down to the following diagrammatic equality.



Let **TortMon** denote the 2-category whose objects are tortile monoidal categories, whose arrows are balanced strong monoidal functors, and whose 2-cells are monoidal natural transformations. Let **TraMon** be the 2-category whose objects are traced monoidal categories, whose arrows are traced strong monoidal functors, and whose 2-cells are monoidal natural transformations. The existence of canonical trace (Proposition 3·1) means we have an inclusion of **TortMon** in **TraMon**.

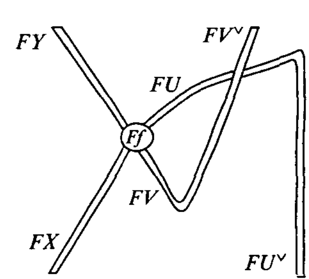
PROPOSITION 5·2. *Suppose $\mathscr{V}$ is a traced monoidal category and $\mathscr{W}$ is a tortile monoidal category. Then, for all traced monoidal functors $F: \mathscr{V} \to \mathscr{W}$, there exists a balanced monoidal functor $K: \operatorname{Int}\mathscr{V} \to \mathscr{W}$ which is unique up to monoidal natural isomorphism with the property $KN \cong F$. Moreover, the inclusion of the 2-category* **TortMon** *in* **TraMon** *has a left biadjoint with unit having component at $\mathscr{V}$ given by $N: \mathscr{V} \to \operatorname{Int}\mathscr{V}$.*

Proof. Given F, we shall construct K. On the object (X, U) of $\operatorname{Int}\mathscr{V}$, put $K(X, U) = FX \otimes (FU)^{\vee}$. For $f: (X, U) \to (Y, V)$ in $\operatorname{Int}\mathscr{V}$, put $K(f): K(X, U) \to K(Y, V)$ equal to the composite

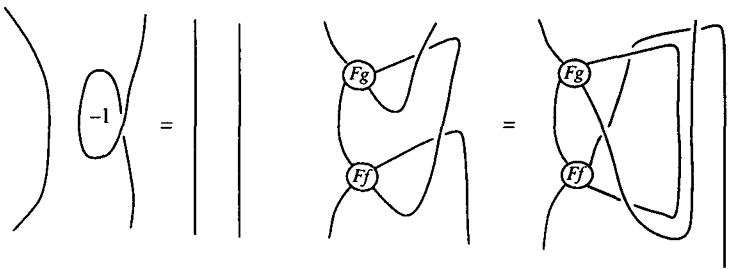
$$FX \otimes FU^{\vee} \xrightarrow{1 \otimes \eta \otimes 1} FX \otimes FV \otimes FV^{\vee} \otimes FU^{\vee} \xrightarrow{Ff \otimes 1 \otimes 1} FY \otimes FU \otimes FV^{\vee} \otimes FU^{\vee}$$

$$\xrightarrow{1 \otimes c^{-1}} FY \otimes FV^{\vee} \otimes FU \otimes FU^{\vee} \xrightarrow{1 \otimes \epsilon'} FY \otimes FV^{\vee}.$$

There is a slight abuse of notation here in that, as well as the usual omission of the associativity constraints, we have omitted the structural isomorphisms $F(X \otimes V) \cong F(X) \otimes F(V), F(Y) \otimes F(U) \cong F(Y \otimes U)$; this simplification can be justified by coherence, but the unconvinced reader should have no problem inserting these isomorphisms. The diagram for $K(f)$ is as follows.



That $K\colon \operatorname{Int}\mathscr{V}\to\mathscr{W}$ is a functor amounts to the following obvious equalities, the second of which uses the fact that F preserves trace



The canonical isomorphism $\phi\colon K(X,U)\otimes K(X',U')\cong K(X\otimes X',U'\otimes X')$ given by the composite

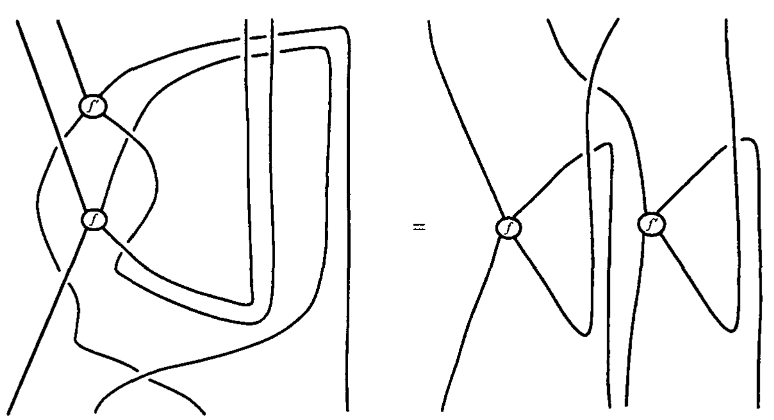
$$FX\otimes FU^{\vee}\otimes FX'\otimes FU'^{\vee}\xrightarrow{1\otimes c\otimes 1}FX\otimes FX'\otimes FU^{\vee}\otimes FU'^{\vee}$$

$$\xrightarrow{\phi\otimes\phi^{\vee}}F(X\otimes X')\otimes F(U'\otimes U)^{\vee},$$

together with the obvious $I\cong K(I,I)$, equip K with the structure of monoidal functor. To see this, we need to prove commutativity of the following diagram.

$$\begin{tikzcd}
K(X,U)\otimes K(X',U') \arrow[r, "1\otimes c\otimes 1"] \arrow[d, "Kf\otimes Kf'"'] & K(X\otimes X',U'\otimes U) \arrow[d, "K(f\otimes f')"] \\
K(Y,V)\otimes K(Y',V') \arrow[r, "1\otimes c\otimes 1"] & K(Y\otimes Y',V'\otimes V).
\end{tikzcd}$$

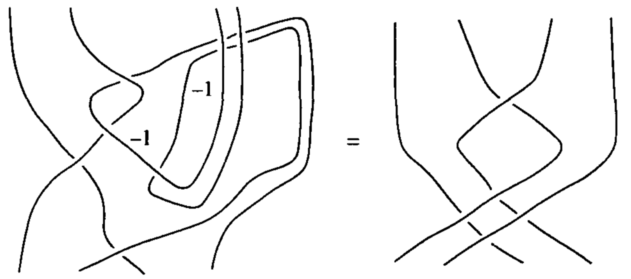
This follows from the obvious diagrammatic equality:



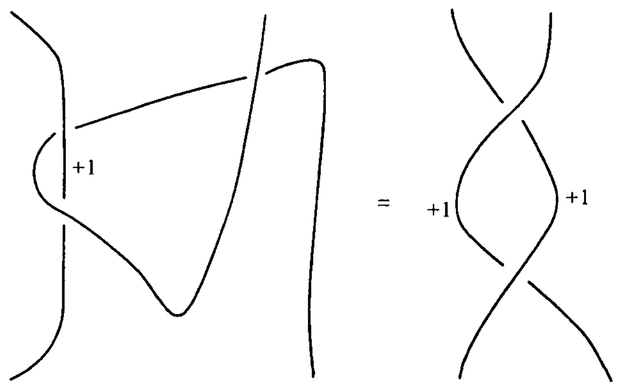
We must see that K is braided; that is, that the following square commutes.

$$\begin{tikzcd}
K(X,U)\otimes K(X',U') \arrow[r, "1\otimes c\otimes 1"] \arrow[d, "c"'] & K(X\otimes X',U'\otimes U) \arrow[d, "Kc"] \\
K(X',U')\otimes K(X,U) \arrow[r, "1\otimes c\otimes 1"] & K(X'\otimes X,U\otimes U')\ .
\end{tikzcd}$$

This follows from the obvious diagrammatic equality:



To complete the proof that K is balanced, we must see that K preserves twists. This follows from the diagrammatic equality:



Clearly $KN \cong F$. We leave it to the reader to check that K is unique up to isomorphism, but note that each object (X, U) of $\mathrm{Int}\,\mathscr{V}$ can be written as $(X, U) = (X, I) \otimes (I, U) = N(X) \otimes N(U)^{\vee}$. The astute reader will realize that this uniqueness follows from the last sentence of the proposition which amounts to the assertion that restriction along N gives an equivalence between the category of balanced monoidal functors $K\colon \mathrm{Int}\,\mathscr{V} \to \mathscr{W}$ and the category of traced monoidal functors $F\colon \mathscr{V} \to \mathscr{W}$. To prove this equivalence it remains to show that, for all balanced monoidal functors K, $L\colon \mathrm{Int}\,\mathscr{V} \to \mathscr{W}$, restriction along N gives a bijection between monoidal natural transformations $\alpha\colon K \to L$ and monoidal natural transformations $\beta\colon KN \to LN$. The inverse bijection takes β to the natural transformation α whose component at (X, U) is the composite

$$K(X, U) \xrightarrow{\phi^{-1}} KN(X) \otimes KN(U)^{\vee} \xrightarrow{\beta_X \otimes \gamma_U{}^{\vee}} LN(X) \otimes LN(U)^{\vee} \xrightarrow{\phi} L(X, U),$$

where $(\gamma_U)^{\vee} = \beta_U$. We leave the few remaining details to the reader who should refer to [**JS1**; proposition 7·1] to see that any such α (and hence any such β) is invertible.

6. *Some examples of traced monoidal categories*

Every full subcategory of a tortile monoidal category (in particular, of a compact closed category) provides an example of a traced monoidal category by Propositions 2·4 and 3·1. In particular, the free traced monoidal category $\mathbb{T}^{+}$ on a single generating object is a full subcategory of the free tortile monoidal category $\mathbb{T}$ on a single generating object. Recall from [**Sm**], [**FY2**], or [**JS4**] that the objects of $\mathbb{T}$ are words in the symbols $-$ and $+$, and that the arrows are tangles on ribbons. Then $\mathbb{T}^+$ is the full subcategory of $\mathbb{T}$ consisting of the objects which are words on the single symbol $+$. The construction of this paper leads us to a tortile monoidal category Int $\mathbb{T}^+$ equivalent to $\mathbb{T}$. The point is that every word in $-, +$ is canonically isomorphic to one in which all the pluses come before the minuses.

It is more interesting to find examples which are not obviously full subcategories of tortile monoidal categories so that the construction of this paper will provide new examples of tortile monoidal categories. For this we are indebted to Steve Bloom for making the connection with iteration and hence with our own notion of iterative bicategory [**CJSV**]. It should suffice here to give an illustrative example where bicategories can be avoided.

Let $\mathscr{R}el$ denote the category whose objects are (small) sets and whose arrows are relations. Recall that a relation $R: X \to Y$ is a subset $R \subseteq X \times Y$, and that composition of relations $R: X \to Y, S: Y \to Z$ is given by $SR = \{(x, z) : \exists_y (x, y) \in R \text{ and } (y, z) \in S\}$. The compositional identities are the diagonal relations Δ_X. The category $\mathscr{R}el$ becomes monoidal with addition $X + Y$ (disjoint union) of sets as the tensor product. In fact, $X + Y$ is both the coproduct and product of the objects X and Y in the category $\mathscr{R}el$. It follows that arrows in $\mathscr{R}el$ between multiple tensor products can be written as matrices of arrows between the components. For example, a relation $R: X + V \to Y + U$ can be written as a matrix

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X + V \to Y + U,$$

where the entries are relations $A: X \to Y, B: V \to Y, C: X \to U, D: V \to U$. Composition of relations between decomposed sets is just given by matrix multiplication using union for addition, and composition for multiplication, on the matrix entries. Note that the braiding $X + Y \to Y + X$ is represented by the matrix

$$\begin{bmatrix} \varnothing & \Delta_Y \\ \Delta_X & \varnothing \end{bmatrix}.$$

This braiding is a symmetry so the twist is the identity.

For a relation $R: U \to U$ on a set U, write R^* for the smallest reflexive, transitive relation containing R. The construction $R \mapsto R^*$ has the following easily verified properties of an iteration operation [**BÉ**].

LEMMA 6·1. *For relations $R, P: U \to V, S: V \to U$, the following equations hold:*

(i) $(R \cup S)^* = (R^*S)^*R^*$ *for* $U = V$;

(ii) $(RS)^* = \Delta_U \cup R(SR)^*S$;

(iii) $R^* = \Delta_U \cup RR^*$ *for* $U = V$;

(iv) $R^*R = RR^*$ *for* $U = V$;

(v) $(RS)^*R = R(SR)^*$;

(vi) $\varnothing^* = \Delta_U$ *for* $\varnothing \subseteq U \times U$;

(vii) $R \subseteq P$ *implies* $R^* \subseteq P^*$.

We require another equality called the pairing identity [**BÉ**].

LEMMA 6·2. *If* $R: X+U \to X+U$ *is a relation given by the matrix*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

then $R^*: X+U \to X+U$ *is the relation given by the matrix*

$$\begin{bmatrix} E & EBD^* \\ D^*CE & D^* \cup D^*CEBD^* \end{bmatrix}$$

where $E = (A \cup BD^*C)^*$.

(Note also that, by Lemma 6·1 (i), $E = (A^*BD^*C)^* A^*$.)

PROPOSITION 6·3. *Consider* $\mathcal{R}el$ *as a symmetric monoidal category where the tensor product is disjoint union of sets. A trace is given as follows:*

$$\mathrm{Tr}^U_{X,Y}(R) = A \cup BD^*C : X \to Y \quad \text{for} \quad R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X+U \to Y+U.$$

Proof. The axioms for a trace are proved by calculations using Lemmas 6·1 and 6·2. Sliding:

$$\mathrm{Tr}\begin{bmatrix} A & BP \\ C & DP \end{bmatrix} = A \cup BP(DP)^*C = A \cup B(PD)^*PC = \mathrm{Tr}\begin{bmatrix} A & B \\ PC & PD \end{bmatrix};$$

tightening:

$$Q\left(\mathrm{Tr}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)P = Q(A \cup BD^*C)P = QAP \cup QBD^*CP = \mathrm{Tr}\begin{bmatrix} QAP & QB \\ CP & D \end{bmatrix};$$

vanishing:

$$\mathrm{Tr}^U\left(\mathrm{Tr}^V\begin{bmatrix} L & M & N \\ P & A & B \\ Q & C & D \end{bmatrix}\right)$$

$$= \mathrm{Tr}^U\left(\begin{bmatrix} L & M \\ P & A \end{bmatrix} \cup \begin{bmatrix} N \\ B \end{bmatrix} D^*[Q \quad C]\right) = \mathrm{Tr}^U\begin{bmatrix} L \cup ND^*Q & M \cup ND^*C \\ P \cup BD^*Q & A \cup BD^*C \end{bmatrix}$$

$$= L \cup ND^*Q \cup (M \cup ND^*C)(A \cup BD^*C)^*(P \cup BD^*Q)$$

$$= L \cup ND^*Q \cup (M \cup ND^*C)E(P \cup BD^*Q) \text{ (using } E \text{ as in Lemma 5·2)}$$

$$= L \cup MEP \cup MEBD^*Q \cup ND^*CEP \cup N(D^* \cup D^*CEBD^*)Q$$

$$= L \cup MEP \cup NGP \cup MFQ \cup NHQ \text{ (for appropriate } F, G, H\text{)}$$

$$= L \cup [ME \cup NG \quad MF \cup NH]\begin{bmatrix} P \\ Q \end{bmatrix}$$

$$= L \cup [M \quad N]\begin{bmatrix} E & F \\ G & H \end{bmatrix}\begin{bmatrix} P \\ Q \end{bmatrix} = \mathrm{Tr}^{U+V}\begin{bmatrix} L & M & N \\ P & A & B \\ Q & C & D \end{bmatrix};$$

superposing:

$$\mathrm{Tr}\left(\begin{bmatrix}\Delta & \varnothing & \varnothing\\ \varnothing & \varnothing & \Delta\\ \varnothing & \Delta & \varnothing\end{bmatrix}\begin{bmatrix}A & B & \varnothing\\ C & D & \varnothing\\ \varnothing & \varnothing & Q\end{bmatrix}\begin{bmatrix}\Delta & \varnothing & \varnothing\\ \varnothing & \varnothing & \Delta\\ \varnothing & \Delta & \varnothing\end{bmatrix}\right) = \mathrm{Tr}\begin{bmatrix}A & \varnothing & B\\ \varnothing & Q & \varnothing\\ C & \varnothing & D\end{bmatrix}$$

$$= \begin{bmatrix}A & \varnothing\\ \varnothing & Q\end{bmatrix} \cup \begin{bmatrix}B\\ \varnothing\end{bmatrix} D^*[C \quad \varnothing] = \begin{bmatrix}\mathrm{Tr}\begin{bmatrix}A & B\\ C & D\end{bmatrix} & \varnothing\\ \varnothing & Q\end{bmatrix};$$

yanking:

$$\mathrm{Tr}\begin{bmatrix}\varnothing & \Delta\\ \Delta & \varnothing\end{bmatrix} = \varnothing \cup \Delta\varnothing^*\Delta = \Delta. \quad \blacksquare$$

It is therefore possible to form the compact closed category Int$\mathcal{R}el$ from the traced monoidal category $\mathcal{R}el$. The objects of Int$\mathcal{R}el$ can be called integer sets in contrast to the objects of $\mathcal{R}el$ which are natural sets. The cardinality of the integer set (X, U) is defined to be the difference $\#(X, U) = \#X - \#U$ of the cardinalities of X, U.

An explicit description of Int$\mathcal{R}el$ may be of interest. The objects are pairs (X, U) of sets. Arrows $R: (X, U) \to (Y, V)$ can be depicted as diagrams of relations:

$$\begin{tikzcd} X \arrow[r, "C"'] \arrow[d, "A"'] & U \\ Y & V \arrow[l, "B"'] \arrow[u, "D"'] \end{tikzcd} .$$

Composition in Int$\mathcal{R}el$ is given by:

$$\begin{tikzcd} X \arrow[r, "C"'] \arrow[d, "A"'] & U \\ Y \arrow[r, bend right, "G"'] \arrow[d, "E"'] & V \arrow[l, bend right, "B"'] \arrow[u, "D"'] \\ Z & W \arrow[l, "F"'] \arrow[u, "H"'] \end{tikzcd} = \begin{tikzcd} X \arrow[r, "C \cup D(GB)^*GA"'] \arrow[d, "E(BG)^*A"'] & U \\ Z & W \arrow[l, "F \cup E(BG)^*BH"'] \arrow[u, "D(GB)^*H"'] \end{tikzcd} .$$

Tensor product is given by:

$$\begin{tikzcd} X \arrow[r, "C"'] \arrow[d, "A"'] & U \\ Y & V \arrow[l, "B"'] \arrow[u, "D"'] \end{tikzcd} \otimes' \begin{tikzcd} X' \arrow[r, "C'"'] \arrow[d, "A'"'] & U' \\ Y' & V' \arrow[l, "B'"'] \arrow[u, "D'"'] \end{tikzcd} = \begin{tikzcd} X+X' \arrow[r, "\begin{bmatrix}\varnothing & C'\\ C & \varnothing\end{bmatrix}"'] \arrow[d, "\begin{bmatrix}A & \varnothing\\ \varnothing & A'\end{bmatrix}"'] & U'+U \\ Y+Y' & V'+V \arrow[l, "\begin{bmatrix}\varnothing & B\\ B' & \varnothing\end{bmatrix}"'] \arrow[u, "\begin{bmatrix}D' & \varnothing\\ \varnothing & D\end{bmatrix}"'] \end{tikzcd} .$$

It follows from Lemma 6·1 (vii) that inclusion of relations is compatible with composition in Int$\mathcal{R}el$, and it is clear that it is also compatible with the tensor product. So Int$\mathcal{R}el$ becomes a monoidal 2-category whose 2-cells are merely inclusions.

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