The May filtration on THH and faithfully flat descent

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Abstract

In this paper, we prove that both topological Hochschild homology and topological cyclic homology are sheaves for the fpqc topology on connective commutative ring spectra, by exploiting the May filtration on topological Hochschild homology.

Contents

1   Introduction

Topological Hochschild homology (THH) and topological cyclic homology (TC), along with their Dennis and cyclotomic trace maps, have proved to be very useful tools for calculating algebraic K-theory. Algebraic K-theory has the nice property of being a sheaf for the Nisnevich topology; in practice, this allows one to decompose calculations into more manageable pieces. However, K-theory fails to satisfy the more computationally useful étale descent. Fortunately, THH and TC do not have this deficiency; they have fpqc descent for commutative rings, by a theorem of Bhatt-Morrow-Scholze in [BMS19]. Additionally, they have étale descent for $\mathbb{E}_\infty$-ring spectra by a theorem of Clausen-Mathew in [CM19]. At this stage, it is reasonable to wonder whether it is possible to promote fpqc descent to ring spectra. Our work answers this in the affirmative, provided we work with connective commutative ring spectra, and our main theorem is the following:

Theorem 1.1. *Let* $F = \text{THH}$, $\text{THH}(-)^{h\mathbb{T}}$, $\text{THH}(-)^{hC_p}$, $\text{THH}(-)_{hC_p}$, $\text{THH}(-)^{tC_p}$, *or* TC *viewed as functors* $\text{CAlg}_{\geq 0} \to \text{Sp}$. *Then* F *satisfies descent for the fpqc topology on* $\text{CAlg}_{\geq 0}$.

One would hope that our proof might follow the same strategy as for commutative rings, but working with spectra excludes this possibility. In [BMS19], THH is shown to have fpqc descent by a series of reductions. First, the authors reduce the problem to the case of Hochschild homology, by

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using the Postnikov tower of $\mathbb{S}$, and then reduce to the case of wedge powers of the cotangent complex, via the Hochschild-Konstant-Rosenberg (HKR) filtration on Hochschild homology. Finally, the case of the cotangent complex is handled by a theorem of Bhatt, a proof of which appears as [BMS19, 3.1]. Since many of the tools used in [BMS19] are unique to working with (simplicial) commutative rings, different methods became necessary. Our strategy exploits structural properties of the May filtration on THH, as introduced in [AKS18]. In particular, this filtration is Hausdorff, and its associated graded can be calculated via a canonical equivalence $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}(A) \simeq \mathrm{THH}(H\pi_* A)$. By proving Theorem 1.1 in the case of generalized Eilenberg-Mac Lane spectra, we bootstrap to the general case using our control of the May filtration.

We now provide a summary of our work. In Section 2, we carefully prove several folklore results on filtered spectra, all of which are undoubtedly known to experts. In particular, we prove that the "evaluate at zero" functor and the associated graded functor $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp}) \rightarrow \mathrm{Sp}$ are symmetric monoidal for the Day convolution, and that the Whitehead tower functor $\mathrm{Sp} \rightarrow \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp})$ is also symmetric monoidal. In Section 3, we review some facts about Čech conerves and prove a special case of our main theorem for generalized Eilenberg-Mac Lane spectra. In Section 4, we present an ∞-categorical treatment of the May filtration on THH and use this to construct a similar filtration on the C_p-homotopy fixed points of THH. Before doing so, we give a brief review of the symmetric monoidal envelope construction which appears in [Lur17, 2.2.4], which is crucial to the ∞-categorical treatment of the May filtration. In Section 5, we turn to the proof of our main theorem, and use the May filtration in conjunction with our work in Section 3 to deduce the general result.

Throughout this work, we freely use the theory of ∞-categories, incarnated via quasicategories, as in [Lur09] and [Lur17]. For consistency, wherever possible, we also follow the notation therein. Throughout, we refer to the ∞-category of spectra by Sp and use $\otimes$ to denote the smash product. Additionally, we will use the notation $\mathrm{Sp}_{\geq 0}$ to denote connective spectra, and $\mathrm{CAlg}_{\geq 0}$ to denote connective commutative ring spectra. We will fix the following convention that a spectrum X is n-connective provided that $\pi_m X \cong 0$ for $m < n$, and a map $f : X \rightarrow Y$ of spectra is n-connective provided that $\pi_m f$ is an isomorphism in degrees $m < n$ and a surjection when $n = m$. We will make frequent use of the groundbreaking work of Nikolaus-Scholze, providing an ∞-categorical treatment of topological Hochschild homology, topological cyclic homology, and cyclotomic spectra, [NS18, III.2.3, II.1.8, II.1.6].

2 Filtered spectra

Our main technical tool is the May filtration on THH, as defined in [AKS18], which we must import into the ∞-categorical context. To do so, we establish several folklore results about filtered (resp. $\mathbb{N}$-filtered) spectra. While the May filtration we consider is indexed by $\mathbb{Z}_{\geq 0}^{op}$, it will be technically useful to work with $\mathbb{Z}^{op}$-indexed filtrations as well. Unless otherwise specified, $\mathbb{Z}^{op}$ and $\mathbb{Z}_{\geq 0}^{op}$ carry the usual ordering, and by abuse, we use the same notation to refer to the nerves of these posets.

Definition 2.1. The ∞-category of filtered spectra is $\mathrm{Fil}(\mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}^{op}, \mathrm{Sp})$, and the ∞-category of $\mathbb{N}$-filtered spectra is $\mathrm{Tow}(\mathrm{Sp}) = \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathrm{Sp})$.

It is readily checked that both of these ∞-categories are presentably symmetric monoidal and stable where the symmetric monoidal product in both cases is Day convolution; see [Lur09, 5.5.3.6], [Lur17, 1.1.3.1], and [Gla13, 2.13]. Throughout, we will let $\circledast$ denote the Day convolution product and allow context to dictate exactly which product we mean. The unit object in $\mathrm{Fil}(\mathrm{Sp})$, denoted

by $\mathbb{1}_{\text{Fil}}$, is the filtered spectrum

$$\cdots \to 0 \to 0 \to \mathbb{S} \to \mathbb{S} \to \cdots$$

which is $\mathbb{S}$ in degrees $n \leq 0$ with identity maps in negative degree. The unit object in Tow(Sp), denoted by $\mathbb{1}_{\text{Tow}}$, is the $\mathbb{N}$-filtered spectrum which is $\mathbb{S}$ in degree 0, and 0 otherwise:

$$\cdots \to 0 \to 0 \to \mathbb{S}$$

Because Fil(Sp) and Tow(Sp) are symmetric monoidal, we may consider the ∞-categories of associative (resp. commutative) algebra objects, which we denote by Alg^{Fil} (resp. CAlg^{Fil}) and Alg^{Tow} (resp. CAlg^{Tow}). These categories will appear later when we work with the May filtration on THH. The following are the technical claims we seek to prove:

1. The evaluation functor $\text{ev}_0 : \text{Tow}(\text{Sp}) \to \text{Sp}$ is symmetric monoidal;

2. the associated graded functor $\text{gr}_* : \text{Tow}(\text{Sp}) \to \text{Sp}$ is colimit-preserving and symmetric monoidal functor; and

3. the Whitehead tower functor $\text{Sp} \to \text{Tow}(\text{Sp})$ is lax monoidal.

Before commencing, we will record several general results regarding Fil(Sp), Tow(Sp), and their relationship to one another.

2.1 Properties of Fil(Sp), Tow(Sp), and Day convolution

Recall, that for C a stable ∞-category, an object $X \in C$ is said to *generate* C, provided that $\pi_0 \text{Map}_C(X, Y) \simeq *$ implies that $Y \simeq 0$. We will say that a collection of objects of C, $\{X_i\}_{i \in I}$, *jointly generates* C, provided that $\pi_0 \text{Map}(X_i, Y) \simeq *$ for all $i \in I$ implies that $Y \simeq 0$.

Lemma 2.2. *Let K be a simplicial set and let $\text{ev}_k : \text{Fun}(K, \text{Sp}) \to \text{Sp}$ denote evaluation at $k \in K$. For all objects $k \in K$, the functors ev_k admit left adjoints, $L_k : \text{Sp} \to \text{Fun}(K, \text{Sp})$ given on vertices by $L_k X : k' \mapsto X^{\otimes \text{Map}_K(k, k')}$. Additionally, the objects $S(k) = L_k \mathbb{S}$ have the following properties:*

1. *$S(k)$ is compact for all $k \in K$;*

2. *the $S(k)$'s jointly generate $\text{Fun}(K, \text{Sp})$; and*

3. *the collection $\{\Sigma^n S(k)\}_{n \in \mathbb{Z}, k \in K}$ generates $\text{Fun}(K, \text{Sp})$ under small colimits.*

Proof. Since ev_k preserves small limits and colimits, it admits a left adjoint $L_k : \text{Sp} \to \text{Fun}(K, \text{Sp})$. The compactness of $S(k)$ follows from the fact that $\mathbb{S}$ is compact in Sp and ev_k preserves colimits. To explicitly identify L_k, we use the following chain of natural equivalences obtained from the end formula, which appears for example, in [GHN17, 5.2]

$$\text{Map}_{\text{Fun}(K, \text{Sp})}(X^{\otimes \text{Map}_K(k, -)}, E_\bullet) \simeq \varprojlim_{i \to j \in \text{Tw}(K)} \text{Map}_{\text{Sp}} \Big(\varinjlim_{\text{Map}_K(k, i)} X, E_j \Big)$$

$$\simeq \varprojlim_{i \to j \in \text{Tw}(K)} \varprojlim_{\text{Map}_K(k, i)} \text{Map}_{\text{Sp}}(X, E_j)$$

$$\simeq \text{Map}_{\text{Sp}}(X, \varprojlim_{i \to j \in \text{Tw}(K)} E_j^{\text{Map}_K(k, i)})$$

As the exponential object $E_j^{\text{Map}_K(k, i)}$ is equivalent to the mapping spectrum $\underline{\text{Map}}_{\text{Sp}}(\Sigma_+^\infty \text{Map}_K(k, i), E_j)$, an application of the end formula and the spectral co-Yoneda lemma yield:

$$\text{Map}_{\text{Sp}}(X, \varprojlim_{i \to j \in \text{Tw}(K)} E_j^{\text{Map}_K(k, i)}) \simeq \text{Map}_{\text{Sp}} \Big(X, \underline{\text{Map}}_{\text{Fun}(K, \text{Sp})}(\Sigma_+^\infty \text{Map}_K(k, -), E_\bullet) \Big)$$

$$\simeq \text{Map}_{\text{Sp}}(X, E_k).$$

This proves that L_k is given on vertices as claimed.

To prove the final assertions, note that as $\mathrm{Map}_{\mathrm{Fun}(K,\mathrm{Sp})}(S(k), E_\bullet) \simeq E_k$, the collection $\{S(k)\}_{k \in K}$ jointly generates the stable ∞-category $\mathrm{Fun}(K, \mathrm{Sp})$. This allows us to mimic the proof of [Lur17, 1.4.4.2] to show that the objects $\Sigma^n S(k)$ generate $\mathrm{Fun}(K, \mathrm{Sp})$ under small colimits. □

Example 2.3. In the case where $K = \mathbb{Z}^{op}$, the object $S(k)$ is the tower with $\mathbb{S}$ in degrees $\leq k$ and 0 otherwise. The maps $S(k)_n \to S(k)_{n-1}$ are the identity if both objects are $\mathbb{S}$. Similarly, if $K = \mathbb{Z}^{op}_{\geq 0}$, the objects $S(k)$ are given by $\mathbb{S}$ with identity maps in the range $[0, k]$ and 0 otherwise.

This next lemma will give us a convenient way to calculate Day convolution products of ($\mathbb{N}$-)filtered spectra.

Lemma 2.4. *Let* $\oplus : \mathbb{Z}^{op} \times \mathbb{Z}^{op} \to \mathbb{Z}^{op}$ *denote the monoidal product and let* $A_k \subseteq B_k = (\mathbb{Z}^{op} \times \mathbb{Z}^{op}) \times_{\mathbb{Z}^{op}} \mathbb{Z}^{op}_{/k}$ *denote the full subcategory of those pairs* (n, m) *such that* $k + 1 \geq n + m \geq k$. *Then the inclusion* $A_k \subseteq B_k$ *is cofinal.*

Proof. This is a straightforward application of the ∞-categorical version of Quillen's Theorem A, [Lur09, 4.1.3.1]. For $(r, s) \in B_k$, where $r + s > k$, the nerve of $A_k \times_{B_k} (B_k)_{(r,s)/}$ is equivalent to finitely many copies of Λ^2_1 glued together along their initial and terminal vertices in a zig-zag pattern, hence contractible. If $r + s = k$, then the nerve of $A_k \times_{B_k} (B_k)_{(r,s)/}$ is contractible. In either case, the nerve is contractible, so we may conclude by Quillen's Theorem A. □

Remark 2.5. A virtually identical proof holds for $\mathbb{Z}^{op}_{\geq 0}$ as well.

Using Lemma 2.4, we can now determine how the $L_k \mathbb{S}$'s interact through Day convolution.

Lemma 2.6. *Let* K *denote either* $\mathbb{Z}^{op}$ *or* $\mathbb{Z}^{op}_{\geq 0}$. *There is a natural equivalence*

$$S(n) \circledast S(m) \simeq S(n + m).$$

Proof. This is a straightforward, though tedious, computation. By Lemma 2.4, we can compute $(S(n) \circledast S(m))_k$ as the colimit of the diagram

The objects are either of the form $S(n)_s \otimes S(m)_{k-s}$ or $S(n)_s \otimes S(m)_{k+1-s}$. When $k > n + m$, both $S(n)_s \otimes S(m)_{k-s}$ and $S(n)_s \otimes S(m)_{k+1-s}$ are trivial, and hence the colimit of the diagram above is 0. In the case where $k \leq n + m$, the diagram is nonzero in a range: between the objects $S(n)_{k-m} \otimes S(m)_{k-(k-m)}$ and $S(n)_n \otimes S(m)_{k-n}$. In fact, in this range, all objects are given by $\mathbb{S}$ and all maps between nonzero objects are equivalent to $id : \mathbb{S} \to \mathbb{S}$, by Example **??**. This implies that the colimit of the relevant diagram is given by $\mathbb{S}$. In summary, $S(n) \circledast S(m)$ is zero in degrees $k > n + m$ and and given by $\mathbb{S}$ with identity maps in degrees $k \leq n + m$. □

Proposition 2.7. *Let* $\mathcal{E} \subseteq \mathrm{Fil}(\mathrm{Sp})$ *denote the full subcategory of those filtered spectra with the property that the maps* $X_n \to X_{n-1}$ *are equivalences for* $n \leq 0$. *Then,* $\mathcal{E}$ *is a symmetric monoidal subcategory of* $\mathrm{Fil}(\mathrm{Sp})$, *and the restriction map* $\mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$ *induced by the inclusion* $\mathbb{Z}^{op}_{\geq 0} \subseteq \mathbb{Z}^{op}$ *is a symmetric monoidal equivalence* $\mathcal{E} \to \mathrm{Tow}(\mathrm{Sp})$.

Proof. By [Lur09, 4.3.2.15], there is an equivalence of ∞-categories $\theta : \mathcal{E} \to \mathrm{Tow}(\mathrm{Sp})$ given by precomposing with the inclusion $\mathbb{Z}^{op}_{\geq 0} \subseteq \mathbb{Z}^{op}$. We will prove that $\mathcal{E}$ is a symmetric monoidal subcategory of $\mathrm{Fil}(\mathrm{Sp})$ and that θ is symmetric monoidal.

Note that $S(n)$ in $\mathcal{E}$ restricts to $S(n)$ in $\mathrm{Tow}(\mathrm{Sp})$. Since the restriction map $\theta : \mathcal{E} \to \mathrm{Tow}(\mathrm{Sp})$ is an equivalence and compatible with the formation of iterated suspensions, we find that the desuspensions of the objects $S(n)$, where $n \geq 0$, generate $\mathcal{E}$ under small colimits. Thus, to show that $\mathcal{E}$ is a symmetric monoidal subcategory, we can immediately reduce to showing that $\Sigma^s S(n) \circledast \Sigma^t S(m) \in \mathcal{E}$

since $\circledast$ commutes with colimits separately in each variable. Additionally, since $\Sigma^r : \text{Fil(Sp)} \to \text{Fil(Sp)}$ is an equivalence for all $r \in \mathbb{Z}$, the result follows from Lemma 2.6.

It remains to prove that the canonical map

$$\theta_{E,F} : \theta(E_\bullet) \circledast \theta(F_\bullet) \to \theta(E_\bullet \circledast F_\bullet)$$

is an equivalence for all objects in $\mathcal{E}$. Fix $E_\bullet$ and let $\mathcal{X}$ denote the full stable subcategory of $\mathcal{E}$ consisting of those towers $F_\bullet$ for which $\theta_{E,F}$ is an equivalence. Since this category is closed with respect to colimits, we can reduce to checking the claim for objects of the form $S(n)$ where $n \geq 0$. However, the natural map

$$\theta(S(n)) \circledast \theta(S(m)) \to \theta(S(n) \circledast S(m)) \simeq \theta(S(n+m))$$

is an equivalence by Lemma 2.6. This implies that θ is symmetric monoidal and hence $\text{Tow(Sp)} \to \mathcal{E} \subseteq \text{Fil(Sp)}$ is symmetric monoidal. $\qquad\square$

For the remainder of Section 2, we proceed with the proofs of the desired folklore claims.

2.2 The evaluation functor

It will be convenient to follow the notation appearing in [Gla13], the relevant parts of which we now recall.

Definition 2.8. Let $C^\otimes$ and $D^\otimes$ be two symmetric monoidal ∞-categories. Define $\overline{\text{Fun}(C,D)^\otimes}$ to be the simplicial set over Fin_* defined by the universal property

$$\text{Fun}_{\text{Fin}_*}(K, \overline{\text{Fun}(C,D)^\otimes}) = \text{Fun}_{\text{Fin}_*}(C_k^\otimes, D^\otimes)$$

where $C_k^\otimes$ is the pullback of $C^\otimes \to \text{Fin}_*$ along the given structure map $k : K \to \text{Fin}_*$. This simplicial set is an ∞-category by [Gla13, 2.3].

Observation 2.9. If $S \in \text{Fin}_*$, we let S^o denote $S \setminus \{*\}$. Let $C^\otimes$ be a symmetric monoidal ∞-category and let $(\text{Fin}_*)_{/S}^{act}$ denote the full subcategory of $(\text{Fin}_*)_{/S}$ spanned by the active morphisms to S. Then for any $S \in \text{Fin}_*$ there is a canonical product decomposition

$$(\text{Fin}_*)_{/S} \simeq \left(\prod_{s \in S^o} (\text{Fin}_*)_{/\{s\}_+}^{act} \right) \times \text{Fin}_*$$

This implies that for any morphism $f : S \to T$ in Fin_*, there is a canonical decomposition

$$C_f^\otimes \simeq \left(\prod_{\Delta^1, t \in T^o} C_{\mu_{f^{-1}(t)_+}}^\otimes \right) \times C_{\beta_{f^{-1}(*)}}^\otimes.$$

For $V \in \text{Fin}_*$, the map μ_V is given by the active map $V \to \langle 1 \rangle$ if V is nonempty, and the inclusion $* \to \langle 1 \rangle$ if V is empty. The map β_V denotes the unique map $V \to *$. One can check that this decomposition of $C_f^\otimes$ is compatible with the decompositions of $C_S^\otimes$ and $C_T^\otimes$.

Definition 2.10. [Gla13, 2.8] Let $C^\otimes$ and $D^\otimes$ be symmetric monoidal ∞-categories so that the tensor product of $D^\otimes$ preserves colimits separately in each variable. The *Day convolution symmetric monoidal ∞-category* is the largest simplicial subset of $\overline{\text{Fun}(C,D)^\otimes}$ whose vertices over S correspond to functors $F : C_S^\otimes \to D_S^\otimes$ which lie in the essential image of the inclusion

$$\prod_{s \in S^o} \text{Fun}(C,D) \to \text{Fun}(C_S^\otimes, D_S^\otimes)$$

and whose edges over $f : S \to T$ correspond to functors $F : C_f^\otimes \to D_f^\otimes$ in the essential image of the inclusion

$$\prod_{t \in T^o} \operatorname{Fun}_{\Delta^1}(C_{\mu_{f^{-1}(t)_+}}^\otimes, D_{\mu_{f^{-1}(t)_+}}^\otimes) \times \operatorname{Fun}_{\Delta^1}(C_{\beta_{f^{-1}(*)}}^\otimes, D_{\beta_{f^{-1}(*)}}^\otimes) \to \operatorname{Fun}_{\Delta^1}(C_f^\otimes, D_f^\otimes)$$

By [Gla13, 2.10], this is a symmetric monoidal ∞-category, and in the case where $C^\otimes$ also preserves colimits separately, Day convolution preserves colimits separately in each variable [Gla13, 2.13].

With our notation set, we are ready to prove that $\operatorname{ev}_0 : \operatorname{Fun}(\mathbb{Z}_{\geq 0}^{op}, \operatorname{Sp}) \to \operatorname{Sp}$ is symmetric monoidal.

Proposition 2.11. *Let $C^\otimes, D^\otimes$ and $E^\otimes$ be symmetric monoidal ∞-categories where the tensor product of $E^\otimes$ preserves colimits separately in each variable, and let $F^\otimes : C^\otimes \to D^\otimes$ be a symmetric monoidal functor. Then there is an induced "pullback" functor*

$$F^* : \operatorname{Fun}(D, E)^\otimes \to \operatorname{Fun}(C, E)^\otimes$$

given by precomposition with $F^\otimes$. The "pullback" functor $\operatorname{ev}_0^\otimes$, which is induced by $0 : \Delta^0 \to \mathbb{Z}_{\geq 0}^{op}$, is symmetric monoidal.

Proof. The universal property of $\overline{\operatorname{Fun}(D, E)^\otimes}$ from [Gla13, 2.1], immediately implies that $F^\otimes$ induces a functor of ∞-categories

$$\overline{F^*} : \overline{\operatorname{Fun}(D, E)^\otimes} \to \overline{\operatorname{Fun}(C, E)^\otimes}.$$

It now suffices to check that the restriction of $\overline{F^*}$ to $\operatorname{Fun}(D, E)^\otimes$ factors through $\operatorname{Fun}(C, E)^\otimes$.

The vertices of $\operatorname{Fun}(D, E)^\otimes$ over $S \in \operatorname{Fin}_*$ are precisely those functors $D_S^\otimes \to E_S^\otimes$ in the essential image of the inclusion $\operatorname{Fun}(D, E)^S \to \operatorname{Fun}(D_S^\otimes, E_S^\otimes)$. Since $F^\otimes$ is symmetric monoidal, $F_S^\otimes \simeq F^S$, so the vertices of $\operatorname{Fun}(D, E)^\otimes$ are sent to the vertices of $\operatorname{Fun}(C, E)^\otimes$. Similarly, the edges are functors $F_f^\otimes : D_f^\otimes \to E_f^\otimes$ in the essential image of the map

$$\left(\prod_{t \in T^o} \operatorname{Fun}_{\Delta^1}(D_{\mu_{S_t^+}}^\otimes, E_{\mu_{S_t^+}}^\otimes) \right) \times \operatorname{Fun}(D_{\beta_{S_*}}^\otimes, E_{\beta_{S_*}}^\otimes) \to \operatorname{Fun}_{\Delta^1}(D_f^\otimes, E_f^\otimes),$$

where $S_t^+ = f^{-1}(\{t\})_+$ and $S_* = f^{-1}(\{*\})$. As before, $F_f^\otimes \simeq \left(\prod_{\Delta^1, t \in T^o} F_{\mu_{S_t^+}}^\otimes \right) \times F_{\beta_{S_*}}^\otimes$, which means the edges in $\operatorname{Fun}(D, E)^\otimes$ are sent to the edges in $\operatorname{Fun}(C, E)^\otimes$. This implies we have a well-defined functor F^* as claimed.

Let $K^\otimes$ denote the nerve of the category of operators of $\mathbb{Z}_{\geq 0}^{op}$ (this construction appears in [Lur17, 2.0.0.2], for example). The map $0 : \Delta^0 \to \mathbb{Z}_{\geq 0}^{op}$ induces a functor $e : \operatorname{Fin}_* \to K^\otimes$ which induces a "pullback" $\operatorname{ev}_0^\otimes : \operatorname{Tow}(E)^\otimes = \operatorname{Fun}(\mathbb{Z}_{\geq 0}^{op}, E)^\otimes \to \operatorname{Fun}(\Delta^0, E)^\otimes \simeq E^\otimes$. To show that $\operatorname{ev}_0^\otimes$ is symmetric monoidal, we must show that it takes cocartesian edges to cocartesian edges. By [Gla13, 2.10], the cocartesian edges over $f : S \to T$ in $\operatorname{Tow}(E)^\otimes$ are precisely those functors $F_f : K_f^\otimes \to E_f^\otimes$ whose decomposition components are p-left Kan extensions. As $\operatorname{ev}_0^\otimes(F_f)$ is the composite

$$\Delta^1 \cong \operatorname{Fin}_* \times_{\operatorname{Fin}_*} \Delta^1 \longrightarrow K^\otimes \times_{\operatorname{Fin}_*} \Delta^1 \xrightarrow{F_f} E_f^\otimes \longrightarrow E^\otimes,$$

by [Lur09, 4.3.1.4] and [Lur09, 4.3.1.15] it suffices to show that the diagram

$$\begin{array}{ccccc}
\Delta^0 & \xrightarrow{e_S} & K_S^\otimes & \xrightarrow{F_S} & E_f^\otimes \\
\downarrow & & \downarrow & \nearrow^{F_f} & \downarrow^p \\
\Delta^1 & \xrightarrow{e_f} & K_f^\otimes & \longrightarrow & \Delta^1
\end{array}$$

exhibits $\mathrm{ev}_0^\otimes(F_f)$ as the p-colimit of $F_S \circ e_S$. By assumption, we may decompose F_f into a product of active and inert pieces, all of which are relative left Kan extensions. Furthermore, since the formation of overcategories commutes with pullbacks, we only need show that $\mathrm{ev}_0^\otimes(F_g)$ is a p-colimit, where g is one of the active or inert parts of f, as in Observation 2.9. Therefore, the result will follow if we can prove the following claim – Let $f : S \to T$ be a morphism in Fin_*, and let $F_f : K_f^\otimes \to E_f^\otimes$ be p-left Kan extension of $F_f|_{K_S^\otimes}$, exhibited by the diagram:

$$
\begin{array}{ccc}
K_S^\otimes & \longrightarrow & E_f^\otimes \\
\downarrow & \nearrow & \downarrow{\scriptstyle p} \\
K_f^\otimes & \longrightarrow & \Delta^1
\end{array}
$$

Then, $F_f \circ e_f : \Delta^1 \to E_f^\otimes$ is a p-left Kan extension of $F_f|_{K_S^\otimes} \circ e_S : \Delta^0 \to E_f^\otimes$ along the source inclusion $\Delta^0 \subseteq \Delta^1$. To prove this, note that the edge $\mathrm{Fin}_* \times_{\mathrm{Fin}_*} \Delta^1 \to K_f^\otimes$ is a lift of $f : S \to T$, with source $(0, \dots, 0) \in N(\mathbb{Z}_{\geq 0}^{op})^S \simeq K_S^\otimes$. The structure of $K^\otimes$ forces the target of $\tilde{f}$ to be $(0, \dots, 0) \in N(\mathbb{Z}_{\geq 0}^{op})^T \simeq K_T^\otimes$. The functors $\Delta_{/0}^0 \to (K_S^\otimes)_{/\tilde{f}(0)}$ and $\Delta_{/1}^0 \to (K_S^\otimes)_{/\tilde{f}(1)}$ are cofinal by [Lur09, 4.1.2.6], as everything in sight is a contractible Kan complex. This shows that all the relevant induced diagrams are p-colimits, so that $\Delta^1 \to E_f^\otimes$ is a p-left Kan extension, and hence a cocartesian arrow. $\qquad\square$

2.3 The associated graded functor

In this section, we establish that the associated graded functor $\mathrm{gr}_* : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$ is symmetric monoidal by exploiting the result for $\mathrm{Fil}(\mathrm{Sp})$, established in [Lur15, 3.2.1]. Let $(\mathbb{Z}^{op})^{\mathrm{disc}}$ be $\mathbb{Z}^{op}$ endowed with the discrete topology, and let $\mathrm{Gr}(\mathrm{Sp}) = \mathrm{Fun}((\mathbb{Z}^{op})^{\mathrm{disc}}, \mathrm{Sp})$ denote the symmetric monoidal ∞-category of graded spectra equipped with the Day convolution product

Proposition 2.12. [Lur15, 3.2.1] *There exists a functor,* $\mathrm{gr} : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$, *given on vertices by* $X_\bullet \mapsto (X_i/X_{i-1})_{i \in \mathbb{Z}}$, *which is symmetric monoidal for the Day convolution and colimit preserving.*

As $\mathrm{Gr}(\mathrm{Sp}) \simeq \prod_{i \in \mathbb{Z}} \mathrm{Sp}$, there is a colimit-preserving symmetric monoidal functor $\bigoplus : \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Sp}$ given on vertices by $(X_i)_{i \in \mathbb{Z}} \mapsto \bigoplus_{i \in \mathbb{Z}} X_i$. Now define $\mathrm{gr}_*^{\mathrm{Fil}} : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$ as $\bigoplus \circ \mathrm{gr}$, and let $\mathrm{gr}_* : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$ denote the composition of $\mathrm{gr}_*^{\mathrm{Fil}}$ with the symmetric monoidal functor, $\mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Fil}(\mathrm{Sp})$, from Proposition 2.7.

Proposition 2.13. *The associated graded functor* $\mathrm{gr}_* : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$, *is colimit preserving and symmetric monoidal.*

Proof. As $\mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Fil}(\mathrm{Sp})$ is symmetric monoidal, it is immediate that gr_* is symmetric monoidal. The fact that gr_* is colimit-preserving follows from the fact that $\mathrm{gr}_*^{\mathrm{Fil}}$ is colimit-preserving and that the functor $\mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Fil}(\mathrm{Sp})$ is given by left Kan extension. $\qquad\square$

Corollary 2.14. gr_* *induces functors* $\mathrm{gr}_* : \mathrm{Alg}^{\mathrm{Tow}} \to \mathrm{Alg}$ *and* $\mathrm{gr}_* : \mathrm{CAlg}^{\mathrm{Tow}} \to \mathrm{CAlg}$ *which preserve sifted colimits.*

Proof. As gr_* is symmetric monoidal, it induces maps functors on commutative and associative algebras by definition ([Lur17, 2.1.3.1]). Additionally, [Lur17, 3.2.3.1] guarantees that gr_* preserves sifted colimits of associative and commutative algebras. $\qquad\square$

2.4 The Whitehead tower

Recall, that for all $n \in \mathbb{Z}$, there are functors $\tau_{\geq n} : \mathrm{Sp} \to \mathrm{Sp}_{\geq n}$, which are right adjoint to the inclusions $i_n : \mathrm{Sp}_{\geq n} \subseteq \mathrm{Sp}$ [Lur17, 1.2.1.7]. The composites $i_n \circ \tau_{\geq n} : \mathrm{Sp} \to \mathrm{Sp}$ are colocalization functors, and the inclusions $\cdots \subseteq \mathrm{Sp}_{\geq n} \subseteq \mathrm{Sp}_{\geq n-1} \subseteq \cdots \subseteq \mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ induce a diagram, $\mathbb{Z}_{\geq 0}^{op} \to \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp})$, given by

$$\cdots i_n \circ \tau_{\geq n} \to i_{n-1} \circ \tau_{\geq n-1} \to \cdots \to i_0 \circ \tau_{\geq 0}.$$

Using this diagram, we can construct a functor $T : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$ which is given on vertices by $\{X_n\}_{n \geq 0} \mapsto \{\tau_{\geq n} X_n\}_{n \geq 0}$.

Definition 2.15. The *Whitehead tower functor* is the composite $W = T \circ \delta : \mathrm{Sp} \to \mathrm{Tow}(\mathrm{Sp})$, where δ is the constant tower functor.

Proposition 2.16. *The Whitehead tower functor* $\mathrm{Sp} \to \mathrm{Tow}(\mathrm{Sp})$ *is lax symmetric monoidal.*

Proof. Let $\mathcal{E} \subseteq \mathrm{Tow}(\mathrm{Sp})$ denote the full subcategory of $\mathbb{N}$-filtered spectra, $X_\bullet$, with the property that $X_n \simeq \tau_{\geq n} X_n$. Additionally, note that the essential image of T is exactly $\mathcal{E}$; this yields a right adjoint $R : \mathrm{Tow}(\mathrm{Sp}) \to \mathcal{E}$, which is a colocalization. It remains to check the hypotheses of [Lur17, 2.2.1.1] are satisfied, and by [Lur17, 2.2.1.2] it suffices to show that $\mathcal{E}$ contains the unit and is closed with respect to the Day convolution for $\mathrm{Tow}(\mathrm{Sp})$. Certainly $\mathbb{1}_{\mathrm{Tow}} \in \mathcal{E}$, and if $X_\bullet, Y_\bullet \in \mathcal{E}$,

$$(X_\bullet \otimes Y_\bullet)_n = \varinjlim_{p+q \geq n} X_p \otimes Y_q \in \mathrm{Sp}_{\geq n}$$

since $\mathrm{Sp}_{\geq n}$ is stable under colimits in Sp. We conclude that $\mathcal{E} \subseteq \mathrm{Tow}(\mathrm{Sp})$ is symmetric monoidal and $R : \mathrm{Tow}(\mathrm{Sp}) \to \mathcal{E}$ is lax monoidal, meaning that T is also lax monoidal. As the constant tower functor $\delta : \mathrm{Sp} \to \mathrm{Tow}(\mathrm{Sp})$ is symmetric monoidal, we may conclude the result. $\square$

Remark 2.17. Since W is lax monoidal, we obtain functors $\mathrm{Alg} \to \mathrm{Alg}^{\mathrm{Tow}}$ and $\mathrm{CAlg} \to \mathrm{CAlg}^{\mathrm{Tow}}$. In particular, for $A \in \mathrm{CAlg}_{\geq 0}$, W produces a multiplicative tower

$$\cdots \to \tau_{\geq n} A \to \tau_{\geq n-1} A \to \cdots \to \tau_{\geq 0} A \simeq A,$$

where the k-th graded term is given by $\Sigma^k H \pi_k A$.

3 Descent for generalized Eilenberg-Mac Lane spectra

In this section, we prove a special case of our main theorem. Before doing so, we establish some facts about Čech conerves and recall a few definitions.

Definition 3.1. Let C be an ∞-category which admits finite colimits, and let $f : X \to Y$ be an arrow in C. Then the *augmented Čech conerve of* f, denoted by $C^\bullet(f)_+ : N(\Delta_+) \to C$ is the left Kan extension of $f : \Delta^1 \to C$ along the inclusion $N(\Delta_+^{\leq 0}) \subseteq N(\Delta_+)$. The *Čech conerve*, denoted by $C^\bullet(f)$ is the restriction of $C^\bullet(f)_+$ to $N(\Delta)$. In what follows, we may abuse notation and conflate $C^\bullet(f)_+$ and $C^\bullet(f)$.

Example 3.2. Let $f : A \to B$ be a morphism in CAlg. As the coproduct in CAlg is given by the relative smash product, the augmented Čech conerve is simply the augmented cobar complex of f:

$$A \longrightarrow B \rightrightarrows B \otimes_A B \underset{\longrightarrow}{\overset{\longrightarrow}{\rightrightarrows}} \cdots$$

We will be primarily interested in this cobar complex when f is faithfully flat.

Definition 3.3. Let $f : A \to B$ be a morphism in CAlg. We say that f is *faithfully flat*, provided the following conditions hold:

1. The map $\pi_0 f : \pi_0 A \to \pi_0 B$ is a faithfully flat map of commutative rings; and

2. the induced map $(\pi_0 B) \otimes_{\pi_0 A} (\pi_* A) \to \pi_* B$ is an isomorphism of graded rings.

As in the discrete case, faithfully flat morphisms determine a Grothendieck topology on both CAlg and $\mathrm{CAlg}_{\geq 0}$, see [Lur18, B.6.1.3, B.6.1.7].

Lemma 3.4. *Let C be an ∞-category which admits finite colimits and let $F : \Delta^1 \times \Delta^1 \to C$ be a commutative square, such that $\alpha = F|_{\{0\} \times \Delta^1}$ and $\beta = F|_{\{1\} \times \Delta^1}$ are equivalences in C, which we depict as*

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$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X' & \xrightarrow{\;g\;} & Y' \end{array}$$

Then, the induced map $C^\bullet(f)_+ \to C^\bullet(g)_+$ is an equivalence of augmented cosimplicial objects.

Proof. Since C admits finite colimits we can take the left Kan extension of F along the inclusion $\Delta_+^{\leq 0} \times \Delta^1 \subseteq \Delta_+ \times \Delta^1$, yielding a morphism of augmented cosimplicial objects $\widetilde{F} : \Delta_+ \times \Delta^1 \to C$. It suffices to check that $\widetilde{F}|_{\{n\}\times\Delta^1}$ is an equivalence for all $n \in \Delta_+$. Since $\widetilde{F}|_{\{n\}\times\Delta^1}$ is given by the map $Y \coprod_X \cdots \coprod_X Y \to Y' \coprod_{X'} \cdots \coprod_{X'} Y'$ induced by α and β, it is an equivalence. $\qquad\square$

Lemma 3.5. *Let C be an ∞-category which admits finite colimits and let $D : \Delta^1 \times \Delta^1 \to C$ be a coCartesian square, depicted as*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \phi}\downarrow & & \downarrow \\ X' & \xrightarrow{\;f'\;} & Y' \end{array}$$

Then, the natural map $C^\bullet(f) \coprod_X X' \to C^\bullet(f')$ is an equivalence of cosimplicial objects of C.

Proof. First, we construct a pushout functor. Let $\mathcal{E}$ be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, C)$ consisting of pushout diagrams. The restriction functor $\mathcal{E} \to \mathrm{Fun}(\Lambda_0^2, C)$ is an acyclic fibration by [Lur09, 4.3.2.15]. However, since $\Lambda_0^2 \cong (\partial\Delta^1)^\triangleleft \cong \Delta^1 \coprod_{\Delta^{\{0\}}} \Delta^1$ we know that $\mathrm{Fun}(\Lambda_0^2, C) \cong \mathrm{Fun}(\Delta^1, C) \times_C \mathrm{Fun}(\Delta^1, C)$. Pulling back along the inclusion of the vertex $\{\phi : X \to X'\}$, we obtain an acyclic fibration

$$q : \mathcal{E} \times_{\mathrm{Fun}(\Delta^1, C)} \{\phi\} \to \mathrm{Fun}(\Delta^1, C) \times_C \{X\}.$$

By choosing a section of q and restricting to the opposite edge, we obtain a functor

$$(-)\coprod_X X' : \mathrm{Fun}(\Delta^1, C) \times_C \{X\} \to \mathcal{E} \times_{\mathrm{Fun}(\Delta^1, C)} \{\phi\} \to \mathrm{Fun}(\Delta^1, C) \times_C \{X'\}$$

which takes $X \to Y$ to $X' \to Y \coprod_X X'$ and preserves colimits on account that the section of q is a left adjoint and colimits in functor categories are computed pointwise. By taking left Kan extensions, we obtain the following commutative diagram:

By the universal property, we obtain a natural map $C^\bullet(f)_+ \coprod_X X' \to C^\bullet(f')_+$. This is an equivalence, since left Kan extensions are calculated pointwise as colimits and $(-)\coprod_X X'$ preserves them. Finally, by projecting down to C, we obtain the desired equivalence of augmented cosimplicial objects of C. $\qquad\square$

We now record two key results used in our proof of Proposition 3.8.

Theorem 3.6. [BMS19, Corollary 3.4, Remark 3.5] *The functors $\mathrm{THH}(-)$ and $\mathrm{THH}(-)_{hC_p}$ are Sp-valued sheaves for the fpqc topology on the category of commutative rings.*

Theorem 3.7. [DR18, Theorem 1.2] *Let $A \to B$ be a 1-connected morphism in $\mathrm{CAlg}_{\geq 0}$, and let $F = \mathrm{THH}$ or $\mathrm{THH}(-)_{hC_p}$. Then, the natural map*

$$F(A) \longrightarrow \varprojlim \left(F(B) \rightrightarrows F(B \otimes_A B) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right)$$

is an equivalence in Sp.

Proposition 3.8. *Let* $f : A \to B$ *be a faithfully flat map in* $\mathrm{CAlg}_{\geq 0}$, *which induces a faithfully flat map* $H\pi_* A \to H\pi_* B$. *The natural map*

$$\mathrm{THH}(H\pi_* A) \longrightarrow \varprojlim \left(\mathrm{THH}(H\pi_* B) \rightrightarrows \mathrm{THH}(H\pi_* B \otimes_{H\pi_* A} H\pi_* B) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right)$$

is an equivalence in Sp.

Proof. For brevity, we will write A_* (resp. B_*) for $H\pi_* A$ (resp. $H\pi_* B$) and A_0 (resp. B_0) for $H\pi_0 A$ (resp. $H\pi_0 B$). Since f is faithfully flat, we find that the commutative square

$$
\begin{array}{ccc}
A_* & \xrightarrow{\ f\ } & B_* \\
{\scriptstyle \phi_{A_*}} \downarrow & & \downarrow {\scriptstyle \phi_{B_*}} \\
A_0 & \xrightarrow{\ g\ } & B_0
\end{array}
$$

is a pushout in $\mathrm{CAlg}_{\geq 0}$ by a Künneth spectral sequence calculation. Viewing this square as a functor $\Delta_+^{\leq 0} \times \Delta_+^{\leq 0} \to \mathrm{CAlg}_{\geq 0}$ we can take its left Kan extension along the inclusion $\Delta_+^{\leq 0} \times \Delta_+^{\leq 0} \subseteq \Delta_+ \times \Delta_+$, which we denote by $X_+^{\bullet,\bullet} : \Delta_+ \times \Delta_+ \to \mathrm{CAlg}_{\geq 0}$. By [Lur09, 4.3.2.8] and the dual of [Lur09, 6.1.2.11], we can realize this diagram as the degreewise Čech conerves of either $C^\bullet(f)_+ \to C^\bullet(g)_+$ or $C^\bullet(\phi_{A_*})_+ \to C^\bullet(\phi_{B_*})_+$. In virtue of this, we may identify the n-th row in the bisimplical diagram, $X_+^{\bullet,n}$, as the augmented Čech conerve of the map $A_0 \otimes_{A_*} \cdots \otimes_{A_*} A_0 \to B_0 \otimes_{B_*} \cdots \otimes_{B_*} B_0$, and the m-th column, $X_+^{m,\bullet}$, as the augmented Čech conerve of the map $B_* \otimes_{A_*} \cdots \otimes_{A_*} B_* \to B_0 \otimes_{A_0} \cdots \otimes_{A_0} B_0$. Note that the diagram, $\mathrm{THH} \circ X_+^{\bullet,\bullet}$, induces the following commutative square

$$
\begin{array}{ccc}
\mathrm{THH}(X_+^{-1,-1}) & \longrightarrow & \varprojlim_{m \in \Delta} \mathrm{THH}(X_+^{m,-1}) \\
\downarrow & & \downarrow \\
\varprojlim_{n \in \Delta} \mathrm{THH}(X_+^{-1,n}) & \longrightarrow & \varprojlim_{n \in \Delta} \varprojlim_{m \in \Delta} \mathrm{THH}(X_+^{m,n})
\end{array}
\tag{3.1}
$$

Since $X_+^{\bullet,-1}$ is the augmented Čech conerve of the map $A_* \to B_*$, the result will follow if we can prove all the other arrows in Diagram 3.1 are equivalences. As the maps $X_+^{m,-1} \to X_+^{m,0}$ are 1-connected for all $m \geq -1$, Theorem 3.7 implies that the maps

$$\mathrm{THH}(X_+^{m,-1}) \to \varprojlim_{n \in \Delta} \mathrm{THH}(X_+^{m,n})$$

are equivalences for all $m \geq -1$. Thus, the vertical maps in the commutative square are equivalences, by commuting the limits in the bottom right corner. It remains to show that

$$\mathrm{THH}(X_+^{-1,n}) \to \varprojlim_{m \in \Delta} \mathrm{THH}(X_+^{m,n})$$

is an equivalence for all $n \geq 0$, which we prove by induction. The case $n = 0$ is precisely Theorem 3.6, so we may assume the result holds for n. As noted above, the n-th row of $X_+^{\bullet,\bullet}$ is given by the augmented Čech conerve of the map $A_0 \otimes_{A_*} \cdots \otimes_{A_*} A_0 \to B_0 \otimes_{B_*} \cdots \otimes_{B_*} B_0$. However, this augmented cobar construction is equivalent to the augmented Čech conerve of the map

$$id \otimes \cdots \otimes id \otimes f_0 : A_0 \otimes_{A_*} \cdots \otimes_{A_*} A_0 \to A_0 \otimes_{A_*} \cdots \otimes_{A_*} A_0 \otimes_{A_*} B_0.$$

Since the following diagram is a retract in CAlg,

$$\begin{array}{ccccc}
A_0 & \longrightarrow & A_* & \longrightarrow & A_0 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_0} \\
B_0 & \longrightarrow & B_* & \longrightarrow & B_0
\end{array}$$

we obtain a retract diagram of augmented Čech conerves

$$C^\bullet(id^{\otimes n} \otimes f_0)_+ \to C^\bullet(id^{\otimes(n-1)} \otimes f_0)_+ \to C^\bullet(id^{\otimes n} \otimes f_0)_+$$

and hence a retract diagram

$$\mathrm{THH}\left(C^\bullet(id^{\otimes n} \otimes f_0)_+\right) \to \mathrm{THH}\left(C^\bullet(id^{\otimes n-1} \otimes f_0)_+\right) \to \mathrm{THH}\left(C^\bullet(id^{\otimes n} \otimes f_0)_+\right).$$

By the inductive hypothesis, we may conclude that $\mathrm{THH}\left(C^\bullet(id^{\otimes n} \otimes f_0)_+\right)$ is a limit diagram. This completes the proof. □

Proposition 3.9. *Let $f : A \to B$ be a faithfully flat map in $\mathrm{CAlg}_{\geq 0}$. Then the natural map*

$$\mathrm{THH}(H\pi_* A)_{hC_p} \to \varprojlim \left(\mathrm{THH}(H\pi_* B^{\otimes_{H\pi_* A}\bullet})_{hC_p}\right)$$

is a limit diagram.

Proof. This argument is identical to the one in Proposition 3.8 since we can apply Theorems 3.7 and 3.6 for $\mathrm{THH}(-)_{hC_p}$. □

4 The May filtration on THH

Let A be a connective algebra spectrum and let $A_\bullet$ be its Whitehead tower. The May filtration on $\mathrm{THH}(A)$, as defined in [AKS18], is the geometric realization of the following diagram;

$$A_\bullet \Longleftarrow A_\bullet \circledast A_\bullet \Lleftarrow A_\bullet \circledast A_\bullet \circledast A_\bullet \Lleftarrow \cdots$$

i.e. the cyclic bar construction of $A_\bullet$ with respect to the Day convolution product on $\mathrm{Tow}(\mathrm{Sp})$. In order to make this construction precise in the ∞-categorical language, we mimic the construction of THH in [NS18]. One prerequisite therein, is the symmetric monoidal envelope of [Lur17, 2.2.4.1]. After a brief discussion of the monoidal envelope, we present a ∞-categorical description of the May filtration, and establish several useful properties.

Remark 4.1. The cyclic bar construction can be performed for any associative algebra in a presentably symmetric monoidal ∞-category. Since we are primarily interested in the Whitehead towers of algebra spectra and the Day convolution product, we will not work in such general terms, though many of the results in this section will hold in greater generality.

4.1 The symmetric monoidal envelope

Definition 4.2. [Lur17, 2.2.4.1] Let $p : C^\otimes \to \mathrm{Fin}_*$ be a symmetric monoidal ∞-category. The *monoidal envelope* of $C^\otimes$, is the fiber product

$$\mathrm{Env}(C)^\otimes = C^\otimes \times_{\mathrm{Fun}(\{0\},\mathrm{Fin}_*)} \mathrm{Act}(\mathrm{Fin}_*),$$

where $\mathrm{Act}(\mathrm{Fin}_*) \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Fin}_*)$ is the full subcategory spanned by the active morphisms.

By [Lur17, 2.2.4.4], evaluation at $\{1\} \subseteq \Delta^1$ induces a map $q : \mathrm{Env}(C)^\otimes \to \mathrm{Fin}_*$, exhibiting $\mathrm{Env}(C)^\otimes$ as a symmetric monoidal ∞-category. If we let $\mathrm{Env}(C) = \mathrm{Env}(C)^\otimes_{\langle 1 \rangle}$ denote the fiber of q over $\langle 1 \rangle$, by unwinding the definition, we may identify $\mathrm{Env}(C)$ with the subcategory $C^\otimes_{\mathrm{act}} \subseteq C^\otimes$ spanned by objects and active morphisms between them. In other words, $C^\otimes_{\mathrm{act}}$ is a symmetric monoidal ∞-category, and its monoidal product, $\oplus : C^\otimes_{\mathrm{act}} \times C^\otimes_{\mathrm{act}} \to C^\otimes_{\mathrm{act}}$, may be described as follows: for $X \in C^\otimes_{\langle n \rangle}$ and $Y \in C^\otimes_{\langle m \rangle}$ corresponding to sequences $(X_i)_{i=1}^n$ and $(Y_j)_{j=1}^m$, the product $X \oplus Y$ corresponds to the concatenation $(X_i)_{i=1}^n \cup (Y_j)_{j=1}^m \in C^\otimes_{\langle n+m \rangle}$.

There is a diagonal embedding $\mathrm{Fin}_* \to \mathrm{Act}(\mathrm{Fin}_*)$, and the pullback along this embedding induces a lax symmetric monoidal inclusion $i_C : C^\otimes \subseteq \mathrm{Env}(C)^\otimes$, [Lur17, 2.2.4.4, 2.2.4.16]. The following proposition provides a useful characterization of symmetric monoidal functors out of $\mathrm{Env}(C)^\otimes$.

Proposition 4.3. [Lur17, 2.2.4.9] *Let $C^\otimes$ and $D^\otimes$ be symmetric monoidal ∞-categories. The inclusion $i_C : C^\otimes \subseteq \mathrm{Env}(C)^\otimes$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^\otimes(\mathrm{Env}(C), D) \simeq \mathrm{Alg}_C(D).$$

Here, $\mathrm{Fun}^\otimes(\mathrm{Env}(C), D)$ *denotes the ∞-category of symmetric monoidal functors* $\mathrm{Env}(C)^\otimes \to D^\otimes$ *and* $\mathrm{Alg}_C(D)$ *is the ∞-category of lax symmetric monoidal functors* $C^\otimes \to D^\otimes$.

Under this equivalence, the identity functor $C \to C$, which is lax monoidal, corresponds to a symmetric monoidal functor, $\otimes_C : \mathrm{Env}(C) = C^\otimes_{\mathrm{act}} \to C$, given on vertices by $(X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n$. In order to establish several key properties of the May filtration on THH, we need a lemma on the interaction of $\otimes_C$ with symmetric monoidal functors. Suppose we are given a symmetric monoidal functor, $F^\otimes : C^\otimes \to D^\otimes$. Pulling this morphism back along $\mathrm{Act}(\mathrm{Fin}_*)$, produces a functor $\mathrm{Env}(F)^\otimes : \mathrm{Env}(C)^\otimes \to \mathrm{Env}(D)^\otimes$ which is also symmetric monoidal by [Lur17, 2.2.4.15]. $\mathrm{Env}(F)^\otimes$ fits into the following diagram,

$$
\begin{array}{ccc}
C^\otimes & \xrightarrow{\ F^\otimes\ } & D^\otimes \\
{\scriptstyle i_C}\downarrow & & \downarrow{\scriptstyle i_D} \\
\mathrm{Env}(C)^\otimes & \xrightarrow[\mathrm{Env}(F)^\otimes]{} & \mathrm{Env}(D)^\otimes
\end{array}
$$

which is commutative in virtue of the fact that i_C and i_D are induced by the diagonal.

Lemma 4.4. *Let $F^\otimes : C^\otimes \to D^\otimes$ be a symmetric monoidal functor. There is an equivalence*

$$\otimes_D \circ \mathrm{Env}(F)^\otimes \simeq F^\otimes \circ \otimes_C.$$

Proof. As the equivalence in Proposition 4.3 is given by precomposing by the lax symmetric monoidal inclusion $i_C : C^\otimes \subseteq \mathrm{Env}(C)^\otimes$, we can check the desired functors are equivalent by showing that

$$\otimes_D \circ \mathrm{Env}(F)^\otimes \circ i_C \simeq F^\otimes \circ \otimes_C \circ i_C.$$

This is immediate since $\mathrm{Env}(F)^\otimes \circ i_C \simeq i_D \circ F^\otimes$, $\otimes_C \circ i_C \simeq id$, and $\otimes_D \circ i_D \simeq id$. $\qquad\square$

Remark 4.5. In the case where $C = \mathrm{Tow}(\mathrm{Sp})$, we will denote $\otimes_C$ by $\circledast$, and in the case where $C = \mathrm{Sp}$, we will denote $\otimes_C$ by $\otimes$.

4.2 The May filtration on THH

Definition 4.6. Let $A_\bullet \in \mathrm{Alg}^{\mathrm{Tow}}$, given by a section $A^\otimes_\bullet : \mathrm{Assoc}^\otimes \to \mathrm{Tow}(\mathrm{Sp})^\otimes$. Let $\mathrm{THH}^{\mathrm{Tow}}(A_\bullet) \in \mathrm{Tow}(\mathrm{Sp})^{B\mathbb{T}}$ denote the colimit of the diagram

$$N(\Lambda^{op}) \xrightarrow{\ V^o\ } \mathrm{Assoc}^\otimes_{\mathrm{act}} \xrightarrow{\ A^\otimes_\bullet\ } \mathrm{Tow}(\mathrm{Sp})^\otimes_{\mathrm{act}} \xrightarrow{\ \circledast\ } \mathrm{Tow}(\mathrm{Sp}),$$

where V^o is the map appearing in [NS18, B.1].

Remark 4.7. This is essentially the definition of THH as given in [NS18], the only salient difference being that we are using the Day convolution of towers to form the cyclic bar construction as opposed to the smash product. Additionally, we know $\mathrm{THH}^{\mathrm{Tow}}(A_\bullet) \in \mathrm{Tow}(\mathrm{Sp})^{B\mathbb{T}}$ since the geometric realization of a cyclic object admits a $\mathbb{T}$-action, [NS18, B.5].

Proposition 4.8. *The functor* $\mathrm{THH}^{\mathrm{Tow}} : \mathrm{Alg}^{\mathrm{Tow}} \to \mathrm{Tow}(\mathrm{Sp})^{B\mathbb{T}}$ *is symmetric monoidal.*

Proof. This essentially follows from the fact that Day convolution preserves sifted colimits. The careful reader should consult [Lur17, 3.2.4.3, 3.2.4.3] to explicitly construct the functor

$$(\mathrm{THH}^{\mathrm{Tow}})^{\otimes} : (\mathrm{Alg}^{\mathrm{Tow}})^{\otimes} \to (\mathrm{Tow}(\mathrm{Sp})^{B\mathbb{T}})^{\otimes}$$

which exhibits $\mathrm{THH}^{\mathrm{Tow}}$ as symmetric monoidal. □

Corollary 4.9. $\mathrm{THH}^{\mathrm{Tow}}$ *induces a functor* $\mathrm{CAlg}^{\mathrm{Tow}} \to (\mathrm{CAlg}^{\mathrm{Tow}})^{B\mathbb{T}}$.

Proof. Since $\mathrm{THH}^{\mathrm{Tow}}$ is symmetric monoidal, it induces a functor

$$\mathrm{CAlg}(\mathrm{Alg}^{\mathrm{Tow}}) \to \mathrm{CAlg}(\mathrm{Tow}(\mathrm{Sp})^{B\mathbb{T}}).$$

As $\mathrm{CAlg}(\mathrm{Alg}^{\mathrm{Tow}}) \simeq \mathrm{CAlg}(\mathrm{Tow}(\mathrm{Sp}))$ and $\mathrm{CAlg}(\mathrm{Tow}(\mathrm{Sp})^{B\mathbb{T}}) \simeq (\mathrm{CAlg}^{\mathrm{Tow}})^{B\mathbb{T}}$, we conclude the result. □

For connective commutative ring spectra, the May filtration on THH can be given an alternative construction, following [AKS18]. Since $\mathrm{CAlg}^{\mathrm{Tow}}$ is presentable, it is automatically tensored over spaces [Lur09, 4.4.4]; we denote this by $X \otimes A_{\bullet}$. Using the simplicial model, $\Delta^1/\partial\Delta^1$, for S^1, a standard argument shows we may identify $S^1 \otimes A_{\bullet}$ in $\mathrm{CAlg}^{\mathrm{Tow}}$ with the cyclic bar construction of $A_{\bullet}$, where $A_{\bullet}$ is the Whitehead tower of $A \in \mathrm{CAlg}_{\geq 0}$. More generally, given $A_{\bullet} \in \mathrm{CAlg}^{\mathrm{Tow}}$ and X a finite space, the tower $X \otimes A_{\bullet} \in \mathrm{CAlg}^{\mathrm{Tow}}$ gives a filtration on $X \otimes A_0$, which is also called the *May filtration* in [AKS18, 3.3.3].

The remainder of this subsection is devoted an ∞-categorical treatment of results in [AKS18]. In particular, Proposition 4.10 allows us to define the May filtration on THH in the ∞-categorical language.

Proposition 4.10. *Let* $A \in \mathrm{Alg}_{\geq 0}$ *and let* $A_{\bullet}$ *denote the Whitehead tower of A. There is a canonical equivalence of spectra with* $\mathbb{T}$-*action,* $\mathrm{ev}_0\, \mathrm{THH}^{\mathrm{Tow}}(A_{\bullet}) \simeq \mathrm{THH}(A)$. *In the case where* $A \in \mathrm{CAlg}_{\geq 0}$, *this is an equivalence in* $\mathrm{CAlg}^{B\mathbb{T}}$.

Proof. Proposition 2.11 states that $\mathrm{ev}_0 : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$ is symmetric monoidal so we may apply Lemma 4.4 to extract the following commutative square.

$$
\begin{array}{ccc}
\mathrm{Tow}(\mathrm{Sp})^{\otimes}_{\mathrm{act}} & \xrightarrow{\circledast} & \mathrm{Tow}(\mathrm{Sp}) \\
{\scriptstyle (\mathrm{ev}_0)^{\otimes}_{\mathrm{act}}} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_0} \\
\mathrm{Sp}^{\otimes}_{\mathrm{act}} & \xrightarrow{\otimes} & \mathrm{Sp}
\end{array}
$$

Additionally, note that ev_0 commutes with colimits and that $\mathrm{ev}_0\, A_{\bullet} \simeq A$. We obtain the following chain of canonical equivalences in $\mathrm{Sp}^{B\mathbb{T}}$:

$$
\begin{aligned}
\mathrm{ev}_0\, \mathrm{THH}^{\mathrm{Tow}}(A_{\bullet}) &= \mathrm{ev}_0 \varinjlim \left(\circledast \circ (A_{\bullet}^{\otimes})_{\mathrm{act}} \circ V^o \right) \\
&\simeq \varinjlim \left(\mathrm{ev}_0 \circ \circledast \circ (A_{\bullet}^{\otimes})_{\mathrm{act}} \circ V^o \right) \\
&\simeq \varinjlim \left(\otimes \circ A_{\mathrm{act}}^{\otimes} \circ V^o \right) \\
&= \mathrm{THH}(A).
\end{aligned}
$$

In the case where $A \in \mathrm{CAlg}_{\geq 0}$, a similar argument works, since the induced functor $\mathrm{ev}_0 : \mathrm{CAlg}^{\mathrm{Tow}} \to \mathrm{CAlg}$ preserves sifted colimits, by [Lur17, 3.2.3.2]. □

Remark 4.11. The proof of Proposition 4.10 yields a slightly more general result; there is a canonical equivalence of spectra with $\mathbb{T}$-action, $\mathrm{ev}_0\, \mathrm{THH}^{\mathrm{Tow}}(A_{\bullet}) \simeq \mathrm{THH}(A_0)$, for any $A_{\bullet} \in \mathrm{Alg}^{\mathrm{Tow}}$.

Definition 4.12. [AKS18, 3.3.3, 3.4.9] Let $A \in \mathrm{Alg}_{\geq 0}$ and let $A_{\bullet}$ denote the Whitehead tower of A. The *May filtration* of $\mathrm{THH}(A)$ is the tower $\mathrm{THH}^{\mathrm{Tow}}(A_{\bullet})$. We will let $\mathrm{Fil}_i^{\mathrm{May}}$ denote $\mathrm{ev}_i\, \mathrm{THH}^{\mathrm{Tow}}(A_{\bullet})$ and let $\mathrm{gr}_i^{\mathrm{May}}$ denote $\mathrm{Fil}_i^{\mathrm{May}}/\mathrm{Fil}_{i+1}^{\mathrm{May}}$. While our notation makes no reference to A, the context of our usage should indicate the underlying connective ring spectrum.

If we view THH as a functor, THH : $\mathrm{Alg} \to \mathrm{Sp}^{B\mathbb{T}}$, we obtain a tower of functors

$$\cdots \to \mathrm{Fil}_i^{\mathrm{May}} \to \cdots \to \mathrm{Fil}_1^{\mathrm{May}} \to \mathrm{THH}\,.$$

We call this tower the *May filtration of the functor* THH, and by taking cofibers, it induces another tower

$$\cdots \to \mathrm{THH}/\mathrm{Fil}_i^{\mathrm{May}} \to \cdots \to \mathrm{THH}/\mathrm{Fil}_1^{\mathrm{May}} \to 0.$$

The following is the ∞-categorical version of [AKS18, 3.3.10], which is crucial for the sequel.

Proposition 4.13. *For $A_\bullet \in \mathrm{Alg}^{\mathrm{Tow}}$, there is a canonical equivalence of spectra with $\mathbb{T}$-action, $\mathrm{gr}_* \mathrm{THH}^{\mathrm{Tow}}(A_\bullet) \simeq \mathrm{THH}(\mathrm{gr}_* A_\bullet)$. In the case where $A_\bullet \in \mathrm{CAlg}^{\mathrm{Tow}}$, this is an equivalence in $\mathrm{CAlg}^{B\mathbb{T}}$.*

Proof. By Proposition 2.13 and Lemma 4.4, it follows that

$$
\begin{array}{ccc}
\mathrm{Tow}(\mathrm{Sp})_{\mathrm{act}}^{\otimes} & \xrightarrow{\;\;\circledast\;\;} & \mathrm{Tow}(\mathrm{Sp}) \\[4pt]
{\scriptstyle (\mathrm{gr}_*^{\otimes})_{\mathrm{act}}}\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{gr}_*} \\[4pt]
\mathrm{Sp}_{\mathrm{act}}^{\otimes} & \xrightarrow[\;\;\otimes\;\;]{} & \mathrm{Sp}
\end{array}
$$

commutes. Additionally, since gr_* preserves,

$$
\begin{aligned}
\mathrm{gr}_* \mathrm{THH}^{\mathrm{Tow}}(A_\bullet) = \mathrm{gr}_* \varinjlim \left(\circledast \circ (A_\bullet^{\otimes})_{\mathrm{act}} \circ V^o \right) \\
\simeq \varinjlim \left(\mathrm{gr}_* \circ \circledast \circ (A_\bullet^{\otimes})_{\mathrm{act}} \circ V^o \right) \\
\simeq \varinjlim \left(\otimes \circ (\mathrm{gr}_* A_\bullet)_{\mathrm{act}}^{\otimes} \circ V^o \right) \\
= \mathrm{THH}(\mathrm{gr}_* A_\bullet).
\end{aligned}
$$

In the case where $A \in \mathrm{CAlg}_{\geq 0}$, a similar argument works, since the induced functor $\mathrm{gr}_* : \mathrm{CAlg}^{\mathrm{Tow}} \to \mathrm{CAlg}$ preserves sifted colimits, by [Lur17, 3.2.3.2]. $\qquad\square$

An immediate consequence of the previous result is the following:

Corollary 4.14. [AKS18, 3.4.11] *For $A \in \mathrm{CAlg}_{\geq 0}$ and $A_\bullet$ the Whitehead tower of A, there is a canonical equivalence in $\mathrm{CAlg}^{B\mathbb{T}}$, $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}(A) \simeq \mathrm{THH}(H\pi_* A)$.*

This result, combined with several others, is used to construct the THH-May spectral sequence sequence $E_{*,*}^1 \cong \pi_* \mathrm{THH}(H\pi_* A) \Rightarrow \pi_* \mathrm{THH}(A)$; see [AKS18, 3.4.8].

Proposition 4.15. [AKS18, 3.5.4] *If $A_\bullet \in \mathrm{Alg}_{\geq 0}^{\mathrm{Tow}}$ with the property that $\pi_m A_n \cong 0$ for all $m < n$, then $\pi_m \mathrm{Fil}_n^{\mathrm{May}} \cong 0$ for all $m < n$ as well.*

Proof. First, recall that if E and F are n and m-connective spectra, respectively, then $E \otimes F$ is $(n+m)$-connective. Next, note that $\mathrm{Fil}_n^{\mathrm{May}}$ is the geometric realization of a simplicial object whose k-th term is given by

$$\mathrm{ev}_n(A_\bullet^{\circledast k}) = \varinjlim_{i_1 + \cdots + i_k \geq n} A_{i_1} \otimes \cdots \otimes A_{i_k}.$$

As each A_{i_m} is i_m-connective, by our recollection, each term in the simplicial object is n-connective, and hence $\mathrm{Fil}_n^{\mathrm{May}}$ is n-connective. $\qquad\square$

Corollary 4.16. *For $A \in \mathrm{Alg}_{\geq 0}$, the May filtration has the property that*

$$\varprojlim_{n \geq 0} \left(\mathrm{Fil}_n^{\mathrm{May}} \right) \simeq 0;$$

i.e. the filtration is Hausdorff. Additionally, the May filtration on the functor THH *is also Hausdorff.*

Proof. By Proposition 4.15 and a Bousfield-Kan spectral sequence calculation, $\mathrm{Fil}_n^{\mathrm{May}}$ is n-connective. Since the connectivity grows linearly in the tower, we conclude $\varprojlim_{n \geq 0} \left(\mathrm{Fil}_n^{\mathrm{May}}\right) \simeq 0$. □

Corollary 4.17. *Let* $A \in \mathrm{Alg}_{\geq 0}$. *Then the tower*

$$\cdots \to \mathrm{THH}(A)/\mathrm{Fil}_i^{\mathrm{May}} \to \cdots \to \mathrm{THH}(A)/\mathrm{Fil}_2^{\mathrm{May}} \to \mathrm{THH}(A)/\mathrm{Fil}_1^{\mathrm{May}}$$

converges to $\mathrm{THH}(A)$.

Proof. For $n \geq 1$ there are fiber sequences $\mathrm{Fil}_i^{\mathrm{May}} \to \mathrm{THH}(A) \to \mathrm{THH}(A)/\mathrm{Fil}_i^{\mathrm{May}}$. Passing to the limit and applying Corollary 4.16, we conclude. □

Remark 4.18. As limits are calculated pointwise in functor categories, the lemma above implies that the tower of functors $\{\mathrm{THH}/\mathrm{Fil}_i^{\mathrm{May}}\}_{i \geq 1}$ converges to THH.

Remark 4.19. While working with $\mathrm{Tow}(\mathrm{Sp})$ suffices for our purposes, by Remark 4.7, we can define $\mathrm{THH}^{\mathrm{Fil}}(A_\bullet)$ for $A_\bullet \in \mathrm{Alg}^{\mathrm{Fil}}$, satisfying many of the same properties. For example, analogues of Propositions 4.10 and 4.13 hold for $\mathrm{THH}^{\mathrm{Fil}}$; instead of using the symmetric monoidality of ev_0, we use the symmetric monoidality of $\varinjlim : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Sp}$, and instead of the associated graded functor $\mathrm{gr}_* : \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Sp}$, we use the functor $\mathrm{gr}_*^{\mathrm{Fil}}$ from Section 2.3.

4.3   The May filtration on THH_{hC_p}

In order to prove our main theorem, we need to establish variants of Proposition 4.10 and Corollaries 4.14, 4.16, and 4.17 for $\mathrm{THH}(-)_{hC_p}$. We proceed by applying C_p-homotopy fixed points to the tower $\mathrm{THH}^{\mathrm{Tow}}(A)$, and mimicking the proofs in the previous subsection.

Proposition 4.20. *Let* $A \in \mathrm{Alg}_{\geq 0}$, *and let* $A_\bullet$ *denote the Whitehead tower of* A. *There is an equivalence of spectra with* $\mathbb{T}$-*action,* $\mathrm{ev}_0 \left(\mathrm{THH}^{\mathrm{Tow}}(A_\bullet)_{hC_p}\right) \simeq \mathrm{THH}(A)_{hC_p}$. *In the case where* $A \in \mathrm{CAlg}_{\geq 0}$, *this is an equivalence in* $\mathrm{CAlg}^{B\mathbb{T}}$.

Proof. Since the functor ev_0 is colimit-preserving, we have:

$$\mathrm{ev}_0 \left(\mathrm{THH}^{\mathrm{Tow}}(A_\bullet)_{hC_p}\right) \simeq \left(\mathrm{ev}_0 \, \mathrm{THH}^{\mathrm{Tow}}(A_\bullet)\right)_{hC_p}$$
$$\simeq \mathrm{THH}(A)_{hC_p}.$$

As in Proposition 4.10, similar considerations yield the commutative variant. □

Definition 4.21. Let $A \in \mathrm{Alg}_{\geq 0}$ and let $A_\bullet$ denote the Whitehead tower of A. We define the *May filtration* on $\mathrm{THH}(A)_{hC_p}$ as $\mathrm{THH}^{\mathrm{Tow}}(A_\bullet)_{hC_p}$. We will denote the i-th piece of this filtration by $\mathrm{Fil}_i^{\mathrm{May}}(p)$, and the i-th graded piece by $\mathrm{gr}_i^{\mathrm{May}} \mathrm{THH}^{\mathrm{Tow}}(A_\bullet)_{hC_p}$.

Proposition 4.22. *Let* $A \in \mathrm{Alg}_{\geq 0}$, *and let* $A_\bullet$ *denote the Whitehead tower of* A. *There is an equivalence of spectra with* $\mathbb{T}$-*action,* $\mathrm{gr}_*^{\mathrm{May}} \left(\mathrm{THH}^{\mathrm{Tow}}(A_\bullet)_{hC_p}\right) \simeq \mathrm{THH}(H\pi_* A)_{hC_p}$. *In the case where* $A \in \mathrm{CAlg}_{\geq 0}$, *this is an equivalence in* $\mathrm{CAlg}^{B\mathbb{T}}$.

Proof. Since the functor gr_* is colimit-preserving, we have:

$$\mathrm{gr}_* \left(\mathrm{THH}^{\mathrm{Tow}}(A_\bullet)_{hC_p}\right) \simeq \left(\mathrm{gr}_* \, \mathrm{THH}^{\mathrm{Tow}}(A_\bullet)\right)_{hC_p}$$
$$\simeq \mathrm{THH}(H\pi_* A)_{hC_p}.$$

As in Proposition 4.13, similar considerations yield the commutative variant. □

Proposition 4.23. *If $A_\bullet \in \mathrm{CAlg}_{\geq 0}^{\mathrm{Tow}}$ with the property that $\pi_m A_n \cong 0$ for all $m < n$, then $\pi_m \mathrm{Fil}_n^{\mathrm{May}}(p) \cong 0$, for all $m < n$ as well.*

Proof. It is standard that homotopy orbits preserve connectivity, so we may conclude by Proposition 4.15. $\qquad\square$

Corollary 4.24. *For $A \in \mathrm{CAlg}_{\geq 0}$, the May filtration of $\mathrm{THH}(A)_{hC_p}$ is Hausdorff.*

Proof. By Proposition 4.23, the connectivity of the tower grows linearly, so

$$\varprojlim_{n \geq 0} \mathrm{Fil}_n^{\mathrm{May}}(p) \simeq 0.$$

$\qquad\square$

5 Proof of Theorem 1.1

In this section we prove our main result. Using Proposition 3.8, we deduce the graded pieces of the May filtration are fpqc sheaves and conclude the result for THH by induction up the tower in Corollary 4.17. By similar methods, we prove the claim for THH_{hC_p}, and use the structure of CycSp to prove it for TC as well.

Lemma 5.1. *Let $f : A \to B$ be a faithfully flat morphism in $\mathrm{CAlg}_{\geq 0}$. Let $f_\bullet : A_\bullet \to B_\bullet$ denote the induced map between their Whitehead towers and let $f_* : H\pi_* A \to H\pi_* B$ denote $\mathrm{gr}_*(f_\bullet)$. There is a canonical equivalence of augmented cosimplicial commutative ring spectra $\mathrm{gr}_* (C^\bullet(f_\bullet)_+) \simeq C^\bullet(f_*)_+$.*

Proof. As Čech conerves are computed by left Kan extensions, there is a natural transformation $C^\bullet(f_*)_+ \to \mathrm{gr}_* C^\bullet(f_\bullet)_+$ since $\mathrm{gr}_* C^\bullet(f_\bullet)_+|_{\Delta_+^{\leq 0}} = f_*$. For $n > 0$, we have a canonical map

$$C^n(f_*)_+ \simeq H\pi_* B \otimes_{H\pi_* A} \cdots \otimes_{H\pi_* A} H\pi_* B \to H\pi_* (B \otimes_A \cdots \otimes_A B) \simeq \mathrm{gr}_* C^n(f)_+.$$

Since $A \to B$ is faithfully flat, the usual Künneth spectral sequence calculation shows that this map is an equivalence. Thus, the morphism of Čech conerves is an equivalence. $\qquad\square$

Corollary 5.2. *The functors $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}^{\mathrm{Tow}}, \mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}^{\mathrm{Tow}}(-)_{hC_p} : \mathrm{CAlg}_{\geq 0} \to \mathrm{Sp}$ are sheaves for the fpqc topology.*

Proof. By Proposition 4.13, there is an equivalence of functors $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH} \simeq \mathrm{THH}(\mathrm{gr}_*(-))$. Now let $f : A \to B$ be a faithfully flat morphism in $\mathrm{CAlg}_{\geq 0}$ and observe that by Lemma 5.1, $C^\bullet(f_*) \simeq \mathrm{gr}_* C^\bullet(f)$. Thus, Proposition 3.8 applied to the commutative diagram

$$
\begin{array}{ccc}
\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}(A) & \longrightarrow & \varprojlim_\Delta \mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}(C^\bullet(f)) \\
\downarrow & & \downarrow \\
\mathrm{THH}(H\pi_* A) & \longrightarrow & \varprojlim_\Delta \mathrm{THH}(C^\bullet(f_*))
\end{array}
$$

proves the claim for $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}^{\mathrm{Tow}}$. An identical proof works for $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}^{\mathrm{Tow}}(-)_{hC_p}$ as well. $\quad\square$

Proposition 5.3. *For all $i \geq 0$, the functors $\mathrm{gr}_i^{\mathrm{May}} \mathrm{THH}(-)$ are fpqc sheaves.*

Proof. By definition, $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH} \simeq \bigoplus_{i \geq 0} \mathrm{gr}_i^{\mathrm{May}} \mathrm{THH}$, so that $\mathrm{gr}_i^{\mathrm{May}} \mathrm{THH}$ is a retract of $\mathrm{gr}_*^{\mathrm{May}} \mathrm{THH}$. This implies that $\mathrm{gr}_i^{\mathrm{May}} \mathrm{THH}$ is an fpqc sheaf for all $i \geq 0$. $\qquad\square$

Corollary 5.4. *For all $i \geq 1$, the functors $\mathrm{THH}/\mathrm{Fil}_i^{\mathrm{May}} : \mathrm{CAlg}_{\geq 0} \to \mathrm{Sp}$ are fpqc sheaves.*

Proof. Induct on i using Proposition 5.3 and the fiber sequences

$$\mathrm{gr}_i^{\mathrm{May}} \mathrm{THH} \to \mathrm{THH}/\mathrm{Fil}_{i+1}^{\mathrm{May}} \to \mathrm{THH}/\mathrm{Fil}_i^{\mathrm{May}}.$$

$\square$

Theorem 5.5. *The functors* $\mathrm{THH}, \mathrm{THH}_{hC_p} : \mathrm{CAlg}_{\geq 0} \to \mathrm{Sp}$ *are sheaves for the fpqc topology.*

Proof. Since limits of Sp-valued sheaves can be computed in the category of Sp-valued presheaves, the equivalence

$$\mathrm{THH} \simeq \varprojlim_{i \geq 1} \left(\mathrm{THH}/\mathrm{Fil}_i^{\mathrm{May}}\right),$$

guarantees THH is an fpqc sheaf. Similarly, since

$$\mathrm{THH}(-)_{hC_p} \simeq \varprojlim_{i \geq 1} \left(\mathrm{THH}(-)_{hC_p}/\mathrm{Fil}_i^{\mathrm{May}}(p)\right),$$

$\mathrm{THH}(-)_{hC_p}$ is an fpqc sheaf as well. $\square$

Corollary 5.6. *The functors* $\mathrm{THH}(-)^{h\mathbb{T}}, \mathrm{THH}(-)^{hC_p}$, *and* $\mathrm{THH}(-)^{tC_p}$ *are fpqc sheaves on* $\mathrm{CAlg}_{\geq 0}$.

Proof. The first two functors are fpqc sheaves because homtopy fixed points are calculated via limits. The third is an fpqc sheaf because there is a cofiber sequence

$$\mathrm{THH}(-)_{hC_p} \to \mathrm{THH}(-)^{hC_p} \to \mathrm{THH}(-)^{tC_p}$$

and $\mathrm{THH}(-)_{hC_p}$ and $\mathrm{THH}(-)^{hC_p}$ are fpqc sheaves. $\square$

Using the results above, we will show that $\mathrm{THH} : \mathrm{CAlg}_{\geq 0} \to \mathrm{CycSp}$ is an fpqc sheaf. As a Corollary, we will show TC is as well. We now recall the construction of CycSp and the definition of topological cyclic homology.

Definition 5.7. [NS18, II.1.6] Let $\mathbb{P}$ denote the set of all primes. The ∞-category of *cyclotomic spectra* is the pullback

$$
\begin{array}{ccc}
\mathrm{CycSp} & \longrightarrow & \mathrm{Fun}(\Delta^1, \prod_{p \in \mathbb{P}} \mathrm{Sp}^{B\mathbb{T}}) \\
\downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0; \mathrm{ev}_1)} \\
\mathrm{Sp}^{B\mathbb{T}} & \xrightarrow{\ F\ } & \prod_p \mathrm{Sp}^{B\mathbb{T}} \times \prod_{p \in \mathbb{P}} \mathrm{Sp}^{B\mathbb{T}}
\end{array}
$$

where the functor F is given by the formula $F = \left(\prod_{p \in \mathbb{P}} id, \prod_{p \in \mathbb{P}}(-)^{tC_p}\right)$. By [NS18, II.1.5], CycSp is stable and presentable, hence enriched over Sp. The *topological cyclic homology* of a cyclotomic spectrum, X, is the mapping spectrum

$$\mathrm{TC}(X) = \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X),$$

where $\mathbb{S}^{\mathrm{triv}}$ is the sphere spectrum with the trivial $\mathbb{T}$-action and Frobenius maps given by the composite $\mathbb{T}$-equivariant maps $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$; see [NS18, I.2.3.i, II.1.2.ii] for further details. As $\mathrm{THH}(A)$ admits the structure of a cyclotomic spectrum for all $A \in \mathrm{Alg}$ ([NS18, III.2]), we define $\mathrm{TC}(A) = \mathrm{TC}(\mathrm{THH}(A))$.

Proposition 5.8. $\mathrm{THH} : \mathrm{CAlg}_{\geq 0} \to \mathrm{CycSp}$ *is a fpqc sheaf.*

Proof. Because CycSp is defined as as lax equalizer, which is a strict pullback of ∞-categories, it suffices to check that the functors obtained from THH by post-composing with the projections are all fpqc sheaves. Indeed, we obtain the following three functors:

1. $\text{THH} : \text{CAlg}_{\geq 0} \to \text{Sp}^{B\mathbb{T}}$;

2. $\prod_p (\varphi_p(-)) : \text{CAlg}_{\geq 0} \to \prod_p \text{Fun}(\Delta^1, \text{Sp}^{B\mathbb{T}})$; and by identifying composites,

3. $\left(\prod_p \text{THH}(-), \prod_p \text{THH}(-)^{tC_p} \right) : \text{CAlg}_{\geq 0} \to \prod_p \text{Sp}^{B\mathbb{T}} \times \prod_p \text{Sp}^{B\mathbb{T}}$.

The functor in (1) is an fpqc sheaf because $\text{THH} : \text{CAlg}_{\geq 0} \to \text{Sp}$ is and limits in $\text{Sp}^{B\mathbb{T}}$ are calculated pointwise. Note that the functor in (2) is a sheaf if the functor in (3) is as well. To show the functor in (3) is an fpqc sheaf, we can project onto the different factors and reduce to checking that THH and THH^{tC_p} are fpqc sheaves on $\text{CAlg}_{\geq 0}$. Thus, the claim is proved. $\qquad\square$

Corollary 5.9. $\text{TC} : \text{CAlg}_{\geq 0} \to \text{Sp}$ *is an fpqc sheaf.*

Proof. As $\text{TC}(A) = \text{Map}_{\text{CycSp}}(\mathbb{S}^{\text{triv}}, \text{THH}(A))$, and THH is an fpqc sheaf valued in CycSp, for $A \to B$ faithfully flat, we have an equivalence

$$\text{TC}(A) \to \text{Map}_{\text{CycSp}}(\mathbb{S}^{\text{triv}}, \varprojlim \text{THH}(B^{\otimes_A \bullet})) \simeq \varprojlim \text{Map}_{\text{CycSp}}(\mathbb{S}^{\text{triv}}, \text{THH}(B^{\otimes_A \bullet})) = \varprojlim \text{TC}(B^{\otimes_A \bullet})$$

$\qquad\square$

References

[AKS18]   Gabe Angelini-Knoll and Andrew Salch, *A May-type spectral sequence for higher topological Hochschild homology*, Algebraic & Geometric Topology **18** (2018), no. 5, 2593–2660.

[BMS19]   Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publications mathématiques de l'IHÉS **129** (2019), no. 1, 199–310.

[CM19]   Dustin Clausen and Akhil Mathew, *Hyperdescent and étale K-theory*, 2019.

[DR18]   Bjørn Ian Dundas and John Rognes, *Cubical and cosimplicial descent*, Journal of the London Mathematical Society **98** (2018), no. 2, 439–460.

[GHN17]   David Gepner, Rune Haugseng, and Thomas Nikolaus, *Lax colimits and free fibrations in ∞-categories*, Documenta Mathematica **22** (2017), 1255–1266.

[Gla13]   Saul Glasman, *Day convolution for ∞-categories*, arXiv preprint arXiv:1308.4940 (2013).

[Lur09]   Jacob Lurie, *Higher topos theory*, Princeton University Press, 2009.

[Lur15]   _____, *Rotation invariance in algebraic K-theory*, Preprint, available at https://www.math.ias.edu/~lurie (2015).

[Lur17]   _____, *Higher algebra*, Preprint, available at https://www.math.ias.edu/~lurie (2017).

[Lur18]   _____, *Spectral algebraic geometry*, Preprint, available at https://www.math.ias.edu/~lurie (2018).

[NS18]   Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, Acta Mathematica **221** (2018), no. 2, 203–409.