ON THE MAY SPECTRAL SEQUENCE AT THE PRIME 2

WEINAN LIN

Abstract. We make a conjecture about all the relations in the E_2 page of the May spectral sequence and prove it in a subalgebra which covers a large range of dimensions. We conjecture that the E_2 page is nilpotent free and also prove it in this subalgebra. For further computations we construct maps of spectral sequences which systematically extend one of the techniques used by May and Tangora.

Contents

1. Introduction

The May spectral sequence

$$\mathrm{Ext}^{***}_{E^0\mathscr{A}}(\mathbb{F}_p,\mathbb{F}_p) \Longrightarrow \mathrm{Ext}^{**}_{\mathscr{A}}(\mathbb{F}_p,\mathbb{F}_p)$$

from [3] is one of the first effective methods to compute the cohomology of the Steenrod algebra. In this paper, we will study this spectral sequence at the prime 2.

We start with the E_2 page of the May spectral sequence. Compared with the cohomology of the Steenrod algebra, the E_2 page of the May spectral sequence can be computed in a much larger range. In addition to Conjecture 2.17 by May about what all the indecomposables of the E_2 page are, we state Conjecture 2.20 about all the relations among these indecomposables and Conjecture 2.21 claiming that the E_2 page is nilpotent free. We will prove all three conjectures in a subalgebra which covers a large range of dimensions (Theorem 2.26). It indicates that it is possible that these indecomposables and relations do in fact describe the whole E_2 page, and the E_2 page is nilpotent free. This is startling because all of the positive elements in the stable homotopy groups of spheres are nilpotent.

1.1. **Organization**

In Section 2 we state our main conjectures and theorems about the E_2 page of the May spectral sequence. We also show how to obtain the indecomposables $h_{S,T}$ from h_i under matric Massey products. In Section 3 we give a formula for the d_2 differentials on the indecomposables $h_{S,T}$ of the E_2 page. In Section 4 we set up some computational tools including the Gröbner bases in order to compute HX_7 which proves Theorem 2.26 in Section 2. Some of the work is aided by computer. We also construct some comparison maps of spectral sequences in this section. Appendix A provides a list of charts of the computational results of Section 4.

1.2. **Acknowledgement**

The author would like to thank his advisor Peter May for many helpful discussions on this topic. The author was very glad to come across this problem and to get the support from his advisor to choose this problem as his thesis topic. May also read many drafts of this paper and offered tremendous help on writing.

2. The E_2 Page of the May Spectral Sequence

The main goal of this section is to state the conjecture which fully describes the E_2 page of the May spectral sequence in terms of generators and relations. We will show that this conjecture holds at least in a big subalgebra of E_2.

2.1. **The May filtration**

Recall May's results in his thesis [3] that we can filter the Steenrod algebra as follows.

Let $I(\mathscr{A}) \subset \mathscr{A}$ be the augmentation ideal. Let

$$\Phi_n : I(\mathscr{A}) \otimes \cdots \otimes I(\mathscr{A}) \longrightarrow I(\mathscr{A})$$

be the n-fold multiplication.

Define

$$F_n\mathscr{A} = \mathscr{A}, n \geq 0;\ F_{-n}\mathscr{A} = \mathrm{Im}\,\Phi_n, n > 0.$$

Then the associated graded Hopf algebra $E^0\mathscr{A}$ of $\mathscr{A}$ is defined by

$$E^0_{p,q}\mathscr{A} = (F_p\mathscr{A}/F_{p-1}\mathscr{A})_{p+q}.$$

A theorem due to Milnor and Moore [5] states that any primitively generated Hopf algebra over a field of characteristic p is isomorphic to the universal enveloping algebra of its restricted Lie algebra of primitive elements. The associated graded algebra $E^0\mathscr{A}$ satisfies the conclusion as follows.

Theorem 2.1 (May). *The associated graded algebra $E^0\mathscr{A}$ can be represented by the associative algebra generated by P^i_j, $i \geq 0, j > 0$ with relations*

$$(P^i_j)^2 = 0;\ [P^i_j, P^k_l] = \delta_{i,k+\ell}P^k_{j+\ell},\ i \geq k.$$

Here $P^i_j \in E^0\mathscr{A}$ corresponds to the projection of the dual of $\xi_j^{2^i}$ in the dual Steenrod algebra

$$\mathscr{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

with monomial basis.

We can also filter the the cobar complex of $\mathscr{A}$ based on this filtration. The resulting spectral sequence is the May spectral sequence.

Theorem 2.2 (May). *There exists a spectral sequence (E_r, d_r) converging to the cohomology of the Steenrod algebra, and having its E_2 term $H^*(E^0\mathscr{A})$. Each E_r is a tri-graded algebra and each d_r is a homomorphism*

$$d_r : E_r^{u,v,t} \to E_r^{u+r,v-r+1,t}$$

which is a derivation with respect to the algebra structure.

2.2. **The cohomology of $E^0\mathscr{A}$**

For any Hopf algebra A, May [3] found a reasonably small complex with which to calculate $H^*(E^0A)$. As an application, for the Steenrod algebra $\mathscr{A}$ we get the following.

Theorem 2.3 (May). *The cohomology of the associated graded algebra $E^0\mathscr{A}$ is isomorphic to the homology of the differential graded algebra*

$$X = \mathbb{F}_2[R^i_j : i \geq 0, j > 0]$$

with differentials given by

$$dR^i_j = \sum_{k=1}^{j-1} R^{i+k}_{j-k}R^i_k. \tag{2.4}$$

Remark 2.5. May proved this theorem by showing that $E^0\mathscr{A} \otimes X^*$ is an $E^0\mathscr{A}$-free resolution of $\mathbb{F}_2$ which is much smaller than the bar construction. In 1970 after May's thesis, Priddy [7][8] conceptualized this method into Koszul resolutions which apply to a more general kind of algebras called Koszul algebras. The complex X can be interpreted as the co-Koszul complex of $E^0\mathscr{A}$ in terms of Priddy's setting.

Definition 2.6. We reindex the generators of X by

$$R_{ij} = \begin{cases} R^i_{j-i}, & \text{if } 0 \leq i < j, \\ 0, & \text{if } 0 \leq j \leq i. \end{cases}$$

With a little rewriting, (2.4) now becomes

$$dR_{ij} = \sum_{k=i+1}^{j-1} R_{ik} R_{kj}. \tag{2.7}$$

If we regard R as the strictly upper triangular matrix (R_{ij}), then $dR = R^2$.

Remark 2.8. The symbol R^i_j is written as R_{ij} by Tangora [10] but as $R_{i,i+j}$ in this paper.

The differential algebra X has interesting connections to matric Massey products. Note that (2.7) takes exactly the same form as that of (5) in [4] which is the formula for the defining system of a matric Massey product. The direct consequences are the following.

Let $X_n = \mathbb{F}_2[R_{ij} : 0 \leq i < j \leq n]$. It is a sub-differential algebra of X.

Theorem 2.9. *If A is a commutative differential algebra, then the decompositions of zero in HA as an n-ary Massey product (together with a defining system)*

$$0 \in \langle a_1, \ldots, a_n \rangle, \qquad a_i \in HA$$

are in one-to-one correspondence to maps of differential algebras:

$$f : X_n \to A$$

where f induces the algebraic map

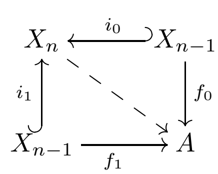
$$f_* : HX_n \to HA$$

with $f_*(h_{i-1}) = a_i$, $1 \leq i \leq n$, where h_{i-1} is the homology class of $R_{i-1,i}$.

Theorem 2.10. *A nontrivial element $a \in HA$ and a defining system for the Massey product*

$$a \in \langle a_1, \ldots, a_n \rangle$$

corresponds to the obstruction to obtaining the dashed map



where f_0 corresponds to the sub-defining system for $0 \in \langle a_1, \ldots, a_{n-1} \rangle$ and f_1 for $0 \in \langle a_2, \ldots, a_n \rangle$. The embeddings i_0 and i_1 are given by $i_0(R_{ij}) = R_{ij}$ and $i_1(R_{ij}) = R_{i+1,j+1}$.

2.3. The indecomposables of $H^*(E^0\mathscr{A})$

Definition 2.11. For two strictly increasing sequences of distinct numbers $S = \{s_1, \ldots, s_n\}$, $T = \{t_1, \ldots, t_n\}$, we define

$$R_{S,T} = \det(R_{s_i t_j}) = \sum_{\sigma \in \Sigma_n} R_{s_1 t_{\sigma(1)}} \cdots R_{s_n t_{\sigma(n)}}.$$

Note that the value of $R_{S,T}$ does not depend on the ordering of numbers in S or T. However we prefer to put them in order, and in the rest of the paper, we assume all sequences S and T are ordered.

Definition 2.12. For two sequences S and T, we write $S < T$ if $\max(S) < \min(T)$ and $S \leq T$ if $\max(S) \leq \min(T)$.

Proposition 2.13. *The determinants $R_{S,T}$ have the following properties*

(1) *$R_{S,T}$ is nonzero if and only if $s_i < t_i$ for $1 \leq i \leq n$.*
(2) *If $T_1 \leq S_2$ or $T_2 \leq S_1$, then*
$$R_{S_1 \cup S_2, T_1 \cup T_2} = R_{S_1,T_1} R_{S_2,T_2}$$
(3) $dR_{S,T} = \sum_{k \in \mathbb{Z}_{\geq 0} \setminus (S \cup T)} R_{S \cup \{k\}, T \cup \{k\}}$. *Note that the summand of the summation is zero when $k < \min(S \cup T)$ or $k > \max(S \cup T)$ because of (1).*
(4) *For any fixed subset I of S,*
$$R_{S,T} = \sum_{|J|=|I|} R_{I,J} R_{S-I,T-J}.$$
Similarly, for any fixed subset J of T,
$$R_{S,T} = \sum_{|I|=|J|} R_{I,J} R_{S-I,T-J}.$$

Proof. We keep using the fact that $R_{S,T}$ is the determinant of $(R_{s_i t_j})$.

(1) If $s_i \geq t_i$ for some i, then $R_{s_j t_k} = 0$ if $j \geq i \geq k$ which yields zero determinant. Thus $s_i < t_i$ for all i.
(2) If $T_1 \leq S_2$ or $T_2 \leq S_1$ we have either an upper or lower triangular block matrix associated to $R_{S_1 \cup S_2, T_1 \cup T_2}$ with determinants of the diagonal blocks being R_{S_1,T_1} and R_{S_2,T_2}.
(3) By the definition of $R_{S,T}$ and property (1), we have
$$\begin{aligned} dR_{S,T} &= \sum_{\sigma \in \Sigma_n} d(R_{s_1 t_{\sigma(1)}} \cdots R_{s_n t_{\sigma(n)}}) \\ &= \sum_{\sigma \in \Sigma_n} \sum_i \sum_k R_{s_1 t_{\sigma(1)}} \cdots \widehat{R_{s_i t_{\sigma(i)}}} \cdots R_{s_n t_{\sigma(n)}} \cdot R_{s_i k} R_{k \sigma(i)} \\ &= \sum_{k \notin S \cup T} R_{S \cup \{k\}, T \cup \{k\}}. \end{aligned}$$
Here $\widehat{R_{s_i t_{\sigma(i)}}}$ means that we skip the factor in the monomial.
(4) This is the expansion of the determinant of $(R_{s_i t_j})$ by the rows corresponding to I.

□

Definition 2.14. Assume we have two sequences $S = \{s_1, \ldots, s_n\}$ and $T = \{t_1, \ldots, t_n\}$ such that $s_k < t_k$ for $1 \leq k \leq n$ and
$$S \cup T = \{i, i+1, \ldots, i+2n-1\}$$
for some integer i. Then $dR_{S,T} = 0$ by (3) of the above proposition. Let $\mathscr{H}'$ be the set of homology classes of all such $R_{S,T}$. Let $\mathscr{H}$ be the set of homology classes of all such $R_{S,T}$ with one extra condition that $s_k < t_{k-1}$ for $2 \leq k \leq n$. For convenience we use $h_{S,T}$ or $h_i(S')$ to denote the homology class of $R_{S,T}$, where $i = s_1$ and $S' = \{s_2 - s_1, \ldots, s_n - s_1\}$. The simplest examples are $h_{i,i+1} = h_i = [R_{i,i+1}]$.

Remark 2.15. By Proposition 2.13.(2) we can see that every element in $\mathscr{H}'$ can be decomposed as a product of elements in $\mathscr{H}$.

Theorem 2.16 (May). *All elements in $\mathscr{H}$ are indecomposables in HX.*

The way May proved this theorem is by studying the dual of X instead of X. The differential graded algebra X is actually a Hopf algebra. May was able to identify all monomial cycles in the dual of X which are primitive in the homology. Each additive summand of the determinant $R_{S,T}$ for $h_{S,T} \in \mathscr{H}$ corresponds to such a monomial cycle in the dual of X and they are homologous to each other. Hence we can get the theorem by dualization. The details can be found in [3, II.5].

Beside elements of $\mathscr{H}$, we can also see that the homology classes of R_{ij}^2 for $j - i \geq 2$ are also indecomposables of HX. Let $b_{S,T}$ denote the homology class of $R_{S,T}^2$. Especially, $b_{ij} = [R_{ij}^2]$ and $b_{i,i+1} = h_i^2$.

The following conjecture suggests that it is possible that these are all the indecomposables we need in HX.

Conjecture 2.17 (May, [3, Conjecture II.5.7]). *The elements of $\mathscr{H}$ and b_{ij} ($j-i \geq 2$) form a basis of indecomposables of HX.*

2.4. The relations in $H^*(E^0\mathscr{A})$

In addition to Conjecture 2.17, we will state a conjecture to describe all the relations in $H^*(E^0\mathscr{A}) \cong HX$. We also conjecture that this algebra is nilpotent free.

Definition 2.18. For $0 \leq m < n$, we define

$$\mathscr{H}_{mn} = \{h_{S,T} \in \mathscr{H} : \min(S) = m,\ \max(T) = n\}$$

and

$$\mathscr{H}'_{mn} = \{h_{S,T} \in \mathscr{H}' : \min(S) = m,\ \max(T) = n\}.$$

Note that $\mathscr{H}_{mn} \subset \mathscr{H}'_{mn}$ and $\mathscr{H}_{mn}$, $\mathscr{H}'_{mn}$ are empty if $n - m$ is even.

Definition 2.19. For a sequence $S = \{s_1, \ldots, s_n\}$, we define $|S|$ to be the length n of S.

Conjecture 2.20. *The algebra HX is generated by $h_{S,T} \in \mathscr{H}$ and b_{ij} ($j - i \geq 2$) with the following relations.*

(1) *For all $0 \leq i < j$,*

$$\sum_k b_{ik} b_{kj} = 0.$$

(2) *Assume $h_{S_1,T_1} \in \mathscr{H}'_{a_1,b_1}$, $h_{S_2,T_2} \in \mathscr{H}'_{a_2,b_2}$, $a_1 < a_2 < b_1 < b_2$ and $b_1 - a_2$ is even. Then*

$$h_{S_1,T_1} h_{S_2,T_2} = 0.$$

(3A) *Assume that $S \subset N = \{a, a+1, \ldots, a+2k-1\}$ and $|S| = k+1$. Let T be the complement of S in N. Then*

$$\sum_{s \in S} b_{sj} h_{S-\{s\},T+\{s\}} = 0$$

for any $j \leq a + 2k$.

(3B) *Assume that* $S \subset N = \{a, a+1, \ldots, a+2k-1\}$ *and* $|S| = k-1$. *Let* T *be the complement of* S *in* N. *Then*

$$\sum_{t \in T} b_{it} h_{S+\{t\}, T-\{t\}} = 0$$

for any $i \geq a-1$.

(4A) *Assume* $h_{S_1,T_1} \in \mathscr{H}'_{a_1,b_1}$, $h_{S_2,T_2} \in \mathscr{H}'_{a_2,b_2}$, $a_1 \leq a_2 < b_1 \leq b_2$ *and* $b_1 - a_2$ *is odd. Then*

$$h_{S_1,T_1} h_{S_2,T_2} = \sum_{\substack{I \subset T_1'' \cap S_2 \\ 2|I| = |T_1''| - |S_1''|}} h_{S_1''+I, T_1''-I} h_{S_1'+S_2-I, T_1'+T_2+I}$$

Where $S_1' = S_1 \backslash N_{a_2,b_1}$, $S_1'' = S_1 \cap N_{a_2,b_1}$, $T_1' = T_1 \backslash N_{a_2,b_1}$, $T_1'' = T_1 \cap N_{a_2,b_1}$.

(4B) *Assume* $h_{S_1,T_1} \in \mathscr{H}'_{a_1,b_1}$, $h_{S_2,T_2} \in \mathscr{H}'_{a_2,b_2}$, $a_1 \leq a_2 < b_1 \leq b_2$ *and* $b_1 - a_2$ *is odd. Then*

$$h_{S_1,T_1} h_{S_2,T_2} = \sum_{\substack{I \subset T_1 \cap S_2'' \\ 2|I| = |S_2''| - |T_2''|}} h_{S_2''-I, T_2''+I} h_{S_1+S_2'+I, T_1+T_2'-I}$$

Where $S_2' = S_2 \backslash N_{a_2,b_1}$, $S_2'' = S_2 \cap N_{a_2,b_1}$, $T_2' = T_2 \backslash N_{a_2,b_1}$, $T_2'' = T_2 \cap N_{a_2,b_1}$.

(5) *Assume* $h_{S_1,T_1} \in \mathscr{H}'_{a_1,b_1}$, $h_{S_2,T_2} \in \mathscr{H}'_{a_2,b_2}$, $a_1 \leq a_2 < b_1 \leq b_2$ *and* $b_1 - a_2$ *is odd. Then*

$$h_{S_1,T_1} h_{S_2,T_2} = \sum_{\substack{I \subset S_1' \\ J \subset T_2'}} h_{S_1'-I, T_1'+I} h_{S_2'+J, T_1'-J} b_{S_1''+I, T_2''+J}$$

Where $S_i' = S_i \backslash N_{a_2,b_1}$, $S_i'' = S_i \cap N_{a_2,b_1}$, $T_i' = T_i \backslash N_{a_2,b_1}$, $T_i'' = T_i \cap N_{a_2,b_1}$, $i = 1, 2$.

(6) *Assume* $h_{S_i,T_i} \in \mathscr{H}_{a,b}$, $i = 1, \ldots, n$, *and*

$$\sum_i x_i h_{S_i-\{a\}, T_i-\{b\}} = 0$$

where x_i *is a product of elements in*

$$\bigcup_{\substack{a<a'<b'<b \\ a_i-a \text{ is odd}}} \mathscr{H}_{a',b'}$$

Then

$$\sum_i x_i h_{S_i,T_i} = 0$$

Conjecture 2.21. *HX is nilpotent free.*

In order to prove Conjecture 2.20, we have to prove that all the relations in the conjecture hold and they imply all the other relations. We are not there yet although we have a great deal of evidence for the conjecture. In the rest of the section we will describe the results we already have, including evidence for Conjecture 2.21.

Theorem 2.22. *The relations (1), (2), (3A), (3B), (4A) and (4B) in Conjecture 2.20 hold in HX. The relations (5) and (6) hold in a large range of dimensions.*

The following proposition for all n shows that the statement (3A) is symmetric to (3B) and (4A) is symmetric to (4B). Hence we only have to prove one for each pair.

Proposition 2.23. *The reflection map*

$$X_n \to X_n$$

$$R_{ij} \mapsto R_{n-j,n-i}$$

is an isomorphism between differential algebras. Therefore HX_n is isomorphic to itself via this reflection map.

The proof is straightforward. Before we prove Theorem 2.22 we need the following lemma.

Lemma 2.24. *Assume that S_1, T_1, S_2, T_2 are four sequences such that $|S_1| = |T_1| - 1$, $|S_2| = |T_2| + 1$, $S_1 \cap T_1 = \emptyset = S_2 \cap T_2$ and*

$$(S_1 \cup T_1)\backslash(S_2 \cup T_2) < (S_1 \cup T_1) \cap (S_2 \cup T_2) < (S_2 \cup T_2)\backslash(S_1 \cup T_1).$$

Then

$$\sum_{\substack{s \in S_1 \cap S_2 \\ i \in T_1 \cap S_2}} R_{S_1, T_1 - \{i\}} R_{S_2 - \{s\}, T_2} R_{s,i} = \sum_{\substack{i \in T_1 \cap S_2 \\ t \in T_1 \cap T_2}} R_{S_1, T_1 - \{t\}} R_{S_2 - \{i\}, T_2} R_{i,t}.$$

Proof. By Proposition 2.13.(4), these are both equal to

$$\sum_{\substack{s \in S_1 \cap S_2 \\ t \in T_1 \cap T_2}} R_{S_1, T_1 - \{t\}} R_{S_2 - \{s\}, T_2} R_{s,t}.$$

□

We now prove Theorem 2.22 by realizing the relations as boundaries via explicit constructions.

Proof of Theorem 2.22. (1). The relation follows from

$$d(R_{ij} dR_{ij}) = (dR_{ij})^2 = \sum_k R_{ik}^2 R_{kj}^2.$$

(2). Let

$$y = \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} R_{S_1 - \{s\}, T_1 - \{i\}} R_{S_2 - \{i\} + \{s\}, T_2}.$$

It suffices to show that $dy = R_{S_1,T_1} R_{S_2,T_2}$. In fact,

$$dy = \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} \Big(R_{S_1, T_1 - \{i\} + \{s\}} R_{S_2 - \{i\} + \{s\}, T_2} + R_{S_1 - \{s\} + \{i\}, T_1} R_{S_2 - \{i\} + \{s\}, T_2} + \\ R_{S_1 - \{s\}, T_1 - \{i\}} R_{S_2 + \{s\}, T_2 + \{i\}} + \sum_{j < a_2} R_{S_1 - \{s\}, T_1 - \{i\}} R_{S_2 - \{i\} + \{j\}, T_2} R_{s,j} \Big).$$

We apply 2.13.(4) and get

$$\sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} \sum_{j < a_2} R_{S_1 - \{s\}, T_1 - \{i\}} R_{S_2 - \{i\} + \{j\}, T_2} R_{s,j} = \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} R_{S_1, T_1 - \{i\} + \{s\}} R_{S_2 - \{i\} + \{s\}, T_2}.$$

Therefore

$$\begin{aligned} dy =& \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} \left(R_{S_1-\{s\},T_1-\{i\}} R_{S_2+\{s\},T_2+\{i\}} + R_{S_1-\{s\}+\{i\},T_1} R_{S_2-\{i\}+\{s\},T_2}\right) \\ =& \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} R_{S_1-\{s\},T_1-\{i\}} R_{S_2,T_2} R_{s,i} + \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2 \\ s_2 \in S_2 \cap S_1}} R_{S_1-\{s\},T_1-\{i\}} R_{S_2-\{s_2\}+\{s\},T_2} R_{s_2,i} \\ &+ \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2 \\ t_1 \in T_1 \cap T_2}} R_{S_1-\{s\},T_1-\{t_1\}} R_{S_2-\{i\}+\{s\},T_2} R_{i,t_1}. \end{aligned}$$

By Lemma 2.24 we have

$$\begin{aligned} &\sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2 \\ s_2 \in S_2 \cap S_1}} R_{S_1-\{s\},T_1-\{i\}} R_{S_2-\{s_2\}+\{s\},T_2} R_{s_2,i} + \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2 \\ t_1 \in T_1 \cap T_2}} R_{S_1-\{s\},T_1-\{t_1\}} R_{S_2-\{i\}+\{s\},T_2} R_{i,t_1} \\ =& \sum_{\substack{S_1 \ni s < a_2 \\ t_1 \in T_1 \cap T_2}} R_{S_1-\{s\},T_1-\{t_1\}} R_{S_2,T_2} R_{s,t_1} \end{aligned}$$

Therefore

$$\begin{aligned} dy &= \sum_{\substack{S_1 \ni s < a_2 \\ i \in T_1 \cap S_2}} R_{S_1-\{s\},T_1-\{i\}} R_{S_2,T_2} R_{s,i} + \sum_{\substack{S_1 \ni s < a_2 \\ t_1 \in T_1 \cap T_2}} R_{S_1-\{s\},T_1-\{t_1\}} R_{S_2,T_2} R_{s,t_1} \\ &= \sum_{\substack{S_1 \ni s < a_2 \\ T_1 \ni t \geq a_2}} R_{S_1-\{s\},T_1-\{t\}} R_{S_2,T_2} R_{s,t} \\ &= R_{S_1,T_1} R_{S_2,T_2}. \end{aligned}$$

The last equality holds because for every monomial α in R_{S_1,T_1} there is an odd number of factors $R_{s,t}$ in α such that $S_1 \ni s < a_2, T_1 \ni t \geq a_2$.

(3A). Let

$$y = \sum_{\{s_1 < s_2\} \subset S} R_{s_1 j} R_{s_2 j} R_{S-\{s_1,s_2\},T}$$

$$\begin{aligned} dy =& \sum_{\{s_1 < s_2\} \subset S} \sum_i \left(R_{s_1 i} R_{ij} R_{s_2 j} + R_{s_2 i} R_{ij} R_{s_1 j}\right) R_{S-\{s_1,s_2\},T} \\ &+ \sum_{\{s_1 < s_2\} \subset S} R_{s_1 j} R_{s_2 j} \left(R_{S-\{s_1\},T+\{s_2\}} + R_{S-\{s_2\},T+\{s_1\}}\right) \\ =& \sum_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} \sum_i R_{s_2 i} R_{ij} R_{s_1 j} R_{S-\{s_1,s_2\},T} + \sum_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} R_{s_1 j} R_{s_2 j} R_{S-\{s_1\},T+\{s_2\}} \\ =& \mathrm{I} + \mathrm{II} \end{aligned}$$

where

$$\mathrm{I} = \sum_{\substack{s_1, s_2 \in S \\ s_1 \neq s_2}} \sum_i R_{s_2 i} R_{ij} R_{s_1 j} R_{S-\{s_1,s_2\},T}$$

and

$$\begin{aligned}\mathrm{II} &= \sum_{\substack{s_1,s_2\in S\\ s_1\neq s_2}} R_{s_1j}R_{s_2j}R_{S-\{s_1\},T+\{s_2\}}\\ &= \sum_{\substack{s_1,s_2,i\in S\\ s_1\neq s_2\\ s_1\neq i}} R_{s_1j}R_{s_2j}R_{is_2}R_{S-\{s_1,i\},T}\\ &= \sum_{\substack{s_1,i,s_2\in S\\ s_1\neq i\\ s_1\neq s_2}} R_{s_1j}R_{ij}R_{s_2i}R_{S-\{s_1,s_2\},T}\end{aligned}$$

The only difference between summations I and II is that i can be equal to s_1 or $i\in T$ in summation I. Therefore

$$\begin{aligned}dy &= \sum_{\substack{s_1,s_2\in S\\ s_1\neq s_2\\ i\in T\cup\{s_1\}}} R_{s_2i}R_{ij}R_{s_1j}R_{S-\{s_1,s_2\},T}\\ &= \sum_{\substack{s_1\in S\\ i\in T\cup\{s_1\}}} R_{ij}R_{s_1j}R_{S-\{s_1\},T+\{i\}}\\ &= \sum_{s_1\in S} R_{s_1j}^2 R_{S-\{s_1\},T+\{s_1\}}\end{aligned}$$

where the right-hand side represents our relation. Hence our relation holds.

(3B). This follows from (3A) because of the symmetry given by Proposition 2.23.

(4A). Let

$$y = \sum_{\substack{I,J\subset T_1''\cap S_2\\ j_0=\max(J\setminus I)>I\setminus J}} R_{S_1''+I,T_1''-J}R_{S_1'+S_2-I-\{j_0\},T_1'+T_2+J'}$$

Then

$$dy = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}$$

where

$$\begin{aligned}\mathrm{I} &= \sum_{\substack{I,J\subset T_1''\cap S_2\\ j_0=\max(J\setminus I)>I\setminus J}} R_{S_1''+I,T_1''-J}R_{S_1'+S_2-I,T_1'+T_2+J}\\ \mathrm{II} &= \sum_{\substack{I,J\subset T_1''\cap S_2\\ j_0=\max(J\setminus I)>I\setminus J}} R_{S_1''+I+\{j_0\},T_1''-J+\{j_0\}}R_{S_1'+S_2-I-\{j_0\},T_1'+T_2+J'}\\ \mathrm{III} &= \sum_{\substack{I,J\subset T_1''\cap S_2\\ j_0=\max(J\setminus I)>I\setminus J\\ j\in J'\setminus I}} R_{S_1''+I+\{j\},T_1''-J+\{j\}}R_{S_1'+S_2-I-\{j_0\},T_1'+T_2+J'}\end{aligned}$$

$$\mathrm{IV} = \sum_{\substack{I,J\subset T_1''\cap S_2\\ j_0=\max(J\backslash I)>I\backslash J\\ i\in I\backslash J}} R_{S_1''+I,T_1''-J}R_{S_1'+S_2-I-\{j_0\}+\{i\},T_1'+T_2+J'+\{i\}}.$$

In summation III, change index by $I_1 = I+\{j\}$, $J_1 = J-\{j\}$. We have

$$\begin{aligned}\mathrm{III} =& \sum_{\substack{I_1,J_1\subset T_1''\cap S_2\\ j_0=\max(J_1\backslash I_1)>I_1\backslash J_1\\ j\in I_1\backslash J_1}} R_{S_1''+I_1,T_1''-J_1}R_{S_1'+S_2-I_1+\{j\}-\{j_0\},T_1'+T_2+J_1'+\{j\}}\\ =& \sum_{\substack{I,J\subset T_1''\cap S_2\\ j_0=\max(J\backslash I)>I\backslash J\\ i\in I\backslash J}} R_{S_1''+I,T_1''-J}R_{S_1'+S_2-I-\{j_0\}+\{i\},T_1'+T_2+J'+\{i\}}\\ =&\mathrm{IV}\end{aligned}$$

In summation II, change index by $I_1 = I+\{j_0\}$, $J_1 = J-\{j_0\}$. We have

$$\mathrm{II} = \sum_{\substack{I_1,J_1\subset T_1''\cap S_2\\ j_0=\max(I_1\backslash J_1)>J_1\backslash I_1}} R_{S_1''+I_1,T_1''-J_1}R_{S_1'+S_2-I_1,T_1'+T_2+J_1}$$

Therefore

$$dy = \mathrm{I}+\mathrm{II} = \sum_{\substack{I,J\subset T_1''\cap S_2\\ I\neq J}} R_{S_1''+I,T_1''-J}R_{S_1'+S_2-I,T_1'+T_2+J}$$

Note that if we instead require $I = J$ in the above summation, we get the right hand side of Relation (4A). Hence in order to prove Relation (4A) it suffices to show that

$$\sum_{I,J\subset T_1''\cap S_2} R_{S_1''+I,T_1''-J}R_{S_1'+S_2-I,T_1'+T_2+J} = R_{S_1,T_1}R_{S_2,T_2}.$$

In fact, if we denote the summation on the left hand side above by V, then

$$\mathrm{V} = \sum_{\substack{I,J\subset T_1''\cap S_2\\ M\subset S_1'\\ L\subset S_1'-M+S_2-I}} R_{S_1''+I,T_1''-J}R_{S_1'-M+S_2-I-L,J}R_{M,T_1'}R_{L,T_2}$$

Fix I, M and L. If

$$(S_1''+I)\cap(S_1'-M+S_2-I-L)=\emptyset \tag{2.25}$$

which is equivalent to

$$(S_1''+I)\cap((S_2\backslash L)-I)=\emptyset \text{ and to } S_1''\cap S_2 = S_1\cap S_2\subset L,$$

then by 2.13.4 we have

$$\begin{aligned}&\sum_{J\subset T_1''\cap S_2} R_{S_1''+I,T_1''-J}R_{S_1'-M+S_2-L-I,J}\\ =&R_{(S_1''+I)+(S_1'-M+S_2-L-I),T_1''}\\ =&R_{S_1-M+S_2-L,T_1''}.\end{aligned}$$

Otherwise if (2.25) does not hold, then

$$\sum_{J\subset T_1''\cap S_2} R_{S_1''+I,T_1''-J}R_{S_1'-M+S_2-L-I,J}=0.$$

Therefore

$$\begin{aligned}
\mathrm{V}=&\sum_{\substack{I\subset T_1''\cap S_2\\ M\subset S_1'\\ S_1\cap S_2\subset L\subset S_1'-M+S_2-I}} R_{S_1-M+S_2-L,T_1''}R_{M,T_1'}R_{L,T_2}\\
=&\sum_{\substack{S_1''\cap S_2\subset L\subset S_1'+S_2\\ M\subset S_1'\backslash L\\ I\subset (T_1''\cap S_2)\backslash L}} R_{S_1-M+S_2-L,T_1''}R_{M,T_1'}R_{L,T_2}\\
=&\sum_{\substack{S_1''\cap S_2\subset L\subset S_1'+S_2\\ M\subset S_1'\backslash L}} 2^{(T_1''\cap S_2)\backslash L}R_{S_1-M+S_2-L,T_1''}R_{M,T_1'}R_{L,T_2}
\end{aligned}$$

The summand is nontrivial only if

$$(T_1''\cap S_2)\backslash L=\emptyset \quad\text{and}\quad S_1-M+S_2-L<\max(T_1'')=b_1,$$

which is equivalent to

$$(T_1''\cap S_2)\subset L \quad\text{and}\quad N_{b_1+1,b_2}\cap S_2\subset L.$$

Note that in the summation we also require $(S_1''\cap S_2)\subset L$. Hence

$$(S_1''+T_1''+N_{b_1+1,b_2})\cap S_2=S_2\subset L$$

which implies $S_2=L$. Therefore

$$\begin{aligned}
\mathrm{V}=&\sum_{M\subset S_1'} R_{S_1-M,T_1''}R_{M,T_1'}R_{S_2,T_2}\\
=&R_{S_1,T_1}R_{S_2,T_2}
\end{aligned}$$

(4B). This follows from (4A) because of the symmetry given by Proposition 2.23. $\square$

Theorem 2.26. *Conjectures 2.20 and 2.21 hold in HX_7.*

We will prove this theorem by computing HX_7 in Section 4. This is strong evidence for the two conjectures since the subalgebra $HX_7\subset HX$ together with h_7 generates a subalgebra isomorphic to HX in stems $t-s\le 285$.

2.5. Massey products in $H^*(E^0\mathscr{A})$

A theorem due to Gugenheim and May [2] states that for a connected algebra A, the cohomology $H^*(A)$ is generated under matric Massey products by $H^1(A)$. As a concrete example, we will show how to obtain the indecomposables $h_{S,T}\in\mathscr{H}$ from h_i under matric Massey products.

Theorem 2.27. *For $h_{S,T}\in\mathscr{H}$ where*

$$S\cup T=\{k,k+1,\dots,k+2n-1\}$$

we have

$$h_{S,T}\in\langle h_k,h_{k+1},\dots,h_{k+2n-2},h_{S-\{k\},T-\{k+2n-1\}}\rangle.$$

Proof. Without loss of generality we assume $k = 0$. By the definition of matric Massey products, we must find a defining system (A_{ij}) with $0 \leq i < j \leq 2n$ and $(i,j) \neq (0,2n)$ such that

$$A_{i,i+1} = \begin{cases} R_{S-\{s_1\},T-\{t_n\}} & \text{if } 0 \leq i < 2n-1 \\ R_{i,i+1} & \text{if } i = 2n-1, \end{cases} \tag{2.28}$$

$$dA_{ij} = \sum_{i<k<j} A_{ik}A_{kj} \tag{2.29}$$

and

$$\widetilde{A}_{0,2n} = \sum_{0<k<2n} A_{0,k}A_{k,2n} = R_{S,T}. \tag{2.30}$$

In fact, for $0 \leq i < j \leq 2n-1$, if we let $A_{ij} = R_{ij}$, then (2.28) and (2.29) are automatically true by (2.7).

We adopt the convention that $R_{S-\{0\},T-\{i\}} = 0$ if $i \notin T$. We let $A_{i,2n} = R_{S-\{0\},T-\{i\}}$ $(i \neq 0)$. Now for (2.29) we only have to show

$$dA_{i,2n} = \sum_{i<k<2n} A_{0,k}A_{k,2n}$$

i.e.

$$dR_{S-\{0\},T-\{i\}} = \sum_{i<k<2n} R_{ik}R_{S-\{0\},T-\{k\}}.$$

If $i \in T$, by (3)(4) of Proposition 2.13.

$$dR_{S-\{0\},T-\{i\}} = R_{S-\{0\}+\{i\},T} = \sum_{i<k<2n} R_{ik}R_{S-\{0\},T-\{k\}}.$$

If $i \notin T$, the right-hand side is zero because $i \in S$ and hence

$$\sum_{i<k<2n} R_{ik}R_{S-\{0\},T-\{i\}} = R_{S-\{0\}+\{i\},T} = 0$$

since $R_{S-\{0\}+\{i\},T}$ is the determinant of a matrix with repeating rows.

Finally, to show (2.30) we have

$$\sum_{0<k<2n} A_{0k}A_{k,2n} = \sum_{0<k<2n} R_{0k}R_{S-\{0\},T-\{k\}} = R_{S,T}.$$

□

Note that in Theorem 2.27 the s degree of $h_{S-\{k\},T-\{k+2n-1\}}$ is one less than the s degree of $h_{S,T}$. The element $h_{S-\{k\},T-\{k+2n-1\}}$ is either an element of $\mathscr{H}$ or a product of elements in $\mathscr{H}$. Hence by induction on s all indecomposables $h_{S,T} \in \mathscr{H}$ can be obtained inductively from h_i under matric Massey products.

Remark 2.31. Although the indecomposables $b_{ij} = [R_{ij}^2]$ are represented by simpler cycles, the decompositions of b_{ij} by matric Massey products are more complicated. The author has followed the proofs in the work of Gugenheim and May [2, Chapter 5] and produced a computer program to write elements in HX by "canonically defined matric Massey products" as defined in [2, Theorem 5.6]. It means that we can generate a sequence of matrices $W_1, W_2, \ldots$ such that we can write everything in HX in terms of

$$\langle W_1, \ldots, W_n, V_{n+1} \rangle$$

with indeterminacies where V_{n+1} is some column matrix (not unique even if the sequence $W_1, W_2, \dots$ is fixed). One can simplify the canonical form if V_{n+1} contains zero entries. Here we list some decompositions of b_{ij} via this method.

$$b_{02} \in \langle h_0, h_1, h_0, h_1 \rangle \subset \left\langle h_0, h_1, \begin{pmatrix} h_0 & h_2 \end{pmatrix}, \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right\rangle$$

$$b_{03} \in \left\langle h_0, h_1, \begin{pmatrix} h_0 & h_2 \end{pmatrix}, \begin{pmatrix} h_2 & 0 \\ h_0 & h_3 \end{pmatrix}, \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, h_2 \right\rangle$$

$$b_{04} \in \left\langle h_0, h_1, \begin{pmatrix} h_0 & h_2 \end{pmatrix}, \begin{pmatrix} h_2 & 0 \\ h_0 & h_3 \end{pmatrix}, \begin{pmatrix} h_1 & h_3 \\ 0 & h_0 \end{pmatrix}, \begin{pmatrix} h_3 \\ h_1 \end{pmatrix}, h_2, h_3 \right\rangle.$$

Here $W_1 = h_0$, $W_2 = h_1$, $W_3 = \begin{pmatrix} h_0 & h_2 \end{pmatrix}$, $\begin{pmatrix} h_2 & 0 \\ h_0 & h_3 \end{pmatrix}$ is a submatrix of W_4, $\begin{pmatrix} h_1 & h_3 \\ 0 & h_0 \end{pmatrix}$ is a submatrix of W_5, $\dots$.

3. The May Spectral Sequence

The main goal of this section is to compute the differentials on $H^*(E^0\mathscr{A})$ in the May spectral sequence.

In this section we use the method of Ravenel [9] to obtain the May spectral sequence. The reason behind this is that the associated graded algebra $E^0_R\mathscr{A}$ of the Steenrod algebra by the filtration suggested by Ravenel is $E^0_K E^0\mathscr{A}$, which is Priddy's associated homogeneous Koszul algebra of May's associated graded algebra of $\mathscr{A}$. When we interact with the cobar complex this filtration is more efficient computationally.

3.1. **The cobar complex**

Recall that if I is the augmentation ideal of the dual Steenrod algebra $\mathscr{A}_*$, then the cobar complex $C(\mathscr{A}_*)$ is the tensor algebra $T^*(I)$ with $d: I^{\otimes n} \to I^{\otimes(n+1)}$ given by

$$d(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_i \sum \alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes \alpha_i' \otimes \alpha_i'' \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n \tag{3.1}$$

where

$$\psi(\alpha_i) = \alpha_i \otimes 1 + 1 \otimes \alpha_i + \sum \alpha_i' \otimes \alpha_i''$$

in $\mathscr{A}_*$. Then $H^*(\mathscr{A}) = \mathrm{Ext}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = HC(\mathscr{A}_*)$.

Definition 3.2. The weight function w on $\mathscr{A}_*$ is given by setting $w(\xi_j^{2^i}) = 2j-1$, i.e.

$$w(\xi_1^{r_1} \cdots \xi_n^{r_n}) = \sum_k (2k-1) a_{k,i}$$

where $r_k = \sum_i a_{k,i} 2^i$ is the 2-adic expansion.

We also define w on $C(\mathscr{A}_*)$ by

$$w(\alpha_1 \otimes \cdots \otimes \alpha_n) = w(\alpha_1) + \cdots + w(\alpha_n).$$

Definition 3.3. The filtrations $F_p(\mathscr{A}_*)$ and $F_p(C(\mathscr{A}_*))$ are given by elements in $\mathscr{A}_*$ and $C(\mathscr{A}_*)$ with weight $\leq p$ respectively. Note that we are using an increasing filtration indexed positively. The associated graded algebra by this filtration is denoted with $E^0_R\mathscr{A}_*$.

It follows that the associated graded algebra $E^0_R\mathscr{A}_*$ is an exterior algebra generated by the projections of $\tilde{R}_{ij} = \xi^{2^i}_{j-i}$ $(0 \leq i < j)$, which are primitive. Therefore we have the following.

Proposition 3.4. *The E_1 page of the spectral sequence determined by the filtration $F_p(C(\mathscr{A}_*))$ is isomorphic to $X = \mathbb{F}_2[R_{ij} : 0 \leq i < j]$ with $d_1(R_{ij}) = \sum_k R_{ik}R_{kj}$. Here R_{ij} corresponds to the primitive generator $\tilde{R}_{ij} = \xi^i_{j-i}$ in the associated graded algebra.*

Remark 3.5. $d_0(x) = \sum x_1 \otimes x_2$ in $E^{p,q}_0 = (F_pC(\mathscr{A}_*)/F_{p-1}C(\mathscr{A}_*))_{s=p+q}$ if x is a monomial in $\mathscr{A}_*$ where the summation is taken over all ordered monomial pairs (x_1, x_2) such that $x = x_1x_2$ in the augmentation ideal of $E^0_R\mathscr{A}_*$. In particular, $d_0(\tilde{R}_{ij}\tilde{R}_{kl}) = \tilde{R}_{ij} \otimes \tilde{R}_{kl} + \tilde{R}_{kl} \otimes \tilde{R}_{ij}$.

Since $w(\xi^{2^i}_j) = 2j - 1$ is odd and the s degree of all differentials in the spectral sequence is 1, all nontrivial differentials d_r in the spectral sequence must have odd index r. The following is the comparison between the spectral sequence obtained by the method of Ravenel and the May spectral sequence.

Table 1

Ravenel	May
$E_1 = X$	$E_1 = C(E^0\mathscr{A}_*)$
(E_{2r-1}, d_{2r-1}), $r \geq 2$	(E_r, d_r), $r \geq 2$
$E_2 = E_3 = H^*(E^0\mathscr{A}_*)$	$E_2 = H^*(E^0\mathscr{A}_*)$

3.2. **The differentials in** $H^*(E^0\mathscr{A})$

We will use the filtration in the previous section and we will therefore use the notations in the left-hand side of Table 1. We want to compute the d_3 differentials on $H^*(E^0\mathscr{A})$.

The following was already proven by May.

- $d_3(b_{02}) = h_1^3 + h_0^2h_2$,
- $d_3(b_{ij}) = h_{i+1}b_{i+1,j} + b_{i,j-1}h_{j+1},\ j - i > 2$,
- $d_3(h_i) = 0$,
- $d_3(h_i(1)) = h_ih^2_{i+2}$,
- $d_3(h_i(1,3)) = h_ih_{i+2}h_{i+2}(1) + h_i(1)h^2_{i+4}$,
- $d_3(h_i(1,2)) = h_{i+3}h_i(1,3)$.

The main goal of this section is to determine the differentials on $h_{S,T} \in \mathscr{H}$. Then all d_3 differentials in $H^*(E^0\mathscr{A})$ will be determined if Conjecture 2.17 is true.

Definition 3.6. We say that $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n \in C(\mathscr{A}_*)$ is a monomial in $C(\mathscr{A}_*)$ if each α_k is a monomial in $\mathscr{A}_*$. Note that all monomials form an additive basis of $C(\mathscr{A}_*)$. We say that the monomial α is simple if each $\alpha_k = \tilde{R}_{i_kj_k}$ for some i_k, j_k. Note that $d_0(\alpha) = 0$ in the E_0 page if α is a simple monomial.

Definition 3.7. We denote the span of simple monomials in $C(\mathscr{A}_*)$ by $S(\mathscr{A}_*)$ and the span of non-simple monomials by $S(\mathscr{A}_*)^\perp$. Note that we have $C(\mathscr{A}_*) = S(\mathscr{A}_*) \oplus S(\mathscr{A}_*)^\perp$.

Proposition 3.8. *The map* $g : (E_0, d_0) \to E_1$ *(with trivial differentials) given by*

$$g(\alpha) = \begin{cases} R_{i_1 j_1} \cdots R_{i_n j_n} & \text{if } \alpha = \tilde{R}_{i_1 j_1} \otimes \cdots \otimes \tilde{R}_{i_n j_n} \in S(\mathscr{A}_*) \\ 0 & \text{if } \alpha \in S(\mathscr{A}_*)^\perp \end{cases}$$

is a homology isomorphism.

Proof. It is clear that the homology classes $[\tilde{R}_{ij}]$ generate E_1 while g is multiplicative. Therefore g induces an isomorphism $g_* : H(E_0, d_0) \to E_1$. □

Remark 3.9. We can project suitable chains in $C(\mathscr{A}_*)$ into cycles in E_r $(r \geq 1)$ via g.

Lemma 3.10. *If* $\alpha \in C(\mathscr{A}_*)$ *is a non-simple monomial and* β *is a simple monomial summand of* $d(\alpha)$, *then either* β *is a summand of* $d_0(\alpha)$ *in* E_0 *or* $w(\beta) \leq w(\alpha) - 2$.

Proof. Write $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n$. If there is a simple summand β of $d(\alpha)$, then there must be at most one factor α_ℓ which is not equal to some $\tilde{R}_{ij}$ by (3.1). Since α is not simple, there must be exactly one such α_ℓ. Assume that β does not appear in $d_0(\alpha)$ in E_0. To obtain the simple summand $\beta_\ell \otimes \beta_{\ell+1}$ in $d(\alpha_\ell)$, we have to replace at least one factor $\tilde{R}_{ij}$ of α_ℓ with $\tilde{R}_{kj} \otimes \tilde{R}_{ik}$ and either $\tilde{R}_{kj}$ or $\tilde{R}_{ik}$ will meet another copy of itself coming from another factor of α_ℓ to become $\tilde{R}_{kj}^2 = \tilde{R}_{k+1,j+1}$ or $\tilde{R}_{ik}^2 = \tilde{R}_{i+1,k+1}$. Noting that $w(\tilde{R}_{kj} \otimes \tilde{R}_{ik}) = w(\tilde{R}_{ij}) - 1$ and in general

$$w((\tilde{R}_{ij})^2) = w(\tilde{R}_{i+1,j+1}) = 2w(\tilde{R}_{ij}) - (2(j-i)-1) \leq 2w(\tilde{R}_{ij}) - 1,$$

we see that $w(\beta) \leq w(\alpha) - 2$. □

Lemma 3.11. *Assume that*

$$d(a_p + a_{p-1}) = a_{p-2} + a_{p-3} + b_{p-3} \mod F_{p-4}C(\mathscr{A}_*)$$

in $C(\mathscr{A}_*)$, *where* a_{p-i} *consists of terms of weight* $p - i$, $i = 0, 1, 2, 3$ *and* b_{p-3} *consists of terms of weight* $p - 3$. *Assume further that* $a_p, b_{p-3} \in S(\mathscr{A}_*)$ *and* $a_{p-1}, a_{p-2}, a_{p-3} \in S(\mathscr{A}_*)^\perp$. *Then* $d_3(a_p) = b_{p-3}$ *in the* E_3 *page of the spectral sequence determined by* $F_pC(\mathscr{A}_*)$.

Proof. Note that $d(a_{p-2} + a_{p-3} + b_{p-3}) = d^2(a_p + a_{p-1}) = 0$. Hence we have $d_0(a_{p-2}) = 0$ in the E_0 page. By Proposition 3.8, $g(a_{p-2}) = 0$ in E_1 implies that a_{p-2} is a boundary in E_0. Therefore we can find $a'_{p-2} \in F_{p-2}C(\mathscr{A}_*) \cap S(\mathscr{A}_*)^\perp$ such that $d_0(a'_{p-2}) = a_{p-2}$ in E_0. By Lemma 3.10, we have

$$d(a'_{p-2}) = a_{p-2} + c_{p-3} \mod F_{p-4}C(\mathscr{A}_*)$$

where $c_{p-3} \in F_{p-3}C(\mathscr{A}_*) \cap S(\mathscr{A}_*)^\perp$. Now consider

$$d(a_p + a_{p-1} + a'_{p-2}) = b_{p-3} + c_{p-3} \mod F_{p-4}C(\mathscr{A}_*).$$

By Remark 3.9 we have $d_3(a_p) = b_{p-3}$ in E_3. □

Now we are ready to prove the main theorem of this section.

Theorem 3.12. *The differentials on* $h_{S,T} \in \mathscr{H}$ *are given by the following*

$$d_3 h_{S,T} = \sum_{s \in S,\ s+1 \in T} h_{s+1,s+2} h_{S-\{s\}+\{s+1\},T-\{s+1\}+\{s\}}.$$

Proof. We are going to compute the differentials via the cobar complex $C(\mathscr{A}_*)$. Note that in $C(\mathscr{A}_*)$, the differentials are given by

$$d(\tilde{R}_{ij}) = \sum_{k=i+1}^{j-1} \tilde{R}_{kj} \otimes \tilde{R}_{ik}.$$

To make the right-hand side look more like matrix multiplications, in this proof we are going to write

$$d(\tilde{R}_{ij}) = \sum_{k=i+1}^{j-1} \tilde{R}_{ik} \mathbin{\bar{\otimes}} \tilde{R}_{kj}$$

where $x \mathbin{\bar{\otimes}} y = y \otimes x$. We also write

$$\overline{\bigotimes_{i=1}^{n}} \alpha_i = \alpha_n \otimes \alpha_{n-1} \otimes \cdots \otimes \alpha_1.$$

The homology class $h_{S,T} \in E_3 = H^*(E^0\mathscr{A})$ can be represented in E_0 by

$$\alpha = \sum_{\sigma \in \Sigma_n} \alpha_\sigma = \sum_{\sigma \in \Sigma_n} \tilde{R}_{s_1 t_{\sigma(1)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_n t_{\sigma(n)}}.$$

Note that $d_1(\alpha) = 0$ in E_1 but $d(\alpha) \neq 0$ in $C(\mathscr{A}_*)$ because $C(\mathscr{A}_*)$ is not commutative. In fact, every monomial summand of $d(\alpha)$ can be paired with another summand the two being equal in the E_1 page. Two typical examples are pairs $(d_{is_j}\alpha_\sigma, d_{is_j}\alpha_{\sigma'})$ and $(d_{it_{\sigma(j)}}\alpha_\sigma, d_{jt_{\sigma(j)}}\alpha_{\sigma'})$ where

$$d_{is_j}\alpha_\sigma = \tilde{R}_{s_1 t_{\sigma(1)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_i s_j} \mathbin{\bar{\otimes}} \tilde{R}_{s_j t_{\sigma(i)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_j t_{\sigma(j)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_n t_{\sigma(n)}}$$

$$d_{js_j}\alpha_{\sigma'} = \tilde{R}_{s_1 t_{\sigma(1)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_i s_j} \mathbin{\bar{\otimes}} \tilde{R}_{s_j t_{\sigma(j)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_j t_{\sigma(i)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_n t_{\sigma(n)}}$$

and

$$d_{it_{\sigma(j)}}\alpha_\sigma = \tilde{R}_{s_1 t_{\sigma(1)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_i t_{\sigma(j)}} \mathbin{\bar{\otimes}} \tilde{R}_{t_{\sigma(j)} t_{\sigma(i)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_j t_{\sigma(j)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_n t_{\sigma(n)}}$$

$$d_{jt_{\sigma(j)}}\alpha_{\sigma'} = \tilde{R}_{s_1 t_{\sigma(1)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_i t_{\sigma(j)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_j t_{\sigma(j)}} \mathbin{\bar{\otimes}} \tilde{R}_{t_{\sigma(j)} t_{\sigma(i)}} \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \tilde{R}_{s_n t_{\sigma(n)}}.$$

Here the permutation σ' is the same as σ but with values $\sigma(i)$ and $\sigma(j)$ swapped and $d_{ik}\alpha_\sigma$ is the summand of $d(\alpha_\sigma)$ which replaces $\tilde{R}_{s_i t_{\sigma(i)}}$ in α_σ with $\tilde{R}_{s_i k} \otimes \tilde{R}_{k t_{\sigma(i)}}$.

Observe the typical example

$$d_0(ab \mathbin{\bar{\otimes}} c \mathbin{\bar{\otimes}} d + b \mathbin{\bar{\otimes}} ac \mathbin{\bar{\otimes}} d + b \mathbin{\bar{\otimes}} c \mathbin{\bar{\otimes}} ad) = a \mathbin{\bar{\otimes}} b \mathbin{\bar{\otimes}} c \mathbin{\bar{\otimes}} d + b \mathbin{\bar{\otimes}} c \mathbin{\bar{\otimes}} d \mathbin{\bar{\otimes}} a$$

where each a, b, c, d is equal to some $\tilde{R}_{st}$. We can find a chain in $C(\mathscr{A}_*)$ whose d_0-boundary is the sum of either typical pair above. In fact, we define

$$\beta = \sum_\sigma \sum_{i<k<j} \gamma_{\sigma,ijk} + \sum_\sigma \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \gamma_{\sigma,ijj} + \sum_\sigma \sum_{\substack{i<k\leq j \\ \sigma(i)>\sigma(j)}} d_{\sigma,ijk}$$

where

$$\gamma_{\sigma,ijk} = \overline{\bigotimes_{l=1}^{n}} \gamma_{\sigma,ijkl}, \quad d_{\sigma,ijk} = \overline{\bigotimes_{l=1}^{n}} d_{\sigma,ijkl}$$

and

$$\gamma_{\sigma,ijkl} = \begin{cases} \tilde{R}_{s_i s_j} & \text{if } l = i \\ \tilde{R}_{s_j t_{\sigma(i)}} \tilde{R}_{s_l t_{\sigma(l)}} & \text{if } l = k \\ \tilde{R}_{s_l t_{\sigma(l)}} & \text{otherwise} \end{cases}$$

$$d_{\sigma,ijkl}=\begin{cases}\tilde{R}_{s_i t_{\sigma(j)}} & \text{if } l=i\\ \tilde{R}_{t_{\sigma(j)}t_{\sigma(i)}}\tilde{R}_{s_l t_{\sigma(l)}} & \text{if } l=k\\ \tilde{R}_{s_l t_{\sigma(l)}} & \text{otherwise}\end{cases}.$$

The careful reader can check that for every $(\sigma,\sigma'=\sigma\circ(ij),i,j)$ with $\sigma(i)>\sigma(j)$,

$$d_0\left(\sum_{k=i+1}^{j-1}(\gamma_{\sigma,ijk}+\gamma_{\sigma',ijk})+\gamma_{\sigma,ijj}\right)=d_{is_j}\alpha_\sigma+d_{is_j}\alpha_{\sigma'}$$

and

$$d_0\left(\sum_{k=i+1}^{j}d_{\sigma,ijk}\right)=d_{it_{\sigma(j)}}\alpha_\sigma+d_{jt_{\sigma(j)}}\alpha_{\sigma'}.$$

Therefore $d_0(\beta)$ agrees with $d(\alpha)$. Here if α is in weight p, β and $d(\alpha)$ are all in weight $p-1$. Noting that $\beta\in S(\mathscr{A}_*)^\perp$, by Lemma 3.10, all simple summands of $d(\alpha+\beta)$ live in weight $\le p-3$ since $d_0(\beta)$ is the same as $d(\alpha)$. Therefore, by Lemma 3.11, in order to compute $d_3(h_i(S'))$ we only have to compute all simple summands of $d(\beta)$ in weight $p-3=w(\beta)-2$. By the proof of Lemma 3.11 such summands can only occur in the d-boundary of

$$\sum_\sigma\sum_{\substack{i<j\\ \sigma(i)>\sigma(j)}}\gamma_{\sigma,ijj}$$

because to get a simple summand of $d(\beta)$ in weight $\le w(\beta)-2$, we can only replace the tensor factor

$$\gamma_{\sigma,ijjj}=\tilde{R}_{s_j t_{\sigma(i)}}\tilde{R}_{s_j t_{\sigma(j)}}$$

of $\gamma_{\sigma,ijj}$ with

$$\tilde{R}^2_{s_j t_{\sigma(j)}}\mathbin{\bar{\otimes}}\tilde{R}_{t_{\sigma(j)}t_{\sigma(i)}}=\tilde{R}_{s_j+1,t_{\sigma(j)}+1}\mathbin{\bar{\otimes}}\tilde{R}_{t_{\sigma(j)}t_{\sigma(i)}}$$

in $d(\gamma_{\sigma,ijjj})$ which has weight $\le w(\gamma_{\sigma,ijjj})-2$. In this typical example,

$$w(\tilde{R}_{s_j+1,t_{\sigma(j)}+1}\mathbin{\bar{\otimes}}\tilde{R}_{t_{\sigma(j)}t_{\sigma(i)}})=w(\gamma_{\sigma,ijjj})-1-(t_{\sigma(j)}-s_j).$$

To reach the equality

$$w(\gamma_{\sigma,ijjj})-1-(t_{\sigma(j)}-s_j)=w(\gamma_{\sigma,ijjj})-2$$

we can further restrict our attention to the terms where $t_{\sigma(j)}-s_j=1$. Hence the simple part in $d(\beta)$ of weight $p-3$ is

$$\gamma=\sum_\sigma\sum_{\substack{i<j\\ \sigma(i)>\sigma(j)\\ t_{\sigma(j)}-s_j=1}}\gamma'_{\sigma,ijj}$$

where

$$\gamma'_{\sigma,ijj}=\overline{\bigotimes_{l=1}^{n}}\gamma'_{\sigma,ijjl}$$

and

$$\gamma'_{\sigma,ijjl}=\begin{cases}\tilde{R}_{s_i s_j} & \text{if } l=i\\ \tilde{R}_{s_j+1,t_{\sigma(j)}+1}\mathbin{\bar{\otimes}}\tilde{R}_{t_{\sigma(j)}t_{\sigma(i)}}=\tilde{R}_{s_j+1,s_j+2}\mathbin{\bar{\otimes}}\tilde{R}_{t_{\sigma(j)},t_{\sigma(i)}} & \text{if } l=j\\ \tilde{R}_{s_l t_{\sigma(l)}} & \text{otherwise}\end{cases}$$

If we pass γ to the E_3 page, we get

$$\gamma = \sum_{j=n \text{ or } s_j < s_{j+1}-1} R_{s_j+1,s_j+2} R_{S-\{s_j\}+\{s_j+1\},T-\{s_j+1\}+\{s_j\}}$$

which is exactly

$$\sum_{s\in S,\ s+1\in T} h_{s+1,s+2} h_{S-\{s\}+\{s+1\},T-\{s+1\}+\{s\}}.$$

By Lemma 3.11 this is $d_3(h_{S,T})$. □

Remark 3.13. If we use the notation $h_i(S')$ instead of $h_{S,T}$, the differential can be written in the following form

$$d_3 h_i(s_1,\ldots,s_{n-1}) = \sum_{\substack{j=n-1 \text{ or} \\ s_j+1<s_{j+1}}} h_{i+s_j+1} h_i(s_1,\ldots,s_{j-1},s_j+1,s_{j+1},\ldots,s_{n-1}).$$

Keep in mind that this is d_2 in May's grading.

4. Gröbner Bases and Computations

In order to do computations in HX, we need the help of Gröbner bases, to which we will give a brief introduction. Gröbner bases are usually used in computer algebra and computational algebraic geometry, where the algebras are usually ungraded. But in algebraic topology most algebras are graded. Therefore we will introduce Gröbner bases in this context. We only consider algebras over $\mathbb{F}_2$.

We also prove a result on polynomial differential graded algebras. We will use this result to compute the algebra HX_7 with an inductive method. The computational results show that Conjectures 2.20 and 2.21 are both true in $HX_7 \subset HX$.

4.1. **Gröbner basis**

In this section we always assume that $P = \mathbb{F}_2[x_1,\ldots,x_n]$ is a connected graded polynomial algebra over $\mathbb{F}_2$.

Definition 4.1. All operations related to Gröbner bases require the choice of a total order on the monomials in each degree, with the following property of compatibility with multiplication. For all monomials M, N, P where M, N are in the same degree,

$$M \le N \Longleftrightarrow MP \le NP.$$

A total order (in each degree) satisfying this condition is called an *admissible ordering.*

Example 4.2. Lexicographical ordering is an obvious example of admissible ordering. In this article we are primarily interested in the reversed lexicographical ordering, where if $M = x_1^{e_1}\cdots x_n^{e_n}$ and $N = x_1^{e_1'}\cdots x_n^{e_n'}$ are in the same degree, then $M < N$ if and only if

$$e_1 = e_1', \ldots, e_{k-1} = e_{k-1}', e_k > e_k'$$

for some k.

Definition 4.3. Once a total ordering is fixed, we let $\mathrm{LM}(f)$ denote the largest monomial in $f \in P$. It is called the leading monomial of f.

Remark 4.4. If we use the reversed lexicographical ordering, then the leading monomial of $f \in P$ is the least monomial of f in the lexicographical ordering.

From now on we assume P is alway equipped with an admissible ordering.

Definition 4.5. Given two polynomials f and g in P, one says that f is *reducible* by g if some monomial M in f is divisible by $\mathrm{LM}(g)$. In this case we define the *one-step* reduction of f by g by

$$\mathrm{red}_1(f,g) = f + \frac{M}{\mathrm{LM}(g)}g.$$

Note that compared with f, $\mathrm{red}_1(f,g)$ replaces M in f with other monomials less than M.

Definition 4.6. For $f \in P$ and a finite subset $S \subset P$, we say that f is *reducible* by S if f is reducible by some $g \in S$. In order to define $\mathrm{red}(f,S)$, if f is reducible by some $g \in S$, we replace f by $\mathrm{red}_1(f,g)$, and we iterate this until f is not reducible by any $g \in S$. The iteration always terminates because there are only finitely many monomials in each degree since P is a connected algebra. The final result depends on the ordering of choices of g, and we define $\mathrm{red}(f,S)$ to be the set of all possible outcomes.

Definition 4.7. A Gröbner basis G of an ideal I in P is a generating set of I such that the set of images of all monomials *not* divisible by $\mathrm{LM}(g)$ for any $g \in G$ under the canonical map $P \to P/I$ form an additive basis for P/I.

Remark 4.8. If G is a Gröbner basis, then $\mathrm{red}(f,G)$ is exactly the standard representation of f in P/I as a linear combination of the additive basis mentioned above. Hence $\mathrm{red}(f,G)$ consists of a single element of P.

Algorithm 4.9 (Buchberger)**.** Given a finite generating set G of an ideal I in P, we can change G into a Gröbner basis of I by doing the following

(1) For $f, g \in G$, let
$$L = \mathrm{lcm}(\mathrm{LM}(f), \mathrm{LM}(g)).$$
Find two monomials m, n such that $\mathrm{LM}(mf) = \mathrm{LM}(ng) = L$. If $\mathrm{red}(mf + ng, G)$ contains a nonzero polynomial, then add it to G.

(2) Repeat (1) until $\mathrm{red}(mf + ng, G)$ is zero for every pair f, g in G.

Remark 4.10. In Step (1), each time we add a new element to G the ideal generated by all leading monomials of G will strictly increase. Therefore the algorithm always terminates in finitely many steps, because P is a Noetherian ring.

Definition 4.11. Let $R = P/I$ for an ideal I of P. For $(a_1, a_2, \dots, a_n) \in R^n$ we define

$$\mathrm{Ann}(a_1, \cdots, a_n) = \{(b_1, \dots, b_n) \in R^n \mid a_1 b_1 + \cdots + a_n b_n = 0\}.$$

This is an R-submodule of R^n. Note that for $1 \le i < j \le n$,

$$(0, \dots, 0, \overset{i}{a_j}, 0, \dots, 0, \overset{j}{a_i}, 0, \dots, 0) \in \mathrm{Ann}(a_1, \cdots, a_n).$$

These are called the *commutators* of $a_1, a_2, \dots, a_n$.

Lemma 4.12. *Assume I is trivial and $R = P$. Then $\mathrm{Ann}(x_1, \dots, x_n)$ is generated by commutators of $x_1, \dots, x_n$.*

Proof. This is a consequence of the fact that $\mathrm{Tor}_P(\mathbb{F}_2, \mathbb{F}_2) \cong E[\sigma x_1, \dots, \sigma x_n]$, so that $\sigma x_i \wedge \sigma x_j$ is an additive basis of $\mathrm{Tor}^2_P(\mathbb{F}_2, \mathbb{F}_2)$. In the Koszul complex this means that all P-linear relations among x_k are generated by $x_i x_j + x_j x_i = 0$ □

Definition 4.13. For $f \in P$, $\bar{f}$ denotes the image of f in P/I.

Theorem 4.14. *Assume P is equipped with the reversed lexicographical ordering and G is the Gröbner basis of an ideal I in P. For the images $\bar{x}_1, \dots, \bar{x}_k$ of the first k generators $x_1, \dots, x_k$ of P in $R = P/I$, $\mathrm{Ann}(\bar{x}_1, \dots, \bar{x}_k)$ is generated as a R-submodule of R^k by commutators of $\bar{x}_1, \dots, \bar{x}_k$ and all $(\bar{f}_1, \dots, \bar{f}_k) \in R^k$ such that $f_i \in P$ and $x_1 f_1 + \cdots + x_k f_k \in G$.*

Proof. Assume that $x_1 g_1 + \cdots + x_k g_k \in I$. By the definition of a Gröbner basis, we can always choose representatives g_i of $\bar{g}_i$ such that no g_i is reducible by G. In order to show that $(\bar{g}_1, \dots, \bar{g}_k)$ is an R-linear combination of commutators of $\bar{x}_1, \dots, \bar{x}_k$ and $(\bar{f}_1, \dots, \bar{f}_k)$ described in the theorem, by Lemma 4.12 it suffices to show that $x_1 g_1 + \cdots + x_k g_k$ is a P-linear combination of elements of G of the form $x_1 f_1 + \cdots + x_k f_k$, i.e. elements of G in which all monomials contain at least one of $x_1, \dots, x_k$.

In fact, since $\mathrm{red}(x_1 g_1 + \cdots + x_k g_k, G) = 0$, for some $1 \le i \le k$, $x_i g_i$ is reducible by some $g \in G$. Since g_i is not reducible by G but $x_i g_i$ is reducible, $\mathrm{LM}(g)$ must contain x_i. Since $\mathrm{LM}(g)$ is the least monomial in g ordered lexicographically, other monomials of g must contain at at least one of $x_1, \dots, x_i$. Therefore if we replace $x_i g_i$ with $\mathrm{red}_1(x_i g_i, g)$, then $x_1 g_1 + \cdots + x_k g_k$ becomes another polynomial of the form $x_1 g_1' + \cdots + x_k g_k'$. We can iterate this until $x_1 g_1 + \cdots + x_k g_k$ becomes zero. Hence $x_1 g_1 + \cdots + x_k g_k$ is a P-linear combination of $g \in G$ in which all monomials contain at least one of $x_1, \dots, x_k$. □

By the theorem for $a_1, \dots, a_k \in R$ we can make an algorithm for finding a generating set of $\mathrm{Ann}(a_1, \cdots, a_n) \in R = P/I$.

Algorithm 4.15. Given an ideal I in P, $R = P/I$ and $f_1, \dots, f_k \in P$, a generating set of $\mathrm{Ann}(\bar{f}_1, \dots, \bar{f}_k)$ as an R-submodule of R^k can be obtained by doing the following

1. Equip $Q = \mathbb{F}_2[y_1, \dots, y_k, x_1, \dots, x_n]$ with the reversed lexicographical ordering.
2. Compute the Gröbner basis G of $I + (y_1 - f_1, \dots, y_k - f_k)$.
3. Find all elements g of G such that $\mathrm{LM}(g)$ contains at least one of $y_1, \dots, y_k$ and write g in the form $g = y_1 h_1 + \cdots + y_k h_k$ where $h_i \in Q$. We can do this because we are using the reversed lexicographical ordering.
4. Replace h_i with a polynomial in $x_1, \dots, x_n$ using the relations $y_1 = f_1, \dots, y_k = f_k$.
5. All images of $(h_1, \dots, h_k)$ in $R = P/I$ together with commutators of $\bar{f}_1, \dots, \bar{f}_k$ form a generating set of $\mathrm{Ann}(\bar{f}_1, \dots, \bar{f}_k)$ as an R-submodule of R^k.

Theorem 4.16. *If the Gröbner basis G of $I \subset P$ with respect to some monomial ordering has the property that all the leading monomials of $g \in G$ are square free, then $R = P/I$ is nilpotent free.*

Proof. By the properties of Gröbner bases, the set of all monomials not reducible by G forms a basis for P/I. If all the leading monomials are square free, we show that this basis is closed under the squaring map.

In fact, given a square free monomial $\alpha = x_{i_1} \cdots x_{i_k}$ $(i_1 < \cdots < i_k)$ in P, another monomial $\beta = x_1^{e_1} x_2^{e_2} \cdots$ is not divisible by α if and only if β^2 is not divisible by α. This is because

$$\alpha | \beta \iff e_{i_j} > 0 \ (1 \le j \le k) \iff 2e_{i_j} > 0 \ (1 \le j \le k) \iff \alpha | \beta^2.$$

Therefore R is nilpotent free since we have a basis closed under the squaring map. $\square$

4.2. **Polynomial differential graded algebras**

Note that the differential graded algebra X is also a polynomial algebra. The following proposition will help us calculate the homology of these kinds of algebras.

Proposition 4.17. *Assume that A is a commutative differential graded algebra over $\mathbb{F}_2$ and $c \in A$ is a cycle. Consider $B = A[x]$ as a differential graded algebra which extends A with $dx = c$.*

If $[c] = 0$ in HA, then $HB \cong HA \otimes \mathbb{F}_2[\tilde{x}]$ where $\tilde{x}$ corresponds to $x + a$ where $da = c$ in A.

If $[c] \neq 0$ in HA, assume that the ideal

$$\mathrm{Ann}_{HA}([c]) = \{y \in HA : y[c] = 0\}$$

of HA is generated by $y_1, \ldots, y_n$ ($n = 0$ if the ideal is zero). If we filter B by

$$F_pB = \{ax^i : a \in A,\ i \leq p\},$$

then the associated graded algebra E^0HB can be represented by

$$HA \otimes \mathbb{F}_2[b, g_1, \ldots, g_n]/\sim$$

where the relations are given by $[c] = 0$ and

(i) *if $a_1y_1 + \cdots + a_ny_n = 0$ in HA for $a_i \in HA$ then*

$$a_1g_1 + \cdots + a_ng_n = 0.$$

(ii) *$g_ig_j = by_iy_j$.*

Proof. Note that x is in filtration 1 and $dx = c$ is in filtration 0. Hence

$$E_1 \cong HA \otimes \mathbb{F}_2[x]$$

with $dx = [c]$.

If $[c] = 0$, then $E_1 = E_\infty$ because x is a permanent cycle represented by $x + a$ for some $a \in A$ such that $da = c$. There is no extensions since there is no relations on x. Hence $HB \cong HA \otimes \mathbb{F}_2[\tilde{x}]$.

If $[c] \neq 0$, noting that $b = [x^2]$ is a permanent cycle, the set of elements in $E_2 = HE_1$ in even filtrations is isomorphic to

$$\bigoplus x^{2i} HA/([c])$$

while the set of elements in odd filtrations is isomorphic to

$$\bigoplus x^{2i-1} \mathrm{Ann}_{HA}([c]).$$

The multiplication by $b = [x^2]$ will map elements in filtration p isomorphically onto elements in filtration $p + 2$. They are both modules over HA and the module structure of $x\mathrm{Ann}_{HA}([c])$ (elements in filtration 1) is precisely given by (i) with $g_i = [xy_i]$. Relations in (ii) are direct consequences of $xy_i \cdot xy_j = x^2 \cdot y_i \cdot y_j$ in E_1. The spectral sequence collapses in E_2 because the $g_i = [xy_i]$ are represented by cycles $xy_i + a_i \in B$ where $da_i = cy_i$ in A. Therefore the g_i are all permanent cycles. $\square$

Remark 4.18. The proposition does not solve the extension problem for computing HB. However, it constrains the number of relations we have to deal with, which is very important for our computation of HX_7 in the next section.

Remark 4.19. A generating set of $(r_1, \dots, r_n) \in (HA)^n$ in (1) in the proposition can be obtained by Algorithm 4.15.

4.3. **The computation of** HX_7

In this section, we are going to compute HX_7 by an inductive method using Proposition 4.17. We will see that Conjectures 2.20 and 2.21 hold in $HX_7 \subset HX$.

It is helpful to see that X has a lot of symmetries. These will be useful in our induction.

Definition 4.20. For $0 \le m < n$, let $X[m,n]$ denote the sub-DGA of X

$$X[m,n] = \mathbb{F}_2[R_{ij} : m \le i < j \le n].$$

Note that $X_n = X[0,n]$. Let $X_{n,k} = \mathbb{F}_2[R_{0i} : i \le k] \otimes X[1,n]$ which is also a sub-DGA of X.

Proposition 4.21. *The map*

$$r : X \to X[m,n]$$

given by

$$r(R_{ij}) = \begin{cases} R_{ij}, & \text{if } m \le i < j \le n \\ 0, & \text{otherwise} \end{cases}$$

is a retraction of DGAs. Therefore the homomorphism in homology $HX[m,n] \to HX$ is injective.

In addition to Proposition 2.23, we have another property of symmetries in X.

Proposition 4.22. *The translation map*

$$f_k : X[m,n] \to X[m+k, n+k]$$
$$R_{ij} \mapsto R_{i+k,j+k}$$

is an isomorphism between differential algebras. Therefore

$$HX[m,n] \cong HX[m+k, n+k]$$

as algebras.

Remark 4.23. The map f_k is actually the same as the squaring operation $(Sq^0)^k$. Here Sq^0 is a power operation in the May spectral sequence (See [6]).

Our strategy to compute HX_7 is to show that Conjecture 2.20 holds in HX_n for $n = 1, 2, \dots, 7$ inductively. For $m < n$, if we can prove that Conjecture 2.20 on $HX[1,n]$ implies Conjecture 2.20 on $HX[0,n] = HX_n$, then by ignoring all R_{ij} with $j > m$ in the proof, we can obtain a proof of the fact that Conjecture 2.20 on $HX[1,m]$ implies Conjecture 2.20 on $HX[0,m] = HX_m$. Moreover, by Proposition 4.22, we have

$$HX[1,n] \cong HX[0, n-1] = HX_{n-1}.$$

Therefore the statement

$$\text{Conjecture 2.20 holds on } HX_6 \Longrightarrow \text{Conjecture 2.20 holds on } HX_7 \tag{4.24}$$

implies the statement

$$\text{Conjecture 2.20 holds on } HX_{k-1} \Longrightarrow \text{Conjecture 2.20 holds on } HX_k$$

for $2 \leq k \leq 5$. Since Conjecture 2.20 holds in $HX_1 = \mathbb{F}_2[h_0]$, it suffices to prove the statement (4.24).

Now we have our assumption on $HX_6 \cong HX[1,7]$. See Appendix A.1 for a list of generators and relations we generate for $HX[1,7]$ according to Conjecture 2.20.

We are going to compute $HX[1,7] = HX_{7,0}$, $HX_{7,1}$, ..., $HX_{7,7} = HX_7$ one by one. Note that $X_{7,i} = X_{7,i-1} \otimes \mathbb{F}_2[R_{0i}]$. We apply Proposition 4.17 to the case where $A = X_{7,i-1}$, $B = X_{7,i}$, $x = R_{0i}$ and $c = \sum_{j=1}^{i-1} R_{0j}R_{ji}$ to obtain the homology $HX_{7,i}$ from $HX_{7,i-1}$.

Recall that Proposition 4.17 does not solve the extension problems for us. I managed to solve all of the extensions via many different approaches, including pure guesses, and to check them with the aid of a computer by realizing the relations as boundaries of chains.

Appendix A.2-A.8 list the generators and relations of $HX_{7,1}, \ldots, HX_{7,7}$ computed by the author. In these charts, the relations are grouped into two parts. Part (i) corresponds to relations (i) in Proposition 4.17 and Part (ii) corresponds to relations (ii) in Proposition 4.17. For Part (i), the author put the extension part of the relations on the right-hand side of the equations.

Appendix A.9 reorganizes the relations of $HX_7 = HX_{7,7}$ in the form of Gröbner bases. We can see that all of the leading monomials are square free. Hence Conjecture 2.21 holds in HX_7 by Theorem 4.16.

Appendix A.10 lists the relations of HX_7 according to Conjecture 2.20. It has been checked by the computer that these relations indeed generate the same Gröbner basis as that in Appendix A.9. Hence we see that Conjecture 2.20 indeed holds in HX_7.

Combining the results above Theorem 2.26 is proved.

4.4. **A localization of the May spectral sequence**

One of the useful tools to compute the May spectral sequence is the Adams vanishing theorem.

Theorem 4.25 (Adams [1])**.** $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) = 0$ *if* $t - s < q(s)$ *where the function* q *is given by*

$$\begin{aligned} q(4k) &= 8k-1; \\ q(4k+1) &= 8k+1; \\ q(4k+2) &= 8k+2; \\ q(4k+3) &= 8k+3. \end{aligned}$$

Note that May [3] and Tangora [10] both used this theorem to compute some differentials in the May spectral sequence. This is based on the fact that all the infinite h_0-structure lines in the May spectral sequence have to be truncated by some differentials in order for the vanishing line to appear in the E_∞ page. One of the examples is the first nontrivial d_6 differential

$$d_6(x) = h_0^5 y$$

where

$$x = h_0 b_{02}^3 b_{03} h_0(1),\ y = h_4 b_{02}^2 h_0(1) + h_0^3 b_{02} b_{13}$$

in E_6. Here $h_0^i y \neq 0$ for all $i \geq 0$ and x is the only thing that can truncate this infinite h_0-structure line supported by y. By computing the filtration degrees this differential is d_6.

These infinite h_0-structure lines inherit structures from the May spectral sequence and form another spectral sequence which converges to zero in positive stems because of Theorem 4.25. A better way to process this information is to invert h_0 in the May spectral sequence and study the localized spectral sequence which converges to $\mathbb{F}_2[h_0^{\pm 1}]$. The following theorem shows the structure of the E_2 page of the localized May spectral sequence. What is surprising is that it contains a subalgebra $HX[2,\infty]$ which is isomorphic to the original $E_2 \cong HX$ with a shift in degree t.

Theorem 4.26.

$$h_0^{-1}HX \cong \mathbb{F}_2[h_0^{\pm 1}, b_{0j} : j \geq 2] \otimes HX[2,\infty]$$

Proof. Note that as a differential algebra,

$$h_0^{-1}HX \cong \mathbb{F}_2[h_0^{\pm 1}] \otimes H(X/(R_{01}-1)),$$

since h_0 is represented by R_{01}. It suffices to show that

$$H(X/(R_{01}-1)) \cong \mathbb{F}_2[b_{0j} : j \geq 2] \otimes HX[2,\infty]. \tag{4.27}$$

Let

$$Y_m = X[2,\infty] \otimes \mathbb{F}_2[R_{0j}, R_{1j} : j \leq m]/(R_{01}-1).$$

Observe that

$$X \cong \operatorname*{colim}_m Y_m \quad \text{and} \quad Y_m \cong Y_{m-1} \otimes \mathbb{F}_2[R_{0m}, R_{1m}].$$

Now it suffices to show by induction that for all m

$$HY_m \cong \mathbb{F}_2[b_{0j} : 2 \leq j \leq m] \otimes HX[2,\infty].$$

The claim is trivial when $m = 0, 1$.

Assume it is true for Y_{m-1}. First we consider $Y_{m-1} \otimes \mathbb{F}_2[R_{1m}]$. Note that dR_{1m} is a boundary in HY_{m-1} since

$$d(e_{0m}) = d(R_{01}R_{1m} + R_{02}R_{2m} + \cdots + R_{0,m-1}R_{m-1,m}) = 0$$

which implies

$$d(R_{1m}) = d(R_{02}R_{2m} + \cdots + R_{0,m-1}R_{m-1,m}).$$

By Proposition 4.17 we have

$$H(Y_{m-1} \otimes \mathbb{F}_2[R_{1m}]) \cong HY_{m-1} \otimes \mathbb{F}_2[e_{0m}].$$

Now we consider $Y_m = Y_{m-1} \otimes \mathbb{F}_2[R_{1m}] \otimes \mathbb{F}_2[R_{0m}]$. Note that $dR_{0m} = e_{0m}$ and $\mathrm{Ann}(e_{0m})$ is trivial in $HX[2,\infty] \otimes \mathbb{F}_2[e_{0m}]$. Therefore by Proposition 4.17,

$$HY_m \cong HY_{m-1} \otimes \mathbb{F}_2[b_{0m}] \cong HX[2,\infty] \otimes \mathbb{F}_2[b_{0j}, 2 \leq j \leq m].$$

Hence the induction is complete. □

By the Adams vanishing theorem on the E_2 page of the Adams spectral sequence we know that

$$h_0^{-1}\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0^{\pm 1}].$$

Hence after inverting h_0 in the May spectral sequence we get a spectral sequence with

$$E_2 = h_0^{-1}HX \Longrightarrow \mathbb{F}_2[h_0^{\pm 1}].$$

By the theorem above this is the same as

$$\mathbb{F}_2[h_0^{\pm 1}, b_{0j} : j \geq 2] \otimes HX[2,\infty] \Longrightarrow \mathbb{F}_2[h_0^{\pm 1}]$$

.

Note that $HX[2,\infty]$ is isomorphic to HX with a shift of degrees. Therefore the following composition is an embedding

$$\varphi : HX \xrightarrow{(Sq^0)^2} HX \longrightarrow h_0^{-1}HX$$

where the second map is the localization. Since the operation Sq^0 (see Remark 4.23) commutes with all d_r differentials in the May spectral sequence we have a comparison map

$$\begin{array}{ccc} HX & \Longrightarrow & \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2,\mathbb{F}_2) \\ \varphi\downarrow & & \downarrow \\ h_0^{-1}HX & \Longrightarrow & \mathbb{F}_2[h_0^{\pm 1}] \end{array} \tag{4.28}$$

The bottom spectral sequence has an advantage in calculation since all elements in positive stems have to be killed by differentials. We intend to use the bottom spectral sequence to aid in computing the top. Interestingly, computations in low degrees lead us to the following conjecture.

Conjecture 4.29. *The localized spectral sequence*

$$E_2 = h_0^{-1}HX \Longrightarrow \mathbb{F}_2[h_0^{\pm 1}]$$

is isomorphic to a sub-spectral sequence

$$E_2 = \mathbb{F}_2[\frac{b_{0j}}{h_0^2} : j \geq 2] \otimes HX[2,\infty] \Longrightarrow \mathbb{F}_2 \tag{4.30}$$

tensored with $\mathbb{F}_2[h_0^{\pm 1}]$.

Although the author cannot yet prove this, there is another spectral sequence with the same E_2 and E_∞ as (4.30). The advantage of the new spectral sequence is that it is also tri-graded.

Theorem 4.31. *Consider the cobar resolution $\tilde{C}(\mathscr{A}_*)$ of $\mathbb{F}_2$ over $\mathscr{A}_*$ where $\tilde{C}_s(\mathscr{A}_*)$ consists of elements $[a_1|\cdots|a_s]a$ and*

$$d[a_1|\cdots|a_s]a = \sum_i \sum [a_1|\cdots|a_i'|a_i''|\cdots|a_s]a + \sum [a_1|\cdots|a_i'|a_i''|\cdots|a_s|\epsilon(a')]a''.$$

There is a filtration on $\tilde{C}(\mathscr{A}_*)$ such the resulting spectral sequence has a E_2 page isomorphic to

$$\mathbb{F}_2[\frac{b_{0j}}{h_0^2} : j \geq 2] \otimes HX[2,\infty]$$

with a degree shift in t, and it converges to $\mathbb{F}_2$.

Proof. We continue the use of Ravenel's filtration in Section 3. Consider the weight function w on $\mathscr{A}_*$ and $C(\mathscr{A}_*)$ in Definition 3.2. We define another linear function w' on $\mathscr{A}_*$ given by

$$w'(\xi_1^{r_1}\cdots\xi_k^{r_k}) = \sum_{i=1}^k 2ir_i.$$

We can define a weight function w on $\tilde{C}_s(\mathscr{A}_*) = C_s(\mathscr{A}_*) \otimes \mathscr{A}_*$ by

$$w[a_1|\cdots|a_s]a = w(a_1) + \cdots + w(a_s) + w'(a).$$

Note that
$$d([]\xi_n) = \sum_{k=0}^{n-1} [\xi_{n-k}^{2^k}]\xi_k$$
and
$$w([]\xi_n) = 2n > w([\xi_{n-k}^{2^k}]\xi_k) = 2k + 2(n-k) - 1 = 2n - 1.$$
Therefore on the E_0 page $d_0([]\xi_n) = 0$. Hence by Proposition 3.4, the E_1-term is isomorphic to $X \otimes \mathscr{A}_*$. The d_1 differentials are given by
$$d_1(R_{ij}) = \sum_k R_{ik}R_{kj},$$
$$d_1(\xi_j) = R_{0j} + \sum_k \xi_k R_{kj}.$$

By (4.27) it suffices to show that
$$X \otimes \mathscr{A}_* \cong X/(R_{01} - 1)$$
as differential algebras. In fact, it is not hard to check that the following map gives the isomorphism.
$$\begin{aligned} X \otimes \mathscr{A}_* &\longrightarrow X/(R_{01} - 1) \\ R_{ij} \otimes 1 &\longmapsto R_{i+1,j+1} \\ 1 \otimes \xi_j &\longmapsto R_{0,j+1}. \end{aligned}$$
□

In contrast to the comparison map in (4.28) we now build another comparison
$$\begin{array}{ccc} HX & \Longrightarrow & \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \\ \big\downarrow \varphi & & \big\downarrow \\ \mathbb{F}_2[\frac{b_{0j}}{h_0^2} : j \geq 2] \otimes HX[2, \infty] & \Longrightarrow & \mathbb{F}_2 \end{array}$$
using the composition of the map of complexes $C(\mathscr{A}_*) \to \tilde{C}(\mathscr{A}_*)$ and the operation Sq^0. The map φ is again an embedding. A stronger version of Conjecture 4.29 includes the claim that (4.30) is isomorphic to the bottom spectral sequence above.

The localization map and other comparison maps with compositions of $(Sq^0)^i$ yield different indeterminacies for computing the May spectral sequence. The author has been collaborating with the computer and feeding these data into the program to obtain higher differentials in the May spectral sequence.

Appendix A. Charts

There is a new symbol in the following charts. Note that for each indecomposable $h_i(S') = h_{S,T} \in \mathscr{H}$,
$$\sum_{j=0}^{i-1} R_{0j} R_{S-\{i\}+\{j\},T}$$
is a cycle in $HX_{7,i}$. We let $r_i(S') = r_{S,T}$ denote the homology class of this cycle in $HX_{7,i}$.

A.1. $HX[1,7]$

Generators.

h_i, $1 \leq i \leq 6$
$h_i(1)$, $1 \leq i \leq 4$
$h_i(1,3)$, $h_i(1,2)$, $1 \leq i \leq 2$
b_{ij}, $1 \leq i < i+2 \leq j \leq 7$.

Relations.

$h_1h_2 = 0$
$h_2h_3 = 0$
$h_3b_{13} = h_1h_1(1)$
$h_3h_4 = 0$
$h_3h_1(1) = h_1b_{24}$
$h_4h_1(1) = 0$
$b_{13}b_{24} = h_2^2b_{14} + h_1(1)^2$
$h_1h_2(1) = 0$
$h_4b_{24} = h_2h_2(1)$
$h_4h_5 = 0$
$h_2(1)b_{13} = h_2h_4b_{14}$
$h_4h_2(1) = h_2b_{35}$
$h_1(1)h_2(1) = 0$
$b_{13}b_{35} = h_1^2b_{25} + h_4^2b_{14}$
$h_1(1)b_{35} = h_1h_3b_{25}$
$h_5h_2(1) = 0$
$b_{24}b_{35} = h_3^2b_{25} + h_2(1)^2$
$h_2h_3(1) = 0$
$h_5b_{35} = h_3h_3(1)$
$h_3(1)b_{13} = h_1h_1(1,3)$
$h_1(1)h_3(1) = h_3h_1(1,3)$
$h_3h_1(1,3) = h_1h_5b_{25}$
$h_3(1)b_{24} = h_3h_5b_{25}$
$h_5h_6 = 0$
$h_1(1)h_1(1,3) = h_2^2h_5b_{15} + h_5b_{13}b_{25}$
$h_3(1)b_{14} = h_1h_1(1,2) + h_3h_5b_{15}$
$h_5h_3(1) = h_3b_{46}$
$h_1(1,3)b_{24} = h_2^2h_1(1,2) + h_5h_1(1)b_{25}$
$h_2(1)h_3(1) = 0$
$h_1(1,2)b_{13} = h_5h_1(1)b_{15} + h_1(1,3)b_{14}$
$h_1(1)b_{46} = h_5h_1(1,3)$
$h_1(1)h_1(1,2) = h_5b_{24}b_{15} + h_5b_{14}b_{25}$
$h_2(1)h_1(1,3) = h_2h_4h_1(1,2)$
$b_{24}b_{46} = h_2^2b_{36} + h_5^2b_{25}$
$b_{46}b_{14} = h_1^2b_{26} + h_5^2b_{15} + b_{13}b_{36}$
$h_1(1,3)b_{35} = h_1h_3(1)b_{25} + h_4^2h_1(1,2)$
$h_1(1)b_{36} = h_1h_3b_{26} + h_5h_1(1,2)$
$h_2(1)b_{46} = h_2h_4b_{36}$
$h_6h_3(1) = 0$
$b_{35}b_{46} = h_4^2b_{36} + h_3(1)^2$

$h_6h_1(1,3) = 0$
$h_3h_4(1) = 0$
$h_3(1)h_1(1,3) = h_1h_4^2b_{26} + h_1b_{46}b_{25}$
$h_6b_{46} = h_4h_4(1)$
$h_1(1)h_4(1) = 0$
$h_6h_1(1,2) = 0$
$b_{13}b_{46}b_{25} = h_2^2h_4^2b_{16} + h_2^2b_{46}b_{15} + h_4^2b_{13}b_{26} + h_1(1,3)^2$
$h_1h_2(1,3) = 0$
$h_4(1)b_{24} = h_2h_2(1,3)$
$h_3(1)h_1(1,2) = h_1b_{35}b_{26} + h_1b_{25}b_{36}$
$h_2(1,3)b_{13} = h_2h_4(1)b_{14}$
$b_{13}b_{25}b_{36} = h_1^2b_{25}b_{26} + h_2^2b_{35}b_{16} + h_2^2b_{36}b_{15} + h_4^2b_{14}b_{26} + h_1(1,3)h_1(1,2)$
$h_2(1)h_4(1) = h_4h_2(1,3)$
$h_4h_2(1,3) = h_2h_6b_{36}$
$h_1(1)h_2(1,3) = 0$
$h_4(1)b_{35} = h_4h_6b_{36}$
$h_1(1,2)b_{46} = h_1h_3(1)b_{26} + h_1(1,3)b_{36}$
$b_{14}b_{25}b_{36} = h_3^2b_{15}b_{26} + h_2(1)^2b_{16} + h_1(1,2)^2 + b_{24}b_{36}b_{15} + b_{35}b_{14}b_{26}$
$h_1h_2(1,2) = 0$
$h_4(1)b_{25} = h_2h_2(1,2) + h_4h_6b_{26}$
$h_2(1)h_2(1,3) = h_3^2h_6b_{26} + h_6b_{24}b_{36}$
$h_6h_4(1) = h_4b_{57}$
$h_2(1,2)b_{13} = h_2h_4h_6b_{16} + h_2h_4(1)b_{15}$
$h_2(1,3)b_{35} = h_3^2h_2(1,2) + h_6h_2(1)b_{36}$
$h_1(1)h_2(1,2) = 0$
$h_3(1)h_4(1) = 0$
$h_2(1,2)b_{24} = h_6h_2(1)b_{26} + h_2(1,3)b_{25}$
$h_2(1,2)b_{14} = h_6h_2(1)b_{16} + h_2(1,3)b_{15}$
$h_4(1)h_1(1,3) = 0$
$h_2(1)b_{57} = h_6h_2(1,3)$
$h_2(1)h_2(1,2) = h_6b_{35}b_{26} + h_6b_{25}b_{36}$
$h_3(1)h_2(1,3) = h_3h_5h_2(1,2)$
$b_{35}b_{57} = h_3^2b_{47} + h_6^2b_{36}$
$h_4(1)h_1(1,2) = 0$
$h_1(1,3)h_2(1,3) = 0$
$b_{57}b_{25} = h_2^2b_{37} + h_6^2b_{26} + b_{24}b_{47}$
$b_{14}b_{47} = h_1^2b_{27} + h_6^2b_{16} + b_{13}b_{37} + b_{57}b_{15}$
$h_2(1)b_{47} = h_2h_4b_{37} + h_6h_2(1,2)$
$h_2(1,3)b_{46} = h_2h_4(1)b_{36} + h_5^2h_2(1,2)$
$h_2(1,3)h_1(1,2) = 0$
$h_3(1)b_{57} = h_3h_5b_{47}$
$h_1(1,3)b_{57} = h_5h_1(1)b_{47}$
$h_1(1,3)h_2(1,2) = 0$
$b_{46}b_{57} = h_5^2b_{47} + h_4(1)^2$
$h_1(1,2)b_{57} = h_1h_3h_5b_{27} + h_5h_1(1)b_{37}$
$h_1(1,2)h_2(1,2) = 0$
$h_4(1)h_2(1,3) = h_2h_5^2b_{37} + h_2b_{57}b_{36}$
$h_1(1,2)b_{47} = h_1h_3(1)b_{27} + h_1(1,3)b_{37}$

$b_{24}b_{57}b_{36} = h_3^2h_5^2b_{27} + h_3^2b_{57}b_{26} + h_5^2b_{24}b_{37} + h_2(1,3)^2$
$h_4(1)h_2(1,2) = h_2b_{46}b_{37} + h_2b_{36}b_{47}$
$b_{24}b_{36}b_{47} = h_2^2b_{36}b_{37} + h_3^2b_{46}b_{27} + h_3^2b_{47}b_{26} + h_5^2b_{25}b_{37} + h_2(1,3)h_2(1,2)$
$h_2(1,2)b_{57} = h_2h_4(1)b_{37} + h_2(1,3)b_{47}$
$b_{25}b_{36}b_{47} = h_4^2b_{26}b_{37} + h_3(1)^2b_{27} + h_2(1,2)^2 + b_{35}b_{47}b_{26} + b_{46}b_{25}b_{37}$

A.2. $HX_{7,1}$

It is obvious that $HX_{7,1} = HX[1,7] \otimes \mathbb{F}_2[h_0]$.

A.3. $HX_{7,2}$

Consider $dR_{02} = R_{01}R_{12}$ whose homology class is $r_1 = h_0h_1$ in $HX_{7,1}$. We have

$$\mathrm{Ann}_{HX_{7,1}}(r_1) = (h_2, h_2(1), h_2(1,3), h_2(1,2))$$

obtained by Algorithm 4.15. Apply Proposition 4.17 on $X_{7,2} = X_{7,1} \otimes R_{02}$. The $E_2 = E_\infty$ page is generated by $R_{02}h_2$, $R_{02}h_2(1)$, $R_{02}h_2(1,3)$, $R_{02}h_2(1,2)$ and R_{02}^2 which are represented by r_2, $r_2(1)$, $r_2(1,3)$, $r_2(1,2)$ and b_{02} in $HX_{7,2}$ respectively. In addition to relations in $HX_{7,1}$, the new relations in $HX_{7,2}$ are $r_1 = 0$ and

Part (i). [1]
$r_2h_1 = 0,$
$r_2h_3 = 0,$
$r_2(1)h_1 = 0,$
$r_2(1)h_1(1) = 0,$
$r_2(1)h_5 = 0,$
$r_2h_3(1) = 0,$
$r_2(1)h_3(1) = 0,$
$r_2(1,3)h_1 = 0,$
$r_2(1,3)h_1(1) = 0,$
$r_2(1,2)h_1 = 0,$
$r_2(1,2)h_1(1) = 0,$
$r_2(1,3)h_1(1,3) = 0,$
$r_2(1,3)h_1(1,2) = 0,$
$r_2(1,2)h_1(1,3) = 0,$
$r_2(1,2)h_1(1,2) = 0,$
$r_2(1)h_2 + r_2h_2(1) = 0,$
$r_2(1)b_{13} + r_2h_4b_{14} = 0,$
$r_2(1)h_4 + r_2b_{35} = 0,$
$r_2(1)h_1(1,3) + r_2h_4h_1(1,2) = 0,$
$r_2(1)b_{46} + r_2h_4b_{36} = 0,$
$r_2(1,3)h_2 + r_2h_2(1,3) = 0,$
$r_2(1,3)b_{13} + r_2h_4(1)b_{14} = 0,$
$r_2(1,3)h_4 + r_2(1)h_4(1) = 0,$
$r_2(1)h_4(1) + r_2h_6b_{36} = 0,$
$r_2(1,2)h_2 + r_2h_2(1,2) = 0,$
$r_2(1,3)h_2(1) + r_2(1)h_2(1,3) = 0,$
$r_2(1,3)h_6 + r_2(1)b_{57} = 0,$
$r_2(1,2)h_2(1) + r_2(1)h_2(1,2) = 0,$

[1] Here there are no nontrivial extensions

$r_2(1,2)h_3h_5 + r_2(1,3)h_3(1) = 0,$
$r_2(1,2)h_2(1,3) + r_2(1,3)h_2(1,2) = 0,$
$r_2(1,2)b_{13} + r_2h_4h_6b_{16} + r_2h_4(1)b_{15} = 0,$
$r_2(1,2)h_3^2 + r_2(1)h_6b_{36} + r_2(1,3)b_{35} = 0,$
$r_2(1,2)b_{24} + r_2(1,3)b_{25} + r_2(1)h_6b_{26} = 0,$
$r_2(1,2)b_{14} + r_2(1,3)b_{15} + r_2(1)h_6b_{16} = 0,$
$r_2(1,2)h_5^2 + r_2h_4(1)b_{36} + r_2(1,3)b_{46} = 0,$
$r_2(1,2)h_6 + r_2(1)b_{47} + r_2h_4b_{37} = 0,$
$r_2(1,3)h_4(1) + r_2h_5^2b_{37} + r_2b_{57}b_{36} = 0,$
$r_2(1,2)h_4(1) + r_2b_{36}b_{47} + r_2b_{46}b_{37} = 0,$
$r_2(1,2)b_{57} + r_2h_4(1)b_{37} + r_2(1,3)b_{47} = 0,$

Part (ii).
$r_{2,3}r_{23,45} = b_{01}b_{1,4}h_{4,5} + b_{02}b_{2,4}h_{4,5},$
$r_{2,3}r_{235,467} = b_{01}b_{1,4}h_{45,67} + b_{02}b_{2,4}h_{45,67},$
$r_{2,3}r_{234,567} = b_{01}b_{1,5}h_{45,67} + b_{02}b_{2,5}h_{45,67} + b_{01}b_{1,6}h_{46,57} + b_{02}b_{2,6}h_{46,57},$
$r_{23,45}r_{235,467} = b_{01}b_{13,46}h_{6,7} + b_{02}b_{23,46}h_{6,7},$
$r_{23,45}r_{234,567} = b_{01}b_{13,56}h_{6,7} + b_{02}b_{23,56}h_{6,7},$
$r_{235,467}r_{234,567} = b_{01}b_{135,567} + b_{02}b_{235,567},$
$r_{2,3}r_{2,3} = b_{01}b_{1,3} + b_{02}b_{2,3},$
$r_{23,45}r_{23,45} = b_{01}b_{13,45} + b_{02}b_{23,45},$
$r_{235,467}r_{235,467} = b_{01}b_{135,467} + b_{02}b_{235,467},$
$r_{234,567}r_{234,567} = b_{01}b_{134,567} + b_{02}b_{234,567}.$

A.4. $HX_{7,3}$

Consider $dR_{03} = R_{01}R_{13} + R_{02}R_{23}$ whose homology class is r_2 in $HX_{7,2}$. We have

$$\mathrm{Ann}_{HX_{7,2}}(r_2) = (h_1, h_3, h_3(1)).$$

Apply Proposition 4.17 on $X_{7,3} = X_{7,2} \otimes R_{03}$. The $E_2 = E_\infty$ page is generated by $R_{03}h_1$, $R_{03}h_3$, $R_{03}h_3(1)$ and R_{03}^2 which are represented by $h_0(1)$, r_3, $r_3(1)$ and b_{03} in $HX_{7,3}$ respectively. In addition to relations in $HX_{7,1}$, the new relations in $HX_{7,3}$ are $r_2 = 0$ and

Part (i).
$h_0(1)h_0 = b_{02}h_2,$
$h_0(1)h_2 = h_0b_{13},$
$r_3h_2 = h_0h_1(1),$
$r_3h_4 = 0,$
$h_0(1)h_2(1) = h_0b_{14}h_4,$
$h_0(1)r_2(1) = 0,$
$r_3(1)h_2 = h_0h_1(1,3),$
$r_3(1)h_2(1) = h_0h_4h_1(1,2),$
$r_3(1)r_2(1) = 0,$
$r_3(1)h_6 = 0,$
$r_3h_4(1) = 0,$
$h_0(1)h_2(1,3) = h_0b_{14}h_4(1),$
$h_0(1)r_2(1,3) = 0,$
$h_0(1)h_2(1,2) = h_0b_{15}h_4(1) + h_0b_{16}h_4h_6,$
$h_0(1)r_2(1,2) = 0,$

$r_3(1)h_4(1) = 0,$
$r_3h_1 + h_0(1)h_3 = 0,$
$r_3b_{13} + h_0(1)h_1(1) = 0,$
$r_3h_1(1) + h_0(1)b_{24} = h_0h_2b_{14},$
$r_3(1)h_1 + h_0(1)h_3(1) = 0,$
$r_3(1)h_3 + r_3h_3(1) = 0,$
$r_3(1)b_{13} + h_0(1)h_1(1,3) = 0,$
$r_3(1)h_1(1) + h_0(1)h_5b_{25} = h_0h_2h_5b_{15},$
$r_3h_1(1,3) + h_0(1)h_5b_{25} = h_0h_2h_5b_{15},$
$r_3(1)b_{24} + r_3h_5b_{25} = h_0h_2h_1(1,2),$
$r_3(1)h_5 + r_3b_{46} = 0,$
$r_3(1)h_2(1,3) + r_3h_5h_2(1,2) = 0,$
$r_3(1)r_2(1,3) + r_3h_5r_2(1,2) = 0,$
$r_3(1)b_{57} + r_3h_5b_{47} = 0,$
$r_3(1)b_{14} + r_3h_5b_{15} + h_0(1)h_1(1,2) = 0,$
$r_3(1)h_1(1,3) + h_0(1)b_{46}b_{25} + h_0(1)h_4^2b_{26} = h_0h_2b_{14,56},$
$r_3(1)h_1(1,2) + h_0(1)b_{25}b_{36} + h_0(1)b_{35}b_{26} = h_0h_2b_{13,56},$

Part (ii).
$h_{01,23}r_{3,4} = b_{0,2}h_{12,34} + b_{0,3}h_{13,24},$
$h_{01,23}r_{34,56} = b_{0,2}h_{124,356} + b_{0,3}h_{134,256},$
$r_{3,4}r_{34,56} = b_{01}b_{1,5}h_{5,6} + b_{02}b_{2,5}h_{5,6} + b_{03}b_{3,5}h_{5,6},$
$h_{01,23}h_{01,23} = b_{01,23},$
$r_{3,4}r_{3,4} = b_{01}b_{1,4} + b_{02}b_{2,4} + b_{03}b_{3,4},$
$r_{34,56}r_{34,56} = b_{01}b_{14,56} + b_{02}b_{24,56} + b_{03}b_{34,56}.$

A.5. $HX_{7,4}$

Consider $dR_{04} = R_{01}R_{14} + R_{02}R_{24} + R_{03}R_{34}$ whose homology class is r_3 in $HX_{7,3}$. We have
$$\mathrm{Ann}_{HX_{7,3}}(r_3) = (h_4, h_4(1)).$$
Apply Proposition 4.17 on $X_{7,4} = X_{7,3} \otimes R_{04}$. The $E_2 = E_\infty$ page is generated by $R_{04}h_4$, $R_{04}h_4(1)$ and R_{04}^2 which are represented by r_4, $r_4(1)$ and b_{04} in $HX_{7,4}$ respectively. In addition to relations in $HX_{7,3}$, the new relations in $HX_{7,4}$ are $r_3 = 0$ and

Part (i).
$r_4h_3 = r_2(1),$
$r_4h_1(1) = 0,$
$r_4h_5 = 0,$
$r_4(1)h_3 = r_2(1,3),$
$r_4(1)h_1(1) = 0,$
$r_4(1)h_3(1) = r_2(1,2)h_5,$
$r_4(1)r_3(1) = 0,$
$r_4(1)h_1(1,3) = 0,$
$r_4(1)h_1(1,2) = 0,$
$r_4(1)r_2(1,2) = h_3(b_{01}b_{14,67} + b_{02}b_{24,67} + b_{03}b_{34,67}),$
$r_4(1)h_4 + r_4h_4(1) = 0,$
$r_4(1)h_2(1) + r_4h_2(1,3) = 0,$
$r_4(1)b_{35} + r_4h_6b_{36} = r_2(1,2)h_3,$

$r_4(1)h_1b_{25} + r_4h_1h_6b_{26} = 0,$
$r_4(1)h_6 + r_4b_{57} = 0,$
$r_4(1)b_{02}b_{25} + r_4h_6b_{02}b_{26} + r_4(1)h_0^2b_{15} + r_4h_0^2h_6b_{16} = r_2(1,2)b_{03}h_3,$
$r_4(1)h_0(1)b_{25} + r_4(1)h_0h_2b_{15} + r_4h_6h_0(1)b_{26} + r_4h_0h_2h_6b_{16} = 0,$
$r_4(1)b_{13}b_{25} + r_4h_6b_{13}b_{26} + r_4h_2^2h_6b_{16} + r_4(1)h_2^2b_{15} = 0,$
$r_4(1)b_{14}b_{25} + r_4h_6b_{14}b_{26} + r_4h_6b_{24}b_{16} + r_4(1)b_{24}b_{15} = 0,$

Part (ii).
$r_{4,5}r_{45,67} = b_{01}b_{1,6}h_{6,7} + b_{02}b_{2,6}h_{6,7} + b_{03}b_{3,6}h_{6,7} + b_{04}b_{4,6}h_{6,7},$
$r_{4,5}r_{4,5} = b_{01}b_{1,5} + b_{02}b_{2,5} + b_{03}b_{3,5} + b_{04}b_{4,5},$
$r_{45,67}r_{45,67} = b_{01}b_{15,67} + b_{02}b_{25,67} + b_{03}b_{35,67} + b_{04}b_{45,67}.$

A.6. $HX_{7,5}$

Consider $dR_{05} = \sum_{i=1}^{r} R_{0i}R_{i5}$ whose homology class is r_4 in $HX_{7,4}$. We have

$$\mathrm{Ann}_{HX_{7,4}}(r_4) = (h_1h_3, h_1(1), h_5).$$

Apply Proposition 4.17 on $X_{7,5} = X_{7,4} \otimes R_{05}$. The $E_2 = E_\infty$ page is generated by $R_{05}h_1h_3$, $R_{05}h_1(1)$, $R_{05}h_5$ and R_{05}^2 which are represented by $h_0(1,3)$, $h_0(1,2)$, r_5 and b_{05} in $HX_{7,5}$ respectively. In addition to relations in $HX_{7,4}$, the new relations in $HX_{7,5}$ are $r_4 = 0$ and

Part (i).
$h_0(1,3)h_0 = b_{02}h_2(1),$
$h_0(1,3)h_2 = h_0b_{14}h_4,$
$h_0(1,3)h_0(1) = b_{01,24}h_4,$
$h_0(1,2)h_0 = b_{03}h_2(1) + b_{04}h_2h_4,$
$h_0(1,3)h_4 = h_0(1)b_{35},$
$h_0(1,2)h_0(1) = b_{01,34}h_4,$
$h_0(1,2)h_4 = h_0h_2b_{15} + h_0(1)b_{25},$
$h_0(1,3)h_2(1) = h_0b_{13,45},$
$r_5h_4 = r_3(1),$
$h_0(1,2)h_2(1) = h_0b_{12,45},$
$r_5h_2(1) = h_0h_1(1,2),$
$r_5h_6 = 0,$
$h_0(1,3)h_4(1) = h_0(1)b_{36}h_6,$
$h_0(1,2)h_4(1) = h_0h_2b_{16}h_6 + h_0(1)b_{26}h_6,$
$h_0(1,3)r_4(1) = 0,$
$h_0(1,3)h_2(1,3) = h_0b_{13,46}h_6,$
$h_0(1,2)r_4(1) = 0,$
$h_0(1,2)h_2(1,3) = h_0b_{12,46}h_6,$
$h_0(1,3)h_2(1,2) = h_0b_{13,56}h_6,$
$h_0(1,3)r_2(1,2) = 0,$
$h_0(1,2)h_2(1,2) = h_0b_{12,56}h_6,$
$h_0(1,2)r_2(1,2) = 0,$
$h_0(1,2)h_1^2 + h_0(1,3)b_{13} = h_0(1)h_4b_{14},$
$h_0(1,2)b_{02} + h_0(1,3)b_{03} = h_0(1)h_4b_{04},$
$h_0(1,2)h_1h_3 + h_0(1,3)h_1(1) = 0,$
$h_0(1,2)h_3^2 + h_0(1,3)b_{24} = h_0h_2(1)b_{14},$
$r_5h_1h_3 + h_0(1,3)h_5 = 0,$

$r_5h_1(1)+h_0(1,2)h_5=0,$
$h_0(1,2)b_{35}+h_0(1,3)b_{25}=h_0h_2(1)b_{15},$
$r_5h_1b_{35}+h_0(1,3)h_3(1)=0,$
$r_5h_1b_{25}+h_0(1,2)h_3(1)=0,$
$h_0(1,2)h_1h_3(1)+h_0(1,3)h_1(1,3)=h_0(1)h_4h_1(1,2),$
$r_5h_1h_3(1)+h_0(1,3)b_{46}=h_0(1)h_4b_{36},$
$r_5h_1(1,3)+h_0(1,2)b_{46}=h_0(1)h_4b_{26}+h_0h_2h_4b_{16},$
$r_5b_{13}b_{25}+r_5h_2^2b_{15}+h_0(1,2)h_1(1,3)=0,$
$r_5b_{35}b_{14}+h_0(1,3)h_1(1,2)+r_5h_3^2b_{15}=0,$
$r_5b_{14}b_{25}+h_0(1,2)h_1(1,2)+r_5b_{24}b_{15}=0,$
$r_5h_1(1,2)+h_0(1,2)b_{36}+h_0(1,3)b_{26}=h_0h_2(1)b_{16},$

Part (ii).
$h_{013,245}h_{012,345}=b_{013,345},$
$h_{013,245}r_{5,6}=b_{0,2}h_{123,456}+b_{0,4}h_{134,256}+b_{0,5}h_{135,246},$
$h_{012,345}r_{5,6}=b_{0,3}h_{123,456}+b_{0,4}h_{124,356}+b_{0,5}h_{125,346},$
$h_{013,245}h_{013,245}=b_{013,245},$
$h_{012,345}h_{012,345}=b_{012,345},$
$r_{5,6}r_{5,6}=b_{01}b_{1,6}+b_{02}b_{2,6}+b_{03}b_{3,6}+b_{04}b_{4,6}+b_{05}b_{5,6}.$

A.7. $HX_{7,6}$

Consider $dR_{06}=\sum_{i=1}^r R_{0i}R_{i6}$ whose homology class is r_5 in $HX_{7,5}$. We have

$$\mathrm{Ann}_{HX_{7,5}}(r_5)=(h_6).$$

Apply Proposition 4.17 on $X_{7,6}=X_{7,5}\otimes R_{06}$. The $E_2=E_\infty$ page is generated by $R_{06}h_6$ and R_{06}^2 which are represented by r_6 and b_{02} in $HX_{7,6}$ respectively. In addition to relations in $HX_{7,5}$, the new relations in $HX_{7,6}$ are $r_5=0$ and

Part (i).
$r_6h_5=r_4(1),$
$r_6h_3(1)=r_2(1,2),$
$r_6h_1(1,3)=0,$
$r_6h_1(1,2)=0,$

Part (ii).
$r_{6,7}r_{6,7}=b_{01}b_{1,7}+b_{02}b_{2,7}+b_{03}b_{3,7}+b_{04}b_{4,7}+b_{05}b_{5,7}+b_{06}b_{6,7}.$

A.8. $HX_{7,7}$

Consider $dR_{07}=\sum_{i=1}^r R_{0i}R_{i7}$ whose homology class is r_6 in $HX_{7,6}$. We have

$$\mathrm{Ann}_{HX_{7,6}}(r_6)=(h_1h_3h_5,h_1(1)h_5,h_1h_3(1),h_1(1,3),h_1(1,2)).$$

Apply Proposition 4.17 on $X_{7,7}=X_{7,6}\otimes R_{07}$. The $E_2=E_\infty$ page is generated by $R_{07}h_1h_3h_5$, $R_{07}h_1(1)h_5$, $R_{07}h_1h_3(1)$, $R_{07}h_1(1,3)$, $R_{07}h_1(1,2)$ and R_{07}^2 which are represented by $h_0(1,3,5)$, $h_0(1,2,5)$, $h_0(1,3,4)$, $h_0(1,2,4)$, $h_0(1,2,3)$ and b_{07} in $HX_{7,7}$ respectively. In addition to relations in $HX_{7,6}$, the new relations in $HX_{7,7}$ are $r_6=0$ and

Part (i).

$h_0(1,3,5)h_0 = b_{02}h_2(1,3)$,
$h_0(1,3,5)h_2 = h_0(1)h_2(1,3)$,
$h_0(1,3,5)h_0(1) = b_{01,24}h_4(1)$,
$h_0(1,2,5)h_0 = b_{03}h_2(1,3) + b_{04}h_2h_4(1)$,
$h_0(1,3,5)h_4 = h_0(1,3)h_4(1)$,
$h_0(1,2,5)h_0(1) = b_{01,34}h_4(1)$,
$h_0(1,2,5)h_4 = h_0(1,2)h_4(1)$,
$h_0(1,3,4)h_0 = b_{02}h_2(1,2)$,
$h_0(1,3,5)h_2(1) = h_0(1,3)h_2(1,3)$,
$h_0(1,3,4)h_2 = h_0(1)h_2(1,2)$,
$h_0(1,3,5)h_0(1,3) = b_{013,246}h_6$,
$h_0(1,3,4)h_0(1) = b_{01,25}h_4(1) + b_{01,26}h_4h_6$,
$h_0(1,2,4)h_0 = b_{03}h_2(1,2) + b_{05}h_2h_4(1) + b_{06}h_2h_4h_6$,
$h_0(1,2,5)h_2(1) = h_0(1,2)h_2(1,3)$,
$h_0(1,3,5)h_0(1,2) = b_{012,246}h_6$,
$h_0(1,2,5)h_0(1,3) = b_{013,346}h_6$,
$h_0(1,2,4)h_0(1) = b_{01,35}h_4(1) + b_{01,36}h_4h_6$,
$h_0(1,2,5)h_0(1,2) = b_{012,346}h_6$,
$h_0(1,2,3)h_0 = b_{0,4}h_2(1,2) + b_{0,5}h_2(1,3) + b_{0,6}h_2(1)h_6$,
$h_0(1,3,5)h_6 = h_0(1,3)b_{57}$,
$h_0(1,2,3)h_0(1) = b_{01,45}h_4(1) + b_{01,46}h_4h_6$,
$h_0(1,3,4)h_2(1) = h_0(1,3)h_2(1,2)$,
$h_0(1,2,5)h_6 = h_0(1,2)b_{57}$,
$h_0(1,3,4)h_0(1,3) = b_{013,256}h_6$,
$h_0(1,3,4)h_0(1,2) = b_{012,256}h_6$,
$h_0(1,2,4)h_0(1,3) = b_{013,356}h_6$,
$h_0(1,2,4)h_0(1,2) = b_{012,356}h_6$,
$h_0(1,3,4)h_6 = h_0(1)h_4b_{3,7} + h_0(1,3)b_{4,7}$,
$h_0(1,2,3)h_0(1,3) = b_{013,456}h_6$,
$h_0(1,2,4)h_6 = h_0h_2h_4b_{17} + h_0(1)h_4b_{27} + h_0(1,2)b_{47}$,
$h_0(1,2,3)h_0(1,2) = b_{012,456}h_6$,
$h_0(1,2,3)h_6 = h_0h_2(1)b_{17} + h_0(1,3)b_{27} + h_0(1,2)b_{37}$,
$h_0(1,3,5)h_4(1) = h_0(1)b_{35,67}$,
$h_0(1,2,5)h_4(1) = h_0h_2b_{15,67} + h_0(1)b_{25,67}$,
$h_0(1,3,5)h_2(1,3) = h_0b_{135,467}$,
$h_0(1,2,5)h_2(1,3) = h_0b_{125,467}$,
$h_0(1,3,4)h_4(1) = h_0(1)b_{34,67}$,
$h_0(1,2,4)h_4(1) = h_0h_2b_{14,67} + h_0(1)b_{24,67}$,
$h_0(1,3,4)h_2(1,3) = h_0b_{134,467}$,
$h_0(1,3,5)h_2(1,2) = h_0b_{135,567}$,
$h_0(1,2,3)h_4(1) = h_0h_2b_{13,67} + h_0(1)b_{23,67}$,
$h_0(1,2,5)h_2(1,2) = h_0b_{125,567}$,
$h_0(1,2,4)h_2(1,3) = h_0b_{124,467}$,
$h_0(1,2,3)h_2(1,3) = h_0b_{123,467}$,
$h_0(1,3,4)h_2(1,2) = h_0b_{134,567}$,
$h_0(1,2,4)h_2(1,2) = h_0b_{124,567}$,
$h_0(1,2,3)h_2(1,2) = h_0b_{123,567}$,

$h_0(1,2,5)h_1^2+h_0(1,3,5)b_{13}=h_0(1)h_4(1)b_{14}$,
$h_0(1,2,5)b_{02}+h_0(1,3,5)b_{03}=h_0(1)h_4(1)b_{04}$,
$h_0(1,2,5)h_1h_3+h_0(1,3,5)h_1(1)=0$,
$h_0(1,2,5)h_3^2+h_0(1,3,5)b_{24}=h_0h_2(1,3)b_{14}$,
$h_0(1,2,4)h_1^2+h_0(1,3,4)b_{13}=h_0(1)h_4(1)b_{15}+h_0(1)h_4h_6b_{16}$,
$h_0(1,2,4)b_{02}+h_0(1,3,4)b_{03}=h_0(1)h_4(1)b_{05}+h_0(1)h_4h_6b_{06}$,
$h_0(1,3,4)h_3^2+h_0(1,3,5)b_{35}=h_0(1,3)h_6b_{36}$,
$h_0(1,2,4)h_1h_3+h_0(1,3,4)h_1(1)=0$,
$h_0(1,3,4)b_{24}+h_0(1,3,5)b_{25}=h_0(1,3)h_6b_{26}$,
$h_0(1,2,5)b_{35}+h_0(1,3,5)b_{25}=h_0h_2(1,3)b_{15}$,
$h_0(1,2,4)h_3^2+h_0(1,2,5)b_{35}=h_0(1,2)h_6b_{36}$,
$h_0(1,3,4)h_3h_5+h_0(1,3,5)h_3(1)=0$,
$h_0(1,2,3)h_2h_4+h_0(1,2,4)h_2(1)=h_0(1,2)h_2(1,2)$,
$h_0(1,2,4)h_3h_5+h_0(1,2,5)h_3(1)=0$,
$h_0(1,2,4)h_1h_3h_5+h_0(1,3,5)h_1(1,3)=0$,
$h_0(1,2,4)h_5h_1(1)+h_0(1,2,5)h_1(1,3)=0$,
$h_0(1,3,4)h_5^2+h_0(1,3,5)b_{46}=h_0(1)h_4(1)b_{36}$,
$h_0(1,2,3)h_1h_3h_5+h_0(1,3,5)h_1(1,2)=0$,
$h_0(1,2,4)h_5^2+h_0(1,2,5)b_{46}=h_0(1)h_4(1)b_{26}+h_0h_2h_4(1)b_{16}$,
$h_0(1,2,3)h_5h_1(1)+h_0(1,2,5)h_1(1,2)=0$,
$h_0(1,2,4)h_1h_3(1)+h_0(1,3,4)h_1(1,3)=0$,
$h_0(1,2,3)h_1h_3(1)+h_0(1,3,4)h_1(1,2)=0$,
$h_0(1,2,3)h_1(1,3)+h_0(1,2,4)h_1(1,2)=0$,
$h_0(1,3,4)b_{57}+h_0(1,3,5)b_{47}=h_0(1)h_4(1)b_{37}$,
$h_0(1,2,4)b_{57}+h_0(1,2,5)b_{47}=h_0(1)h_2(1)b_{27}+h_0h_2h_4(1)b_{17}$,
$h_0(1,2,3)h_1^2+h_0(1,3,4)b_{14}+h_0(1,3,5)b_{15}=h_0(1,3)h_6b_{16}$,
$h_0(1,2,3)b_{02}+h_0(1,3,4)b_{04}+h_0(1,3,5)b_{05}=h_0(1,3)h_6b_{06}$,
$h_0(1,2,3)h_2^2+h_0(1,2,4)b_{24}+h_0(1,2,5)b_{25}=h_0(1,2)h_6b_{26}$,
$h_0(1,2,3)b_{13}+h_0(1,2,4)b_{14}+h_0(1,2,5)b_{15}=h_0(1,2)h_6b_{16}$,
$h_0(1,2,3)b_{03}+h_0(1,2,5)b_{05}+h_0(1,2,4)b_{04}=h_0(1,2)h_6b_{06}$,
$h_0(1,2,3)h_4^2+h_0(1,2,4)b_{35}+h_0(1,3,4)b_{25}=h_0h_2(1,2)b_{15}$,
$h_0(1,2,3)h_5^2+h_0(1,2,5)b_{36}+h_0(1,3,5)b_{26}=h_0h_2(1,3)b_{16}$,
$h_0(1,2,3)b_{46}+h_0(1,2,4)b_{36}+h_0(1,3,4)b_{26}=h_0h_2(1,2)b_{16}$,
$h_0(1,2,3)b_{57}+h_0(1,3,5)b_{27}+h_0(1,2,5)b_{37}=h_0h_2(1,3)b_{17}$,
$h_0(1,2,3)b_{47}+h_0(1,2,4)b_{37}+h_0(1,3,4)b_{27}=h_0h_2(1,2)b_{17}$,

Part (ii).

$h_{0135,2467}h_{0134,2567}=b_{0135,2567}$,
$h_{0135,2467}h_{0125,3467}=b_{0135,3467}$,
$h_{0135,2467}h_{0124,3567}=b_{0135,3567}$,
$h_{0135,2467}h_{0123,4567}=b_{0135,4567}$,
$h_{0134,2567}h_{0125,3467}=b_{0134,3467}$,
$h_{0134,2567}h_{0124,3567}=b_{0134,3567}$,
$h_{0134,2567}h_{0123,4567}=b_{0134,4567}$,
$h_{0125,3467}h_{0124,3567}=b_{0125,3567}$,
$h_{0125,3467}h_{0123,4567}=b_{0125,4567}$,
$h_{0124,3567}h_{0123,4567}=b_{0124,4567}$,
$h_{0135,2467}h_{0135,2467}=b_{0135,2467}$,
$h_{0134,2567}h_{0134,2567}=b_{0134,2567}$,

$h_{0125,3467}h_{0125,3467} = b_{0125,3467}$,
$h_{0124,3567}h_{0124,3567} = b_{0124,3567}$,
$h_{0123,4567}h_{0123,4567} = b_{0123,4567}$.

A.9. **Gröbner basis of HX_7**

Monomial ordering.

The monomial ordering we use here is the reversed lexicographical ordering by the sequence of the following generators

name	degree (s,t,v)	range of i
h_i	$(1, 2^i, 1)$	$0 \le i \le 6$
$h_i(1)$	$(2, 9 \cdot 2^i, 4)$	$0 \le i \le 4$
$h_i(1,3)$	$(3, 41 \cdot 2^i, 7)$	$0 \le i \le 2$
$h_i(1,2)$	$(3, 49 \cdot 2^i, 9)$	$0 \le i \le 2$
$h_0(1,3,5)$	$(4, 169, 10)$	
$h_0(1,2,5)$	$(4, 177, 12)$	
$h_0(1,3,4)$	$(4, 201, 12)$	
$h_0(1,2,4)$	$(4, 209, 14)$	
$h_0(1,2,3)$	$(4, 225, 16)$	
b_{ij}	$(2, 2(2^j - 2^i), 2(j-i))$	$0 \le i \le j-2 < j \le 7$

Here b_{ij} is ordered first by $j - i$ and then by i.

Gröbner basis. [2]

$h_0h_1 = 0$
$h_1h_2 = 0$
$h_2b_{02} = h_0h_0(1)$
$h_2h_3 = 0$
$h_2h_0(1) = h_0b_{13}$
$h_3h_0(1) = 0$
$b_{02}b_{13} = h_1^2b_{03} + h_0(1)^2$
$h_0h_1(1) = 0$
$h_3b_{13} = h_1h_1(1)$
$h_3h_4 = 0$
$h_1(1)b_{02} = h_1h_3b_{03}$
$h_3h_1(1) = h_1b_{24}$
$h_0(1)h_1(1) = 0$
$b_{02}b_{24} = h_0^2b_{14} + h_3^2b_{03}$
$h_0(1)b_{24} = h_0h_2b_{14}$
$h_4h_1(1) = 0$
$b_{13}b_{24} = h_2^2b_{14} + h_1(1)^2$
$h_1h_2(1) = 0$
$h_4b_{24} = h_2h_2(1)$
$h_2(1)b_{02} = h_0h_0(1,3)$

[2]An element g of the Gröbner basis here is presented in the form $\mathrm{LM}(g) = g - \mathrm{LM}(g)$

$h_0(1)h_2(1) = h_0h_4b_{14}$
$h_2h_0(1,3) = h_0h_4b_{14}$
$h_4h_5 = 0$
$h_2(1)b_{13} = h_2h_4b_{14}$
$h_2(1)b_{03} = h_0h_0(1,2) + h_2h_4b_{04}$
$h_0(1)h_0(1,3) = h_1^2h_4b_{04} + h_4b_{02}b_{14}$
$h_4h_2(1) = h_2b_{35}$
$h_0(1,3)b_{13} = h_1^2h_0(1,2) + h_4h_0(1)b_{14}$
$h_1(1)h_2(1) = 0$
$h_0(1,2)b_{02} = h_4h_0(1)b_{04} + h_0(1,3)b_{03}$
$h_0(1)b_{35} = h_4h_0(1,3)$
$h_0(1)h_0(1,2) = h_4b_{13}b_{04} + h_4b_{03}b_{14}$
$h_1(1)h_0(1,3) = h_1h_3h_0(1,2)$
$b_{13}b_{35} = h_1^2b_{25} + h_4^2b_{14}$
$h_2h_4b_{03}b_{14} = h_0h_0(1,2)b_{13} + h_2h_4b_{13}b_{04}$
$b_{35}b_{03} = h_0^2b_{15} + h_4^2b_{04} + b_{02}b_{25}$
$h_0(1,3)b_{24} = h_0h_2(1)b_{14} + h_3^2h_0(1,2)$
$h_0(1)b_{25} = h_0h_2b_{15} + h_4h_0(1,2)$
$h_1(1)b_{35} = h_1h_3b_{25}$
$h_5h_2(1) = 0$
$h_2h_4h_0(1,2) = h_0h_2^2b_{15} + h_0b_{13}b_{25}$
$b_{24}b_{35} = h_3^2b_{25} + h_2(1)^2$
$h_5h_0(1,3) = 0$
$h_2h_3(1) = 0$
$h_2(1)h_0(1,3) = h_0h_3^2b_{15} + h_0b_{35}b_{14}$
$h_5b_{35} = h_3h_3(1)$
$h_0(1)h_3(1) = 0$
$h_5h_0(1,2) = 0$
$b_{02}b_{35}b_{14} = h_1^2h_3^2b_{05} + h_1^2b_{35}b_{04} + h_3^2b_{02}b_{15} + h_0(1,3)^2$
$h_0h_1(1,3) = 0$
$h_3(1)b_{13} = h_1h_1(1,3)$
$h_2(1)h_0(1,2) = h_0b_{24}b_{15} + h_0b_{14}b_{25}$
$h_1(1,3)b_{02} = h_1h_3(1)b_{03}$
$h_1(1)h_3(1) = h_1h_5b_{25}$
$b_{02}b_{14}b_{25} = h_0^2b_{14}b_{15} + h_1^2b_{24}b_{05} + h_1^2b_{25}b_{04} + h_3^2b_{03}b_{15} + h_0(1,3)h_0(1,2)$
$h_3h_1(1,3) = h_1h_5b_{25}$
$h_0(1)h_1(1,3) = 0$
$h_5b_{02}b_{25} = h_0^2h_5b_{15} + h_3h_3(1)b_{03}$
$h_5h_6 = 0$
$h_3(1)b_{24} = h_3h_5b_{25}$
$h_0(1,2)b_{35} = h_0h_2(1)b_{15} + h_0(1,3)b_{25}$
$b_{03}b_{14}b_{25} = h_2^2b_{04}b_{15} + h_1(1)^2b_{05} + h_0(1,2)^2 + b_{13}b_{25}b_{04} + b_{24}b_{03}b_{15}$

$h_0 h_1(1,2) = 0$
$h_3(1) b_{14} = h_1 h_1(1,2) + h_3 h_5 b_{15}$
$h_1(1) h_1(1,3) = h_2^2 h_5 b_{15} + h_5 b_{13} b_{25}$
$h_0 h_5 b_{13} b_{25} = h_0 h_2^2 h_5 b_{15}$
$h_1(1,2) b_{02} = h_1 h_3 h_5 b_{05} + h_1 h_3(1) b_{04}$
$h_5 h_3(1) = h_3 b_{46}$
$h_1(1,3) b_{24} = h_2^2 h_1(1,2) + h_5 h_1(1) b_{25}$
$h_0(1) h_1(1,2) = 0$
$h_2(1) h_3(1) = 0$
$h_1(1,2) b_{13} = h_5 h_1(1) b_{15} + h_1(1,3) b_{14}$
$h_1(1,2) b_{03} = h_5 h_1(1) b_{05} + h_1(1,3) b_{04}$
$h_3(1) h_0(1,3) = 0$
$h_1(1) b_{46} = h_5 h_1(1,3)$
$h_1(1) h_1(1,2) = h_5 b_{24} b_{15} + h_5 b_{14} b_{25}$
$h_0 h_5 b_{14} b_{25} = h_0 h_5 b_{24} b_{15}$
$h_2(1) h_1(1,3) = h_2 h_4 h_1(1,2)$
$b_{24} b_{46} = h_2^2 b_{36} + h_5^2 b_{25}$
$h_3(1) h_0(1,2) = 0$
$h_0(1,3) h_1(1,3) = 0$
$h_1(1,3) b_{03} b_{14} = h_5 h_1(1) b_{13} b_{05} + h_5 h_1(1) b_{03} b_{15} + h_1(1,3) b_{13} b_{04}$
$h_3 h_5 b_{14} b_{25} = h_1 h_1(1,2) b_{24} + h_3 h_5 b_{24} b_{15}$
$b_{46} b_{14} = h_1^2 b_{26} + h_5^2 b_{15} + b_{13} b_{36}$
$b_{03} b_{36} = h_0^2 b_{16} + h_5^2 b_{05} + b_{02} b_{26} + b_{46} b_{04}$
$h_1(1,3) b_{35} = h_1 h_3(1) b_{25} + h_4^2 h_1(1,2)$
$h_1(1) b_{36} = h_1 h_3 b_{26} + h_5 h_1(1,2)$
$h_1(1,3) h_0(1,2) = 0$
$h_2(1) b_{46} = h_2 h_4 b_{36}$
$h_6 h_3(1) = 0$
$h_0(1,3) b_{46} = h_4 h_0(1) b_{36}$
$h_3 h_5 h_1(1,2) = h_1 h_3^2 b_{26} + h_1 b_{24} b_{36}$
$h_0(1,3) h_1(1,2) = 0$
$b_{35} b_{46} = h_4^2 b_{36} + h_3(1)^2$
$h_0(1,2) b_{46} = h_0 h_2 h_4 b_{16} + h_4 h_0(1) b_{26}$
$h_6 h_1(1,3) = 0$
$h_0(1,2) h_1(1,2) = 0$
$h_3 h_4(1) = 0$
$h_3(1) h_1(1,3) = h_1 h_4^2 b_{26} + h_1 b_{46} b_{25}$
$b_{02} b_{46} b_{25} = h_0^2 h_4^2 b_{16} + h_0^2 b_{46} b_{15} + h_4^2 b_{02} b_{26} + h_3(1)^2 b_{03}$
$h_6 b_{46} = h_4 h_4(1)$
$h_0(1,2) b_{36} = h_0 h_2(1) b_{16} + h_0(1,3) b_{26}$
$h_1(1) h_4(1) = 0$
$h_6 h_1(1,2) = 0$

$b_{13}b_{46}b_{25} = h_2^2h_4^2b_{16} + h_2^2b_{46}b_{15} + h_4^2b_{13}b_{26} + h_1(1,3)^2$
$h_1h_2(1,3) = 0$
$h_4(1)b_{24} = h_2h_2(1,3)$
$h_2(1,3)b_{02} = h_0h_0(1,3,5)$
$h_3(1)h_1(1,2) = h_1b_{35}b_{26} + h_1b_{25}b_{36}$
$h_0(1)h_2(1,3) = h_0h_4(1)b_{14}$
$h_2h_0(1,3,5) = h_0h_4(1)b_{14}$
$b_{02}b_{25}b_{36} = h_0^2b_{35}b_{16} + h_0^2b_{36}b_{15} + h_3^2b_{46}b_{05} + h_3(1)^2b_{04} + b_{02}b_{35}b_{26}$
$h_2(1,3)b_{13} = h_2h_4(1)b_{14}$
$h_2(1,3)b_{03} = h_0h_0(1,2,5) + h_2h_4(1)b_{04}$
$h_4(1)b_{02}b_{14} = h_1^2h_4(1)b_{04} + h_0(1)h_0(1,3,5)$
$h_2(1)h_4(1) = h_2h_6b_{36}$
$b_{13}b_{25}b_{36} = h_1^2b_{25}b_{26} + h_2^2b_{35}b_{16} + h_2^2b_{36}b_{15} + h_4^2b_{14}b_{26} + h_1(1,3)h_1(1,2)$
$h_4h_2(1,3) = h_2h_6b_{36}$
$h_0(1,3,5)b_{13} = h_1^2h_0(1,2,5) + h_0(1)h_4(1)b_{14}$
$h_1(1)h_2(1,3) = 0$
$h_0(1,2,5)b_{02} = h_0(1)h_4(1)b_{04} + h_0(1,3,5)b_{03}$
$h_4(1)h_0(1,3) = h_4h_0(1,3,5)$
$h_6h_0(1)b_{36} = h_4h_0(1,3,5)$
$h_4(1)b_{03}b_{14} = h_0(1)h_0(1,2,5) + h_4(1)b_{13}b_{04}$
$h_1(1)h_0(1,3,5) = h_1h_3h_0(1,2,5)$
$h_6b_{13}b_{36} = h_1^2h_6b_{26} + h_4h_4(1)b_{14}$
$h_4(1)b_{35} = h_4h_6b_{36}$
$h_0(1,3,5)b_{24} = h_0h_2(1,3)b_{14} + h_3^2h_0(1,2,5)$
$h_4(1)h_0(1,2) = h_4h_0(1,2,5)$
$h_6h_0(1)b_{26} = h_0h_2h_6b_{16} + h_4h_0(1,2,5)$
$h_1(1,2)b_{46} = h_1h_3(1)b_{26} + h_1(1,3)b_{36}$
$b_{14}b_{25}b_{36} = h_3^2b_{15}b_{26} + h_2(1)^2b_{16} + h_1(1,2)^2 + b_{24}b_{36}b_{15} + b_{35}b_{14}b_{26}$
$h_2h_4h_0(1,2,5) = h_0h_2^2h_6b_{16} + h_0h_6b_{13}b_{26}$
$h_1h_2(1,2) = 0$
$h_4(1)b_{25} = h_2h_2(1,2) + h_4h_6b_{26}$
$h_2(1)h_2(1,3) = h_3^2h_6b_{26} + h_6b_{24}b_{36}$
$h_1h_6b_{24}b_{36} = h_1h_3^2h_6b_{26}$
$h_2(1,2)b_{02} = h_0h_0(1,3,4)$
$h_0(1)h_2(1,2) = h_0h_4h_6b_{16} + h_0h_4(1)b_{15}$
$h_0(1,3)h_2(1,3) = h_0h_3^2h_6b_{16} + h_0h_6b_{14}b_{36}$
$h_2h_0(1,3,4) = h_0h_4h_6b_{16} + h_0h_4(1)b_{15}$
$h_2(1)h_0(1,3,5) = h_0h_3^2h_6b_{16} + h_0h_6b_{14}b_{36}$
$h_2(1,2)b_{13} = h_2h_4h_6b_{16} + h_2h_4(1)b_{15}$
$h_6h_4(1) = h_4b_{57}$
$h_0(1,3)b_{25}b_{36} = h_0h_2(1)b_{35}b_{16} + h_0h_2(1)b_{36}b_{15} + h_0(1,3)b_{35}b_{26}$
$h_2(1,2)b_{03} = h_0h_0(1,2,4) + h_2h_4h_6b_{06} + h_2h_4(1)b_{05}$

$h_4(1)b_{02}b_{15} = h_1^2h_4h_6b_{06} + h_1^2h_4(1)b_{05} + h_4h_6b_{02}b_{16} + h_0(1)h_0(1,3,4)$
$h_0(1,3)h_0(1,3,5) = h_1^2h_3^2h_6b_{06} + h_1^2h_6b_{36}b_{04} + h_3^2h_6b_{02}b_{16} + h_6b_{02}b_{14}b_{36}$
$h_2(1,3)b_{35} = h_3^2h_2(1,2) + h_6h_2(1)b_{36}$
$h_0(1,3,4)b_{13} = h_1^2h_0(1,2,4) + h_4h_6h_0(1)b_{16} + h_0(1)h_4(1)b_{15}$
$h_2(1,3)h_0(1,2) = h_0h_6b_{24}b_{16} + h_0h_6b_{14}b_{26}$
$h_2(1)h_0(1,2,5) = h_0h_6b_{24}b_{16} + h_0h_6b_{14}b_{26}$
$h_1(1)h_2(1,2) = 0$
$h_0(1,2,4)b_{02} = h_4h_6h_0(1)b_{06} + h_0(1)h_4(1)b_{05} + h_0(1,3,4)b_{03}$
$h_3(1)h_4(1) = 0$
$h_0(1,3,5)b_{35} = h_3^2h_0(1,3,4) + h_6h_0(1,3)b_{36}$
$h_0(1,2)h_0(1,3,5) = h_0^2h_6b_{14}b_{16}+h_1^2h_6b_{24}b_{06}+h_1^2h_6b_{04}b_{26}+h_3^2h_6b_{03}b_{16}+h_6b_{02}b_{14}b_{26}$
$h_4(1)b_{03}b_{15} = h_4h_6b_{13}b_{06} + h_4h_6b_{03}b_{16} + h_0(1)h_0(1,2,4) + h_4(1)b_{13}b_{05}$
$h_0(1,3)h_0(1,2,5) = h_0^2h_6b_{14}b_{16}+h_1^2h_6b_{24}b_{06}+h_1^2h_6b_{04}b_{26}+h_3^2h_6b_{03}b_{16}+h_6b_{02}b_{14}b_{26}$
$h_1(1)h_0(1,3,4) = h_1h_3h_0(1,2,4)$
$h_2(1,2)b_{24} = h_6h_2(1)b_{26} + h_2(1,3)b_{25}$
$h_2(1,2)b_{14} = h_6h_2(1)b_{16} + h_2(1,3)b_{15}$
$h_0(1,3,4)b_{24} = h_0h_6h_2(1)b_{16} + h_0h_2(1,3)b_{15} + h_3^2h_0(1,2,4)$
$h_0(1,2,5)b_{35} = h_0h_6h_2(1)b_{16} + h_3^2h_0(1,2,4) + h_6h_0(1,3)b_{26}$
$h_0(1,3,5)b_{25} = h_0h_6h_2(1)b_{16} + h_0h_2(1,3)b_{15} + h_3^2h_0(1,2,4) + h_6h_0(1,3)b_{26}$
$h_4(1)h_1(1,3) = 0$
$h_2(1,2)b_{04} = h_0h_0(1,2,3) + h_6h_2(1)b_{06} + h_2(1,3)b_{05}$
$h_0(1,2)h_0(1,2,5) = h_2^2h_6b_{04}b_{16}+h_6h_1(1)^2b_{06}+h_6b_{13}b_{04}b_{26}+h_6b_{24}b_{03}b_{16}+h_6b_{03}b_{14}b_{26}$
$h_2(1)b_{57} = h_6h_2(1,3)$
$h_0(1,3,4)b_{14} = h_1^2h_0(1,2,3) + h_6h_0(1,3)b_{16} + h_0(1,3,5)b_{15}$
$h_0(1,2,3)b_{02} = h_6h_0(1,3)b_{06} + h_0(1,3,5)b_{05} + h_0(1,3,4)b_{04}$
$h_2(1)h_2(1,2) = h_6b_{35}b_{26} + h_6b_{25}b_{36}$
$h_0(1,2,4)b_{24} = h_2^2h_0(1,2,3) + h_6h_0(1,2)b_{26} + h_0(1,2,5)b_{25}$
$h_0(1,3)b_{57} = h_6h_0(1,3,5)$
$h_1h_6b_{25}b_{36} = h_1h_6b_{35}b_{26}$
$h_4(1)b_{04}b_{15} = h_4h_6b_{14}b_{06} + h_4h_6b_{04}b_{16} + h_0(1)h_0(1,2,3) + h_4(1)b_{14}b_{05}$
$h_3(1)h_2(1,3) = h_3h_5h_2(1,2)$
$h_0(1,2,3)b_{13} = h_6h_0(1,2)b_{16} + h_0(1,2,5)b_{15} + h_0(1,2,4)b_{14}$
$h_0(1,3)h_2(1,2) = h_0h_6b_{35}b_{16} + h_0h_6b_{36}b_{15}$
$h_2(1)h_0(1,3,4) = h_0h_6b_{35}b_{16} + h_0h_6b_{36}b_{15}$
$h_0(1,2,3)b_{03} = h_6h_0(1,2)b_{06} + h_0(1,2,5)b_{05} + h_0(1,2,4)b_{04}$
$b_{35}b_{57} = h_3^2b_{47} + h_6^2b_{36}$
$h_0(1,2)b_{57} = h_6h_0(1,2,5)$
$h_3(1)h_0(1,3,5) = h_3h_5h_0(1,3,4)$
$h_6h_0(1)h_0(1,3,5) = h_1^2h_4b_{57}b_{04} + h_4b_{02}b_{57}b_{14}$
$h_4(1)h_1(1,2) = 0$
$h_0(1,3)h_0(1,3,4) = h_1^2h_6b_{35}b_{06} + h_1^2h_6b_{36}b_{05} + h_6b_{02}b_{35}b_{16} + h_6b_{02}b_{36}b_{15}$
$h_3h_0(1,2,4)b_{14} = h_1h_1(1)h_0(1,2,3) + h_3h_6h_0(1,2)b_{16} + h_3h_0(1,2,5)b_{15}$

$h_0(1,2,5)b_{13}b_{25} = h_2^2h_6h_0(1,2)b_{16}+h_2^2h_0(1,2,5)b_{15}+h_6h_0(1,2)b_{13}b_{26}+h_1(1)^2h_0(1,2,4)$

$h_0(1,2)h_2(1,2) = h_0h_6b_{25}b_{16} + h_0h_6b_{15}b_{26}$

$h_2(1)h_0(1,2,4) = h_0h_6b_{25}b_{16} + h_0h_6b_{15}b_{26} + h_2h_4h_0(1,2,3)$

$h_1(1,3)h_2(1,3) = 0$

$h_2(1,3)b_{14}b_{25} = h_6h_2(1)b_{24}b_{16} + h_6h_2(1)b_{14}b_{26} + h_2(1,3)b_{24}b_{15}$

$h_4h_6b_{25}b_{36} = h_2h_2(1,2)b_{35} + h_4h_6b_{35}b_{26}$

$b_{57}b_{25} = h_2^2b_{37} + h_6^2b_{26} + b_{24}b_{47}$

$h_4h_0(1,2,5)b_{25} = h_0h_2h_6b_{25}b_{16} + h_0h_2h_6b_{15}b_{26} + h_4h_6h_0(1,2)b_{26}$

$h_3(1)h_0(1,2,5) = h_3h_5h_0(1,2,4)$

$h_6h_0(1)h_0(1,2,5) = h_4b_{13}b_{57}b_{04} + h_4b_{57}b_{03}b_{14}$

$h_2(1,3)b_{25}b_{04} = h_0h_0(1,2,3)b_{24} + h_6h_2(1)b_{24}b_{06} + h_6h_2(1)b_{04}b_{26} + h_2(1,3)b_{24}b_{05}$

$h_0(1,2)h_0(1,3,4) = h_1^2h_6b_{25}b_{06} + h_1^2h_6b_{26}b_{05} + h_6b_{02}b_{25}b_{16} + h_6b_{02}b_{15}b_{26}$

$h_0(1,3)h_0(1,2,4) = h_1^2h_6b_{25}b_{06} + h_1^2h_6b_{26}b_{05} + h_4h_0(1)h_0(1,2,3) + h_6b_{02}b_{25}b_{16} + h_6b_{02}b_{15}b_{26}$

$h_0(1,2,4)b_{03}b_{14} = h_6h_0(1,2)b_{13}b_{06}+h_6h_0(1,2)b_{03}b_{16}+h_0(1,2,5)b_{13}b_{05}+h_0(1,2,5)b_{03}b_{15}+h_0(1,2,4)b_{13}b_{04}$

$h_1(1,3)h_0(1,3,5) = h_1h_3h_5h_0(1,2,4)$

$b_{14}b_{47} = h_1^2b_{27} + h_6^2b_{16} + b_{13}b_{37} + b_{57}b_{15}$

$h_2(1,3)b_{04}b_{15} = h_0h_0(1,2,3)b_{14} + h_6h_2(1)b_{14}b_{06} + h_6h_2(1)b_{04}b_{16} + h_2(1,3)b_{14}b_{05}$

$b_{47}b_{04} = h_0^2b_{17} + h_6^2b_{06} + b_{02}b_{27} + b_{57}b_{05} + b_{03}b_{37}$

$h_0(1,2,4)b_{35} = h_0h_2(1,2)b_{15} + h_4^2h_0(1,2,3) + h_0(1,3,4)b_{25}$

$h_0(1,2)h_0(1,2,4) = h_2^2h_6b_{15}b_{06}+h_2^2h_6b_{05}b_{16}+h_6b_{13}b_{25}b_{06}+h_6b_{13}b_{26}b_{05}+h_6b_{03}b_{25}b_{16}+h_6b_{03}b_{15}b_{26}$

$h_1(1,3)h_0(1,2,5) = h_5h_1(1)h_0(1,2,4)$

$h_2(1,3)b_{46} = h_2h_4(1)b_{36} + h_5^2h_2(1,2)$

$h_2(1)b_{47} = h_2h_4b_{37} + h_6h_2(1,2)$

$h_0(1,2,5)b_{14}b_{25} = h_6h_0(1,2)b_{24}b_{16}+h_6h_0(1,2)b_{14}b_{26}+h_1(1)^2h_0(1,2,3)+h_0(1,2,5)b_{24}b_{15}$

$h_6b_{25}b_{36}b_{04} = h_0h_2(1)h_0(1,2,3)+h_3^2h_6b_{26}b_{05}+h_6h_2(1)^2b_{06}+h_6b_{24}b_{36}b_{05}+h_6b_{35}b_{04}b_{26}$

$h_2(1,3)h_1(1,2) = 0$

$h_3(1)b_{57} = h_3h_5b_{47}$

$h_0(1,3,5)b_{46} = h_5^2h_0(1,3,4) + h_0(1)h_4(1)b_{36}$

$h_0(1,3)b_{47} = h_4h_0(1)b_{37} + h_6h_0(1,3,4)$

$h_0(1,3)h_0(1,2,3) = h_3^2h_6b_{15}b_{06}+h_3^2h_6b_{05}b_{16}+h_6b_{35}b_{14}b_{06}+h_6b_{35}b_{04}b_{16}+h_6b_{14}b_{36}b_{05}+h_6b_{36}b_{04}b_{15}$

$h_1(1,2)h_0(1,3,5) = h_1h_3h_5h_0(1,2,3)$

$h_2h_6b_{36}b_{04}b_{15} = h_0h_4h_0(1,2,3)b_{14}+h_2h_6b_{35}b_{14}b_{06}+h_2h_6b_{35}b_{04}b_{16}+h_2h_6b_{14}b_{36}b_{05}$

$h_0(1,2,5)b_{46} = h_0h_2h_4(1)b_{16} + h_5^2h_0(1,2,4) + h_0(1)h_4(1)b_{26}$

$h_0(1,2)b_{47} = h_0h_2h_4b_{17} + h_4h_0(1)b_{27} + h_6h_0(1,2,4)$

$h_6h_0(1)h_0(1,3,4) = h_1^2h_4h_6^2b_{06} + h_1^2h_4b_{57}b_{05} + h_4h_6^2b_{02}b_{16} + h_4b_{02}b_{57}b_{15}$

$h_1(1,3)b_{57} = h_5h_1(1)b_{47}$

$h_0(1,2)h_0(1,2,3) = h_6b_{24}b_{15}b_{06}+h_6b_{24}b_{05}b_{16}+h_6b_{14}b_{25}b_{06}+h_6b_{14}b_{26}b_{05}+h_6b_{25}b_{04}b_{16}+h_6b_{04}b_{15}b_{26}$

$h_1(1,2)h_0(1,2,5) = h_5h_1(1)h_0(1,2,3)$

$h_4h_6h_2(1,2) = h_2h_4^2b_{37} + h_2b_{35}b_{47}$

$h_2h_4h_6b_{03}b_{15}b_{26} = h_0h_2^2h_0(1,2,4)b_{15} + h_0h_0(1,2,4)b_{13}b_{25} + h_2^3h_4h_6b_{15}b_{06} + h_2^3h_4h_6b_{05}b_{16} + h_2h_4h_6b_{13}b_{25}b_{06} + h_2h_4h_6b_{13}b_{26}b_{05} + h_2h_4h_6b_{03}b_{25}b_{16}$

$h_1(1,3)h_2(1,2) = 0$

$b_{03}b_{14}b_{37} = h_0^2b_{14}b_{17} + h_1^2b_{04}b_{27} + h_6^2b_{14}b_{06} + h_6^2b_{04}b_{16} + b_{02}b_{14}b_{27} + b_{13}b_{04}b_{37} + b_{57}b_{14}b_{05} + b_{57}b_{04}b_{15}$

$h_6h_0(1)h_0(1,2,4) = h_4h_6^2b_{13}b_{06} + h_4h_6^2b_{03}b_{16} + h_4b_{13}b_{57}b_{05} + h_4b_{57}b_{03}b_{15}$

$h_1(1,3)h_0(1,3,4) = h_1h_3(1)h_0(1,2,4)$

$h_2h_4b_{57}b_{03}b_{15} = h_0h_6h_0(1,2,4)b_{13} + h_2h_4h_6^2b_{13}b_{06} + h_2h_4h_6^2b_{03}b_{16} + h_2h_4b_{13}b_{57}b_{05}$

$b_{46}b_{57} = h_5^2b_{47} + h_4(1)^2$

$h_5h_0(1,3,4)b_{25} = h_0h_5h_2(1,2)b_{15} + h_3h_3(1)h_0(1,2,4)$

$h_0(1,2,5)b_{36} = h_0h_2(1,3)b_{16} + h_5^2h_0(1,2,3) + h_0(1,3,5)b_{26}$

$h_0(1,2)b_{37} = h_0h_2(1)b_{17} + h_6h_0(1,2,3) + h_0(1,3)b_{27}$

$h_6b_{24}b_{36}b_{04}b_{15} = h_0h_2(1)h_0(1,2,3)b_{14} + h_3^2h_6b_{14}b_{26}b_{05} + h_3^2h_6b_{04}b_{15}b_{26} + h_6h_2(1)^2b_{14}b_{06} + h_6h_2(1)^2b_{04}b_{16} + h_6b_{24}b_{14}b_{36}b_{05}$

$h_1(1,2)b_{57} = h_1h_3h_5b_{27} + h_5h_1(1)b_{37}$

$h_2h_4h_6h_0(1,2,4) = h_0h_2^2h_4^2b_{17} + h_0h_2^2b_{47}b_{15} + h_0h_4^2b_{13}b_{27} + h_0b_{13}b_{25}b_{47}$

$h_2h_4h_6b_{04}b_{15}b_{26} = h_0h_2^2h_0(1,2,3)b_{15} + h_0h_6h_0(1,2)b_{25}b_{16} + h_0h_0(1,2,5)b_{25}b_{15} + h_0h_0(1,2,4)b_{14}b_{25} + h_2^2h_6h_2(1)b_{15}b_{06} + h_2^2h_6h_2(1)b_{05}b_{16} + h_2h_4h_6b_{14}b_{25}b_{06} + h_2h_4h_6b_{14}b_{26}b_{05} + h_2h_4h_6b_{25}b_{04}b_{16}$

$h_1(1,2)h_2(1,2) = 0$

$h_0(1,3)b_{03}b_{37} = h_0^2h_0(1,3)b_{17} + h_4h_0(1)b_{04}b_{37} + h_6^2h_0(1,3)b_{06} + h_6h_0(1,3,5)b_{05} + h_6h_0(1,3,4)b_{04} + h_0(1,3)b_{02}b_{27}$

$h_6h_0(1)h_0(1,2,3) = h_4h_6^2b_{14}b_{06} + h_4h_6^2b_{04}b_{16} + h_4b_{57}b_{14}b_{05} + h_4b_{57}b_{04}b_{15}$

$h_1(1,2)h_0(1,3,4) = h_1h_3(1)h_0(1,2,3)$

$h_6h_2(1)b_{36}b_{04}b_{15} = h_0h_3^2h_0(1,2,3)b_{15} + h_0h_0(1,2,3)b_{35}b_{14} + h_3^2h_6h_2(1)b_{15}b_{06} + h_3^2h_6h_2(1)b_{05}b_{16} + h_6h_2(1)b_{35}b_{14}b_{06} + h_6h_2(1)b_{35}b_{04}b_{16} + h_6h_2(1)b_{14}b_{36}b_{05}$

$h_2h_4b_{57}b_{04}b_{15} = h_0h_6^2h_0(1,2)b_{16} + h_0h_6h_0(1,2,5)b_{15} + h_0h_6h_0(1,2,4)b_{14} + h_2h_4h_6^2b_{14}b_{06} + h_2h_4h_6^2b_{04}b_{16} + h_2h_4b_{57}b_{14}b_{05}$

$h_1(1,2)h_0(1,2,4) = h_1(1,3)h_0(1,2,3)$

$h_4(1)h_2(1,3) = h_2h_5^2b_{37} + h_2b_{57}b_{36}$

$h_2h_4h_6h_0(1,2,3) = h_0h_2^2b_{35}b_{17} + h_0h_2^2b_{15}b_{37} + h_0h_4^2b_{14}b_{27} + h_0b_{13}b_{25}b_{37}$

$h_6h_2(1)b_{04}b_{15}b_{26} = h_0h_0(1,2,3)b_{24}b_{15} + h_0h_0(1,2,3)b_{14}b_{25} + h_6h_2(1)b_{24}b_{15}b_{06} + h_6h_2(1)b_{24}b_{05}b_{16} + h_6h_2(1)b_{14}b_{25}b_{06} + h_6h_2(1)b_{14}b_{26}b_{05} + h_6h_2(1)b_{25}b_{04}b_{16}$

$h_4(1)h_0(1,3,5) = h_5^2h_0(1)b_{37} + h_0(1)b_{57}b_{36}$

$b_{13}b_{57}b_{36} = h_1^2h_5^2b_{27} + h_1^2b_{57}b_{26} + h_5^2b_{13}b_{37} + h_4(1)^2b_{14}$

$h_0(1,2,3)b_{46} = h_0h_2(1,2)b_{16} + h_0(1,3,4)b_{26} + h_0(1,2,4)b_{36}$

$h_4(1)h_0(1,2,5) = h_0h_2h_5^2b_{17} + h_0h_2b_{57}b_{16} + h_5^2h_0(1)b_{27} + h_0(1)b_{57}b_{26}$

$h_1(1,2)b_{47} = h_1h_3(1)b_{27} + h_1(1,3)b_{37}$

$h_6h_2(1)h_0(1,2,3) = h_0h_3^2b_{15}b_{27} + h_0h_2(1)^2b_{17} + h_0b_{24}b_{15}b_{37} + h_0b_{35}b_{14}b_{27} + h_0b_{14}b_{25}b_{37}$

$h_2h_6b_{35}b_{04}b_{15}b_{26} = h_0h_2h_2(1)h_0(1,2,3)b_{15} + h_0h_4h_0(1,2,3)b_{14}b_{25} + h_2h_6h_2(1)^2b_{15}b_{06} + h_2h_6h_2(1)^2b_{05}b_{16} + h_2h_6b_{35}b_{14}b_{25}b_{06} + h_2h_6b_{35}b_{14}b_{26}b_{05} + h_2h_6b_{35}b_{25}b_{04}b_{16}$

$b_{24}b_{57}b_{36} = h_3^2h_5^2b_{27} + h_3^2b_{57}b_{26} + h_5^2b_{24}b_{37} + h_2(1,3)^2$

$h_2(1,3)h_0(1,3,5) = h_0h_3^2h_5^2b_{17} + h_0h_3^2b_{57}b_{16} + h_0h_5^2b_{14}b_{37} + h_0b_{57}b_{14}b_{36}$

$h_0(1,3)b_{25}b_{37} = h_0h_2(1)b_{35}b_{17} + h_0h_2(1)b_{15}b_{37} + h_6h_0(1,2,3)b_{35} + h_0(1,3)b_{35}b_{27}$

$b_{02}b_{57}b_{14}b_{36} = h_1^2h_3^2h_5^2b_{07} + h_1^2h_3^2b_{57}b_{06} + h_1^2h_5^2b_{04}b_{37} + h_1^2b_{57}b_{36}b_{04} + h_3^2h_5^2b_{02}b_{17} + h_3^2b_{02}b_{57}b_{16} + h_5^2b_{02}b_{14}b_{37} + h_0(1,3,5)^2$

$h_4(1)h_2(1,2) = h_2b_{46}b_{37} + h_2b_{36}b_{47}$

$h_2h_6h_0(1,2,3)b_{35} = h_0h_2h_2(1)b_{35}b_{17} + h_0h_2h_2(1)b_{15}b_{37} + h_0h_4b_{35}b_{14}b_{27} + h_0h_4b_{14}b_{25}b_{37}$

$h_2(1,3)h_0(1,2,5) = h_0h_5^2b_{24}b_{17} + h_0h_5^2b_{14}b_{27} + h_0b_{24}b_{57}b_{16} + h_0b_{57}b_{14}b_{26}$

$h_4(1)h_0(1,3,4) = h_0(1)b_{46}b_{37} + h_0(1)b_{36}b_{47}$

$b_{02}b_{57}b_{14}b_{26} = h_0^2h_5^2b_{14}b_{17} + h_0^2b_{57}b_{14}b_{16} + h_1^2h_5^2b_{24}b_{07} + h_1^2h_5^2b_{04}b_{27} + h_1^2b_{24}b_{57}b_{06} + h_1^2b_{57}b_{04}b_{26} + h_3^2h_5^2b_{03}b_{17} + h_3^2b_{57}b_{03}b_{16} + h_5^2b_{02}b_{14}b_{27} + h_0(1,3,5)h_0(1,2,5)$

$b_{13}b_{36}b_{47} = h_1^2b_{46}b_{27} + h_1^2b_{47}b_{26} + h_4^2b_{57}b_{16} + h_4(1)^2b_{15} + b_{13}b_{46}b_{37}$

$h_0(1,3,4)b_{46}b_{25} = h_0h_4^2h_2(1,2)b_{16} + h_0h_2(1,2)b_{46}b_{15} + h_4^2h_0(1,3,4)b_{26} + h_3(1)^2h_0(1,2,4)$

$h_4(1)h_0(1,2,4) = h_0h_2b_{46}b_{17} + h_0h_2b_{47}b_{16} + h_0(1)b_{46}b_{27} + h_0(1)b_{47}b_{26}$

$b_{57}b_{03}b_{14}b_{26} = h_2^2h_5^2b_{04}b_{17} + h_2^2b_{57}b_{04}b_{16} + h_5^2h_1(1)^2b_{07} + h_5^2b_{13}b_{04}b_{27} + h_5^2b_{24}b_{03}b_{17} + h_5^2b_{03}b_{14}b_{27} + h_1(1)^2b_{57}b_{06} + h_0(1,2,5)^2 + b_{13}b_{57}b_{04}b_{26} + b_{24}b_{57}b_{03}b_{16}$

$b_{24}b_{36}b_{47} = h_2^2b_{36}b_{37} + h_3^2b_{46}b_{27} + h_3^2b_{47}b_{26} + h_5^2b_{25}b_{37} + h_2(1,3)h_2(1,2)$

$h_2(1,2)h_0(1,3,5) = h_0h_3^2b_{46}b_{17} + h_0h_3^2b_{47}b_{16} + h_0h_5^2b_{15}b_{37} + h_0h_6^2b_{36}b_{16} + h_0b_{57}b_{36}b_{15}$

$h_2(1,3)h_0(1,3,4) = h_0h_3^2b_{46}b_{17} + h_0h_3^2b_{47}b_{16} + h_0h_5^2b_{15}b_{37} + h_0h_6^2b_{36}b_{16} + h_0b_{57}b_{36}b_{15}$

$h_0(1,3,4)b_{25}b_{36} = h_0h_2(1,2)b_{35}b_{16} + h_0h_2(1,2)b_{36}b_{15} + h_3(1)^2h_0(1,2,3) + h_0(1,3,4)b_{35}b_{26}$

$h_4(1)h_0(1,2,3) = h_0h_2b_{36}b_{17} + h_0h_2b_{37}b_{16} + h_0(1)b_{36}b_{27} + h_0(1)b_{26}b_{37}$

$b_{02}b_{57}b_{36}b_{15} = h_1^2h_3^2b_{46}b_{07} + h_1^2h_3^2b_{47}b_{06} + h_1^2h_5^2b_{37}b_{05} + h_1^2h_6^2b_{36}b_{06} + h_1^2b_{57}b_{36}b_{05} + h_3^2b_{02}b_{46}b_{17} + h_3^2b_{02}b_{47}b_{16} + h_5^2b_{02}b_{15}b_{37} + h_6^2b_{02}b_{36}b_{16} + h_0(1,3,5)h_0(1,3,4)$

$h_2(1,2)h_0(1,2,5) = h_0h_2^2b_{37}b_{16} + h_0h_5^2b_{25}b_{17} + h_0h_5^2b_{15}b_{27} + h_0h_6^2b_{26}b_{16} + h_0b_{24}b_{47}b_{16} + h_0b_{57}b_{15}b_{26}$

$h_2(1,3)h_0(1,2,4) = h_0h_2^2b_{36}b_{17} + h_0h_5^2b_{25}b_{17} + h_0h_5^2b_{15}b_{27} + h_0h_6^2b_{26}b_{16} + h_0b_{13}b_{36}b_{27} + h_0b_{13}b_{26}b_{37} + h_0b_{24}b_{47}b_{16} + h_0b_{57}b_{15}b_{26}$

$b_{02}b_{57}b_{15}b_{26} = h_0^2h_5^2b_{15}b_{17} + h_0^2h_6^2b_{16}^2 + h_0^2b_{13}b_{36}b_{17} + h_0^2b_{13}b_{37}b_{16} + h_0^2b_{57}b_{15}b_{16} + h_1^2h_5^2b_{25}b_{07} + h_1^2h_5^2b_{05}b_{27} + h_1^2h_6^2b_{26}b_{06} + h_1^2b_{24}b_{47}b_{06} + h_1^2b_{57}b_{26}b_{05} + h_3^2b_{46}b_{03}b_{17} + h_3^2b_{03}b_{47}b_{16} + h_5^2b_{02}b_{15}b_{27} + h_6^2b_{02}b_{26}b_{16} + h_0(1)^2b_{36}b_{27} + h_0(1)^2b_{26}b_{37} + h_0(1,3,5)h_0(1,2,4)$

$h_0(1,2,5)h_0(1,3,4) = h_0^2b_{13}b_{36}b_{17} + h_0^2b_{13}b_{37}b_{16} + h_0(1)^2b_{36}b_{27} + h_0(1)^2b_{26}b_{37} + h_0(1,3,5)h_0(1,2,4)$

$h_6h_0(1,2,4)b_{36} = h_0h_2h_4b_{36}b_{17} + h_0h_2h_4b_{37}b_{16} + h_0h_6h_2(1,2)b_{16} + h_4h_0(1)b_{36}b_{27} + h_4h_0(1)b_{26}b_{37} + h_6h_0(1,3,4)b_{26}$

$b_{57}b_{03}b_{15}b_{26} = h_2^2h_5^2b_{15}b_{07} + h_2^2h_5^2b_{05}b_{17} + h_2^2h_6^2b_{16}b_{06} + h_2^2b_{57}b_{05}b_{16} + h_2^2b_{03}b_{37}b_{16} + h_5^2b_{13}b_{25}b_{07} + h_5^2b_{13}b_{05}b_{27} + h_5^2b_{03}b_{25}b_{17} + h_5^2b_{03}b_{15}b_{27} + h_6^2b_{13}b_{26}b_{06} + h_6^2b_{03}b_{26}b_{16} + h_1(1)^2b_{47}b_{06} + h_0(1,2,5)h_0(1,2,4) + b_{13}b_{57}b_{26}b_{05} + b_{24}b_{03}b_{47}b_{16}$

$h_2(1,2)b_{57} = h_2h_4(1)b_{37} + h_2(1,3)b_{47}$

$h_2(1,3)h_0(1,2,3) = h_0h_3^2b_{26}b_{17} + h_0h_3^2b_{16}b_{27} + h_0b_{24}b_{36}b_{17} + h_0b_{24}b_{37}b_{16} + h_0b_{14}b_{36}b_{27} + h_0b_{14}b_{26}b_{37}$

$b_{25}b_{36}b_{47} = h_4^2b_{26}b_{37} + h_3(1)^2b_{27} + h_2(1,2)^2 + b_{35}b_{47}b_{26} + b_{46}b_{25}b_{37}$

$h_0(1,3,4)b_{57} = h_0(1)h_4(1)b_{37} + h_0(1,3,5)b_{47}$

$b_{57}b_{36}b_{04}b_{15} = h_0^2h_3^2b_{16}b_{17} + h_1^2h_3^2b_{26}b_{07} + h_1^2h_3^2b_{27}b_{06} + h_1^2b_{24}b_{36}b_{07} + h_1^2b_{24}b_{37}b_{06} + h_3^2h_5^2b_{05}b_{17} + h_3^2b_{02}b_{16}b_{27} + h_3^2b_{46}b_{04}b_{17} + h_3^2b_{03}b_{37}b_{16} + h_5^2b_{14}b_{37}b_{05} + h_5^2b_{04}b_{15}b_{37} + h_6^2b_{14}b_{36}b_{06} + h_6^2b_{36}b_{04}b_{16} + h_0(1,3,5)h_0(1,2,3) + b_{57}b_{14}b_{36}b_{05}$

$b_{02}b_{14}b_{26}b_{37} = h_0^2h_3^2b_{16}b_{17} + h_0^2b_{14}b_{36}b_{17} + h_0^2b_{14}b_{37}b_{16} + h_1^2h_3^2b_{26}b_{07} + h_1^2h_3^2b_{27}b_{06} + h_1^2b_{24}b_{36}b_{07} + h_1^2b_{24}b_{37}b_{06} + h_1^2b_{36}b_{04}b_{27} + h_1^2b_{04}b_{26}b_{37} + h_3^2h_5^2b_{05}b_{17} + h_3^2b_{02}b_{16}b_{27} + h_3^2b_{46}b_{04}b_{17} + h_3^2b_{03}b_{37}b_{16} + h_0(1,3,5)h_0(1,2,3) + b_{02}b_{14}b_{36}b_{27}$

$h_2(1,2)h_0(1,3,4) = h_0h_4^2b_{37}b_{16} + h_0h_3(1)^2b_{17} + h_0b_{35}b_{47}b_{16} + h_0b_{46}b_{15}b_{37} + h_0b_{36}b_{47}b_{15}$

$h_0(1,3)b_{26}b_{37} = h_0h_2(1)b_{36}b_{17} + h_0h_2(1)b_{37}b_{16} + h_6h_0(1,2,3)b_{36} + h_0(1,3)b_{36}b_{27}$

$h_0(1,2,4)b_{57} = h_0h_2h_4(1)b_{17} + h_0(1)h_4(1)b_{27} + h_0(1,2,5)b_{47}$

$b_{02}b_{36}b_{47}b_{15} = h_1^2h_4^2b_{37}b_{06} + h_1^2h_3(1)^2b_{07} + h_1^2b_{35}b_{47}b_{06} + h_1^2b_{46}b_{37}b_{05} + h_1^2b_{36}b_{47}b_{05} + h_4^2b_{02}b_{37}b_{16} + h_3(1)^2b_{02}b_{17} + h_0(1,3,4)^2 + b_{02}b_{35}b_{47}b_{16} + b_{02}b_{46}b_{15}b_{37}$

$b_{57}b_{04}b_{15}b_{26} = h_0^2b_{24}b_{16}b_{17} + h_0^2b_{14}b_{16}b_{27} + h_1^2b_{24}b_{27}b_{06} + h_2^2b_{04}b_{37}b_{16} + h_3^2b_{03}b_{16}b_{27} + h_5^2b_{24}b_{15}b_{07} + h_5^2b_{24}b_{05}b_{17} + h_5^2b_{14}b_{25}b_{07} + h_5^2b_{14}b_{05}b_{27} + h_5^2b_{25}b_{04}b_{17} + h_5^2b_{04}b_{15}b_{27} + h_6^2b_{14}b_{26}b_{06} + h_6^2b_{04}b_{26}b_{16} + h_1(1)^2b_{37}b_{06} + h_0(1,2,5)h_0(1,2,3) + b_{24}b_{03}b_{37}b_{16} + b_{57}b_{14}b_{26}b_{05}$

$h_2h_6h_0(1,2,3)b_{36} = h_0h_2h_2(1)b_{36}b_{17} + h_0h_2h_2(1)b_{37}b_{16} + h_0h_4b_{14}b_{36}b_{27} + h_0h_4b_{14}b_{26}b_{37}$

$h_2(1,2)h_0(1,2,4) = h_0h_4^2b_{26}b_{17} + h_0h_4^2b_{16}b_{27} + h_0b_{46}b_{25}b_{17} + h_0b_{46}b_{15}b_{27} + h_0b_{25}b_{47}b_{16} + h_0b_{47}b_{15}b_{26}$

$h_0(1,3,5)h_0(1,2,4)b_{14} = h_0^2h_6^2b_{14}b_{16}^2 + h_0^2b_{13}b_{14}b_{36}b_{17} + h_0^2b_{13}b_{14}b_{37}b_{16} + h_1^2h_6^2b_{24}b_{16}b_{06} + h_1^2h_6^2b_{04}b_{26}b_{16} + h_1^2h_0(1,2,5)h_0(1,2,3) + h_3^2h_6^2b_{03}b_{16}^2 + h_6^2b_{02}b_{14}b_{26}b_{16} + h_0(1)^2b_{14}b_{36}b_{27} + h_0(1)^2b_{14}b_{26}b_{37} + h_0(1,3,5)h_0(1,2,5)b_{15}$

$b_{02}b_{47}b_{15}b_{26} = h_0^2h_4^2b_{16}b_{17} + h_0^2b_{46}b_{15}b_{17} + h_1^2h_4^2b_{26}b_{07} + h_1^2h_4^2b_{27}b_{06} + h_1^2b_{46}b_{25}b_{07} + h_1^2b_{46}b_{05}b_{27} + h_1^2b_{25}b_{47}b_{06} + h_1^2b_{47}b_{26}b_{05} + h_4^2b_{02}b_{16}b_{27} + h_3(1)^2b_{03}b_{17} + h_0(1,3,4)h_0(1,2,4) + b_{02}b_{46}b_{15}b_{27} + b_{02}b_{25}b_{47}b_{16}$

$h_0(1,2,3)b_{57} = h_0h_2(1,3)b_{17} + h_0(1,3,5)b_{27} + h_0(1,2,5)b_{37}$

$h_1(1,3)b_{36}b_{47} = h_1h_3(1)b_{46}b_{27} + h_1h_3(1)b_{47}b_{26} + h_1(1,3)b_{46}b_{37}$

$b_{03}b_{47}b_{15}b_{26} = h_2^2h_4^2b_{06}b_{17} + h_2^2b_{46}b_{05}b_{17} + h_2^2b_{47}b_{15}b_{06} + h_2^2b_{47}b_{05}b_{16} + h_4^2b_{13}b_{27}b_{06} + h_4^2b_{03}b_{26}b_{17} + h_4^2b_{03}b_{16}b_{27} + h_1(1,3)^2b_{07} + h_0(1,2,4)^2 + b_{13}b_{46}b_{05}b_{27} + b_{13}b_{25}b_{47}b_{06} + b_{13}b_{47}b_{26}b_{05} + b_{46}b_{03}b_{25}b_{17} + b_{46}b_{03}b_{15}b_{27} + b_{03}b_{25}b_{47}b_{16}$

$h_2(1,2)h_0(1,2,3) = h_0b_{35}b_{26}b_{17} + h_0b_{35}b_{16}b_{27} + h_0b_{25}b_{36}b_{17} + h_0b_{25}b_{37}b_{16} + h_0b_{36}b_{15}b_{27} + h_0b_{15}b_{26}b_{37}$

$h_6h_0(1,2,3)b_{24}b_{36} = h_0h_3^2h_2(1)b_{26}b_{17} + h_0h_3^2h_2(1)b_{16}b_{27} + h_0h_2(1)b_{24}b_{36}b_{17} + h_0h_2(1)b_{24}b_{37}b_{16} + h_0h_2(1)b_{14}b_{36}b_{27} + h_0h_2(1)b_{14}b_{26}b_{37} + h_3^2h_6h_0(1,2,3)b_{26}$

$b_{02}b_{15}b_{26}b_{37} = h_0^2b_{35}b_{16}b_{17} + h_0^2b_{36}b_{15}b_{17} + h_1^2b_{35}b_{26}b_{07} + h_1^2b_{35}b_{27}b_{06} + h_1^2b_{25}b_{36}b_{07} + h_1^2b_{25}b_{37}b_{06} + h_1^2b_{36}b_{05}b_{27} + h_1^2b_{26}b_{37}b_{05} + h_3^2b_{46}b_{05}b_{17} + h_3(1)^2b_{04}b_{17} + h_0(1,3,4)h_0(1,2,3) + b_{02}b_{35}b_{16}b_{27} + b_{02}b_{25}b_{37}b_{16} + b_{02}b_{36}b_{15}b_{27}$

$b_{03}b_{15}b_{26}b_{37} = h_0^2b_{25}b_{16}b_{17} + h_0^2b_{15}b_{26}b_{17} + h_1^2b_{25}b_{27}b_{06} + h_2^2b_{35}b_{06}b_{17} + h_2^2b_{36}b_{05}b_{17} + h_2^2b_{15}b_{37}b_{06} + h_2^2b_{37}b_{05}b_{16} + h_4^2b_{14}b_{27}b_{06} + h_4^2b_{04}b_{26}b_{17} + h_4^2b_{04}b_{16}b_{27} + h_5^2b_{25}b_{05}b_{17} + h_5^2b_{15}b_{05}b_{27} + h_1(1,3)h_1(1,2)b_{07} + h_0(1,2,4)h_0(1,2,3) + b_{02}b_{25}b_{16}b_{27} + b_{02}b_{15}b_{26}b_{27} + b_{13}b_{25}b_{37}b_{06} + b_{13}b_{36}b_{05}b_{27} + b_{13}b_{26}b_{37}b_{05} + b_{46}b_{25}b_{04}b_{17} + b_{46}b_{04}b_{15}b_{27} + b_{03}b_{25}b_{37}b_{16}$

$h_0(1,2,3)b_{47} = h_0h_2(1,2)b_{17} + h_0(1,3,4)b_{27} + h_0(1,2,4)b_{37}$

$b_{04}b_{15}b_{26}b_{37} = h_3^2b_{15}b_{27}b_{06} + h_3^2b_{26}b_{05}b_{17} + h_3^2b_{05}b_{16}b_{27} + h_2(1)^2b_{06}b_{17} + h_1(1,2)^2b_{07} + h_0(1,2,3)^2 + b_{24}b_{36}b_{05}b_{17} + b_{24}b_{15}b_{37}b_{06} + b_{24}b_{37}b_{05}b_{16} + b_{35}b_{14}b_{27}b_{06} + b_{35}b_{04}b_{26}b_{17} + b_{35}b_{04}b_{16}b_{27} + b_{14}b_{25}b_{37}b_{06} + b_{14}b_{36}b_{05}b_{27} + b_{14}b_{26}b_{37}b_{05} + b_{25}b_{36}b_{04}b_{17} + b_{25}b_{04}b_{37}b_{16} + b_{36}b_{04}b_{15}b_{27}$

$h_6h_0(1,2,3)b_{25}b_{36} = h_0h_2(1)b_{35}b_{26}b_{17} + h_0h_2(1)b_{35}b_{16}b_{27} + h_0h_2(1)b_{25}b_{36}b_{17} + h_0h_2(1)b_{25}b_{37}b_{16} + h_0h_2(1)b_{36}b_{15}b_{27} + h_0h_2(1)b_{15}b_{26}b_{37} + h_6h_0(1,2,3)b_{35}b_{26}$

$h_0(1,3,5)h_0(1,2,4)b_{36} = h_0^2h_3^2b_{46}b_{16}b_{17} + h_0^2h_3^2b_{47}b_{16}^2 + h_0^2h_5^2b_{15}b_{37}b_{16} + h_0^2h_6^2b_{36}b_{16}^2 + h_0^2b_{13}b_{36}^2b_{17} + h_0^2b_{13}b_{36}b_{37}b_{16} + h_0^2b_{57}b_{36}b_{15}b_{16} + h_5^2h_0(1,3,4)h_0(1,2,3) + h_0(1)^2b_{36}^2b_{27} + h_0(1)^2b_{36}b_{26}b_{37} + h_0(1,3,5)h_0(1,3,4)b_{26}$

$h_0(1,2,4)b_{36}b_{47} = h_0h_2(1,2)b_{46}b_{17}+h_0h_2(1,2)b_{47}b_{16}+h_0(1,3,4)b_{46}b_{27}+h_0(1,3,4)b_{47}b_{26}+h_0(1,2,4)b_{46}b_{37}$

A.10. Relations of HX_7 organized by patterns

This section is coordinating with Conjecture 2.20 and Theorem 2.22.

Relations (1).

$h_0^2b_{14} + h_3^2b_{03} + b_{02}b_{24} = 0$
$h_0^2b_{15} + h_4^2b_{04} + b_{02}b_{25} + b_{35}b_{03} = 0$
$h_0^2b_{16} + h_5^2b_{05} + b_{02}b_{26} + b_{46}b_{04} + b_{03}b_{36} = 0$
$h_0^2b_{17} + h_6^2b_{06} + b_{02}b_{27} + b_{57}b_{05} + b_{03}b_{37} + b_{47}b_{04} = 0$
$h_0^2b_{18} + h_7^2b_{07} + b_{02}b_{28} + b_{68}b_{06} + b_{03}b_{38} + b_{58}b_{05} + b_{04}b_{48} = 0$

Relations (2).

$h_0h_1 = 0$
$h_3h_0(1) = 0$
$h_0h_1(1) = 0$
$h_0(1)h_1(1) = 0$
$h_5h_0(1,3) = 0$
$h_0(1)h_3(1) = 0$
$h_5h_0(1,2) = 0$
$h_0h_1(1,3) = 0$
$h_0(1)h_1(1,3) = 0$
$h_0h_1(1,2) = 0$
$h_0(1)h_1(1,2) = 0$
$h_3(1)h_0(1,3) = 0$
$h_3(1)h_0(1,2) = 0$
$h_0(1,3)h_1(1,3) = 0$
$h_1(1,3)h_0(1,2) = 0$
$h_0(1,3)h_1(1,2) = 0$
$h_0(1,2)h_1(1,2) = 0$
$h_7h_0(1,3,5) = 0$
$h_7h_0(1,2,5) = 0$
$h_5(1)h_0(1,3) = 0$
$h_7h_0(1,3,4) = 0$
$h_0(1)h_3(1,3) = 0$
$h_5(1)h_0(1,2) = 0$
$h_7h_0(1,2,4) = 0$
$h_0h_1(1,3,5) = 0$
$h_0(1)h_1(1,3,5) = 0$
$h_7h_0(1,2,3) = 0$
$h_0h_1(1,2,5) = 0$
$h_0(1)h_1(1,2,5) = 0$

$h_0(1,3)h_3(1,3) = 0$
$h_3(1,3)h_0(1,2) = 0$
$h_0(1,3)h_1(1,3,5) = 0$
$h_0(1,2)h_1(1,3,5) = 0$
$h_0(1,3)h_1(1,2,5) = 0$
$h_0(1)h_3(1,2) = 0$
$h_0h_1(1,3,4) = 0$
$h_0(1,2)h_1(1,2,5) = 0$
$h_0(1)h_1(1,3,4) = 0$
$h_0h_1(1,2,4) = 0$
$h_0(1)h_1(1,2,4) = 0$
$h_0(1,3)h_3(1,2) = 0$
$h_0(1,2)h_3(1,2) = 0$
$h_0(1,3)h_1(1,3,4) = 0$
$h_0h_1(1,2,3) = 0$
$h_0(1,2)h_1(1,3,4) = 0$
$h_5(1)h_0(1,3,5) = 0$
$h_0(1)h_1(1,2,3) = 0$
$h_0(1,3)h_1(1,2,4) = 0$
$h_5(1)h_0(1,2,5) = 0$
$h_0(1,2)h_1(1,2,4) = 0$
$h_5(1)h_0(1,3,4) = 0$
$h_0(1,3)h_1(1,2,3) = 0$
$h_3(1,3)h_0(1,3,5) = 0$
$h_5(1)h_0(1,2,4) = 0$
$h_0(1,2)h_1(1,2,3) = 0$
$h_3(1,3)h_0(1,2,5) = 0$
$h_0(1,3,5)h_1(1,3,5) = 0$
$h_5(1)h_0(1,2,3) = 0$
$h_1(1,3,5)h_0(1,2,5) = 0$
$h_0(1,3,5)h_1(1,2,5) = 0$
$h_3(1,3)h_0(1,3,4) = 0$
$h_0(1,2,5)h_1(1,2,5) = 0$
$h_3(1,3)h_0(1,2,4) = 0$
$h_1(1,3,5)h_0(1,3,4) = 0$
$h_1(1,3,5)h_0(1,2,4) = 0$
$h_3(1,3)h_0(1,2,3) = 0$
$h_1(1,2,5)h_0(1,3,4) = 0$
$h_3(1,2)h_0(1,3,5) = 0$
$h_1(1,2,5)h_0(1,2,4) = 0$
$h_1(1,3,5)h_0(1,2,3) = 0$
$h_3(1,2)h_0(1,2,5) = 0$

$h_0(1,3,5)h_1(1,3,4)=0$
$h_0(1,2,5)h_1(1,3,4)=0$
$h_1(1,2,5)h_0(1,2,3)=0$
$h_0(1,3,5)h_1(1,2,4)=0$
$h_3(1,2)h_0(1,3,4)=0$
$h_0(1,2,5)h_1(1,2,4)=0$
$h_3(1,2)h_0(1,2,4)=0$
$h_0(1,3,4)h_1(1,3,4)=0$
$h_1(1,3,4)h_0(1,2,4)=0$
$h_3(1,2)h_0(1,2,3)=0$
$h_0(1,3,5)h_1(1,2,3)=0$
$h_0(1,3,4)h_1(1,2,4)=0$
$h_0(1,2,5)h_1(1,2,3)=0$
$h_0(1,2,4)h_1(1,2,4)=0$
$h_1(1,3,4)h_0(1,2,3)=0$
$h_1(1,2,4)h_0(1,2,3)=0$
$h_0(1,3,4)h_1(1,2,3)=0$
$h_0(1,2,4)h_1(1,2,3)=0$
$h_0(1,2,3)h_1(1,2,3)=0$

Relations (3A).

$h_0h_2b_{14}+h_0(1)b_{24}=0$
$h_0h_2(1)b_{14}+h_3^2h_0(1,2)+h_0(1,3)b_{24}=0$
$h_0h_2(1)b_{15}+h_0(1,3)b_{25}+h_0(1,2)b_{35}=0$
$h_4h_0(1)b_{36}+h_0(1,3)b_{46}=0$
$h_0h_2h_4b_{16}+h_4h_0(1)b_{26}+h_0(1,2)b_{46}=0$
$h_0h_2(1)b_{16}+h_0(1,3)b_{26}+h_0(1,2)b_{36}=0$
$h_0h_2(1,3)b_{14}+h_3^2h_0(1,2,5)+h_0(1,3,5)b_{24}=0$
$h_0h_2(1,2)b_{14}+h_3^2h_0(1,2,4)+h_0(1,3,4)b_{24}=0$
$h_0h_2(1,3)b_{15}+h_0(1,3,5)b_{25}+h_0(1,2,5)b_{35}=0$
$h_0h_2(1,2)b_{15}+h_4^2h_0(1,2,3)+h_0(1,3,4)b_{25}+h_0(1,2,4)b_{35}=0$
$h_5^2h_0(1,3,4)+h_0(1)h_4(1)b_{36}+h_0(1,3,5)b_{46}=0$
$h_0h_2h_4(1)b_{16}+h_5^2h_0(1,2,4)+h_0(1)h_4(1)b_{26}+h_0(1,2,5)b_{46}=0$
$h_0h_2(1,3)b_{16}+h_5^2h_0(1,2,3)+h_0(1,3,5)b_{26}+h_0(1,2,5)b_{36}=0$
$h_0h_2(1,2)b_{16}+h_0(1,3,4)b_{26}+h_0(1,2,4)b_{36}+h_0(1,2,3)b_{46}=0$
$h_0(1)h_4(1)b_{37}+h_0(1,3,5)b_{47}+h_0(1,3,4)b_{57}=0$
$h_0h_2h_4(1)b_{17}+h_0(1)h_4(1)b_{27}+h_0(1,2,5)b_{47}+h_0(1,2,4)b_{57}=0$
$h_0h_2(1,3)b_{17}+h_0(1,3,5)b_{27}+h_0(1,2,5)b_{37}+h_0(1,2,3)b_{57}=0$
$h_0h_2(1,2)b_{17}+h_0(1,3,4)b_{27}+h_0(1,2,4)b_{37}+h_0(1,2,3)b_{47}=0$
$h_6h_0(1,3)b_{58}+h_0(1,3,5)b_{68}=0$
$h_6h_0(1,2)b_{58}+h_0(1,2,5)b_{68}=0$
$h_4h_6h_0(1)b_{38}+h_6h_0(1,3)b_{48}+h_0(1,3,4)b_{68}=0$
$h_0h_2h_4h_6b_{18}+h_4h_6h_0(1)b_{28}+h_6h_0(1,2)b_{48}+h_0(1,2,4)b_{68}=0$

$h_0h_6h_2(1)b_{18} + h_6h_0(1,3)b_{28} + h_6h_0(1,2)b_{38} + h_0(1,2,3)b_{68} = 0$
$h_0(1)h_4(1)b_{38} + h_0(1,3,5)b_{48} + h_0(1,3,4)b_{58} = 0$
$h_0h_2h_4(1)b_{18} + h_0(1)h_4(1)b_{28} + h_0(1,2,5)b_{48} + h_0(1,2,4)b_{58} = 0$
$h_0h_2(1,3)b_{18} + h_0(1,3,5)b_{28} + h_0(1,2,5)b_{38} + h_0(1,2,3)b_{58} = 0$
$h_0h_2(1,2)b_{18} + h_0(1,3,4)b_{28} + h_0(1,2,4)b_{38} + h_0(1,2,3)b_{48} = 0$

Relations (3B).

$h_1h_3b_{03} + h_1(1)b_{02} = 0$
$h_1^2h_0(1,2) + h_4h_0(1)b_{14} + h_0(1,3)b_{13} = 0$
$h_4h_0(1)b_{04} + h_0(1,3)b_{03} + h_0(1,2)b_{02} = 0$
$h_1h_3(1)b_{03} + h_1(1,3)b_{02} = 0$
$h_1h_3h_5b_{05} + h_1h_3(1)b_{04} + h_1(1,2)b_{02} = 0$
$h_5h_1(1)b_{05} + h_1(1,3)b_{04} + h_1(1,2)b_{03} = 0$
$h_1^2h_0(1,2,5) + h_0(1)h_4(1)b_{14} + h_0(1,3,5)b_{13} = 0$
$h_0(1)h_4(1)b_{04} + h_0(1,3,5)b_{03} + h_0(1,2,5)b_{02} = 0$
$h_1^2h_0(1,2,4) + h_4h_6h_0(1)b_{16} + h_0(1)h_4(1)b_{15} + h_0(1,3,4)b_{13} = 0$
$h_4h_6h_0(1)b_{06} + h_0(1)h_4(1)b_{05} + h_0(1,3,4)b_{03} + h_0(1,2,4)b_{02} = 0$
$h_3^2h_0(1,3,4) + h_6h_0(1,3)b_{36} + h_0(1,3,5)b_{35} = 0$
$h_6h_0(1,3)b_{26} + h_0(1,3,5)b_{25} + h_0(1,3,4)b_{24} = 0$
$h_1^2h_0(1,2,3) + h_6h_0(1,3)b_{16} + h_0(1,3,5)b_{15} + h_0(1,3,4)b_{14} = 0$
$h_6h_0(1,3)b_{06} + h_0(1,3,5)b_{05} + h_0(1,3,4)b_{04} + h_0(1,2,3)b_{02} = 0$
$h_2^2h_0(1,2,3) + h_6h_0(1,2)b_{26} + h_0(1,2,5)b_{25} + h_0(1,2,4)b_{24} = 0$
$h_6h_0(1,2)b_{16} + h_0(1,2,5)b_{15} + h_0(1,2,4)b_{14} + h_0(1,2,3)b_{13} = 0$
$h_6h_0(1,2)b_{06} + h_0(1,2,5)b_{05} + h_0(1,2,4)b_{04} + h_0(1,2,3)b_{03} = 0$
$h_1h_3(1,3)b_{03} + h_1(1,3,5)b_{02} = 0$
$h_1h_3h_5(1)b_{05} + h_1h_3(1,3)b_{04} + h_1(1,2,5)b_{02} = 0$
$h_1(1)h_5(1)b_{05} + h_1(1,3,5)b_{04} + h_1(1,2,5)b_{03} = 0$
$h_1h_3(1,2)b_{03} + h_1(1,3,4)b_{02} = 0$
$h_1h_3h_5h_7b_{07} + h_1h_3h_5(1)b_{06} + h_1h_3(1,2)b_{04} + h_1(1,2,4)b_{02} = 0$
$h_5h_7h_1(1)b_{07} + h_1(1)h_5(1)b_{06} + h_1(1,3,4)b_{04} + h_1(1,2,4)b_{03} = 0$
$h_1h_7h_3(1)b_{07} + h_1h_3(1,3)b_{06} + h_1h_3(1,2)b_{05} + h_1(1,2,3)b_{02} = 0$
$h_7h_1(1,3)b_{07} + h_1(1,3,5)b_{06} + h_1(1,3,4)b_{05} + h_1(1,2,3)b_{03} = 0$
$h_7h_1(1,2)b_{07} + h_1(1,2,5)b_{06} + h_1(1,2,4)b_{05} + h_1(1,2,3)b_{04} = 0$

Relations (4A).

$h_2h_0(1,3) + h_0(1)h_2(1) = 0$
$h_2h_0(1,3,5) + h_0(1)h_2(1,3) = 0$
$h_4h_0(1,3,5) + h_4(1)h_0(1,3) = 0$
$h_4h_0(1,2,5) + h_4(1)h_0(1,2) = 0$
$h_2h_0(1,3,4) + h_0(1)h_2(1,2) = 0$
$h_2(1)h_0(1,3,5) + h_0(1,3)h_2(1,3) = 0$
$h_2(1)h_0(1,2,5) + h_2(1,3)h_0(1,2) = 0$
$h_2(1)h_0(1,3,4) + h_0(1,3)h_2(1,2) = 0$
$h_2h_4h_0(1,2,3) + h_2(1)h_0(1,2,4) + h_0(1,2)h_2(1,2) = 0$
$h_2(1,3)h_0(1,3,4) + h_2(1,2)h_0(1,3,5) = 0$
$h_2h_4(1)h_0(1,2,3) + h_2(1,3)h_0(1,2,4) + h_2(1,2)h_0(1,2,5) = 0$

Relations (4B).

$h_0(1,3)h_0(1,2,5)+h_0(1,2)h_0(1,3,5)=0$

$h_4h_0(1)h_0(1,2,3)+h_0(1,3)h_0(1,2,4)+h_0(1,2)h_0(1,3,4)=0$

Relations (5).

$h_0h_0(1)+h_2b_{02}=0$

$h_0b_{13}+h_2h_0(1)=0$

$h_1^2b_{03}+h_0(1)^2+b_{02}b_{13}=0$

$h_0h_0(1,3)+h_2(1)b_{02}=0$

$h_0h_4b_{14}+h_0(1)h_2(1)=0$

$h_0h_0(1,2)+h_2h_4b_{04}+h_2(1)b_{03}=0$

$h_1^2h_4b_{04}+h_4b_{02}b_{14}+h_0(1)h_0(1,3)=0$

$h_4h_0(1,3)+h_0(1)b_{35}=0$

$h_4b_{13}b_{04}+h_4b_{03}b_{14}+h_0(1)h_0(1,2)=0$

$h_0h_2b_{15}+h_4h_0(1,2)+h_0(1)b_{25}=0$

$h_0h_3^2b_{15}+h_0b_{35}b_{14}+h_2(1)h_0(1,3)=0$

$h_1^2h_3^2b_{05}+h_1^2b_{35}b_{04}+h_3^2b_{02}b_{15}+h_0(1,3)^2+b_{02}b_{35}b_{14}=0$

$h_0b_{24}b_{15}+h_0b_{14}b_{25}+h_2(1)h_0(1,2)=0$

$h_3^2b_{13}b_{05}+h_3^2b_{03}b_{15}+h_0(1,3)h_0(1,2)+b_{13}b_{35}b_{04}+b_{35}b_{03}b_{14}=0$

$h_2^2b_{14}b_{05}+h_2^2b_{04}b_{15}+h_0(1,2)^2+b_{13}b_{24}b_{05}+b_{13}b_{25}b_{04}+b_{24}b_{03}b_{15}+b_{03}b_{14}b_{25}=0$

$h_0h_0(1,3,5)+h_2(1,3)b_{02}=0$

$h_0h_4(1)b_{14}+h_0(1)h_2(1,3)=0$

$h_0h_0(1,2,5)+h_2h_4(1)b_{04}+h_2(1,3)b_{03}=0$

$h_1^2h_4(1)b_{04}+h_0(1)h_0(1,3,5)+h_4(1)b_{02}b_{14}=0$

$h_6h_0(1)b_{36}+h_4(1)h_0(1,3)=0$

$h_0(1)h_0(1,2,5)+h_4(1)b_{13}b_{04}+h_4(1)b_{03}b_{14}=0$

$h_0h_2h_6b_{16}+h_6h_0(1)b_{26}+h_4(1)h_0(1,2)=0$

$h_0h_0(1,3,4)+h_2(1,2)b_{02}=0$

$h_0h_4h_6b_{16}+h_0h_4(1)b_{15}+h_0(1)h_2(1,2)=0$

$h_0h_3^2h_6b_{16}+h_0h_6b_{14}b_{36}+h_0(1,3)h_2(1,3)=0$

$h_0h_0(1,2,4)+h_2h_4h_6b_{06}+h_2h_4(1)b_{05}+h_2(1,2)b_{03}=0$

$h_1^2h_4h_6b_{06}+h_1^2h_4(1)b_{05}+h_4h_6b_{02}b_{16}+h_0(1)h_0(1,3,4)+h_4(1)b_{02}b_{15}=0$

$h_1^2h_3^2h_6b_{06}+h_1^2h_6b_{36}b_{04}+h_3^2h_6b_{02}b_{16}+h_6b_{02}b_{14}b_{36}+h_0(1,3)h_0(1,3,5)=0$

$h_0h_6b_{24}b_{16}+h_0h_6b_{14}b_{26}+h_2(1,3)h_0(1,2)=0$

$h_4h_6b_{13}b_{06}+h_4h_6b_{03}b_{16}+h_0(1)h_0(1,2,4)+h_4(1)b_{13}b_{05}+h_4(1)b_{03}b_{15}=0$

$h_3^2h_6b_{13}b_{06}+h_3^2h_6b_{03}b_{16}+h_6b_{13}b_{36}b_{04}+h_6b_{03}b_{14}b_{36}+h_0(1,3)h_0(1,2,5)=0$

$h_0h_0(1,2,3)+h_6h_2(1)b_{06}+h_2(1,3)b_{05}+h_2(1,2)b_{04}=0$

$h_2^2h_6b_{14}b_{06}+h_2^2h_6b_{04}b_{16}+h_6b_{13}b_{24}b_{06}+h_6b_{13}b_{04}b_{26}+h_6b_{24}b_{03}b_{16}+h_6b_{03}b_{14}b_{26}+h_0(1,2)h_0(1,2,5)=0$

$h_6h_0(1,3,5)+h_0(1,3)b_{57}=0$

$h_4h_6b_{14}b_{06}+h_4h_6b_{04}b_{16}+h_0(1)h_0(1,2,3)+h_4(1)b_{14}b_{05}+h_4(1)b_{04}b_{15}=0$

$h_0h_6b_{35}b_{16}+h_0h_6b_{36}b_{15}+h_0(1,3)h_2(1,2)=0$

$h_6h_0(1,2,5)+h_0(1,2)b_{57}=0$

$h_1^2h_6b_{35}b_{06} + h_1^2h_6b_{36}b_{05} + h_6b_{02}b_{35}b_{16} + h_6b_{02}b_{36}b_{15} + h_0(1,3)h_0(1,3,4) = 0$

$h_0h_6b_{25}b_{16} + h_0h_6b_{15}b_{26} + h_0(1,2)h_2(1,2) = 0$

$h_6b_{13}b_{35}b_{06} + h_6b_{13}b_{36}b_{05} + h_6b_{35}b_{03}b_{16} + h_6b_{03}b_{36}b_{15} + h_0(1,3)h_0(1,2,4) = 0$

$h_2^2h_6b_{15}b_{06} + h_2^2h_6b_{05}b_{16} + h_6b_{13}b_{25}b_{06} + h_6b_{13}b_{26}b_{05} + h_6b_{03}b_{25}b_{16} + h_6b_{03}b_{15}b_{26} + h_0(1,2)h_0(1,2,4) = 0$

$h_4h_0(1)b_{37} + h_6h_0(1,3,4) + h_0(1,3)b_{47} = 0$

$h_3^2h_6b_{15}b_{06} + h_3^2h_6b_{05}b_{16} + h_6b_{35}b_{14}b_{06} + h_6b_{35}b_{04}b_{16} + h_6b_{14}b_{36}b_{05} + h_6b_{36}b_{04}b_{15} + h_0(1,3)h_0(1,2,3) = 0$

$h_0h_2h_4b_{17} + h_4h_0(1)b_{27} + h_6h_0(1,2,4) + h_0(1,2)b_{47} = 0$

$h_6b_{24}b_{15}b_{06} + h_6b_{24}b_{05}b_{16} + h_6b_{14}b_{25}b_{06} + h_6b_{14}b_{26}b_{05} + h_6b_{25}b_{04}b_{16} + h_6b_{04}b_{15}b_{26} + h_0(1,2)h_0(1,2,3) = 0$

$h_0h_2(1)b_{17} + h_6h_0(1,2,3) + h_0(1,3)b_{27} + h_0(1,2)b_{37} = 0$

$h_5^2h_0(1)b_{37} + h_0(1)b_{57}b_{36} + h_4(1)h_0(1,3,5) = 0$

$h_0h_2h_5^2b_{17} + h_0h_2b_{57}b_{16} + h_5^2h_0(1)b_{27} + h_0(1)b_{57}b_{26} + h_4(1)h_0(1,2,5) = 0$

$h_0h_3^2h_5^2b_{17} + h_0h_3^2b_{57}b_{16} + h_0h_5^2b_{14}b_{37} + h_0b_{57}b_{14}b_{36} + h_2(1,3)h_0(1,3,5) = 0$

$h_1^2h_3^2h_5^2b_{07} + h_1^2h_3^2b_{57}b_{06} + h_1^2h_5^2b_{04}b_{37} + h_1^2b_{57}b_{36}b_{04} + h_3^2h_5^2b_{02}b_{17} + h_3^2b_{02}b_{57}b_{16} + h_5^2b_{02}b_{14}b_{37} + h_0(1,3,5)^2 + b_{02}b_{57}b_{14}b_{36} = 0$

$h_0h_5^2b_{24}b_{17} + h_0h_5^2b_{14}b_{27} + h_0b_{24}b_{57}b_{16} + h_0b_{57}b_{14}b_{26} + h_2(1,3)h_0(1,2,5) = 0$

$h_0(1)b_{46}b_{37} + h_0(1)b_{36}b_{47} + h_4(1)h_0(1,3,4) = 0$

$h_3^2h_5^2b_{13}b_{07} + h_3^2h_5^2b_{03}b_{17} + h_3^2b_{13}b_{57}b_{06} + h_3^2b_{57}b_{03}b_{16} + h_5^2b_{13}b_{04}b_{37} + h_5^2b_{03}b_{14}b_{37} + h_0(1,3,5)h_0(1,2,5) + b_{13}b_{57}b_{36}b_{04} + b_{57}b_{03}b_{14}b_{36} = 0$

$h_0h_2b_{46}b_{17} + h_0h_2b_{47}b_{16} + h_0(1)b_{46}b_{27} + h_0(1)b_{47}b_{26} + h_4(1)h_0(1,2,4) = 0$

$h_2^2h_5^2b_{14}b_{07} + h_2^2h_5^2b_{04}b_{17} + h_2^2b_{57}b_{14}b_{06} + h_2^2b_{57}b_{04}b_{16} + h_5^2b_{13}b_{24}b_{07} + h_5^2b_{13}b_{04}b_{27} + h_5^2b_{24}b_{03}b_{17} + h_5^2b_{03}b_{14}b_{27} + h_0(1,2,5)^2 + b_{13}b_{24}b_{57}b_{06} + b_{13}b_{57}b_{04}b_{26} + b_{24}b_{57}b_{03}b_{16} + b_{57}b_{03}b_{14}b_{26} = 0$

$h_0h_5^2b_{35}b_{17} + h_0h_5^2b_{15}b_{37} + h_0b_{35}b_{57}b_{16} + h_0b_{57}b_{36}b_{15} + h_2(1,2)h_0(1,3,5) = 0$

$h_0h_2b_{36}b_{17} + h_0h_2b_{37}b_{16} + h_0(1)b_{36}b_{27} + h_0(1)b_{26}b_{37} + h_4(1)h_0(1,2,3) = 0$

$h_1^2h_5^2b_{35}b_{07} + h_1^2h_5^2b_{37}b_{05} + h_1^2b_{35}b_{57}b_{06} + h_1^2b_{57}b_{36}b_{05} + h_5^2b_{02}b_{35}b_{17} + h_5^2b_{02}b_{15}b_{37} + h_0(1,3,5)h_0(1,3,4) + b_{02}b_{35}b_{57}b_{16} + b_{02}b_{57}b_{36}b_{15} = 0$

$h_0h_5^2b_{25}b_{17} + h_0h_5^2b_{15}b_{27} + h_0b_{57}b_{25}b_{16} + h_0b_{57}b_{15}b_{26} + h_2(1,2)h_0(1,2,5) = 0$

$h_5^2b_{13}b_{35}b_{07} + h_5^2b_{13}b_{37}b_{05} + h_5^2b_{35}b_{03}b_{17} + h_5^2b_{03}b_{15}b_{37} + h_0(1,3,5)h_0(1,2,4) + b_{13}b_{35}b_{57}b_{06} + b_{13}b_{57}b_{36}b_{05} + b_{35}b_{57}b_{03}b_{16} + b_{57}b_{03}b_{36}b_{15} = 0$

$h_3^2b_{13}b_{46}b_{07} + h_3^2b_{13}b_{47}b_{06} + h_3^2b_{46}b_{03}b_{17} + h_3^2b_{03}b_{47}b_{16} + h_0(1,2,5)h_0(1,3,4) + b_{13}b_{46}b_{04}b_{37} + b_{13}b_{36}b_{47}b_{04} + b_{46}b_{03}b_{14}b_{37} + b_{03}b_{14}b_{36}b_{47} = 0$

$h_2^2h_5^2b_{15}b_{07} + h_2^2h_5^2b_{05}b_{17} + h_2^2b_{57}b_{15}b_{06} + h_2^2b_{57}b_{05}b_{16} + h_5^2b_{13}b_{25}b_{07} + h_5^2b_{13}b_{05}b_{27} + h_5^2b_{03}b_{25}b_{17} + h_5^2b_{03}b_{15}b_{27} + h_0(1,2,5)h_0(1,2,4) + b_{13}b_{57}b_{25}b_{06} + b_{13}b_{57}b_{26}b_{05} + b_{57}b_{03}b_{25}b_{16} + b_{57}b_{03}b_{15}b_{26} = 0$

$h_0h_3^2b_{26}b_{17} + h_0h_3^2b_{16}b_{27} + h_0b_{24}b_{36}b_{17} + h_0b_{24}b_{37}b_{16} + h_0b_{14}b_{36}b_{27} + h_0b_{14}b_{26}b_{37} + h_2(1,3)h_0(1,2,3) = 0$

$h_3^2h_5^2b_{15}b_{07} + h_3^2h_5^2b_{05}b_{17} + h_3^2b_{57}b_{15}b_{06} + h_3^2b_{57}b_{05}b_{16} + h_5^2b_{35}b_{14}b_{07} + h_5^2b_{35}b_{04}b_{17} + h_5^2b_{14}b_{37}b_{05} + h_5^2b_{04}b_{15}b_{37} + h_0(1,3,5)h_0(1,2,3) + b_{35}b_{57}b_{14}b_{06} + b_{35}b_{57}b_{04}b_{16} + b_{57}b_{14}b_{36}b_{05} + b_{57}b_{36}b_{04}b_{15} = 0$

$h_0h_4^2b_{36}b_{17} + h_0h_4^2b_{37}b_{16} + h_0b_{35}b_{46}b_{17} + h_0b_{35}b_{47}b_{16} + h_0b_{46}b_{15}b_{37} + h_0b_{36}b_{47}b_{15} + h_2(1,2)h_0(1,3,4) = 0$

$h_1^2h_4^2b_{36}b_{07}+h_1^2h_4^2b_{37}b_{06}+h_1^2b_{35}b_{46}b_{07}+h_1^2b_{35}b_{47}b_{06}+h_1^2b_{46}b_{37}b_{05}+h_1^2b_{36}b_{47}b_{05}+h_4^2b_{02}b_{36}b_{17}+h_4^2b_{02}b_{37}b_{16}+h_0(1,3,4)^2+b_{02}b_{35}b_{46}b_{17}+b_{02}b_{35}b_{47}b_{16}+b_{02}b_{46}b_{15}b_{37}+b_{02}b_{36}b_{47}b_{15}=0$

$h_5^2b_{24}b_{15}b_{07}+h_5^2b_{24}b_{05}b_{17}+h_5^2b_{14}b_{25}b_{07}+h_5^2b_{14}b_{05}b_{27}+h_5^2b_{25}b_{04}b_{17}+h_5^2b_{04}b_{15}b_{27}+h_0(1,2,5)h_0(1,2,3)+b_{24}b_{57}b_{15}b_{06}+b_{24}b_{57}b_{05}b_{16}+b_{57}b_{14}b_{25}b_{06}+b_{57}b_{14}b_{26}b_{05}+b_{57}b_{25}b_{04}b_{16}+b_{57}b_{04}b_{15}b_{26}=0$

$h_0h_4^2b_{26}b_{17}+h_0h_4^2b_{16}b_{27}+h_0b_{46}b_{25}b_{17}+h_0b_{46}b_{15}b_{27}+h_0b_{25}b_{47}b_{16}+h_0b_{47}b_{15}b_{26}+h_2(1,2)h_0(1,2,4)=0$

$h_4^2b_{13}b_{36}b_{07}+h_4^2b_{13}b_{37}b_{06}+h_4^2b_{03}b_{36}b_{17}+h_4^2b_{03}b_{37}b_{16}+h_0(1,3,4)h_0(1,2,4)+b_{13}b_{35}b_{46}b_{07}+b_{13}b_{35}b_{47}b_{06}+b_{13}b_{46}b_{37}b_{05}+b_{13}b_{36}b_{47}b_{05}+b_{35}b_{46}b_{03}b_{17}+b_{35}b_{03}b_{47}b_{16}+b_{46}b_{03}b_{15}b_{37}+b_{03}b_{36}b_{47}b_{15}=0$

$h_2^2h_4^2b_{16}b_{07}+h_2^2h_4^2b_{06}b_{17}+h_2^2b_{46}b_{15}b_{07}+h_2^2b_{46}b_{05}b_{17}+h_2^2b_{47}b_{15}b_{06}+h_2^2b_{47}b_{05}b_{16}+h_4^2b_{13}b_{26}b_{07}+h_4^2b_{13}b_{27}b_{06}+h_4^2b_{03}b_{26}b_{17}+h_4^2b_{03}b_{16}b_{27}+h_0(1,2,4)^2+b_{13}b_{46}b_{25}b_{07}+b_{13}b_{46}b_{05}b_{27}+b_{13}b_{25}b_{47}b_{06}+b_{13}b_{47}b_{26}b_{05}+b_{46}b_{03}b_{25}b_{17}+b_{46}b_{03}b_{15}b_{27}+b_{03}b_{25}b_{47}b_{16}+b_{03}b_{47}b_{15}b_{26}=0$

$h_0b_{35}b_{26}b_{17}+h_0b_{35}b_{16}b_{27}+h_0b_{25}b_{36}b_{17}+h_0b_{25}b_{37}b_{16}+h_0b_{36}b_{15}b_{27}+h_0b_{15}b_{26}b_{37}+h_2(1,2)h_0(1,2,3)=0$

$h_3^2h_4^2b_{16}b_{07}+h_3^2h_4^2b_{06}b_{17}+h_3^2b_{46}b_{15}b_{07}+h_3^2b_{46}b_{05}b_{17}+h_3^2b_{47}b_{15}b_{06}+h_3^2b_{47}b_{05}b_{16}+h_4^2b_{14}b_{36}b_{07}+h_4^2b_{14}b_{37}b_{06}+h_4^2b_{36}b_{04}b_{17}+h_4^2b_{04}b_{37}b_{16}+h_0(1,3,4)h_0(1,2,3)+b_{35}b_{46}b_{14}b_{07}+b_{35}b_{46}b_{04}b_{17}+b_{35}b_{14}b_{47}b_{06}+b_{35}b_{47}b_{04}b_{16}+b_{46}b_{14}b_{37}b_{05}+b_{46}b_{04}b_{15}b_{37}+b_{14}b_{36}b_{47}b_{05}+b_{36}b_{47}b_{04}b_{15}=0$

$h_4^2b_{24}b_{16}b_{07}+h_4^2b_{24}b_{06}b_{17}+h_4^2b_{14}b_{26}b_{07}+h_4^2b_{14}b_{27}b_{06}+h_4^2b_{04}b_{26}b_{17}+h_4^2b_{04}b_{16}b_{27}+h_0(1,2,4)h_0(1,2,3)+b_{24}b_{46}b_{15}b_{07}+b_{24}b_{46}b_{05}b_{17}+b_{24}b_{47}b_{15}b_{06}+b_{24}b_{47}b_{05}b_{16}+b_{46}b_{14}b_{25}b_{07}+b_{46}b_{14}b_{05}b_{27}+b_{46}b_{25}b_{04}b_{17}+b_{46}b_{04}b_{15}b_{27}+b_{14}b_{25}b_{47}b_{06}+b_{14}b_{47}b_{26}b_{05}+b_{25}b_{47}b_{04}b_{16}+b_{47}b_{04}b_{15}b_{26}=0$

$h_3^2b_{25}b_{16}b_{07}+h_3^2b_{25}b_{06}b_{17}+h_3^2b_{15}b_{26}b_{07}+h_3^2b_{15}b_{27}b_{06}+h_3^2b_{26}b_{05}b_{17}+h_3^2b_{05}b_{16}b_{27}+h_0(1,2,3)^2+b_{24}b_{35}b_{16}b_{07}+b_{24}b_{35}b_{06}b_{17}+b_{24}b_{36}b_{15}b_{07}+b_{24}b_{36}b_{05}b_{17}+b_{24}b_{15}b_{37}b_{06}+b_{24}b_{37}b_{05}b_{16}+b_{35}b_{14}b_{26}b_{07}+b_{35}b_{14}b_{27}b_{06}+b_{35}b_{04}b_{26}b_{17}+b_{35}b_{04}b_{16}b_{27}+b_{14}b_{25}b_{36}b_{07}+b_{14}b_{25}b_{37}b_{06}+b_{14}b_{36}b_{05}b_{27}+b_{14}b_{26}b_{37}b_{05}+b_{25}b_{36}b_{04}b_{17}+b_{25}b_{04}b_{37}b_{16}+b_{36}b_{04}b_{15}b_{27}+b_{04}b_{15}b_{26}b_{37}=0$

Relations (6).

$h_1h_3h_0(1,2)+h_1(1)h_0(1,3)=0$

$h_1h_3h_0(1,2,5)+h_1(1)h_0(1,3,5)=0$

$h_1h_3h_0(1,2,4)+h_1(1)h_0(1,3,4)=0$

$h_3h_5h_0(1,3,4)+h_3(1)h_0(1,3,5)=0$

$h_3h_5h_0(1,2,4)+h_3(1)h_0(1,2,5)=0$

$h_1h_3h_5h_0(1,2,4)+h_1(1,3)h_0(1,3,5)=0$

$h_5h_1(1)h_0(1,2,4)+h_1(1,3)h_0(1,2,5)=0$

$h_1h_3h_5h_0(1,2,3)+h_1(1,2)h_0(1,3,5)=0$

$h_5h_1(1)h_0(1,2,3)+h_1(1,2)h_0(1,2,5)=0$

$h_1h_3(1)h_0(1,2,4)+h_1(1,3)h_0(1,3,4)=0$

$h_1h_3(1)h_0(1,2,3)+h_1(1,2)h_0(1,3,4)=0$

$h_1(1,3)h_0(1,2,3)+h_1(1,2)h_0(1,2,4)=0$

References

[1] J. F. Adams, *Stable homotopy theory.* Lecture notes. Berlin-Göttingen-Heidelberg: Springer 1964.

[2] V.K.A.M. Gugenheim, J. Peter May, *On the theory and applications of differential torsion products*, Memoirs of the AMS, vol. 142, American Mathematical Society, Providence, R.I., 1974.

[3] J. Peter May, *The Cohomology of Restricted Lie Algebras and of Hopf Algebras: Application to the Steenrod Algebra*, Thesis (Ph.D.)–Princeton University, 1964.

[4] J. Peter May, *Matric Massey products*, Journal of algebra 12, 533-568 (1969)

[5] John W. Milnor, John C. Moore, *On the structure of Hopf algebras*, Ann. Math. (2) 81 (1965) 211–264.

[6] Osamu Nakamura, *On the squaring operations in the May spectral sequence*, Mem. Fac. Sci., Kyushu Univ., Ser. A, Math. 26 (1972) 293–308.

[7] S. B. Priddy, *Koszul resolutions and the Steenrod algebra*, Bull. Amer. Math. Soc. 76 (1970),

[8] S. B. Priddy, *Koszul resolutions*, Trans. Am. Math. Soc. 152 (1970) 39–60. MR0265437834-83 839.

[9] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, Pure and Applied Mathematics, 121. Academic Press, Inc., Orlando, FL, 1986.

[10] M. C. Tangora, *On the cohomology of the Steenrod algebra*, Math. Z. 116 (1970), 18–64.