ROTATION INVARIANCE IN ALGEBRAIC K-THEORY

Contents

1. Overview

1.1. **Introduction.** Let $\mathcal{C}$ be a stable ∞-category. We let $K_0(\mathcal{C})$ denote the *Grothendieck group* of $\mathcal{C}$: that is, the quotient of the free group generated by symbols $[X]$ (where X ranges over the set of objects of $\mathcal{C}$) by the relations $[X] = [X'] + [X'']$, where

$$X' \to X \to X''$$

ranges over all fiber sequences in $\mathcal{C}$. Every exact functor $f : \mathcal{C} \to \mathcal{D}$ between stable ∞-categories induces a homomorphism of abelian groups $f_* : K_0(\mathcal{C}) \to K_0(\mathcal{D})$.

For every object $X \in \mathcal{C}$, we have a fiber sequence

$$X \to 0 \to \Sigma X,$$

which yields the identity $[\Sigma X] = -[X]$ in $K_0(\mathcal{C})$. Applying this identity twice, we obtain $[\Sigma^2 X] = [X]$. In other words, the exact functor $\Sigma^2 : \mathcal{C} \to \mathcal{C}$ induces the identity map from $K_0(\mathcal{C})$ to itself. Our goal in this paper is to show that this identity is a consequence of a more general fact concerning the "rotation invariance" of algebraic K-theory.

The collection of all (small) stable ∞-categories can itself be organized into an ∞-category, which we will denote by $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ (the morphisms in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ are given by *exact* functors between stable ∞-categories). Moreover, we can regard $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ as a *symmetric monoidal* ∞-category: for every pair of stable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ is the universal recipient of a bifunctor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which is exact separately in each variable. The unit object of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ is the ∞-category $\mathrm{Sp}^{\mathrm{fin}}$ of finite spectra, and its automorphism group can be identified with the classifying space $\mathrm{Pic}(S)$ for invertible spectra. We may therefore identify the (connected) delooping $\mathrm{BPic}(S)$ with a symmetric monoidal subcategory of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ (the subcategory consisting of the unit object together with its invertible endomorphisms). Let ϕ denote the composite map

$$S^1 \simeq U(1) \to U \overset{\beta}{\to} \Omega^{-1}(\mathbf{Z} \times \mathrm{BU}) \overset{J_{\mathbf{C}}}{\to} \mathrm{BPic}(S)$$

where $U = \varinjlim U(n)$ denotes the infinite unitary group, the map β is given by Bott periodicity, and $J_{\mathbf{C}}$ is the complex J-homomorphism. Then ϕ determines an action of the circle S^1 on the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ of stable ∞-categories. If $\mathcal{C}$ is a stable ∞-category, then the Grothendieck group $K_0(\mathcal{C})$ can be realized as the set of connected components of an infinite loop space $K(\mathcal{C})$, which we will refer to as the *algebraic K-theory space of* $\mathcal{C}$. The main result of this paper is the following:

Theorem 1.1.1. *The K-theory construction $\mathcal{C} \mapsto K(\mathcal{C})$ is invariant under the action of S^1. In other words, the functor $K : \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \to \mathcal{S}$ can be promoted to a S^1-equivariant functor, where S^1 acts on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ via the map ϕ and acts trivially on the ∞-category $\mathcal{S}$ of spaces.*

Remark 1.1.2. In what follows, it will be convenient to abuse notation by identifying S^1 with its singular simplicial set (or any homotopy equivalent Kan complex, such as the simplicial abelian group $B\mathbf{Z}$). The action of S^1 on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ determines a map $e : S^1 \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$. For every stable ∞-category $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$, evaluation on $\mathcal{C}$ yields a map $e_{\mathcal{C}} : S^1 \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$, given by $e_{\mathcal{C}}(\theta) = \phi(\theta) \otimes \mathcal{C}$. Unwinding the definitions, we see that $e_{\mathcal{C}}$ is characterized by the fact that it carries the base point of S^1 to the ∞-category $\mathcal{C}$, and carries a generator of $\pi_1 S^1 \simeq \mathbf{Z}$ to

the double suspension equivalence $\Sigma^2 : \mathcal{C} \to \mathcal{C}$. Theorem 1.1.1 immediately implies that the composite map

$$S^1 \overset{e_{\mathcal{C}}}{\to} \mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty} \overset{K}{\to} \mathcal{S}$$

is nullhomotopic, so that the double suspension equivalence $\Sigma^2 : \mathcal{C} \to \mathcal{C}$ induces a map $K(\mathcal{C}) \to K(\mathcal{C})$ which is homotopic to the identity. We may therefore regard Theorem 1.1.1 as a "delooping" of the observation that double suspension induces the identity map from $K_0(\mathcal{C})$ to itself.

Remark 1.1.3. The results of this paper were inspired by Dyckerhoff-Kapranov theory of 2-Segal spaces ([2]) and its relationship with algebraic K-theory ([3]). Our main results generalize some of the constructions given in [4] and [8] to the setting of an arbitrary stable ∞-category, which need not be 2-periodic or linear over a commutative ring.

1.2. **Outline.** The main obstacle to proving Theorem 1.1.1 is that the action of the circle S^1 on $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$ is defined in a very geometric way (using Bott periodicity and the complex J-homomorphism), but the definition of the algebraic K-theory space $K(\mathcal{C})$ is purely combinatorial. The proof divides naturally into two parts:

(a) In the first part of this paper, we give a purely combinatorial construction of a monoidal functor $\phi : S^1 \to \mathrm{BPic}(S)$ and show that the algebraic K-theory functor $\mathcal{C} \mapsto K(\mathcal{C})$ is S^1-equivariant (with respect to the resulting action of S^1 on $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$).

(b) In the second part of this paper, we will show that the monoidal functor ϕ constructed in (a) agrees (up to homotopy) with the composite map

$$S^1 \simeq U(1) \to U \overset{\beta}{\to} \Omega^{-1}(\mathbf{Z} \times \mathrm{BU}) \overset{J_{\mathbf{S}}}{\to} \mathrm{BPic}(S).$$

Let us first outline our approach to (a). We will begin in §3 by studying the notion of a *filtered spectrum*: that is, a diagram of spectra of the form

$$\cdots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots.$$

The collection of filtered spectra can be organized into an ∞-category $\mathrm{Rep}(\mathbf{Z})$ which can be identified with the Ind-completion of a full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ of *finite* filtered spectra (this is a special case of a general categorical construction which we will discuss in §2). The group $\mathbf{Z}$ of integers acts on both $\mathrm{Rep}(\mathbf{Z})$ and $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ by "shifting" the filtrations. This action "deloops" to an action of the circle $S^1 \simeq B\mathbf{Z}$ on the ∞-category $\mathrm{Mod}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty})$ of stable ∞-categories tensored over $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$. In §3.6, we will construct a functor

$$\mathrm{MF} : \mathrm{Mod}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}) \to \mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty},$$

which can be regarded as an ∞-categorical version of the theory of equivariant matrix factorizations. Moreover, we will use the same ideas to construct a monoidal functor $\phi : S^1 \to \mathrm{BPic}(S)$ (hence an action of S^1 on $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$) for which the functor MF is S^1-equivariant.

In §4, we will review the definition of the K-theory space $K(\mathcal{C})$ associated to a stable ∞-category $\mathcal{C}$. Recall that $K(\mathcal{C})$ is given explicitly by the formula $K(\mathcal{C}) = \Omega|S_{\bullet}(\mathcal{C})^{\simeq}|$, where $S_{\bullet}(\mathcal{C})$ denotes the simplicial ∞-category given by the *Waldhausen construction* of $\mathcal{C}$. The Waldhausen construction makes sense for very general classes of categories and ∞-categories (see [9] and [1]). However, the setting of stable ∞-categories has some special features: if $\mathcal{C}$ is stable, then the simplicial ∞-category $S_{\bullet}(\mathcal{C})$ can be refined to a *paracyclic* ∞-category $S_{\bullet}^{\circlearrowleft}(\mathcal{C})$ (that is, a functor $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\circlearrowleft}) \to \mathcal{C}\mathrm{at}_{\infty}$ for a certain category $\mathbf{\Delta}_{\circlearrowleft}$; see Definition 4.3.4). In §4.5, we will show that the construction $\mathcal{C} \mapsto S_{\bullet}^{\circlearrowleft}(\mathcal{C})$ is corepresentable: that is, there exists a (co)paracyclic object $\mathrm{Quiv}^{\bullet} : \mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}) \to \mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$ such that $S_{\Lambda}^{\circlearrowleft}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^{\Lambda}, \mathcal{C})$ for any object $\Lambda \in \mathbf{\Delta}_{\circlearrowleft}$ and

any stable ∞-category $\mathcal{C}$ (here $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^{\Lambda}, \mathcal{C})$ denotes the ∞-category of exact functors from Quiv^{Λ} to $\mathcal{C}$). Moreover, we show that each ∞-category Quiv^{Λ} can be realized as an ∞-category of equivariant matrix factorizations $\mathrm{MF}(\mathrm{Rep}^{\mathrm{fin}}(\Lambda))$. Using the S^1-equivariance of the functor MF, we deduce that the paracyclic Waldhausen construction

$$S_{\bullet}^{\circlearrowleft} : \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{S})$$

is S^1-equivariant, where the circle S^1 acts on the left hand side via the monoidal functor ϕ and on the right hand side via an action on $\mathbf{\Delta}_{\circlearrowleft}$. From this we will formally deduce that the K-theory functor $K : \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}} \to \mathcal{S}$ is also equivariant, where the circle S^1 acts trivially on $\mathcal{S}$.

The last three sections of this paper are devoted to the proof of (b). We first note that by passing to loop spaces, we can identify the monoidal functor $\phi : S^1 \to \mathrm{BPic}(S)$ with an $\mathbb{E}_2$-monoidal functor $\Phi : \mathbf{Z} \to \mathrm{Pic}(S)$. To describe this functor geometrically, it will be convenient to choose a particularly simple model for the notion of $\mathbb{E}_2$-algebra: we will think of Φ as supplying a construction which assigns an invertible spectrum to every pair (D, n), where $D \subseteq \mathbf{C}$ is an open disk and $n \in \mathbf{Z}$ is an integer (and exhibits appropriate functorial behavior with respect to embeddings of disks). In §5, we will show that when restricted to nonnegative integers n, the functor Φ can be described concretely by the construction $(D, n) \mapsto \Sigma^{\infty}(\mathrm{Sym}^n D)^c$: here $(\mathrm{Sym}^n D)^c$ denotes the one-point compactification of the nth symmetric power of D, which homeomorphic to a sphere of dimension $2n$. The proof involves Koszul duality for $\mathbb{E}_2$-algebras and an analysis of the skeletal filtration

$$\mathbf{CP}^0 \subseteq \mathbf{CP}^1 \subseteq \mathbf{CP}^2 \subseteq \cdots$$

of $\mathbf{CP}^{\infty} \simeq K(\mathbf{Z}, 2)$.

To complete the proof of (b), we wish to show that the $\mathbb{E}_2$-monoidal functor Φ described is homotopic to a composition

$$\mathbf{Z} \simeq \Omega^2(\mathrm{BU}(1)) \to \Omega^2(\mathrm{BU}) \xrightarrow{\beta} \mathbf{Z} \times \mathrm{BU} \xrightarrow{J_{\mathbf{C}}} \mathrm{Pic}(S),$$

as a map of $\mathbb{E}_2$ spaces. Recall that the complex J-homomorphism $J_{\mathbf{C}} : \mathbf{Z} \times \mathrm{BU} \to \mathrm{Pic}(S)$ arises concretely from the construction which carries a complex vector space V to the invertible spectrum $\Sigma^{\infty} V^c$. This bears a close resemblance to the geometric description of Φ given above: note that if $D \subseteq \mathbf{C}$ is an open disk, then the symmetric power $\mathrm{Sym}^n D$ does not canonically inherit the structure of a complex vector space, but it is a contractible complex manifold and is therefore *essentially* equivalent to its tangent space at any point (see §7.1 for a more precise statement). From this, it is not hard to see that Φ factors as a composition

$$\mathbf{Z} \xrightarrow{\Phi'} \mathbf{Z} \times \mathrm{BU} \xrightarrow{J_{\mathbf{C}}} \mathrm{Pic}(S).$$

The key point is to show that after delooping twice and applying Bott periodicity, the map Φ corresponds to the natural map $\mathrm{BU}(1) \to \mathrm{BU}$. In order to prove this, we will need to understand the Bott map β *as a map of $\mathbb{E}_2$-spaces*: in other words, we will need a description of β which relates the $\mathbb{E}_2$-structure on $\Omega^2(\mathrm{BU})$ (coming from its presentation as a 2-fold loop space) to the $\mathbb{E}_2$-structure on $\mathbf{Z} \times \mathrm{BU}$ (arising from direct sums of complex vector spaces). We will give such a description in §6, and apply it in §7 to complete the proof of (b).

1.3. **Further Motivation: Fukaya Categories of Surfaces.** Let Σ be a symplectic manifold. In favorable cases, one can associate to Σ an A_{∞}-category $\mathrm{Fuk}(\Sigma)$, called the *Fukaya category* of Σ. We will not attempt to give a definition of $\mathrm{Fuk}(\Sigma)$ here, except to recall that the definition requires the consideration of many subtle and difficult analytic questions. However, in the special case where $\dim(\Sigma) = 2$, Dyckerhoff-Kapranov ([4]) and Nadler ([8]) have introduced

purely "combinatorial" versions of the Fukaya category. Let us briefly describe the construction of [4] (using a slightly different language) and its relationship to the work described here.

(a) Let $\mathcal{C}$ be an ∞-category which admits small limits and let $\boldsymbol{\Delta}_{\mathrm{cyc}}$ denote the cyclic category (see Definition 4.2.13). A *cyclic object of* $\mathcal{C}$ is a functor $\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}}^{\mathrm{op}}) \to \mathcal{C}$. Dyckerhoff and Kapranov introduce a special class of cyclic objects of $\mathcal{C}$, which they call 2-*Segal cyclic objects*. Furthermore, they show that if Σ is a connected, oriented surface with nonempty boundary, then one can canonically associate to each 2-Segal cyclic object X of $\mathcal{C}$ an invariant RX_Σ, which they call the *derived membrane space* of Σ.

(b) Let $\mathbf{A}$ denote the category of 2-periodic differential graded categories over a field k. Using the theory of equivariant matrix factorizations, Dyckerhoff and Kapranov construct a cyclic object $\mathcal{E}^\bullet$ of $\mathbf{A}^{\mathrm{op}}$, which determines a 2-Segal simplicial object of the associated ∞-category.

(c) Let $\mathcal{A}$ be a (pretriangulated) 2-periodic differential graded category and let Σ be a surface with boundary together with a (sufficiently large) set of marked points $M \subseteq \Sigma$. The formation of derived mapping spaces then gives a cyclic object $\mathrm{Map}(\mathcal{E}^\bullet, \mathcal{A})$ in the underlying ∞-category of $\mathbf{A}$. Applying their derived membrane space construction, Dyckerhoff and Kapranov obtain 2-periodic differential graded category $R\,\mathrm{Map}(\mathcal{E}^\bullet, \mathcal{A})_\Sigma$ which they refer to as the *topological Fukaya category of* (Σ, M) *with coefficients in* $\mathcal{A}$.

In essence, our goal in this paper is to carry out step (b) in the case where the field k is replaced by a ring spectrum, and to understand the analogue of "2-periodicity" in this more general context. To understand the role of periodicity in the outline above, let us recall that in addition to the myriad analytic difficulties one encounters in defining the Fukaya category $\mathrm{Fuk}(\Sigma)$ due to potential non-smoothness and non-compactness of various moduli spaces of pseudo-holomorphic disks, one also encounters *topological* obstructions coming from the need to choose *compatible* orientations of those moduli spaces.

Let R be an $\mathbb{E}_2$-ring, and let $\mathrm{Pic}(R)$ denote a classifying for invertible R-module spectra; then $\mathrm{Pic}(R)$ inherits a monoidal structure and can be written as the loop space of a space $\mathrm{BPic}(R)$. This space carries an action of $\mathrm{BPic}(S)$; we will denote the homotopy quotient by $\mathrm{BPic}(R)/\mathrm{BPic}(S)$, so that there is a fiber sequence

$$\mathrm{BPic}(R) \to \mathrm{BPic}(R)/\mathrm{BPic}(S) \xrightarrow{\epsilon} \Omega^{-2}\,\mathrm{Pic}(S).$$

If Σ is a symplectic manifold of dimension $2d$, then to choose coherent R-orientations of all of the relevant moduli spaces involved in the definition of $\mathrm{Fuk}(\Sigma)$ one needs a dotted arrow so that the diagram

$$
\begin{array}{ccc}
\Sigma & \dashrightarrow & \mathrm{BPic}(R)/\mathrm{BPic}(S) \\
\downarrow & & \downarrow{\scriptstyle \epsilon} \\
\mathrm{BU}(d) \longrightarrow \mathrm{BU} \xrightarrow{\ \beta\ } \Omega^{-2}(\mathbf{Z} \times \mathrm{BU}) \xrightarrow{\ J_{\mathbf{C}}\ } & & \Omega^{-2}\,\mathrm{Pic}(S)
\end{array}
$$

commutes up to homotopy, where the left vertical map classifies the tangent bundle of Σ. In the special case $d = 1$, the composite map appearing in the lower part of this diagram is given by delooping the monoidal functor $\phi : S^1 \to \mathrm{BPic}(S)$ appearing in the statement of Theorem 1.1.1. Consequently, if we hope to be able to define the Fukaya category $\mathrm{Fuk}(\Sigma)$ "over R" for a general surface Σ, then we should demand that ϕ factors (as a monoidal functor) through the quotient $\mathrm{Pic}(R)/\mathrm{Pic}(S)$. There is a universal choice for such an R: it is given by applying the Thom construction to the $\mathbb{E}_2$-map

$$\mathbf{Z} \simeq \Omega^2\,\mathrm{BU}(1) \to \Omega^2\,\mathrm{BU} \xrightarrow{\ \beta\ } \mathrm{BU} \times \mathbf{Z}.$$

We will denote the resulting ring spectrum by S^{per}, and refer to it as the 2-*periodic sphere spectrum* (see Remark 3.5.13).

In §4.5, we will construct a family of stable ∞-categories $\text{Quiv}^{\bullet}$ which corepresent the Waldhausen $S_{\bullet}$-construction. These ∞-categories are closely related to the differential graded categories $\mathcal{E}^{\bullet}$ appearing in [4], with one important difference: rather than getting a cocyclic object of $\mathcal{C}\text{at}_{\infty}^{\text{St}}$, we only get a *coparacyclic* object of $\mathcal{C}\text{at}_{\infty}^{\text{St}}$. To descend $\text{Quiv}^{\bullet}$ to a cocyclic object of $\mathcal{C}\text{at}_{\infty}^{\text{St}}$, we would need it to be invariant under the circle group S^1 (which acts on the collection of paracyclic objects of any ∞-category $\mathcal{C}$). However, the main results of this paper assert that $\text{Quiv}^{\bullet}$ is *almost* invariant under the action of S^1: when regarded as a functor from $\text{N}(\boldsymbol{\Delta}_{\circlearrowleft})$ to $\mathcal{C}\text{at}_{\infty}^{\text{St}}$, it is *equivariant* if we allow S^1 to act on $\mathcal{C}\text{at}_{\infty}^{\text{St}}$ via the monoidal functor ϕ. It follows that $\text{Quiv}^{\bullet}$ descends to a cocyclic stable ∞-category after extension of scalars to the 2-periodic sphere spectrum S^{per}. The resulting cocyclic S^{per}-linear ∞-category can then be regarded as a refinement of the cocyclic differential graded category $\mathcal{E}^{\bullet}$ (one can obtain the latter from the former extending scalars to the field k). Moreover, it can employed for the same purpose: that is, the main results of this paper allow us extend the definition of the topological Fukaya categories of Dyckerhoff-Kapranov to allow coefficients in an arbitrary S^{per}-linear stable ∞-category (in the place of 2-periodic differential graded category).

1.4. **Acknowledgements.** This paper grew out of conversations with Tobias Dyckerhoff and David Nadler; I am grateful to both of them for sharing their ideas with me. The idea of realizing the ∞-categories Quiv^{Λ} using the formalism of equivariant matrix factorizations was suggested to me by Anatoly Preygel and David Nadler (and discovered independently by Dyckerhoff-Kapranov, in the setting of differential graded categories). I am also grateful to Søren Galatius for suggesting many helpful ways of thinking about Bott periodicity, some of which have made their way into §6. I would also like to thank the National Science Foundation for supporting this work under Grant No. 0906194.

1.5. **Notation and Terminology.** Throughout this paper, we will make extensive use of the theory of ∞-categories (also known as *quasicategories* and *weak Kan complexes* in the literature) as described in [6] and [7]. We will indicate references to [6] using the letters HTT and references to [7] using the letters HA. For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [6].

We let Sp denote the ∞-category of spectra. We will regard Sp as endowed with the smash product symmetric monoidal structure (see §H.4.8.2). We will indicate the smash product by

$$\wedge : \text{Sp} \times \text{Sp} \to \text{Sp}.$$

We let S denote the sphere spectrum (the unit object of Sp with respect to the smash product).

If $\mathcal{C}$ is an ∞-category containing a pair of objects X and Y, we let $\text{Map}_{\mathcal{C}}(X, Y)$ denote the space of maps from X to Y. If $\mathcal{C}$ is stable then we will regard $\mathcal{C}$ as enriched over the ∞-category of spectra and we denote the spectrum of maps from X to Y by $\underline{\text{Map}}_{\mathcal{C}}(X, Y)$. This spectrum can be described more concretely by the formula

$$\Omega^{\infty - n} \underline{\text{Map}}_{\mathcal{C}}(X, Y) = \text{Map}_{\mathcal{C}}(X, \Sigma^n Y).$$

If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, we let $\text{Fun}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$. If both $\mathcal{C}$ and $\mathcal{D}$ are stable, we let $\text{Fun}^{\text{ex}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\text{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the exact functors.

2. Categorical Background

2.1. **Monoidal ∞-Categories.** In this section, we will review some basic facts about the theory of monoidal ∞-categories and recall some notation which will be needed in the body of

this paper. For a more detailed discussion of the theory of monoidal ∞-categories, we refer the reader to [7].

Let $\mathcal{C}$ be a monoidal ∞-category. We let $\mathrm{Alg}(\mathcal{C})$ denote the ∞-category of associative algebra objects of $\mathcal{C}$. Given a pair of associative algebras $A, B \in \mathrm{Alg}(\mathcal{C})$, we let $\mathrm{LMod}_A(\mathcal{C})$, $\mathrm{RMod}_B(\mathcal{C})$, and ${}_A\mathrm{BMod}_B(\mathcal{C})$ denote the ∞-categories of left A-module, right B-module, and A-B-bimodule objects of $\mathcal{C}$, respectively. If the monoidal structure on $\mathcal{C}$ is symmetric and A is a commutative algebra object of $\mathcal{C}$, then the ∞-categories $\mathrm{LMod}_A(\mathcal{C})$ and $\mathrm{RMod}_A(\mathcal{C})$ are canonically equivalent to one another and we will denote either simply by $\mathrm{Mod}_A(\mathcal{C})$.

Let $\mathcal{C}\mathrm{at}_\infty$ denote the ∞-category of (small) ∞-categories. We will generally identify monoidal ∞-categories with associative algebra objects of $\mathcal{C}\mathrm{at}_\infty$. If $\mathcal{C}$ and $\mathcal{D}$ are monoidal ∞-categories, we let $\mathrm{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ denote the ∞-category of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$; the underlying Kan complex of $\mathrm{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ can be identified with the mapping space $\mathrm{Map}_{\mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty)}(\mathcal{C}, \mathcal{D})$.

Remark 2.1.1. Let $\mathcal{C}$ be an ∞-category. Suppose that $\mathcal{C}$ has the structure of a simplicial monoid: that is, a map of simplicial sets $m : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is unital and strictly associative. Then m exhibits $\mathcal{C}$ as a monoidal ∞-category. Moreover, every monoidal ∞-category is equivalent to one which arises in this way (Example H.4.1.4.7).

Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$. We will say that a morphism $f : C \to D$ in $\mathcal{C}$ is an L-*equivalence* if $L(f)$ is an equivalence in $\mathcal{C}_0$. We say that the functor L is *compatible with the monoidal structure on* $\mathcal{C}$ if, for every L-equivalence $f : C \to D$ and every object $E \in \mathcal{C}$, the induced maps

$$C \otimes E \to D \otimes E \qquad E \otimes C \to E \otimes D$$

are also L-equivalences. If this condition is satisfied, then there is an essentially unique monoidal structure on $\mathcal{C}_0$ for which the functor $L : \mathcal{C} \to \mathcal{C}_0$ is monoidal (see §H.2.2.1). The tensor product on $\mathcal{C}_0$ is then given by $(C, D) \mapsto L(C \otimes D)$.

Let $\mathcal{C}$ be a monoidal ∞-category with unit object $\mathbf{1}$. We will say that an object $C \in \mathcal{C}$ is *invertible* if there exists another object $C^{-1} \in \mathcal{C}$ and equivalences

$$C \otimes C^{-1} \simeq \mathbf{1} \simeq C^{-1} \otimes C.$$

In this case, we will refer to C^{-1} as the *inverse* of C; it is uniquely determined up to equivalence. We let $\mathcal{C}^{\mathrm{inv}}$ denote the subcategory of $\mathcal{C}$ spanned by the invertible objects and equivalences between them. The collection of invertible objects of $\mathcal{C}$ contains the unit object $\mathbf{1}$ and is closed under tensor products, so that $\mathcal{C}^{\mathrm{inv}}$ inherits the structure of a monoidal ∞-category.

Let $\mathcal{C}$ be a monoidal ∞-category equipped with a unit object $\mathbf{1}$. We say that an object $C \in \mathcal{C}$ is *right dualizable* if there exists another object $C^\vee \in \mathcal{C}$ together with morphisms

$$e : C^\vee \otimes C \to \mathbf{1} \qquad c : \mathbf{1} \to C \otimes C^\vee$$

such that the composite maps

$$C \simeq \mathbf{1} \otimes C \xrightarrow{c \otimes \mathrm{id}} C \otimes C^\vee \otimes C \xrightarrow{\mathrm{id} \otimes e} C \otimes \mathbf{1} \simeq C$$

$$C^\vee \simeq C^\vee \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes c} C^\vee \otimes C \otimes C^\vee \xrightarrow{e \otimes \mathrm{id}} \mathbf{1} \otimes C^\vee \simeq C^\vee$$

are homotopic to the identity. In this case, the object $C^\vee$ (and the morphisms e and c) are uniquely determined up to equivalence. We will refer to $C^\vee$ as the *right dual* of C, to the morphism e as the *evaluation map* of C, and to the morphism c as the *coevaluation map* of C. The collection of right-dualizable objects of $\mathcal{C}$ spans a full subcategory $\mathcal{C}^{\mathrm{dual}} \subseteq \mathcal{C}$ which contains the unit object $\mathbf{1}$ and is closed under the formation of tensor products, and therefore

inherits the structure of a monoidal ∞-category. Note that every invertible object $C \in \mathcal{C}$ is right dualizable (with right dual given by C^{-1}), so we have inclusions $\mathcal{C}^{\mathrm{inv}} \subseteq \mathcal{C}^{\mathrm{dual}} \subseteq \mathcal{C}$.

Let $\mathcal{C}$ be a monoidal ∞-category. An *action* of $\mathcal{C}$ on an ∞-category $\mathcal{M}$ is a monoidal functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{M}, \mathcal{M})$. Giving action of $\mathcal{C}$ on $\mathcal{M}$ is equivalent to exhibiting $\mathcal{M}$ as an ∞-category left-tensored over $\mathcal{C}$, or exhibiting $\mathcal{M}$ as a left $\mathcal{C}$-module in the ∞-category $\mathcal{C}\mathrm{at}_\infty$.

Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ and $\mathcal{N}$ be ∞-categories which are left-tensored over $\mathcal{C}$. We let $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ denote the ∞-category of $\mathcal{C}$-linear functors from $\mathcal{M}$ to $\mathcal{N}$. The underlying Kan complex $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})^{\simeq}$ can be identified with the mapping space

$$\mathrm{Map}_{\mathrm{LMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty)}(\mathcal{M}, \mathcal{N}).$$

Let $\mathcal{M}$ and $\mathcal{N}$ be ∞-categories acted on by a monoidal ∞-category $\mathcal{C}$, and let $F : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{C}$-linear functor. Suppose that the functor F admits a right adjoint G. Then G can be regarded as a *lax* $\mathcal{C}$-linear functor from $\mathcal{N}$ to $\mathcal{M}$: in particular, the functor G comes equipped with canonical maps

$$\phi_{C,N} : C \otimes G(N) \to G(C \otimes N)$$

for $C \in \mathcal{C}$, $N \in \mathcal{N}$. If each of the maps $\phi_{C,N}$ is an equivalence, then G can be regarded as a $\mathcal{C}$-linear functor from $\mathcal{N}$ to $\mathcal{M}$ (and the unit and counit of the adjunction between F and G are given by $\mathcal{C}$-linear natural transformations). This condition is automatically satisfied if the object C is right-dualizable: in this case, the map $\phi_{C,N}$ has a homotopy inverse given by the composition

$$
\begin{array}{rcl}
G(C \otimes N) & \simeq & \mathbf{1} \otimes G(C \otimes N) \\
 & \overset{c}{\to} & C \otimes C^{\vee} \otimes G(C \otimes N) \\
 & \overset{\phi_{C^{\vee}, C \otimes N}}{\to} & C \otimes G(C^{\vee} \otimes C \otimes N) \\
 & \overset{e}{\to} & C \otimes G(N).
\end{array}
$$

It follows that the right adjoint G can always be regarded as a $\mathcal{C}^{\mathrm{dual}}$-linear functor from $\mathcal{N}$ to $\mathcal{M}$.

Suppose that $\mathcal{C}$ is a monoidal ∞-category which admits geometric realizations for simplicial objects, and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. For every algebra object $A \in \mathrm{Alg}(\mathcal{C})$, the ∞-category $\mathrm{RMod}_A(\mathcal{C})$ is left-tensored over $\mathcal{C}$. Given another algebra object $B \in \mathrm{Alg}(\mathcal{C})$ and a bimodule $K \in {}_A\mathrm{BMod}_B(\mathcal{C})$, the construction $M \mapsto M \otimes_A K$ determines a $\mathcal{C}$-linear functor ρ_K from $\mathrm{RMod}_A(\mathcal{C})$ to $\mathrm{RMod}_B(\mathcal{C})$. It follows from Theorem H.4.8.4.1 that the construction $K \mapsto \rho_K$ induces a fully faithful embedding from ${}_A\mathrm{BMod}_B(\mathcal{C})$ to $\mathrm{Fun}_{\mathcal{C}}(\mathrm{RMod}_A(\mathcal{C}), \mathrm{RMod}_B(\mathcal{C}))$, whose essential image consists of those $\mathcal{C}$-linear functors which preserve geometric realizations of simplicial objects.

Remark 2.1.2. Let $\mathcal{C}$ be a monoidal ∞-category which admits geometric realizations of simplicial objects and for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let A and B be associative algebra objects of $\mathcal{C}$ and let $K \in {}_A\mathrm{BMod}_B(\mathcal{C})$. Suppose that the functor $\rho_K : \mathrm{RMod}_A(\mathcal{C}) \to \mathrm{RMod}_B(\mathcal{C})$ admits a right adjoint. In this case, we will denote that right adjoint by $N \mapsto \mathrm{Mor}_A(K, N)$. Since ρ_K is a $\mathcal{C}$-linear functor, the construction $N \mapsto \mathrm{Mor}_A(K, N)$ is a lax $\mathcal{C}$-linear functor, which is automatically $\mathcal{C}^{\mathrm{dual}}$-linear.

For the remainder of this section, let us fix a symmetric monoidal ∞-category $\mathcal{C}$. We let $\mathrm{Alg}(\mathcal{C})$ denote the ∞-category of associative algebra objects of $\mathcal{C}$ and $\mathrm{LMod}(\mathcal{C})$ the ∞-category of pairs (A, M), where $A \in \mathrm{Alg}(\mathcal{C})$ and M is a left A-module. Then $\mathrm{Alg}(\mathcal{C})$ and $\mathrm{LMod}(\mathcal{C})$ inherit the structure of symmetric monoidal ∞-categories. We let $\mathrm{Alg}^{(2)}(\mathcal{C}) \simeq \mathrm{Alg}(\mathrm{Alg}(\mathcal{C}))$ denote the ∞-category of $\mathbb{E}_2$-algebra objects of $\mathcal{C}$, and we let $\mathrm{LMod}^{(2)}(\mathcal{C}) = \mathrm{Alg}(\mathrm{LMod}(\mathcal{C}))$. We

will refer to $\mathrm{LMod}^{(2)}(\mathcal{C})$ as the ∞-*category of central actions in* $\mathcal{C}$. The objects of $\mathrm{LMod}^{(2)}(\mathcal{C})$ can identified with pairs (A, M) where A is an $\mathbb{E}_2$-algebra object of $\mathcal{C}$ and M is an A-algebra: that is, an associative algebra object of $\mathrm{LMod}_A(\mathcal{C})$. In this case, we will say that A *acts centrally on* M.

Remark 2.1.3. Let M be an associative algebra object of $\mathcal{C}$. We will say that a central action $(A, M) \in \mathrm{LMod}^{(2)}(\mathcal{C})$ *exhibits* A *as a center of* M if, for every $\mathbb{E}_2$-algebra B in $\mathcal{C}$, the canonical map

$$\mathrm{Map}_{\mathrm{Alg}^{(2)}(\mathcal{C})}(B, A) \to \mathrm{LMod}^{(2)}(\mathcal{C}) \times_{\mathrm{Alg}(\mathcal{C})} \{M\}$$

is a homotopy equivalence. If $\mathbf{1}$ denotes the unit object of $\mathcal{C}$, then $\mathbf{1}$ can be regarded either as an associative algebra object of $\mathcal{C}$ or as an $\mathbb{E}_2$-algebra object of $\mathcal{C}$, and the latter can be identified with the center of the former (more precisely, the unit object $(\mathbf{1}, \mathbf{1})$ of $\mathrm{LMod}^{(2)}(\mathcal{C})$ exhibits $\mathbf{1}$ as the center of itself; this follows immediately from Proposition H.5.3.1.8). In other words, if A is an $\mathbb{E}_2$-algebra object of $\mathcal{C}$, then the data of a central action of A on $\mathbf{1}$ is equivalent to the data of a morphism of $\mathbb{E}_2$-algebras $A \to \mathbf{1}$.

Remark 2.1.4. The ∞-category $\mathrm{LMod}^{(2)}(\mathcal{C})$ can be identified with $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for a suitable ∞-operad $\mathcal{O}$ (given by the operadic tensor product of the ∞-operad governing associative algebras with the ∞-operad governing left module objects; see §H.2.2.5). In the special case where $\mathcal{C} = \mathcal{C}\mathrm{at}_\infty$, we can identify $\mathrm{LMod}^{(2)}(\mathcal{C})$ with the ∞-category of $\mathcal{O}$-*monoidal* ∞-*categories*. The objects of $\mathrm{LMod}^{(2)}(\mathcal{C}\mathrm{at}_\infty)$ are pairs $(\mathcal{A}, \mathcal{M})$, where $\mathcal{M}$ is a monoidal ∞-category and $\mathcal{A}$ is an $\mathbb{E}_2$-monoidal ∞-category equipped with a monoidal functor $a : \mathcal{A} \to \mathcal{M}$ which is central in a suitable sense. If $\mathcal{M}_0 \subseteq \mathcal{M}$ is a full subcategory for which the inclusion $\mathcal{M}_0 \hookrightarrow \mathcal{M}$ admits a left adjoint $L : \mathcal{M} \to \mathcal{M}_0$ which is compatible with the symmetric monoidal structure on $\mathcal{M}$, then the results of §H.2.2.1 show that $(\mathcal{A}, \mathcal{M}_0)$ can be regarded as an object of $\mathrm{LMod}^{(2)}(\mathcal{C}\mathrm{at}_\infty)$ (and that L induces a map $(\mathcal{A}, \mathcal{M}) \to (\mathcal{A}, \mathcal{M}_0)$ in $\mathrm{LMod}^{(2)}(\mathcal{C}\mathrm{at}_\infty)$). In other words, any central action of $\mathcal{A}$ on $\mathcal{M}$ induces a central action of $\mathcal{A}$ on $\mathcal{M}_0$.

2.2. **Representations of ∞-Categories.** Let $\mathcal{C}$ be an essentially small ∞-category. A *representation* of $\mathcal{C}$ is a functor $\rho : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$, where Sp denotes the ∞-category of spectra. The collection of all representations of $\mathcal{C}$ can be organized into an ∞-category $\mathrm{Rep}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$. We will refer to $\mathrm{Rep}(\mathcal{C})$ as the ∞-*category of representations of* $\mathcal{C}$.

Note that $\mathrm{Rep}(\mathcal{C})$ can be identified with the stabilization of the ∞-category

$$\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

of space-valued presheaves on $\mathcal{C}$. Let $j : \mathcal{C} \to \mathrm{Rep}(\mathcal{C})$ denote the composition of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the infinite suspension functor $\Sigma_+^\infty : \mathcal{P}(\mathcal{C}) \to \mathrm{Rep}(\mathcal{C})$; concretely, the functor j is given by the formula

$$j(C)(D) = \Sigma_+^\infty \mathrm{Map}_{\mathcal{C}}(D, C).$$

We will refer to the functor j as the *stable Yoneda embedding*.

We let $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$ denote the smallest stable subcategory of $\mathrm{Rep}(\mathcal{C})$ which contains the essential image of the stable Yoneda embedding j. We will say that a representation of $\mathcal{C}$ is *finite* if it belongs to $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$.

Warning 2.2.1. The term "stable Yoneda embedding" is possibly misleading: the functor $j : \mathcal{C} \to \mathrm{Rep}(\mathcal{C})$ is never fully faithful (unless $\mathcal{C}$ is empty).

Remark 2.2.2. Let $\mathcal{C}$ be an essentially small ∞-category. Then the essential image of the Yoneda embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ consists of compact objects of $\mathcal{P}(\mathcal{C})$, and the infinite suspension functor $\Sigma_+^\infty : \mathcal{P}(\mathcal{C}) \to \mathrm{Rep}(\mathcal{C})$ preserves compact objects (since the right adjoint $\Omega^\infty : \mathrm{Rep}(\mathcal{C}) \to$

$\mathcal{P}(\mathcal{C})$ commutes with filtered colimits). Consequently, every object of $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$ is compact as an object of $\mathrm{Rep}(\mathcal{C})$.

Proposition 2.2.3. *Let $\mathcal{C}$ be an essentially small ∞-category. Then the inclusion $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}) \hookrightarrow \mathrm{Rep}(\mathcal{C})$ extends to an equivalence of ∞-categories $\mathrm{Ind}(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})) \simeq \mathrm{Rep}(\mathcal{C})$.*

Proof. Using Remark 2.2.2 and Proposition T.5.3.5.11, we see that the inclusion $f : \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}) \to \mathrm{Rep}(\mathcal{C})$ extends to a fully faithful embedding $F : \mathrm{Ind}(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})) \to \mathrm{Rep}(\mathcal{C})$ which preserves filtered colimits. Since f is right exact, the functor F preserves small colimits and therefore admits a right adjoint $G : \mathrm{Rep}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}))$ (Corollary T.5.5.2.9). To show that F is an equivalence, it will suffice to show that the functor G is conservative. Since G is an exact functor between stable ∞-categories, it will suffice to show that if V is a nonzero object of $\mathrm{Rep}(\mathcal{C})$, then $G(V) \not\simeq 0$. Replacing V by a suspension if necessary, we may assume that there exists an object $C \in \mathcal{C}$ such that $\pi_0 V(C) \neq 0$. Any nonzero element of $\pi_0 V(C)$ classifies a nonzero map $j(C) \to V$, where $j : \mathcal{C} \to \mathrm{Rep}(\mathcal{C})$ is the stable Yoneda embedding, so that $G(V) \neq 0$. □

Corollary 2.2.4. *Let $\mathcal{C}$ be an essentially small ∞-category. Then an object of $\mathrm{Rep}(\mathcal{C})$ is compact if and only if it is a retract of some object of $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$.*

In general, not every compact object of $\mathrm{Rep}(\mathcal{C})$ belongs to $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$. However, this is true if we assume that $\mathcal{C}$ has a particularly simple form.

Notation 2.2.5. Let Q be a partially ordered set. We let $\mathrm{Rep}(Q)$ denote the ∞-category $\mathrm{Rep}(\mathrm{N}(Q)) = \mathrm{Fun}(\mathrm{N}(Q)^{\mathrm{op}}, \mathrm{Sp})$. We will refer to the objects of $\mathrm{Rep}(Q)$ as *representations* of Q. For $V \in \mathrm{Rep}(Q)$, we will indicate the value of V on an element $\lambda \in Q$ by V_λ.

Proposition 2.2.6. *Let Q be a finite partially ordered set and let $V \in \mathrm{Rep}(Q)$. The following conditions are equivalent:*

(1) *The representation V belongs to $\mathrm{Rep}^{\mathrm{fin}}(Q)$.*
(2) *The representation V is a compact object of $\mathrm{Rep}(Q)$.*
(3) *For each object $\lambda \in Q$, the spectrum V_λ is finite.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Remark 2.2.2. We next prove that (2) $\Rightarrow$ (3). Let $\mathcal{D} \subseteq \mathrm{Rep}(Q)$ denote the full subcategory spanned by those objects V for which V_λ is finite for each $\lambda \in Q$. Note that the stable Yoneda embedding $j : \mathrm{N}(Q) \to \mathrm{Rep}(Q)$ is given by the formula

$$j(\lambda)_\mu = \begin{cases} S & \text{if } \mu \leq \lambda \\ 0 & \text{otherwise.} \end{cases}$$

From this, we immediately deduce that j carries $\mathrm{N}(Q)$ into $\mathcal{D}$. Since $\mathcal{D}$ is a stable subcategory of $\mathrm{Rep}(Q)$, we conclude that $\mathrm{Rep}^{\mathrm{fin}}(P) \subseteq \mathcal{D}$. Because $\mathcal{D}$ is closed under retracts, Corollary 2.2.4 implies that $\mathcal{D}$ contains all compact objects of $\mathrm{Rep}(P)$. This completes the proof that (2) $\Rightarrow$ (3).

We now show that (3) $\Rightarrow$ (1). For every downward-closed subset $P \subseteq Q$, consider the following assertion:

$(*_P)$ If $V \in \mathcal{D}$ satisfies $V_\lambda \simeq 0$ for $\lambda \notin P$, then $V \in \mathrm{Rep}^{\mathrm{fin}}(Q)$.

Note that the implication (3) $\Rightarrow$ (1) is equivalent to $(*_Q)$. We will show that $(*_P)$ holds for all downward-closed subsets $P \subseteq Q$ using induction on the cardinality of P. If P is empty, then there is nothing to prove. Otherwise, the set P contains a maximal element λ, so that $P' = P - \{\lambda\}$ is also closed downwards. Let $V \in \mathcal{D}$ satisfy $V_\lambda \simeq 0$ for $\lambda \notin P$. Let V' be

the representation given by $V'_\mu = \begin{cases} V_\lambda & \text{if } \mu \leq \lambda \\ 0 & \text{otherwise.} \end{cases}$, so that the identity map $\text{id} : V'_\lambda \to V_\lambda$ determines a fiber sequence

$$V' \to V \to V''$$

in the ∞-category $\mathcal{D}$. Then $V''_\mu \simeq 0$ for $\mu \notin P'$, so that $V'' \in \mathcal{D}$ by virtue of the inductive hypothesis. Since $\text{Rep}^{\text{fin}}(Q)$ is closed under extensions, we are reduced to proving that $V' \in \text{Rep}^{\text{fin}}(Q)$. This is clear, since V' belongs to the smallest stable subcategory of $\text{Rep}(Q)$ which contains $j(\lambda)$. $\qquad\square$

The ∞-category $\text{Rep}^{\text{fin}}(\mathcal{C})$ can be characterized by the following universal property:

Proposition 2.2.7. *Let $\mathcal{C}$ be an essentially small ∞-category. For any stable ∞-category $\mathcal{D}$, composition with the stable Yoneda embedding $j : \mathcal{C} \to \text{Rep}^{\text{fin}}(\mathcal{C})$ induces an equivalence of ∞-categories*

$$\text{Fun}^{\text{ex}}(\text{Rep}^{\text{fin}}(\mathcal{C}), \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Without loss of generality, we may assume that $\mathcal{D}$ is small. Let $\mathcal{E} = \text{Ind}(\mathcal{D})$, let $\text{Fun}'(\text{Rep}(\mathcal{C}), \mathcal{E})$ denote the full subcategory of $\text{Fun}(\text{Rep}(\mathcal{C}), \mathcal{E})$ spanned by those functors which preserve small colimits, and define $\text{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{E})$ similarly. We have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\text{Fun}'(\text{Rep}(\mathcal{C}), \mathcal{E}) & \longrightarrow & \text{Fun}'(\mathcal{P}(\mathcal{C}), \mathcal{E}) \\
\downarrow & & \downarrow \\
\text{Fun}^{\text{ex}}(\text{Rep}^{\text{fin}}(\mathcal{C}), \mathcal{E}) & \longrightarrow & \text{Fun}(\mathcal{C}, \mathcal{E}).
\end{array}
$$

The upper horizontal map is an equivalence by virtue of Proposition 2.2.3 and Corollary H.1.4.4.5, the left vertical map is an equivalence by virtue of Propositions T.5.3.5.10 and T.5.5.1.9, and the right vertical map is an equivalence by virtue of Theorem T.5.1.5.6. It follows that composition with j induces an equivalence $\text{Fun}^{\text{ex}}(\text{Rep}^{\text{fin}}(\mathcal{C}), \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})$. To complete the proof, it will suffice to show that if $g : \text{Rep}^{\text{fin}}(\mathcal{C}) \to \mathcal{E}$ is an exact functor and $g \circ f$ factors through the essential image of the Yoneda embedding $\iota : \mathcal{D} \hookrightarrow \mathcal{E}'$, then g factors through the essential image of ι. This is clear: because g is exact, the collection of objects

$$\{V \in \text{Rep}^{\text{fin}}(\mathcal{C}) : g(V) \text{ belongs to the essential image of } \iota\}$$

spans a stable subcategory of $\text{Rep}^{\text{fin}}(\mathcal{C})$ which contains $j(\mathcal{C})$ and is closed under equivalence, and therefore contains every object of $\text{Rep}^{\text{fin}}(\mathcal{C})$. $\qquad\square$

Corollary 2.2.8. *Let $\mathcal{C}$ be an essentially small ∞-category, let $\mathcal{D}$ be a stable ∞-category, and let $f : \mathcal{C} \to \mathcal{D}$ be a functor. The following conditions are equivalent:*

(a) For every stable ∞-category $\mathcal{E}$, composition with f induces an equivalence

$$\text{Fun}^{\text{ex}}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E}).$$

(b) The functor f factors as a composition

$$\mathcal{C} \xrightarrow{j} \text{Rep}^{\text{fin}}(\mathcal{C}) \xrightarrow{f'} \mathcal{D},$$

where j denotes the stable Yoneda embedding and f' is an equivalence of ∞-categories.

(c) The functor f satisfies the following conditions:

(i) *For every pair of objects $C, C' \in \mathcal{C}$, the canonical map*

$$\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Map}_{\mathcal{D}}(f(C), f(C')) \simeq \Omega^{\infty}\underline{\mathrm{Map}}_{\mathcal{D}}(f(C), f(C'))$$

induces a homotopy equivalence of spectra

$$\Sigma^{\infty}_{+} \mathrm{Map}_{\mathcal{C}}(C, C') \to \underline{\mathrm{Map}}_{\mathcal{D}}(f(C), f(C')).$$

(ii) *The essential image of f generates $\mathcal{D}$ as a stable ∞-category. In other words, if $\mathcal{D}_0 \subseteq \mathcal{D}$ is a stable subcategory which is closed under equivalence and contains the essential image of f, then $\mathcal{D}_0 = \mathcal{D}$.*

Proof. Note that Proposition 2.2.7 implies that f admits an essentially unique factorization

$$\mathcal{C} \xrightarrow{j} \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}) \xrightarrow{f'} \mathcal{D},$$

where f' is exact. For any stable ∞-category $\mathcal{E}$, Proposition 2.2.7 allows us to identify the forgetful functor $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ with the map $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}), \mathcal{E})$ given by composition with f', from which we immediately see that (a) and (b) are equivalent. The implication (b) $\Rightarrow$ (c) follows from an elementary calculation. We now prove that (c) $\Rightarrow$ (b).

Assume first that f satisfies condition (i). For every pair of objects $V, W \in \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$, the exact functor f' induces a map of spectra

$$\theta_{V,W} : \underline{\mathrm{Map}}_{\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})}(V, W) \to \underline{\mathrm{Map}}_{\mathcal{D}}(f'(V), f'(W)).$$

We will prove that $\theta_{V,W}$ is a homotopy equivalence of spectra for each pair of objects $V, W \in \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$. Let us first regard W as fixed. The collection of those objects $V \in \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$ for which $\theta_{V,W}$ is an equivalence is a stable subcategory of $\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$. It will therefore suffice to show that $\theta_{V,W}$ is an equivalence when V has the form $j(C)$ for some $C \in \mathcal{C}$. Similarly, we may assume that $W = j(C')$ for some $C' \in \mathcal{C}$. In this case, the desired result follows immediately from assumption (i). Applying the functor Ω^{∞}, we deduce that f' induces a homotopy equivalence of spaces

$$\mathrm{Map}_{\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})}(V, W) \to \mathrm{Map}_{\mathcal{D}}(f'(V), f'(W))$$

for $V, W \in \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$, so that f' is fully faithful. In this case, the essential image of f' is a stable subcategory of $\mathcal{D}$ which contains the essential image of f and is closed under equivalence. If f satisfies condition (ii), then f' is essentially surjective and therefore an equivalence of ∞-categories. $\qquad\square$

2.3. **Tensor Products of Stable ∞-Categories.** Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be a finite collection of stable ∞-categories and let $\mathcal{D}$ be a stable ∞-category. We will say that a functor

$$f : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$$

exhibits $\mathcal{D}$ as a tensor product of the stable ∞-categories $\{\mathcal{C}_i\}_{1 \leq i \leq n}$ if it satisfies the following condition:

($*$) For every stable ∞-category $\mathcal{E}$, composition with f induces a fully faithful embedding

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{E}),$$

whose essential image is spanned by those functors which are exact separately in each variable (in particular, the functor f itself is exact separately in each variable).

Remark 2.3.1. If f satisfies condition ($*$), then the ∞-category $\mathcal{D}$ (and the functor f) are determined uniquely up to equivalence by the tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$. We will typically denote $\mathcal{D}$ by $\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_n$, and refer to it as the tensor product of the ∞-categories $\{\mathcal{C}_i\}_{1 \leq i \leq n}$.

Proposition 2.3.2. *Let* $\mathcal{C}_1, \ldots, \mathcal{C}_n$ *be a finite collection of stable* ∞*-categories. Then there exists a stable* ∞*-category* $\mathcal{D}$ *and a functor* $f : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ *which exhibits* $\mathcal{D}$ *as a tensor product of the stable* ∞*-categories* $\{\mathcal{C}_i\}_{1 \le i \le n}$*. Moreover, if each* $\mathcal{C}_i$ *is essentially small, then* $\mathcal{D}$ *is essentially small.*

The proof of Proposition 2.3.2 will require a few general remarks.

Notation 2.3.3. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are colimit-preserving functors. We will regard $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ as endowed with the symmetric monoidal structure described in §H.4.8.1. Given a pair of presentable ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we denote their tensor product by $\mathcal{C} \widehat{\otimes} \mathcal{D}$. It is characterized by the existence of a bifunctor $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \widehat{\otimes} \mathcal{D}$ with the following universal property: for any ∞-category $\mathcal{E}$ which admits small colimits, composition with the functor $\otimes$ induces an equivalence from the full subcategory of $\mathrm{Fun}(\mathcal{C} \widehat{\otimes} \mathcal{D}, \mathcal{E})$ spanned by those functors which preserve small colimits to the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by those functors which preserve small colimits separately in each variable.

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the presentable stable ∞-categories. Then $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ inherits a symmetric monoidal structure from the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see §H.4.8.2), with the same tensor product (but a different unit). Moreover, the stabilization functor $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ determines a symmetric monoidal functor from $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ to $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, which is left adjoint to the lax symmetric monoidal inclusion $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Proof of Proposition 2.3.2. Without loss of generality we may assume that each $\mathcal{C}_i$ is small, so that $\mathrm{Ind}(\mathcal{C}_i)$ is a presentable stable ∞-category. Let us abuse notation by identifying each $\mathcal{C}_i$ with its essential image in $\mathrm{Ind}(\mathcal{C}_i)$. Let $\widehat{\mathcal{D}}$ denote the tensor product

$$\mathrm{Ind}(\mathcal{C}_1) \widehat{\otimes} \cdots \widehat{\otimes} \mathrm{Ind}(\mathcal{C}_n).$$

(here the tensor product is formed in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$, though it agrees with the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ for $n > 0$). Let $\mathcal{D}$ denote the smallest stable subcategory of $\widehat{\mathcal{D}}$ which is closed under equivalence and contains the essential image of the composite map

$$f : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \hookrightarrow \mathrm{Ind}(\mathcal{C}_1) \times \cdots \times \mathrm{Ind}(\mathcal{C}_n) \overset{\otimes}{\to} \widehat{\mathcal{D}}.$$

It follows immediately from the construction that $\mathcal{D}$ is essentially small and that the functor $f : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ is exact separately in each variable.

To complete the proof, we must show that for every stable ∞-category $\mathcal{E}$, composition with f induces a fully faithful embedding

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{E}),$$

whose essential image is spanned by those functors which are separately exact in each variable. In the case $n = 0$, this follows immediately from Proposition 2.2.7. We will therefore assume that $n > 0$. In this case, Lemma H.5.3.2.11 implies that objects of the form $\{C_1 \otimes \cdots \otimes C_n\}_{C_i \in \mathcal{C}_i}$ form a set of compact generators for $\widehat{\mathcal{D}}$. It follows that $\mathcal{D}$ is comprised of compact objects of $\widehat{\mathcal{D}}$, and that the induced map $\mathrm{Ind}(\mathcal{D}) \hookrightarrow \widehat{\mathcal{D}}$ is an equivalence of ∞-categories.

Without loss of generality, we may assume that $\mathcal{E}$ is essentially small. Let $\mathcal{E}' = \mathrm{Ind}(\mathcal{E})$, and let us abuse notation by identifying $\mathcal{E}$ with its essential image in $\mathcal{E}'$. Let $\mathrm{Fun}'(\widehat{\mathcal{D}}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}(\widehat{\mathcal{D}}, \mathcal{E}')$ spanned by those functors which preserves small colimits, let $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}_1) \times \cdots \times \mathrm{Ind}(\mathcal{C}_n), \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}_1) \times \cdots \times \mathrm{Ind}(\mathcal{C}_n), \mathcal{E}')$ spanned by those functors which preserve small colimits separately in each variable, and let $\mathrm{Fun}'(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{E}')$ spanned by those functors

which are exact separately in each variable. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Fun}'(\widehat{\mathcal{D}}, \mathcal{E}') & \longrightarrow & \mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}_1) \times \cdots \times \mathrm{Ind}(\mathcal{C}_n), \mathcal{E}') \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E}') & \longrightarrow & \mathrm{Fun}'(\mathcal{C}_1 \times \cdots \mathcal{C}_n, \mathcal{E}') \end{array}$$

where the vertical maps and the upper horizontal map are equivalences of ∞-categories. It follows that the upper horizontal map is also an equivalence of ∞-categories. To complete the proof, it will suffice to show that an exact functor $g : \mathcal{D} \to \mathcal{E}'$ factors through the stable subcategory $\mathcal{E} \subseteq \mathcal{E}'$ if and only if the composite functor

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}'$$

factors through $\mathcal{E}$. This is clear, since $g^{-1}(\mathcal{E})$ is a stable subcategory of $\mathcal{D}$ which is closed under equivalence and contains the essential image of f. $\qquad\square$

Remark 2.3.4. Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be essentially small stable ∞-categories, let $\mathcal{D}$ be an essentially small stable ∞-category, and let

$$f : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$$

be a functor which is exact separately in each variable. The proof of Proposition 2.3.2 shows that f exhibits $\mathcal{D}$ as a tensor product of the stable ∞-categories $\{\mathcal{C}_i\}_{1 \leq i \leq n}$ if and only if the following pair of conditions is satisfied:

(a) The ∞-category $\mathcal{D}$ is generated (as a stable ∞-category) by the essential image of f (in other words, if $\mathcal{D}' \subseteq \mathcal{D}$ is a stable subcategory which contains the essential image of f and is closed under equivalence, then $\mathcal{D}' = \mathcal{D}$).

(b) The functor f induces an equivalence of presentable stable ∞-categories

$$\mathrm{Ind}(\mathcal{C}_1) \widehat{\otimes} \cdots \widehat{\otimes} \mathrm{Ind}(\mathcal{C}_n) \to \mathrm{Ind}(\mathcal{D}).$$

Remark 2.3.5. Let $\{\mathcal{D}\}_{1 \leq j \leq n}$ be a finite collection of stable ∞-categories. Suppose that, for each $1 \leq j \leq n$, we are given a functor

$$f_j : \prod_{1 \leq i \leq m_j} \mathcal{C}_{i,j} \to \mathcal{D}_j$$

which exhibits $\mathcal{D}_j$ as the tensor product of a collection of stable ∞-categories $\{\mathcal{C}_{i,j}\}_{1 \leq i \leq m_j}$. Let $g : \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \to \mathcal{E}$ be a functor which exhibits $\mathcal{E}$ as the tensor product of the $\mathcal{D}_j$. Using the criterion of Remark 2.3.4, we see that the composite functor

$$\prod_{i,j} \mathcal{C}_{i,j} \xrightarrow{\{f_j\}} \prod_j \mathcal{D}_j \xrightarrow{g} \mathcal{E}$$

exhibits $\mathcal{E}$ as a tensor product of the collection of stable ∞-categories $\{\mathcal{C}_{i,j}\}$.

Construction 2.3.6. Let $\mathcal{C}\mathrm{at}_\infty$ denote the ∞-category of small ∞-categories. We will regard $\mathcal{C}\mathrm{at}_\infty$ as endowed with the symmetric monoidal structure given by the Cartesian product of ∞-categories. Let $\mathcal{C}\mathrm{at}_\infty^\times$ denote the associated ∞-operad: the objects of $\mathcal{C}\mathrm{at}_\infty^\times$ are finite sequences $\{\mathcal{C}_i\}_{1 \leq i \leq m}$ of small ∞-categories, and a morphism from $\{\mathcal{C}_i\}_{1 \leq i \leq m}$ to $\{\mathcal{D}_j\}_{1 \leq j \leq n}$ in $\mathcal{C}\mathrm{at}_\infty^\times$ is a map of finite pointed sets $\alpha : \{1, \ldots, m, *\} \to \{1, \ldots, n, *\}$ together with a collection of functors

$$f_j : \prod_{\alpha(i)=j} \mathcal{C}_i \to \mathcal{D}_j.$$

We let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St} \otimes}$ denote the subcategory of $\mathcal{C}\mathrm{at}_\infty^\times$ whose objects are sequences $\{\mathcal{C}_i\}$ where each $\mathcal{C}_i$ is stable, and whose morphisms are pairs $(\alpha, \{f_j\})$ where each of the functors f_j is exact separately

in each variable. Then $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}\,\otimes}$ is an ∞-operad whose underlying ∞-category is the subcategory $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \subseteq \mathcal{C}\mathrm{at}_\infty$ spanned by the stable ∞-categories and exact functors. It follows from Proposition 2.3.2 and Remark 2.3.5 that the ∞-operad $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}\,\otimes}$ determines a symmetric monoidal structure on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$. In other words, the forgetful functor q from the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}\,\otimes}$ to the ordinary category of finite pointed sets is a coCartesian fibration (Proposition 2.3.2 implies that q is a locally coCartesian fibration, and Remark 2.3.5 implies that the collection of q-coCartesian morphisms is closed under composition). The underlying tensor product functor

$$\otimes : \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$$

is the tensor product of stable ∞-categories defined above.

By construction, the inclusion $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ is a lax symmetric monoidal functor (where we regard $\mathcal{C}\mathrm{at}_\infty$ as a symmetric monoidal ∞-category via the Cartesian product).

Proposition 2.3.7. *The construction* $\mathcal{C} \mapsto \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C})$ *determines a symmetric monoidal functor* $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$, *whose right adjoint agrees with the inclusion* $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ *(as a lax symmetric monoidal functor).*

Proof. It will suffice to show that the inclusion $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}\,\otimes} \hookrightarrow \mathcal{C}\mathrm{at}_\infty^\times$ admits a left adjoint relative to the category of finite pointed sets (see §H.7.3.2). We claim that such a left adjoint exists and is given on the level of objects by the formula

$$(\mathcal{C}_1, \ldots, \mathcal{C}_n) \mapsto (\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_1), \ldots, \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_n)).$$

To prove this, it will suffice to show that for every collection of small ∞-categories $\{\mathcal{C}_i\}_{1 \le i \le n}$ and every small stable ∞-category $\mathcal{D}$, if $\mathrm{Fun}'(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_1) \times \cdots \times \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_n), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_1) \times \cdots \times \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_n), \mathcal{D})$ spanned by those functors which are exact separately in each variable, then composition with the stable Yoneda embeddings $\mathcal{C}_i \to \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_i)$ induces an equivalence of ∞-categories

$$\mathrm{Fun}'(\mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_1) \times \cdots \times \mathrm{Rep}^{\mathrm{fin}}(\mathcal{C}_n), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D}).$$

This follows from Proposition 2.2.7. $\qquad\square$

Corollary 2.3.8. *The construction* $\mathcal{C} \mapsto \mathrm{Rep}(\mathcal{C})$ *determines a symmetric monoidal functor* $\mathcal{C}\mathrm{at}_\infty \to \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{L}}$.

Proof. Combine Proposition 2.3.7 with Remark 2.3.4. Alternatively, one can argue that the construction $\mathcal{C} \mapsto \mathrm{Rep}(\mathcal{C})$ is given by composing the symmetric monoidal functor

$$\mathcal{C}\mathrm{at}_\infty \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$$
$$\mathcal{C} \mapsto \mathcal{P}(\mathcal{C})$$

(see Corollary H.4.8.1.12) with the stabilization functor $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$. $\qquad\square$

Corollary 2.3.9. *Let* $\mathcal{C}$ *be an essentially small (symmetric) monoidal* ∞-*category. Then the* ∞-*category* $\mathrm{Rep}(\mathcal{C})$ *inherits the structure of a (symmetric) monoidal* ∞-*category.*

Remark 2.3.10. In the situation of Corollary 2.3.9, we will indicate the (symmetric) monoidal structure on $\mathrm{Rep}(\mathcal{C})$ by

$$\star : \mathrm{Rep}(\mathcal{C}) \times \mathrm{Rep}(\mathcal{C}) \to \mathrm{Rep}(\mathcal{C}).$$

We will refer to $\star$ as the *Day convolution product*. It is characterized up to equivalence by the following:

(a) The convolution product $\star$ preserves small colimits separately in each variable.
(b) The stable Yoneda embedding $j : \mathcal{C} \to \mathrm{Rep}(\mathcal{C})$ is a (symmetric) monoidal functor.

Concretely, the Day convolution product is given by the formula

$$(V \star W)_C = \varinjlim_{C \to C' \otimes C''} V_{C'} \wedge W_{C''},$$

where $\wedge$ indicates the smash product of spectra.

Note that the full subcategory $\operatorname{Rep}^{\mathrm{fin}}(\mathcal{C}) \subseteq \operatorname{Rep}(\mathcal{C})$ contains the unit object and is closed under the convolution product, and therefore inherits the structure of a (symmetric) monoidal ∞-category.

Variant 2.3.11. Let $\mathcal{C}$ be a monoidal ∞-category and let $\mathcal{M}$ be an ∞-category which is left tensored over $\mathcal{C}$. Then $\operatorname{Rep}(\mathcal{M})$ inherits the structure of an ∞-category left-tensored over $\operatorname{Rep}(\mathcal{C})$. We will indicate the action of $\operatorname{Rep}(\mathcal{C})$ on $\operatorname{Rep}(\mathcal{M})$ by

$$\star : \operatorname{Rep}(\mathcal{C}) \times \operatorname{Rep}(\mathcal{M}) \to \operatorname{Rep}(\mathcal{M}).$$

Concretely, this convolution product is given by

$$(V \star W)_M = \varinjlim_{M \to C \otimes N} V_C \wedge W_N.$$

2.4. **Graded Spectra.** Let A be an arbitrary set. We let A^{ds} denote the constant simplicial set associated to A (so that the n-simplices of A^{ds} can be identified with A for every integer $n \geq 0$). Note that there is a canonical isomorphism of simplicial sets

$$A^{\mathrm{ds}} \simeq (A^{\mathrm{ds}})^{\mathrm{op}}.$$

We will regard A^{ds} as an ∞-category and we let

$$\operatorname{Rep}(A^{\mathrm{ds}}) = \operatorname{Fun}((A^{\mathrm{ds}})^{\mathrm{op}}, \operatorname{Sp}) \simeq \operatorname{Fun}(A^{\mathrm{ds}}, \operatorname{Sp}) \simeq \prod_{\alpha \in A} \operatorname{Sp}$$

denote the ∞-category of representations of A^{ds}. We will refer to the objects of $\operatorname{Rep}(A^{\mathrm{ds}})$ as *A-graded spectra* and to $\operatorname{Rep}(A^{\mathrm{ds}})$ as *the ∞-category of A-graded spectra*. If X is an A-graded spectrum and $\alpha \in A$, we will denote the value of X on α by X_α. In the special case $A = \mathbf{Z}$, we will simply refer to the objects of $\operatorname{Rep}(\mathbf{Z}^{\mathrm{ds}})$ as *graded spectra* and to $\operatorname{Rep}(\mathbf{Z}^{\mathrm{ds}})$ as *the ∞-category of graded spectra*.

Remark 2.4.1. Let A be a set. An A-graded spectrum $X = \{X_\alpha\}_{\alpha \in A}$ belongs to the full subcategory $\operatorname{Rep}^{\mathrm{fin}}(A^{\mathrm{ds}}) \subseteq \operatorname{Rep}(A^{\mathrm{ds}})$ if and only if it satisfies the following pair of conditions:

   (*a*) For each index $\alpha \in A$, the spectrum X_α is finite.
   (*b*) For all but finitely many indices $\alpha \in A$, the spectrum X_α vanishes.

Equivalently, an A-graded spectrum X is finite if and only if the sum $\bigoplus_{\alpha \in A} X_\alpha$ is a finite spectrum.

Let G be a group. Then the multiplication on G endows G^{ds} with the structure of a monoidal ∞-category, so that $\operatorname{Rep}(G^{\mathrm{ds}})$ inherits the structure of a monoidal ∞-category (Corollary 2.3.9). We will indicate the monoidal structure on $\operatorname{Rep}(G^{\mathrm{ds}})$ by

$$\otimes : \operatorname{Rep}(G^{\mathrm{ds}}) \times \operatorname{Rep}(G^{\mathrm{ds}}) \to \operatorname{Rep}(G^{\mathrm{ds}}).$$

Concretely, it is given by the formula

$$(X \otimes Y)_g = \bigoplus_{g = g' g''} X_{g'} \wedge Y_{g''}.$$

Note that the ∞-category $\operatorname{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})$ of finite G-graded spectra contains the unit object of $\operatorname{Rep}(G^{\mathrm{ds}})$ and is closed under tensor products, and therefore inherits the structure of a monoidal

∞-category. If the group G is abelian, then $\mathrm{Rep}(G^{\mathrm{ds}})$ and $\mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})$ can be regarded as symmetric monoidal ∞-categories.

The construction $G \mapsto \mathrm{Rep}(G^{\mathrm{ds}})$ depends functorially on G. There are two special cases of this observation which will be of interest to us:

- For any group G, the inclusion of the identity element $\{e\} \hookrightarrow G$ induces monoidal functors

$$\rho : \mathrm{Sp} \simeq \mathrm{Rep}(\{e\}) \to \mathrm{Rep}(G^{\mathrm{ds}}) \qquad \rho^{\mathrm{fin}} : \mathrm{Sp}^{\mathrm{fin}} \simeq \mathrm{Rep}^{\mathrm{fin}}(\{e\}) \to \mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}}),$$

given concretely by

$$\rho^{\mathrm{fin}}(X)_g = \rho(X)_g = \begin{cases} X & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

These functors are fully faithful and are symmetric monoidal when G is abelian. We will generally abuse notation by identifying each spectrum X with its image $\rho(X)$ in $\mathrm{Rep}(G^{\mathrm{ds}})$.

- For any group G, the projection map $G \to \{e\}$ induces monoidal functors

$$\mathrm{Und} : \mathrm{Rep}(G^{\mathrm{ds}}) \to \mathrm{Rep}(\{e\}) \simeq \mathrm{Sp} \qquad \mathrm{Und}^{\mathrm{fin}} : \mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}}) \to \mathrm{Rep}^{\mathrm{fin}}(\{e\}) \simeq \mathrm{Sp}^{\mathrm{fin}},$$

given concretely by $\mathrm{Und}^{\mathrm{fin}}(X) = \mathrm{Und}(X) = \bigoplus_{g \in G} X_g$. We will refer to $\mathrm{Und}(X)$ as the *underlying spectrum* of a G-graded spectrum X. If G is abelian, then Und and $\mathrm{Und}^{\mathrm{fin}}$ can be regarded as symmetric monoidal functors.

Definition 2.4.2. Let $\mathcal{C}$ be a stable ∞-category. A *local grading* of $\mathcal{C}$ is an equivalence from $\mathcal{C}$ to itself. We will use the term *locally graded ∞-category* to refer to a pair $(\mathcal{C}, T)$, where T is a local grading of $\mathcal{C}$.

We will prove the following result at the end of this section:

Proposition 2.4.3. *Let $\mathcal{C}$ be a monoidal ∞-category. Then the evaluation functor $F \mapsto F(1)$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\otimes}(\mathbf{Z}^{\mathrm{ds}}, \mathcal{C}) \simeq \mathcal{C}^{\mathrm{inv}}.$$

Corollary 2.4.4. *Let $\mathcal{C}$ be a stable ∞-category. Then the following types of data are equivalent:*

- *Equivalences $T : \mathcal{C} \to \mathcal{C}$.*
- *Monoidal functors $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$.*
- *(Left or right) actions of $\mathbf{Z}^{\mathrm{ds}}$ on the ∞-category $\mathcal{C}$.*
- *Monoidal functors $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C})$.*
- *Exact monoidal functors $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C})$.*
- *(Left or right) actions of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ on $\mathcal{C}$ for which the action map $\mathcal{C} \times \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}) \to \mathcal{C}$ is exact in each variable.*

Remark 2.4.5. In what follows, we will refer to any of the types of data described in Corollary 2.4.4 as a *local grading* of $\mathcal{C}$.

Notation 2.4.6. Let $\mathcal{C}$ be a stable ∞-category equipped with a local grading, given by a monoidal functor $\alpha : \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$. For each integer n and each object $C \in \mathcal{C}$, we let $C(n)$ denote the image of C under the functor $\alpha(-n) \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

Remark 2.4.7. Let $\mathcal{C}$ be a stable ∞-category. Heuristically, we should think that a local grading of $\mathcal{C}$ is a structure which allows us to view each object $C \in \mathcal{C}$ as equipped with an "internal grading" of some sort. For each integer $n \in \mathbf{Z}$, we should imagine that the object $C(n) \in \mathcal{C}$ is obtained by from C by "shifting the grading by n".

Example 2.4.8. The monoidal structure on $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ determines a left action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ on $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$, which we can identify with a local grading of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. If X is a graded spectrum and n is an integer, then the graded spectrum $X(n)$ is given by the formula $X(n)_m = X_{m+n}$.

Remark 2.4.9. Let $\mathcal{C}$ be a (locally small) stable ∞-category equipped with a local grading. Then the action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ allows us to view $\mathcal{C}$ as enriched over the ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ of graded spectra. Concretely, if C and D are objects of $\mathcal{C}$, then there is a graded spectrum of maps from C to D given by the prescription

$$n \mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(C, D(n)).$$

Notation 2.4.10. The collection of all locally graded stable ∞-categories can itself be organized into an ∞-category $\mathcal{C}\mathrm{at}^{\mathrm{gd}}_\infty$, which we define as the ∞-category $\mathrm{Mod}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty)$ of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$-module objects of $\mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty$. The symmetric monoidal structure on $\mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty$ induces a symmetric monoidal structure on $\mathcal{C}\mathrm{at}^{\mathrm{gd}}_\infty$, whose tensor product is given by

$$(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})} \mathcal{D}.$$

We now supply some examples of locally graded stable ∞-categories.

Definition 2.4.11. Let G be a group. A *G-torsor* is a set X equipped with a right action of G for which there exists a G-equivariant isomorphism $G \simeq X$ (equivalently, the action of G on X is free and has only one orbit). The collection of all G-torsors forms a groupoid; we will denote the nerve of this groupoid by $\mathrm{Tors}(G)$.

Remark 2.4.12. Let G be a group and let $\mathrm{Tors}_0(G)$ denote the full subcategory of $\mathrm{Tors}(G)$ spanned by G (regarded as a torsor via the right translation action of G on itself). Note that the nerve of $\mathrm{Tors}_0(G)$ is isomorphic (as a simplicial set) to the classifying space BG and that the inclusion $\mathrm{Tors}_0(G) \hookrightarrow \mathrm{Tors}(G)$ is a homotopy equivalence of Kan complexes. In particular, the Kan complex $\mathrm{Tors}(\mathbf{Z})$ is homotopy equivalent to (the singular simplicial set of) the circle S^1.

Construction 2.4.13. Let G be a group and let T be a G-torsor. The right action of G on T induces a right action of the ∞-category $\mathrm{Rep}(G^{\mathrm{ds}})$ on $\mathrm{Rep}(T^{\mathrm{ds}})$. Concretely, this action supplies a functor

$$\otimes : \mathrm{Rep}(T^{\mathrm{ds}}) \times \mathrm{Rep}(G^{\mathrm{ds}}) \to \mathrm{Rep}(T^{\mathrm{ds}})$$

$$(X \otimes Y)_t = \bigoplus_{g \in G}(X_{tg^{-1}} \wedge Y_g).$$

Note that the induced action of $\mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})$ on $\mathrm{Rep}(T^{\mathrm{ds}})$ restricts to an action of $\mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})$ on $\mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}})$. The construction $T \mapsto \mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}})$ determines a functor

$$\mathrm{Tors}(G) \to \mathrm{RMod}_{\mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty).$$

Construction 2.4.14. Let G be an abelian group and suppose that we are given a pair of G-torsors T and T'. We let $T +_G T'$ denote the quotient of $T \times T'$ by the equivalence relation given by

$$(s, s') \simeq (t, t') \text{ if } (\exists g \in G)[s = tg, s' = t'g^{-1}].$$

There are evident commutativity and associativity isomorphisms

$$T +_G T' \simeq T' +_G T \qquad (T +_G T') +_G T'' \simeq T +_G (T' +_G T'')$$

which endow $\mathrm{Tors}(G)$ with the structure of a symmetric monoidal category.

For $T, T' \in \mathrm{Tors}(G)$, we can identify $(T +_G T')^{\mathrm{ds}}$ with the relative tensor product of T^{ds} with T'^{ds} over G^{ds} (in the ∞-category $\mathcal{C}\mathrm{at}_\infty$). Applying Proposition 2.3.7, we obtain canonical equivalences

$$\mathrm{Rep}^{\mathrm{fin}}((T +_G T')^{\mathrm{ds}}) \simeq \mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}}) \otimes_{\mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})} \mathrm{Rep}^{\mathrm{fin}}(T'^{\mathrm{ds}}).$$

More precisely, we can view the construction $T \mapsto \mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}})$ as a symmetric monoidal functor from $\mathrm{Tors}(G)$ into $\mathrm{Mod}_{\mathrm{Rep}^{\mathrm{fin}}(G^{\mathrm{ds}})}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$.

Specializing to the case $G = \mathbf{Z}$, we obtain a symmetric monoidal functor

$$\mu : \mathrm{Tors}(\mathbf{Z}) \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}.$$

Remark 2.4.15. Let $\mathcal{C}$ be an associative algebra object of the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ of graded stable ∞-categories: that is, a stable monoidal ∞-category for which the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in each variable which is equipped with a central action of $\mathbf{Z}$ (or equivalently an exact central action of the symmetric monoidal ∞-category $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$). Then we can regard the ∞-category

$$\mathrm{RMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}) \simeq \mathrm{RMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}})$$

as left-tensored over $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$. Composing with the map $\mu : \mathrm{Tors}(\mathbf{Z}) \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ of Construction 2.4.14, we obtain an action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathrm{RMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$, which we will denote by $\mu_{\mathcal{C}}$.

Example 2.4.16. Taking $\mathcal{C} = \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ in Remark 2.4.15, we obtain an action of the monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$ on the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ itself. This action determines a map

$$a : \mathrm{Tors}(\mathbf{Z}) \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}.$$

Composing with the forgetful functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$, we obtain a map

$$\rho : \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}} \to \mathrm{Fun}(\mathrm{Tors}(\mathbf{Z}), \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}),$$

given by $\rho(\mathcal{E})(T) = \mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}}) \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})} \mathcal{E}$. Recall that the Kan complex $\mathrm{Tors}(\mathbf{Z})$ is homotopy equivalent to the classifying space $B\mathbf{Z} \simeq S^1$, so that $\mathrm{Fun}(\mathrm{Tors}(\mathbf{Z}), \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$ can be identified with the free loop space of $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ (whose objects are given by a stable ∞-category $\mathcal{E}$ together with an equivalence from $\mathcal{E}$ to itself). Using Corollary 2.4.4, we see that ρ is an equivalence of ∞-categories.

Remark 2.4.17. The map $\rho : \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}} \to \mathrm{Fun}(\mathrm{Tors}(\mathbf{Z}), \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$ of Example 2.4.16 is equivariant with respect to the action of $\mathrm{Tors}(\mathbf{Z})$, which acts on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ via the map μ of Remark 2.4.15 and on $\mathrm{Fun}(\mathrm{Tors}(\mathbf{Z}), \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$ via its translation action on itself. It follows that the (homotopy) fixed points for the action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ can be identified with the *constant* maps from $\mathrm{Tors}(\mathbf{Z})$ into $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$.

Remark 2.4.18. Let $\mathcal{C}$ be as in Remark 2.4.15. Using the description of the action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ given by Remark 2.4.17, we conclude that the (homotopy) fixed points for the action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathrm{RMod}_{\mathcal{C}}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}})$ can be identified with $\mathrm{RMod}_{\mathcal{C}'}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$, where $\mathcal{C}' \simeq \mathrm{Sp}^{\mathrm{fin}} \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})} \mathcal{C}$ denotes the image of $\mathcal{C}$ in $\mathrm{Alg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$ under the map induced by the symmetric monoidal functor $\mathrm{Und}^{\mathrm{fin}} : \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \to \mathrm{Sp}^{\mathrm{fin}}$.

We conclude this section with the proof of Proposition 2.4.3.

Proof of Proposition 2.4.3. We first claim that $\mathrm{Fun}^{\otimes}(\mathbf{Z}^{\mathrm{ds}}, \mathcal{C})$ is a Kan complex. Suppose that $\alpha : F \to F'$ is a (monoidal) natural transformation between monoidal functors $F, F' : \mathbf{Z}^{\mathrm{ds}} \to \mathcal{C}$.

For every integer n, the induced map $\alpha(n) : F(n) \to F'(n)$ is an equivalence; it has a homotopy inverse is given by the composition

$$F'(n) \simeq F'(n) \otimes F(-n) \otimes F(n) \overset{\alpha(-n)}{\to} F'(n) \otimes F'(-n) \otimes F(n) \simeq F(n).$$

Note that for any monoidal functor $F : \mathbf{Z}^{\mathrm{ds}} \to \mathcal{C}$, each object $F(n) \in \mathcal{C}$ is invertible with inverse $F(-n)$. It follows that the evaluation functor $F \mapsto F(1)$ takes values in the subcategory $\mathcal{C}^{\mathrm{inv}} \subseteq \mathcal{C}$. We may therefore replace $\mathcal{C}$ by $\mathcal{C}^{\mathrm{inv}}$ and thereby reduce to the case where $\mathcal{C}$ is a Kan complex and every object of $\mathcal{C}$ is invertible. In this case, we can identify $\mathcal{C}$ and $\mathbf{Z}^{\mathrm{ds}}$ grouplike associative algebra objects of the ∞-category $\mathcal{S} \subseteq \mathcal{C}\mathrm{at}_\infty$ of spaces. Recall that the formation of loop spaces induces a fully faithful embedding

$$\Omega : \mathcal{S}^{\geq 1}_* \to \mathrm{Alg}(\mathcal{S})$$

whose essential image spanned by the grouplike associative algebra objects of $\mathcal{S}$ (here $\mathcal{S}^{\geq 1}_*$ denotes the ∞-category of pointed connected spaces). Let $B\mathbf{Z}, B\mathcal{C} \in \mathcal{S}^{\geq 1}_*$ denote preimages of $\mathbf{Z}^{\mathrm{ds}}$ and $\mathcal{C}$ under these equivalences. Since the element $1 \in \mathbf{Z} \simeq \pi_1 B\mathbf{Z}$ classifies a homotopy equivalence $S^1 \to B\mathbf{Z}$, it follows that the composite map

$$\mathrm{Fun}^{\otimes}(\mathbf{Z}^{\mathrm{ds}}, \mathcal{C}) \simeq \mathrm{Map}_{\mathcal{S}^{\geq 1}_*}(B\mathbf{Z}, B\mathcal{C}) \simeq \mathrm{Map}_{\mathcal{S}^{\geq 1}_*}(S^1, B\mathcal{C}) \simeq \Omega B\mathcal{C} \simeq \mathcal{C}$$

is also a homotopy equivalence. $\square$

3. Equivariant Matrix Factorizations

Let X be a smooth algebraic variety defined over a field k equipped with an action of the multiplicative group $\mathbf{G}_m$, and let $f : X \to \mathbb{A}^1$ be a nonconstant $\mathbf{G}_m$-equivariant map. One can associate to the pair (X, f) a triangulated category $\mathrm{MF}(X, f)$, called the *category of equivariant matrix factorizations of X*. Concretely, it is given by the Verdier quotient

$$\mathcal{D}^b_{\mathbf{G}_m}(X_0) / \mathcal{D}^{\mathrm{perf}}_{\mathbf{G}_m}(X_0)$$

where X_0 denotes the fiber $f^{-1}\{0\}$, $\mathcal{D}^b_{\mathbf{G}_m}(X_0)$ denotes the $\mathbf{G}_m$-equivariant bounded derived category of coherent sheaves on X_0, and $\mathcal{D}^{\mathrm{perf}}_{\mathbf{G}_m}(X_0)$ denotes the full subcategory of $\mathcal{D}^b_{\mathbf{G}_m}(X_0)$ spanned by the perfect complexes.

Our goal in this section is to discuss a generalization of the construction $(X, f) \mapsto \mathrm{MF}(X, f)$ to the setting of stable ∞-categories. Our generalization will take the following form:

(a) In place of the algebraic variety X, we consider a stable ∞-category $\mathcal{C}$, which should be viewed as playing the role of the equivariant derived category $\mathcal{D}^b_{\mathbf{G}_m}(X)$. However, we do not assume that $\mathcal{C}$ arises in this way. Consequently, our construction is more general in two respects: we do not work over a ground field k or even over the integers (for our applications in §4, it is important that our construction works "over the sphere spectrum"), and we do not assume that $\mathcal{C}$ arises from an algebro-geometric object (loosely speaking, we permit X to be a "noncommutative" variety).

(b) In place of the $\mathbf{G}_m$-equivariant map $f : X \to \mathbb{A}^1$, we consider an action of the monoidal ∞-category $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ on $\mathcal{C}$, where $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ is the ∞-category of finite filtered spectra (see §3.1). After extension of scalars to a field k, the monoidal ∞-category $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ can be identified with (an enhancement of) the equivariant derived category $\mathcal{D}^b_{\mathbf{G}_m}(\mathbf{A}^1)$, which acts on the equivariant derived category $\mathcal{D}^b_{\mathbf{G}_m}(X)$ whenever we have a $\mathbf{G}_m$-equivariant map $f : X \to \mathbf{A}^1$.

(c) In place of the equivariant derived category $\mathcal{D}^b_{\mathbf{G}_m}(X_0)$, we consider the ∞-category of modules $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$. Here $\mathbb{A}$ is a certain commutative algebra object of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ which (after extension of scalars to a field k) corresponds to the $\mathbf{G}_m$-equivariant sheaf of commutative algebras on $\mathbf{A}^1$ which corresponds to the affine morphism $\{0\} \hookrightarrow \mathbf{A}^1$.

(d) In place of the the Verdier quotient $\mathcal{D}^b_{\mathbf{G}_m}(X_0)/\mathcal{D}^{\mathrm{perf}}_{\mathbf{G}_m}(X_0)$, we define $\mathrm{MF}(\mathcal{C})$ to be the ∞-category obtained from $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$ by extension of scalars along a certain morphism of graded $\mathbb{E}_2$-rings $S[\beta] \to S[\beta^{\pm 1}]$.

We begin in §3.1 by studying the ∞-category $\mathrm{Rep}(\mathbf{Z})$ of *filtered spectra* and its full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ (both of which can be obtained as a special case of the constructions of §2.2). In §3.2 we will discuss the relationship between the theories of filtered spectra and graded spectra (where the latter were defined in §2.4) and define the algebra $\mathbb{A}$. In §3.3 we will gather some general facts about bimodules over $\mathbb{A}$, and in §3.4 we will apply these facts to construct a graded $\mathbb{E}_2$-ring $S[\beta]$ which "acts" on any ∞-category of the form $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$. In §3.5 we will consider the localization $S[\beta^{\pm 1}]$ and use it to construct a monoidal functor $\phi : \mathrm{Tors}(\mathbf{Z}) \to \mathrm{BPic}(S)$ (which we will eventually prove to be equivalent to map appearing in the statement of Theorem 1.1.1). In §3.6 we will use extension of scalars along the map $S[\beta] \to S[\beta^{\pm 1}]$ to define a functor

$$\mathrm{MF} : \mathrm{RMod}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty) \to \mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty$$

and show that it is equivariant with respect to suitable actions of $\mathrm{Tors}(\mathbf{Z})$ on both sides.

3.1. **Filtered Spectra.** Let $\mathbf{Z}$ denote the set of integers. We let $\mathrm{N}(\mathbf{Z})$ denote the nerve of $\mathbf{Z}$ as a linearly ordered set: concretely, it is the simplicial set whose n-simplices are sequences of integers $(a_0, \ldots, a_n)$ which satisfy $a_0 \le a_1 \le \cdots \le a_n$. We let

$$\mathrm{Rep}(\mathbf{Z}) = \mathrm{Rep}(\mathrm{N}(\mathbf{Z})) = \mathrm{Fun}(\mathrm{N}(\mathbf{Z})^{\mathrm{op}}, \mathrm{Sp})$$

denote the ∞-category of representations of $\mathrm{N}(\mathbf{Z})$. We will refer to $\mathrm{Rep}(\mathbf{Z})$ as the ∞-*category of filtered spectra*, and we will refer to the objects of $\mathrm{Rep}(\mathbf{Z})$ as *filtered spectra*.

Remark 3.1.1. By definition, a filtered spectrum is a diagram

$$\cdots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

in the ∞-category Sp of spectra. We can think of this diagram as supplying a *decreasing* filtration of the colimit $\varinjlim X_{-n}$.

Notation 3.1.2. Since the group structure on $\mathbf{Z}$ is compatible with its linear ordering, the nerve $\mathrm{N}(\mathbf{Z})$ inherits the structure of a simplicial commutative monoid; in particular, it can be regarded as a symmetric monoidal ∞-category. It follows that the ∞-category $\mathrm{Rep}(\mathbf{Z})$ of filtered spectra inherits the structure of a symmetric monoidal ∞-category (Corollary 2.3.9). We will indicate the symmetric monoidal structure on $\mathrm{Rep}(\mathbf{Z})$ by $\circledast : \mathrm{Rep}(\mathbf{Z}) \times \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z})$; concretely, it is given by the formula

$$(X \circledast Y)_n = \varinjlim_{n \le n' + n''} X_{n'} \wedge Y_{n''}.$$

We will denote the unit object of $\mathrm{Rep}(\mathbf{Z})$ by $\mathbb{S}$; concretely, it is given by the formula

$$\mathbb{S}_n = \begin{cases} S & \text{if } n \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.1.3. There is an evident inclusion of simplicial sets $\iota : \mathbf{Z}^{\mathrm{ds}} \hookrightarrow \mathrm{N}(\mathbf{Z})$, whose image is the simplicial subset of $\mathrm{N}(\mathbf{Z})$ whose n-simplices are tuples $(a_0, a_1, \ldots, a_n)$ satisfying $a_0 = \cdots = a_n$. This inclusion induces a symmetric monoidal functor $I : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Rep}(\mathbf{Z})$, given by left Kan

extension along ι. Concretely, the functor I is given by the formula $(IX)_n = \bigoplus_{m \geq n} X_m$. The functor I determines an action of the ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ on $\mathrm{Rep}(\mathbf{Z})$. In particular, we can regard $\mathrm{Rep}(\mathbf{Z})$ as a graded ∞-category, with shift functors $X \mapsto X(n)$ described by the formula $X(n)_m = X_{m+n}$.

Composition with the inclusion map $\iota : \mathbf{Z}^{\mathrm{ds}} \hookrightarrow \mathrm{N}(\mathbf{Z})$ determines a restriction functor $\mathrm{Res} : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. This functor is right adjoint to I, and therefore inherits the structure of a lax symmetric monoidal functor. In particular, the restriction functor Res carries commutative algebra objects of $\mathrm{Rep}(\mathbf{Z})$ to commutative algebra objects of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$.

Notation 3.1.4. We let $S[t] \in \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ denote the graded spectrum $\mathrm{Res}(\mathbb{S})$, given concretely by the formula

$$S[t]_n = \begin{cases} S & \text{if } n \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since the restriction functor Res is lax symmetric monoidal, we can regard $S[t]$ as a commutative algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$.

Remark 3.1.5. The graded spectrum $S[t]$ can be identified with the free associative algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ generated by the shifted sphere spectrum $S(1)$.

Proposition 3.1.6. *The restriction functor* $\mathrm{Res} : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ *induces a symmetric monoidal equivalence of ∞-categories*

$$\theta : \mathrm{Rep}(\mathbf{Z}) \simeq \mathrm{Mod}_{\mathbb{S}}(\mathrm{Rep}(\mathbf{Z})) \to \mathrm{Mod}_{S[t]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})).$$

Proof. We first show that θ is fully faithful. Let X and Y be filtered spectra; we wish to show that the induced map

$$\phi_{X,Y} : \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z})}(X, Y) \to \underline{\mathrm{Map}}_{\mathrm{Mod}_{S[t]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(\theta(X), \theta(Y))$$

is a homotopy equivalence of spectra. Let us regard Y as fixed. The construction $X \mapsto \phi_{X,Y}$ carries colimits in $\mathrm{Rep}(\mathbf{Z})$ to limits in $\mathrm{Fun}(\Delta^1, \mathrm{Sp})$. Consequently, the collection of those filtered spectra X for which $\phi_{X,Y}$ is a stable subcategory of $\mathrm{Rep}(\mathbf{Z})$ which is closed under colimits. It will therefore suffice to show that $\phi_{X,Y}$ is an equivalence when X has the form $\mathbb{S}(n)$ for some integer n. In this case, we observe that both sides can be identified with the spectrum Y_{-n}.

Since θ is fully faithful and commutes with colimits, the essential image of θ is closed under colimits in $\mathrm{Mod}_{S[t]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$. Since $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ is generated under small colimits by objects of the form $S(n)$, it follows that $\mathrm{Mod}_{S[t]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ is generated under small colimits by objects of the form $S[t](n) \simeq \theta(\mathbb{S}(n))$. From this we deduce that θ is essentially surjective.

To complete the proof, it will suffice to show that the lax symmetric monoidal functor θ is actually symmetric monoidal. Since θ preserves unit objects by construction, it suffices to verify that for each pair of filtered spectra X and Y, the canonical map

$$\psi_{X,Y} : \theta(X) \otimes_{S[t]} \theta(Y) \to \theta(X \circledast Y)$$

is an equivalence of $S[t]$-modules in $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. Regarding Y as fixed, we observe that the collection of those filtered spectra X for which $\psi_{X,Y}$ is an equivalence is a stable subcategory of $\mathrm{Rep}(\mathbf{Z})$ which is closed under colimits. We may therefore assume without loss of generality that X has the form $\mathbb{S}(m)$ for some integer m. Similarly, we may assume that Y has the form $\mathbb{S}(n)$ for some integer n. The desired result now follows from a simple computation (note that the domain and codomain of $\psi_{X,Y}$ can both be identified with $S[t](m+n)$). $\qquad\square$

We now turn our attention to the full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \subseteq \mathrm{Rep}(\mathbf{Z})$ spanned by the *finite* filtered spectra.

Notation 3.1.7. For each integer n, let $\mathrm{Rep}(\mathbf{Z})_{\leq n}$ denote the full subcategory of $\mathrm{Rep}(\mathbf{Z})$ spanned by those filtered spectra X for which $X_m \simeq 0$ for $m > n$. Note that $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$ is contains the unit object of $\mathrm{Rep}(\mathbf{Z})$ and is closed under tensor products, and therefore inherits the structure of a symmetric monoidal ∞-category (one can describe $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$ as the essential image of the fully faithful embedding $\mathrm{Rep}(\mathbf{Z}_{\leq 0}) \to \mathrm{Rep}(\mathbf{Z})$ induced by the inclusion of partially ordered monoids $\mathbf{Z}_{\leq 0} \hookrightarrow \mathbf{Z}$).

Proposition 3.1.8. *Let* $X \in \mathrm{Rep}(\mathbf{Z})$ *be a filtered spectrum. Then* $X \in \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ *if and only if the following three conditions are satisfied:*

    (*a*) *There exists an integer n such that* $X \in \mathrm{Rep}(\mathbf{Z})_{\leq n}$.
    (*b*) *For each integer n, the spectrum X_n is finite.*
    (*c*) *For $n \ll 0$, the map $X_n \to X_{n-1}$ is an equivalence of spectra.*

Proof. Let $\mathcal{C} \subseteq \mathrm{Rep}(\mathbf{Z})$ denote the full subcategory spanned by those filtered spectra X which satisfy conditions (a), (b), and (c). It is easy to see that $\mathcal{C}$ is a stable subcategory of $\mathrm{Rep}(\mathbf{Z})$ which contains each $\mathbb{S}(n)$, so that $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \subseteq \mathcal{C}$. Conversely, suppose that $X \in \mathcal{C}$; we will show that $X \in \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$. Replacing X by a shift if necessary, we may assume that the map $X_i \to X_{i-1}$ is an equivalence for $i \leq 0$. Choose an integer n such that $X \in \mathrm{Rep}(\mathbf{Z})_{\leq n}$. We proceed by induction on n; if $n < 0$, then $X \simeq 0$ and there is nothing to prove. Otherwise, we define a filtered spectrum X' by the formula

$$X'_m = \begin{cases} X_n & \text{if } m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Since X_n is finite, it is easy to see that $X' \in \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$. We have a fiber sequence

$$X' \to X \to X''$$

where $X'' \in \mathrm{Rep}(\mathbf{Z})_{\leq n-1}$. Applying the inductive hypothesis, we conclude that $X'' \in \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$. Since $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ is closed under extensions, it follows that $X \in \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ as desired. $\qquad\square$

Remark 3.1.9. It follows from Proposition 3.1.8 that the full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \subseteq \mathrm{Rep}(\mathbf{Z})$ is closed under retracts, and therefore contains all compact objects of $\mathrm{Rep}(\mathbf{Z})$.

Definition 3.1.10. Let $\mathcal{C}$ be a stable ∞-category. A *local filtration* on $\mathcal{C}$ is a right action of the monoidal ∞-category $\mathrm{N}(\mathbf{Z})$ on $\mathcal{C}$. We will say that $\mathcal{C}$ is *locally filtered* if it is equipped with a local filtration.

Warning 3.1.11. The notion of locally filtered stable ∞-category (introduced in Definition 3.1.10) is completely unrelated to the notion of filtered ∞-category (studied in §T.5.3.1).

Remark 3.1.12. Let $\mathcal{C}$ be a stable ∞-category. The following types of data are equivalent:

    (*a*) (Right or left) actions of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ on $\mathcal{C}$ for which the action map $\mathcal{C} \times \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \to \mathcal{C}$ is separately exact in each variable.
    (*b*) Exact monoidal functors $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C})$.
    (*c*) Monoidal functors $\mathrm{N}(\mathbf{Z}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{C})$.
    (*d*) Monoidal functors $\mathrm{N}(\mathbf{Z}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$.
    (*e*) (Right or left) actions of the monoidal ∞-category $\mathrm{N}(\mathbf{Z})$ on $\mathcal{C}$.

We will generally abuse terminology and refer to any of these types of data as a *local filtration* on $\mathcal{C}$.

Notation 3.1.13. We let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$ denote the ∞-category $\mathrm{Mod}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}})$. We will refer to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$ as the ∞-*category of locally filtered stable ∞-categories.*

Remark 3.1.14. Using the symmetric monoidal inclusion functor $\mathbf{Z}^{\mathrm{ds}} \hookrightarrow \mathrm{N}(\mathbf{Z})$, we can regard every locally filtered stable ∞-category $\mathcal{C}$ as a locally graded stable ∞-category $\mathcal{C}$. However, a local filtration on $\mathcal{C}$ supplies more data: in addition to the shift functors $C \mapsto C(n)$ for $n \in \mathbf{Z}$, it supplies natural maps $C(n) \to C(m)$ for $m \leq n$ (together with a copious amount of additional coherence data).

Remark 3.1.15. Let $\mathcal{C}$ be a stable ∞-category. Heuristically, one can think of a local filtration on $\mathcal{C}$ a datum which allows us to view the objects $C \in \mathcal{C}$ as equipped with filtrations. The operation $C \mapsto C(n)$ should be thought of as "shifting filtrations by n", and the natural maps $C(n) \to C(m)$ for $m \leq n$ reflect the idea that each stage of the filtration of C is contained in each earlier stage.

Example 3.1.16. Let T be a $\mathbf{Z}$-torsor. Then there is a canonical linear ordering on T, where we write $t \leq t'$ if $t' = t + n$ for some nonnegative integer n. We let $\mathrm{N}(T)$ denote the nerve of T as a linearly ordered set: that is, the simplicial set whose k-simplices are tuples $(t_0, \ldots, t_k) \in T^{k+1}$ satisfying $t_0 \leq t_1 \leq \cdots \leq t_k$. We let $\mathrm{Rep}(T)$ denote the ∞-category $\mathrm{Rep}(\mathrm{N}(T)) = \mathrm{Fun}(\mathrm{N}(T)^{\mathrm{op}}, \mathrm{Sp})$ of representations of $\mathrm{N}(T)$ and $\mathrm{Rep}^{\mathrm{fin}}(T) \subseteq \mathrm{Rep}(T)$ the full subcategory spanned by the finite respresentations. The action of $\mathbf{Z}$ on T induces an action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ on $\mathrm{Rep}^{\mathrm{fin}}(T)$, which we can regard as a local filtration on the stable ∞-category $\mathrm{Rep}^{\mathrm{fin}}(T)$. Note that we can identify $\mathrm{Rep}^{\mathrm{fin}}(T)$ with the relative tensor product

$$\mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}}) \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})} \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}),$$

formed in the symmetric monoidal ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$.

Remark 3.1.17. By virtue of Remark 2.4.15, we can regard the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$ as equipped with an action of the monoidal ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$, so that the monoidal functor $\mu : \mathrm{Tors}(\mathbf{Z}) \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}}$ of Construction 2.4.14 induces an action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$. This action determines a map

$$\mathrm{Tors}(\mathbf{Z}) \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$$

which is given concretely by $(T, \mathcal{C}) \mapsto \mathrm{Rep}^{\mathrm{fin}}(T) \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})} \mathcal{C}$ (where the relative tensor product is formed in the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ of stable ∞-categories).

3.2. **Associated Graded Spectra.** Let X be a filtered spectrum given by a diagram

$$\cdots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

For each integer n, we let $\mathrm{gr}(X)_n$ denote the cofiber of the map $X_{n+1} \to X_n$. The collection $\{\mathrm{gr}(X)_n\}_{n \in \mathbf{Z}}$ can be regarded as a graded spectrum, which we will denote by $\mathrm{gr}(X)$. We will refer to $\mathrm{gr}(X)$ as the *associated graded spectrum* of X. We will regard the construction $X \mapsto \mathrm{gr}(X)$ as a functor from $\mathrm{Rep}(\mathbf{Z})$ to $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$.

Our goal in this section is to prove the following result:

Proposition 3.2.1. *There exists a symmetric monoidal structure on the functor* $\mathrm{gr} : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. *Moreover, this symmetric monoidal structure can be chosen in such a way that the composite map*

$$\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \overset{I}{\to} \mathrm{Rep}(\mathbf{Z}) \overset{\mathrm{gr}}{\to} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$$

is homotopic to the identity (as a symmetric monoidal functor), where I is the functor described in Remark 3.1.3.

The proof of Proposition 3.2.1 will require some preliminaries.

Definition 3.2.2. Let X be a filtered spectrum and let n be an integer. We will say that X is *concentrated in degree n* if $X_m \simeq 0$ for $m \neq n$.

Proposition 3.2.3. *Let n be an integer and let $\mathrm{Rep}(\mathbf{Z})_{=n}$ denote the full subcategory of $\mathrm{Rep}(\mathbf{Z})$ spanned by those filtered spectra which are concentrated in degree n. Then the construction $X \mapsto X_n$ induces an equivalence of ∞-categories $\mathrm{Rep}(\mathbf{Z})_{=n} \to \mathrm{Sp}$.*

Proof. Without loss of generality we may assume that $n = 0$. Note that the inclusion $\iota : \{0\} \hookrightarrow \mathbf{Z}_{\leq 0}$ induces a functor $\iota^* : \mathrm{Rep}(\mathbf{Z}_{\leq 0}) \simeq \mathrm{Rep}(\mathbf{Z})_{\leq 0} \to \mathrm{Sp}$, given by $X \mapsto X_0$. The functor ι^* admits a left adjoint $\iota_!$ (given by left Kan extension along ι) and ι_* (given by right Kan extension along ι), both of which are fully faithful. Concretely, these are given by the formulae

$$\iota_!(E)_n = \begin{cases} E & \text{if } n \leq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad \iota_*(E)_n = \begin{cases} E & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that for any object $X \in \mathrm{Rep}(\mathbf{Z})_{\leq 0}$, the unit map $u : X \to \iota_* \iota^* X$ induces an equivalence $X_0 \simeq (\iota_* \iota^* X)_0$, so that u is an equivalence if and only if X is concentrated in degree zero. It follows that ι_* is homotopy inverse to the restriction $\iota^*|_{\mathrm{Rep}(\mathbf{Z})_{=0}}$. $\square$

Notation 3.2.4. Let S denote the sphere spectrum. We let $\mathbb{A}$ denote an inverse image of S under the equivalence of ∞-categories $\mathrm{Rep}(\mathbf{Z})_{=0} \to \mathrm{Sp}$. In other words, $\mathbb{A}$ is a filtered spectrum which is characterized up to equivalence by the formula

$$\mathbb{A}_n = \begin{cases} S & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.2.5. *There exists an (essentially unique) commutative algebra structure on the filtered spectrum $\mathbb{A}$ for which the unit map $\mathbb{S} \to \mathbb{A}$ restricts to an equivalence $\mathbb{S}_0 \to \mathbb{A}_0$.*

Proof. Let ι_*, ι^*, and $\iota_!$ be as in the proof of Proposition 3.2.3. The composite functor $\iota_* \iota^*$ is left adjoint to the inclusion $\mathrm{Rep}(\mathbf{Z})_{=0} \hookrightarrow \mathrm{Rep}(\mathbf{Z})_{\leq 0}$. We may therefore regard $\mathrm{Rep}(\mathbf{Z})_{=0}$ as a localization of $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$. Using the description of the convolution product on $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$ given in Notation 3.1.2, we see that the symmetric monoidal structure on $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$ is compatible with the localization $\iota_* \iota^*$ and therefore induces a symmetric monoidal structure on $\mathrm{Rep}(\mathbf{Z})_{=0}$ for which the equivalence $\mathrm{Rep}(\mathbf{Z})_{=0} \simeq \mathrm{Sp}$ of Proposition 3.2.3 is symmetric monoidal. By construction, the filtered spectrum $\mathbb{A} \in \mathrm{Rep}(\mathbf{Z})_{=0}$ can be identified with the image of $\mathbb{S}$ under this symmetric monoidal functor. $\square$

Remark 3.2.6. Under the identification $\mathrm{Rep}(\mathbf{Z}) \simeq \mathrm{Mod}_{S[t]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ of Proposition 3.1.6, the map $\mathbb{S} \to \mathbb{A}$ of Proposition 3.2.5 corresponds to a map $\epsilon : S[t] \to S$ of commutative algebra objects of in the ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ of graded spectra. This map exhibits $S[t]$ as an augmented commutative algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$; roughly speaking, the augmentation ϵ is given by "sending t to zero."

In what follows, we will always regard $\mathbb{A}$ as equipped with the commutative algebra structure described in Proposition 3.2.5. Using this commutative algebra structure, we will regard $\mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ as a symmetric monoidal ∞-category.

Proposition 3.2.7. *Let $I : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Rep}(\mathbf{Z})$ be the functor described in Remark 3.1.3. Then the composite functor*

$$\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \xrightarrow{I} \mathrm{Rep}(\mathbf{Z}) \xrightarrow{\otimes \mathbb{A}} \mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$$

is an equivalence of symmetric monoidal ∞-categories.

Proof. Proposition 3.1.6 supplies equivalences

$$\mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \simeq \mathrm{Mod}_{\mathbb{A}}(\mathrm{Mod}_{S[t]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))) \simeq \mathrm{Mod}_{\mathrm{Res}(\mathbb{A})}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})).$$

The desired result now follows from the observation that the unit map $S \to \mathrm{Res}(\mathbb{A})$ is an equivalence of graded spectra. □

Remark 3.2.8. The unit map $\mathbb{S} \to \mathbb{A}$ fits into a fiber sequence

$$\mathbb{S}(1) \xrightarrow{\gamma} \mathbb{S} \to \mathbb{A},$$

where $\gamma : \mathbb{S}(1) \to \mathbb{S}$ denotes the image under the stable Yoneda embedding of the unique morphism $-1 \to 0$ in $\mathrm{N}(\mathbf{Z})$. It follows that for any filtered spectrum X, we obtain a fiber sequence

$$X(1) \xrightarrow{\gamma_X} X \to \mathbb{A} \otimes X.$$

In particular, we have fiber sequences of spectra $X_{n+1} \to X_n \to (\mathbb{A} \otimes X)_n$, so that we can identify the associated graded spectrum $\mathrm{gr}(X)$ with $\mathrm{Res}(\mathbb{A} \otimes X)$.

Example 3.2.9. The filtered spectrum $\mathbb{A} \otimes \mathbb{A}$ is given levelwise by

$$(\mathbb{A} \otimes \mathbb{A})_n = \begin{cases} S^0 & \text{if } n = 0 \\ S^1 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2.10. For every integer n, the object $\mathbb{S}(n)$ is invertible in $\mathrm{Rep}(\mathbf{Z})$, with inverse given by $\mathbb{S}(-n)$. In particular, each $\mathbb{S}(n)$ is a (right and left) dualizable object of $\mathrm{Rep}(\mathbf{Z})$. Using the fiber sequence

$$\mathbb{S}(1) \xrightarrow{\gamma} \mathbb{S} \to \mathbb{A}$$

of Remark 3.2.8, we see that $\mathbb{A}$ is also a (right and left) dualizable object of $\mathrm{Rep}(\mathbf{Z})$, with dual

$$\Sigma^{-1} \mathbb{A}(-1) \simeq \mathrm{fib}(\gamma^{\vee} : \mathbb{S} \to \mathbb{S}(-1)).$$

Note that the left action of $\mathbb{A}$ on itself determines a right action of $\mathbb{A}$ on its dual $\Sigma^{-1} \mathbb{A}(-1)$ (see §H.4.6.2). By virtue of Proposition 3.2.7, this action is unique.

Remark 3.2.11. The equivalence $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \simeq \mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ admits a right adjoint, given by the composition

$$\mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \to \mathrm{Rep}(\mathbf{Z}) \xrightarrow{\mathrm{Res}} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}).$$

It follows that this composite functor is also an equivalence of symmetric monoidal ∞-categories.

Proof of Proposition 3.2.1. Using Remark 3.2.8, we see that the associated graded functor gr is given by the composition

$$\mathrm{Rep}(\mathbf{Z}) \xrightarrow{\mathbb{A} \otimes} \mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \to \mathrm{Rep}(\mathbf{Z}) \xrightarrow{\mathrm{Res}} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}),$$

where the first map is evidently symmetric monoidal and the composition of the second and third map is symmetric monoidal by virtue of Remark 3.2.11. The functor $\mathrm{gr} \circ I$ can be obtained by composing the equivalence of Proposition 3.2.7 with its right adjoint, and is therefore homotopic to the identity. □

Remark 3.2.12. Proposition 3.2.7 supplies an equivalence of ∞-categories

$$\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \simeq \mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})).$$

Under this equivalence, the full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}) \subseteq \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ corresponds to the full subcategory $\mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})) \subseteq \mathrm{Mod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$. This follows immediately from the characterization of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ supplied by Proposition 3.1.8.

3.3. **Koszul Duality and $\mathbb{A}$-Bimodules.** Let $\mathcal{C}$ be a monoidal ∞-category. Assume that for every associative algebra object $A \in \mathrm{Alg}(\mathcal{C})$ and every pair of right A-modules $M, N \in \mathrm{RMod}_A(\mathcal{C})$, there exists another object $\mathrm{Mor}_A(M, N) \in \mathcal{C}$ which classifies morphisms from M to N in the following sense: for every object $C \in \mathcal{C}$ we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, \mathrm{Mor}_A(M, N)) \simeq \mathrm{Map}_{\mathrm{RMod}_A(\mathcal{C})}(C \otimes M, N).$$

In the special case $M = N$, the object $B = \mathrm{Mor}_A(M, M)$ inherits the structure of an associative algebra object of $\mathcal{C}$. Moreover, there is a canonical left action of B on $M \in \mathrm{RMod}_A(\mathcal{C})$, so that M can be regarded as an B-A-bimodule object of $\mathcal{C}$.

Example 3.3.1. Let A be an associative algebra object of $\mathcal{C}$ equipped with a map of associative algebras $\epsilon : A \to \mathbf{1}$. Using ϵ, we can regard the unit object $\mathbf{1}$ as a right A-module. We let $\mathbf{D}(A) = \mathrm{Mor}_A(\mathbf{1}, \mathbf{1})$. We will refer to $\mathbf{D}(A)$ as the *Koszul dual* of A. By construction, the unit object $\mathbf{1}$ is equipped with the structure of an $\mathbf{D}(A)$-A-bimodule.

Remark 3.3.2. Our convention in this paper differs slightly from that of [7], in which the opposite algebra $\mathbf{D}(A)^{\mathrm{rev}}$ is referred to as the Koszul dual of A.

Definition 3.3.3. Let $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{gd}}$ be the ∞-category of locally graded stable ∞-categories, with unit object $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ and the symmetric monoidal structure on $\mathcal{C}$ is given by the relative tensor product

$$(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})} \mathcal{N}.$$

We will regard the ∞-category $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ of finite filtered spectra as an associative algebra object of $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{gd}}$, equipped with an augmentation given by the associated graded functor $\mathrm{gr} : \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \to \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ of §3.2. We let Θ denote the Koszul dual of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ (as an augmented associative algebra in $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{gd}}$).

Remark 3.3.4. By construction, Θ is an associative algebra object of $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{gd}}$: in other words, it is a stable monoidal ∞-category equipped with a central and exact action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$. It follows from Remark 2.4.15 that the ∞-category $\mathrm{RMod}_{\Theta}(\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}}) \simeq \mathrm{RMod}_{\Theta}(\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{gd}})$ inherits an action of the monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$.

It follows immediately from the definitions that as a monoidal ∞-category, Θ can be identified with the ∞-category $\mathrm{Fun}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}^{\mathrm{ex}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}), \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}))$ of exact $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$-linear functors from $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ to itself. The next result will be helpful in describing this ∞-category more explicitly.

Proposition 3.3.5. *Let $\mathcal{C}$ be a stable ∞-category equipped with a local filtration, which we will identify with a right action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ on $\mathcal{C}$. Then there is a canonical equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}^{\mathrm{ex}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}), \mathcal{C}) \simeq \mathrm{RMod}_{\mathbb{A}}(\mathcal{C}).$$

Proof. Proposition 3.2.7 supplies a $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$-linear equivalence of ∞-categories

$$\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}) \simeq \mathrm{LMod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})).$$

In particular, the unit object $S \in \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ can be regarded as a right $\mathbb{A}$-module in $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$. Consequently, for any $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$-linear functor $F : \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}) \to \mathcal{C}$, the image $F(S)$ inherits the structure of a right $\mathbb{A}$-module. We therefore have an evaluation functor

$$\mathrm{Fun}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}^{\mathrm{ex}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}), \mathcal{C}) \to \mathrm{RMod}_{\mathbb{A}}(\mathcal{C}).$$

We claim that this evaluation functor is always an equivalence. To prove this, we are free to enlarge $\mathcal{C}$ (replacing it by $\mathrm{Ind}(\mathcal{C})$ if necessary) to reduce to the case where $\mathcal{C}$ admits small colimits. In this case, we have equivalences

$$\mathrm{Fun}^{\mathrm{ex}}_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}), \mathcal{C}) \simeq \mathrm{Fun}^{L}_{\mathrm{Rep}(\mathbf{Z})}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}), \mathcal{C}) \simeq \mathrm{RMod}_{\mathbb{A}}(\mathcal{C}),$$

where $\mathrm{Fun}^{L}_{\mathrm{Rep}(\mathbf{Z})}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}), \mathcal{C})$ denotes the ∞-category of colimit-preserving $\mathrm{Rep}(\mathbf{Z})$-linear functors from $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ to $\mathcal{C}$, and the second equivalence is supplied by Theorem H.4.8.4.1. $\square$

Combining Propositions 3.3.5 and 3.2.7, we obtain the following:

Corollary 3.3.6. *There is a canonical equivalence of ∞-categories*

$$\Theta \simeq {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})).$$

Remark 3.3.7. Under the equivalence of ∞-categories given by Corollary 3.3.6, the monoidal structure on Θ corresponds to the relative tensor product $(M, N) \mapsto M \otimes_{\mathbb{A}} N$ on the ∞-category

$${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})) \subseteq {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})).$$

Variant 3.3.8. We can view the ∞-category $\mathrm{Rep}(\mathbf{Z})$ of *all* filtered spectra as an associative algebra in the ∞-category $\mathrm{Mod}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of presentable ∞-categories tensored over $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$, with an augmentation given by the functor $\mathrm{gr}: \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. In this setting, the Koszul dual of $\mathrm{Rep}(\mathbf{Z})$ can be identified with the ∞-category ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ of *all* $\mathbb{A}$-$\mathbb{A}$-bimodule objects of $\mathrm{Rep}(\mathbf{Z})$. In particular, the monoidal ∞-category ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ is equipped with a central action of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$.

Combining Proposition 3.3.5 and Remark 2.1.2 (and using the fact that the monoidal functor $\mu: \mathrm{Tors}(\mathbf{Z}) \to \mathcal{C}\mathrm{at}^{\mathrm{gd}}_{\infty}$ of Construction 2.4.14 takes invertible values), we obtain the following:

Corollary 3.3.9. *Let $\mathcal{C}$ be a locally filtered stable ∞-category. Then the ∞-category $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$ inherits a right action of the monoidal ∞-category Θ. Moreover, the construction*

$$\mathcal{C} \mapsto \mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$$

is $\mathrm{Tors}(\mathbf{Z})$-equivariant, where the actions of $\mathrm{Tors}(\mathbf{Z})$ on $\mathcal{C}\mathrm{at}^{\mathrm{filt}}_{\infty}$ and $\mathrm{RMod}_{\Theta}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty})$ are given in Remarks 3.1.17 and 3.3.4, respectively.

Remark 3.3.10. One can show that the monoidal ∞-categories Θ and $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ are equivalent to one another. We will not need this fact and will not prove it here. For our purposes, an identification of Θ with $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ would merely create the potential for unnecessary confusion; the ∞-categories Θ and $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ will have very different roles to play in the constructions of the next few sections. We should also note that although Θ and $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ are equivalent as monoidal ∞-categories, they are not equivalent as *algebras* over the symmetric monoidal ∞-category $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$.

3.4. **The Graded Algebra $S[\beta]$.** Let A be a graded $\mathbb{E}_2$-ring: that is, an $\mathbb{E}_2$-algebra object of the ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ of graded spectra. Then the ∞-category $\mathrm{RMod}_A(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ of graded A-module spectra is a monoidal ∞-category equipped with a central action of $\mathbf{Z}^{\mathrm{ds}}$, as is the full subcategory

$$\mathrm{RMod}^{\mathrm{fin}}_A(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \subseteq \mathrm{RMod}_A(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$$

spanned by the compact objects. Our goal in this section is to show that the monoidal ∞-category Θ of §3.3 is of this form (Proposition 3.4.8).

We begin by studying the larger ∞-category ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ of *all* $\mathbb{A}$-$\mathbb{A}$ bimodules. According to Variant 3.3.8, we can view ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ as equipped with a central action

of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. This central action exhibits ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ as an ∞-category *enriched* over $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. In other words, we can associate to each pair of bimodules $M, N \in {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ a graded spectrum $\underline{\mathrm{Map}}^{\mathrm{gd}}(M, N)$ classifying morphisms from M to N: concretely, it can be described by the formula

$$\underline{\mathrm{Map}}^{\mathrm{gd}}(M, N)_k = \underline{\mathrm{Map}}_{{}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(M, N(k)).$$

Definition 3.4.1. We define a graded spectrum $S[\beta]$ by the formula

$$S[\beta] = \underline{\mathrm{Map}}^{\mathrm{gd}}(\mathbb{A}, \mathbb{A}).$$

Note that since ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ can be regarded as an associative algebra object of the ∞-category $\mathrm{Mod}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$, the graded spectrum $S[\beta]$ inherits the structure of an $\mathbb{E}_2$-algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ (see §H.4.8.5).

Our first goal is to analyze the structure of the graded spectrum $S[\beta]$.

Proposition 3.4.2. *Let $M, N \in {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$. Suppose that M is concentrated in degrees ≤ 0 and that N is concentrated in degrees ≥ 0. Then the canonical map*

$$\phi_{M,N} : \underline{\mathrm{Map}}_{{}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(M, N) \to \underline{\mathrm{Map}}_{\mathrm{Sp}}(M_0, N_0)$$

is an equivalence of spectra.

Proof. Let $g : {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ be the composition of the forgetful functor with the restriction functor, let f be its left adjoint, and let $T = f \circ g : {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \to {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ be the corresponding monad. Then M is given by the geometric realization of a simplicial object $[n] \mapsto T^{n+1}M$. It will therefore suffice to show that each of the maps $\phi_{T^{n+1}M,N}$ is an equivalence of spectra. We may therefore assume without loss of generality that M has the form $f(M')$, where $M' \in \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})_{\leq 0}$. In this case, the domain of $\phi_{M,N}$ is given by

$$
\begin{aligned}
\underline{\mathrm{Map}}_{{}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(f(M'), N) &\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(M', g(N)) \\
&\simeq \prod_{k \in \mathbf{Z}} \underline{\mathrm{Map}}_{\mathrm{Sp}}(M'_k, N_k)
\end{aligned}
$$

from which the desired result follows immediately. $\square$

Corollary 3.4.3. *For every integer n, the forgetful functor*

$$\theta : {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \times_{\mathrm{Rep}(\mathbf{Z})} \mathrm{Rep}(\mathbf{Z})_{=n} \to \mathrm{Sp}$$

$$X \mapsto X_n$$

is an equivalence of ∞-categories.

Proof. It follows from Proposition 3.4.2 that θ is fully faithful. Since θ preserves small colimits, it follows that the essential image of θ is closed under colimits and desuspensions. It will therefore suffice to show that the sphere spectrum S belongs to the essential image of θ. This is clear, since $S \simeq \theta(\mathbb{A}(-n))$. $\square$

It follows from Corollary 3.4.3 that each of the filtered spectra $\mathbb{A}(n)$ admits an essentially unique $\mathbb{A}$-$\mathbb{A}$-bimodule structure.

Construction 3.4.4. Let $m : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ denote the multiplication map. It follows from Example 3.2.9 that the cofiber of m can be described by the formula

$$\mathrm{cofib}(m)_n = \begin{cases} S^2 & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the bimodule $\mathrm{cofib}(m)$ is concentrated in degree -1 and is equivalent to $\Sigma^2 \mathbb{A}(1)$ as a graded spectrum, and is therefore equivalent to $\Sigma^2 \mathbb{A}(1)$ as a bimodule (Corollary 3.4.3). We therefore have a fiber sequence of $\mathbb{A}$-bimodules

$$\mathbb{A} \circledast \mathbb{A} \overset{m}{\to} \mathbb{A} \overset{\beta}{\to} \Sigma^2 \mathbb{A}(1).$$

We will refer to β as the *anchor map*. Note that β classifies a map of spectra $S^{-2} \to S[\beta]_1$, which we will also denote by β. Using the algebra structure on $S[\beta]$, we obtain maps

$$\beta^n : S^{-2n} \to S[\beta]_n$$

for $n \geq 0$.

Our next result describes the structure of $S[\beta]$ as a graded spectrum:

Proposition 3.4.5. *For each integer $n \geq 0$, the map $\beta^n : S^{-2n} \to S[\beta]_n$ is an equivalence of spectra. For $n < 0$, the spectrum $S[\beta]_n$ vanishes.*

Proof. For $n \leq 0$, this follows immediately from Proposition 3.4.2. To complete the proof, it will suffice to show that for $n \geq 0$, multiplication by β induces an equivalence $\Sigma^{-2}S[\beta]_n \to S[\beta]_{n+1}$. This follows immediately from the existence of a fiber sequence

$$\underline{\mathrm{Map}}_{\mathbb{A}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(\Sigma^2 \mathbb{A}(1), \mathbb{A}(n+1))$$

$$\downarrow$$

$$\underline{\mathrm{Map}}_{\mathbb{A}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(\mathbb{A}, \mathbb{A}(n+1))$$

$$\downarrow$$

$$\underline{\mathrm{Map}}_{\mathbb{A}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(\mathbb{A} \circledast \mathbb{A}, \mathbb{A}(n+1)),$$

since the third term vanishes for $n \neq -1$. $\qquad\qquad\square$

Remark 3.4.6. Proposition 3.4.5 is equivalent to the assertion that the map β exhibits $S[\beta]$ as the free associative algebra generated by $S^{-2}(-1)$ in the ∞-category of graded spectra. However, it does not supply a description of the $\mathbb{E}_2$-algebra structure on $S[\beta]$.

We will say that an object $X \in {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ is *finite* if it is finite when regarded as an object of $\mathrm{Rep}(\mathbf{Z})$. Note that this is equivalent to the condition that the sum $\bigoplus_{n \in \mathbf{Z}} X_n$ is a finite spectrum, by virtue of Remark 3.2.12. In what follows, we will identify the monoidal ∞-category Θ with the full subcategory of ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ spanned by the finite objects.

Warning 3.4.7. Every compact object of ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ is finite, but the converse fails: for example, the filtered spectrum $\mathbb{A}$ itself is finite, but is not compact when regarded as an object of ${}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$.

Note that the construction $M \mapsto M \otimes_{S[\beta]} \mathbb{A}$ determines a monoidal functor

$$F : \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \to {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})).$$

The functor F has a right adjoint $G : {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z})) \to \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$, given by the formula

$$G(N) = \underline{\mathrm{Map}}^{\mathrm{gd}}(\mathbb{A}, N).$$

Proposition 3.4.8. *Let* $\mathrm{RMod}^{\mathrm{fin}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ *be the full subcategory of* $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ *spanned by the compact objects. Then the functor F induces a (monoidal) equivalence of ∞-categories*

$$F' : \mathrm{RMod}^{\mathrm{fin}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \to \Theta.$$

Proof. We first show that F' is fully faithful. Fix a pair of objects

$$M, N \in \mathrm{RMod}^{\mathrm{fin}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}));$$

we wish to show that the canonical map

$$\phi_{M,N} : \underline{\mathrm{Map}}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(M, N) \to \underline{\mathrm{Map}}_{\mathbb{A}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))}(FM, FN)$$

is an equivalence of spectra. Let us regard N as fixed. The collection of those M for which $F(M) \in \Theta$ and the map $\phi_{M,N}$ is an equivalence is a stable subcategory of $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ which is closed under retracts. Consequently, it will suffice to show that this subcategory contains a set of compact generators for $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$. We may therefore assume without loss of generality that M has the form $\Sigma^k S[\beta](n)$ for some integers k and n, in which case both sides can be identified with the spectrum $\Sigma^{-k} N_{-n}$.

We now prove that F' is essentially surjective. Let $M \in \Theta$; we wish to show that M belongs to the essential image of F'. Replacing M by a shift $M(k)$ if necessary, we may assume that M is concentrated in degrees ≥ 0. If $M \simeq 0$, there is nothing to prove. Otherwise, there exists some largest integer $n \geq 0$ such that $M_n \neq 0$. We will proceed by induction on n. Let M'' denote the graded spectrum given by

$$M''_k = \begin{cases} M_k & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Corollary 3.4.3 that M'' can be regarded as an $\mathbb{A}$-bimodule object of $\mathrm{Rep}(\mathbf{Z})$ in an essentially unique way. Using Proposition 3.4.2, we conclude that the identity map from M_n to M''_n extends (in an essentially unique way) to a morphism of bimodules $M \to M''$. This map of bimodules fits into a fiber sequence

$$M' \to M \to M''.$$

Since M' is concentrated in degrees $< n$, it belongs to the essential image of F' by virtue of the inductive hypothesis. It will therefore suffice to show that M'' belongs to the essential image of F'. This is clear: we have $M'' = F'(M''_n(-n) \otimes S[\beta])$. $\qquad\square$

3.5. **The Monoidal Functor ϕ: Categorical Description.** Let X be a graded spectrum equipped with the structure of a right module over the graded $\mathbb{E}_2$-ring $S[\beta]$ of §3.4. We will say that M is *periodic* if multiplication by β induces an equivalence of graded spectra

$$\beta : M \to \Sigma^2 M(1).$$

We let $\mathrm{RMod}^{\mathrm{per}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ denote the full subcategory of $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ spanned by the periodic right $S[\beta]$-modules.

Construction 3.5.1. Since $S[\beta]$ is an $\mathbb{E}_2$-algebra, we can regard the sequence

$$S[\beta] \xrightarrow{\beta} \Sigma^2 S[\beta](1) \xrightarrow{\beta} \Sigma^4 S[\beta](2) \xrightarrow{\beta} \cdots$$

as a diagram of right $S[\beta]$-modules. Let us denote the colimit of this diagram by $S[\beta^{\pm 1}]$.

If M is a right $S[\beta]$-module, we let $M[\beta^{-1}]$ denote the right $S[\beta]$-module given by $M \otimes_{S[\beta]} S[\beta^{\pm 1}]$ (where the tensor product is formed in the monoidal ∞-category $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$).

Remark 3.5.2. Using the description of $S[\beta]$ supplied by Proposition 3.4.5, we conclude that the graded spectrum $S[\beta^{\pm 1}]$ is given by $S[\beta^{\pm 1}]_n \simeq S^{-2n}$ for every integer n.

Lemma 3.5.3. *The construction $M \mapsto M[\beta^{-1}]$ is left adjoint to the inclusion*

$$\mathrm{RMod}_{S[\beta]}^{\mathrm{per}}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \hookrightarrow \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})).$$

Proof. Note that $M[\beta^{-1}]$ can be identified with the colimit of the sequence

$$M \xrightarrow{\beta} \Sigma^2 M(1) \xrightarrow{\beta} \Sigma^4 M(2) \xrightarrow{\beta} \cdots$$

In particular, its homotopy groups are given by

$$\pi_s M[\beta^{-1}]_t \simeq \varinjlim \pi_{s-2k} M_{t+k}.$$

From this description, we immediately deduce that $M[\beta^{-1}]$ is periodic. To complete the proof, it will suffice to show that if N is periodic, then the canonical map

$$\mathrm{Map}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(M[\beta^{-1}], N) \to \mathrm{Map}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(M, N)$$

is a homotopy equivalence. In fact, we claim that each of the maps

$$\mathrm{Map}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(\Sigma^{2k+2} M(k+1), N) \to \mathrm{Map}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(\Sigma^{2k} M(k), N),$$

is a homotopy equivalence; this follows from our assumption that β induces an equivalence $\Sigma^{-2} N(-1) \to N$. $\square$

Lemma 3.5.4. *The localization functor $M \mapsto M[\beta^{-1}]$ is compatible with the monoidal structure on $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$, in the sense of Definition H.2.2.1.6.*

Proof. Let $L : \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \to \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ be the functor given by $M \mapsto M[\beta^{-1}]$, and let $f : M \to M'$ be a morphism of $S[\beta]$-modules which is an L-equivalence. We wish to show that for every right $S[\beta]$-module N, the induced maps

$$g : M \otimes_{S[\beta]} N \to M' \otimes_{S[\beta]} N \qquad h : N \otimes_{S[\beta]} M \to N \otimes_{S[\beta]} M'$$

are also L-equivalences. We will show that h is an L-equivalence; the proof for g is similar. Without loss of generality we may assume that $M' = M[\beta^{-1}]$. In this case, we can identify h with the canonical map

$$N \times_{S[\beta]} M \to N \otimes_{S[\beta]} M \otimes_{S[\beta]} S[\beta^{\pm 1}],$$

which is an L-equivalence by virtue of Lemma 3.5.3. $\square$

It follows from Lemma 3.5.4 that the ∞-category $\mathrm{RMod}_{S[\beta]}^{\mathrm{per}}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ inherits a monoidal structure, whose tensor product

$$(M, N) \mapsto (M \otimes_{S[\beta]} N)[\beta]^{-1} \simeq M \otimes_{S[\beta]} N[\beta^{-1}] \simeq M \otimes_{S[\beta]} N$$

coincides with the tensor product on $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ (however, it has a different unit object: the graded spectrum $S[\beta^{\pm 1}]$). The construction $M \mapsto M[\beta^{-1}]$ can be regarded as a monoidal functor from $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ to $\mathrm{RMod}_{S[\beta]}^{\mathrm{per}}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$, or as a lax monoidal functor from $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ to itself. In particular, $S[\beta^{\pm 1}] = S[\beta][\beta^{-1}]$ inherits the structure of an associative algebra object of $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$.

Proposition 3.5.5. *The forgetful functor*

$$g : \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \to \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$$

is a fully faithful embedding, whose essential image consists of the periodic right $S[\beta]$-modules.

Proof. The functor g admits a right adjoint f, given by $f(M) = M \otimes_{S[\beta]} S[\beta^{\pm 1}] \simeq M[\beta^{-1}]$. To prove that g is fully faithful, it will suffice to show that the counit map $v : f \circ g \to \mathrm{id}$ is an equivalence. In other words, we must show that if M admits the structure of an $S[\beta^{\pm 1}]$-module, then the action map

$$v_M : M[\beta^{-1}] \simeq M \otimes_{S[\beta]} S[\beta^{\pm 1}] \to M$$

is an equivalence. This is clear, since M is periodic as a right $S[\beta]$-module and v_M is left homotopy inverse to the map $M \to M[\beta^{-1}]$. To complete the proof, it will suffice to show that a right $S[\beta]$-module N is periodic if and only if the unit map $N \to (g \circ f)(N) = N[\beta^{-1}]$ is an equivalence, which follows from Lemma 3.5.3. $\qquad\square$

Remark 3.5.6. Since $S[\beta]$ is an $\mathbb{E}_2$-algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$, the symmetric monoidal ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ acts centrally on $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$. Combining Lemma 3.5.4 with Remark 2.1.4, we conclude that $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ also acts centrally on the ∞-category

$$\mathrm{RMod}^{\mathrm{per}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \simeq \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})).$$

Consequently, we can view $S[\beta^{\pm 1}]$ as an $\mathbb{E}_2$-algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$, and the natural map $S[\beta] \to S[\beta^{\pm 1}]$ as a morphism of $\mathbb{E}_2$-algebras (see Corollary H.5.1.2.6).

Proposition 3.5.7. *The composite functor*

$$f : \mathrm{Sp} \hookrightarrow \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \overset{\otimes S[\beta^{\pm 1}]}{\longrightarrow} \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$$

is an equivalence of monoidal ∞-categories.

Proof. Note that the functor f admits a right adjoint g, given on objects by the formula $g(M) = M_0$. For every spectrum X, the unit map $X \to (g \circ f)(X) = (X \otimes S[\beta^{\pm 1}])_0$ is given by the smash product of X with the unit map $S \to S[\beta^{\pm 1}]_0$, and is therefore an equivalence. This proves that f is fully faithful. To complete the proof, it will suffice to show that g is conservative. Since g is an exact functor between stable ∞-categories, this is equivalent to the assertion that if $M \in \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ satisfies $g(M) \simeq 0$, then $M \simeq 0$. This is clear: for each integer n, multiplication by a suitable power of β induces an equivalence $M_n \simeq \Sigma^{-2n} M_0 = \Sigma^{-2n} g(M) \simeq 0$. $\qquad\square$

Combining Proposition 3.5.7 with Remark 2.1.3, we obtain the following result:

Corollary 3.5.8. *The central action of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ on the ∞-category $\mathrm{RMod}^{\mathrm{per}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ factors (in an essentially unique way) through an $\mathbb{E}_2$-monoidal functor $\Phi : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Sp}$.*

Remark 3.5.9. When regarded as a monoidal functor, Φ is given by composing the map

$$(\bullet \otimes S[\beta^{\pm 1}]) : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$$

with the equivalence $\mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \simeq \mathrm{Sp}$ of Proposition 3.5.7. Concretely, we have

$$\Phi(X) = (X \otimes S[\beta^{\pm 1}])_0 \simeq \bigoplus_{n \in \mathbf{Z}} \Sigma^{2n} X_n.$$

Remark 3.5.10. The data of the functor Φ of Corollary 3.5.8 is equivalent to the data of an $\mathbb{E}_2$-monoidal functor $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Sp}$. Note that such a functor automatically factors through the Kan complex $\mathrm{Pic}(S) = \mathrm{Sp}^{\mathrm{inv}} \subseteq \mathrm{Sp}$. Using Remark 3.5.9, we see that the map $\mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(S)$ is given, at the level of objects, by $n \mapsto S^{2n}$.

The next result characterizes the $\mathbb{E}_2$-monoidal functor Φ:

Proposition 3.5.11. *The functor* $\Phi : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Sp}$ *of Corollary 3.5.8 admits a right adjoint* $\Psi : \mathrm{Sp} \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. *Moreover, the functor* Ψ *is given (as a lax* $\mathbb{E}_2$*-monoidal functor) by* $X \mapsto X \otimes S[\beta^{\pm 1}]$.

Proof. The functor Φ preserves small colimits and therefore admits a right adjoint Ψ by Corollary T.5.5.2.9. Since Φ is an $\mathbb{E}_2$-monoidal functor, the functor Ψ is lax $\mathbb{E}_2$-monoidal. In particular, $\Psi(S)$ can be regarded as an $\mathbb{E}_2$-algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. It follows immediately from Remark 3.5.9 that the functor Ψ preserves small colimits and is therefore Sp-linear; it is therefore given by $\Psi(X) = X \otimes \Psi(S)$. We will complete the proof by showing that $\Psi(S)$ is equivalent to $S[\beta^{\pm 1}]$ as an $\mathbb{E}_2$-algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$.

Let A denote the algebra $S[\beta^{\pm 1}]$ regarded as an regarded as an associative algebra object of $\mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$. Then we have an evident central action α of $S[\beta^{\pm 1}]$ on A. The equivalence $\mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \simeq \mathrm{Sp}$ of Proposition 3.5.7 carries A to the sphere spectrum, so that α determines a central action of $\Phi(S[\beta^{\pm 1}])$ on S in the ∞-category of spectra. By virtue of Remark 2.1.3, the central action of $\Phi(S[\beta^{\pm 1}])$ on S is classified by a map of $\mathbb{E}_2$-rings $\Phi(S[\beta^{\pm 1}]) \to S$, which we can identify with a map $S[\beta^{\pm 1}] \to \Psi(S)$ of $\mathbb{E}_2$-algebras in $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. The map ρ exhibits $\Psi(S)$ as a $S[\beta^{\pm 1}]$-module; by virtue of Proposition 3.5.7 it will suffice to show that the unit map

$$S \simeq S[\beta^{\pm 1}]_0 \to \Psi(S)_0$$

is an equivalence. This is clear, since the construction $X \mapsto \Psi(X)_0$ is right adjoint to the composite functor

$$\mathrm{Sp} \hookrightarrow \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \xrightarrow{\Phi} \mathrm{Sp}$$

and is therefore an equivalence of ∞-categories. $\square$

Remark 3.5.12. From the description of the functor Φ supplied by Remark 3.5.9, we see that Φ restricts to an $\mathbb{E}_2$-monoidal functor $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Sp}^{\mathrm{fin}}$. Passing to ∞-categories of modules, we obtain a monoidal functor $\mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$. We let ϕ denote the composite map

$$\mathrm{Tors}(\mathbf{Z}) \xrightarrow{\mu} \mathcal{C}\mathrm{at}_\infty^{\mathrm{gd}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}},$$

where μ is the symmetric monoidal functor of Construction 2.4.14. More informally, the functor ϕ is given by

$$\phi(T) = \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}})).$$

Then the action of $\mathrm{Tors}(\mathbf{Z})$ on the ∞-category

$$\mathrm{RMod}_{\mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}) \simeq \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$$

described in Remark 2.4.15 is given pointwise by the formula

$$(T, \mathcal{C}) \mapsto \phi(T) \otimes \mathcal{C}.$$

Remark 3.5.13. Let S^{per} denote the $\mathbb{E}_2$-algebra given by the image of $S[\beta^{\pm 1}]$ under the symmetric monoidal functor $\mathrm{Und} : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Sp}$, so that as a spectrum S^{per} is given by the sum $\bigoplus_{n \in \mathbf{Z}} S^{-2n}$. We will refer to S^{per} as the 2-*periodic sphere spectrum*. We let $\mathrm{RMod}_{S^{\mathrm{per}}}$ denote the monoidal ∞-category of right S^{per}-module spectra, and $\mathrm{RMod}_{S^{\mathrm{per}}}^{\mathrm{fin}}$ the smallest full subcategory of $\mathrm{RMod}_{S^{\mathrm{per}}}$ that contains S^{per} itself. We will say that a stable ∞-category $\mathcal{C}$ is 2-*periodic* if it is right-tensored over the ∞-category $\mathrm{RMod}_{S^{\mathrm{per}}}$. Note that if $\mathcal{C}$ is a 2-periodic stable ∞-category, then the double suspension functor $\Sigma^2 : \mathcal{C} \to \mathcal{C}$ is homotopic to the identity (but the converse need not be true).

We have canonical equivalences

$$
\begin{aligned}
\mathrm{RMod}_{S^{\mathrm{per}}} \quad &\simeq \quad \mathrm{Sp}\,\widehat{\otimes}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(\mathrm{RMod}_{S[\beta^{\pm 1}]}\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \\
&\simeq \quad \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}})\widehat{\otimes}_{\mathrm{Ind}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}))}\mathrm{Ind}(\mathrm{RMod}^{\mathrm{fin}}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))) \\
&\simeq \quad \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}}\otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})}\mathrm{RMod}^{\mathrm{fin}}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))).
\end{aligned}
$$

We therefore obtain a fully faithful embedding

$$
\mathrm{Sp}^{\mathrm{fin}}\otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})}\mathrm{RMod}^{\mathrm{fin}}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))\hookrightarrow \mathrm{RMod}_{S^{\mathrm{per}}},
$$

and it is not difficult to see that the essential image of this embedding is the ∞-category $\mathrm{RMod}^{\mathrm{fin}}_{S^{\mathrm{per}}}$. Using Remark 2.4.18, we see that the ∞-category $\mathrm{RMod}_{\mathrm{RMod}^{\mathrm{fin}}_{S^{\mathrm{per}}}}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty})$ of 2-periodic stable ∞-categories can be identified with the ∞-category of (homotopy) fixed points for the action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$ via the monoidal functor ϕ of Remark 3.5.12.

3.6. **The Construction $\mathcal{C}\mapsto \mathrm{MF}(\mathcal{C})$.** In this section, we will describe a construction which associates to each locally filtered stable ∞-category $\mathcal{C}$ another stable ∞-category $\mathrm{MF}(\mathcal{C})$, which we call the ∞-*category of equivariant matrix factorizations in $\mathcal{C}$* (Construction 3.6.6). Our goals are twofold:

   (*a*) To describe the structure of $\mathrm{MF}(\mathcal{C})$ in sufficiently concrete terms that it is possible to do calculations in $\mathrm{MF}(\mathcal{C})$ (Remark 3.6.7).
   (*b*) To show that the formation of equivariant matrix factorizations is given by a functor

$$
\mathrm{MF}:\mathcal{C}\mathrm{at}^{\mathrm{filt}}_{\infty}\to \mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}
$$

which is *equivariant* with respect to the action of $S^1\simeq \mathrm{Tors}(\mathbf{Z})$ (see Proposition 3.6.10).

We begin with some general remarks. Recall that the constructionn $M\mapsto M_0$ determines a monoidal equivalence of ∞-categories

$$
\mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))\simeq \mathrm{Sp}
$$

(Proposition 3.5.7). It follows that the construction

$$
M\mapsto M[\beta^{-1}]_0\simeq \varinjlim \Sigma^{-2k}M_k
$$

determines a monoidal functor from $\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))\to \mathrm{Sp}$. Note that this functor preserves compact objects, and therefore restricts to a monoidal functor

$$
\Theta\simeq \mathrm{RMod}^{\mathrm{fin}}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))\to \mathrm{Sp}^{\mathrm{fin}},
$$

which we will view as a morphism between associative algebra objects of the ∞-category $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$.

Notation 3.6.1. Let $\mathcal{C}$ be a stable ∞-category equipped with a right action of Θ. We will indicate this action by a functor

$$
\otimes_{S[\beta]}:\mathcal{C}\times \Theta \to \mathcal{C}.
$$

Then $\mathcal{C}$ inherits an action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ (via the central action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}})$ on Θ), and may therefore be regarded as a graded stable ∞-category. Moreover, for every object $C\in \mathcal{C}$, multiplication by $\beta \in \pi_{-2}S[\beta]_1$ induces a map $C\to \Sigma^2 C(1)$, which we will also denote by β.

Construction 3.6.2. Let $\mathcal{C}$ be a stable ∞-category equipped with a right action of Θ. We let $\mathcal{C}[\beta^{-1}]$ denote the relative tensor product $\mathcal{C}\otimes_{\Theta}\mathrm{Sp}^{\mathrm{fin}}$ formed in the ∞-category $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$. Here we regard $\mathrm{Sp}^{\mathrm{fin}}$ as a left Θ-module via the monoidal functor $M\mapsto \varinjlim \Sigma^{-2k}M_k$.

Remark 3.6.3. In the situation of Construction 3.6.2, there is an evident functor $\mathcal{C}\to \mathcal{C}[\beta^{-1}]$ which we will denote by $C\mapsto C[\beta^{-1}]$. Since the monoidal functor $\Theta \to \mathrm{Sp}^{\mathrm{fin}}$ is essentially surjective, the functor $C\mapsto C[\beta^{-1}]$ is also essentially surjective.

Remark 3.6.3 allows us to understand what the objects of the ∞-category $\mathcal{C}[\beta^{-1}]$ look like. Our next goal is to understand the morphisms in $\mathcal{C}[\beta^{-1}]$. For this, it will be convenient to work in the Ind-completion $\mathrm{Ind}(\mathcal{C}[\beta^{-1}])$. Recall that the construction $\mathcal{C} \mapsto \mathrm{Ind}(\mathcal{C})$ determines a symmetric monoidal functor from $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$ to the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ of presentable stable ∞-categories. In particular, if $\mathcal{C}$ is an essentially small stable ∞-category equipped with a right action of Θ, then $\mathrm{Ind}(\mathcal{C})$ inherits an action of $\mathrm{Ind}(\Theta) \simeq \mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$. Moreover, using Theorem H.4.8.4.6 we obtain an equivalence

$$
\begin{aligned}
\mathrm{Ind}(\mathcal{C}[\beta^{-1}]) \quad &= \quad \mathrm{Ind}(\mathcal{C} \otimes_{\Theta} \mathrm{Sp}^{\mathrm{fin}}) \\
&\simeq \quad \mathrm{Ind}(\mathcal{C}) \widehat{\otimes}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))} \mathrm{Sp} \\
&\simeq \quad \mathrm{Ind}(\mathcal{C}) \widehat{\otimes}_{\mathrm{RMod}_{S[\beta]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))} \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \\
&\simeq \quad \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Ind}(\mathcal{C})).
\end{aligned}
$$

Note that since the multiplication map $S[\beta^{\pm 1}] \otimes_{S[\beta]} S[\beta^{\pm 1}] \to S[\beta^{\pm 1}]$ is an equivalence, the forgetful functor $\mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$ is a fully faithful embedding. It follows that there exists a fully faithful embedding $\mathcal{C}[\beta^{-1}] \to \mathrm{Ind}(\mathcal{C})$, given by the composition

$$
\mathcal{C}[\beta^{-1}] \hookrightarrow \mathrm{Ind}(\mathcal{C}[\beta^{-1}]) \simeq \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C}).
$$

Unwinding the definitions, we see that this fully faithful embedding carries an object $C[\beta^{-1}] \in \mathcal{C}[\beta^{-1}]$ to the Ind-object of $\mathcal{C}$ represented by the direct system

$$
C \xrightarrow{\beta} \Sigma^2 C(1) \xrightarrow{\beta} \Sigma^4 C(2) \xrightarrow{\beta} \cdots.
$$

In what follows, we will often identify $\mathcal{C}[\beta^{-1}]$ with its essential image in $\mathrm{Ind}(\mathcal{C})$, and for $C \in \mathcal{C}$ we identify $C[\beta^{-1}]$ with the Ind-object given by

$$
C \xrightarrow{\beta} \Sigma^2 C(1) \xrightarrow{\beta} \Sigma^4 C(2) \xrightarrow{\beta} \cdots.
$$

Remark 3.6.4. Let $\mathcal{C}$ be a stable ∞-category equipped a right action of Θ. Then the above discussion supplies the following informal description of the ∞-category $\mathcal{C}[\beta^{-1}]$:

- The objects of $\mathcal{C}[\beta^{-1}]$ have the form $C[\beta^{-1}]$, where C is an object of $\mathcal{C}$.
- Morphism spaces in $\mathcal{C}[\beta^{-1}]$ are given by

$$
\begin{aligned}
\mathrm{Map}_{\mathcal{C}[\beta^{-1}]}(C[\beta^{-1}], D[\beta^{-1}]) \quad &\simeq \quad \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C[\beta^{-1}], D[\beta^{-1}]) \\
&\simeq \quad \mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(C, D[\beta^{-1}]) \\
&\simeq \quad \varinjlim \mathrm{Map}_{\mathcal{C}}(C, \Sigma^{2n} D(n)).
\end{aligned}
$$

Remark 3.6.5. The construction $\mathcal{C} \mapsto \mathcal{C}[\beta^{-1}]$ determines a functor $\mathrm{RMod}_{\Theta}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}) \to \mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$. This functor is equivariant with respect to the action of the monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$, where $\mathrm{Tors}(\mathbf{Z})$ acts on $\mathrm{RMod}_{\Theta}(\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty})$ as in Remark 3.3.4 and on $\mathcal{C}\mathrm{at}^{\mathrm{St}}_{\infty}$ via the monoidal functor ϕ of Remark 3.5.12.

Construction 3.6.6. Let $\mathcal{C}$ be a locally filtered stable ∞-category (which we identify with a right action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ on $\mathcal{C}$), so that $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$ inherits an action of the monoidal ∞-category $\Theta \simeq {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}^{\mathrm{ds}}))$. We define a new stable ∞-category $\mathrm{MF}(\mathcal{C})$ by the formula

$$
\mathrm{MF}(\mathcal{C}) = \mathrm{RMod}_{\mathbb{A}}(\mathcal{C})[\beta^{-1}].
$$

We will refer to $\mathrm{MF}(\mathcal{C})$ as the ∞-*category of equivariant matrix factorizations in $\mathcal{C}$*.

Remark 3.6.7. Let $\mathcal{C}$ be as in Construction 3.6.6. Using Remark 3.6.4, we can describe the ∞-category $\mathrm{MF}(\mathcal{C})$ more informally as follows:

- The objects of $\mathrm{MF}(\mathcal{C})$ are right $\mathbb{A}$-module objects of $\mathcal{C}$. If C is an object of $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$, we will denote its image in $\mathrm{MF}(\mathcal{C})$ by $C[\beta^{-1}]$.
- Given a pair of right $\mathbb{A}$-module objects $C, D \in \mathcal{C}$, the mapping space
$$\mathrm{Map}_{\mathrm{MF}(\mathcal{C})}(C[\beta^{-1}], D[\beta^{-1}])$$
is given by the filtered colimit
$$\varinjlim_{n} \mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})}(C, \Sigma^{2n} D(n)),$$
where the transition maps are determined by the morphism $\beta : \mathbb{A} \to \Sigma^2 \mathbb{A}(1)$ in $_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$.

Remark 3.6.8. Let $\mathcal{C}$ be a locally filtered stable ∞-category, and let us indicate the right action of $\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})$ on $\mathcal{C}$ by
$$\circledast : \mathcal{C} \times \mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z}) \to \mathcal{C}.$$
Recall that the anchor map $\beta : \mathbb{A} \to \Sigma^2 \mathbb{A}(1)$ in $_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$ fits into a fiber sequence
$$\mathbb{A} \circledast \mathbb{A} \to \mathbb{A} \xrightarrow{\beta} \Sigma^2 \mathbb{A}(1).$$
It follows that for every object $C \in \mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$, we have a fiber sequence
$$C \circledast \mathbb{A} \to C \xrightarrow{\beta_C} \Sigma^2 C(1)$$
in the ∞-category $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$.

Remark 3.6.9. Let $\mathcal{C}$ be as in Remark 3.6.8. Suppose that $C \in \mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$ is a *free* $\mathbb{A}$-module: that is, it has the form $C_0 \circledast \mathbb{A}$ for some object $C_0 \in \mathcal{C}$. In that case, the natural map $C \circledast \mathbb{A} \to C$ admits a section (as a morphism of right $\mathbb{A}$-modules). Combining this observation with Remark 3.6.8, we conclude that the map
$$\beta_C : C \to \Sigma^2 C(1)$$
is nullhomotopic. It follows that *all* of the maps in the sequence
$$C \to \Sigma^2 C(1) \to \Sigma^4 C(2) \to \cdots$$
are nullhomotopic in $\mathrm{RMod}_{\mathbb{A}}(\mathcal{C})$, so that $C[\beta^{-1}]$ is a zero object of the ∞-category $\mathrm{MF}(\mathcal{C})$.

Combining Remark 3.6.5 with Corollary 3.3.9, we obtain the following:

Proposition 3.6.10. *The construction $\mathcal{C} \mapsto \mathrm{MF}(\mathcal{C})$ determines a functor $\mathrm{MF} : \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{filt}} \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}}$. Moreover, this functor is $\mathrm{Tors}(\mathbf{Z})$-equivariant, where $\mathrm{Tors}(\mathbf{Z})$ acts on $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{filt}}$ as in Remark 3.1.17 and on $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}}$ via the monoidal functor $\phi : \mathrm{Tors}(\mathbf{Z}) \to \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}}$ of Remark 3.5.12.*

More informally, Proposition 3.6.10 asserts that for every $\mathbf{Z}$-torsor T and every locally filtered stable ∞-category $\mathcal{C}$, there is a canonical equivalence of stable ∞-categories
$$\mathrm{MF}(\mathrm{Rep}^{\mathrm{fin}}(T) \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})} \mathcal{C}) \simeq \phi(T) \otimes \mathrm{MF}(\mathcal{C}) = \mathrm{RMod}_{S[\beta^{\pm 1}]}(\mathrm{Rep}^{\mathrm{fin}}(T^{\mathrm{ds}})) \otimes \mathrm{MF}(\mathcal{C}).$$

4. K-Theory

Let $\mathcal{C}$ be an arbitrary ∞-category. For each integer $n \geq 0$, we can consider the Kan complex $\mathrm{Fun}(\Delta^n, \mathcal{C})^{\simeq}$ whose objects are diagrams
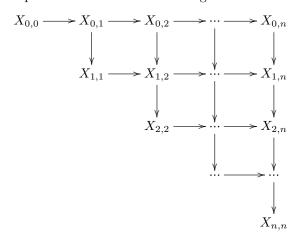$$X_0 \to X_1 \to \cdots \to X_n$$
in $\mathcal{C}$. This Kan complex depends contravariantly on the simplex Δ^n; we therefore obtain a simplicial Kan complex
$$T_{\bullet}(\mathcal{C}) = \mathrm{Fun}(\Delta^{\bullet}, \mathcal{C})^{\simeq}.$$

The simplicial Kan complex $T_\bullet(\mathcal{C})$ is an example of a *complete Segal space*, and the construction $\mathcal{C} \mapsto T_\bullet(\mathcal{C})$ determines an equivalence between the homotopy theory of ∞-categories and the homotopy theory of complete Segal spaces (see [5]).

Assume now that the ∞-category $\mathcal{C}$ is pointed and admits finite colimits. In this case, one can introduce a refinement of the simplicial space $T_\bullet(\mathcal{C})$. For each $n \geq 0$, one can associate an ∞-category $S_n(\mathcal{C})$ which parametrizes commutative diagrams

$$\begin{array}{ccccccccc}
X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,2} & \longrightarrow & \cdots & \longrightarrow & X_{0,n} \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & X_{1,1} & \longrightarrow & X_{1,2} & \longrightarrow & \cdots & \longrightarrow & X_{1,n} \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & X_{2,2} & \longrightarrow & \cdots & \longrightarrow & X_{2,n} \\
& & & & & & \downarrow & & \downarrow \\
& & & & & & \cdots & \longrightarrow & \cdots \\
& & & & & & & & \downarrow \\
& & & & & & & & X_{n,n}
\end{array}$$

in which each square is required to be a pushout and each object $X_{i,i}$ appearing along the diagonal is required to be a zero object. Such a diagram canonically determined by the chain of morphisms

$$X_{0,1} \to X_{0,2} \to \cdots \to X_{0,n},$$

so we have canonical equivalences

$$S_n(\mathcal{C}) \simeq \operatorname{Fun}(\Delta^{n-1}, \mathcal{C}) \qquad S_n(\mathcal{C})^{\simeq} \simeq T_{n-1}(\mathcal{C}).$$

However, the above description exhibits some extra functoriality: the simplicial Kan complex $T_\bullet(\mathcal{C})$ is weakly equivalent to the *shift* of another simplicial set $S_\bullet(\mathcal{C})^{\simeq}$, which contain face and degeneracy maps which are not visible in $T_\bullet(\mathcal{C})$: the "extra" face maps are given by the construction which assign to a diagram

$$X_1 \to X_2 \to X_2 \to \cdots \to X_n$$

the associated diagram of cofibers

$$X_2/X_1 \to X_2/X_1 \to \cdots \to X_n/X_1.$$

Following [9] and [1], we define the *Waldhausen K-theory* of the ∞-category $\mathcal{C}$ to be the geometric realization $|S_\bullet(\mathcal{C})^{\simeq}|$ of the simplicial Kan complex $S_\bullet(\mathcal{C})^{\simeq}$.

If the ∞-category $\mathcal{C}$ is stable, then there is even more functoriality. Given a point of $S_n(\mathcal{C})$ corresponding to a sequence

$$X_1 \to X_2 \to \cdots \to X_n$$

as above, we note that each cofiber X_i/X_1 is equipped with a natural map to the suspension of X_1. This construction is functorial and gives rise to a map $\sigma : S_n(\mathcal{C}) \to S_n(\mathcal{C})$, which carries a diagram

$$X_1 \to X_2 \to X_2 \to \cdots \to X_n$$

to another diagram

$$X_2/X_1 \to X_2/X_1 \to \cdots \to X_n/X_1 \to \Sigma(X_1).$$

The map σ is an equivalence of ∞-categories: in fact, the $(n+1)$st iterate of σ is given by the construction which assigns to each diagram

$$X_1 \to X_2 \to X_2 \to \cdots \to X_n$$

the induced diagram of double suspensions

$$\Sigma^2(X_1) \to \Sigma^2(X_2) \to \cdots \to \Sigma^2(X_n),$$

which gives a homotopy equivalence from $S_n(\mathcal{C})$ to itself because the suspension functor on $\mathcal{C}$ is assumed to be an equivalence. Elaborating on this argument, one can show that the simplicial ∞-category $S_\bullet(\mathcal{C})$ can be refined to a *paracyclic ∞-category* $S_\bullet^{\circlearrowleft}(\mathcal{C})$, which we will call the *paracyclic Waldhausen construction* on $\mathcal{C}$.

Our main goal in this section is to show that the paracyclic Waldhausen construction $S_\bullet^{\circlearrowleft}$ determines an S^1-equivariant functor from the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ of stable ∞-categories (equipped with the S^1-action via the monoidal functor ϕ of Remark 3.5.12) to the ∞-category of paracyclic ∞-categories (equipped with an S^1-action which we will describe in §4.3). This has two immediate corollaries:

(a) The formation of algebraic K-theory $\mathcal{C} \mapsto K(\mathcal{C})$ is invariant under the action of S^1 on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ (this is the statement of Theorem 1.1.1, modulo the geometric description of the monoidal functor ϕ): this follows from the fact that the geometric realization of paracyclic spaces is an S^1-equivariant construction (see Remark 4.2.12).

(b) If $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ is a homotopy fixed point for the action of S^1 (in other words, if $\mathcal{C}$ is linear over the 2-periodic sphere spectrum S^{per} of Remark 3.5.13), then the paracyclic Kan complex $S_\bullet^{\circlearrowleft}(\mathcal{C})$ is a homotopy fixed point for the action of S^1 on paracyclic spaces: that is, it descends to a cyclic space (which is a suitable input for the 2-Segal machinery described in [4]; see §1.3 for a brief discussion).

Let us now outline the contents of this section. We will begin in §4.1 with a discussion the Waldhausen construction in the setting of ∞-categories. In §4.2 we review the theory of paracyclic spaces, and in §4.3 we define the paracyclic Waldhausen construction $S_\bullet^{\circlearrowleft}$ and show that it refines the classical Waldhausen construction $S_\bullet$. To analyze the paracyclic Waldhausen construction, we observe that it is corepresentable: that is, there is a coparacyclic object $\mathrm{Quiv}^\bullet$ of the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ such that the paracyclic Waldhausen construction $S_\bullet^{\circlearrowleft}$ is given by

$$S_\bullet^{\circlearrowleft}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^\bullet, \mathcal{C}).$$

To show that the functor $S_\bullet^{\circlearrowleft}(\mathcal{C})$ is S^1-equivariant, it will suffice to show that the the construction $\bullet \mapsto \mathrm{Quiv}^\bullet$ is S^1-equivariant. We will prove this in §4.6 using the fact that $\mathrm{Quiv}^\bullet$ has a description in terms of equivariant matrix factoriations (which we establish in §4.5) applied to suitable locally filtered stable ∞-categories (which we study in §4.4).

4.1. **The Waldhausen Construction.** In this section, we give a brief review of the construction of the Waldhausen K-theory of a pointed ∞-category $\mathcal{C}$ which admits finite colimits. For a more detailed discussion, we refer the reader to [1].

Definition 4.1.1. Let P be a linearly ordered set. We let $P^{[1]}$ denote the set $\{(x,y) \in P \times P : x \le y\}$. We regard $P^{[1]}$ as a partially ordered subset of the Cartesian product $P \times P$.

Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. An P-*gapped object* of $\mathcal{C}$ is a map $X : \mathrm{N}(P^{[1]}) \to \mathcal{C}$ with the following properties:

(a) For every element $p \in P$, $X(p,p)$ is a zero object of $\mathcal{C}$.

(b) For every triple $p, q, r \in P$ satisfying $p \le q \le r$, the diagram

$$
\begin{array}{ccc}
X(p,q) & \longrightarrow & X(p,r) \\
\downarrow & & \downarrow \\
X(q,q) & \longrightarrow & X(q,r)
\end{array}
$$

is a pushout square in $\mathcal{C}$. In other words, we have a fiber sequence

$$X(p,q) \to X(p,r) \to X(q,r).$$

We let $S_P(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(P^{[1]}), \mathcal{C})$ spanned by the P-gapped objects of $\mathcal{C}$. When P has the form $[n] = \{0 < 1 < \cdots < n\}$ for some nonnegative integer n, we will denote $S_P(\mathcal{C})$ simply by $S_n(\mathcal{C})$. The construction $[n] \mapsto S_n(\mathcal{C})$ depends functorially on the linearly ordered set $[n]$, and therefore determines a simplicial ∞-category $S_\bullet(\mathcal{C})$. We will refer to $S_\bullet(\mathcal{C})$ as the *Waldhausen construction* on $\mathcal{C}$, and we let $S_\bullet(\mathcal{C})^\simeq$ denote its underlying simplicial Kan complex.

Remark 4.1.2. Let $n \ge 0$ be an integer. Then the construction $i \mapsto (0, i+1)$ determines a map of partially ordered sets $\iota : [n] \to [n+1]^{[1]}$. Composition with ι induces a map of ∞-categories $S_{n+1}(\mathcal{C}) \to \mathrm{Fun}(\Delta^n, \mathcal{C})$ which depends functorially on $[n]$. According to Lemma H.1.2.2.4, this functor is an equivalence of ∞-categories. It follows that the simplicial space $S_{\bullet+1}(\mathcal{C})^\simeq$ can be identified with the complete Segal space associated to the ∞-category $\mathcal{C}$.

Remark 4.1.3. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then $S_0(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ spanned by the zero objects. In particular, it is a contractible Kan complex.

Definition 4.1.4. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. We let $K(\mathcal{C})$ denote the fiber product

$$S_0(\mathcal{C})^\simeq \times_{|S_\bullet(\mathcal{C})^\simeq|} S_0(\mathcal{C})^\simeq,$$

formed in the ∞-category $\mathcal{S}$ of spaces. We will refer to $K(\mathcal{C})$ as the *K-theory space* associated to $\mathcal{C}$.

Remark 4.1.5. In the situation of Definition 4.1.4, the space $S_0(\mathcal{C})^\simeq$ is contractible (Remark 4.1.3). We may therefore identify the natural map $S_0(\mathcal{C})^\simeq \to |S_\bullet(\mathcal{C})^\simeq|$ with a base point of $|S_\bullet(\mathcal{C})^\simeq|$, and $K(\mathcal{C})$ with the based loop space $\Omega|S_\bullet(\mathcal{C})^\simeq|$.

4.2. **Cyclic and Paracyclic Objects.** In his section, we review the theory of cyclic and paracyclic objects of an ∞-category $\mathcal{C}$ (such as the ∞-category $\mathcal{S}$ of spaces).

Definition 4.2.1. A *parasimplex* is a nonempty linearly ordered set Λ equipped with an action of the group $\mathbf{Z}$ (which we will indicate by $+ : \Lambda \times \mathbf{Z} \to \Lambda$) which satisfies the following axioms:

(a) For each $\lambda \in \Lambda$, we have $\lambda < \lambda + 1$.
(b) For every pair of elements $\lambda, \lambda' \in \Lambda$, the set $\{\mu \in \Lambda : \lambda \le \mu \le \lambda'\}$ is finite.

If Λ and Λ' are parasimplices, then we will say that a map $f : \Lambda \to \Lambda'$ is *paracyclic* if it is $\mathbf{Z}$-equivariant and nondecreasing.

We let $\mathbf{\Delta}_{\circlearrowleft}$ denote the category whose objects are parasimplices and whose morphisms are paracyclic maps. We will refer to $\mathbf{\Delta}_{\circlearrowleft}$ as the *parasimplex category*.

Definition 4.2.2. Let $\mathcal{C}$ be an ∞-category. A *paracyclic object* of $\mathcal{C}$ is a functor $X : \mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{op}}) \to \mathcal{C}$. A *paracyclic space* is a paracyclic object of the ∞-category $\mathcal{S}$ of spaces, and a *paracyclic ∞-category* is a paracyclic object in the ∞-category $\mathcal{C}\mathrm{at}_\infty$ of ∞-categories.

Example 4.2.3. Let n be a positive integer. Then the set $\frac{1}{n}\mathbf{Z} = \{\frac{a}{n} : a \in \mathbf{Z}\}$ is a parasimplex (when equipped with its usual ordering and action of $\mathbf{Z}$ by translation). Conversely, every parasimplex Λ is isomorphic to $\frac{1}{n}\mathbf{Z}$ for a unique positive integer n (the integer n can be characterized as the cardinality of the set $\{\mu \in \Lambda : \lambda \le \mu < \lambda + 1\}$ for any choice of element $\lambda \in \Lambda$).

Remark 4.2.4. Let Λ be a parasimplex. We can define a new parasimplex $\Lambda^\vee$ as follows:

- The elements of $\Lambda^\vee$ are symbols $\lambda^\vee$, where $\lambda \in \Lambda$.
- The linear ordering of $\Lambda^\vee$ is given by

$$\lambda^\vee \le \mu^\vee \Leftrightarrow \mu \le \lambda.$$

- The action of $\mathbf{Z}$ on $\Lambda^\vee$ is given by

$$\lambda^\vee + n = (\lambda - n)^\vee.$$

The construction $\Lambda \mapsto \Lambda^\vee$ determines a covariant equivalence from the parasimplex category $\mathbf{\Delta}_\circlearrowleft$ to itself. Note that there is a canonical isomorphism $\Lambda \simeq (\Lambda^\vee)^\vee$ which depends functorially on Λ (given by $\lambda \mapsto (\lambda^\vee)^\vee$).

Remark 4.2.5. For every parasimplex Λ, the map $\lambda \mapsto \lambda + 1$ is a paracyclic map from Λ to itself. This construction determines a natural transformation from the identity functor of $\mathbf{\Delta}_\circlearrowleft$ to itself.

Example 4.2.6. Let Q be a nonempty finite linearly ordered set. We let $Q_\mathbf{Z}$ denote the product $Q \times \mathbf{Z}$, which we regard as equipped with the reverse lexicographic ordering (so that $(q, n) \le (q', n')$ if and only if either $n < n'$, or $n = n'$ and $q \le q'$). Then $Q_\mathbf{Z}$ is a parasimplex, if we regard $Q_\mathbf{Z}$ as equipped with the action of $\mathbf{Z}$ given by the formula $(q, m) + n = (q, m + n)$.

Let $\mathbf{\Delta}$ denote the category of combinatorial simplices: that is, the objects of $\mathbf{\Delta}$ are the linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \ge 0$, and the morphisms are nondecreasing functions. The construction $Q \mapsto Q_\mathbf{Z}$ determines a faithful functor $\mathbf{\Delta} \to \mathbf{\Delta}_\circlearrowleft$. It follows from Example 4.2.3 that this functor is essentially surjective (note that $[n]_\mathbf{Z} \simeq \frac{1}{n+1}\mathbf{Z}$ for each $n \ge 0$).

Remark 4.2.7. Let $\mathcal{C}$ be an ∞-category. The construction of Example 4.2.6 determines a forgetful functor from paracyclic objects of $\mathcal{C}$ to simplicial objects of $\mathcal{C}$.

Proposition 4.2.8. *The construction $Q \mapsto Q_\mathbf{Z}$ induces a right cofinal functor $\mathrm{N}(\mathbf{\Delta}) \to \mathrm{N}(\mathbf{\Delta}_\circlearrowleft)$.*

Corollary 4.2.9. *Let $\mathcal{C}$ be an ∞-category which admits small colimits, let X be a paracyclic object of $\mathcal{C}$, and let $Y_\bullet$ be the associated simplicial object of $\mathcal{C}$. Then the canonical map*

$$|Y_\bullet| \to \varinjlim_{\Lambda \in \mathbf{\Delta}_\circlearrowleft} X(\Lambda)$$

is an equivalence in $\mathcal{C}$.

Proof of Proposition 4.2.8. Fix a parasimplex Λ. We wish to show that the category

$$\mathcal{C} = \mathbf{\Delta} \times_{\mathbf{\Delta}_\circlearrowleft} (\mathbf{\Delta}_\circlearrowleft)_{/\Lambda}$$

has weakly contractible nerve. Unwinding the definitions, we can identify the objects of $\mathcal{C}$ with pairs (Q, α), where Q is a nonempty finite linearly ordered set and $\alpha : Q_\mathbf{Z} \to \Lambda$ is nondecreasing $\mathbf{Z}$-equivariant map. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by those pairs (Q, α) for which the restriction $\alpha|_{Q \times \{0\}}$ is injective. The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint and therefore induces a weak homotopy equivalence $\mathrm{N}(\mathcal{C}_0) \to \mathrm{N}(\mathcal{C})$. It will therefore suffice to show that $\mathrm{N}(\mathcal{C}_0)$ is weakly contractible.

Let P denote the collection of all nonempty subsets $S \subseteq \Lambda$ such that $\mu \leq \lambda + 1$ for each $\lambda, \mu \subseteq S$. We regard P as a partially ordered set with respect to inclusions. The construction

$$(Q, \alpha) \mapsto \alpha(Q \times \{0\})$$

determines an equivalence from the category $\mathcal{C}_0$ to the partially ordered set P. It will therefore suffice to show that $\mathrm{N}(P)$ is weakly contractible.

For every pair of elements $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, let $\Lambda_{\lambda,\mu}$ denote the set $\{\nu \in \Lambda : \lambda \leq \nu \leq \mu\}$ and let $P_{\lambda,\mu}$ denote the subset of P consisting of those sets S which are contained in $\Lambda_{\lambda,\mu}$. Then $\mathrm{N}(P)$ can be written as a filtered colimit of simplicial sets of the form $\mathrm{N}(P_{\lambda,\mu})$. It will therefore suffice to show that each $\mathrm{N}(P_{\lambda,\mu})$ is weakly contractible. We proceed by induction on the size of the set $\Lambda_{\lambda,\mu}$. If $\mu \leq \lambda + 1$, then $P_{\lambda,\mu}$ has a largest element (given by the set $S = \Lambda_{\lambda,\mu}$) and there is nothing to prove. Otherwise, let μ' denote the predecessor of μ in Λ, and observe that there is a pushout diagram

$$
\begin{array}{ccc}
\mathrm{N}(P_{\mu-1,\mu'}) & \longrightarrow & \mathrm{N}(P_{\mu-1,\mu}) \\
\downarrow & & \downarrow \\
\mathrm{N}(P_{\lambda,\mu'}) & \longrightarrow & \mathrm{N}(P_{\lambda,\mu}).
\end{array}
$$

It follows from the inductive hypothesis that the simplicial sets $\mathrm{N}(P_{\mu-1,\mu'})$, $\mathrm{N}(P_{\mu-1,\mu})$, and $\mathrm{N}(P_{\lambda,\mu'})$ are weakly contractible, so that $\mathrm{N}(P_{\lambda,\mu})$ is also weakly contractible. $\square$

Construction 4.2.10. Let Λ be a parasimplex and let T be a $\mathbf{Z}$-torsor. We let $T +_{\mathbf{Z}} \Lambda$ denote the quotient of $T \times \Lambda$ by the equivalence relation

$$(t, \lambda) \simeq (t', \lambda') \text{ if } (\exists n \in \mathbf{Z})[t = t' + n \text{ and } \lambda = \lambda' - n]$$

We will denote the image of a pair (t, λ) in $T +_{\mathbf{Z}} \Lambda$ by $[t, \lambda]$. Then $T +_{\mathbf{Z}} \Lambda$ can be regarded as a parasimplex, with the action of $\mathbf{Z}$ given by $[t, \lambda] + n = [t + n, \lambda]$ and linear ordering given by

$$[t, \lambda] \leq [t', \lambda'] \text{ if and only if } (\exists n \in \mathbf{Z})[t = t' + n \text{ and } \lambda + n \leq \lambda'].$$

The construction $(T, \Lambda) \mapsto T +_{\mathbf{Z}} \Lambda$ determines a (left) action of the monoidal category of $\mathbf{Z}$-torsors on the category $\mathbf{\Delta}_{\circlearrowleft}$ of parasimplices. For any ∞-category $\mathcal{C}$, we obtain an induced left action of the monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$ on the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C})$ of paracyclic objects of $\mathcal{C}$, given by the formula

$$(T \otimes X)_\Lambda = X_{-T +_{\mathbf{Z}} \Lambda}$$

where $-T$ denotes an inverse of T in the symmetric monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$.

Remark 4.2.11. The action of the monoidal category of $\mathbf{Z}$-torsors on $\mathbf{\Delta}_{\circlearrowleft}$ induces an action the monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$ on the ∞-category $\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft})$. In fact, we can be more precise: the simplicial set $\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft})$ carries a strict action of the simplicial abelian group $B\mathbf{Z}$, which is homotopy equivalent to $\mathrm{Tors}(\mathbf{Z})$. This action is determined by a functor

$$a : B\mathbf{Z} \times \mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}) \to \mathrm{N}(\mathbf{\Delta}_{\circlearrowleft})$$

which is the identity on objects, and is given on morphisms by the formula

$$(n, f : \Lambda \to \Lambda') \mapsto f + n$$

where $f + n : \Lambda \to \Lambda'$ is the map of parasimplices given by $(f + n)(\lambda) = f(\lambda) + n$.

Remark 4.2.12. Let $\mathcal{C}$ be an ∞-category which admits small colimits. Then the colimit functor

$$\varinjlim : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C}) \to \mathcal{C}$$

is $\mathrm{Tors}(\mathbf{Z})$-equivariant, where we endow $\mathcal{C}$ with the trivial action of $\mathrm{Tors}(\mathbf{Z})$. This follows immediately from the observation that the projection map $\boldsymbol{\Delta}_{\circlearrowleft} \to *$ is $\mathrm{Tors}(\mathbf{Z})$-equivariant.

Definition 4.2.13 (The Cyclic Category). For every pair of parasimplices Λ and Λ', the set $\mathrm{Hom}_{\boldsymbol{\Delta}_{\circlearrowleft}}(\Lambda, \Lambda')$ carries an action of the group $\mathbf{Z}$, given by the formula $(f + n)(\lambda) = f(\lambda) + n$.

We define a category $\boldsymbol{\Delta}_{\mathrm{cyc}}$ as follows:

- The objects of $\boldsymbol{\Delta}_{\mathrm{cyc}}$ are parasimplices Λ.
- Given a pair of objects $\Lambda, \Lambda' \in \boldsymbol{\Delta}_{\mathrm{cyc}}$, we let $\mathrm{Hom}_{\boldsymbol{\Delta}_{\mathrm{cyc}}}(\Lambda, \Lambda')$ be the quotient set

$$\mathrm{Hom}_{\boldsymbol{\Delta}_{\circlearrowleft}}(\Lambda, \Lambda')/\mathbf{Z}.$$

We will refer to $\boldsymbol{\Delta}_{\mathrm{cyc}}$ as the *cyclic category*. If $\mathcal{C}$ is an arbitrary ∞-category, we will refer to $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}}^{\mathrm{op}}), \mathcal{C})$ as the ∞-category of *cyclic objects* of $\mathcal{C}$. In the special case $\mathcal{C} = \mathcal{S}$, we will refer to $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}}^{\mathrm{op}}), \mathcal{C})$ as the ∞-*category of cyclic spaces*.

Remark 4.2.14. Unwinding the definitions, we see that the nerve $\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}})$ can be identified with the quotient of $\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft})$ by the action of the simplicial abelian group $B\mathbf{Z}$ described in Remark 4.2.11. Since the action of $B\mathbf{Z}$ on $\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft})$ is free, we can identify $\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}})$ with the (homotopy) quotient of $\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft})$ by the action of the monoidal ∞-category $\mathrm{Tors}(\mathbf{Z})$. It follows that for any ∞-category $\mathcal{C}$, we can identify the ∞-category of cyclic objects $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}}^{\mathrm{op}}), \mathcal{C})$ with the (homotopy) fixed points for the action of $\mathrm{Tors}(\mathbf{Z})$ on the ∞-category of paracyclic objects $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C})$.

Remark 4.2.15. Let $\mathcal{C}$ be an ∞-category which admits small colimits, so that the construction

$$\varinjlim : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C}) \to \mathcal{C}$$

is $\mathrm{Tors}(\mathbf{Z})$-equivariant (Remark 4.2.12). Passing to homotopy fixed points on both sides (and using the fact that the action of $\mathrm{Tors}(\mathbf{Z}) \simeq S^1$ on $\mathcal{C}$ is trivial), we obtain a map

$$\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}_{\mathrm{cyc}}^{\mathrm{op}}, \mathcal{C})) \to \mathrm{Fun}(\mathrm{BS}^1, \mathcal{C}).$$

More informally: if X is a cyclic object of $\mathcal{C}$, then the geometric realization of X (which can be computed as the colimit of the underlying simplicial object) can be regarded as an S^1-equivariant object of $\mathcal{C}$.

4.3. **The Paracyclic Waldhausen Construction.** Let $\mathcal{C}$ be a stable ∞-category. Our goal in this section is to define the paracyclic Waldhausen construction $S_{\bullet}^{\circlearrowleft}(\mathcal{C})$ (Definition 4.3.4) and to show that it refines the usual Waldhausen construction of §4.1 (Remark 4.3.5).

Definition 4.3.1. Let Λ be a parasimplex. We let $\Lambda_{\circlearrowleft}^{[1]}$ denote the subset

$$\{(\lambda, \mu) \in \Lambda \times \Lambda : \lambda \le \mu \le \lambda + 1\} \subseteq \Lambda \times \Lambda.$$

We regard $\Lambda_{\circlearrowleft}^{[1]}$ as a partially ordered subset of $\Lambda \times \Lambda$.

Let $\mathcal{C}$ be a stable ∞-category. A *paracyclic Λ-gapped object* of $\mathcal{C}$ is a functor $X : \mathrm{N}(\Lambda_{\circlearrowleft}^{[1]}) \to \mathcal{C}$ which satisfies the following conditions:

(a) For every element $\lambda \in \Lambda$, the objects $X(\lambda, \lambda), X(\lambda, \lambda + 1) \in \mathcal{C}$ are zero.

(b) For every triple of elements $\lambda, \mu, \nu \in \Lambda$ satisfying $\lambda \leq \mu \leq \nu \leq \lambda + 1$, the diagram

$$
\begin{array}{ccc}
X(\lambda, \mu) & \longrightarrow & X(\lambda, \nu) \\
\downarrow & & \downarrow \\
X(\mu, \mu) & \longrightarrow & X(\mu, \nu)
\end{array}
$$

is a pushout square. In other words,we have a fiber sequence

$$X(\lambda, \mu) \to X(\lambda, \nu) \to X(\mu, \nu)$$

in the ∞-category $\mathcal{C}$.

We let $S_\Lambda^{\circlearrowleft}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Lambda_{\circlearrowleft}^{[1]}), \mathcal{C})$ spanned by the paracyclic Λ-gapped objects.

Remark 4.3.2. Let Q be a nonempty finite linearly ordered set. The construction $(q, q') \mapsto ((q, 0), (q', 0))$ determines a map of partially ordered sets $\iota : Q^{[1]} \to (Q_{\mathbf{Z}})_{\circlearrowleft}^{[1]}$. For every stable ∞-category $\mathcal{C}$, composition with ι induces a functor

$$\theta : S_{Q_{\mathbf{Z}}}^{\circlearrowleft}(\mathcal{C}) \to S_Q(\mathcal{C}).$$

Proposition 4.3.3. *Let Q be a nonempty finite linearly ordered set and let $\mathcal{C}$ be a stable ∞-category. Then the map $\theta : S_{Q_{\mathbf{Z}}}^{\circlearrowleft}(\mathcal{C}) \to S_Q(\mathcal{C})$ of Remark 4.3.2 is a trivial Kan fibration.*

Proof. It follows immediately from the definitions that θ is a categorical fibration. It will therefore suffice to show that θ is an equivalence of ∞-categories. Choose a least element $q_0 \in Q$, and write $Q = \{q_0\} \sqcup Q_+$. Let $\theta' : S_Q(\mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(Q_+), \mathcal{C})$ be the functor given by the formula $(\theta' X)(q) = (q_0, q)$, so that θ' is an equivalence of ∞-categories by virtue of Remark 4.1.2. It will therefore suffice to show that the composite map

$$\theta' \circ \theta : S_{Q_{\mathbf{Z}}}^{\circlearrowleft}(\mathcal{C}) \to \mathrm{Fun}(\mathrm{N}(Q_+), \mathcal{C})$$

is a trivial Kan fibration.

Set $\Lambda = Q_{\mathbf{Z}}$, let $\lambda_0 = (q_0, 0) \in \Lambda$, and write

$$\Lambda = \{\cdots < \lambda_{-1} < \lambda_0 < \lambda_1 < \cdots\}.$$

For every pair of integer $i, j \in \mathbf{Z}$ with $i \leq j$, let $\Lambda_{i,j}^{[1]}$ denote the subset of $\Lambda_{\circlearrowleft}^{[1]}$ consisting of those pairs (μ, ν) such that $\lambda_i \leq \mu \leq \lambda_j$. Let $S_{i,j}^{\circlearrowleft}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Lambda_{i,j}^{[1]}), \mathcal{C})$ spanned by those functors $X : \mathrm{N}(\Lambda_{i,j}^{[1]}) \to \mathcal{C}$ which satisfy the following pair of conditions:

(a) For each $\mu \in \Lambda$ with $\lambda_i \leq \mu \leq \lambda_j$, the objects $X(\mu, \mu), X(\mu, \mu + 1) \in \mathcal{C}$ are trivial.
(b) For every triple $\mu, \mu', \mu'' \in \Lambda$ with $\lambda_i \leq \mu \leq \mu' \leq \lambda_j$ and $\mu \leq \mu' \leq \mu'' \leq 1 + \mu$, the diagram

$$
\begin{array}{ccc}
X(\mu, \mu') & \longrightarrow & X(\mu, \mu'') \\
\downarrow & & \downarrow \\
X(\mu', \mu') & \longrightarrow & X(\mu', \mu'')
\end{array}
$$

is a pushout square in $\mathcal{C}$.

Then the ∞-category $S_\Lambda^{\circlearrowleft}(\mathcal{C})$ is given by the inverse limit of a tower of restriction maps

$$\cdots \to S_{-1,2}^{\circlearrowleft}(\mathcal{C}) \overset{\phi_2}{\to} S_{-1,1}^{\circlearrowleft}(\mathcal{C}) \overset{\phi_1}{\to} S_{0,1}^{\circlearrowleft}(\mathcal{C}) \overset{\phi_0}{\to} S_{0,0}^{\circlearrowleft}(\mathcal{C}).$$

Moreover, the map $\theta' \circ \theta$ factors as a composition

$$S_\Lambda^\circlearrowleft(\mathcal{C}) \to S_{0,0}^\circlearrowleft(\mathcal{C}) \overset{\psi}{\to} \mathrm{Fun}(\mathrm{N}(Q_+), \mathcal{C}).$$

It follows from Proposition T.4.3.2.15 that the functor ψ is a trivial Kan fibration. It will therefore suffice to show that each of the maps ϕ_i is a trivial Kan fibration. We will treat the case where $i = 2n$ is even; the case where i is odd follows by a similar argument. We wish to show that the map $S_{-n,n+1}^\circlearrowleft(\mathcal{C}) \to S_{-n,n}^\circlearrowleft(\mathcal{C})$ is a trivial Kan fibration. For this, let $P \subseteq \Lambda_{-n,n+1}^{[1]}$ denote the union of $\Lambda_{-n,n}^{[1]}$ with $\{(\lambda_{n+1}, \lambda_{n+1})\}$, and let $P' = \Lambda_{-n,n+1}^{[1]} - \{(\lambda_{n+1}, 1 + \lambda_{n+1})\}$. We observe that $S_{-n,n+1}^\circlearrowleft(\mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Lambda_{-n,n+1}^{[1]}), \mathcal{C})$ spanned by those functors X which satisfy the following conditions:

- The restriction $X|_{\mathrm{N}(\Lambda_{-n,n}^{[1]})}$ belongs to $S_{-n,n}^\circlearrowleft(\mathcal{C})$.
- The restriction $X|_{\mathrm{N}(P)}$ is a right Kan extension of $X|_{\mathrm{N}(\Lambda_{-n,n}^{[1]})}$.
- The restriction $X|_{\mathrm{N}(P')}$ is a left Kan extension of $X|_{\mathrm{N}(P)}$.
- The functor X is a right Kan extension of $X|_{\mathrm{N}(P')}$.

The assertion that ϕ_i is a trivial Kan fibration now follows from three applications of Proposition T.4.3.2.15. $\qquad\square$

Definition 4.3.4. Let $\mathcal{C}$ be a stable ∞-category. The construction

$$\Lambda \mapsto S_\Lambda^\circlearrowleft(\mathcal{C})$$

determines a functor from $\mathbf{\Delta}_\circlearrowleft^{\mathrm{op}}$ to the category of Kan complexes, which we will denote by $S_\bullet^\circlearrowleft(\mathcal{C})$. We will refer to $S_\bullet^\circlearrowleft(\mathcal{C})$ as the *paracyclic Waldhausen construction* on $\mathcal{C}$.

Remark 4.3.5. Composition with the functor $[n] \mapsto [n]_\circlearrowleft$ determines a map

$$F : \mathrm{Fun}(\mathbf{\Delta}_\circlearrowleft^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty) \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty).$$

For any stable ∞-category $\mathcal{C}$, the construction of Remark 4.3.2 supplies a map of simplicial ∞-categories

$$F(S_\bullet^\circlearrowleft(\mathcal{C})) \to S_\bullet(\mathcal{C}).$$

It follows from Proposition 4.3.3 that this map is a trivial Kan fibration in each degree. We may therefore regard the paracyclic Waldhausen construction $S_\bullet^\circlearrowleft(\mathcal{C})$ as a *refinement* of the classical Waldhausen construction $S_\bullet(\mathcal{C})$.

Remark 4.3.6. Let $\mathcal{C}$ be a stable ∞-category. Combining Remark 4.3.5 with Corollary 4.2.9, we see that the K-theory space of $\mathcal{C}$ can be computed from the paracyclic Waldhausen construction using the formula

$$K(\mathcal{C}) \simeq \Omega(\varinjlim_{\Lambda \in \mathbf{\Delta}_\circlearrowleft} S_\Lambda^\circlearrowleft(\mathcal{C})^\simeq).$$

4.4. **Representations of a Parasimplex.** Let Λ be a parasimplex, which we regard as fixed throughout this section. We let $\mathrm{N}(\Lambda)$ denote the nerve of Λ as a linearly ordered set (that is, the simplicial set whose n-simplices are sequences $(\lambda_0, \ldots, \lambda_n)$ in Λ satisfying $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n$). We let $\mathrm{Rep}(\Lambda) = \mathrm{Fun}(\mathrm{N}(\Lambda)^{\mathrm{op}}, \mathrm{Sp})$ denote the ∞-category of representations of $\mathrm{N}(\Lambda)$, and $j_\Lambda : \mathrm{N}(\Lambda) \to \mathrm{Rep}(\Lambda)$ the stable Yoneda embedding. We will refer to the objects of $\mathrm{Rep}(\Lambda)$ as

Λ-*filtered spectra.* For $V \in \text{Rep}(\Lambda)$, we let V_λ denote the value of V on an element $\lambda \in \Lambda$, so that the stable Yoneda embedding is given by

$$j_\Lambda(\mu)_\lambda = \begin{cases} S & \text{if } \lambda \le \mu \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.4.1. The action of $\mathbf{Z}$ on Λ determines an action of $\text{N}(\mathbf{Z})$ on the ∞-category $\text{Rep}(\Lambda)$. In other words, we can view $\text{Rep}(\Lambda)$ as a locally filtered stable ∞-category. In particular, we have shift functors $V \mapsto V(n)$ which can be described concretely by the formula

$$V(n)_\lambda = V_{\lambda+n}.$$

Since $\text{Rep}(\Lambda)$ is a presentable stable ∞-category, the action of $\text{N}(\mathbf{Z})$ on $\text{Rep}(\Lambda)$ extends to a (right) action of $\text{Rep}(\mathbf{Z})$ on $\text{Rep}(\Lambda)$. We will denote the action map by

$$\circledast : \text{Rep}(\Lambda) \times \text{Rep}(\mathbf{Z}) \to \text{Rep}(\Lambda).$$

On objects, this action is given by the formula

$$(V \circledast W)_\lambda = \varinjlim_{\lambda \le \mu+n} V_\mu \wedge W_n.$$

Remark 4.4.2. Every parasimplex Λ is (noncanonically) isomorphic to $\mathbf{Z}$ as a linearly ordered set, so that $\text{Rep}(\Lambda)$ is equivalent to $\text{Rep}(\mathbf{Z})$ as an ∞-category. However, this equivalence is usually not compatible with the structure described in Remark 4.4.1.

We next show that there is a good supply of $\mathbb{A}$-modules in the ∞-category $\text{Rep}(\Lambda)$.

Notation 4.4.3. Fix an object $\lambda \in \Lambda$. We let $\text{Rep}(\Lambda)_{\ge\lambda}$, $\text{Rep}(\Lambda)_{>\lambda}$, $\text{Rep}(\Lambda)_{\le\lambda}$, and $\text{Rep}(\Lambda)_{<\lambda}$ denote the full subcategories of $\text{Rep}(\Lambda)$ spanned by those objects V for which $V_\mu \simeq 0$ unless $\mu \ge \lambda$, $\mu > \lambda$, $\mu \le \lambda$, or $\mu < \lambda$, respectively.

Proposition 4.4.4. *Let X be an object of $\text{Rep}(\Lambda)$. Suppose that there exists an element $\lambda \in \Lambda$ such that $X \in \text{Rep}(\Lambda)_{\le\lambda} \cap \text{Rep}(\Lambda)_{>\lambda-1}$. Then X admits the structure of a right $\mathbb{A}$-module.*

Proposition 4.4.5. *Let $X, Y \in \text{RMod}_\mathbb{A}(\text{Rep}(\Lambda))$. Suppose that there exists an element $\lambda \in \Lambda$ such that $Y \in \text{Rep}(\Lambda)_{\le\lambda}$ and $Y' \in \text{Rep}(\Lambda)_{>\lambda-1}$. Then the canonical map*

$$\text{Map}_{\text{RMod}_\mathbb{A}(\text{Rep}(\Lambda))}(Y, Y') \to \text{Map}_{\text{Rep}(\Lambda)}(Y, Y')$$

is a homotopy equivalence.

Corollary 4.4.6. *Fix $\lambda \in \Lambda$, and let $\mathcal{C} = \text{Rep}(\Lambda)_{\le\lambda} \cap \text{Rep}(\Lambda)_{>\lambda-1}$. Then the projection map*

$$\text{RMod}_\mathbb{A}(\text{Rep}(\Lambda)) \times_{\text{Rep}(\Lambda)} \mathcal{C} \to \mathcal{C}$$

is an equivalence of ∞-categories. In other words, every object of $\mathcal{C}$ admits an essentially unique $\mathbb{A}$-module structure.

Proof. Full and faithfulness follows from Proposition 4.4.5, and essential surjectivity follows from Proposition 4.4.4. $\qquad\square$

Proof of Proposition 4.4.4. Note that for $X \in \text{Rep}(\mathbf{Z})_{\le 0}$ and $Y \in \text{Rep}(\Lambda)_{\le\lambda}$, the convolution product $Y \circledast X$ also belongs to $\text{Rep}(\Lambda)_{\le\lambda}$. We may therefore regard $\text{Rep}(\Lambda)_{\le\lambda}$ as right-tensored over the ∞-category $\text{Rep}(\mathbf{Z})_{\le 0}$.

Let $P = \{\mu \in \Lambda : \lambda - 1 < \mu \le \lambda\}$, and let $\mathcal{C} = \text{Fun}(\text{N}(P)^{\text{op}}, \text{Sp})$. Consider the restriction functor $L : \text{Rep}(\Lambda)_{\le\lambda} \to \mathcal{C}$. Note that L has a fully faithful right adjoint whose essential image is the intersection $\text{Rep}(\Lambda)_{\le\lambda} \cap \text{Rep}(\Lambda)_{>\lambda-1}$. Moreover, a map $Y \to Y'$ in $\text{Rep}(\Lambda)_{\le\lambda}$ is an L-equivalence if and only if it induces equivalences $Y_\mu \to Y'_\mu$ for $\lambda - 1 < \mu \le \lambda$. Using the description of the

action of $\mathrm{Rep}(\mathbf{Z})$ on $\mathrm{Rep}(\Lambda)$ given in Remark 4.4.1, we see that for each $X \in \mathrm{Rep}(\mathbf{Z})_{\leq 0}$, the functor $Y \mapsto Y \circledast X$ carries L-equivalences to L-equivalences, so that the ∞-category $\mathcal{C}$ inherits an action of $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$. Remark 4.4.1 shows that this action factors through the monoidal functor

$$\mathrm{Rep}(\mathbf{Z})_{\leq 0} \to \mathrm{Sp}$$
$$X \mapsto X_0.$$

Since this functor carries $\mathbb{A}$ to the sphere spectrum, it follows that each object of $\mathcal{C}$ admit an (essentially unique) action of $\mathbb{A}$. The right adjoint to L is weakly $\mathrm{Rep}(\mathbf{Z})_{\leq 0}$-enriched, and therefore carries $\mathbb{A}$-modules to $\mathbb{A}$-modules; it follows that every object of the essential image $\mathrm{Rep}(\Lambda)_{\leq \lambda} \cap \mathrm{Rep}(\Lambda)_{> \lambda - 1}$ admits the structure of an $\mathbb{A}$-module. $\qquad\square$

The proof of Proposition 4.4.5 will require the following lemma:

Lemma 4.4.7. *Let* $Y, Z \in \mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$. *Suppose that there exists an element* $\lambda \in \Lambda$ *such that* $Z \in \mathrm{Rep}(\Lambda)_{\leq \lambda}$ *and* $Y \in \mathrm{Rep}(\Lambda)_{> \lambda}$. *Then the mapping space* $\mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(Z, Y)$ *is contractible.*

Proof. The collection of those objects $Z \in \mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda)_{\leq \lambda})$ for which the mapping space $\mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(Z, Y)$ is contractible is closed under small colimits. Since $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda)_{\leq 0})$ is generated under small colimits by objects of the form $\mathbb{A} \circledast Z_0$ for $Z_0 \in \mathrm{Rep}(\Lambda)_{\leq \lambda}$, we may assume without loss of generality that Z has the form $Z_0 \circledast \mathbb{A}$. In this case, we have a homotopy equivalence

$$\mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(Z, Y) \simeq \mathrm{Map}_{\mathrm{Rep}(\Lambda)}(Z_0, Y).$$

The latter space vanishes, because Z_0 is a left Kan extension of its restriction to the subset $\{\mu \in \Lambda : \mu \leq \lambda\} \subseteq \Lambda$, on which the functor Y vanishes. $\qquad\square$

Proof of Proposition 4.4.5. Let $Y \in \mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$. Then we have a cofiber sequence

$$\Sigma^1 Y(1) \to Y \circledast \mathbb{A} \to Y$$

in $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$. Consequently, for any object $Y' \in \mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$, we obtain a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(Y, Y') \to \mathrm{Map}_{\mathrm{Rep}(\Lambda)}(Y, Y') \to \mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\Sigma^1 Y(1), Y').$$

It will therefore suffice to show that if $Y \in \mathrm{Rep}(\Lambda)_{\leq \lambda}$ and $Y' \in \mathrm{Rep}(\Lambda)_{> \lambda - 1}$, then the mapping space $\mathrm{Map}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\Sigma^1 Y(1), Y')$ is contractible. This follows from Lemma 4.4.7. $\qquad\square$

Notation 4.4.8. Let E be a spectrum. For each $\lambda \in \Lambda$, we let $E[\lambda]$ denote the Λ-filtered spectrum given by the formula

$$E[\lambda]_\mu = \begin{cases} E & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Corollary 4.4.6 that $E[\lambda]$ can be regarded as an $\mathbb{A}$-module in an essentially unique way.

Remark 4.4.9. Let $\lambda \in \Lambda$ and let λ_- denote its predecessor in Λ (that is, the largest element $\mu \in \Lambda$ such that $\mu < \lambda$). Then we have a fiber sequence

$$j_\Lambda(\lambda_-) \to j_\Lambda(\lambda) \to S[\lambda].$$

It follows that $S[\lambda]$ belongs to $\mathrm{Rep}^{\mathrm{fin}}(\Lambda)$.

Proposition 4.4.10. *Let* $X \in \mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$. *The following conditions are equivalent:*

(a) *Each of the spectra* X_λ *is finite and* $X_\lambda \simeq 0$ *for all but finitely many values of* λ.

(b) *The object X belongs to the smallest stable subcategory of $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$ which is closed under equivalence and contains the objects $\{S[\lambda]\}_{\lambda \in \Lambda}$.*

(c) *The image of X in $\mathrm{Rep}(\Lambda)$ belongs to the full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\Lambda)$.*

Proof. We first show that $(a) \Rightarrow (b)$. Let $\mathcal{D}$ denote the smallest stable subcategory of the ∞-category $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$ which is closed under equivalence and contains the objects $\{S[\lambda]\}_{\lambda \in \Lambda}$. We will show that if X satisfies (a), then $X \in \mathcal{D}$. Let $P = \{\lambda \in \Lambda : X_\lambda \not\simeq 0\}$. Condition (a) implies that P is finite; we will proceed by induction on the cardinality of P. If $P = \varnothing$, then $X \simeq 0$ and there is nothing to prove. Otherwise, let λ be the largest element of P, and let $E = X_\lambda$. Then condition (a) implies that E is a finite spectrum, so that $E[\lambda]$ belongs to $\mathcal{D}$. Using Proposition 4.4.5, we see that the identity map $\mathrm{id} : E[\lambda]_\lambda \to X_\lambda$ extends (in an essentially unique way) to a morphism $u : E[\lambda] \to X$ in $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))$. We are therefore reduced to proving that the cofiber $\mathrm{cofib}(u)$ belongs to $\mathcal{D}$, which follows from the inductive hypothesis.

The implication $(b) \Rightarrow (c)$ follows immediately from Remark 4.4.9. We will complete the proof by showing that $(c) \Rightarrow (a)$. Note that since X is a left $\mathbb{A}$-module, the image of X in $\mathrm{Rep}(\Lambda)$ is a retract of $X \circledast \mathbb{A}$. It will therefore suffice to show that for every object $Y \in \mathrm{Rep}^{\mathrm{fin}}(\Lambda)$, the spectra $(Y \circledast \mathbb{A})_\lambda$ are finite for all $\lambda \in \Lambda$, and vanish for all but finitely many values of λ. The collection of those objects $Y \in \mathrm{Rep}(\Lambda)$ which satisfy this condition spans a stable subcategory of $\mathrm{Rep}(\Lambda)$. We may therefore assume without loss of generality that Y has the form $j_\Lambda(\mu)$ for some $\mu \in \Lambda$, in which case the result follows immediately from the formula

$$(j_\Lambda(\mu) \circledast \mathbb{A})_\lambda = \begin{cases} S & \text{if } \mu - 1 < \lambda \leq \mu \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

4.5. **Quiver Representations.** Let Λ denote a parasimplex. The construction $\mathcal{C} \mapsto S_\Lambda^{\circlearrowleft}(\mathcal{C})$ determines a functor from the ∞-category $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ of stable ∞-categories to the ∞-category of spaces. It is not hard to show that this functor is corepresentable: that is, there exists a stable ∞-category Quiv^Λ and a point $\eta \in S_\Lambda^{\circlearrowleft}(\mathrm{Quiv}^\Lambda)$ which induces homotopy equivalences

$$S_\Lambda^{\circlearrowleft}(\mathcal{C}) \simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}}(\mathrm{Quiv}^\Lambda, \mathcal{C})$$

for all every stable ∞-category $\mathcal{C}$. To see this, note that we may assume without loss of generality that $\Lambda = \frac{1}{n+1}\mathbf{Z}$ for some positive integer n (Example 4.2.3) in which case $S_\Lambda^{\circlearrowleft}(\mathcal{C})$ is a classifying space for diagrams

$$X_1 \to \cdots \to X_n$$

in $\mathcal{C}$; it then follows from Proposition 2.2.7 that the functor $S_\Lambda^{\circlearrowleft}$ is corepresented by the ∞-category $\mathrm{Rep}^{\mathrm{fin}}(Q)$ where Q is the finite linearly ordered set $\{1 < 2 < \ldots < n\}$. However, this answer is somewhat unsatisfying because it fails to reflect the symmetry inherent in the problem. For example, the parasimplex Λ has a large automorphism group (which we can identify with $\frac{1}{n+1}\mathbf{Z}$), but none of the resulting automorphisms of $\mathrm{Rep}^{\mathrm{fin}}(Q)$ arise from automorphisms of Q itself.

Our goal in this section is to offer a different construction of the corepresenting object for the functor $S_\Lambda^{\circlearrowleft}$, for which the functoriality in Λ will be more readily apparent.

Definition 4.5.1. Let Λ be a parasimplex and let $\mathrm{Rep}^{\mathrm{fin}}(\Lambda)$ denote the ∞-category of Λ-filtered spectra, which we regard as a locally filtered stable ∞-category as in Remark 4.4.1. Note that the shift operation $X \mapsto X(1)$ carries the full subcategory $\mathrm{Rep}^{\mathrm{fin}}(\Lambda)$ to itself, so that

$\operatorname{Rep}^{\operatorname{fin}}(\Lambda)$ inherits the structure of a locally filtered stable ∞-category. We let $\operatorname{Quiv}^\Lambda$ denote the ∞-category $\operatorname{MF}(\operatorname{Rep}^{\operatorname{fin}}(\Lambda))$ of equivariant matrix factorizations of $\operatorname{Rep}^{\operatorname{fin}}(\Lambda)$.

The main result of this section asserts that the stable ∞-category $\operatorname{Quiv}^\Lambda$ corepresents the functor $S_\Lambda^{\circlearrowleft} : \mathcal{C}\mathrm{at}_\infty^{\operatorname{St}} \to \mathcal{C}\mathrm{at}_\infty$. The first step will be to construct a point of the space $S_\Lambda^{\circlearrowleft}(\operatorname{Quiv}^\Lambda)$. This will require us to construct some objects of $\operatorname{Quiv}^\Lambda$.

Construction 4.5.2. Let Λ be a parasimplex and let $j_\Lambda : N(\Lambda) \to \operatorname{Rep}(\Lambda)$ denote the stable Yoneda embedding. Given a pair of elements $\lambda, \mu \in \Lambda$ with $\lambda \le \mu$, we let $J_\Lambda^\circ(\lambda, \mu)$ denote the cofiber of the induced map $j_\Lambda(\lambda) \to j_\Lambda(\mu)$. More concretely, $J_\Lambda^\circ(\lambda, \mu)$ is the Λ-filtered spectrum given by the formula

$$J_\Lambda^\circ(\lambda, \mu)_\nu = \begin{cases} S & \text{if } \lambda < \nu \le \mu. \\ 0 & \text{otherwise.} \end{cases}$$

We will regard the construction $(\lambda, \mu) \mapsto J_\Lambda^\circ(\lambda, \mu)$ as defining a functor

$$J_\Lambda^\circ : N(\Lambda_{\circlearrowleft}^{[1]}) \to \operatorname{Rep}(\Lambda).$$

Note that we have canonical equivalences

$$J_\Lambda^\circ(\lambda, \lambda) \simeq 0 \qquad J_\Lambda^\circ(\lambda - 1, \lambda) \simeq j_\Lambda(\lambda) \circledast \mathbb{A}.$$

Proposition 4.5.3. *Let J_Λ° be as in Construction 4.5.2. Then the fiber over J_Λ° of the forgetful functor*

$$\operatorname{Fun}(N(\Lambda_{\circlearrowleft}^{[1]}), \operatorname{RMod}_{\mathbb{A}}(\operatorname{Rep}(\Lambda))) \to \operatorname{Fun}(N(\Lambda_{\circlearrowleft}^{[1]}), \operatorname{Rep}(\Lambda))$$

is a contractible Kan complex.

More informally, Proposition 4.5.3 asserts that the functor J_Λ° admits an essentially unique lift to a functor $\overline{J}_\Lambda^\circ : N(\Lambda_{\circlearrowleft}^{[1]}) \to \operatorname{RMod}_{\mathbb{A}}(\operatorname{Rep}(\Lambda))$.

Proof of Proposition 4.5.3. We show more generally that for every simplicial set K equipped with a map $N(\Lambda_{\circlearrowleft}^{\vee[1]})$, every lifting problem of the form

$$
\begin{array}{ccc}
K_0 & \longrightarrow & \operatorname{RMod}_{\mathbb{A}}(\operatorname{Rep}(\Lambda)) \\
\downarrow & \nearrow & \downarrow \\
K & \longrightarrow & \operatorname{Rep}(\Lambda)
\end{array}
$$

admits a solution provided that the left vertical map is injective. Proceeding one simplex at a time, we may reduce to the case where $K = \Delta^n$ and $K_0 = \partial\Delta^n$. In this case, the desired result follows from either Proposition 4.4.4 (if $n = 0$) or Proposition 4.4.5 (if $n > 0$). $\qquad\square$

Construction 4.5.4. Let Λ be a parasimplex and let us denote the canonical map

$$\operatorname{RMod}_{\mathbb{A}}(\operatorname{Rep}^{\operatorname{fin}}(\Lambda)) \to \operatorname{RMod}_{\mathbb{A}}(\operatorname{Rep}^{\operatorname{fin}}(\Lambda))[\beta^{-1}] = \operatorname{Quiv}^\Lambda$$

by $X \mapsto X[\beta^{-1}]$. Note that the functor $\overline{J}_\Lambda^\circ$ carries $N(\Lambda_{\circlearrowleft}^{[1]})$ into $\operatorname{RMod}_{\mathbb{A}}(\operatorname{Rep}^{\operatorname{fin}}(\Lambda))$. We $J_\Lambda : N(\Lambda_{\circlearrowleft}^{[1]}) \to \operatorname{Quiv}^\Lambda$ denote the functor given by

$$J_\Lambda(\lambda, \mu) = \overline{J}_\Lambda^\circ(\lambda, \mu)[\beta^{-1}].$$

Proposition 4.5.5. *The functor J_Λ of Construction 4.5.4 is a paracyclic Λ-gapped object of the stable ∞-category $\operatorname{Quiv}^\Lambda$, in the sense of Definition 4.3.1.*

Proof. We will verify that J_Λ satisfies conditions (a) and (b) of Definition 4.3.1:

(b) Suppose we are given $\lambda, \mu, \nu \in \Lambda$ satisfying

$$\lambda \le \mu \le \nu \le \lambda + 1;$$

we must show that the diagram

$$
\begin{array}{ccc}
J_\Lambda(\lambda, \mu) & \longrightarrow & J_\Lambda(\lambda, \nu) \\
\downarrow & & \downarrow \\
J_\Lambda(\mu, \mu) & \longrightarrow & J_\Lambda(\mu, \nu)
\end{array}
$$

is a pushout square in Quiv^Λ. In fact, we claim that the diagram

$$
\begin{array}{ccc}
\overline{J}{}^\circ_\Lambda(\lambda, \mu) & \longrightarrow & \overline{J}{}^\circ_\Lambda(\lambda, \nu) \\
\downarrow & & \downarrow \\
\overline{J}{}^\circ_\Lambda(\mu, \mu) & \longrightarrow & \overline{J}{}^\circ_\Lambda(\mu, \nu)
\end{array}
$$

is already a pushout square in $\mathrm{RMod}_\mathbb{A}(\mathrm{Rep}(\Lambda))$. This is clear, because the diagram

$$
\begin{array}{ccc}
J^\circ_\Lambda(\lambda, \mu) & \longrightarrow & J^\circ_\Lambda(\lambda, \nu) \\
\downarrow & & \downarrow \\
J^\circ_\Lambda(\mu, \mu) & \longrightarrow & J^\circ_\Lambda(\mu, \nu)
\end{array}
$$

is a pushout square in $\mathrm{Rep}(\Lambda)$.

(a) Let $\lambda \in \Lambda$; we wish to prove that the objects $J_\Lambda(\lambda, \lambda)$ and $J_\Lambda(\lambda - 1, \lambda)$ vanish. In the first case, this is clear (since $J^\circ_\Lambda(\lambda, \lambda)$ is already a zero object of $\mathrm{Rep}(\Lambda)$). To handle the second case, we first note that $J^\circ_\Lambda(\lambda - 1, \lambda)$ is equivalent to $j_\Lambda(\lambda) \circledast \mathbb{A}$ as an object of $\mathrm{Rep}(\Lambda)$. Using Corollary 4.4.6, we see that $\overline{J}{}^\circ_\Lambda(\lambda - 1, \lambda)$ is a free left $\mathbb{A}$-module, so that

$$J_\Lambda(\lambda - 1, \lambda) \simeq \overline{J}{}^\circ(\lambda - 1, \lambda)[\beta^{-1}] \simeq 0$$

by virtue of Remark 3.6.9.

$\square$

We can now formulate the main result of this section:

Theorem 4.5.6. *Let $\mathcal{C}$ be a stable ∞-category and let Λ be a parasimplex. Then composition with the functor J_Λ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^\Lambda, \mathcal{C}) \to S^{\circlearrowleft}_\Lambda(\mathcal{C}),$$

and therefore a homotopy equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^\Lambda, \mathcal{C})^{\simeq} \to S^{\circlearrowleft}_\Lambda(\mathcal{C})^{\simeq}.$$

Remark 4.5.7. The stable ∞-category Quiv^Λ depends functorially on Λ: that is, the construction $\Lambda \mapsto \mathrm{Quiv}^\Lambda$ determines a functor $\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft}) \to \mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty$. The map $J_\Lambda : \mathrm{N}(\Lambda^{[1]}_{\circlearrowleft}) \to \mathrm{Quiv}^\Lambda$ given by Construction 4.5.4 also depends functorially on Λ (that is, the maps J_Λ can be extended to a natural transformation of functors from $\mathrm{N}(\boldsymbol{\Delta}_{\circlearrowleft})$ into $\mathcal{C}\mathrm{at}^{\mathrm{St}}_\infty$). Consequently, if $\mathcal{C}$ is a stable

∞-category, then the homotopy equivalence $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^{\Lambda}, \mathcal{C})^{\simeq} \simeq S_{\Lambda}^{\circlearrowleft}(\mathcal{C})$ supply an identification $S_{\bullet}^{\circlearrowleft}(\mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Quiv}^{\bullet}, \mathcal{C})$, where $S_{\bullet}^{\circlearrowleft}(\mathcal{C})$ denotes the paracyclic Waldhausen construction of §4.

The proof of Theorem 4.5.6 will require some preliminaries.

Proposition 4.5.8. *Let $\lambda \in \Lambda$, and let X be an object of $\mathrm{LMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda)_{<\lambda+1})$. For $\lambda \leq \mu \leq \lambda+1$, composition with the anchor map $\beta_X : X \to \Sigma^2 X(1)$ induces an equivalence of spectra*

$$\underline{\mathrm{Map}}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\overline{J}_{\Lambda}^{\circ}(\lambda, \mu), X) \to \underline{\mathrm{Map}}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\overline{J}_{\Lambda}^{\circ}(\lambda, \mu), \Sigma^2 X(1)).$$

Proof. We have a fiber sequence of right $\mathbb{A}$-modules

$$X \circledast \mathbb{A} \to X \overset{\beta_X}{\to} \Sigma^2 X(1).$$

It will therefore suffice to show that the spectrum

$$\underline{\mathrm{Map}}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\overline{J}_{\Lambda}^{\circ}(\lambda, \mu), X \circledast \mathbb{A})$$

is trivial. Note that the dual of $\mathbb{A}$ is equivalent to $\Sigma^{-1}\mathbb{A}(-1)$ as a left $\mathbb{A}$-module (Remark 3.2.10). We therefore have

$$
\begin{aligned}
\underline{\mathrm{Map}}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\overline{J}_{\Lambda}^{\circ}(\lambda, \mu), X \circledast \mathbb{A}) &\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\Lambda)}(\Sigma^{-1}\mathbb{A}(-1) \circledast_{\mathbb{A}} \overline{J}_{\Lambda}^{\circ}(\lambda, \mu), X) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\Lambda)}(J_{\Lambda}^{\circ}(\lambda, \mu), \Sigma X(1)) \\
&\simeq \mathrm{fib}(\Sigma X_{\mu+1} \to \Sigma X_{\lambda+1}).
\end{aligned}
$$

This spectrum vanishes by virtue of our assumption that $X \in \mathrm{Rep}(\Lambda)_{<\lambda+1}$. $\qquad \square$

Corollary 4.5.9. *Let $\lambda \in \Lambda$ and let X be an object of $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\Lambda))$ whose image in $\mathrm{Rep}(\Lambda)$ belongs to $\mathrm{Rep}(\Lambda)_{<1+\lambda}$. For $\lambda \leq \mu \leq 1 + \lambda$, the canonical map*

$$\underline{\mathrm{Map}}_{\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}(\Lambda))}(\overline{J}_{\Lambda}^{\circ}(\lambda, \mu), X) \to \underline{\mathrm{Map}}_{\mathrm{Quiv}^{\Lambda}}(J_{\Lambda}(\lambda, \mu), X[\beta^{-1}]).$$

is a homotopy equivalence of spectra.

Corollary 4.5.10. *Let $\lambda, \mu \in \Lambda$ satisfy $\lambda \leq \mu \leq \lambda+1$. Let X be an object of $\mathrm{RMod}_{\mathbb{A}}(\mathrm{Rep}^{\mathrm{fin}}(\Lambda))$ whose image in $\mathrm{Rep}(\Lambda)$ is contained in the intersection $\mathrm{Rep}(\Lambda)_{<\lambda+1} \cap \mathrm{Rep}(\Lambda)_{>\lambda}$. Then we have a canonical homotopy equivalence of spectra*

$$\underline{\mathrm{Map}}_{\mathrm{Quiv}^{\Lambda}}(J_{\Lambda}(\lambda, \mu), X[\beta^{-1}]) \simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\Lambda)}(J_{\Lambda}^{\circ}(\lambda, \mu), X).$$

Proof. Combine Corollary 4.5.9 with Proposition 4.4.5. $\qquad \square$

Remark 4.5.11. In the situation of Corollary 4.5.10, we have a canonical fiber sequence of spectra

$$\underline{\mathrm{Map}}_{\mathrm{Rep}(\Lambda)}(J_{\Lambda}^{\circ}(\lambda, \mu), X) \to X_{\mu} \to X_{\lambda}.$$

Since $X_{\lambda} \simeq 0$ by assumption, Corollary 4.5.10 supplies an equivalence

$$\underline{\mathrm{Map}}_{\mathrm{Quiv}^{\Lambda}}(J_{\Lambda}(\lambda, \mu), X[\beta^{-1}]) \simeq X_{\mu}.$$

Proof of Theorem 4.5.6. Without loss of generality, we may assume that $\Lambda = \frac{1}{n}\mathbf{Z}$ for some positive integer n (Example 4.2.3). Using Proposition 4.3.3 and Remark 4.1.2, we deduce that the construction

$$S_{\Lambda}^{\circlearrowleft}(\mathcal{C}) \to \mathrm{Fun}(\Delta^{n-2}, \mathcal{C})$$

$$X \mapsto (X(0, \frac{1}{n}) \to X(0, \frac{2}{n}) \to \cdots \to X(0, \frac{n-1}{n}))$$

is an equivalence of ∞-categories (where we agree to the convention that $\Delta^{n-2} = \varnothing$ when $n = 1$). It will therefore suffice to show that the sequence

$$J_\Lambda(0, \frac{1}{n}) \to J_\Lambda(0, \frac{2}{n}) \to \cdots \to J_\Lambda(0, \frac{n-1}{n})$$

determines a map $\Delta^{n-2} \to \mathrm{Quiv}^\Lambda$ which satisfies conditions (i) and (ii) of Corollary 2.2.8:

(i) Let $0 \le i, j \le n - 2$. We wish to show that the canonical map

$$\Sigma_+^\infty \mathrm{Map}_{\Delta^{n-2}}(i, j) \to \underline{\mathrm{Map}}_{\mathrm{Quiv}^\Lambda}(J_\Lambda(0, \frac{i+1}{n}), J_\Lambda(0, \frac{j+1}{n}))$$

is an equivalence of spectra. This follows immediately from the calculation of Remark 4.5.11.

(ii) Let $\mathcal{C}$ denote the smallest stable subcategory of Quiv^Λ which contains the objects $\{J_\Lambda(0, \frac{i}{n})\}_{1 \le i \le n-1}$ and is closed under equivalence. We wish to prove that $\mathcal{C} = \mathrm{Quiv}^\Lambda$. To prove this, it will suffice to show that for each object $M \in \mathrm{RMod}_\mathbb{A}(\mathrm{Rep}^{\mathrm{fin}}(\Lambda))$, the object $M[\beta^{-1}] \in \mathrm{Quiv}^\Lambda$ is contained in $\mathcal{C}$. Using Proposition 4.4.10, we may assume without loss of generality that M has the form $\overline{J}_\Lambda^\circ(\frac{i}{n}, \frac{i+1}{n})$ for some integer i. Note that for each i, the canonical map

$$\beta : \overline{J}_\Lambda^\circ(\frac{i}{n}, \frac{i+1}{n}) \to \Sigma^2 \overline{J}^\circ(\frac{i-n}{n}, \frac{i+1-n}{n})$$

induces an equivalence in Quiv^Λ. We are therefore free to modify i by a multiple of n and thereby reduce to the case where $0 \le i < n$. Using the fiber sequence

$$J_\Lambda(0, \frac{i}{n}) \to J_\Lambda(0, \frac{i+1}{n}) \to M[\beta^{-1}],$$

we are reduced to proving that $\mathcal{C}$ contains the objects $J_\Lambda(0, \frac{i}{n})$ for $0 \le i \le n$. This follows from the definition of $\mathcal{C}$ if $0 < i < n$; to handle the exceptional cases we note that $J_\Lambda(0, 0)$ and $J_\Lambda(0, 1)$ are zero objects of Quiv^Λ (Proposition 4.5.5).

$\square$

4.6. **Equivariance for Z-Torsors.** Let Λ be a parasimplex and let T be a $\mathbf{Z}$-torsor. Then the simplicial set $\mathrm{N}(T +_{\mathbf{Z}} \Lambda)$ can be identified with the relative tensor product of $\mathrm{N}(T)$ with $\mathrm{N}(\Lambda)$ over $\mathrm{N}(\mathbf{Z})$ (formed in the ∞-category $\mathcal{C}\mathrm{at}_\infty$). This induces an equivalence of stable ∞-categories

$$\mathrm{Rep}^{\mathrm{fin}}(T +_{\mathbf{Z}} \Lambda) \simeq \mathrm{Rep}^{\mathrm{fin}}(T) \otimes_{\mathrm{Rep}^{\mathrm{fin}}(\mathbf{Z})} \mathrm{Rep}^{\mathrm{fin}}(\Lambda).$$

In other words, we can regard the construction $\Lambda \mapsto \mathrm{Rep}^{\mathrm{fin}}(\Lambda)$ as a $\mathrm{Tors}(\mathbf{Z})$-equivariant functor from $\mathrm{N}(\boldsymbol{\Delta}_\circlearrowleft)$ to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$, where the action of $\mathrm{Tors}(\mathbf{Z})$ on $\mathrm{N}(\boldsymbol{\Delta}_\circlearrowleft)$ and $\mathcal{C}\mathrm{at}_\infty^{\mathrm{filt}}$ are given by Construction 4.2.10 and Remark 3.1.17, respectively. Combining this observation with Proposition 3.6.10, we obtain the following:

Proposition 4.6.1. *The construction $\Lambda \mapsto \mathrm{Quiv}^\Lambda$ determines a $\mathrm{Tors}(\mathbf{Z})$-equivariant functor from $\mathrm{N}(\boldsymbol{\Delta}_\circlearrowleft)$ to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ (where $\mathrm{Tors}(\mathbf{Z})$ acts on $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ via the monoidal functor ϕ of Remark 3.5.12.*

More informally, Proposition 4.6.1 asserts that for every $\mathbf{Z}$-torsor T and every parasimplex Λ, we have a canonical equivalence of stable ∞-categories

$$\mathrm{Quiv}^{T +_{\mathbf{Z}} \Lambda} \simeq \phi(T) \otimes \mathrm{Quiv}^\Lambda.$$

Corollary 4.6.2. *The paracyclic Waldhausen construction* $S_\bullet^{\circlearrowleft} : \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C}\mathrm{at}_\infty)$
can be promoted to a $\mathrm{Tors}(\mathbf{Z})$*-equivariant functor, where* $\mathrm{Tors}(\mathbf{Z})$ *acts on the* ∞*-categories* $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$
and $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C}\mathrm{at}_\infty)$ *via Remark 3.5.12 and Construction 4.2.10, respectively.*

Proof. At the level of objects we have

$$
\begin{aligned}
(T \otimes S_\bullet^{\circlearrowleft}(\mathcal{C}))_\Lambda &\simeq S_{-T+\mathbf{z}\Lambda}^{\circlearrowleft}(\mathcal{C}) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}}(\mathrm{Quiv}^{-T+\mathbf{z}\Lambda}, \mathcal{C}) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}}(\phi(-T) \otimes \mathrm{Quiv}^\Lambda, \mathcal{C}) \\
&\simeq \mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}}(\mathrm{Quiv}^\Lambda, \phi(T) \otimes \mathcal{C}) \\
&\simeq S_\Lambda^{\circlearrowleft}(\phi(T) \otimes \mathcal{C}).
\end{aligned}
$$

$\square$

Corollary 4.6.3. *The* K*-theory construction* $\mathcal{C} \mapsto K(\mathcal{C})$ *determines a* $\mathrm{Tors}(\mathbf{Z})$*-equivariant*
functor $K : \mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} \to \mathcal{S}$, *where* $\mathrm{Tors}(\mathbf{Z})$ *acts on* $\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}$ *via Remark 3.5.12 and acts trivially on* $\mathcal{S}$.

Proof. Combine Corollary 4.2.9, Corollary 4.6.2, and Remark 4.2.12. $\square$

Corollary 4.6.4. *Let* $\mathcal{C}$ *be a 2-periodic stable* ∞*-category (see Remark 3.5.13). Then the*
paracyclic Waldhausen construction $S_\bullet^{\circlearrowleft}(\mathcal{C})$ *can be refined to a cyclic* ∞*-category* $S_\bullet^{\mathrm{cyc}}(\mathcal{C})$. *More*
precisely, there exists a commutative diagram of ∞*-categories*

$$
\begin{array}{ccc}
\mathrm{RMod}_{\mathrm{RMod}_{\mathcal{S}\mathrm{per}}^{\mathrm{fin}}}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}}) & \xrightarrow{S_\bullet^{\mathrm{cyc}}} & \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\mathrm{cyc}}^{\mathrm{op}}), \mathcal{C}\mathrm{at}_\infty) \\
\downarrow & & \downarrow \\
\mathcal{C}\mathrm{at}_\infty^{\mathrm{St}} & \xrightarrow{S_\bullet^{\circlearrowleft}} & \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{\circlearrowleft}^{\mathrm{op}}), \mathcal{C}\mathrm{at}_\infty).
\end{array}
$$

Proof. Combine Corollary 4.6.2 with Remark 3.5.13. $\square$

Remark 4.6.5. It follows from Corollary 4.6.4 and Remark 4.2.15 that if $\mathcal{C}$ is a 2-periodic
stable ∞-category, then the K-theory space $K(\mathcal{C})$ comes equipped with a canonical action of
the circle group S^1.

5. Symmetric Powers of Disks

In §3.5, we constructed an $\mathbb{E}_2$-monoidal functor

$$
\Phi : \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(S).
$$

Our definition of Φ was essentially combinatorial: it arose from the observation that the ∞-
category of spectra is equivalent to (graded) modules over the graded $\mathbb{E}_2$-algebra $S[\beta^{\pm 1}]$ (Propo-
sition 3.5.7), and therefore inherits a central action of the ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ of graded
spectra.

Our goal in this section is to supply an alternative description of the $\mathbb{E}_2$-monoidal functor
Φ, or at least of the restriction $\Phi_{\geq 0} = \Phi|_{\mathbf{Z}_{\geq 0}^{\mathrm{ds}}}$, which is more geometric in nature. Heuristically,
we can think of an $\mathbb{E}_2$-monoidal functor F from $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ to $\mathrm{Pic}(S)$ as a family of functors $\{F_D : \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{Pic}(S)\}$, where D ranges over all 2-dimensional open disks, having the property that
every embedding

$$
D_1 \sqcup D_2 \sqcup \cdots \sqcup D_n \hookrightarrow D
$$

determines a commutative diagram

$$
\begin{array}{ccc}
(\mathbf{Z}_{\geq 0}^{\mathrm{ds}})^n & \xrightarrow{\ +\ } & \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \\
{\scriptstyle \Pi\, F_{D_i}} \Big\downarrow & & \Big\downarrow {\scriptstyle F_D} \\
\mathrm{Pic}(S)^n & \xrightarrow{\ \wedge\ } & \mathrm{Pic}(S).
\end{array}
$$

The main result of this section (Theorem 5.1.14) asserts that the $\mathbb{E}_2$-monoidal functor $\Phi_{\geq 0}$ corresponds to the family of functors $\{F_D\}$ given by the formula

$$F_D(n) = \Sigma^\infty (\mathrm{Sym}^n D)^c,$$

where $\mathrm{Sym}^n(D)$ denotes the nth symmetric power of D and $(\mathrm{Sym}^n D)^c \simeq S^{2n}$ denotes its one-point compactification.

We begin in §5.1 by making the above description of $\mathbb{E}_2$-algebras more explicit. For this, we introduce a colored operad $\mathcal{O}$ (whose colors are open disks D in the complex plane) whose algebras in a symmetric monoidal ∞-category $\mathcal{C}$ are closely related to $\mathbb{E}_2$-algebras in $\mathcal{C}$ (Proposition 5.1.5). Using the colored operad $\mathcal{O}$, we can turn the heuristic construction above into a precise definition of an $\mathbb{E}_2$-monoidal functor $F : \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{Pic}(S)$.

Our goal for the remainder of the section is to show that the $\mathbb{E}_2$-monoidal functors F and $\Phi_{\geq 0}$ are equivalent to one another. The functor $\Phi_{\geq 0}$ is closely related to the graded $\mathbb{E}_2$-ring $S[\beta]$ and its localization $S[\beta^{\pm 1}]$ (see Proposition 3.5.11). Consequently, the problem of describing the functor $\Phi_{\geq 0}$ in a "geometric" way is related to the problem of describing $S[\beta]$ in a geometric way. Recall that the graded spectrum $S[\beta]$ can be described by the formula

$$
S[\beta]_n \simeq \begin{cases} S^{-2n} & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}
$$

It follows that $S[\beta]$ can be identified as the "associated graded" of the function spectrum $S^{\mathbf{CP}_+^\infty}$, where $S^{\mathbf{CP}_+^\infty}$ is equipped with the filtration induced by the skeletal filtration

$$\mathbf{CP}^0 \hookrightarrow \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^2 \hookrightarrow \cdots.$$

The fundamental observation is that this identification $S[\beta] \simeq \mathrm{gr}\, S^{\mathbf{CP}^\infty}$ can be promoted to an equivalence of graded $\mathbb{E}_2$-rings (Proposition 5.4.9). Here the $\mathbb{E}_2$-structure on $\mathrm{gr}\, S^{\mathbf{CP}^\infty}$ arises from an $\mathbb{E}_2$-coalgebra structure on the filtered space $\{\mathbf{CP}^n\}_{n \geq 0}$, which we will discuss in §5.2. The proof of Proposition 5.4.9 will be given in §5.4 using a concrete description of $\{\mathbf{CP}^n\}_{n \geq 0}$ as obtained from a 2-fold bar construction in the setting of filtered spaces, which we discuss in §5.3. In §5.5, we will exploit the identification of $S[\beta]$ with $\mathrm{gr}\, S^{\mathbf{CP}^\infty}$ to show that the $\mathbb{E}_2$-monoidal functors F and $\Phi_{\geq 0}$ are equivalent to one another.

5.1. **The Colored Operad $\mathcal{O}$.** Let ${}^t\mathbb{E}_2$ denote the topological operad of little 2-cubes: that is, the operad whose n-ary operations is the space of rectilinear embeddings

$$\coprod_{1 \leq i \leq n} (-1,1)^2 \hookrightarrow (-1,1)^2.$$

This topological operad has many equivalent incarnations (for example, we can replace cubes by disks). Note however that for any topological operad which is weakly equivalent to ${}^t\mathbb{E}_2$, the space of n-ary operations must be weakly homotopy equivalent to the space $\mathrm{Conf}_n(\mathbf{R}^2)$ of (labelled) configurations of n distinct points in $\mathbf{R}^2$. Since these configuration spaces have nontrivial fundamental groups (given by the pure braid groups), the operad ${}^t\mathbb{E}_2$ cannot be equivalent to a topological operad whose spaces of operations are discrete. This is often a technical nuisance: for many applications, it is much easier to work with operads in sets (where

one does not need to worry about continuity) than operads in topological spaces. Our goal in this section is to review a construction from §H.5.4.5 which will enable us to replace the operad $^{t}\mathbb{E}_2$ in topological spaces with a (colored) operad $\mathcal{O}$ in sets (in fact, in sets of cardinality ≤ 1) whose categories (and ∞-categories) of algebras are almost the same (Proposition 5.1.5).

Definition 5.1.1. Let $\mathbf{C}$ be the set of complex numbers. We will say that an open subset $D \subseteq \mathbf{C}$ is a *disk* if it is homeomorphic to $\mathbf{R}^2$. We define a colored operad $\mathcal{O}$ as follows:

- The colors of $\mathcal{O}$ are open disks $D \subseteq \mathbf{C}$.
- Given a collection of colors $\{D_i\}_{1 \leq i \leq n}$ and another color D', there is at most one operation $\{D_i\} \to D'$ in $\mathcal{O}$, which exists if and only if the disks D_i are disjoint and contained in D'.

Remark 5.1.2. An open subset $D \subseteq \mathbf{C}$ is a disk (in the sense of Definition 5.1.1) if and only if it is simply connected. In particular, we do *not* require that D is an open ball with respect to the usual metric on $\mathbf{C}$ (though, for our applications, restriction to disks of this special type would make no difference).

Notation 5.1.3. In what follows, we will not distinguish between the colored operad $\mathcal{O}$ of Definition 5.1.1 and the associated ∞-operad (given by Construction H.2.1.1.7). For any symmetric monoidal ∞-category (or ordinary category) $\mathcal{C}$, we let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ denote the ∞-category (or ordinary category of $\mathcal{O}$-algebra objects of $\mathcal{C}$).

Remark 5.1.4. If $\mathcal{C}$ is a symmetric monoidal ∞-category, then a $\mathcal{O}$-algebra object A of $\mathcal{C}$ is given by the following data:

(1) For every open disk $D \subseteq \mathbf{C}$, an object $A(D) \in \mathcal{C}$.
(2) For every open disk $D \subseteq \mathbf{C}$ and every finite collection of disjoint open disks $\{D_i \subseteq D\}$, a morphism $\mu : \bigotimes_i A(D_i) \to A(D)$ in $\mathcal{C}$.

The multiplication maps μ are required to satisfy an associative law up to coherent homotopy.

The following result is a special case of Theorem H.5.4.5.9:

Proposition 5.1.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then there is a fully faithful embedding $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, whose essential image consists of those $\mathcal{O}$-algebra objects $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ which satisfy the following condition:*

(∗) *For every inclusion $D \subseteq D'$ of open disks in $\mathbf{C}$, the induced map $A(D) \to A(D')$ is an equivalence in $\mathcal{C}$.*

Example 5.1.6. Using Proposition 5.1.5, we can identify the ∞-category $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$ of $\mathbb{E}_2$-spaces with the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathcal{O}^{\otimes}), \mathcal{S})$ spanned by those functors with the following properties:

(a) For every inclusion $D \subseteq D'$ of open disks in $\mathbf{C}$, the map $A(D) \to A(D')$ is a homotopy equivalence.
(b) For every object $(D_1, \ldots, D_n) \in \mathcal{O}^{\otimes}$, the canonical maps $(D_1, \ldots, D_n) \to (D_i)$ in $\mathcal{O}^{\otimes}$ induce a homotopy equivalence

$$A(D_1, \ldots, D_n) \to A(D_1) \times \cdots \times A(D_n).$$

Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. For any ∞-operad $\mathcal{O}$, we can view the Cartesian product $\mathcal{C} \times \mathcal{O}$ as an ∞-operad. Moreover, the ∞-category of lax $\mathcal{O}$-monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ is *isomorphic* to the ∞-category $\mathrm{Alg}_{\mathcal{C} \times \mathcal{O}}(\mathcal{D})$. We will be interested in the special case where $\mathcal{O}$ is the colored operad of Definition 5.1.1 and $\mathcal{C}$ is the discrete simplicial set $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$, regarded as a symmetric monoidal ∞-category via the addition on $\mathcal{C}$.

Definition 5.1.7. We define a colored operad $\mathcal{O}[\mathbf{Z}_{\geq 0}]$ as follows:
- The colors of $\mathcal{O}[\mathbf{Z}_{\geq 0}]$ are pairs (D, n) where $D \subseteq \mathbf{C}$ is an open disk and $n \geq 0$ is a nonnegative integer.
- Given a set of objects $\{(D_i, n_i)\}_{1 \leq i \leq m}$ and another color (D', n), there is at most one operation $\{(D_i, n_i)\} \to (D', n)$ in $\mathcal{O}[\mathbf{Z}_{\geq 0}]$, which exists if and only if the disks D_i are disjoint, each disk D_i is contained in D, and $n = n_1 + \cdots + n_m$.

We will need the following consequence of Proposition 5.1.5:

Proposition 5.1.8. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the ∞-category of $\mathbb{E}_2$-monoidal functors from $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ to $\mathcal{C}$ can be identified with the full subcategory of $\mathrm{Alg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\mathcal{C})$ spanned by those algebras A which satisfy the following condition:*

($*$) *For every open disk $D \subseteq \mathbf{C}$ containing smaller disjoint open disks $\{D_i\}_{i \in I}$ and any collection of nonnegative integers $\{n_i\}_{i \in I}$, the induced map*

$$\bigotimes_{i \in I} A(D_i, n_i) \to A(D, \sum_{i \in I} n_i)$$

is an equivalence in $\mathcal{C}$.

Proof. It follows from the above discussion that we can identify $\mathrm{Alg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\mathcal{C})$ with the ∞-category of lax $\mathcal{O}$-monoidal functors from $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ to $\mathcal{C}$. Under this identification, the algebras which satisfy ($*$) correspond to the $\mathcal{O}$-monoidal functors from $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ to $\mathcal{C}$: that is, the morphisms from $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ to $\mathcal{C}$ in the ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$. The desired result now follows from Proposition 5.1.5 (note that any $\mathcal{O}$-algebra which arises from a commutative algebra automatically satisfies condition ($*$) appearing in the statement of Proposition 5.1.5). $\qquad\square$

Propositions 5.1.5 and 5.1.8 can be useful for building examples of $\mathbb{E}_2$-algebras and $\mathbb{E}_2$-monoidal functors. The basic paradigm is this: suppose we are given a symmetric monoidal ∞-category $\mathcal{C}$ (such as the ∞-category of spaces, with the Cartesian product) which receives a symmetric monoidal functor F from an ordinary category $\mathcal{C}_0$ (such as the ordinary category of topological spaces, which the Cartesian product). If A is a $\mathcal{O}$-algebra object of $\mathcal{C}_0$ having the property that $F(A) \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ satisfies condition ($*$) of Proposition 5.1.5, then we can identify $F(A)$ with an $\mathbb{E}_2$-algebra object of $\mathcal{C}$. We will now apply this idea in a simple special case; more elaborate examples will appear in the sections which follow.

Definition 5.1.9. Let Top_* denote the category whose objects are pointed topological spaces (X, x) for which there exists an open set $U \subseteq X$ which contains $\{x\}$ as a (strong) deformation retract. We regard Top_* as a symmetric monoidal category with respect to the formation of smash products.

The construction $(X, x) \mapsto (\mathrm{Sing}(X), x)$ determines a symmetric monoidal functor from the ordinary category Top_* to the ∞-category $\mathcal{S}_*$ of pointed spaces (equipped with the symmetric monoidal structure given by the formation of smash products). Consequently, we can regard the construction $(X, x) \mapsto \Sigma^\infty \mathrm{Sing}(X)$ as determining a symmetric monoidal functor $\mathrm{N}(\mathrm{Top}_*) \to \mathrm{Sp}$.

Remark 5.1.10. The technical requirement that our pointed spaces have base points which are strong deformation retracts of neighborhoods of themselves guarantees that the natural maps

$$\mathrm{Sing}(X) \wedge \mathrm{Sing}(Y) \to \mathrm{Sing}(X \wedge Y)$$

are weak homotopy equivalences of simplicial sets, so that the construction $X \mapsto \Sigma^\infty \mathrm{Sing}(X)$ determines a symmetric monoidal functor $\mathrm{N}(\mathrm{Top}_*) \to \mathrm{Sp}$ as asserted (of course, this requirement is almost always satisfied in practice).

Notation 5.1.11. Let X be a locally compact topological space. We let X^c denote the one-point compactification of X, and we let v_X denote the "point at ∞" in X^c (that is, the unique point of X^c which does not belong to X).

For any open embedding $U \hookrightarrow X$, we let $\mathrm{Coll} : X^c \to U^c$ denote the "collapse" map given by

$$\mathrm{Coll}(x) = \begin{cases} x & \text{if } x \in U \\ v_U & \text{otherwise.} \end{cases}$$

Construction 5.1.12. We define a $\mathcal{O}[\mathbf{Z}_{\geq 0}]$-coalgebra object T of Top_* as follows:

- If $D \subseteq \mathbf{C}$ is an open disk and $n \geq 0$ is an integer, we let $T(D, n)$ denote the pointed topological space $\mathrm{Sym}^n(D)^c$ (which is homeomorphic to a sphere of dimension $2n$), with base point given by the "point at infinity."
- If $D \subseteq \mathbf{C}$ is an open disk containing disjoint disks $\{D_i\}_{i \in I}$ and we are given nonnegative integers $\{n_i\}_{i \in I}$ with sum n, then we associate the map of pointed topological spaces

$$T(D, n) \to \bigwedge_{i \in I} T(D_i, n_i)$$

given by the collapse determined by the natural open immersion

$$\prod_{i \in I} \mathrm{Sym}^{n_i}(D_i) \hookrightarrow \mathrm{Sym}^n(D).$$

Composing with the symmetric monoidal functor $\mathrm{N}(\mathrm{Top}_*) \to \mathrm{Sp}$ of Definition 5.1.9, we obtain an algebra object $\Sigma^\infty(T) \in \mathrm{Alg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\mathrm{Sp}^{\mathrm{op}})$.

Proposition 5.1.13. *The algebra $\Sigma^\infty(T) \in \mathrm{Alg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\mathrm{Sp}^{\mathrm{op}})$ of Construction 5.1.12 satisfies condition $(*)$ of Proposition 5.1.8, and therefore classifies an $\mathbb{E}_2$-monoidal functor $\rho : \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{Sp}^{\mathrm{op}}$.*

Proof. It suffices to show that for every open disk $D \subseteq \mathbf{C}$ containing smaller disjoint disks $\{D_i\}_{i \in I}$ and every collection of nonnegative integers $\{n_i\}_{i \in I}$ with sum n, the collapse map

$$f : \mathrm{Sym}^n(D)^c \to \bigwedge_{i \in I} \mathrm{Sym}^{n_i}(D_i)^c$$

is a homotopy equivalence. This follows from the observation that f is equivalent to the collapse map associated to an open immersion $\mathbf{R}^{2n} \hookrightarrow \mathbf{R}^{2n}$. $\qquad\square$

Note that the discrete simplicial set $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ is canonically isomorphic to its opposite. We may therefore identify ρ with an $\mathbb{E}_2$-monoidal functor from $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$ into Sp, which assigns to each integer n the sphere S^{2n}. It follows that ρ factors through the subcategory $\mathrm{Pic}(S) = \mathrm{Sp}^{\mathrm{inv}}$. We can now give a precise formulation of our main result:

Theorem 5.1.14. *The $\mathbb{E}_2$-monoidal functor ρ of Proposition 5.1.13 is homotopic to the composition*

$$\mathbf{Z}_{\geq 0}^{\mathrm{ds}} \hookrightarrow \mathbf{Z}^{\mathrm{ds}} \xrightarrow{j} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \xrightarrow{\Phi} \mathrm{Sp},$$

where Φ is the $\mathbb{E}_2$-monoidal functor appearing in the statement of Corollary 3.5.8.

The proof of Theorem 5.1.14 will be given in §5.5.

5.2. **The Skeletal Filtration of $\mathbf{CP}^\infty$.** For each $n \geq 0$, we let $\mathbf{CP}^n$ denote complex projective space of dimension n: that is, the quotient of $\mathbf{C}^{n+1} - \{0\}$ by the action of the multiplicative group $\mathbf{C}^\times$. For each $n > 0$, there is a canonical embedding $\mathbf{CP}^{n-1} \hookrightarrow \mathbf{CP}^n$; we let $\mathbf{CP}^\infty$ denote the direct limit of the sequence

$$\mathbf{CP}^0 \hookrightarrow \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^2 \hookrightarrow \cdots$$

We will regard $\mathbf{CP}^\infty$ as a CW complex with a single cell of every even dimension $2n$, given by the complement of $\mathbf{CP}^{n-1}$ in $\mathbf{CP}^n$ (where we agree to the convention that $\mathbf{CP}^{-1} = \varnothing$.

The diagonal map

$$\delta : \mathbf{CP}^\infty \to \mathbf{CP}^\infty \times \mathbf{CP}^\infty$$

exhibits $\mathbf{CP}^\infty$ as a commutative coalgebra in the category of topological spaces (and therefore also in the ∞-category $\mathcal{S}$ of spaces). However, the map δ is not cellular (with respect to the product cell decomposition of $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$): for $n > 0$, it does not carry $\mathbf{CP}^n$ into the union $\bigcup_{a+b=n} \mathbf{CP}^a \times \mathbf{CP}^b \subseteq \mathbf{CP}^\infty \times \mathbf{CP}^\infty$. Of course, the cellular approximation theorem guarantees that δ is *homotopic* to a cellular map

$$\delta' : \mathbf{CP}^\infty \to \mathbf{CP}^\infty \times \mathbf{CP}^\infty .$$

Since the comultiplication δ is commutative and associative, the comultiplication δ' is commutative and associative up to homotopy. For example, if we let δ'' denote the composition of δ' with the automorphism of $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$ given by swapping the two factors, then δ' and δ'' are both homotopic to δ and we can therefore choose a homotopy $h : \mathbf{CP}^\infty \times [0,1] \to \mathbf{CP}^\infty \times \mathbf{CP}^\infty$. Using cellular approximation again, we can assume that h is a cellular map. Using the fact that the product $\mathbf{CP}^\infty \times \mathbf{CP}^\infty$ has only even-dimensional cells, we conclude that h restricts to a homotopy

$$h_n : \mathbf{CP}^n \times [0,1] \to \bigcup_{a+b=n} \mathbf{CP}^a \times \mathbf{CP}^b$$

for each $n \geq 0$. We can summarize the situation by saying that δ' yields a homotopy commutative comultiplication on the *filtered* space

$$\mathbf{CP}^0 \hookrightarrow \mathbf{CP}^1 \hookrightarrow \mathbf{CP}^2 \hookrightarrow \cdots$$

(and a similar argument shows that δ' is associative up to homotopy).

In this section, we will study the problem of promoting δ' to a *coherently* commutative and associative multiplication on the filtered space $\{\mathbf{CP}^n\}_{n \geq 0}$. Before we can state our main result, we need to introduce a bit of terminology.

Definition 5.2.1. A *filtered space* is a functor $\mathrm{N}(\mathbf{Z}) \to \mathcal{S}$, where $\mathrm{N}(\mathbf{Z})$ denotes the nerve of $\mathbf{Z}$ regarded as a linearly ordered set. The collection of all filtered spaces can be organized into an ∞-category $\mathcal{S}^{\mathrm{filt}} = \mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathcal{S})$. Note that we can identify $\mathcal{S}^{\mathrm{filt}}$ with the ∞-category of presheaves $\mathcal{P}(\mathrm{N}(\mathbf{Z})^{\mathrm{op}})$, which inherits a symmetric monoidal structure from the addition on $\mathbf{Z}$. Concretely, this symmetric monoidal structure is given by the Day convolution product

$$\circledast : \mathcal{S}^{\mathrm{filt}} \times \mathcal{S}^{\mathrm{filt}} \to \mathcal{S}^{\mathrm{filt}}$$

$$(X \circledast Y)_n = \varinjlim_{a+b \leq n} X_a \times Y_b.$$

Example 5.2.2. The construction

$$n \mapsto \begin{cases} \mathrm{Sing}_\bullet \, \mathbf{CP}^n & \text{if } n \geq 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

determines an object of $\mathcal{S}^{\mathrm{filt}}$; we will abuse notation by denoting this object by $\{\mathbf{CP}^n\}_{n \geq 0}$.

We can now state the first main result of this section:

Theorem 5.2.3. *The filtered space* $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$ *has the structure of an* $\mathbb{E}_2$-*coalgebra in the* ∞-*category* $\mathcal{S}^{\mathrm{filt}}$ *of filtered spaces.*

For our purposes in this paper, we will be interested less in the statement of Theorem 5.2.3 than in its proof, which will produce a *specific* $\mathbb{E}_2$-coalgebra structure on $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$, which is closely related to the $\mathbb{E}_2$-monoidal functor ρ introduced in §5.1.

Remark 5.2.4. The construction $\{X_n\}_{n \in \mathbf{Z}} \mapsto \varinjlim X_n$ determines a symmetric monoidal functor $\mathcal{S}^{\mathrm{filt}} \to \mathcal{S}$. Consequently, any $\mathbb{E}_2$-coalgebra structure on the filtered space $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$ determines an $\mathbb{E}_2$-coalgebra structure on the space $\mathbf{CP}^\infty$. However, Proposition H.2.4.3.9 implies that $\mathbf{CP}^\infty$ admits a *unique* $\mathbb{E}_2$-coalgebra structure (up to a contractible space of choices). It follows that any $\mathbb{E}_2$-coalgebra structure on the filtered space $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$ can be regarded as a refinement of the standard coalgebra structure on $\mathbf{CP}^\infty$, given by the diagonal embedding $\delta : \mathbf{CP}^\infty \to \mathbf{CP}^\infty \times \mathbf{CP}^\infty$.

Remark 5.2.5. The amount of commutativity described in Theorem 5.2.3 is optimal: one can show that there does not exist an $\mathbb{E}_3$-coalgebra structure on the filtered space $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$.

We will prove Theorem 5.2.3 using the general paradigm described in §5.1. Namely, we will first construct a $\mathcal{O}$-coalgebra in an ordinary category and use it to produce a $\mathcal{O}$-coalgebra in the ∞-category $\mathcal{S}^{\mathrm{filt}}$ which satisfies the hypotheses of Proposition 5.1.5. We begin by describing the ordinary category we will use.

Definition 5.2.6. A *filtered topological space* is a topological space X equipped with an increasing sequence

$$\cdots \subseteq X_{-2} \subseteq X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

of closed subspaces with the following properties:
   (a) The spaces X_n are empty for $n \ll 0$.
   (b) The space X is a direct limit of the subspaces X_i (in the category of topological spaces).
   (c) Each of the closed embeddings $X_i \hookrightarrow X_{i+1}$ exhibits X_i as a deformation retract of some open neighborhood of its image.

If $Y = \bigcup Y_n$ is another filtered topological space, then we will say that a continuous map $f : X \to Y$ is *filtration-preserving* if it carries each X_n into Y_n. We let $\mathrm{Top}^{\mathrm{filt}}$ denote the category whose objects are filtered topological spaes and whose morphisms are filtration-preserving maps. We will regard $\mathrm{Top}^{\mathrm{filt}}$ as a symmetric monoidal category, where the tensor product of a filtered topological space $X = \bigcup X_n$ with $Y = \bigcup Y_n$ is given by the product $X \times Y$ with the filtration given by subspaces

$$(X \times Y)_n = \bigcup_{a+b=n} X_a \times Y_b \subseteq X \times Y;$$

here we endow $X \times Y$ with the topology which is dictated by condition (b) (in all cases of interest to us here, this topology will coincide with the product topology).

There is an evident symmetric monoidal functor $\mathrm{N}(\mathrm{Top}^{\mathrm{filt}}) \to \mathcal{S}^{\mathrm{filt}}$, which carries a filtered topological space $X = \bigcup X_n$ to the diagram $\{\mathrm{Sing}_\bullet X_n\}_{n \in \mathbf{Z}}$. In what follows, we will generally abuse notation by not distinguishing between a filtered topological space and its its image in $\mathcal{S}^{\mathrm{filt}}$; in particular, we will regard $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$ as a filtered space (here by convention we agree that $\mathbf{CP}^n = \varnothing$ for $n < 0$).

Remark 5.2.7. In the situation of Definition 5.2.6, assumption (c) is not needed to define the category of filtered topological spaces or to construct the forgetful functor $\theta : \mathrm{N}(\mathrm{Top}^{\mathrm{filt}}) \to \mathcal{S}^{\mathrm{filt}}$.

However, it is needed to ensure that the functor θ is symmetric monoidal (it follows from condition (c) that each $(X \times Y)_n$ can be identified with the *homotopy* colimit of the diagram $\{X_a \times Y_b\}_{a+b \le n}$).

Notation 5.2.8. For any topological space X and any integer $n \ge 0$, we let $\mathrm{Sym}^n(X)$ denote the topological space obtained by forming quotient of X^n by the action of the symmetric group Σ_n. By convention, we define $\mathrm{Sym}^n(X)$ to be the empty set if $n < 0$.

We will identify the points of $\mathrm{Sym}^n(X)$ with (non-continuous) functions $\chi : X \to \mathbf{Z}_{\ge 0}$ satisfying $\sum_{x \in X} \chi(x) = n$. Note that a choice of base point $v \in X$ determines continuous maps

$$j : \mathrm{Sym}^n(X) \to \mathrm{Sym}^{n+1}(X),$$

given by

$$j(\chi)(x) = \begin{cases} \chi(x) & \text{if } x \ne v \\ \chi(x) + 1 & \text{if } x = v. \end{cases}$$

Example 5.2.9. Let $\mathbf{C}[x]$ denote a polynomial ring in one variable over $\mathbf{C}$. For each integer $n \ge 0$, let $\mathrm{Poly}_{\le n}$ denote the subspace of $\mathbf{C}[x]$ consisting of polynomials of degree n. We will identify $\mathbf{CP}^n$ with the quotient of $\mathrm{Poly}_{\le n} - \{0\}$ by the action of $\mathbf{C}^\times$. The multiplication on $\mathbf{C}[x]$ determines maps

$$\mathbf{CP}^{a_1} \times \mathbf{CP}^{a_2} \times \cdots \times \mathbf{CP}^{a_n} \to \mathbf{CP}^{a_1 + \cdots + a_n} .$$

Taking $a_1 = a_2 = \cdots = a_n = 1$ and making use of the commutativity of $\mathbf{C}[x]$, we obtain a continuous map $h_n : \mathrm{Sym}^n(\mathbf{CP}^1) \to \mathbf{CP}^n$ which is easily seen to be a homeomorphism. Moreover, the homeomorphisms h_n fit into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sym}^n(\mathbf{CP}^1) & \xrightarrow{\ h_n\ } & \mathbf{CP}^n \\
\downarrow & & \downarrow \\
\mathrm{Sym}^{n+1}(\mathbf{CP}^1) & \xrightarrow{\ h_{n+1}\ } & \mathbf{CP}^{n+1}
\end{array}
$$

where the left vertical map is determined by the base point of $\mathbf{CP}^1$ (that is, the point given by the image of $\mathbf{CP}^0 \to \mathbf{CP}^1$, represented by the constant polynomial $1 \in \mathbf{C}[x]$).

Remark 5.2.10. Let X be a topological space equipped with a base point $v \in X$. We let $\mathrm{Sym}^*(X)$ denote the direct system of topological spaces

$$\mathrm{Sym}^0(X) \hookrightarrow \mathrm{Sym}^1(X) \hookrightarrow \mathrm{Sym}^2(X) \hookrightarrow \cdots,$$

and we let $\mathrm{Sym}^\infty(X)$ denote the direct limit $\varinjlim \mathrm{Sym}^n(X)$. In all of the examples that we consider below, this sequence will satisfy condition (c) of Definition 5.2.6 so that we can regard $\mathrm{Sym}^*(X)$ as a filtered topological space.

In the special case where X can be written as the disjoint union of closed subspaces X_0 and X_1 with $X_0 \cap X_1 = \{v\}$, we have canonical closed embeddings

$$\mathrm{Sym}^a(X_0) \times \mathrm{Sym}^b(X_1) \hookrightarrow \mathrm{Sym}^{a+b}(X)$$

which induce a homeomorphism $\mathrm{Sym}^\infty(X_0) \times \mathrm{Sym}^\infty(X_1) \simeq \mathrm{Sym}^\infty(X)$. Under this homeomorphism, we can identify each individual symmetric power $\mathrm{Sym}^n(X)$ with the union

$$\bigcup_{a+b=n} \mathrm{Sym}^a(X_0) \times \mathrm{Sym}^b(X_1),$$

so that $\mathrm{Sym}^*(X)$ is the tensor product of $\mathrm{Sym}^*(X_0)$ with $\mathrm{Sym}^*(X_1)$ in the category $\mathrm{Top}^{\mathrm{filt}}$.

Construction 5.2.11. We define a $\mathcal{O}$-coalgebra C in the category $\mathrm{Top}^{\mathrm{filt}}$ of filtered topological spaces as follows:

- For any open disk $D \subseteq \mathbf{C}$, we let $C(D)$ be the filtered topological space $\mathrm{Sym}^*(D^c)$. (note that Example 5.2.9 shows that $C(D)$ is noncanonically homeomorphic to the filtered topological space $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$ and therefore satisfies condition (c) of Definition 5.2.6).
- Given an open disk $D \subseteq \mathbf{C}$ and a collection of disjoint open disks D_i contained in D, the induced map $C(D) \to \bigotimes_i C(D_i)$ is given by the composition

$$C(D) = \mathrm{Sym}^*(D^c) \xrightarrow{\alpha} \mathrm{Sym}^*((\sqcup_i D_i)^c) \overset{\beta}{\simeq} \bigotimes_i \mathrm{Sym}^*(D_i^c) = \bigotimes_i C(D_i),$$

where α is induced by the collapse map $\mathrm{Coll} : D^c \to (\sqcup_i D_i)^c$ and β is the identification given by Remark 5.2.10.

Proof of Theorem 5.2.3. Let $C \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Top}^{\mathrm{filt\,op}})$ be the coalgebra of Construction 5.2.11. Applying the symmetric monoidal functor $\mathrm{N}(\mathrm{Top}^{\mathrm{filt}}) \to \mathcal{S}^{\mathrm{filt}}$, we obtain a $\mathcal{O}$-coalgebra object $\mathcal{S}^{\mathrm{filt}}$ whose underlying filtered space is $\mathrm{Sym}^*(\mathbf{C}^c) = \mathrm{Sym}^*(\mathbf{CP}^1) \simeq \{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$. To complete the proof, it will suffice to show that this $\mathcal{O}$-coalgebra satisfies condition $(*)$ of Proposition 5.1.5 (so that it arises from an $\mathbb{E}_2$-coalgebra object of $\mathcal{S}^{\mathrm{filt}}$). To prove this, we must show that for every inclusion $D \hookrightarrow D'$ of open disks in $\mathbf{C}$, the collapse map $\mathrm{Coll} : D'^c \to D^c$ induces a homotopy equivalence $\mathrm{Sym}^n(D'^c) \to \mathrm{Sym}^n(D^c)$. This follows from the fact that the collapse map $\mathrm{Coll} : D'^c \to D^c$ is itself a homotopy equivalence. $\square$

5.3. Koszul Duality and the Skeletal Filtration.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We let $\mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C})$ denote the ∞-category of augmented $\mathbb{E}_2$-algebra objects of $\mathcal{C}$, and we let $\mathrm{CoAlg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C}) = \mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ denote the ∞-category of augmented $\mathbb{E}_2$-coalgebra objects of $\mathcal{C}$. If $\mathcal{C}$ admits geometric realizations of simplicial objects, then the ∞-categories $\mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C})$ and $\mathrm{CoAlg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C})$ are related by the *2-fold bar construction*

$$\mathrm{Bar}^{(2)} : \mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C}) \to \mathrm{CoAlg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{C});$$

we refer the reader to §H.5.2.3 for more details.

In the special case where $\mathcal{C} = \mathcal{S}$, the two-fold bar construction $\mathrm{Bar}^{(2)}$ assigns to each $\mathbb{E}_2$-space X the two-fold delooping of its group completion. In particular, if we regard the set of nonnegative integers $\mathbf{Z}_{\geq 0}$ as an $\mathbb{E}_2$-space (with the discrete topology and $\mathbb{E}_2$-structure given by addition of integers), then we have

$$\mathrm{Bar}^{(2)}(\mathbf{Z}_{\geq 0}) \simeq K(\mathbf{Z}, 2) \simeq \mathbf{CP}^\infty.$$

Our goal in this section is to prove a filtered analogue of this statement. To formulate our result, let us regard $\mathbf{Z}_{\geq 0}$ as a *filtered* topological space by considering the family of subsets

$$\{0\} \hookrightarrow \{0, 1\} \hookrightarrow \{0, 1, 2\} \hookrightarrow \cdots$$

The addition law on $\mathbf{Z}_{\geq 0}$ respects this filtration, so that we obtain a commutative algebra in the category $\mathrm{Top}^{\mathrm{filt}}$ of filtered topological spaces, hence also in the ∞-category $\mathcal{S}^{\mathrm{filt}}$. Let $A \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}^{\mathrm{filt}})$ denote the underlying $\mathbb{E}_2$-algebra of this commutative algebra. Note that A admits a unique augmentation. Our goal in this section is to prove the following result:

Theorem 5.3.1. *The two-fold bar construction* $\mathrm{Bar}^{(2)}(A) \in \mathrm{CoAlg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{S}^{\mathrm{filt}})$ *is equivalent to* $\{\mathbf{CP}^n\}_{n \geq 0}$, *equipped with the $\mathbb{E}_2$-coalgebra structure given by Construction 5.2.11.*

To prove Theorem 5.3.1, it will be convenient to replace A by an equivalent $\mathbb{E}_2$-algebra object of $\mathcal{S}^{\mathrm{filt}}$ for which the relationship with Construction 5.2.11 is more apparent.

Construction 5.3.2. For every topological space X, we let X_+ denote the disjoint union $X \sqcup \{v\}$. We define a $\mathcal{O}$-algebra object A' of $\mathrm{Top}^{\mathrm{filt}}$ as follows:

- For every open disk $D \subseteq \mathbf{C}$, we let $A'(D)$ denote the filtered topological space given by $\mathrm{Sym}^*(D_+)$.
- For every open disk $D \subseteq \mathbf{C}$ and every finite collection $\{D_i\}$ of disjoint open disks contained in D, the map $\bigotimes_i A'(D_i) \to A'(D)$ is given by the composition

$$\bigotimes_i A'(D_i) = \bigotimes_i \mathrm{Sym}^*(D_{i+}) \overset{\beta}{\simeq} \mathrm{Sym}^*((\sqcup_i D_i)_+) \overset{\gamma}{\to} \mathrm{Sym}^*(D_+) = A'(D),$$

where β is the identification described in Definition 5.2.6 and γ is induced by the continuous map $\sqcup_i D_i \to D$.

Remark 5.3.3. Let us abuse notation by identifying the $\mathcal{O}$-algebra A' of Construction 5.3.2 with its image in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{S}^{\mathrm{filt}})$. This $\mathcal{O}$-algebra satisfies condition $(*)$ of Proposition 5.1.5: that is, every inclusion of open disks $D \hookrightarrow D'$ induces a homotopy equivalence $\mathrm{Sym}^n(D_+) \simeq \mathrm{Sym}^n(D'_+)$. Consequently, the $\mathcal{O}$-algebra A' arises from an $\mathbb{E}_2$-algebra object of $\mathcal{S}^{\mathrm{filt}}$, which we will also denote by A'.

Remark 5.3.4. For any topological space X, we have canonical homeomorphisms

$$\mathrm{Sym}^n(X_+) \simeq \sqcup_{0 \leq m \leq n} \mathrm{Sym}^m X.$$

In particular, the projection map $X \to *$ induces a continuous map $\mathrm{Sym}^n(X_+) \to \mathrm{Sym}^n(*_+) \simeq \{0, \ldots, n\}$, which is a homotopy equivalence whenever X is contractible. Since any open disk in $\mathbf{C}$ is contractible, we conclude that $A' \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}^{\mathrm{filt}})$ is equivalent to the $\mathbb{E}_2$-algebra A whose definition precedes the statement of Theorem 5.3.1.

Our next step is to relate the algebra A' of Construction 5.3.2 with the coalgebra C of Construction 5.2.11. Recall that for any category $\mathcal{C}$, the collection of all morphisms in $\mathcal{C}$ can be organized into a category $\mathrm{TwArr}(\mathcal{C})$ (called the *twisted arrow category of* $\mathcal{C}$), where a morphism from $(f : X \to Y)$ to $(f' : X' \to Y')$ in $\mathrm{TwArr}(\mathcal{C})$ is a commutative diagram

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ \downarrow & & \uparrow \\ X' & \overset{f'}{\longrightarrow} & Y'. \end{array}$$

If $\mathcal{C}$ is a symmetric monoidal category, then $\mathrm{TwArr}(\mathcal{C})$ inherits a symmetric monoidal structure; moreover, we have a symmetric monoidal forgetful functor

$$\mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$$

$$(f : X \to Y) \mapsto (X, Y).$$

Construction 5.3.5. For every locally compact Hausdorff space X, there is canonical continuous bijection $X_+ \to X^c$, which is a homeomorphism if and only if X is compact. Note that if $j : U \hookrightarrow X$ is an open immersion, then the diagram

$$\begin{array}{ccc} U_+ & \longrightarrow & U^c \\ \downarrow & & \uparrow{\scriptstyle\mathrm{Coll}} \\ X_+ & \longrightarrow & X^c \end{array}$$

commutes.

We define a $\mathcal{O}$-algebra object T in the symmetric monoidal category $\mathrm{TwArr}(\mathrm{Top}^{\mathrm{filt}})$ as follows:

- For every open disk $D \subseteq \mathbf{C}$, we let $T(D)$ denote the map of filtered spaces given by applying Sym^* to the canonical map $D_+ \to D^c$.
- Given an open disk $D \subseteq \mathbf{C}$ and a collection of disjoint open subdisks $D_i \subseteq D$, we assign the map $\otimes_i T(D_i) \to T(D)$ in $\mathrm{TwArr}(\mathrm{Top}^{\mathrm{filt}})$ given by the commutative diagram of filtered topological spaces

$$
\begin{array}{ccc}
\otimes_i \mathrm{Sym}^*(D_{i+}) & \longrightarrow & \otimes_i \mathrm{Sym}^*(D_i^c) \\
\downarrow & & \uparrow \\
\mathrm{Sym}^*(D_+) & \longrightarrow & \mathrm{Sym}^*(D^c)
\end{array}
$$

obtained by applying Sym^* to the commutative diagram

$$
\begin{array}{ccc}
(\sqcup D_i)_+ & \longrightarrow & (\sqcup D_i)^c \\
\downarrow & & \uparrow \\
D_+ & \longrightarrow & D^c.
\end{array}
$$

Remark 5.3.6. The forgetful functor $\mathrm{TwArr}(\mathrm{Top}^{\mathrm{filt}}) \to \mathrm{Top}^{\mathrm{filt}} \times (\mathrm{Top}^{\mathrm{filt}})^{\mathrm{op}}$ carries the $\mathcal{O}$-algebra T of Construction 5.3.5 to the pair $(A', C) \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{Top}^{\mathrm{filt}}) \times \mathrm{Alg}_{\mathcal{O}}(\mathrm{Top}^{\mathrm{filt}\,\mathrm{op}})$.

The construction $\mathcal{C} \mapsto \mathrm{TwArr}(\mathcal{C})$ can be generalized to the setting of ∞-categories; we refer the reader to §H.5.2.1 for details. Applying the symmetric monoidal functor

$$
\mathrm{N}(\mathrm{TwArr}(\mathrm{Top}^{\mathrm{filt}})) \to \mathrm{TwArr}(\mathcal{S}^{\mathrm{filt}})
$$

to the $\mathcal{O}$-algebra T, we obtain a $\mathcal{O}$-algebra object of $\mathrm{TwArr}(\mathcal{S}^{\mathrm{filt}})$ which we will also denote by T. It is clear that this algebra satisfies condition $(*)$ of Proposition 5.1.5 and can therefore be identified with an $\mathbb{E}_2$-algebra object of $\mathrm{TwArr}(\mathcal{S}^{\mathrm{filt}})$. Let

$$
\mathrm{Bar}^{(2)} : \mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{S}^{\mathrm{filt}}) \to \mathrm{CoAlg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{S}^{\mathrm{filt}})
$$

denote the 2-fold bar construction described in §H.5.2.3, so that the algebra T can be identified with a morphism $\psi : \mathrm{Bar}^{(2)} A' \to C$ in $\mathrm{CoAlg}_{\mathbb{E}_2}^{\mathrm{aug}}(\mathcal{S}^{\mathrm{filt}})$. Theorem 5.3.1 is an immediate consequence of the following more precise result:

Theorem 5.3.7. *The map $\psi : \mathrm{Bar}^{(2)} A' \to C$ described above is an equivalence in the ∞-category* $\mathrm{CoAlg}_{\mathbb{E}_2}(\mathcal{S}^{\mathrm{filt}})$ *of $\mathbb{E}_2$-coalgebras in filtered spaces.*

Proof. Let C' denote the filtered space given by $\mathrm{Bar}^{(2)}(A')$. Note that since $A'_n = \varnothing$ for $n < 0$, the filtered space C' has the same property; we may therefore identify C' with a diagram

$$
C_0' \to C_1' \to C_2' \to \cdots
$$

in the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{Z}_{\geq 0}), \mathcal{S})$. We wish to show that the map $\psi : C' \to C$ is an equivalence of filtered spaces. The proof will take place in several steps.

Step (1): We claim that each of the spaces C_i' is simply connected. Note that as an associative algebra object of $\mathcal{S}^{\mathrm{filt}}$, A is freely generated by the filtered space

$$
\varnothing \subseteq \{1\} \subseteq \{1\} \subseteq \{1\} \subseteq \cdots.
$$

It follows that A' has the same property. Invoking Corollary H.5.2.2.13, we see that the 1-fold bar construction $B = \mathrm{Bar}^{(1)}(A')$ can be identified with the filtered space

$$* \to S^1 \to S^1 \to S^1 \to \cdots.$$

Let $\mathbf{1}$ denote the unit object of $\mathcal{S}^{\mathrm{filt}}$, given by

$$\mathbf{1}_n = \begin{cases} * & \text{if } n \geq 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

We will identify B with the relative tensor product $\mathbf{1} \otimes_{A'} \mathbf{1}$, so that B can be regarded as a commutative algebra object of $\mathcal{S}^{\mathrm{filt}}$.

For each $n \geq 0$, let $\tau_{\leq n} : \mathcal{S}^{\mathrm{filt}} \to \mathcal{S}^{\mathrm{filt}}$ be the n-truncation functor, which carries a filtered space $\{X_m\}_{m \in \mathbf{Z}}$ to its truncation $\{\tau_{\leq n} X_m\}_{m \in \mathbf{Z}}$. The explicit description of B supplied above shows that the canonical map $B \to \mathbf{1}$ induces an equivalence $\tau_{\leq 0} B \simeq \mathbf{1}$. From this, we deduce the existence of equivalences

$$\tau_{\leq 0}(B \otimes B \otimes \cdots \otimes B) \simeq \mathbf{1}$$

for arbitrarily many tensor factors. It follows that if X is any 1-truncated filtered space, then the canonical map

$$X_0 \simeq \mathrm{Map}_{\mathcal{S}^{\mathrm{filt}}}(\mathbf{1}, X) \to \mathrm{Map}_{\mathcal{S}^{\mathrm{filt}}}(B \otimes B \otimes \cdots \otimes B, X)$$

is fully faithful. If X is a commutative algebra object of $\mathcal{S}^{\mathrm{filt}}$, we conclude that the induced map

$$* \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}^{\mathrm{filt}})}(\mathbf{1}, X) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{S}^{\mathrm{filt}})}(B, X)$$

is also a fully faithful: that is, the diagram

$$\begin{array}{ccc} B & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

is a pushout square in $\mathrm{CAlg}(\tau_{\leq 1} \mathcal{S}^{\mathrm{filt}})$. This shows that $\tau_{\leq 1}(\mathbf{1} \otimes_B \mathbf{1})$ is equivalent to $\mathbf{1}$, which is equivalent to the simple connectivity of the spaces C'_n.

Step (2): Let $C'_\infty = \varinjlim_n C'_n$. For every filtered space X, let $\mathrm{H}_*(X)$ denote the bigraded abelian group given by $(a, b) \mapsto \mathrm{H}_a(X_b; \mathbf{Z})$. We will regard $\mathrm{H}_*(X)$ as a module over $\mathbf{Z}[x]$ (where x has degree $(0, 1)$), with multiplication by x given by the maps $X_b \to X_{b+1}$. The explicit description of B above supplies an isomorphism $\mathrm{H}_*(B) = \mathbf{Z}[x, \epsilon]$, where ϵ has bidegree $(1, 1)$ and satisfies $\epsilon^2 = 0$. The identification $C' \simeq \mathbf{1} \otimes_B \mathbf{1}$ yields a spectral sequence

$$\mathrm{Tor}_s^{\mathbf{Z}[x, \epsilon]}(\mathbf{Z}[x], \mathbf{Z}[x])_t \Rightarrow \mathrm{H}_*(C').$$

Using the projective resolution

$$\cdots \to \mathbf{Z}[x, \epsilon] \xrightarrow{\epsilon} \mathbf{Z}[x, \epsilon] \xrightarrow{\epsilon} \mathbf{Z}[x, \epsilon]$$

of $\mathbf{Z}[x, \epsilon]$, we see that the groups $\mathrm{Tor}_s^{\mathbf{Z}[x, \epsilon]}(\mathbf{Z}[x], \mathbf{Z}[x])_t$ are given by $\mathbf{Z}[x]$ when $t = s$ and zero otherwise. Using this, we deduce that our spectral sequence degenerates and yields isomorphisms

$$\mathrm{H}_*(C'_n; \mathbf{Z}) \simeq \begin{cases} \mathrm{H}_*(C'_\infty; \mathbf{Z}) & \text{if } * \leq 2n \\ 0 & \text{if } * > 2n. \end{cases}$$

Since C'_n is simply-connected, the map $\psi_n : C'_n \to C_n = \mathbf{CP}^n$ is a homotopy equivalence if and only if the upper vertical map in the commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}_*(C'_n; \mathbf{Z}) & \longrightarrow & \mathrm{H}_*(\mathbf{CP}^n; \mathbf{Z}) \\
\downarrow & & \downarrow \\
\mathrm{H}_*(C'_\infty; \mathbf{Z}) & \longrightarrow & \mathrm{H}_*(\mathbf{CP}^\infty; \mathbf{Z})
\end{array}
$$

is an isomorphism. This is automatic for $* > 2n$ (since the domain and codomain both vanish). For $* \le 2n$, the vertical maps are isomorphisms. Consequently, to show that ψ is an equivalence of filtered spaces, it will suffice to show that the induced map $C'_\infty \to \mathbf{CP}^\infty$ is a homotopy equivalence.

Step (3): For every filtered space $\{X_n\}$, let X_∞ denote the colimit $\varinjlim X_n$. The construction $\{X_n\} \mapsto X_\infty$ determines a colimit-preserving symmetric monoidal functor $\mathcal{S}^{\mathrm{filt}} \to \mathcal{S}$. Consequently, we can identify C'_∞ with the 2-fold bar construction

$$
\mathrm{Bar}^{(2)}(A'_\infty) \simeq \mathrm{Bar}^{(2)}(A_\infty) \simeq \mathrm{Bar}^{(2)}(\mathbf{Z}_{\ge 0})
$$

(formed in the ∞-category $\mathcal{S}$). Note that the inclusion $\mathbf{Z}_{\ge 0} \hookrightarrow \mathbf{Z}$ induces a homotopy equivalence on 1-fold bar constructions (and therefore also on 2-fold bar constructions), so we can identify C'_∞ with the Eilenberg-MacLane space $\mathrm{Bar}^{(2)}(\mathbf{Z}) \simeq K(\mathbf{Z}, 2)$. Consequently, to show that $\psi_\infty : C'_\infty \to C_\infty = \mathbf{CP}^\infty$ is a homotopy equivalence, it will suffice to show that it induces an isomorphism $\mathrm{H}_2(C'_\infty; \mathbf{Z}) \to \mathrm{H}_2(\mathbf{CP}^\infty, \mathbf{Z})$.

Step (4): Let $\widetilde{T} \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{TwArr}(\mathcal{S}^{\mathrm{filt}}))$ be a $\mathcal{O}$-algebra object which witnesses an identification of C' with the 2-fold bar construction on A' (so that $\widetilde{T}$ has image $(A', C') \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{S}^{\mathrm{filt}} \times \mathcal{S}^{\mathrm{filt\,op}}))$. Let $D \subseteq \mathbf{C}$ be the standard unit disk. For every disk $D' \subseteq \mathbf{C}$ disjoint from D, the composite map

$$
A'(D')_\infty \to A'(\mathbf{C})_\infty \to C'(\mathbf{C})_\infty \to C'(D)_\infty
$$

can be identified with the composition

$$
A'(D')_\infty \hookrightarrow A'(D)_\infty \times A'(D')_\infty \to C'(D)_\infty \times C'(D')_\infty \to C'(D)_\infty
$$

and is therefore canonically nullhomotopic. Composing with the inclusions

$$
D' \hookrightarrow \mathrm{Sym}^1(D'_+) = A'(D')_1 \qquad \mathbf{C} \hookrightarrow \mathrm{Sym}^1(\mathbf{C}_+) = A'(\mathbf{C})_1,
$$

we obtain a canonical map

$$
\theta : \mathrm{cofib}((\varinjlim_{D'} D') \to \mathbf{C}) \to C'(D)_\infty,
$$

where the colimit on the left is taken over all open disks in $\mathbf{C}$ which are disjoint from D. Consequently, to show that ψ_∞ induces a surjection on H_2, it will suffice to show that the composite map

$$
\mathrm{cofib}((\varinjlim_{D'} D') \to \mathbf{C}) \to C'(D)_\infty \to C(D)_\infty
$$

induces a surjection on H_2. This composite map is obtained by applying the same procedure to the $\mathcal{O}$-algebra T of $\mathrm{TwArr}(\mathcal{S}^{\mathrm{filt}})$. Unwinding the definitions, we see that it is given by a composition

$$
\mathrm{cofib}((\varinjlim_{D'} D') \to \mathbf{C}) \overset{\alpha}{\to} D^c = C(D)_1 \overset{\beta}{\to} C(D)_\infty.
$$

Note that β can be identified with the canonical inclusion $\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty$ (Example 5.2.9), and is therefore an isomorphism on H_2. We note that D^c is given by the cofiber of the canonical inclusion $\mathbf{C} - D \hookrightarrow \mathbf{C}$; consequently, to prove that α is an equivalence it suffices to show that the natural map $\varinjlim_{D'} D' \to \mathbf{C} - D$ is an equivalence in the ∞-category of spaces. This is equivalent to the assertion that $\mathbf{C} - \overline{D}$ is a homotopy colimit of the diagram $\{D'\}_{D' \cap D = \varnothing}$; here $\overline{D}$ denotes the closed unit disk in $\mathbf{C}$. This follows from the observation that for every point $x \in \mathbf{C} - \overline{D}$, the partially ordered set of open disks containing x and disjoint from D has weakly contractible nerve (in fact, it is the opposite of a filtered partially ordered set); see Theorem H.A.3.1.

$\square$

5.4. **A Geometric Description of $S[\beta]$.** Let $S[\beta]$ denote the graded $\mathbb{E}_2$ algebra introduced in §3.4. Our goal in this section is to prove that $S[\beta]$ can be identified with the associated graded algebra of a filtered $\mathbb{E}_2$-algebra, obtained by applying Spanier-Whitehead duality to the $\mathbb{E}_2$-algebra in filtered spaces $\{\mathbf{CP}^n\}$ introduced in §5.2 (Proposition 5.4.9). We begin with a general discussion of Spanier-Whitehead duality.

Notation 5.4.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. Suppose that the monoidal structrue on $\mathcal{C}$ is closed. Then for each object $C \in \mathcal{C}$, the functor $D \mapsto \mathrm{Map}_\mathcal{C}(C \otimes D, \mathbf{1})$ is representable by an object of $\mathcal{C}$. We will denote such an object by $C^\vee$ and refer to it as the *weak dual* of C. The construction $C \mapsto C^\vee$ determines a lax symmetric monoidal functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}$ (see §H.5.2.5); in particular, for every pair of objects $X, Y \in \mathcal{C}$ we have a canonical map

$$X^\vee \otimes Y^\vee \to (X \otimes Y)^\vee$$

which is an equivalence if either X or Y is dualizable.

Example 5.4.2 (Duality in Spectra). If $\mathcal{C}$ is the ∞-category of spectra, then the construction $X \mapsto X^\vee$ of Notation 5.4.1 is given by Spanier-Whitehead duality.

Example 5.4.3 (Duality in Graded Spectra). Let $\mathcal{C}$ be the ∞-category $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ of graded spectra. For any graded spectrum X, we have canonical equivalences

$$
\begin{aligned}
(X^\vee)_n &\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(S(-n), X^\vee) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(S(-n) \otimes X, S) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(X, S(n)) \\
&\simeq (X_{-n})^\vee.
\end{aligned}
$$

Example 5.4.4. Let $\mathrm{Rep}(\mathbf{Z})$ be the ∞-category of filtered spectra and let $j : \mathrm{N}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z})$ denote the stable Yoneda embedding. We let $\underline{S} = \varinjlim_n j(n) \in \mathrm{Rep}(\mathbf{Z})$ be the constant diagram taking the value S. For each integer n, we let $E(n) \in \mathrm{Rep}(\mathbf{Z})$ be the cofiber of the canonical map $j(n-1) \to \underline{S}$, given concretely by the formula

$$E(n)_m = \begin{cases} S & \text{if } m \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Note that as a functor, $E(n)$ is a right Kan extension of its restriction to $\{m\} \subseteq \mathrm{N}(\mathbf{Z})^{\mathrm{op}}$, so for any filtered spectrum X we have a canonical equivalence

$$\underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z})}(X, E(n)) \simeq X_n^\vee.$$

Writing $\underline{S} = \varprojlim_n E(-n)$, we obtain an equivalence $\underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z})}(X, \underline{S}) \simeq X_\infty^\vee$, where $X_\infty = \varinjlim_n X_n$. Using the fiber sequences

$$j(-n) \to \underline{S} \to E(1-n),$$

we obtain equivalences

$$\begin{aligned}
(X^\vee)_n &\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z})}(j(n), X^\vee) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z})}(j(n) \circledast X; j(0)) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z})}(X, j(-n)) \\
&\simeq \mathrm{fib}(X_\infty^\vee \to X_{1-n}^\vee).
\end{aligned}$$

Remark 5.4.5. Let X be a filtered spectrum and let $X^\vee$ be the weak dual of X in the ∞-category $\mathrm{Rep}(\mathbf{Z})$. It follows from Example 5.4.4 that the associated graded spectrum of $X^\vee$ is given by

$$\begin{aligned}
\mathrm{gr}(X^\vee)_n &= \mathrm{cofib}((X^\vee)_{n+1} \to (X^\vee)_n) \\
&\simeq \mathrm{fib}(X_{-n}^\vee \to X_{1-n}^\vee) \\
&\simeq \mathrm{cofib}(X_{1-n} \to X_{-n})^\vee \\
&\simeq \mathrm{gr}(X)_n^\vee.
\end{aligned}$$

It is not difficult to see that this identification is induced by the symmetric monoidal structure on the functor $\mathrm{gr} : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$: in other words, the associated graded functor gr commutes with (weak) duality.

Construction 5.4.6. Composition with the functor $\Sigma_+^\infty : \mathcal{S} \to \mathrm{Sp}$ induces a symmetric monoidal functor

$$\mathcal{S}^{\mathrm{filt}} \to \mathrm{Fun}(\mathrm{N}(\mathbf{Z}), \mathrm{Sp}) \simeq \mathrm{Fun}(\mathrm{N}(\mathbf{Z})^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{Rep}(\mathbf{Z}).$$

Composing this functor with the weak duality functor $\mathrm{Rep}(\mathbf{Z})^{\mathrm{op}} \to \mathrm{Rep}(\mathbf{Z})$, we obtain a lax symmetric monoidal functor $\psi : (\mathcal{S}^{\mathrm{filt}})^{\mathrm{op}} \to \mathrm{Rep}(\mathbf{Z})$.

Remark 5.4.7. Using the description of the duality functor on $\mathrm{Rep}(\mathbf{Z})$ given in Example 5.4.4, we see that ψ can be described concretely by the formula

$$\psi(\{X_n\}_{n \in \mathbf{Z}})_m = (\Sigma^\infty X/X_{m-1})^\vee,$$

where X/X_{m-1} denotes the cofiber $X \amalg_{X_{m-1}} *$. It follows that the associated graded spectrum is given by

$$(\mathrm{gr}\,\psi(\{X_n\}_{n \in \mathbf{Z}}))_m = \Sigma^\infty (X_m/X_{m-1})^\vee.$$

Example 5.4.8. Let C denote the filtered space $\{\mathbf{CP}^n\}_{n \in \mathbf{Z}}$. For each $n \geq 0$, the cofiber C_n/C_{n-1} can be identified with the sphere S^{2n}. It follows that the associated graded spectrum of $\psi(C)$ is given by

$$\mathrm{gr}(\psi(C))_n \simeq \begin{cases} S^{-2n} & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since the functor ψ is lax symmetric monoidal, Construction 5.2.11 supplies the graded spectrum $\mathrm{gr}(\psi(C))$ of Example 5.4.8 with the structure of a graded $\mathbb{E}_2$-algebra. We can now state the main result of this section:

Proposition 5.4.9. *Let $C = \{\mathbf{CP}^n\}$ be as in Example 5.4.8. Then there is a canonical equivalence of graded $\mathbb{E}_2$-rings $\mathrm{gr}(\psi(C)) \simeq S[\beta]$, where $S[\beta]$ is the graded $\mathbb{E}_2$-algebra introduced in §3.4.*

Proof. Let $S[t]$ be the $\mathbb{E}_\infty$-algebra in $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ described in Notation 3.1.4: that is, the image of the unit object of $\mathrm{Rep}(\mathbf{Z})$ under the forgetful functor $G : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. Let F be left adjoint to G (given by left Kan extension along the inclusion $(\mathbf{Z}^{\mathrm{ds}})^{\mathrm{op}} \hookrightarrow \mathrm{N}(\mathbf{Z})^{\mathrm{op}}$), so that $S[t]$ is the image of the unit object under the composition functor

$$\mathrm{Rep}(\mathbf{Z}) \overset{G}{\to} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \overset{F}{\to} \mathrm{Rep}(\mathbf{Z}) \overset{\mathrm{gr}}{\to} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}).$$

Note that the restriction map $g : \mathcal{S}^{\mathrm{filt}} \to \prod_{n \in \mathbf{Z}} \mathcal{S}$ admits a left adjoint f (given by left Kan extension along the inclusion $\mathbf{Z}^{\mathrm{ds}} \hookrightarrow \mathrm{N}(\mathbf{Z})$) which fits into a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{S}^{\mathrm{filt}} & \overset{g}{\longrightarrow} & \prod_{n \in \mathbf{Z}} \mathcal{S} & \overset{f}{\longrightarrow} & \mathcal{S}^{\mathrm{filt}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Rep}(\mathbf{Z}) & \overset{G}{\longrightarrow} & \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) & \overset{F}{\longrightarrow} & \mathrm{Rep}(\mathbf{Z})
\end{array}
$$

where the vertical maps are given by levelwise composition with the functor Σ_+^∞ (together with the identification of $\mathrm{N}(\mathbf{Z})$ with $\mathrm{N}(\mathbf{Z})^{\mathrm{op}}$). It follows that $S[t]$ can be identified with the image of the unit object of $\mathcal{S}^{\mathrm{filt}}$ under the (lax symmetric monoidal) composite functor

$$\mathcal{S}^{\mathrm{filt}} \overset{g}{\to} \prod_{n \in \mathbf{Z}} \mathcal{S} \overset{f}{\to} \mathcal{S}^{\mathrm{filt}} \overset{\Sigma_+^\infty}{\to} \mathrm{Rep}(\mathbf{Z}) \overset{\mathrm{gr}}{\to} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}).$$

Since unit object $\mathbf{1}$ of $\mathcal{S}^{\mathrm{filt}}$ is the filtered space given by the formula

$$
\mathbf{1}(n) = \begin{cases} * & \text{if } n \geq 0 \\ \varnothing & \text{if } n < 0, \end{cases}
$$

an easy calculation gives an identification $(f \circ g)(\mathbf{1}) \simeq A$ of $\mathbb{E}_\infty$-algebras of $\mathcal{S}^{\mathrm{filt}}$, where A is the commutative algebra described in §5.3. We may therefore write $S[t] \simeq \mathrm{gr}(\Sigma_+^\infty A)$. The unique augmentation $\epsilon_0 : S[t] \to S$ in $\mathrm{CAlg}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ can be obtained by applying the symmetric monoidal functor $\mathrm{gr} \circ \psi_0$ to the augmentation $A \to \mathbf{1}$ in $\mathcal{S}^{\mathrm{filt}}$.

Let $\mathrm{Bar}^{(2)}(S[t])$ denote the $\mathbb{E}_2$-coalgebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ obtained by applying a 2-fold bar construction to $S[t]$ (regarded as an augmented $\mathbb{E}_2$-algebra in $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$). Since both $\Sigma_+^\infty : \mathcal{S}^{\mathrm{filt}} \to \mathrm{Rep}(\mathbf{Z})$ and $\mathrm{gr} : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ are symmetric monoidal functors which preserve small colimits, Theorem 5.3.1 supplies an equivalence of $\mathbb{E}_2$-coalgebras

$$\mathrm{Bar}^{(2)}(S[t]) \simeq \mathrm{Bar}^{(2)}(\mathrm{gr}(\Sigma_+^\infty A)) \simeq \mathrm{gr}(\Sigma_+^\infty \mathrm{Bar}^{(2)}(A))) = \mathrm{gr}(\Sigma_+^\infty C).$$

Remark 5.4.5 implies that we can identify the graded algebra $\mathrm{gr}(\psi(C))$ with the dual of $\mathrm{gr}(\Sigma_+^\infty C) \simeq \mathrm{Bar}^{(2)}(S[t])$. Applying the results of §H.5.2.5, we see that the $\mathbb{E}_2$-algebra $\mathrm{gr}(\psi(C))$ can be identified with the *Koszul dual* of $S[t]$: that is, $\mathrm{gr}(\psi(C))$ is a final object of the ∞-category of $\mathbb{E}_2$-algebra objects B of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ equipped with maps $B \otimes S[t] \to S$ (note that compatibility with ϵ_0 is automatic, since the mapping space $\mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))}(S[t], S)$ is contractible). Using the results of §H.4.8.5, we can instead regard $\mathrm{gr}(\psi(C))$ as universal among those $\mathbb{E}_2$-algebra objects B of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ for which there is a monoidal functor

$$\mathrm{Rep}(\mathbf{Z}) \otimes \mathrm{RMod}_B(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$$

extending the associated graded functor $\mathrm{gr} : \mathrm{Rep}(\mathbf{Z}) \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$ of Proposition 3.2.1; here the tensor product is formed in the ∞-category $\mathrm{Mod}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$ of locally graded presentable stable ∞-categories. This is equivalent to the data of a colimit-preserving monoidal functor

$$\mathrm{RMod}_B(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})) \to {}_{\mathbb{A}}\mathrm{BMod}_{\mathbb{A}}(\mathrm{Rep}(\mathbf{Z}))$$

(see Variant 3.3.8), which are classified by morphisms $B \to S[\beta]$ of $\mathbb{E}_2$-algebras in $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. $\quad\square$

5.5. **A Geometric Model for** Φ. Our goal in this section is to prove Theorem 5.1.14, which asserts that the diagram of $\mathbb{E}_2$-monoidal functors

$$
\begin{array}{ccc}
\mathbf{Z}^{\mathrm{ds}}_{\geq 0} & \xrightarrow{\quad\rho\quad} & \mathrm{Pic}(S) \\
\downarrow & & \downarrow \\
\mathbf{Z}^{\mathrm{ds}} \longrightarrow \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \,{}^{\Phi}\,[r] & & \mathrm{Sp}
\end{array}
$$

commutes up to (canonical) homotopy. First, we need to introduce a bit of terminology.

Notation 5.5.1. Let $\mathrm{Top}^{\mathrm{gr}}_*$ denote the full subcategory of the product $\prod_{n \in \mathbf{Z}} \mathrm{Top}_*$ spanned by those sequences of pointed spaces $\{X_n\}_{n \in \mathbf{Z}}$ such that $X_n \simeq *$ for $n \ll 0$. We will regard $\mathrm{Top}^{\mathrm{gr}}_*$ as a symmetric monoidal category with the tensor product given by

$$
\{X_i\}_{i \in \mathbf{Z}} \otimes \{Y_j\}_{j \in \mathbf{Z}} = \{ \bigvee_{i+j=n} X_i \wedge Y_j \}_{n \in \mathbf{Z}}.
$$

There is an evident symmetric monoidal functor

$$
\mathrm{gr} : \mathrm{Top}^{\mathrm{filt}} \to \mathrm{Top}^{\mathrm{gr}}_*
$$

which is given on objects by the formula

$$
\mathrm{gr}(\{X_n\}_{n \geq 0})_m = X_m / X_{m-1}.
$$

Remark 5.5.2. There is an evident lax symmetric monoidal functor

$$
\psi_{\mathrm{gr}} : \mathrm{N}(\mathrm{Top}^{\mathrm{gr}}_*)^{\mathrm{op}} \to \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}),
$$

given on objects by the formula

$$
\psi_{\mathrm{gr}}(\{X_n\}_{n \in \mathbf{Z}})_m = \Sigma^{\infty}(\mathrm{Sing}(X_m))^{\vee}.
$$

It is not hard to see that this functor fits into a commutative diagram of lax symmetric monoidal functors

$$
\begin{array}{ccc}
\mathrm{N}(\mathrm{Top}^{\mathrm{filt}})^{\mathrm{op}} & \xrightarrow{\psi} & \mathrm{Rep}(\mathbf{Z}) \\
\downarrow{\scriptstyle\mathrm{gr}} & & \downarrow{\scriptstyle\mathrm{gr}} \\
\mathrm{N}(\mathrm{Top}^{\mathrm{gr}}_*)^{\mathrm{op}} & \xrightarrow{\psi_{\mathrm{gr}}} & \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}),
\end{array}
$$

where ψ is as in Construction 5.4.6.

Let C be the $\mathcal{O}$-coalgebra object of $\mathrm{Top}^{\mathrm{filt}}$ described in Construction 5.2.11. We let C^+ denote the $\mathcal{O}[\mathbf{Z}_{\geq 0}]$-coalgebra in $\mathrm{Top}^{\mathrm{filt}}$ given by the formula

$$
C^+(D, m)_n = C(D)_{n-m} = \mathrm{Sym}^{n-m}(D^c).
$$

In other words, C^+ carries each color (D, m) to the filtered space

$$
\mathrm{Sym}^0(D^c) \hookrightarrow \mathrm{Sym}^1(D^c) \hookrightarrow \mathrm{Sym}^2(D^c) \hookrightarrow \cdots.
$$

where the filtration begins in degree $-m$. The associated graded coalgebra $\mathrm{gr}\, C^+$ is then given by the formula

$$
(\mathrm{gr}\, C^+(D, m))_n = \mathrm{Sym}^{n-m}(D)^c \in \mathrm{Top}_* .
$$

Let T be the $\mathcal{O}[\mathbf{Z}_{\geq 0}]$-coalgebra of Construction 5.1.12. Using the commutative diagram of Remark 5.5.2, we deduce that $(\Sigma^{\infty} T)^{\vee} \in \mathrm{Alg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\mathrm{Sp})$ is equivalent to the image of C^+ under the composition of lax symmetric monoidal functors

$$\mathrm{N}(\mathrm{Top}^{\mathrm{filt}})^{\mathrm{op}} \to (\mathcal{S}^{\mathrm{filt}})^{\mathrm{op}} \overset{\psi}{\to} \mathrm{Rep}(\mathbf{Z}) \overset{\mathrm{gr}}{\to} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \overset{e}{\to} \mathrm{Sp},$$

where e is given by evaluation at $0 \in \mathbf{Z}$. Combining this observation with Proposition 5.4.9, we obtain the following:

Lemma 5.5.3. *Let $B \in \mathrm{Alg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$ be the algebra given on objects by the formula*

$$B(D, n) = S[\beta](D) \otimes j(-n);$$

here we abuse notation by identifying $S[\beta]$ with the corresponding $\mathcal{O}$-algebra object of $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})$. Then the image of B under the lax symmetric monoidal evaluation functor

$$\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Sp}$$

$$X \mapsto X_0$$

corresponds, under the equivalence of Proposition 5.1.8, to the $\mathbb{E}_2$-monoidal functor $\mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{Sp}$ obtained by composing ρ with Spanier-Whitehead duality.

Notation 5.5.4. For any $\mathbb{E}_2$-algebra $R \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}))$, the functor

$$X \mapsto \underline{\mathrm{Map}}(X, R)$$

determines a lax $\mathbb{E}_2$-monoidal functor from $\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})^{\mathrm{op}}$ to Sp, which we will denote by θ_R. Note that θ_R is a covariant functor of R.

If B is as in Lemma 5.5.3, then we have

$$\begin{aligned}
B(D, n)_0 &\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(j(0), S[\beta](D) \otimes j(-n)) \\
&\simeq \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(j(n), S[\beta](D)) \\
&\simeq \theta_{S[\beta]}(j(n))(D).
\end{aligned}$$

Consequently, Lemma 5.5.3 can be reformulated as follows:

Lemma 5.5.5. *The composition of ρ with Spanier-Whitehead duality is homotopic (as a lax $\mathbb{E}_2$-monoidal functor) to the composition*

$$(\mathbf{Z}_{\geq 0}^{\mathrm{ds}})^{\mathrm{op}} \overset{j}{\to} \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \overset{\theta_{S[\beta]}}{\longrightarrow} \mathrm{Sp}.$$

Similarly, we can use Notation 5.5.4 to reformulate Proposition 3.5.11:

Lemma 5.5.6. *The composition*

$$\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})^{\mathrm{op}} \overset{\Phi}{\longrightarrow} \mathrm{Sp}^{\mathrm{op}} \overset{\vee}{\to} \mathrm{Sp}$$

is equivalent, as a lax $\mathbb{E}_2$-monoidal functor, to the map $\theta_{S[\beta^{\pm 1}]}$.

Proof of Theorem 5.1.14. Since the functors $\Phi \circ j$ and ρ both take values in finite spectra, it will suffice to show that they are equivalent (on $\mathbf{Z}_{\geq 0}^{\mathrm{ds}}$) after applying Spanier-Whitehead duality. By virtue of Lemmas 5.5.5 and 5.5.6, we are reduced to proving that the functors

$$n \mapsto \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(j(n), S[\beta]) \qquad n \mapsto \underline{\mathrm{Map}}_{\mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}})}(j(n), S[\beta^{\pm 1}])$$

are equivalent (as lax $\mathbb{E}_2$-monoidal functors from $(\mathbf{Z}_{\geq 0}^{\mathrm{ds}})^{\mathrm{op}}$ to Sp). The localization map $S[\beta] \to S[\beta^{\pm 1}]$ induces a lax $\mathbb{E}_2$-monoidal natural transformation from the left hand side to the right hand side. To show that it is an equivalence, it suffices to observe that the natural map

$$S[\beta]_n \to S[\beta^{\pm 1}]_n \simeq \varprojlim_m \Sigma^{2m} S[\beta]_{n+m}$$

is an equivalence for $n \geq 0$, which follows from the description of $S[\beta]$ given in Proposition 3.4.5. $\qquad\qquad\square$

6. A Model for Bott Periodicity

Let K denote the (periodic) complex K-theory spectrum. For every finite CW complex X, the abelian group $K^0(X)$ can be identified with the Grothendieck group of complex vector bundles on X. In particular, every complex vector bundle $\mathcal{E}$ on X determines an element $[\mathcal{E}] \in K^0(X)$. Let $\mathcal{O}(1)$ denote the (holomorphic) complex line bundle on $S^2 = \mathbf{CP}^1$. Then $[\mathcal{O}(1)] - 1$ can be regarded as an element of the group $\beta \in K_{\mathrm{red}}^0(S^2) \simeq \pi_2 K$. The celebrated *Bott periodicity theorem* asserts that multiplication by β induces an equivalence from the spectrum K to $\Omega^2 K$.

The zeroth space of K is given by $\Omega^\infty K = \mathbf{Z} \times \mathrm{BU}$. Bott periodicity then supplies a canonical homotopy equivalence of spaces

$$(\mathbf{Z} \times \mathrm{BU}) \to \Omega^2(\mathbf{Z} \times \mathrm{BU}).$$

In fact, this is a map of infinite loop spaces. In order to establish Theorem 1.1.1, we will need a concrete understanding of the Bott periodicity map as a map of 2-fold loop spaces. More precisely, let us regard the disjoint union $\sqcup_{n \geq 0} \mathrm{BU}(n)$ as an $\mathbb{E}_\infty$-space and let us identify $\mathbf{Z} \times \mathrm{BU}$ with its group completion. For every positive integer m, we can identify the m-fold bar construction

$$\mathrm{Bar}^{(m)}(\sqcup_{n \geq 0} \mathrm{BU}(n)) \simeq \mathrm{Bar}^{(m)}(\mathbf{Z} \times \mathrm{BU})$$

with the m-connective cover $\tau_{\geq m}(\Omega^{\infty - m} K)$. Using Bott periodicity, we obtain homotopy equivalences of pointed spaces

$$
\begin{aligned}
\mathrm{Bar}^{(2)}(\sqcup_{n \geq 0} \mathrm{BU}(n)) \quad &\simeq \quad \tau_{\geq 2}(\Omega^{\infty - 2} K) \\
&\overset{\beta}{\simeq} \quad \tau_{\geq 2}(\Omega^\infty K) \\
&\simeq \quad \tau_{\geq 2}(\mathbf{Z} \times \mathrm{BU}) \\
&\simeq \quad \mathrm{BU}.
\end{aligned}
$$

Let us denote the composite map by β, and refer to it as the *Bott map*.

Our goal in this section is to give an explicit geometric construction of β. The main obstacle is that the $\mathbb{E}_2$-structures on the spaces $\sqcup_{n \geq 0} \mathrm{BU}(n)$ and $\Omega^2(\mathbf{Z} \times \mathrm{BU})$ are of very different natures: in the first case, the $\mathbb{E}_2$-structure comes from the formation of direct sums of complex vector spaces, and in the second it comes the description of $\Omega^2(\mathbf{Z} \times \mathrm{BU})$ as a 2-fold loop space. Consequently, the concrete description of the map $\mathbf{Z} \times \mathrm{BU} \to \Omega^2(\mathbf{Z} \times \mathrm{BU})$ given above (obtained by tensor product with $\mathcal{O}(1)$) is not immediately useful for describing the delooping β. We will instead proceed indirectly in two steps:

(a) In §6.1 we show that the Bott map β is characterized (up to a scalar factor) by the fact that it "commutes" (up to homotopy) with the multiplicative action of $\mathrm{BU}(1)$ (Proposition 6.1.1).

(b) In §6.5, we give an *a priori* unrelated construction of a map

$$\mathfrak{B} : \mathrm{Bar}^{(2)}(\sqcup_{n \geq 0} \mathrm{BU}(n)) \to \mathbf{Z} \times \mathrm{BU},$$

which commutes with the action of BU(1). To make this construction explicit, we will need to choose particular models for the space $\mathbf{Z} \times \mathrm{BU}$ (which we describe in §6.2) and for the two-fold bar construction (which we describe in §6.4, using some notation which we establish in §6.3).

It will then follow from (a) that we have $\mathfrak{B} = k\beta$ for some integer k; the proof that $k = 1$ will be given in §7 (see Corollary 7.2.3).

6.1. **Abstract Characterization of the Bott Map.** The formation of tensor products of complex vector spaces determines an action of the space $\mathrm{BU}(1) \simeq \mathbf{CP}^{\infty}$ on the disjoint union $\amalg_{n \geq 0} \mathrm{BU}(n)$. Since tensor products distribute over direct sums, this action is given by a map of spaces

$$\mathrm{BU}(1) \to \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_{\infty}}}(\amalg_{n \geq 0} \mathrm{BU}(n), \amalg_{n \geq 0} \mathrm{BU}(n)).$$

It follows that BU(1) also acts on the group completion $\mathbf{Z} \times \mathrm{BU}$, its identity component BU, and the 2-fold bar construction $\mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n))$. Since the Bott map $\beta : K \to \Omega^2 K$ is a map of K-modules, it is equivariant with respect to the action of BU(1): in particular, it gives a homotopy commutative diagram

$$\begin{array}{ccc}
\mathrm{BU}(1) \times \mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n)) & \xrightarrow{\mathrm{id} \times \beta} & \mathrm{BU}(1) \times \mathrm{BU} \\
\downarrow{\scriptstyle a} & & \downarrow \\
\mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n)) & \xrightarrow{\beta} & \mathrm{BU}.
\end{array}$$

This is almost sufficient to characterize the Bott map, up to homotopy:

Proposition 6.1.1. *Suppose that* $\theta : \mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n)) \to \mathbf{Z} \times \mathrm{BU}$ *is a map of spaces for which the diagram*

$$\begin{array}{ccc}
\mathrm{BU}(1) \times \mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n)) & \xrightarrow{\mathrm{id} \times \theta} & \mathrm{BU}(1) \times (\mathbf{Z} \times \mathrm{BU}) \\
\downarrow{\scriptstyle a} & & \downarrow \\
\mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n)) & \xrightarrow{\theta} & \mathbf{Z} \times \mathrm{BU}.
\end{array}$$

commutes up to homotopy. Then γ *is homotopic to an integral multiple of the Bott map* β.

Proof. Since β is a homotopy equivalence, we can assume without loss of generality that θ factors as a composition

$$\mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n)) \xrightarrow{\beta} \mathrm{BU} \xrightarrow{\gamma} \mathbf{Z} \times \mathrm{BU}.$$

Let ι denote the canonical inclusion of BU into $\mathbf{Z} \times \mathrm{BU}$, and let us abuse notation by identifying γ and ι with the classes that they represent in the abelian group $K^0(\mathrm{BU})$. We wish to prove that γ is an integral multiple of ι.

Let $K_{\mathbf{Q}}$ denote the rationalization of complex K-theory. For each integer $n \geq 0$, let ι_n and γ_n denote the images of ι and γ in the group $K_{\mathbf{Q}}^0(\mathrm{BU}(n))$. We will prove the following:

$(*)$ For each $n > 0$, there exists a rational number q such that $\gamma_n = q\iota_n$ in $K_{\mathbf{Q}}^0(\mathrm{BU}(n))$.

Since ι_n is a nonvanishing element of the vector space $K_{\mathbf{Q}}^0(\mathrm{BU}(n))$ for $n > 0$, it follows that the rational number q appearing in $(*)$ is independent of n. Note that the image of ι under the composite map

$$K^0(\mathrm{BU}) \to K^0(\mathrm{BU}(1)) = K^0(\mathbf{CP}^{\infty}) \to K^0(\mathbf{CP}^1) \simeq \pi_0 K \oplus \pi_2 K$$

is equal to the Bott class $\beta \in \pi_2 K$. Since the map $K^0(\mathbf{CP}^1) \to K_{\mathbf{Q}}^0(\mathbf{CP}^1)$ is injective, it follows that the image of γ in $K^0(\mathbf{CP}^1)$ is given by the product $q\beta$, from which it follows that q is an integer. Since the map

$$K^0(\mathrm{BU}) \to \varprojlim K_{\mathbf{Q}}^0(\mathrm{BU}(n))$$

is injective, we conclude that $\gamma = q\iota$ as desired.

It remains to prove $(*)$. It follows from our hypothesis on θ that the diagram

$$
\begin{array}{ccc}
\mathrm{BU}(1) \times \mathrm{BU} & \xrightarrow{\mathrm{id} \times \gamma} & \mathrm{BU}(1) \times (\mathbf{Z} \times \mathrm{BU}) \\
\downarrow & & \downarrow \\
\mathrm{BU} & \xrightarrow{\gamma} & \mathbf{Z} \times \mathrm{BU}
\end{array}
$$

commutes up to homotopy, where the vertical maps are given by the multiplication on K-theory. For each complex vector bundle $\mathcal{E}$ on a space X, let $[\mathcal{E}]$ denote the associated element of $K^0(X)$. Let $\mathcal{E}_n$ denote the tautological vector bundle on $\mathrm{BU}(n)$ and let $\mathcal{L}$ denote the tautological line bundle on $\mathrm{BU}(1)$. Then $K_{\mathbf{Q}}^0(\mathrm{BU}(n))$ can be identified with the power series ring $\mathbf{Q}[[x_1, \ldots, x_n]]$, where each x_i is obtained by applying the Adams operation ψ^i to the class $[\mathcal{E}_n] - n$. Under this identification, γ_n corresponds to some power series $f(x_1, \ldots, x_n)$. Invoking the homotopy commutativity of the above diagram (after applying the canonical map $K^0(\mathrm{BU}(1) \times \mathrm{BU}) \to K_{\mathbf{Q}}^0(\mathrm{BU}(1) \times \mathrm{BU}(n)))$, we obtain the relation

$$f(x_1, \ldots, x_n)[\mathcal{L}] = f(x_1[\mathcal{L}], x_2[\mathcal{L}]^2, \ldots, x_n[\mathcal{L}]^n)$$

in the power series ring $\mathbf{Q}[[t, x_1, \ldots, x_n]] \simeq K_{\mathbf{Q}}^0(\mathrm{BU}(1) \times \mathrm{BU}(n))$, where $t = [\mathcal{L}] - 1$. For every sequence of integers $e_1, \ldots, e_n \geq 0$, let $c_{e_1, \ldots, e_n}$ denote the coefficient $x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$ in the power series $f(x_1, \ldots, x_n)$. Extracting coefficients, we obtain the equation

$$c_{e_1, \ldots, e_n}(1 + t) = c_{e_1, \ldots, e_n}(1 + t)^{e_1 + 2e_2 + \cdots + ne_n}$$

in the power series ring $\mathbf{Q}[[t]]$. In particular, the coefficient $c_{e_1, \ldots, e_n}$ must vanish unless $e_1 = 1$ and $e_i = 0$ for $i > 1$. It follows that we have $f(x_1, \ldots, x_n) = qx_1 = q\iota_n$ for some rational number $q \in \mathbf{Q}$, which completes the proof of $(*)$. $\qquad\square$

6.2. **The Homotopy Type $\mathbf{Z} \times \mathrm{BU}$.** Let $\mathrm{Vect}_{\mathbf{C}}$ denote the category whose objects are finite-dimensional complex vector spaces and whose morphisms are $\mathbf{C}$-linear maps. We let $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ denote the subcategory of $\mathrm{Vect}_{\mathbf{C}}$ whose morphisms are $\mathbf{C}$-linear isomorphisms. We will regard $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ as a topologically enriched category: for every pair of complex vector spaces V and W, we endow $\mathrm{Iso}_{\mathbf{C}}(V, W) = \mathrm{Hom}_{\mathrm{Vect}_{\mathbf{C}}^{\simeq}}(V, W)$ with the topology determined by its inclusion as a subspace of the vector space $\mathrm{Hom}_{\mathbf{C}}(V, W)$ of linear maps from V to W. Let $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ denote the homotopy coherent nerve of $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$; it is a Kan complex (since every morphism in $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ is invertible) whose geometric realization is homotopy equivalent to the disjoint union $\coprod_{n \geq 0} \mathrm{BU}(n)$. Our goal in this section is to give an analogous description of the space $\mathbf{Z} \times \mathrm{BU} \simeq \Omega^\infty K$ obtained from $\sqcup_{n \geq 0} \mathrm{BU}(n)$ by group completion. Roughly speaking, the idea is to allow pairs of vector spaces (V, W) which represent the "formal difference" $W - V$.

Construction 6.2.1. We define a category $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ as follows:

- The objects of $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ are pairs (V, W), where V and W are finite-dimensional complex vector spaces.
- A morphism from (V, W) to (V', W') in $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ is given by a triple (f, g, U) where $f : V \hookrightarrow V'$ and $g : W \hookrightarrow W'$ are injective $\mathbf{C}$-linear maps and $U \subseteq V' \oplus W'$ is a subspace

with the property that the maps

$$f \oplus \pi_1|_U : V \oplus U \to V'$$

$$g \oplus \pi_2|_U : W \oplus U \to W'$$

are isomorphisms; here $\pi_1 : V' \oplus W' \to V'$ and $\pi_2 : V' \oplus W' \to W'$ denote the projection maps onto the first and second factor, respectively.

- Given a triple of objects $(V, W), (V', W'), (V'', W'') \in \text{Vect}_{\mathbf{C}}^{\pm}$, a morphism (f, g, U) from (V, W) to (V', W'), and a morphism (f', g', U') from (V', W') to (V'', W''), the composition of (f', g', U') with (f, g, U) is defined to be the triple

$$(f' \circ f, g' \circ g, U' + \text{Im}(U)),$$

where $\text{Im}(U)$ denotes the image of U under the injective linear map

$$f' \oplus g' : V' \oplus W' \to V'' \oplus W''.$$

For every pair of objects $(V, W), (V', W') \in \text{Vect}_{\mathbf{C}}^{\pm}$, we regard the set of morphisms

$$\text{Hom}_{\text{Vect}_{\mathbf{C}}^{\pm}}((V, W), (V', W'))$$

as a subset of the product

$$\text{Hom}_{\mathbf{C}}(V, V') \times \text{Hom}_{\mathbf{C}}(W, W') \times \text{Gr}_d(V' \oplus W'),$$

where $\text{Gr}_d(V' \oplus W')$ denotes the Grassmannian parametrizing subspaces of $V' \oplus W'$ having dimension $d = \dim(V') - \dim(V)$. We will endow $\text{Hom}_{\text{Vect}_{\mathbf{C}}^{\pm}}((V, W), (V', W'))$ with the subspace topology. The composition law on $\text{Vect}_{\mathbf{C}}^{\pm}$ is continuous with respect to these topologies, so we can regard $\text{Vect}_{\mathbf{C}}^{\pm}$ as topologically enriched category. We will denote the homotopy coherent nerve of $\text{Vect}_{\mathbf{C}}^{\pm}$ by $\text{N}(\text{Vect}_{\mathbf{C}}^{\pm})$.

Remark 6.2.2. We will think of objects $(V, W) \in \text{Vect}_{\mathbf{C}}^{\pm}$ as "formal differences" $W - V$ of finite-dimensional vector spaces V and W. Note that any morphism $(V, W) \to (V', W')$ in $\text{Vect}_{\mathbf{C}}^{\pm}$ supplies isomorphisms

$$V' \simeq V \oplus U \qquad W' \simeq W \oplus U$$

for some finite-dimensional vector space U, so that the "formal differences" $W' - V'$ and $W - V$ should be identified.

Remark 6.2.3. The construction $W \mapsto (0, W)$ induces an equivalence from the topologically enriched category $\text{Vect}_{\mathbf{C}}^{\simeq}$ to the full subcategory of $\text{Vect}_{\mathbf{C}}^{\pm}$ spanned by those objects (V, W) where $V \simeq 0$. In particular, we obtain a natural map of simplicial sets

$$\text{N}(\text{Vect}_{\mathbf{C}}^{\simeq}) \to \text{N}(\text{Vect}_{\mathbf{C}}^{\pm}),$$

where the left hand side is homotopy equivalent to $\amalg_{n \geq 0} \text{BU}(n)$.

Warning 6.2.4. The simplicial set $\text{N}(\text{Vect}_{\mathbf{C}}^{\pm})$ is an ∞-category, but it is not a Kan complex (unlike the simplicial set $\text{N}(\text{Vect}_{\mathbf{C}}^{\simeq})$).

Remark 6.2.5. Let (V, W) and (V', W') be objects of $\text{Vect}_{\mathbf{C}}^{\pm}$. There are no morphisms from (V, W) to (V', W') in $\text{Vect}_{\mathbf{C}}^{\pm}$ unless $\dim(V') - \dim(V) = \dim(W') - \dim(W) \geq 0$. If this conditions is satisfied, then the mapping space $\text{Map}_{\text{Vect}_{\mathbf{C}}^{\pm}}((V, W), (V', W'))$ is acted on transitively by the product $\text{GL}(V') \times \text{GL}(W')$; moreover, the stabilizer of any morphism

$$(f, g, U) : (V, W) \to (V', W')$$

can be identified with $\mathrm{GL}(U)$ (embedded diagonally in the product $\mathrm{GL}(V') \times \mathrm{GL}(W')$). In particular, there is a homeomorphism

$$\mathrm{Map}_{\mathrm{Vect}_{\mathbf{C}}^{\pm}}((V,W),(V',W')) \simeq (\mathrm{GL}(V') \times \mathrm{GL}(W'))/\mathrm{GL}_n(\mathbf{C}).$$

where $n = \dim(V') - \dim(V) = \dim(W') - \dim(W)$.

Remark 6.2.6. The topologically enriched category $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ admits a symmetric monoidal structure, given on objects by the construction

$$(V,W),(V',W') \mapsto (V \oplus V', W \oplus W').$$

Consequently, we can view $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ as a symmetric monoidal ∞-category.

We can now state the main result of this section:

Proposition 6.2.7. *The identification* $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \simeq \amalg_{n \geq 0} \mathrm{BU}(n)$ *extends to a weak homotopy equivalence* $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm}) \simeq \mathbf{Z} \times \mathrm{BU}$.

Proof. For each integer d, let $\mathrm{Vect}_{\mathbf{C}}^{(d)}$ denote the full subcategory of $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ spanned by those pairs (V,W) where $\dim(W) - \dim(V) = d$. Then $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})$ is given by the disjoint union $\amalg_{d \in \mathbf{Z}} \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})$. We will show that each of the simplicial sets $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})$ is weakly homotopy equivalent to BU.

Let us now regard d as fixed. We define a category $\mathcal{C}$ as follows:

- The objects of $\mathcal{C}$ are finite-dimensional complex vector spaces having dimension $\geq d$.
- If V and W are finite-dimensional complex vector spaces, a morphism from V to W in $\mathcal{C}$ is an injective linear map $f: V \to W$ together with an ordered collection of elements $w_1, \ldots, w_n \in W$ whose images form a basis for the quotient $\mathrm{coker}(f) = W/f(V)$.
- Given morphisms $(f, v_1, \ldots, v_m): U \to V$ and $(g, w_1, \ldots, w_n): V \to W$ of $\mathcal{C}$, we define their composition to be the morphism

$$(g \circ f, g(v_1), \ldots, g(v_m), w_1, \ldots, w_n): U \to W.$$

We will regard $\mathcal{C}$ as a topologically enriched category and denote its homotopy coherent nerve by $\mathrm{N}(\mathcal{C})$. Note that the construction $V \mapsto \dim(V)$ determines a left fibration of simplicial sets $\psi: \mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathbf{Z}_{\geq d})$ whose fiber over an integer n can be identified with the classifying space $\mathrm{BGL}_n(\mathbf{C}) \simeq \mathrm{BU}(n)$, so that $\mathrm{N}(\mathcal{C})$ is weakly homotopy equivalent to BU.

The construction $V \mapsto (\mathbf{C}^{\dim(V)-d}, V)$ determines a functor $\mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{(d)}$. We claim that the induced map of ∞-categories $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})$ is left cofinal. To prove this, we must show that for each object $(V,W) \in \mathrm{Vect}_{\mathbf{C}}^{(d)}$, the fiber product

$$\mathcal{D} = \mathrm{N}(\mathcal{C}) \times_{\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})} \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})_{(V,W)/}$$

is weakly contractible. Note that ψ induces a left fibration $\psi': \mathcal{D} \to \mathbf{Z}_{\geq d}$, so that $\mathcal{D}$ is weakly homotopy equivalent to the homotopy colimit of the diagram $n \mapsto \psi'^{-1}\{n\}$. Unwinding the definitions, we see that $\psi'^{-1}\{n\}$ can be identified with the quotient of the mapping space $\mathrm{Map}_{\mathrm{Vect}_{\mathbf{C}}^{\pm}}((V,W),(\mathbf{C}^{n-d}, \mathbf{C}^n))$ by the (free) action of the general linear group $\mathrm{GL}_n(C)$. Using Remark 6.2.5, for $n \geq \dim(V)$ we obtain a homotopy equivalence

$$\psi'^{-1}\{n\} \simeq \mathrm{GL}_{n-d}(\mathbf{C})/\mathrm{GL}_{n-\dim(V)}(\mathbf{C}).$$

It follows that $\psi'^{-1}\{n\}$ is $2(n - \dim(V))$-connected, so that $\mathcal{D}^{\mathrm{gpd}} \simeq \varinjlim_n \psi'^{-1}\{n\}$ is contractible.

Since the map $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})$ is left cofinal, it induces a homotopy equivalence from $\mathrm{N}(\mathcal{C})^{\mathrm{gpd}} \simeq \mathrm{BU}$ to $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{(d)})^{\mathrm{gpd}}$; let us regard this homotopy equivalence as a map $\iota_d: \mathrm{BU} \to$

$N(\text{Vect}^{\pm}_{\mathbf{C}})^{\text{gpd}}$. If $d \geq 0$, then the functor $\mathcal{C} \to \text{Vect}^{(d)}_{\mathbf{C}}$ carries the full subcategory of $\mathcal{C}$ spanned by vector spaces of dimension d into the full subcategory $\text{Vect}^{\simeq}_{\mathbf{C}} \subseteq \text{Vect}^{\pm}_{\mathbf{C}}$, so that the diagram

$$
\begin{array}{ccc}
\text{BU}(d) & \longrightarrow & \text{BU} \\
\downarrow & & \downarrow{\scriptstyle \iota_d} \\
N(\text{Vect}^{\simeq}_{\mathbf{C}}) & \longrightarrow & N(\text{Vect}^{\pm}_{\mathbf{C}})^{\text{gpd}}
\end{array}
$$

commutes up to homotopy. Passing to the disjoint union as d varies, we obtain a homotopy commutative diagram

$$
\begin{array}{ccc}
\amalg_{d \geq 0} \text{BU}(d) & \longrightarrow & \text{BU} \times \mathbf{Z} \\
\downarrow & & \downarrow \\
N(\text{Vect}^{\simeq}_{\mathbf{C}}) & \longrightarrow & N(\text{Vect}^{\pm}_{\mathbf{C}})^{\text{gpd}}
\end{array}
$$

where the vertical maps are homotopy equivalences. $\qquad\square$

6.3. **Digression: Topological Categories.** In §6.2, we proved that the codomain $\mathbf{Z} \times \text{BU}$ of the Bott map β can be represented (up to weak homotopy equivalence) as the homotopy coherent nerve of a topologically enriched category $\text{Vect}^{\pm}_{\mathbf{C}}$ (Proposition 6.2.7). In §6.5, we will use this description to construct a map

$$
\mathfrak{B} : \text{Bar}^{(2)}(\amalg_{n \geq 0} \text{BU}(n)) \to \mathbf{Z} \times \text{BU}
$$

which we will later show to be homotopic to β. Unfortunately, it is not so easy to realize the domain of the map $\mathfrak{B}$ as the homotopy coherent nerve of a topologically enriched category. Instead, it will be convenient to consider the following more general notion:

Definition 6.3.1. A *topological category* is category $\mathcal{C}$ in which the collection of objects $\text{Ob}(\mathcal{C})$ and the collection of morphisms $\text{Mor}(\mathcal{C})$ have been equipped with topologies for which the source and target functions define continuous maps $\text{Mor}(\mathcal{C}) \to \text{Ob}(\mathcal{C})$, the formation of identity maps is given by a continuous function $\text{Ob}(\mathcal{C}) \to \text{Mor}(\mathcal{C})$, and the composition law $\text{Mor}(\mathcal{C}) \times_{\text{Ob}(\mathcal{C})} \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C})$ is continuous.

If $\mathcal{C}$ is a topological category, we let $N_\bullet(\mathcal{C})$ denote the simplicial set given by the nerve of the underlying discrete category: that is, the simplicial set whose n-simplices are given by *strictly commutative* diagrams

$$
C_0 \to C_1 \to \cdots \to C_n
$$

in the category $\mathcal{C}$. The topologies on $\text{Ob}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$ determine topologies on the sets $N_n(\mathcal{C})$, which allow us to regard $N_\bullet(\mathcal{C})$ as a simplicial topological space. We let $\text{Sing}_\bullet N_\bullet(\mathcal{C})$ denote the simplicial Kan complex obtained by levelwise application of the singular complex functor $\text{Sing}_\bullet : \text{Top} \to \mathcal{S}\text{et}_\Delta$. Let $|\text{Sing}_\bullet N_\bullet(\mathcal{C})|$ denote the geometric realization of this simplicial Kan complex: that is, the simplicial set whose n-simplices are given by $\text{Sing}_n N_n(\mathcal{C})$.

Warning 6.3.2. Let $\mathcal{C}$ be a category enriched in topological spaces. We can (and will) identify $\mathcal{C}$ with a topological category for which the space of objects $\text{Ob}(\mathcal{C})$ is equipped with the discrete topology. We will use the notation $N(\mathcal{C})$ to denote the homotopy coherent nerve of $\mathcal{C}$. This is a simplicial set (in fact, an ∞-category) rather than a simplicial space, and is different from the nerve $N_\bullet(\mathcal{C})$ of Definition 6.3.1. However, there is a close relationship between the two:

(a) By definition, an n-simplex of the simplicial set $N(\mathcal{C})$ consists of a sequence of objects $X_0, X_1, \ldots, X_n \in \mathcal{C}$ together with a collection of continuous maps $|N(P_{i,j})| \to \text{Map}_{\mathcal{C}}(X_i, X_j)$, where $P_{i,j}$ denotes the partially ordered set of subsets of $[n]$ having

least element i and greatest element j. These maps are required to be compatible in the sense that the diagrams

$$
\begin{array}{ccc}
|\mathrm{N}(P_{i,j})| \times |\mathrm{N}(P_{j,k})| & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X_i, X_j) \times \mathrm{Map}_{\mathcal{C}}(X_j, X_k) \\
\downarrow{\scriptstyle\cup} & & \downarrow \\
|\mathrm{N}(P_{i,k})| & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X_i, X_k)
\end{array}
$$

must commute (in the category of topological spaces).

(b) By definition, an n-simplex of the simplicial set $|\mathrm{Sing}_\bullet \mathrm{N}_\bullet(\mathcal{C})|$ consists of a sequence of objects $X_0, X_1, \ldots, X_n$ together with continuous maps

$$|\Delta^n| \to \mathrm{Map}_{\mathcal{C}}(X_{i-1}, X_i)$$

for $1 \le i \le n$.

There is a canonical map of simplicial sets $|\mathrm{Sing}_\bullet \mathrm{N}_\bullet \mathcal{C}| \to \mathrm{N}(\mathcal{C})$, which is given on n-simplies by carrying data of type (b) to data of type (a) using the maps of partially ordered sets

$$P_{i,j} \to \prod_{i < k \le j} [n]$$

$$S \mapsto \{\max S \cap [k]\}_{i < k \le j}.$$

This map is always a weak homotopy equivalence of simplicial sets (see [**?**]).

Notation 6.3.3. Let $\mathcal{C}$ be a topological category, and suppose we are given a functor of topological categories $F : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$. Then F induces a map

$$|\mathrm{Sing}_\bullet \mathrm{N}_\bullet \mathcal{C}| \to |\mathrm{Sing}_\bullet \mathrm{N}_\bullet \mathrm{Vect}_{\mathbf{C}}^{\pm}| \xrightarrow{\sim} \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm}) \simeq \mathbf{Z} \times \mathrm{BU},$$

which determines an element of the abelian group $K^0(|\mathrm{Sing}_\bullet \mathrm{N}_\bullet \mathcal{C}|)$. We will denote this element by $[F]$.

The following result will be needed in §7.3:

Proposition 6.3.4. *Let $\mathcal{C}$ be a topological category. Suppose that we are given topological functors $F', F, F'' : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$ which fit into an exact sequence*

$$0 \to F' \to F \to F'' \to 0$$

in the following sense:

(a) *For every object $C \in \mathcal{C}$, if we set*

$$F'(C) = (V_C', W_C') \qquad F(C) = (V_C, W_C) \qquad F''(C) = (V_C'', W_C''),$$

then we are given exact sequences of vector spaces

$$0 \to V_C' \to V_C \to V_C'' \to 0$$

$$0 \to W_C' \to W_C \to W_C'' \to 0$$

where the maps depend continuously on the object C (note that since the collection of objects of $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ is equipped with the discrete topology, the constructions

$$C \mapsto V_C', V_C, V_C'', W_C', W_C, W_C''$$

are locally constant functions on the space of objects of $\mathcal{C}$).

(b) *For every morphism $C \to D$ in $\mathcal{C}$, the associated maps*

$$(f', g', U') : (V'_C, W'_C) \to (V'_D, W'_D)$$

$$(f, g, U) : (V_C, W_C) \to (V_D, W_D)$$

$$(f'', g'', U'') : (V''_C, W''_C) \to (V''_D, W''_D)$$

in $\mathrm{Vect}^{\pm}_{\mathbf{C}}$ *have the property that the diagrams*

$$
\begin{array}{ccccc}
V'(C) & \longrightarrow & V(C) & \longrightarrow & V''(C) \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\
V'(D) & \longrightarrow & V(D) & \longrightarrow & V''(D)
\end{array}
$$

$$
\begin{array}{ccccc}
W'(C) & \longrightarrow & W(C) & \longrightarrow & W''(C) \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g''} \\
W'(D) & \longrightarrow & W(D) & \longrightarrow & W''(D)
\end{array}
$$

*commute, the maps $V'_D \oplus W'_D \to V_D \oplus W_D \to V''_D \oplus W''_D$ carry U' into U and U into U''
(it then follows that the the sequence $0 \to U' \to U \to U'' \to 0$ is also exact).*

Then $[F] = [F'] + [F'']$ *in the K-group $K^0(|\mathrm{Sing}_{\bullet} \mathrm{N}_{\bullet} \mathcal{C}|)$.*

Warning 6.3.5. In the situation described in Proposition 6.3.4, the maps $F' \to F$ and $F \to F''$ are usually *not* natural transformations between functors from $\mathcal{C}$ to $\mathrm{Vect}^{\pm}_{\mathbf{C}}$.

Proof of Proposition 6.3.4. The exact sequence of functors $0 \to F' \to F \to F'' \to 0$ determines a topological functor $\alpha : \mathcal{C} \to \mathcal{E}$, where $\mathcal{E}$ is the topological category category whose objects are pairs of exact sequences

$$(0 \to V' \to V \to V'' \to 0, 0 \to W' \to W \to W'' \to 0)$$

and whose morphisms are given by the type of data described in (b) above. Then the functors F', F, and F'' can be identified with $G' \circ \alpha$, $G \circ \alpha$, and $G'' \circ \alpha$, where

$$G', G, G'' : \mathcal{E} \to \mathrm{Vect}^{\pm}_{\mathbf{C}}$$

are the topological functors given on objects by

$$G'(0 \to V' \to V \to V'' \to 0, 0 \to W' \to W \to W'' \to 0) = (V', W')$$

$$G(0 \to V' \to V \to V'' \to 0, 0 \to W' \to W \to W'' \to 0) = (V, W)$$

$$G''(0 \to V' \to V \to V'' \to 0, 0 \to W' \to W \to W'' \to 0) = (V'', W'').$$

It will therefore suffice to establish the identity $[G] = [G'] + [G'']$ in $K^0(|\mathrm{N}_{\bullet} \mathrm{Sing}_{\bullet} \mathcal{E}|)$.

There is an evident topological functor $\gamma : \mathrm{Vect}^{\pm}_{\mathbf{C}} \times \mathrm{Vect}^{\pm}_{\mathbf{C}} \to \mathcal{E}$, given on objects by

$$\gamma((V', W'), (V'', W'')) = (0 \to V' \to V' \oplus V'' \to V'' \to 0, 0 \to W' \to W' \oplus W'' \to W'' \to 0).$$

We have equaltiies of topological functors

$$G' \circ \gamma = \pi_1 \qquad G \circ \gamma = \oplus \qquad G'' \circ \gamma = \pi_2,$$

where $\pi_1, \pi_2 : \mathrm{Vect}^{\pm}_{\mathbf{C}} \times \mathrm{Vect}^{\pm}_{\mathbf{C}} \to \mathrm{Vect}^{\pm}_{\mathbf{C}}$ are projection onto the first and second factor respectively, and $\oplus$ denotes the functor from $\mathrm{Vect}^{\pm}_{\mathbf{C}} \times \mathrm{Vect}^{\pm}_{\mathbf{C}}$ to $\mathrm{Vect}^{\pm}_{\mathbf{C}}$ given by the direct sum. We therefore tautologically have the identity

$$[G \circ \gamma] = [G' \circ \gamma] + [G'' \circ \gamma]$$

in the K-group $K^0(|\operatorname{Sing}_\bullet \operatorname{N}_\bullet(\operatorname{Vect}_\mathbf{C}^\pm \times \operatorname{Vect}_\mathbf{C}^\pm)|)$. Consequently, to complete the proof, it will suffice to show that the functor γ induces a weak homotopy equivalence of simplicial sets

$$|\operatorname{Sing}_\bullet \operatorname{N}_\bullet(\operatorname{Vect}_\mathbf{C}^\pm \times \operatorname{Vect}_\mathbf{C}^\pm)| \to |\operatorname{Sing}_\bullet \operatorname{N}_\bullet \mathcal{E}|.$$

To prove this, it is convenient to factor γ as a composition

$$\operatorname{Vect}_\mathbf{C}^\pm \times \operatorname{Vect}_\mathbf{C}^\pm \xrightarrow{\gamma'} \mathcal{D} \xrightarrow{\gamma''} \mathcal{E},$$

where $\mathcal{D}$ is a topological category whose objects are pairs of exact sequences

$$0 \to V' \to V \to V'' \to 0$$
$$0 \to W' \to W \to W'' \to 0$$

together with chosen sections of the projection maps $V \to V''$ and $W \to W''$. We claim that the maps γ' and γ'' both induce weak homotopy equivalences at the level of classifying spaces. We conclude by applying following pair of observations:

- For each integer $m \geq 0$, the functor γ' induces a weak homotopy equivalence of simplicial sets

  $$\operatorname{Sing}_m \operatorname{N}_\bullet(\operatorname{Vect}_\mathbf{C}^\pm \times \operatorname{Vect}_\mathbf{C}^\pm) \to \operatorname{Sing}_m \operatorname{N}_\bullet(\mathcal{D}).$$

  This follows from the fact that the associated map

  $$\operatorname{Sing}_m(\operatorname{Vect}_\mathbf{C}^\pm \times \operatorname{Vect}_\mathbf{C}^\pm) \to \operatorname{Sing}_m(\mathcal{D})$$

  is an equivalence of ordinary categories.
- For each integer $n \geq 0$, the functor γ'' induces a homotopy equivalence of Kan complexes

  $$\operatorname{Sing}_\bullet \operatorname{N}_n(\mathcal{D}) \to \operatorname{Sing}_\bullet \operatorname{N}_n(\mathcal{E}).$$

  This follows from the fact that the map of topological spaces $\operatorname{N}_n(\mathcal{D}) \to \operatorname{N}_n(\mathcal{E})$ is a fiber bundle whose fibers are affine spaces.

$\square$

6.4. **The Double Bar Construction.** For every pointed space X, we can regard the 2-fold loop space $\Omega^2 X$ as an $\mathbb{E}_2$-space. The construction $X \mapsto \Omega^2 X$ determines a functor

$$\Omega^2 : \mathcal{S}_* \to \operatorname{Alg}_{\mathbb{E}_2}(\mathcal{S}).$$

This functor admits a left adjoint, given by the two-fold bar construction $\operatorname{Bar}^{(2)} : \operatorname{Alg}_{\mathbb{E}_2}(\mathcal{S}) \to \mathcal{S}_*$. Our goal in this section is to give an explicit description of this 2-fold bar construction which will be convenient for our applications in §6.5 and §7. Let us begin by introducing a bit of terminology.

Definition 6.4.1. Let $\mathcal{O}$ denote the colored operad introduced in Definition 5.1.1. We let $\mathcal{O}^\otimes$ denote the category obtained by applying Construction H.2.1.1.7 to $\mathcal{O}$. In other words, we let $\mathcal{O}^\otimes$ denote the nerve of the category whose objects are finite sequences $(D_1, \ldots, D_m)$, where each D_i is an open disk in $\mathbf{C}$, where a morphism from $(D_1, D_2, \ldots, D_m)$ to $(D_1', \ldots, D_n')$ is given by a map of pointed finite sets $\alpha : \{1, \ldots, m, *\} \to \{1, \ldots, n, *\}$ with the property that for $1 \leq j \leq n$, the disks $\{D_i\}_{\alpha(i)=j}$ are disjoint and contained in D_j'.

A *topological $\mathcal{O}$-operad* is a topologically enriched category $\mathcal{C}^\otimes$ equipped with a functor $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ for which the induced map of homotopy coherent nerves $q : \operatorname{N}(\mathcal{C}^\otimes) \to \operatorname{N}(\mathcal{O}^\otimes)$ is a coCartesian fibration which exhibits $\operatorname{N}(\mathcal{C}^\otimes)$ as an ∞-operad. In this case, the map q is classified by an algebra $A \in \operatorname{Alg}_{\mathcal{O}}(\mathcal{C}\mathrm{at}_\infty)$. We will refer to A as the *classifying $\mathcal{O}$-algebra* of $\mathcal{C}^\otimes$. We will say that a topological $\mathcal{O}$-operad is *special* if its classifying $\mathcal{O}$-algebra satisfies condition $(*)$ of Proposition 5.1.5 (so that we can identify A with an $\mathbb{E}_2$-algebra object of $\mathcal{C}\mathrm{at}_\infty$).

Example 6.4.2. Let $\mathcal{C}$ be a topologically enriched category. Suppose that $\mathcal{C}$ is equipped with a symmetric monoidal structure which is compatible with its topological enrichment (that is, for which the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is continuous with respect to the topologies on the morphism spaces in $\mathcal{C}$). We can then define a topological $\mathcal{O}$-monoidal category $\mathcal{O}[\mathcal{C}]^{\otimes}$ as follows:

- The objects of $\mathcal{O}[\mathcal{C}]^{\otimes}$ are finite sequences $((C_1, D_1), \ldots, (C_m, D_m))$, where each C_i is an object of $\mathcal{C}$ and each $D_i \subseteq \mathbf{C}$ is an open disk.
- The space of morphisms from an object

$$((C_1, D_1), \ldots, (C_m, D_m))$$

to another object

$$((C_1', D_1'), \ldots, (C_n', D_n'))$$

in $\mathcal{O}[\mathcal{C}]^{\otimes}$ is given by the disjoint union

$$\coprod_{\alpha} \prod_{1 \leq j \leq n} \mathrm{Map}_{\mathcal{C}}(\otimes_{\alpha(i)=j} C_i, C_j')$$

where the coproduct is taken over all morphisms α from $(D_1, \ldots, D_m)$ to $(D_1', \ldots, D_n')$ in the category $\mathcal{O}^{\otimes}$.

A topological $\mathcal{O}$-monoidal category of this form is automatically special.

Remark 6.4.3. The inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_{\infty}$ admits a left adjoint, which assigns to every ∞-category $\mathcal{C}$ its *groupoid completion* $\mathcal{C}^{\mathrm{gpd}}$. The groupoid completion $\mathcal{C}^{\mathrm{gpd}}$ can be characterized (up to homotopy equivalence) by the fact that it is a Kan complex equipped with a weak homotopy equivalence $\mathcal{C} \to \mathcal{C}^{\mathrm{gpd}}$. Note that the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{gpd}}$ commutes with products; consequently, if $\mathcal{C}$ is an $\mathbb{E}_n$-monoidal ∞-category for some $0 \leq n \leq \infty$, then $\mathcal{C}^{\mathrm{gpd}}$ can be regarded as an $\mathbb{E}_n$-space.

Remark 6.4.4. Since $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})$ is a symmetric monoidal ∞-category, the groupoid completion $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})^{\mathrm{gpd}}$ admits the structure of an $\mathbb{E}_{\infty}$-space. The calculation $\pi_0 \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})^{\mathrm{gpd}} \simeq \mathbf{Z}$ shows that the $\mathbb{E}_{\infty}$-structure on $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})^{\mathrm{gpd}}$ is grouplike. It follows that the canonical map $\sqcup_{n \geq 0} \mathrm{BU}(n) \simeq \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \to \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})^{\mathrm{gpd}}$ factors through the group completion of $\sqcup_{n \geq 0} \mathrm{BU}(n)$, which is the product $\mathbf{Z} \times \mathrm{BU}$. It is not difficult to see that the underlying map $\mathbf{Z} \times \mathrm{BU} \to \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})^{\mathrm{gpd}}$ is the homotopy equivalence constructed in Proposition 6.2.7. We may therefore summarize Proposition 6.2.7 as follows: the inclusion of symmetric monoidal ∞-categories $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \hookrightarrow \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})$ exhibits the groupoid completion of $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\pm})$ as the group completion of $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$.

Construction 6.4.5 (Explicit 2-Fold Bar Construction)**.** We let $\mathcal{O}^{\circ}$ denote the subcategory of $\mathcal{O}^{\otimes}$ defined as follows:

- An object $(D_1, \ldots, D_m) \in \mathcal{O}^{\otimes}$ belongs to $\mathcal{O}^{\circ}$ if and only if the disks $(D_1, \ldots, D_m)$ are disjoint.
- A morphism $(D_1, \ldots, D_m) \to (D_1', \ldots, D_n')$ in $\mathcal{O}^{\otimes}$ between objects of $\mathcal{O}^{\circ}$ is a morphism in $\mathcal{O}^{\circ}$ if and only if the map of pointed finite sets $\alpha : \{1, \ldots, m, *\} \to \{1, \ldots, n, *\}$ has the property that if $0 \in D_i$, then $\alpha(i) \neq *$.

Suppose that we are given a functor of topological categories $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ (here we need not assume that the topology on the space of objects of $\mathcal{C}^{\otimes}$ is discrete). We let $\Theta(\mathcal{C}^{\otimes})$ denote the simplicial set

$$|\mathrm{Sing}_{\bullet} \mathrm{N}_{\bullet}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes})|.$$

The significance of Construction 6.4.5 is contained in the following result:

Proposition 6.4.6. *Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a special topological $\mathcal{O}$-monoidal category and let $A \in$ $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}\mathrm{at}_{\infty})$ be its classifying algebra. Then the simplicial set $\Theta(\mathcal{C}^{\otimes})$ is naturally weakly homotopy equivalent to the 2-fold bar construction $\mathrm{Bar}^{(2)}(A^{\mathrm{gpd}})$ on the groupoid completion of A.*

To prove Proposition 6.4.6, we note that Warning 6.3.2 supplies a canonical weak homotopy equivalence
$$\Theta(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}).$$
The simplicial set $\mathrm{N}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes})$ is an ∞-category equipped with a coCartesian fibration $q :$ $\mathrm{N}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}) \to \mathcal{O}^{\circ}$ which is classified by a map $A' : \mathcal{O}^{\circ} \to \mathcal{C}\mathrm{at}_{\infty}$ given by the restriction of the $\mathcal{O}$-algebra A. Using Proposition T.3.3.4.2, we see that the weak homotopy type of $\mathrm{N}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes})$ can be identified with the groupoid completion $(\varinjlim A')^{\mathrm{gpd}} \simeq \varinjlim A^{\mathrm{gpd}}|_{\mathrm{N}(\mathcal{O}^{\circ})}$. Consequently, Proposition 6.4.6 is an immediate consequence of the following more general statement:

Proposition 6.4.7. *Let A be an $\mathbb{E}_2$-space, which we will identify with a functor from $\mathrm{N}(\mathcal{O}^{\otimes})$ to the ∞-category $\mathcal{S}$ which satisfies conditions (a) and (b) of Example 5.1.6. Then the two-fold bar construction $\mathrm{Bar}^{(2)}(A)$ is canonically equivalent to the colimit of the functor $A|_{\mathcal{O}^{\circ}}$.*

Proof. For each object $A \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}) \subseteq \mathrm{Fun}(\mathrm{N}(\mathcal{O}^{\otimes}), \mathcal{S})$, we let $F(A)$ denote the colimit of $A|_{\mathcal{O}^{\circ}}$. We regard $F(A)$ as a pointed space via the canonical map $A(\varnothing) \to F(A)$. The construction $A \mapsto F(A)$ determines a functor $F : \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}) \to \mathcal{S}_*$; we wish to show that this functor coincides with the 2-fold bar construction.

We first construct a natural transformation of functors $F \to \mathrm{Bar}^{(2)}$. Recall that the 2-fold bar construction $\mathrm{Bar}^{(2)} : \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}) \to \mathcal{S}_*$ is left adjoint to the construction $\Omega^2 : \mathcal{S}_* \to \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$. If (X, x) is a pointed Kan complex, we can identify $\Omega^2 X$ with the functor $\mathrm{N}(\mathcal{O}^{\otimes}) \to \mathcal{S}$ which assigns to each n-tuple of disks $(D_1, \ldots, D_n)$ the product $\prod_{1 \le i \le n} \mathrm{Map}_*(\mathrm{Sing}(D_i^c), X))$, where $\mathrm{Map}_*(\mathrm{Sing}(D_i^c), X)$ denotes the simplicial set of pointed maps from the singular simplicial set of D_i^c into X (here $D_i^c \simeq S^2$ denotes the 1-point compactification of D_i). For each tuple $(D_1, \ldots, D_n) \in \mathcal{O}^{\otimes}$, the construction

$$(f_1, \ldots, f_n) \mapsto \begin{cases} f_i(0) & \text{if } 0 \in D_i \\ x & \text{otherwise} \end{cases}$$

determines a map $\Omega^2(X)(D_1, \ldots, D_n) \to X$, which is functorial with respect to morphisms in $\mathcal{O}^{\circ}$. We therefore obtain a map of pointed spaces $F(\Omega^2 X) \to X$. Specializing to the case $X = \mathrm{Bar}^{(2)}(A)$ and composing with the unit map $A \to \Omega^2 \mathrm{Bar}^{(2)}(A)$, we obtain a map $\lambda_A :$ $F(A) \to \mathrm{Bar}^{(2)}(A)$ which depends functorially on $A \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$.

We wish to prove that the map λ_A is a homotopy equivalence for every $\mathbb{E}_2$-space A. Note that the fully faithful embedding $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathrm{N}(\mathcal{O}^{\otimes}), \mathcal{S})$ preserves sifted colimits, from which we immediately deduce that the functor F commutes with sifted colimits. The 2-fold bar construction $\mathrm{Bar}^{(2)}$ commutes with all colimits (since it is a left adjoint). It follows that the collection of those objects $A \in \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$ for which λ_A is an equivalence is closed under sifted colimits. Let $\mathrm{Free} : \mathcal{S} \to \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$ be a left adjoint to the forgetful functor (which assigns to every space S the free $\mathbb{E}_2$-space $\mathrm{Free}(S)$ generated by S). The ∞-category $\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{S})$ is generated under sifted colimits by objects of the form $\mathrm{Free}(S)$, where S is a finite set. It will therefore suffice to show that λ_A is an equivalence in the special case where $A = \mathrm{Free}(S)$ for some finite set S, which we will henceforth assume.

For every Hausdorff topological space X, let $X^{(n)}$ denote the open subset of X^n consisting of n-tuples $(x_1, \ldots, x_n)$ of *distinct* points of X. Let $\mathrm{Conf}_S^n(X)$ denote the quotient of $S^n \times X^{(n)}$ by the (free) action of the symmetric group Σ_n. More informally, $\mathrm{Conf}_S^n(X)$ is the configuration

space which parametrizes unordered n-tuples of points of X which are labelled by the elements of S, and we let $\mathrm{Conf}_S(X)$ denote the disjoint union $\amalg_{n \geq 0} \mathrm{Conf}_S^n(X)$. Note that the free $\mathbb{E}_2$-algebra A generated by S can be identified with the functor $\mathrm{N}(\mathcal{O}^\otimes) \to \mathcal{S}$ given by

$$A(D_1, \ldots, D_m) = \prod_{1 \leq i \leq m} \mathrm{Sing}\,\mathrm{Conf}_S(D_i) \simeq \mathrm{Sing}\,\mathrm{Conf}_S(\amalg_{1 \leq i \leq m} D_i).$$

Let $\mathcal{J}$ be the subcategory of $\mathcal{O}^\circ$ containing all objects and those morphisms $(D_1, \ldots, D_m) \to (D_1', \ldots, D_n')$ for which the underlying map of pointed finite sets $\alpha : \{1, \ldots, m, *\} \to \{1, \ldots, n, *\}$ has the property that $\alpha^{-1}\{j\}$ has exactly one element for $1 \leq j \leq n$. Note that the construction $(D_1, \ldots, D_m) \mapsto \prod_{1 \leq i \leq m} \mathrm{Sing}\,\mathrm{Conf}_S^1(D_i)$ determines a functor $A_0 : \mathcal{J} \to \mathcal{S}$. We first claim:

($*$) The functor $A|_{\mathcal{O}^\circ}$ is a left Kan extension of A_0 (along the inclusion $\mathcal{J} \hookrightarrow \mathcal{O}^\circ$).

To prove ($*$), we must show that for every object $(D_1, \ldots, D_m)$ in $\mathcal{O}^\circ$, we can identify $\mathrm{Conf}(D_1 \cup \ldots \cup D_m)$ with the homotopy colimit of the diagram of spaces

$$\varinjlim_{D_1', \ldots, D_n'} \prod_{1 \leq j \leq n} \mathrm{Conf}_S^1(D_j'),$$

where the colimit is taken over the partially ordered set P of all collections of disjoint disks $D_1', \ldots, D_n'$ which are contained in the union $D_1 \cup \ldots \cup D_m$. Note that each of the products $\prod_{1 \leq j \leq n} \mathrm{Conf}_S^1(D_j')$ can be identified with the open subset of $\mathrm{Conf}(D_1 \cup \ldots \cup D_m)$ consisting of those S-labelled configurations which contain exactly one point belonging to each D_j'. Using Theorem H.A.3.1, we are reduced to showing that for every point $\gamma \in \mathrm{Conf}_S(D_1 \cup \ldots \cup D_m)$, the partially ordered set $P_\gamma = \{(D_1', \ldots, D_n') \in P : \gamma \in \prod_{1 \leq j \leq n} \mathrm{Conf}_S^1(D_j')\}$ is weakly contractible. This follows from the observation that the opposite partially ordered set P_γ^{op} is filtered.

It follows from ($*$) that we can identify the space $F(A)$ with the colimit

$$\varinjlim_{(D_1, \ldots, D_m) \in \mathcal{J}} A_0(D_1, \ldots, D_m).$$

For any Hausdorff topological space X, we can identify $\mathrm{Conf}_S^1(X)$ with the product $S \times X$. Using Theorem H.A.3.1 again, we see that for every open set $U \subseteq \mathbf{C}$ the colimit $\varinjlim_{D \subseteq U} A_0(D)$ (taken over the partially ordered set of all disks D contained in U) can be identified with $S \times \mathrm{Sing}(U) \in \mathcal{S}$. In particular, we can identify the 2-fold suspension $\Sigma^2(S_+)$ with the pushout of the diagram

$$A_0(\varnothing) \leftarrow \varinjlim_{D \subseteq \mathbf{C} - \{0\}} A_0(D) \to \varinjlim_{D \subseteq \mathbf{C}} A_0(D),$$

which is the colimit of the restriction $A_0|_{\mathcal{J}_0}$ where $\mathcal{J}_0 \subseteq \mathcal{J}$ is the full subcategory spanned by those tuples $(D_1, \ldots, D_m)$ where $m \leq 1$. It follows from Proposition H.5.2.3.15 that the 2-fold bar construction $\mathrm{Bar}^{(2)}(A)$ can also be identified with $\Sigma^2(S_+)$. Moreover, an analysis of the proof of Proposition H.5.2.3.15 shows that this identification is given by the composite map

$$\varinjlim_{(D_1, \ldots, D_m) \in \mathcal{J}_0} A_0(D_1, \ldots, D_m) \xrightarrow{\alpha} \varinjlim_{(D_1, \ldots, D_m) \in \mathcal{J}_0} A_0(D_1, \ldots, D_m) \xrightarrow{\lambda_A} \mathrm{Bar}^{(2)}(A).$$

To complete the proof, it will suffice to show that the map α is a homotopy equivalence. In fact, we claim that the inclusion $\mathcal{J}_0 \subseteq \mathcal{J}$ is left cofinal. To prove this, consider an arbitrary tuple $(D_1, \ldots, D_m) \in \mathcal{J}$; we wish to prove that the ∞-category $\mathcal{I} = \mathcal{J}_0 \times_{\mathcal{J}} \mathcal{J}_{(D_1, \ldots, D_m)/}$ is weakly contractible. We consider two cases:

- If one of the disks D_i contains $0 \in \mathbf{C}$, then $\mathcal{I}$ can be identified with the nerve of the partially ordered set of all disks $D \subseteq \mathbf{C}$ which contain D_i; this partially ordered set has a smallest element (given by the disk D_i itself) and is therefore weakly contractible.

- Suppose none of the disks D_i contains $0 \in \mathbf{C}$. For $1 \le i \le m$, let Q_i denote the partially ordered set of disks $D \subseteq \mathbf{C}$ which contain D_i, and let Q_i' be the subset of Q consisting of those disks D which do not contain 0. Unwinding the definitions, we see that $\mathcal{I}$ is given by the join

$$\bigvee_{1 \le i \le n} \mathrm{N}(Q_i) \sqcup_{\mathrm{N}(Q_i')} \mathrm{N}(Q_i')^{\triangleright}.$$

It will therefore suffice to show that each $\mathrm{N}(Q_i) \sqcup_{\mathrm{N}(Q_i')} \mathrm{N}(Q_i')^{\triangleleft}$ is weakly contractible. This follows from the fact that $\mathrm{N}(Q_i)$ and $\mathrm{N}(Q_i')$ are individually weakly contractible (since Q_i and Q_i' both have a smallest element given by the disk D_i).

$\square$

6.5. **A Model for the Bott Map.** Our goal in this section is to construct a map

$$\mathfrak{B} : \mathrm{Bar}^{(2)}(\sqcup_{n \ge 0} \mathrm{BU}(n)) \to \mathbf{Z} \times \mathrm{BU},$$

and to prove that it is homotopic to a constant multiple $k\,\beta$ of the Bott map β (Remark 6.5.6). Our first goal is to describe the domain of $\mathfrak{B}$ in a convenient way. We have already noted that the space $\sqcup_{n \ge 0} \mathrm{BU}(n)$ can be modeled explicitly by the homotopy coherent nerve $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$. However, it will be convenient to use a slightly different (but weakly equivalent) simplicial set.

Notation 6.5.1. Let V be a finite-dimensional vector space over $\mathbf{C}$ and let $\mathrm{End}(V)$ denote the vector space of endomorphisms of V. For every subset $D \subseteq \mathbf{C}$, we let $\mathrm{End}_D(V)$ denote the open subset of $\mathrm{End}(V)$ consisting of those endomorphisms $\phi : V \to V$ such that every eigenvector of ϕ lies in D.

Proposition 6.5.2. *Let $D \subseteq \mathbf{C}$ be an open disk and let V be a finite-dimensional vector space over $\mathbf{C}$. Then the subset $\mathrm{End}_D(V) \subseteq \mathrm{End}(V)$ is contractible.*

Proof. Without loss of generality, we may assume that $0 \in D$. Since D is contractible, we can choose a homotopy $h : D \times [0,1] \to D$ such that $h|_{D \times \{0\}} = \mathrm{id}_D$ and $h|_{D \times \{1\}} = 0$. Let $\phi \in \mathrm{End}_D(V)$. Then V decomposes as a direct sum of generalized eigenspaces $V_1 \oplus \cdots \oplus V_m$ where each V_i is the generalized eigenspace for some eigenvalue $\lambda_i \in V$. For $t \in [0,1]$, let ϕ_t be the endomorphism of V which is given on V_i by $\phi|_{V_i} + (h(\lambda_i, t) - \lambda_i)\,\mathrm{id}_{V_i}$. It is not difficult to see that ϕ_t depends continuously on t and f, so that the construction $(t, \phi) \mapsto \phi_t$ determines a homotopy from the identity map on $\mathrm{End}_D(V)$ to a map $\mathrm{End}_D(V) \to \mathrm{End}_{\{0\}}(V)$. It will therefore suffice to show that the space $\mathrm{End}_{\{0\}}(V)$ is contractible. This is clear: for each $\phi \in \mathrm{End}_{\{z\}}(V)$, the construction $t \mapsto t\phi$ determines a continuous path from 0 to ϕ in $\mathrm{End}_{\{0\}}(V)$. $\square$

Construction 6.5.3. We define a category $(\overline{\mathrm{Vect}}_{\mathbf{C}}^{\simeq})^{\otimes}$ as follows:

- The objects of $(\overline{\mathrm{Vect}}_{\mathbf{C}}^{\simeq})^{\otimes}$ are finite sequences

$$((V_1, D_1, \phi_1), \ldots, (V_m, D_m, \phi_m))$$

where each V_i is a finite-dimensional vector space over $\mathbf{C}$, each D_i is an open disk in $\mathbf{C}$, and each ϕ_i is an endomorphism V_i whose eigenvalues belong to D_i.
- A morphism from $((V_1, D_1, \phi_1), \ldots, (V_m, D_m, \phi_m))$ to $((V_1', D_1', \phi_1'), \ldots, (V_n', D_n', \phi_n'))$ consists of a map of finite pointed sets $\alpha : \{1, \ldots, m, *\} \to \{1, \ldots, n, *\}$ together with a collection of vector space isomorphisms

$$f_j : \bigoplus_{\alpha(i) = j} V_i \to V_j'$$

for which the diagrams

$$
\begin{CD}
\oplus_{\alpha(i)=j} V_i @>f_j>> V_j' \\
@V{\oplus \phi_i}VV @VV{\phi_j'}V \\
\oplus_{\alpha(i)=j} V_i @>f_j>> V_j'
\end{CD}
$$

commute.

We regard the collection of objects of $(\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes}$ as equipped with the topology given by its description as the disjoint union of the spaces

$$
\prod_{1 \leq i \leq m} \mathrm{End}_{D_i}(V_i)
$$

where $((D_1, V_1), \ldots, (D_m, V_m))$ ranges over all objects of $\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes}$. Similarly, we regard the collection of morphisms in $(\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes}$ as endowed with the topology given by its description as a disjoint union of spaces of the form

$$
(\prod_{1 \leq j \leq n} \mathrm{Iso}_{\mathbf{C}}(\bigoplus_{\alpha(i)=j} V_i, V_j')) \times (\prod_{1 \leq i \leq m} \mathrm{End}_{D_i}(V_i)).
$$

These topologies allow us to view $(\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes}$ as a topological category equipped with a forgetful functor $(\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes} \to \mathcal{O}^{\otimes}$.

Remark 6.5.4. There is an evident forgetful functor

$$
(\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes} \to \mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes},
$$

which induces a map of simplicial sets

$$
\Theta(\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes} \to \Theta(\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes}).
$$

It follows immediately from Proposition 6.5.2 that this map is a weak homotopy equivalence. Using Proposition 6.4.6, we see that $\Theta(\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes})$ is canonically weakly homotopy equivalent to the 2-fold bar construction $\mathrm{Bar}^{(2)}(\amalg_{n \geq 0} \mathrm{BU}(n))$.

Construction 6.5.5 (The Bott Map). We define a functor of topological categories

$$
\mathfrak{B} : \mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} (\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes} \to (\mathrm{Vect}_{\mathbf{C}}^{\pm})^{\mathrm{op}}
$$

as follows:

- On objects, the functor $\mathfrak{B}$ is given by the formula

$$
\mathfrak{B}((V_1, D_1, \phi_1), \ldots, (V_m, D_m, \phi_m)) = (\bigoplus_{1 \leq i \leq m} V_i, \bigoplus_{1 \leq i \leq m} V_i).
$$

- Suppose we are given a morphism

$$
((V_1, D_1, \phi_1), \ldots, (V_m, D_m, \phi_m)) \to ((V_1', D_1', \phi_1'), \ldots, (V_n', D_n', \phi_n'))
$$

in the category $\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} (\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^{\otimes}$, given by a map of pointed finite sets $\alpha : \{1, \ldots, m, *\} \to \{1, \ldots, n, *\}$ and a collection of $\mathbf{C}$-linear isomorphisms $f_j : \bigoplus_{\alpha(i)=j} V_i \to V_j'$. We let $\mathfrak{B}(\alpha, \{f_j\})$ denote the morphism

$$
(\bigoplus_{1 \leq j \leq n} V_j', \bigoplus_{1 \leq j \leq n} V_j') \to (\bigoplus_{1 \leq i \leq m} V_i, \bigoplus_{1 \leq i \leq m})
$$

in $\mathrm{Vect}_{\mathbf{C}}^{\pm}$ given by the triple (ι, ι, E), where

$$
\iota : \bigoplus_{1 \leq j \leq n} V_j' \hookrightarrow \bigoplus_{1 \leq i \leq m} V_i
$$

is the direct sum of the isomorphisms $\{f_j^{-1}\}_{1 \le j \le n}$ and E is the graph of the map

$$\bigoplus_{\alpha(i)=*} \phi_i \colon \bigoplus_{\alpha(i)=*} V_i \to \bigoplus_{\alpha(i)=*} V_i$$

(note that this map is an isomorphism, since all the eigenvalues of ϕ_i belong to the disk D_i which is forbidden to contain $0 \in \mathbf{C}$ when $\alpha(i) = *$).

Passing to classifying spaces, we obtain a map of simplicial sets

$$\Theta((\overline{\mathrm{Vect}}_{\mathbf{C}}^{\simeq})^{\otimes}) \to |\operatorname{Sing}_{\bullet} \mathrm{N}_{\bullet}(\mathrm{Vect}_{\mathbf{C}}^{\pm})|.$$

We can identify the domain of this map with $\mathrm{Bar}^{(2)}(\amalg_{n \ge 0} \mathrm{BU}(n))$ (Remark 6.5.4) and its codomain with $\mathbf{Z} \times \mathrm{BU}$ (Proposition 6.2.7 and Warning 6.3.2). We therefore obtain a map of spaces $\mathrm{Bar}^{(2)}(\amalg_{n \ge 0} \mathrm{BU}(n)) \to \mathbf{Z} \times \mathrm{BU}$, which we will also denote by $\mathfrak{B}$.

Remark 6.5.6. All of the linear-algebraic constructions above are compatible with the formation of tensor products by 1-dimensional complex vector spaces. It follows that the map $\mathfrak{B} \colon \mathrm{Bar}^{(2)}(\amalg_{n \ge 0} \mathrm{BU}(n)) \to \mathbf{Z} \times \mathrm{BU}$ satisfies the hypothesis of Proposition 6.1.1, and is therefore homotopic to an integral multiple of the Bott map β.

7. Comparing the Geometric and Combinatorial Definitions of ϕ

In §3.5, we introduced a monoidal functor

$$\phi \colon \mathrm{Tors}(\mathbf{Z}) \to \mathrm{BPic}(S)$$

which determines an action of the circle group $S^1 \simeq \mathrm{Tors}(\mathbf{Z})$ on the ∞-category $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}}$ of stable ∞-categories. In §4, we proved that the algebraic K-theory functor $\mathcal{C} \mapsto K(\mathcal{C})$ determines an S^1-equivariant map $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}} \to \mathcal{S}$, where the circle group acts on $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{St}}$ via ϕ and trivially on $\mathcal{S}$. To complete the proof of Theorem 1.1.1, it remains only to show that the monoidal functor ϕ is homotopic (as a map of 1-fold loop spaces) to the composition

$$\mathrm{Tors}(\mathbf{Z}) \simeq U(1) \to U(\infty) \overset{\beta}{\to} \Omega^{-1}(\mathbf{Z} \times \mathrm{BU}) \overset{J_{\mathbf{C}}}{\to} \mathrm{BPic}(S),$$

where β denotes the Bott periodicity map and $J_{\mathbf{C}}$ the complex J-homomorphism.

In what follows, it will be convenient to take loop spaces and consider instead the $\mathbb{E}_2$-monoidal functor $\Phi = \Omega(\phi) \colon \mathbf{Z}^{\mathrm{ds}} \to \mathrm{Pic}(S)$. In §5, we proved that on nonnegative integers, the functor Φ could be identified with the $\mathbb{E}_2$-map $\rho \colon \mathbf{Z}_{\ge 0}^{\mathrm{ds}} \to \mathrm{Pic}(S)$ of Proposition 5.1.13, given informally by

$$n \mapsto \Sigma^{\infty} \mathrm{Sym}^n(D)^c$$

where D ranges over open disks in the complex numbers. In §7.1, we will use this description to show that the map ρ factors (up to homotopy) through the complex J-homomorphism $J_{\mathbf{C}} \colon \mathbf{Z} \times \mathrm{BU} \to \mathrm{Pic}(S)$: roughly speaking, the idea is that each $\mathrm{Sym}^n(D)$ is a contractible complex manifold and can therefore be exchanged (without loss of homotopic-theoretic information) for its tangent space at any point. From this, it will follow that $\Phi|_{\mathbf{Z}_{\ge 0}^{\mathrm{ds}}}$ is homotopic (as a morphism of $\mathbb{E}_2$-spaces) to a composition

$$\mathbf{Z}_{\ge 0} \overset{\alpha}{\to} \mathbf{Z} \times \mathrm{BU} \overset{J_{\mathbf{C}}}{\to} \mathrm{Pic}(S).$$

Using the Bott periodicity, we can identify α with a map

$$\overline{\alpha} \colon \mathrm{BU}(1) \simeq \mathrm{Bar}^2 \mathbf{Z}_{\ge 0} \to \mathrm{BU}.$$

Our goal will then be to show that $\overline{\alpha}$ is homotopic to the one induced by the inclusion $U(1) \hookrightarrow U$: that is, to show that $\overline{\alpha}$ corresponds to the element $[\mathcal{O}(1)] - 1 \in K^0(\mathrm{BU}(1))$, where $\mathcal{O}(1)$ denotes the tautological line bundle on $\mathrm{BU}(1) \simeq \mathbf{CP}^{\infty}$. We will give a proof of this statement in §7.2,

using our explicit description of Bott periodicity from §6 together with a certain commutative diagram which we construct in §7.3.

7.1. **Complex Disks.** Let $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ denote the topologically enriched category whose objects are finite-dimensional complex vector spaces and whose morphisms are $\mathbf{C}$-linear isomorphisms, as in §6.2. For every finite-dimensional complex vector space V, let V^c denote the one-point compactification of V. We regard the construction $V \mapsto V^c$ determines a symmetric monoidal functor from $\mathrm{Vect}_{\mathbf{C}}^{\simeq}$ to the (topologically enriched) category whose objects are pointed spaces which are homeomorphic to a sphere (with the symmetric monoidal structure given by the formation of smash products). It follows that the construction $V \mapsto \Sigma^{\infty} V^c$ can be regarded as a symmetric monoidal functor from the ∞-category $\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})^{\mathrm{op}}$ to the ∞-category $\mathrm{Pic}(S)$. We will denote this functor by
$$J_{\mathbf{C}} : \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \to \mathrm{Pic}(S);$$
it is a model for the complex J-homomorphism.

Note that if V is complex vector space, then the spectrum $J_{\mathbf{C}}(V) = \Sigma^{\infty} V^c$ depends only on the underlying topological space of V, not on its vector space structure. We will use this observation to enlarge the domain of definition of the functor $J_{\mathbf{C}}$.

Definition 7.1.1. An *embeddable complex disk* is a complex analytic manifold D which satisfies the following conditions:

- As a topological space, D is homeomorphic to $\mathbf{R}^{2n}$ for some integer n (we will refer to n as the *dimension* of D).
- There exists a holomorphic open embedding $D \hookrightarrow \mathbf{C}^n$.

If D and D' are embeddable complex disks, we let $\mathrm{Emb}(D, D')$ denote the collection of all holomorphic open embeddings of D into D'. We will endow $\mathrm{Emb}(D, D')$ with the compact-open topology. We let $\mathrm{Disk}_{\mathbf{C}}$ denote the category whose objects are embeddable complex disks and whose morphisms are holomorphic open embeddings, which we regard as a topologically enriched category.

Remark 7.1.2. If D and D' are embeddable complex disks of dimensions n and n', respectively, then the Cartesian product $D \times D'$ is an embeddable complex disk of dimension $n + n'$. The formation of products determines a symmetric monoidal structure on the category $\mathrm{Disk}_{\mathbf{C}}$ which is compatible with its topological enrichment. Consequently, the homotopy coherent nerve $\mathrm{N}(\mathrm{Disk}_{\mathbf{C}})$ inherits the structure of a symmetric monoidal ∞-category.

Every finite-dimensional vector space V can be regarded as an embeddable complex disk. Moreover, if V and W are finite-dimensional complex vector spaces, then we can regard the space $\mathrm{Iso}_{\mathbf{C}}(V, W)$ of $\mathbf{C}$-linear isomorphisms from V to W as a subspace of the space $\mathrm{Emb}(W, V)$ of holomorphic open embeddings of W into V. Consequently, we have a faithful topologically-enriched forgetful functor $\mathrm{Vect}_{\mathbf{C}}^{\simeq} \to \mathrm{Disc}_{\mathbf{C}}^{\mathrm{op}}$ which induces a symmetric monoidal functor of ∞-categories $\theta : \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \to \mathrm{Disc}_{\mathbf{C}}^{\mathrm{op}}$.

Construction 7.1.3. For every embeddable complex disk D, we let D^c denote the one-point compactificaton of D. The construction $D \mapsto \Sigma^{\infty} D^c$ determines a symmetric monoidal functor from the ∞-category $\mathrm{N}(\mathrm{Disc}_{\mathbf{C}}^{\mathrm{op}})$ to the ∞-category $\mathrm{Pic}(S) \subseteq \mathrm{Sp}^{\mathrm{op}}$. Since $\mathrm{Pic}(S)$ is a Kan complex, this induces a symmetric monoidal functor from $\mathrm{N}(\mathrm{Disk}_{\mathbf{C}})^{\mathrm{gpd}}$ to $\mathrm{Pic}(S)$ which we will denote by $J_{\mathbf{C}}^{+} : \mathrm{N}(\mathrm{Disc}_{\mathbf{C}}^{\mathrm{op}})^{\mathrm{gpd}} \to \mathrm{Pic}(S)$. We will refer to $J_{\mathbf{C}}^{+}$ as the *enhanced complex J-homomorphism*.

Remark 7.1.4. Note that the composite map
$$\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) \hookrightarrow \mathrm{N}(\mathrm{Disc}_{\mathbf{C}}^{\mathrm{op}}) \overset{J_{\mathbf{C}}^{+}}{\to} \mathrm{Pic}(S)$$

agrees with the usual complex J-homomorphism $J_{\mathbf{C}}$.

Remark 7.1.5. Let $D \subseteq \mathbf{C}$ be an open disk in the complex plane. For each integer $n \geq 0$, the complex structure on D induces a complex structure on the symmetric power $\operatorname{Sym}^n D$. We note that $\operatorname{Sym}^n \mathbf{C}$ is biholomorphic to $\mathbf{C}^n$, so that the inclusion $D \subseteq \mathbf{C}$ induces a holomorphic open immersion $\operatorname{Sym}^n D \hookrightarrow \mathbf{C}^n$ and a homeomorphism of D with $\mathbf{C}$ induces a homeomorphism $\operatorname{Sym}^n D \simeq \mathbf{R}^{2n}$. We may therefore regard $\operatorname{Sym}^n D$ as an embeddable complex disk.

The coalgebra $T \in \operatorname{CoAlg}_{\mathcal{O}[\mathbf{Z}_{\geq 0}]}(\operatorname{Top}_*)$ of Construction 5.1.12 arises from a $\mathcal{O}[\mathbf{Z}_{\geq 0}]$-coalgebra in the ordinary category $\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}}$, or in the ∞-category $\mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})$. The image of this coalgebra in the groupoid completion $(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})^{\mathrm{gpd}}$ satisfies condition $(*)$ of Proposition 5.1.8, and can therefore be identified with an $\mathbb{E}_2$-monoidal functor $\rho_0 : \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{N}(\operatorname{Disk}_{\mathbf{C}})^{\mathrm{gpd}}$. It follows from Theorem 5.1.14 that we have a commutative diagram of $\mathbb{E}_2$-monoidal functors

$$
\begin{array}{ccc}
\mathbf{Z}_{\geq 0}^{\mathrm{ds}} & \xrightarrow{\quad \rho_0 \quad} & \mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})^{\mathrm{gpd}} \\
\downarrow & & \downarrow{\scriptstyle J_{\mathbf{C}}^+} \\
\mathbf{Z}^{\mathrm{ds}} \longrightarrow \operatorname{Rep}(\mathbf{Z}^{\mathrm{ds}}) & \xrightarrow{\quad \Phi \quad} & \operatorname{Pic}(S),
\end{array}
$$

where Φ is the $\mathbb{E}_2$-monoidal functor of Corollary 3.5.8. More informally: the restriction $\Phi|_{\mathbf{Z}_{\geq 0}^{\mathrm{ds}}}$ factors (as an $\mathbb{E}_2$-monoidal functor) through the enhanced complex J-homomorphism $J_{\mathbf{C}}^+$.

We can now state the main result of this section:

Proposition 7.1.6. *The canonical map $\theta : \mathrm{N}(\operatorname{Vect}_{\mathbf{C}}^{\simeq}) \to \mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})$ is a weak homotopy equivalence of simplicial sets.*

Warning 7.1.7. The functor $\theta : \mathrm{N}(\operatorname{Vect}_{\mathbf{C}}^{\simeq}) \to \mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})$ is *not* an equivalence of ∞-categories, because the ∞-category $\mathrm{N}(\operatorname{Disk}_{\mathbf{C}})$ is not a Kan complex. For example, there exists an open embedding from the standard unit disk $\{z \in \mathbf{C} : |z| < 1\}$ to the complex plane $\mathbf{C}$, but there does not exist an open embedding from $\mathbf{C}$ to the standard unit disk.

Remark 7.1.8. Proposition 7.1.6 is equivalent to the assertion that the composite map

$$
\mathrm{N}(\operatorname{Vect}_{\mathbf{C}}^{\simeq}) \xrightarrow{\theta} \mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}}) \xrightarrow{\theta'} \mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})^{\mathrm{gpd}}
$$

is a homotopy equivalence of Kan complexes.

It follows from Proposition 7.1.6 that the enhanced complex J-homomorphism $J_{\mathbf{C}}^+$ is determined by the usual J-homomorphism $J_{\mathbf{C}}$ (up to contractible choice). In particular, we have the following:

Corollary 7.1.9. *Let $\Phi : \operatorname{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \operatorname{Pic}(S)$ be the $\mathbb{E}_2$-monoidal functor of Corollary 3.5.8. Then the restriction $\Phi_{\geq 0} = \Phi|_{\mathbf{Z}_{\geq 0}^{\mathrm{ds}}}$ is given by the composition*

$$
\mathbf{Z}_{\geq 0}^{\mathrm{ds}} \xrightarrow{\rho_0} \mathrm{N}(\operatorname{Disc}_{\mathbf{C}}^{\mathrm{op}})^{\mathrm{gpd}} \xleftarrow{\sim} \mathrm{N}(\operatorname{Vect}_{\mathbf{C}}^{\simeq}) \xrightarrow{J_{\mathbf{C}}} \operatorname{Pic}(S),
$$

where ρ_0 is defined in Remark 7.1.5 and the equivalence is supplied by Remark 7.1.8.

The remainder of this section is devoted to the proof of Proposition 7.1.6. First, we need to introduce an auxiliary notion.

Definition 7.1.10. We will say that an complex disk D of dimension n is *small* if there exists a holomorphic open embedding $D \hookrightarrow \mathbf{C}^n$ whose image is a bounded convex set. Let $\operatorname{Disk}_{\mathbf{C}\mathrm{sm}}$ denote the the full subcategory of $\operatorname{Disk}_{\mathbf{C}}$ spanned by the small complex disks.

Let $\mathcal{M}$ denote the ∞-category

$$\mathrm{N}(\mathrm{Disk}_{\mathbf{C}\mathrm{sm}}) \times_{\mathrm{Fun}(\{0\},\mathrm{N}(\mathrm{Disk}_{\mathbf{C}}))} \mathrm{Fun}(\Delta^1, \mathrm{N}(\mathrm{Disk}_{\mathbf{C}}))$$

whose objects are pairs $(\phi : D_0 \hookrightarrow D)$ where ϕ is a holomorphic open embedding of embeddable complex disks and D_0 is small. The construction $(\phi : D_0 \hookrightarrow D) \mapsto D$ determines a forgetful functor $\beta : \mathcal{M} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}})$; we let $\mathcal{N}$ denote the fiber product $\mathcal{M} \times_{\mathrm{N}(\mathrm{Disk}_{\mathbf{C}})} \mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq})$ whose objects are pairs $(\phi : D_0 \hookrightarrow V)$ where D_0 is a small complex disk and V is a finite-dimensional complex vector space. We will deduce Proposition 7.1.6 from the following pair of assertions:

Lemma 7.1.11. *The forgetful functor $\beta : \mathcal{M} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}})$ is a trivial Kan fibration.*

Lemma 7.1.12. *The forgetful functor $(\phi : D_0 \hookrightarrow V) \mapsto D_0$ determines a trivial Kan fibration $\mathcal{N} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}\mathrm{sm}})$.*

Proof of Proposition 7.1.6. We have a pullback diagram

$$
\begin{array}{ccc}
\mathcal{N} & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle\beta} \\
\mathrm{N}(\mathrm{Vect}_{\mathbf{C}}^{\simeq}) & \xrightarrow{\ \theta\ } & \mathrm{N}(\mathrm{Disk}_{\mathbf{C}}).
\end{array}
$$

Lemma 7.1.11 implies that the vertical maps are trivial Kan fibrations. Consequently, to prove that θ is weak homotopy equivalence, it will suffice to show that the inclusion $\mathcal{N} \hookrightarrow \mathcal{M}$ is a weak homotopy equivalence. Note that the forgetful functor

$$(\phi : D_0 \hookrightarrow D) \mapsto D_0$$

$$\mathcal{M} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}\mathrm{sm}})$$

admits a right adjoint (given on objects by $D_0 \mapsto (\mathrm{id}_{D_0} : D_0 \hookrightarrow D_0)$) and is therefore a weak homotopy equivalence. It will therefore suffice to show that the composite map

$$\mathcal{N} \hookrightarrow \mathcal{M} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}\mathrm{sm}})$$

is a weak homotopy equivalence, which follows from Lemma 7.1.12. $\square$

We will deduce Lemmas 7.1.11 and 7.1.12 from the following:

Lemma 7.1.13. *Let V and W be complex vector spaces of the same dimension n. Let D be a bounded convex open subset of V and let D' be an arbitrary open subset of W. Then differentiation at any point $v \in D$ induces a homotopy equivalence*

$$\mathrm{Emb}(D, D') \to D' \times \mathrm{Iso}_{\mathbf{C}}(V, W).$$

Proof of Lemma 7.1.11. Since the map β is a coCartesian fibration, it will suffice to show that the fibers of β are contractible Kan complexes. Fix an embeddable complex disk D of dimension n, choose a holomorphic embedding of D into a vector space W, and set $\mathcal{C} = \beta^{-1}\{D\}$. We first observe that $\mathcal{C}$ is nonempty (for any point $z \in D$, any sufficiently small open ball around z determines an object of $\mathcal{C}$). It will therefore suffice to show that for any pair of objects $C, C' \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(C, C')$ is contractible. Let us write

$$C = (\phi : D_0 \hookrightarrow D) \qquad C' = (\phi' : D_0' \hookrightarrow D),$$

so that we have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Emb}(D_0, D_0') \xrightarrow{\gamma} \mathrm{Emb}(D_0, D).$$

It will therefore suffice to show that γ (which is induced by composition with ϕ') is a homotopy equivalence. Since D_0 is small, it can be identified with a convex bounded open subset of some

complex vector space V. Using Lemma 7.1.13, we see that γ is a homotopy equivalence if and only if the induced map

$$D_0' \times \mathrm{Iso}_{\mathbf{C}}(V, W) \to D \times \mathrm{Iso}_{\mathbf{C}}(V, W)$$

is a homotopy equivalence, which is clear. □

Proof of Lemma 7.1.12. The forgetful functor $\alpha : \mathcal{N} \to \mathrm{Disk}_{\mathbf{C}_{\mathrm{sm}}}$ is a Cartesian fibration; it will therefore suffice to show that each fiber of α is a contractible Kan complex. Let us therefore fix a a small complex disk D and set $\mathcal{C} = \alpha^{-1}\{D\}$. Since D is small, there exists a holomorphic open embedding $D \hookrightarrow V$ where n is the dimension of D; this proves that $\mathcal{C}$ is nonempty. Moreover, we may assume without loss of generality that the image of D in V is bounded and convex. To complete the proof, it will suffice to show that the mapping space $\mathrm{Map}_{\mathcal{C}}(C, C')$ is contractible for every pair of objects $C, C' \in \mathcal{C}$. Let us write

$$C = (\phi : D \hookrightarrow W,) \qquad C' = (\phi' : D \hookrightarrow W'),$$

for some complex vector spaces W and W', so that we have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{C}}(C, C') \to \mathrm{Iso}_{\mathbf{C}}(W, W') \xrightarrow{\gamma} \mathrm{Emb}(D, W').$$

Consequently, to complete the proof it will suffice to show that γ (which is given by composition with ϕ') is a homotopy equivalence.

Fix a point $v \in V$, and let $w = \phi(v) \in W$. The desired result now follows from Lemma 7.1.13 together with the observation that the map

$$\mathrm{Iso}_{\mathbf{C}}(W, W') \to W' \times \mathrm{Iso}_{\mathbf{C}}(V, W')$$

$$f \mapsto (f(w), f \circ \phi_v')$$

is a homotopy equivalence, where $\phi_v' : V \to W$ is the map given by differentiating ϕ at the point v. □

Proof of Lemma 7.1.13. It will suffice to show that for every compact Hausdorff space K, every closed subset $A \subseteq B$, and every commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;f\;\;} & \mathrm{Emb}(D, D') \\
\downarrow & {}_{\overline{f}} \nearrow & \downarrow \\
B & \xrightarrow[\;\;g\;\;]{} & D' \times \mathrm{Iso}_{\mathbf{C}}(V, W),
\end{array}
$$

it is possible to compatibly modify f and g by a homotopy so that there exists an extension as indicated by the dotted arrow in the diagram. Let us identify g with a pair of maps $g' : B \to D'$ and $g'' : B \to \mathrm{Iso}_{\mathbf{C}}(V, W)$.

We may assume without loss of generality that $v = 0$. For each real number $t \in (0, 1]$, let

$$f_t : A \to \mathrm{Emb}(D, D') \qquad g_t : B \to D' \times \mathrm{Iso}_{\mathbf{C}}(V, W)$$

be given by

$$f_t(a)(z) = f(a)(tz) \qquad g_t(b) = (g'(b), tg''(b)).$$

The pair (f_t, g_t) is homotopic to (f, g) (the homotopy given by $\{(f_{t'}, g_{t'})\}_{t' \in [t, 1]}$). Consider the map

$$A \times D \times [0, 1] \to W$$

$$(a, z, s) \mapsto s f_t(a)(z) + (1 - s) g'(a) + t(1 - s) g''(a)(z).$$

For t sufficiently small, this map classifies a homotopy from f_t to another map $f' : A \to \mathrm{Emb}(D, D')$ satisfying $f'(a)(z) = g'(a) + tg''(a)(z)$, compatible with the identity homotopy from g_t to itself. We may therefore replace (f, g) by (f', g_t) and thereby reduce to the case

where f satisfies $f(a)(z) = g'(a) + g''(a)(z)$, in which case we can define $\overline{f}$ by the formula $\overline{f}(b)(z) = g'(b) + g''(b)(z)$. $\qquad\square$

7.2. A Commutative Diagram.

Let $\mathrm{Disk}_{\mathbf{C}}$ denote the (topologically enriched) category of embeddable complex disks (Definition 7.1.1) and let $\mathbf{Z}_{\geq 0}$ be the set of nonnegative integers. In the previous section, we constructed a homotopy equivalence of $\mathbb{E}_\infty$-spaces $\mathrm{N}(\mathrm{Disk}_{\mathbf{C}})^{\mathrm{gpd}} \simeq \amalg_{n\geq 0} \mathrm{BU}(n)$ (Proposition 7.1.6) and an $\mathbb{E}_2$-monoidal functor $\rho_0 : \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}})^{\mathrm{gpd}}$. Taken together, these determine a map of spaces

$$\mathbf{C}\mathrm{P}^\infty \simeq \mathrm{Bar}^{(2)}(\mathbf{Z}_{\geq 0}^{\mathrm{ds}}) \overset{\rho_0}{\to} \mathrm{Bar}^{(2)}(\mathrm{N}(\mathrm{Disk}_{\mathbf{C}})^{\mathrm{gpd}}) \simeq \mathrm{Bar}^{(2)}(\amalg_{n\geq 0} \mathrm{BU}(n)) \simeq \mathrm{BU}.$$

Our goal in this section is to show that this composite map classifies the K-theory class $[\mathcal{O}(1)] - 1 \in K^0(\mathbf{C}\mathrm{P}^\infty)$, thereby completing the proof of Theorem 1.1.1.

Let $\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes$ and $\mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes$ be the categories (topologically enriched in the former case) be defined as in Example 6.4.2. The objects of the category $\mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes$ are given by finite sequences $((D_1, n_1), \ldots, (D_m, n_m))$ where each D_i is an open disk in $\mathbf{C}$ and each n_i is a nonnegative integer, and the objects of $\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes$ are given by finite sequences $((D_1, U_1), \ldots, (D_m, U_m))$ where each D_i is an open disk in $\mathbf{C}$ and each U_i is an embeddable complex disk. The $\mathbb{E}_2$-monoidal functor $\rho_0 : \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}})$ of Remark 7.1.5 arises from a map of topological $\mathcal{O}$-monoidal categories $\mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes \to \mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes$ which we will also denote by ρ_0, given on objects by the formula

$$\rho_0((D_1, n_1), \ldots, (D_m, n_m)) = ((D_1, \mathrm{Sym}^{n_1} D_1), \ldots, (D_m, \mathrm{Sym}^{n_m} D_m)).$$

We will prove the following result in §7.3:

Proposition 7.2.1. *There is a commutative diagram of topological categories (over $\mathcal{O}^\otimes$)*

$$
\begin{array}{ccccc}
\mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes & \xleftarrow{\;\pi'\;} & \mathrm{Conf}^\otimes & \xrightarrow{\;\xi\;} & (\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^\otimes \\
\Big\downarrow{\scriptstyle\rho_0} & & \Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\
\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes & \xleftarrow{\;\pi''\;} & \overline{\mathrm{Disk}_{\mathbf{C}}}^\otimes & \xrightarrow{\;\xi_0\;} & \mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]
\end{array}
$$

with the following properties:

(a) *The functors π, π', and π'' induce maps of simplicial spaces*

$$\mathrm{N}_\bullet \mathrm{Conf}^\otimes \to \mathrm{N}_\bullet \mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes \quad \mathrm{N}_\bullet \overline{\mathrm{Disk}_{\mathbf{C}}}^\otimes \to \mathrm{N}_\bullet \mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes \quad \mathrm{N}_\bullet (\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^\otimes \to \mathrm{N}_\bullet \mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]$$

which are are levelwise homotopy equivalences. In particular, they induce weak homotopy equivalences of simplicial sets

$$\Theta(\mathrm{Conf}^\otimes) \to \Theta(\mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes) \quad \Theta(\overline{\mathrm{Disk}_{\mathbf{C}}}^\otimes) \to \Theta(\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes) \quad \Theta((\overline{\mathrm{Vect}_{\mathbf{C}}^{\simeq}})^\otimes) \to \Theta(\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}])$$

(b) *The induced map*

$$\Theta(\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^\otimes) \overset{\sim}{\leftarrow} \Theta(\overline{\mathrm{Disk}_{\mathbf{C}}}^\otimes) \to \Theta(\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}])$$

is a left homotopy inverse to the map induced by the inclusion of symmetric monoidal topological categories $\mathrm{Vect}_{\mathbf{C}}^{\simeq} \hookrightarrow \mathrm{Disk}_{\mathbf{C}}$.

(c) *Let $\mathfrak{B} : \Theta(\overline{\mathrm{Vect}}^\otimes) \to |\mathrm{Sing}_\bullet \mathrm{N}_\bullet \mathrm{Vect}_{\mathbf{C}}^\pm|$ be as in Construction 6.5.5. Under the weak homotopy equivalence*

$$\Theta(\mathrm{Conf}^\otimes) \overset{\sim}{\to} \Theta(\mathcal{O}[\mathbf{Z}_{\geq 0}]^\otimes) \simeq \mathrm{Bar}^{(2)} \mathbf{Z}_{\geq 0} \simeq \mathbf{C}\mathrm{P}^\infty,$$

the composite map

$$\Theta(\mathrm{Conf}^{\otimes}) \overset{\Theta(\xi)}{\to} \Theta(\overline{\mathrm{Vect}}^{\otimes}) \to |\,\mathrm{Sing}_{\bullet}\, \mathrm{N}_{\bullet}\, \mathrm{Vect}_{\mathbf{C}}^{\pm}|\simeq \mathbf{Z} \times \mathrm{BU}$$

corresponds to the element of $K^0(\mathbf{CP}^{\infty})$ given by $[\mathcal{O}(1)] - 1$, where $\mathcal{O}(1)$ denotes the tautological line bundle on $\mathbf{CP}^{\infty}$.

The proof of Proposition 7.2.1 will be given in §7.3. Let us assume Proposition 7.2.1 for the moment, and show that it completes the proof of our main result.

Corollary 7.2.2. *The composite map*

$$\mathrm{Bar}^{(2)}\,\mathbf{Z}_{\geq 0} \overset{\rho_0}{\to} \mathrm{Bar}^{(2)}\,\mathrm{Disk}_{\mathbf{C}}^{\mathrm{gpd}} \overset{\sim}{\leftarrow} \mathrm{Bar}^{(2)}(\amalg_{n\geq 0}\mathrm{BU}(n)) \overset{\mathfrak{B}}{\to} \mathrm{BU}$$

is homotopic to the canonical map

$$\mathbf{CP}^{\infty} \simeq \mathrm{BU}(1) \to \mathrm{BU},$$

corresponding to the element $[\mathcal{O}(1)] - 1$ in $K^0(\mathbf{CP}^{\infty})$.

Proof. Combine Proposition 7.2.1 with Proposition 6.4.6. $\qquad\square$

Corollary 7.2.3. *The map $\mathfrak{B} : (\amalg_{n\geq 0}\mathrm{BU}(n)) \to \mathrm{BU}$ is homotopic to the Bott periodicity map β.*

Proof. According to Remark 6.5.6, we have $\mathfrak{B} = k\beta$ for some integer k. Let us identify $\mathfrak{B}$ and β with elements of the group $K^0(\mathrm{BU})$. We have normalized the Bott map so the adjoint map

$$\amalg_{n\geq 0}\mathrm{BU}(n) \to \Omega^2\,\mathrm{BU}$$

carries $\mathrm{BU}(n)$ to the connected component of $\Omega^2\,\mathrm{BU}$ corresponding to the element

$$n\beta \in \pi_0\Omega^2\,\mathrm{BU} \simeq K_{\mathrm{red}}^0(S^2),$$

where β is the K-theory class $[\mathcal{O}(1)] - 1$. In particular, the composition

$$\mathbf{Z}_{\geq 0} \overset{\rho_0}{\to} \pi_0(\amalg_{n\geq 0}\mathrm{BU}(n)) \overset{\beta}{\to} \pi_0(\Omega^2\,\mathrm{BU}) = K_{\mathrm{red}}^0(S^2)$$

is given by $n \mapsto n\beta$. It follows that the composite map

$$\mathbf{Z}_{\geq 0} \overset{\rho_0}{\to} \pi_0(\amalg_{n\geq 0}\mathrm{BU}(n)) \overset{\mathfrak{B}}{\to} \pi_0(\Omega^2\,\mathrm{BU}) \simeq K_{\mathrm{red}}^0(S^2)$$

is given by $n \mapsto kn\beta$. Using Corollary 7.2.2, we see that this composition is also given by

$$\mathbf{Z}_{\geq 0} \hookrightarrow \mathbf{Z} \to \pi_0\Omega^2(\mathbf{CP}^{\infty}) \to \pi_0\Omega^2(\mathrm{BU}) \simeq K_{\mathrm{red}}^0(S^2),$$

where the the map $\mathbf{CP}^{\infty} \to \mathrm{BU}$ classifies the line bundle $\mathcal{O}(1)$ on $\mathbf{CP}^{\infty}$, which is given by $n \mapsto n\beta$. It follows that $k = 1$, as desired. $\qquad\square$

Corollary 7.2.4. *The $\mathbb{E}_2$-monoidal functor $\rho_0 : \mathbf{Z}_{\geq 0} \to \mathrm{N}(\mathrm{Disk}_{\mathbf{C}})^{\mathrm{gpd}}$ of Remark 7.1.5 fits into a commutative diagram*

$$
\begin{array}{ccc}
\mathbf{Z}_{\geq 0} & \overset{\rho_0}{\longrightarrow} & \mathrm{N}(\mathrm{Disk}_{\mathbf{C}})^{\mathrm{gpd}} \\
\downarrow & & \downarrow \\
\Omega^2(\mathrm{BU}(1)) & \longrightarrow & \Omega^2(\mathrm{BU})
\end{array}
$$

where the right vertical map is given by Bott periodicity.

Proof. Combine Corollaries 7.2.2 and 7.2.3. $\qquad\square$

Corollary 7.2.5. *The $\mathbb{E}_2$-monoidal functor $\Phi : \mathrm{Rep}(\mathbf{Z}^{\mathrm{ds}}) \to \mathrm{Sp}$ of Corollary 3.5.8 is given by the composite map*

$$\mathbf{Z}^{\mathrm{ds}} \simeq \Omega^2(\mathrm{BU}(1)) \to \Omega^2(\mathrm{BU}) \xrightarrow{\beta} \mathbf{Z} \times \mathrm{BU} \to J_{\mathbf{C}} \to \mathrm{Pic}(S)$$

where β is given by Bott periodicity and $J_{\mathbf{C}}$ denotes the complex J-homomorphism.

Proof. Combine Corollaries 7.2.4 and 7.1.9. □

7.3. **Configuration Spaces.** Our goal in this section is to prove Proposition 7.2.1. We begin by constructing the auxiliary categories and functors which appear in the statement of Proposition 7.2.1. The verification of assertions (a) and (b) is relatively straightforward, while the proof of (c) is more involved.

Construction 7.3.1 (Construction of the Topological Category $\overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}$). The objects of $\overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}$ are given by finite sequences

$$((D_1, U_1, p_1), (D_2, U_2, p_2), \ldots, (D_m, U_m, p_m))$$

where each D_i is an open disk in $\mathbf{C}$, each U_i is an embeddable complex disk, and each p_i is a point of disk U_i. We endow the collection of all objects of $\overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}$ with the topology given by its presentation as a disjoint union

$$\coprod U_1 \times U_2 \times \cdots \times U_m,$$

indexed by the collection of all sequences $\{(D_i, U_i)\}$.

A morphism from $\{(D_i, U_i, p_i)\}_{1 \leq i \leq m}$ to $\{(D'_j, U'_i, p_j)\}_{1 \leq j \leq n}$ in $\overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}$ is a morphism from $\{(D_i, U_i)\}$ to $\{(D'_j, U'_j)\}$ in the category $\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^{\otimes}$ having the property that for each $1 \leq j \leq n$, the associated holomorphic embedding

$$\prod_{i \mapsto j} U_i \hookrightarrow V_j$$

carries the point $\{(p_i)\}$ to p'_j. We endow the collection of all morphisms in $\overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}$ with the topology given by its presentation as a disjoint union of topological spaces of the form

$$\mathrm{Map}_{\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^{\otimes}}(\{(D_i, U_i)\}, \{(D'_j, U'_j)\}) \times \prod U_i.$$

There is an evident forgetful functor $\pi'' : \overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes} \to \mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^{\otimes}$, given on objects by the formula

$$\pi''(\{(D_i, U_i, p_i)\}) = \{(D_i, U_i)\}.$$

The induced map of simplicial spaces

$$\mathrm{N}_{\bullet} \overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes} \to \mathrm{N}_{\bullet} \mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^{\otimes}$$

is levelwise a fiber bundle with contractible fibers (given by products of complex disks) and therefore a levelwise homotopy equivalence.

To define the functor $\xi_0 : \overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes} \to \mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes}$, we need to make a collection of arbitrary (but harmless) choices. Let us choose, for each embeddable complex disk $U \in \mathrm{Disk}_{\mathbf{C}}$, a finite-dimensional complex vector space V_U and a holomorphic open embedding $\iota_U : U \hookrightarrow V_U$. The functor ξ_0 is then given on objects by the formula $\xi_0(\{(D_i, U_i, p_i)\} = \{D_i, V_{U_i}\}$. Suppose we are given a morphism

$$f : \{(D_i, U_i, p_i)\}_{1 \leq i \leq m} \to \{(D'_j, U'_j, p'_j)\}_{1 \leq j \leq n}$$

in $\overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}$ whose underlying map of pointed finite sets we denote by $\alpha : \{1,\ldots,m,*\} \to \{1,\ldots,n,*\}$. We then define $\xi_0(f)$ to be the morphism in $\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes}$ given by α together with the vector space isomorphisms

$$\bigoplus_{\alpha(i)=j} V_{U_i} \simeq V_{U_j'}$$

which are obtained by differentiating the holomorphic open embedding $\prod_{\alpha(i)=j} U_i \hookrightarrow U_j'$ at the point $p = \{p_i\}_{\alpha(i)=j} \in \prod_{\alpha(i)=j} U_i$; here we identify the tangent space to $\prod_{\alpha(i)=j} U_i$ with $\bigoplus_{\alpha(i)=j} V_{U_i}$ using the embeddings ι_{U_i}, and the tangent space of U_j' at p_j' with $V_{U_j'}$ using the embedding $\iota_{U_j'}$.

Proof of Part (b) of Proposition 7.2.1. The inclusion $\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes} \hookrightarrow \mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^{\otimes}$ lifts to a topological functor

$$\nu : \mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes} \to \overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes},$$

given on objects by the formula $\nu(\{(D_i, V_i)\}) = \{(D_i, V_i, 0)\}$. It now suffices to observe that the composite functor $\xi_0 \circ \nu$ is canonically isomorphic to the identity functor on $\mathcal{O}[\mathrm{Vect}_{\mathbf{C}}^{\simeq}]^{\otimes}$ (the isomorphism being given by the differentials of the embeddings $\iota_V : V \hookrightarrow V_V$ at zero). $\qquad\square$

Construction 7.3.2 (Construction of the Topological Category $\mathrm{Conf}^{\otimes}$)**.** We define $\mathrm{Conf}^{\otimes}$ to be the fiber product

$$\mathcal{O}[\mathbf{Z}_{\geq 0}]^{\otimes} \times_{\mathcal{O}[\mathrm{Disk}_{\mathbf{C}}]^{\otimes}} \overline{\mathrm{Disk}_{\mathbf{C}}}^{\otimes}.$$

More concretely, the objects of $\mathrm{Conf}^{\otimes}$ are given by sequences

$$((D_1, n_1, P_1), (D_2, n_2, P_2), \ldots, (D_m, n_m, P_m))$$

where each D_i is an open disk in $\mathbf{C}$, each n_i is a nonnegative integer, and each P_i is a point of the symmetric power $\mathrm{Sym}^{n_i} D_i$. Let $\pi' : \mathrm{Conf}^{\otimes} \to \mathcal{O}[\mathbf{Z}_{\geq 0}]^{\otimes}$ be the projection onto the first factor. As in Construction 7.3.1, we see that the induced map of simplicial topological spaces

$$\mathrm{N}_{\bullet} \mathrm{Conf}^{\otimes} \to \mathrm{N}_{\bullet} \mathcal{O}[\mathbf{Z}_{\geq 0}]^{\otimes}$$

is levelwise a fiber bundle whose fibers are complex disks, and therefore a levelwise homotopy equivalence.

At this point, it will be convenient to be a bit more specific about some of the arbitrary choices made in Construction 7.3.1. For each integer $n \geq 0$, let $\mathrm{Poly}_{<n}$ be the subspace of the polynomial ring $\mathbf{C}[z]$ consisting of polynomials having degree $< n$. Let us identify the symmetric power $\mathrm{Sym}^n(\mathbf{C})$ with the space $z^n + \mathrm{Poly}_{<n}$ of monic polynomials of degree n (via the Σ_n-equivariant map $(a_1, \ldots, a_n) \mapsto \prod(z - a_n)$). We will henceforth assume that for every open disk $D \subseteq \mathbf{C}$ and every integer $n \geq 0$, the vector space $V_{\mathrm{Sym}^n(D)}$ appearing in Construction 7.3.1 has been chosen to be $\mathrm{Poly}_{<n}$ and the embedding $\iota_{\mathrm{Sym}^n(D)}$ is chosen to be the composite map

$$\mathrm{Sym}^n(D) \hookrightarrow \mathrm{Sym}^n(\mathbf{C}) \simeq z^n + \mathrm{Poly}_{<n} \xrightarrow{s} \mathrm{Poly}_{<n},$$

where s is given by subtracting z^n.

If $P \in z^n + \mathrm{Poly}_{<n}$ is a monic polynomial of degree n, we let ϕ_P denote the unique endomorphism of $\mathrm{Poly}_{<n}$ which fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Poly}_{<n} & \xrightarrow{P^{-1}} & \frac{1}{P}\mathbf{C}[z]/\mathbf{C}[z] \\
\downarrow{\scriptstyle \phi_P} & & \downarrow{\scriptstyle z} \\
\mathrm{Poly}_{<n} & \xrightarrow{P^{-1}} & \frac{1}{P}\mathbf{C}[z]/\mathbf{C}[z].
\end{array}
$$

More concretely, the endomorphism ϕ_P is given by the formula

$$\phi_P(z^i) = \begin{cases} z^{i+1} & \text{if } i < n-1 \\ z^n - P & \text{if } i = n-1. \end{cases}$$

The composite functor

$$\mathrm{Conf}^\otimes \to \overline{\mathrm{Disk}_\mathbf{C}}^\otimes \overset{\xi_0}{\to} \mathcal{O}[\mathrm{Vect}_\mathbf{C}^{\tilde{\simeq}}]^\otimes$$

is then given on objects by the formula

$$\{(D_i, n_i, P_i)\} \mapsto \{(D_i, \mathrm{Poly}_{<n_i})\}.$$

A simple calculation shows that this functor lifts to a map $\xi : \mathrm{Conf}^\otimes \to (\overline{\mathrm{Vect}_\mathbf{C}^{\tilde{\simeq}}})^\otimes$ given on objects by $\{(D_i, n_i, P_i)\} \mapsto \{(D_i, \mathrm{Poly}_{<n_i}, \phi_{P_i})\}$. This completes the construction of the commutative diagram described in Proposition 7.2.1.

The proof of part (c) of Proposition 7.2.1 will require some preliminaries.

Proof of Part (c) of Proposition 7.2.1. Let $\mathcal{C}$ denote the opposite of the topological category given by the fiber product $\mathcal{O}^\circ \times_{\mathcal{O}^\otimes} \mathrm{Conf}^\otimes$. In what follows, we will use Proposition 6.4.6 to identify $\mathbf{CP}^\infty$ with the classifying space $|\mathrm{Sing}_\bullet \mathrm{N}_\bullet \mathcal{C}|$. Let $F : \mathcal{C} \to \mathrm{Vect}_\mathbf{C}^\pm$ denote the topological functor given by the composition

$$\mathcal{C} \overset{\xi}{\to} (\mathcal{O}^\circ \times_{\mathcal{O}^\otimes} (\overline{\mathrm{Vect}_\mathbf{C}^{\tilde{\simeq}}})^\otimes)^{\mathrm{op}} \overset{\mathfrak{B}}{\to} \mathrm{Vect}_\mathbf{C}^\pm.$$

We wish to prove that $[F] = [\mathcal{O}(1)] - 1 \in K^0(\mathbf{CP}^\infty)$. We will verify this identity by constructing four other topological functors $G, H, L, L_0 : \mathcal{C} \to \mathrm{Vect}_\mathbf{C}^\pm$ which satisfy the following requirements:

(i) There is an exact sequence $0 \to L_0 \to G \to F \to 0$ (in the sense of Proposition 6.3.4).
(ii) There is an exact sequence $0 \to H \to G \to L \to 0$ (in the sense of Proposition 6.3.4).
(iii) We have an identity $[L_0] = 1 \in K^0(\mathbf{CP}^\infty)$.
(iv) We have an identity $[L] = [\mathcal{O}(1)] \in K^0(\mathbf{CP}^\infty)$.
(v) We have an identity $[H] = 0$ in $K^0(\mathbf{CP}^\infty)$.

Assuming that (i) through (v) have been verified, we invoke Proposition 6.3.4 twice to compute

$$\begin{aligned} [F] &= [G] - [L_0] \\ &= ([H] + [L]) - [L_0] \\ &= [\mathcal{O}(1)] - 1. \end{aligned}$$

Let us begin by describing the functor F a bit more explicitly. Unwinding the definitions, we can identify objects of the topological category $\mathcal{C}$ with finite sequences $\{(D_i, n_i, P_i)\}_{1 \le i \le m}$, where the D_i are disjoint open disks in $\mathbf{C}$, the n_i are nonnegative integers, and each P_i is a point of $\mathrm{Sym}^{n_i} D_i$ which we will identify with a monic polynomial of degree n_i in one variable $\mathbf{Z}$. The functor F is given on objects by the formula

$$F(\{(D_i, n_i, P_i)\}_{1 \le i \le m}) = (\bigoplus \mathrm{Poly}_{<n_i}, \bigoplus \mathrm{Poly}_{<n_i}).$$

For every monic polynomial $P \in \mathbf{C}[z]$, we let V_P denote the vector space $\frac{1}{P}\mathbf{C}[z]/\mathbf{C}[z]$. We will regard V_P as a finite-dimensional subspace of $\mathbf{C}(z)/\mathbf{C}[z]$. Note that if we are given a pair of monic polynomials P and Q, then we have natural inclusion maps

$$V_P \hookrightarrow V_{PQ} \hookleftarrow V_Q,$$

which induce an isomorphism $V_P \oplus V_Q \to V_{PQ}$ if P and Q are relatively prime to one another. Note that if $C = \{(D_i, n_i, P_i)\}$ is an object of $\mathcal{C}$, then the polynomials P_i are pairwise relatively prime; we therefore have a canonical isomorphism of complex vector spaces

$$\psi_C : \bigoplus \mathrm{Poly}_{<n_i} \to V_P$$

$$(\{Q_i\}_{1 \leq i \leq m}) \mapsto \sum \frac{Q_i}{P_i}$$

where $P = \prod P_i$. A morphism from $C = \{(D_i, n_i, P_i)\}_{1 \leq i \leq m}$ to $C' = \{(D'_j, n'_j, P'_j)\}_{1 \leq j \leq m'}$ consists of a map of pointed sets $\alpha : \{1, \ldots, m', *\} \to \{1 \ldots, m, *\}$ satisfying some additional conditions; we note that $F(\alpha)$ is the morphism from $(\bigoplus \mathrm{Poly}_{<n_i}, \bigoplus \mathrm{Poly}_{<n_i})$ to $(\bigoplus \mathrm{Poly}_{<n'_j}, \bigoplus \mathrm{Poly}_{<n'_j})$ given by (ρ, ρ, E), where ρ fits into a commutative diagram

$$
\begin{array}{ccc}
\bigoplus \mathrm{Poly}_{<n_i} & \xrightarrow{\rho} & \bigoplus \mathrm{Poly}_{<n'_j} \\
\downarrow{\scriptstyle \psi_C} & & \downarrow{\scriptstyle \psi_{C'}} \\
V_P & \longrightarrow & V_{P'}
\end{array}
$$

where $P = \prod P_i$ and $P' = \prod P'_j$ and E is the graph of the automorphism of $\bigoplus_{\alpha(j)=*} \mathrm{Poly}_{<n'_j}$ given by multiplication by z on the vector space $V_{P'/P}$.

Let us now consider a slight variation on the above. For every monic polynomial P in $\mathbf{C}[z]$, let V_P^+ denote the quotient $\frac{1}{P}\mathbf{C}[z]/z\mathbf{C}[z]$, so that we have a canonical exact sequence

$$0 \to \mathbf{C} \to V_P^+ \to V_P \to 0.$$

For each object $C = \{(D_i, n_i, P_i)\}$ in $\mathcal{C}$ with $P = \prod P_i$, we have a vector space isomorphism

$$\psi_C^+ : \mathbf{C} \oplus \bigoplus \mathrm{Poly}_{<n_i} \to V_P^+$$

$$(c, \{Q_i\}) \mapsto c + \sum \frac{Q_i}{P_i}.$$

We define a functor $G : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$ as follows:

- On objects, the functor G is given by

$$G(\{(D_i, n_i, P_i)\}) = (\bigoplus \mathrm{Poly}_{<n_i}, \mathbf{C} \oplus \bigoplus \mathrm{Poly}_{<n_i}).$$

- Suppose that we are given a morphism from an object $C = \{(D_i, n_i, P_i)\}_{1 \leq i \leq m}$ to another object $C' = \{(D'_j, n'_j, P'_j)\}_{1 \leq j \leq m'}$ as above, corresponding to a map of pointed finite sets $\alpha : \{1, \ldots, m', *\} \to \{1 \ldots, m, *\}$. Then $G(\alpha)$ is given by the triple (ρ, ρ^+, E), where ρ and E are defined as above for the functor F and the injection ρ^+ is chosen to guarantee the commutativity of the diagram

$$
\begin{array}{ccc}
\mathbf{C} \oplus \bigoplus \mathrm{Poly}_{<n_i} & \xrightarrow{\rho^+} & \bigoplus \mathrm{Poly}_{<n'_j} \\
\downarrow{\scriptstyle \psi_C^+} & & \downarrow{\scriptstyle \psi_{C'}^+} \\
V_P^+ & \longrightarrow & V_{P'}^+
\end{array}
$$

where $P = \prod P_i$, $P' = \prod P'_j$, and the lower horizontal map is induced by the inclusion $\frac{1}{P}\mathbf{C}[z] \hookrightarrow \frac{1}{P'}\mathbf{C}[z]$.

It follows immediately from the construction that the functor G fits into an exact sequence

$$0 \to L_0 \to G \to F \to 0,$$

where $L_0 : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$ denotes the constant functor taking the value $(0, \mathbf{C})$. This completes the verification of (i) and (iii).

We next define the functor $L : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$:

- On objects, the functor L is given by $L(\{(D_i, n_i, P_i)\}) = (0, \mathbf{C})$.
- Suppose that we are given a morphism from an object $C = \{(D_i, n_i, P_i)\}_{1 \le i \le m}$ to another object $C' = \{(D'_j, n'_j, P'_j)\}_{1 \le j \le m'}$ as above, corresponding to a map of pointed finite sets $\alpha : \{1, \ldots, m', *\} \to \{1 \ldots, m, *\}$. Then the induced map $L(\alpha) : (0, \mathbf{C}) \to (0, \mathbf{C})$ is the isomorphism given by multiplication by the complex number $\prod_{\alpha(j) = *} P_j(0)$. Note that this complex number is invertible, since the disk D'_j is forbidden to contain $0 \in \mathbf{C}$ when $\alpha(j) = *$.

Since the functor L factors through the full subcategory of $\mathrm{Vect}_{\mathbf{C}}^{\simeq} \subseteq \mathrm{Vect}_{\mathbf{C}}^{\pm}$ consisting of pairs $(0, W)$ where W has dimension 1, the induced map $[L] : \mathbf{CP}^{\infty} \to \mathbf{Z} \times \mathrm{BU}$ factors through $\mathrm{BU}(1)$. It therefore classifies a complex line bundle on $\mathbf{CP}^{\infty}$ which is given by some power $\mathcal{O}(r) = \mathcal{O}(1)^{\otimes r}$ of the tautological line bundle $\mathcal{O}(1)$. We claim that $r = 1$; this follows (after suitably unwinding the definitions) from the fact that for any open disk $D \subseteq \mathbf{C}$ not containing zero, the value of L on the morphism $\{(D, 1, \lambda)\} \to \{\varnothing\}$ in $\mathcal{C}$ is given by multiplication by $-\lambda$, and the map

$$\mathbf{C}^{\times} \to \mathbf{C}^{\times}$$

$$\lambda \mapsto -\lambda$$

has degree 1 (we can obtain a slightly weaker result without doing any calculation at all; see Remark 7.3.3).

We define a functor $H : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$ as follows:

- On objects, the functor H is given by

$$H(\{(D_i, n_i, P_i)\}) = (\bigoplus \mathrm{Poly}_{<n_i}, \bigoplus \mathrm{Poly}_{<n_i}).$$

- Suppose that we are given a morphism from an object $C = \{(D_i, n_i, P_i)\}_{1 \le i \le m}$ to another object $C' = \{(D'_j, n'_j, P'_j)\}_{1 \le j \le m'}$ as above, corresponding to a map of pointed finite sets $\alpha : \{1, \ldots, m', *\} \to \{1 \ldots, m, *\}$. Then $H(\alpha)$ is given by the triple (ρ, ρ, E_0), where ρ is defined as above and E_0 is the graph of the identity map from $\bigoplus_{\alpha(j) = *} \mathrm{Poly}_{<n'_j}$ to itself.

Let $H' : \mathcal{C} \to \mathrm{Vect}_{\mathbf{C}}^{\pm}$ be the constant functor taking the value 0. There is an evident natural transformation of topological functors $H' \to H$, given on objects by the maps

$$(0, 0, U) : (0, 0) \to (\bigoplus \mathrm{Poly}_{<n_i}, \bigoplus \mathrm{Poly}_{<n_i})$$

where U is the graph of the identity map from $\bigoplus \mathrm{Poly}_{<n_i}$ to itself. From this, we see that $[H] = [H'] = 0$, which proves (v).

We now complete the proof by constructing an exact sequence

$$0 \to H \to G \to L \to 0.$$

To every object $C = (\{D_i, n_i, P_i\})_{1 \le i \le m} \in \mathcal{C}$ with $P = \prod P_i$, we assign the pair of exact sequences

$$0 \to \bigoplus \mathrm{Poly}_{<n_i} \overset{\mathrm{id}}{\to} \bigoplus \mathrm{Poly}_{<n_i} \to 0 \to 0$$

$$0 \to \bigoplus \mathrm{Poly}_{<n_i} \overset{u}{\to} \mathbf{C} \oplus \bigoplus \mathrm{Poly}_{<n_i} \overset{v}{\to} \mathbf{C} \to 0$$

where u fits into a commutative diagram

$$\begin{array}{ccc} \oplus \operatorname{Poly}_{<n_i} & \xrightarrow{\;u\;} & \mathbf{C} \oplus \oplus \operatorname{Poly}_{<n_i} \\ \downarrow{\scriptstyle \psi_C} & & \downarrow{\scriptstyle \psi_C^+} \\ V_P & \xrightarrow{\;z\;} & V_P^+ \end{array}$$

and v is given by the composition

$$\mathbf{C} \oplus \bigoplus_{\operatorname{Poly}_{<n_i}} \overset{\psi_C^+}{\to} \frac{1}{P}\mathbf{C}[z]/z\mathbf{C}[z] \overset{\mu}{\to} \mathbf{C},$$

where μ carries (the residue class of) a rational function $\frac{Q}{P}$ to the complex number $Q(0)$. An elementary calculation shows that this construction satisfies requirements (a) and (b) of Proposition 6.3.4, so that (ii) is satisfied. $\qquad\square$

Remark 7.3.3. The proof of Proposition 7.2.1 requires us to calculate the degree r of a certain line bundle on $\mathbf{C}\mathrm{P}^\infty$. If we omit this calculation, then the rest of the proof shows that $[F] = [\mathcal{O}(r)] - 1 \in K^0(\mathbf{C}\mathrm{P}^\infty)$ for some integer r. The proof of Corollary 7.2.3 then shows that $kr = 1$, where k is the unique integer such that $\mathfrak{B} = k\beta$. From this, we can conclude that $k = r \in \{1, -1\}$, so the map

$$\mathbf{C}\mathrm{P}^\infty \simeq \operatorname{Bar}^{(2)} \mathbf{Z}_{\geq 0}^{\mathrm{ds}} \overset{\rho_0}{\to} \operatorname{Bar}^{(2)}(\amalg_n \operatorname{BU}(n)) \overset{\beta}{\to} \operatorname{BU}$$

corresponds either to $[\mathcal{O}(1)] - 1$ or $1 - [\mathcal{O}(-1)]$ in $K^0(\mathbf{C}\mathrm{P}^\infty)$. These maps are not homotopic to one another, but become so after taking 2-fold loop spaces and composing with the complex J-homomorphism $J_\mathbf{C}$ (since complex conjugation on K-theory carries $[\mathcal{O}(1)]$ to $[\mathcal{O}(-1)]$ and anticommutes with the Bott map). Consequently, to prove Corollary 7.2.5, it is not necessary to verify that $r = 1$.

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