The Cyclotomic Trace in Algebraic K-Theory

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1. Introduction

This paper reviews the relations between algebraic K-theory and topological cyclic homology given by cyclotomic trace. If one, very superficially, views algebraic K-theory as classifying invertible matrices, then the cyclotomic trace records the trace of all powers of matrices. In a more relevant formulation, the topological cyclic homology has the same relationship to Bökstedt's topological Hochschild homology as Connes' cyclic homology has to Hochschild homology, and the cyclotomic trace is a topological cyclic version of the Dennis trace map.

The topological cyclic homology was initially defined in [BHM], and used there to show the K-theory analogue of Novikov's conjecture. It associates to every ring R and every prime p an infinite loop space (or spectrum) $\mathrm{TC}(R,p)$. The cyclotomic trace is a natural transformation in the category of finite loop spaces.

$$\mathrm{Trc} : K(R) \rightarrow \mathrm{TC}(R,p). \tag{1.1}$$

(The construction uses edgewise subdivision, or subdivision of the circle; hence the name "cyclotomic").

The basic theme of this paper is to discuss when one may reasonably expect (1.1) to induce isomorphism on *mod p* or *p-completed* homotopy groups. Actually, it is better to consider a relative situation, and ask for a surjection of rings $R \to S$, when one expects the diagram

$$\begin{array}{ccc} K(R)^{\wedge}_p & \longrightarrow & \mathrm{TC}(R,p)^{\wedge}_p \\ \downarrow & & \downarrow \\ K(S)^{\wedge}_p & \longrightarrow & \mathrm{TC}(S,p)^{\wedge}_p \end{array} \tag{1.2}$$

to be homotopy Cartesian. The decoration $(-)^{\wedge}_p$ indicates p-adic completion in the sense of Bousfield–Kan. For rings which are finitely generated over $\mathbb{Z}$,

$$\mathrm{TC}(R,p)^{\wedge}_p \simeq \mathrm{TC}(R \otimes \mathbb{Z}_p, p)^{\wedge}_p$$

so one does not expect the cyclotomic trace to carry very much number theoretic information. For example, in the basic situation of $R = \mathbb{Z}$, the

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numerators of the Bernoulli numbers enter (at least conjecturally) into the structure of $K_*(\mathbb{Z})$. They do not enter the description of $\mathrm{TC}_*(\mathbb{Z},p)$, and should also not enter into $K_*(\mathbb{Z}_p)$. The basic situation in which one might hope for (1.2) to be of help is when $S = R/I$ is a semi-simple finite dimensional $\mathbb{F}_p$-algebra and $R = \varprojlim R/I^n$. This is discussed in Sections 3 and 4 below. The cyclotomic trace works equally well for Waldhausen's K-theory of spaces (A-theory). In fact, for simply connected spaces, it gives an equivalence of the reduced theories. This, together with some other conjectures and speculations are discussed in Section 5.

With the exception maybe of Section 4, this paper is expository. Details of many of the claims can be found in a series of preprints, [BM], [BHM], [BCCGHM], and [HM].

Added in March 1994. S. Tsalidis, in his 1994 Ph.D thesis from Brown University, seems to have removed the assumption (*) in Theorem 3.11 below, or what amounts to the same thing, to have proved Assertion (*). R. McCarthy has recently proved that the diagram

$$\begin{array}{ccc} K(R)^\wedge_p & \longrightarrow & \mathrm{TC}(R,p)^\wedge_p \\ \downarrow & & \downarrow \\ K(R/I)^\wedge_p & \longrightarrow & \mathrm{TC}(R/I,p)^\wedge_p \end{array}$$

is homotopy Cartesian for any ring R with a square zero ideal I. This implies the continuous version of Conjecture 3.1(i), and shows that $K(A)^\wedge_p \simeq \mathrm{TC}(\mathbb{Z}_p,p)^\wedge_p$ is given in Theorem 3.11.

2. Topological cyclic homology

Let F be a functor from pointed spaces to itself equipped with two extra pieces of structure, a product and a unit,

$$\mu_{X,Y} : F(X) \wedge F(Y) \to F(X \wedge Y),\ \iota_X : X \to F(X)$$

both natural transformations. The functor F is called a functor *with smash product,* for short FSP, cf. [B1], if

$$\begin{aligned} &\mu_{X,Y}(\iota_X \wedge \iota_Y) = \iota_{X\wedge Y} \\ &\mu_{X\wedge Y,Z}(\mu_{X,Y} \wedge id_{F(Z)}) = \mu_{X,Y\wedge Z}(id_{F(X)} \wedge \mu_{Y,Z}) \\ &F(T)\,\mu_{X,Y}(\iota_X \wedge id_{F(Y)}) = \mu_{Y\wedge X}(id_{F(Y)} \wedge \iota_X) \circ T \end{aligned}$$

where T switches factors. Such a functor has a stabilization map, $\sigma =$

$\mu\,(\iota \wedge id)$

$$\sigma_X : S^1 \wedge F\,(X) \to F(S^1 \wedge X)$$

which induces maps $\Omega^i F(S^i X) \to \Omega^{i+1} F(S^{i+1} X)$ with limit $F^s\,(X)$. We assume that $F\,(X)$ is n-connected when X is n-connected, and that the limit system

$$\pi_{n+i} F(S^i X) \to \pi_{n+i+1} F(S^{i+1} X)$$

stabilizes for each n. We notice that the spaces $F(S^i)$ form a unitial associative ring spectrum.

Given a ring R we can form the *cyclic construction.* It is a simplicial abelian group with n-simplices $Z_n\,(R) = R^{\otimes(n+1)}$ and face and degeneracy operators given by

$$\begin{aligned} d_i(r_o \otimes \ldots \otimes r_n) &= r_o \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n &&,\quad i < n \\ &= r_n r_o \otimes \ldots \otimes r_{n-1} &&,\quad i = n\ . \\ s_i(r_o \otimes \ldots \otimes r_n) &= r_o \otimes \ldots \otimes r_i \otimes 1 \otimes \ldots \otimes r_n \end{aligned}$$

Its topological realization is denoted $\mathrm{HH}\,(R)$. It is an Eilenberg–MacLane space whose homotopy groups are the Hochshild homology groups $\mathrm{HH}_i\,(R)$. In [B1] the analogous construction is introduced for FSP's. The space of n-simplices is now

$$Z_n^{\mathrm{top}}\,(F) = \mathrm{ho}\varinjlim \mathrm{Map}\,(S^{k_o} \wedge \ldots \wedge S^{k_n}, F(S^{k_o}) \wedge \ldots \wedge F(S^{k_n})) \tag{2.1}$$

With the limit running over all $(n+1)$-tuples of positive numbers, and with Map denoting the space of based maps. The resulting space $\mathrm{THH}(F)$ is called the *topological Hochschild homology* of F. It is an infinite loop space (alias the zeroth part of a connected spectrum (*denoted* $\mathrm{tHH}(F)$) via the theory of Γ-spaces, cf. [BHM, Section 4]. The *spectrum* homology of $\mathrm{tHH}(F)$ is approximated by a spectral sequence, cf. [B2], [BCCGHM]

$$\mathrm{HH}_p(H_*(F^s)) \underset{p}{\Rightarrow} H_*\,(\mathrm{tHH}\,(F))\ . \tag{2.2}$$

Here F^s is the spectrum of the $F\left(S^i\right)$, and we have used *field coefficients.*

The simplicial space $Z_\bullet^{\mathrm{top}}\,(F)$ is a cyclic space in the sense of Connes, so its realization $\mathrm{THH}\,(F)$ has an action of the circle group S^1. We shall consider the fixed sets $\mathrm{THH}\,(F)^C$ for the finite subgroups $C \subseteq S^1$. To this end it is convenient to use the concept of *edgewise subdivision,* cf. [BMM, Section 1]. Let $Z_\bullet$ be a cyclic space, and C a cyclic group. We form a new simplicial space, with a *simplicial* action of C,

$$\begin{aligned}
&(sd_C Z)_n = Z_{(n+1)|C|-1} \\
&d_i^C = d_i \circ d_{i+(n+1)} \circ \ldots \circ d_{i+(|C|-1)(n+1)}, \qquad 0 \le i \le n \ . \\
&s_i^C = s_{i+(|C|-1)(n+1)} \circ \ldots \circ s_{i+(n+1)} \circ s_i, \qquad 0 \le i \le n
\end{aligned}$$

The topological realization of $sd_C Z_\bullet$ is homeomorphic to that of $Z_\bullet$, and the cyclic structure of $Z_\bullet$, that is, the endomorphism $t_n : Z_n \to Z_n$ of order $n+1$, induces a simplicial C-action on $sd_C Z_\bullet$. Moreover, there is a C-homeomorphism

$$(| Z_\bullet |, C) = (| sd_C Z_\bullet |, C)$$

where on the left hand side the action is the restriction of Connes' S^1-action. In particular the C-fixed sets are homeomorphic.

For the construction $Z_\bullet^{\text{top}}(F)$, the subdivision $sd_C Z_\bullet^{\text{top}}(F)$ has n-simplices

$$sd_C Z_\bullet^{\text{top}}(F) = \ \operatorname{holim}_{\to} \ \operatorname{Map}(S^{k_o R} \wedge \ldots \wedge S^{k_n R}, F(S^{k_o})^{(c)} \wedge \ldots \wedge F(S^{k_n})^{(c)}), \tag{2.3}$$

where $c = | C |$, $R = \mathbb{R}[C]$ is the regular representation, $kR = R \oplus \ldots \oplus R$ and where S^{kR} is the one point compactification of kR with the induced action of C. In (2.3) we use the conjugation action, where the c smash factors in each $F(S^{k_i})^{(c)} = F(S^{k_i}) \wedge \ldots \wedge F(S^{k_i})$ are being permuted by C.

Given C-spaces X and Y and a subgroup $\Gamma \subseteq C$ there is the map

$$\text{fix} : \ \operatorname{Hom}_C(X, Y) \to \ \operatorname{Hom}_{C/\Gamma}(X^\Gamma, Y^\Gamma),$$

which restricts a C-map f to the Γ-fixed set. Applied to (2.3) we get a simplicial map

$$\Phi : [sd_C Z_\bullet^{\text{top}}(F)]^C \to [sd_{C/\Gamma} Z_\bullet^{\text{top}}(F)]^{C/\Gamma}$$

This in turn induces a map Φ from $\text{THH}(F)^C$ to $\text{THH}(F)^{C/\Gamma}$. When we restrict to cyclic p-groups we then have two sets of commuting maps

$$D, \Phi : \ \text{THH}(F)^{C_{p^n}} \to \text{THH}(F)^{C_{p^{n-1}}}$$

The first one is the inclusion of fixed sets (using $C_{p^{n-1}} \subset C_{p^n}$), the second is the map above (using $C_{p^{n-1}}$ is also a quotient of C_{p^n}). Since Φ and D

commute one has self-maps

$$\Phi : \underset{D}{\operatorname{holim}} \operatorname{THH}(F)^{C_{p^n}} \to \underset{D}{\operatorname{holim}} \operatorname{THH}(F)^{C_{p^n}}$$
$$D : \underset{\Phi}{\operatorname{holim}} \operatorname{THH}(F)^{C_{p^n}} \to \underset{\Phi}{\operatorname{holim}} \operatorname{THH}(F)^{C_{p^n}}$$

and one can form the homotopy fixed sets, i.e., the homotopy fibers of Φ-id, resp. D-id. (In order to get connected spectra we consider Φ-id and D-id to be maps into the connected covers).

The C_{p^n} action on THH(F) lifts to an action of the spectrum, cf. [BHM, Section 4], and tHH(F) becomes an equivariant C_{p^n} spectrum. In particular each fixed set $\operatorname{THH}(F)^{C_{p^n}}$ is an ordinary spectrum and the maps D and Φ are stable.

Definition 2.5. ([BHM]) The topological cyclic homology at p of the FSP F is the functor

$$\operatorname{TC}(F,p) = [\underset{D}{\operatorname{holim}} \operatorname{THH}(F)^{C_{p^n}}]^{h\Phi} \simeq [\underset{\Phi}{\operatorname{holim}} \operatorname{THH}(F)^{C_{p^n}}]^{hD}.$$

Let me distill the relevant abstract properties of THH (F) by introducing the concept of a *cyclotomic spectrum.* First recall from [LMS , p.12] that an S^1-prespectrum T associates to each S^1-representation (in a "complete universe") a space T_V, and that there are S^1-equivariant structure maps

$$\sigma : S^V \wedge T_W \to T_{V\oplus W}$$

which satisfy the obvious conditions.

Definition 2.6 A p-cyclotomic spectrum consists of an S^1-prespectrum together with S^1-equivariant maps

$$\varphi_V : T_V^{C_p} \to T_{V^{C_p}}$$

for each prime order subgroup C_p of S^1 such that

(i) The adjoins $\hat{\sigma} : T_V \to \Omega^V T_{W\oplus V}$ are equivariant homotopy equivalences for all *finite* subgroups of S^1

(ii) The induced maps

$$\varinjlim \Omega^{V^{C_p}} \varphi_V : \varinjlim \Omega^{V^{C_p}} T_V^{C_p} \to \varinjlim \Omega^{V^{C_p}} T_{V^{C_p}}$$

are equivariant homotopy equivalences for all finite subgroups of S^1. Here the limit runs over all representations of S^1.

In (2.6) we have identified S^1/C_p with S^1 (via the obvious isomorphism). This identifies the given S^1/C_p action on $T_V^{C_p}$ with an S^1-action, and it is with respect to this action, φ_V is assumed to be S^1-equivariant. In (2.6), (i) and (2.6), (ii) we only require equivariant homotopy equivalence with respect to finite cyclic subgroups. This might seem a little odd, but the explanation is that the S^1-fixed set of THH(F) is contractible. Let me finally remark that there is assumed in (2.6) some evident relation between the φ and $\hat{\sigma}$ which I leave for the reader to explain.

Proposition 2.7 ([BM]) THH(F) *is a p-cyclotomic spectrum.*

There are several possible concrete models for the deloops $\mathrm{THH}_V(F)$. The simplest one is the topological realization of the cyclic space whose n-simplices are given by

$$\operatorname{holim}_{\to} \mathrm{Map}\,(S^{k_o} \wedge \ldots \wedge S^{k_n}, S^V \wedge F(S^{k_o}) \wedge \ldots \wedge F(S^{k_n}))$$

The S^1-action on $\mathrm{THH}_V(F)$ is the composition of the S^1-action associated to the cyclic structure and the action on S^V. For each finite subgroup $C \subset S^1$ this turns out to be equivalent to the deloop of THH(F) associated to the C-equivariant Γ-space structure of THH(F) defined in [BHM, Section 4]. The mapping

$$\varphi_V : \mathrm{THH}_V(F)^{C_p} \to \mathrm{THH}_{V^{C_p}}(F)$$

can be seen explicitly upon using the subdivided model with n-simplices

$$\operatorname{holim}_{\to} \mathrm{Map}\,(S^{k_o R} \wedge \ldots \wedge S^{k_n R}, S^V \wedge F(S^{k_o})^{(p)} \wedge \ldots \wedge F(S^{k_n})^{(c)})$$

with $R = \mathbb{R}C_p$ to be : $\varphi_V(f) =$ fix (f), the restriction to the C_p fixed set. The S^1-equivariance property follows from [BHM, Lemma 1.10].

Given any p-cyclotomic spectrum T we have a stable map

$$\Phi : T^{C_{p^n}} \to T^{C_{p^{n-1}}}$$

whose homotopy fiber is the homotopy orbit spectrum $T_{hC_{p^n}} = \mathbb{H}_\bullet(C_{p^n}, T)$, cf. (4.1) below. Indeed Φ is the composition

$$[\lim_{\to} \Omega^V T_V]^{C_{p^n}} \xrightarrow{\text{fix}} [\lim_{\to} \Omega^{V^{C_p}} T_V^{C_p}]^{C_{p^{n-1}}} \xrightarrow{\varphi^{C_{p^{n-1}}}} [\lim_{\to} \Omega^{V^{C_p}} T_{V^{C_p}}]^{C_{p^{n-1}}}$$

The first map has homotopy fiber

$$\lim_{\to} \Omega^V (EC_{p^n +} \wedge T_V)^{C_{p^n}} \simeq EC_{p^n +} \wedge_{C_{p^n}} T = \mathbb{H}_\bullet(C_{p^n}, T)$$

and the second map is a homotopy equivalence, so Φ has the stated homotopy fiber, cf. [BM] for more details.

Example 2.8 For the identity FSP, the topological Hochschild homology can be identified with the spheres pectrum. The identification is C_{p^n}-equivariant for every n, so that $(\text{THH}(\text{Id}), C_{p^n}) \simeq (Q_{C_{p^n}}(S^o), C_{p^n})$ with $Q_{C_{p^n}}(S^o) = \varinjlim \Omega^V S^V$ over all representations ofC_{p^n}. It follows from [tD] that

$$\text{THH (Id)}^{C_{p^n}} \simeq \prod_{i=o}^{n} Q_+(BC_{p^i}) \quad , Q_+(B) = \Omega^\infty S^\infty(B\coprod\{+\})$$

The maps D and Φ are correspondingly given by

$$\begin{aligned} D(x_o, \ldots, x_n) &= (x_o + t(x_1), t(x_2), \ldots, t(x_n)) \\ \Phi(x_o, \ldots, x_n) &= (x_o, \ldots, x_{n-1}) \end{aligned}$$

where $t : Q_+(BC_{p^i}) \to Q_+(BC_{p^{i-1}})$ is the transfer mapping associated to the cover $BC_{p^{i-1}} \to BC_{p^i}$. In this case one can evaluate the p-completion of TC (Id, p) to be

$$\text{TC}(\text{Id}, p)_p^\wedge \simeq [QS^o \times \text{ hofib}(Q(\Sigma_+ \mathbb{C}P^\infty) \xrightarrow{\hat{t}} QS^o)]_p^\wedge$$

with $\hat{t}$ being the S^1-transfer, cf. [BHM, Section 5].

Let $L_{K\mathbb{F}_P}(-)$ denote the localization functor in the sense of Bousfield w.r.t. *mod p* topological K-theory. It follows from the above formula for TC $(\text{Id}, p)_p^\wedge$ that $L_{K\mathbb{F}_p}(\text{ TC (Id}, p))$ surjects onto $(J \times BJ \times SU)_p^\wedge$, but there is a large kernel. Here SU is the infinite special unitary group, J is the homotopy fiber of the stable Adams operation $\Psi^g - \text{id} : K_p^\wedge \to K_p^\wedge$ and K represents periodic topological K-theory. The (-1) −connected cover of J is $K(\mathbb{F}_g)_p^\wedge$ by [Q], where g is a prime which generates $(\mathbb{Z}/p^2)^\times$.

Given an FSP F we can form the corresponding matrix FSP

$$M_m(F)(X) = \text{ Map}([m], [m] \wedge F(X))$$

where $[m] = \{0, \ldots, m\}$ with 0 as base point. Let $GL_m(F)$ be the invertible components of $\varinjlim \Omega^k M_m(F)(S^k)$. It is a group-like topological monoid, and one can construct its K-theory:

$$K(F) = \Omega B(\coprod_m BGL_m(F)) \simeq BGL(F)^+$$

There are two cases of special interest. First, if $F(X) = X \wedge \Gamma_+$ with Γ a group-like topological monoid ($\pi_0\Gamma$ is a group) then $K(F) = A(B\Gamma)$, Waldhausen's A-theory of $B\Gamma$ (in the version where $A_0 = \mathbb{Z}$). Since $X \simeq B\Omega X$ for every connected X this defines $A(X)$ in general. Second, for a ring R consider $F_R(X) = | R\Delta_\bullet(X)/R\Delta_\bullet(*) |$ where $\Delta_\bullet(X) = \mathrm{Map}(\Delta^\bullet, X)$ is the singular simplicial set, and the bars indicate topological realization. This $F_R(X)$ is a generalized Eilenberg–MacLane complex with $\pi_i F_R(X) = \tilde{H}_i(X, R)$ and $K(F_R) \simeq K(R)$, the Quillen K-theory of R. Indeed, evaluating on homology gives a homotopy equivalence $GL_m(F_R) \to GL_m(R)$, and in turn an equivalence from $BGL_m(F_R)$ to $BGL_m(R)$, etc.

The topological Dennis trace is a map (of spectra)

$$\mathrm{Tr} : K(F) \to \mathrm{THH}(F) \tag{2.9}$$

We sketch the construction and refer to [BHM] for details. First let us recall the constructions $N_\bullet(\Gamma)$ and $N_\bullet^{cy}(\Gamma)$ for a monoid Γ. These are simplicial spaces with

$$N_n(\Gamma) = \Gamma^n, N_n^{cy}(\Gamma) = \Gamma^{n+1}$$

and face operators

$$d_i(\gamma_1, .., \gamma_n) = \begin{cases} (\gamma_2, .., \gamma_n) & , i = 0 \\ (\gamma_1, .., \gamma_i\gamma_{i+1}, .., \gamma_n) & , 0 < i < n \\ (\gamma_1, .., \gamma_{n-1}) & , i = n \end{cases}$$

$$d_i^{cy}(\gamma_0, .., \gamma_n) = \begin{cases} (\gamma_0, .., \gamma_i\gamma_{i+1}, .., \gamma_n) & , 0 \le i < n \\ (\gamma_n\gamma_0, .., \gamma_{n-1}) & , i = n \end{cases}$$

and degeneracy operators by inserting an extra $1 \in \Gamma$. The topological realizations are $B\Gamma = | N_\bullet(\Gamma) |$ and $B^{cy}(\Gamma) = | N_\bullet^{cy}(\Gamma) |$. If Γ is a group then we have the inclusion

$$I_\bullet : N_\bullet(\Gamma) \to N_\bullet^{cy}(\Gamma), I(\gamma_1, \ldots, \gamma_n) = ((\textstyle\prod \gamma_i)^{-1}, \gamma_1, \ldots, \gamma_n).$$

Its realization I is homotopic to the inclusion of $B\Gamma$ into the free loop space $\Lambda B\Gamma$ as the constant loops. One can also define $I_\bullet$ for group-like monoids; again it corresponds to the inclusion $B\Gamma \subseteq \Lambda B\Gamma$. Consider now for $\Gamma = GL_m(F)$ the composition

$$N_\bullet(GL_m(F)) \xrightarrow{I_\bullet} N_\bullet^{cy}(GL_m(F)) \xrightarrow{S_\bullet} \mathrm{THH}_\bullet(M_m(F)) \tag{2.10}$$

with $S(f_o, \ldots, f_n) = f_o \wedge \ldots \wedge f_n, f_i : S^{k_i} \to M_m(F)(S^{k_i})$.

Lemma 2.11 (Morita invariance) *For each FSP*

$$\Omega B(\coprod_m \mathrm{THH}(M_m(F))) \simeq \mathrm{THH}(F) \times \mathbb{Z}.$$

The realization of the composition (2.10) together with (2.11) produces the Dennis trace map (2.9). The *cyclotomic trace*

$$\mathrm{Trc} : K(F) \to \mathrm{TC}(F,p) \tag{2.12}$$

is a variant of the above construction. As mentioned before, I is homotopic to the inclusion of $BGL_m(F)$ into its free loop space. In order to get into the fixed set, one can consider the composition

$$BGL_m(F) \xrightarrow{I} \Lambda BGL_m(F) \xrightarrow{\Delta_{p^n}} (\Lambda BGL_m(F))^{C_{p^n}}$$

with $\Delta_{p^n}(\lambda)(z) = \lambda\left(z^{p^n}\right)$ for a free loop λ. On the simplicial level this corresponds to

$$\Delta_{p^n,\bullet} : N^{cy}_\bullet(\Gamma) \to [sd_{C_{p^n}} N^{cy}_\bullet(\Gamma)]^{C_{p^n}}$$

with $\Delta_{p^n,k}(\underline{\gamma}) = (\underline{\gamma}, \ldots, \underline{\gamma}), p^n$ repetitions. We can now insert $\Delta_{p^n,\bullet}$ between $S_\bullet$ and $I_\bullet$ in (2.10) to get a mapping

$$N_\bullet(GL_m(F)) \to [sd_{C_{p^n}} \mathrm{THH}_\bullet(M_m(F))]^{C_{p^m}}.$$

There is an equivariant version of Morita invariance (2.9) so we obtain by the above procedure a map from $K(F)$ to $\mathrm{THH}(F)^{C_{p^n}}$ for each n. A closer look at the situation, cf. [BHM, Section 2], produces a canonical homotopy between the two ways around in the diagram

$$\begin{array}{ccc} K(F) & \to & \mathrm{THH}(F)^{C_{p^n}} \\ & \searrow & \downarrow D \\ & & \mathrm{THH}(F)^{C_{p^{n-1}}} \end{array}$$

and in turn a well-defined mapping Trc, into the homotopy fiber of Φ-id. We point out that also $\mathrm{TC}(-,p)$ is a Morita invariant:

$$\mathrm{TC}(M_m(F),p) \simeq \mathrm{TC}(F,p). \tag{2.13}$$

There is a version of the TC-functor, and of the cyclotomic trace, where one does not in advance single out a specific prime. Instead of forming limits over prime power cyclic groups one forms limits over all cyclic groups.

However, this functor, TC (R), is no stronger than the set of functors TC (R,p); this is analogous to the fact that the finite completion of the integers is just the product of the p-adic integers, cf. [HM].

3. On the K-theory of complete local rings — a conjecture

In this section we consider rings A with an ideal I such that

(i) $A = \varprojlim A/I^n$

(ii) A/I is a finite semi-simple $\mathbb{F}_p$-algebra (3.0)

(iii) A finitely generated as $\mathbb{Z}_p$-module

This class of p-complete semi local rings includes the ring of integers in local fields with positive residue characteristic but also p-adic group rings of finite groups. A celebrated result of Gabber and Suslin, cf. [G], [S1] implies that the projection $K(A) \to K(A/I)$ induces an equivalence of ℓ-adic completions when $\ell \neq p$. We are interested in completions at p. Define

$$K^c(A) = \operatorname{holim}_{\leftarrow} K(A/I^n)$$

This is the continuous K-theory of A. Let us write TC (A,p) for the topological cyclic homology of the FSP

$$F_A(X) = | A\Delta_\bullet(X)/A\Delta_\bullet(*) |$$

We can define the continuous cyclic homology by

$$\mathrm{TC}^c(A,p) = \operatorname{holim}_{\leftarrow} \mathrm{TC}(A/I^n, p),$$

but this is probably not a new functor for the rings in (3.0). At least for discrete valuation rings with finite residue fields of characteristic p one has from [HM]:

$$\mathrm{TC}\,(A,p)_p^\wedge \xrightarrow{\simeq} \mathrm{TC}^c(A,p)_p^\wedge\,.$$

Before I state the main conjecture, it is in order to remark that working with homotopy groups with coefficients and ordinary homotopy groups of completions amount to the same thing. There is an exact sequence, valid for any spectrum K,

$$0 \to \varprojlim{}^{(1)} \pi_{n+1}(K;\mathbb{Z}/p^n) \to \pi_n(K_p^\wedge) \to \varprojlim \pi_{n+1}(K;\mathbb{Z}/p^n) \to 0$$

cf. [BK]. Moreover, if K has finite type then $\pi_{n+1}(K;\mathbb{Z}/p^n)$ is finite, hence compact, and the $\lim{}^{(1)}$-term vanishes.

Conjecture 3.1 (i) *The cyclotomic trace induces a homotopy equivalence*

$$\mathrm{Trc} : K(A)_p^\wedge \xrightarrow{\simeq} \mathrm{TC}(A,p)_p^\wedge$$

for the rings in (3.0)

(ii) *The natural map* $K(A)_p^\wedge \xrightarrow{\simeq} K^c(A)_p^\wedge$ *is ahomotopy equivalence for the rings in (3.0).*

We note from [P] that Conjecture 3.1 (ii) is true when A is the ring of integers in a local number field. In this case then, 3.1 (i) follows from its continuous version which might be easier to prove, since it (via Theorem 3.8 below) reduces it to the relative situation (1.2) with $S = R/I$ and $I^2 = 0$.

The topological cyclic homology is not easy to calculate and there is at the time of writing only limited information available about the right hand side of (3.1) (i). But something is known, and it would appear at any rate that $\mathrm{TC}(A,p)$ lends itself to analysis by algebraic topological methods more readily than does $K(A)$. We give examples of this later in this section.

The steps involved in the calculation of TC (F,p) are first the evaluation of $\mathrm{THH}(F)$, second the fixed sets THH $(F)^{C_{p^n}}$ and third the homotopy fiber of Φ-id. In the linear case at hand, $F = F_R$, so far the only way to get at the fixed sets THH $(R)^{C_{p^n}}$ is through the homotopy fixed sets:

$$\mathrm{THH}\ (R)^{hC_{p^n}} = \mathrm{Map}_{C_{p^n}}\left(EC_{p^n+}, \mathrm{THH}\ (R)\right).$$

Here one has a spectral sequence to facilitate the calculation of homotopy groups, namely the spectral sequence associated to skeleton filtration of the free contractible C_{p^n}-space EC_{p^n}. We have the inclusion of spectra

$$\Gamma : \mathrm{THH}\ (F)^{C_{p^n}} \to \mathrm{THH}\ (F)^{hC_{p^n}}\ .$$

The homotopy fixed sets usually have non-zero homotopy groups in negative degrees whereas the actual fixed set does not; THH $(F)^{C_{p^n}}$ is (-1)-connected by definition. Thus it is too much to ask for Γ to be a homotopy equivalence in general, but we can reasonably pose

Problem 3.2 *For which FSPs does* Γ : THH $(F)^{C_{p^n}} \to$ THH $(F)^{hC_{p^n}}$ *induce a p-adic homotopy equivalence onto the* (-1)*-connected cover of the target?*

One would hope that (3.2) has a positive answer for the rings which appear in (3.1). For the identity FSP, THH(Id) is equal to the (equivariant) sphere spectrum, and (3.2) is satisfied according to the affirmed Sullivan conjecture (the homotopy fixed set is even (-1)-connected in this case).

Remark 3.3 It would be tempting to initially bypass (3.2) by replacing THH $(F)^{C_{p^n}}$ by THH $(F)^{hC_{p^n}}$ in the definition of TC (F,p). The problem with this is that no one so far has been able to extend the map Φ : THH $(F)^{C_{p^n}} \to$ THH $(F)^{C_{p^{n-1}}}$ to a corresponding map between homotopy fixed sets.

Let me next recall the basic calculational results which have been proved for the functors THH (R) and TC (R,p).

Theorem 3.4 ([B2]) *THH$(\mathbb{F}_p)$ and THH$(\mathbb{Z}_p)$ are generalized Eilenberg–MacLane spectra with homotopy groups*

(*a*) $\mathrm{THH}_{2i}(\mathbb{F}_p) = \mathbb{Z}/p$ *for* $i \geq 0$ *and* $\mathrm{THH}_{2i-1}(\mathbb{F}_p) = 0$

(*b*) $\mathrm{THH}_{2i-1}(\mathbb{Z}_p) = \mathbb{Z}/i \otimes \mathbb{Z}_p$, $\mathrm{THH}_0(\mathbb{Z}_p) = \mathbb{Z}$ *and* $\mathrm{THH}_{2i}(\mathbb{Z}_p) = 0$

for $i > 0$

The proof of (3.4) given in [B2] uses the spectral sequence (2.2). Quite recently Pirashvili and Waldhausen have shown that for discrete rings the topological Hochschild homology is equal to MacLane homology as defined in [ML], i.e.

$$\mathrm{THH}_i(R) \simeq H_i^{\mathrm{ML}}(R)\ . \tag{3.5}$$

This result might help to calculate $\mathrm{THH}_i\,(R)$ for the rings in (3.1). For any discrete monoid G ,

$$\mathrm{THH}\,(R[G]) \simeq \mathrm{THH}\ (R) \wedge \mid N_\bullet^{cy}\,(G) \mid_+\ . \tag{3.6}$$

When G is a group, $\mid N_\bullet^{cy}(G) \mid \simeq \Lambda BG$ by a theorem of Goodwillie. The equivalence in (3.6) is valid in the category of C_{p^n}-equivariant spectra (for each n). In particular $\mathrm{THH}_i\,(R\,[G])$ is the i'th THH (R)-homology group of $\mid N_\bullet^{cy}\,(G) \mid$, i.e.

$$\mathrm{THH}\ (R\,[G])^{C_{p^n}} \simeq \operatorname{holim}_{\to} \mathrm{Map}_{C_{p^n}}(S^V, \mathrm{THH}_V\,(R) \wedge \mid N_\bullet^{cy}\,(G) \mid_+)\,. \tag{3.7}$$

In the next section we prove:

Theorem 3.8 *Let HA denote the Eilenberg–Maclane spectrum associated to the ring A. Then we have $TC(\mathbb{F}_p,p)_p^\wedge \simeq H\mathbb{Z}_p$.*

Let us note that this result is in agreement with (3.1) since $K(\mathbb{F}_p)_p^\wedge = H\mathbb{Z}_p$ by [Q].

Theorem 3.9 ([H1]) *After p-completion, there are homotopy equivalences*

$$(i) \qquad TC(\mathbb{F}_p[v, v^{-1}], p) \simeq H\mathbb{Z}_p \vee \Sigma H\mathbb{Z}_p$$

$$(ii) \qquad \mathrm{TC}^c(\mathbb{F}_p[[v]], p) \simeq H\mathbb{Z}_p \vee \Sigma H(\mathbb{F}_p[[v]]^\times)$$

It follows from (3.8) and (3.9), (i) that the topological cyclic homology satisfies the "fundamental theorem" for the ring $\mathbb{F}_p$. This is somewhat atypical, see the discussion following Theorem 5.1 below. The quotient field of $\mathbb{F}_p\,[[v]]$ is a local field, so (3.1) and (3.9) (ii) together predict that

$$K_i^c(\mathbb{F}_p\,[[v]])_p^\wedge = 0 \text{ for } i \geq 2\ .$$

This is indeed known to be the case for $i = 2$ by [S2], Theorem 1.10, where it was even proved that $K_i\,(\mathbb{F}_p\,[[v]])_p^\wedge = 0$ for $i = 2$.

Let $\mathbb{F}_p[\varepsilon]$ denote the ring of dual numbers over $\mathbb{F}_p$ or in other words the exterior algebra in one generator. The algebraic K-groups of $\mathbb{F}_p[\varepsilon]$ were examined in [EF] where $K_i(\mathbb{F}_p[\varepsilon])$ was calculated for $i \leq 4$ and $p \geq 5$. In order to describe $\mathrm{TC}\,(\mathbb{F}_p[\varepsilon], p)$ let us introduce the following notation. If k is prime to p , let $s\,(k, n)$ be the number determined by the inequalities

$$kp^{s(k,n)-1} \leq n < kp^{s(k,n)}\,.$$

If we write $\mathrm{TC}_n\,(R, p) = \pi_n(\mathrm{TC}\ (R, p)_p^\wedge)$ then we have:

Theorem 3.10. ([HM]) *For the dual numbers* $\mathbb{F}_p\,[\epsilon]$ *the only non-zero homotopy groups of* $\mathrm{TC}\,(\mathbb{F}_p\,[\epsilon]\,, p)$ *are*

$$(i)\ \ p > 2: \qquad \mathrm{TC}_{2n-1}\,(\mathbb{F}_p\,[\epsilon]) = \bigoplus \left\{\mathbb{Z}/p^{s(k,n)} \mid (k, 2p) = 1, \quad 1 \leq k \leq n\right\}$$

$$(ii)\ \ p = 2: \qquad \mathrm{TC}_{2n-1}\,(\mathbb{F}_2\,[\epsilon]) = \mathbb{F}_2^{\oplus n}$$

together with $\mathrm{TC}_0\,(\mathbb{F}_p\,[\epsilon]) = \mathbb{Z}_p$ *in both cases.*

M. Bökstedt has pointed out the following attractive formulation of (3.10). Let $\widehat{W}_n\,(\mathbb{F}_p)$ denote the big Witt vectors in $\mathbb{F}_p$ of length n , i.e.

$$[\mathbb{F}_p[x]/(x^n)]^\times = \mathbb{F}_p^\times \oplus \widehat{W}_n(\mathbb{F}_p)$$

and let α be the endomorphism of $\widehat{W}_n(\mathbb{F}_p)$ induced from the ring endomorphism of $\mathbb{F}_p[[x]]$ which sends x to $-x$. The subgroup generated by $1 - x^k$ in $\widehat{W}_n(\mathbb{F}_p)$ is cyclic of order $p^{s(k,n)}$, and thus

$$\mathrm{TC}_n(\mathbb{F}_p[\varepsilon], p) \simeq \widehat{W}_n(\mathbb{F}_p)^{<-1>},$$

the (-1) -eigenspace of $\alpha : \widehat{W}_n(\mathbb{F}_p) \to \widehat{W}_n(\mathbb{F}_p)$.

Since $\mathrm{TC}(\mathbb{F}_p[\varepsilon], p)$ is a module spectrum over $\mathrm{TC}(\mathbb{F}_p, p)$ and the latter is an Eilenberg–MacLane spectrum, so is $\mathrm{TC}(\mathbb{F}_p[\varepsilon], p)$. The groups $\mathrm{TC}_n(\mathbb{F}_p[\varepsilon], p)$ listed in (3.10) agree with the Evens–Friedlander calculations of $K_n(\mathbb{F}_p[\varepsilon]; \mathbb{Z}_p)$ for $n \leq 4$, supporting Conjecture 3.1. They evaluated in [EF] the spectral sequence of the fibration (in low dimensions)

$$BM_k(\mathbb{F}_p) \to BGL_k(\mathbb{F}_p[\varepsilon]) \to BGL_k(\mathbb{F}_p), \quad k \to \infty$$

with E^2 -term

$$H_*(GL_k(\mathbb{F}_p); E\{M_k(\mathbb{F}_p)\} \otimes P\{M_k(\mathbb{F}_p)\}), \quad k \to \infty$$

and converging to $H_*(K(\mathbb{F}_p[\varepsilon]); \mathbb{F}_p)$. Here $GL_k(\mathbb{F}_p)$ acts on $M_k(\mathbb{F}_p)$ by conjugation. In low dimensions the homology determines the homotopy, but in general the connection between homology and homotopy might be less tight. In this connection it is interesting to observe that

$$H_n(GL(R); M(R)) \simeq \bigoplus_{i+j=n} H_i(GL(R); \pi_j K^s(R))$$

where $K^s(R)$ is the *stable* K-theory of R and the action of $GL(R)$ on $\pi_* K^s(R)$ is trivial, cf. [K]. It was conjectured by Waldhausen and proved in [DMc] that $K^s(R) \simeq \mathrm{THH}(R)$. This is in agreement with the fact from [H] that $\mathrm{TC}^s(R, p) \simeq \mathrm{THH}(R)$, and implies for example that

$$H_n(GL(\mathbb{F}_p); M(\mathbb{F}_p)) = \mathbb{F}_p[\iota_2] .$$

Let us turn to the simplest characteristic zero case of (3.1). The calculation of $\mathrm{TC}(\mathbb{Z}_p, p)$ is far more complicated than that of (3.8) or (3.9). For p odd, the homotopy type of $\mathrm{TC}(\mathbb{Z}_p, p)^\wedge_p$ is determined in [BM], modulo (for the time being at least) a certain (standard) assertion.

Let us recall that the sphere spectrum QS^0 fibers over the connected image of J spectrum $\mathrm{Im} J$ with fiber Coker J. The completion $\mathrm{Im}\, J^\wedge_p$ is equivalent to $K(\mathbb{F}_\ell)^\wedge_p$ when ℓ is a prime which generates $(\mathbb{Z}/p^2)^\times$. Let

$$\iota : QS^0 \to K(\mathbb{Z}_p)$$

be the unit. One knows from [Mi] that the restriction of ι to Coker$J$ is null-homotopy as a map of spaces, but *not* that this is the case at the spectrum level, although it is strongly expected. Consider the diagram:

$$
\begin{array}{ccc}
QS^o & \overset{\mathrm{Trc}\circ\iota}{\rightarrow} & \mathrm{TC}\,(\mathbb{Z}_p, p) \\
\downarrow \hat{L} & & \downarrow \mathrm{proj} \\
T(\mathbb{Z}_p)^{hC_{p^n}} & \overset{\Gamma}{\leftarrow} & \mathrm{THH}\,(\mathbb{Z}_p)^{C_{p^n}}
\end{array}
$$

where $\hat{L}$ is the composition

$$
QS^o \xrightarrow{\mathrm{incl}} (QS^o)^{hC_{p^n}} \simeq \mathrm{THH}\ (\mathrm{id})^{hC_{p^n}} \xrightarrow{L^{hC_{p^n}}} \mathrm{THH}\,(\mathbb{Z}_p)^{hC_{p^n}}
$$

induced from the linearization $L : \mathrm{THH}\ (\mathrm{Id}) \to \mathrm{THH}\ (\mathbb{Z}_p)$.

Assertion (*) *The restriction of* $\hat{L}$ *to* Coker$J$ *is null-homotopic as a map of spectra.*

Theorem 3.11 ([BM]) *Assuming assertion* (*),

$$
\mathrm{TC}\,(\mathbb{Z}_p, p)^\wedge_p \simeq SU^\wedge_p \times \mathrm{Im}\, J^\wedge_p \times B\ \mathrm{Im}\, J^\wedge_p
$$

for odd primes p.

The role of the assertion in the proof of (3.11) is to supply the diagonal arrow in the diagram

$$
\begin{array}{ccc}
\mathrm{THH}\ (\mathrm{Id})^{hC_{p^n}} & \rightarrow & \mathrm{THH}\ (\mathbb{Z}_p)\, hC_{p^n} \\
\downarrow & \nearrow & \\
\mathrm{THH}\ (\mathrm{Im}\, J)^{hC_{p^n}} & &
\end{array}
\qquad (**)
$$

The spectral sequence

$$
H^*(BC_{p^n}; \pi_*(\mathrm{Im}\, J; \mathbb{F}_p)) \Rightarrow \pi_*(\mathrm{Im}\, J^{hC_{p^n}}; \mathbb{F}_p)
$$

can be completely worked out because of our extensive knowledge of K-theory, and the dotted arrow is then used to give the basic differentials in the corresponding spectral sequence for $\pi_*(\mathrm{THH}\ (\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$. Of course, there might be other ways of getting at these differentials. For example, for small n ($n \leq 3$) it suffices to use the horizontal arrow in (**). In general however, not enough seems to be known about the interaction of π_* Coker J and π_* Im J in the spectral sequence $H^*(BC_{p^n};\pi_*(QS^o;\mathbb{F}_p) \Rightarrow \pi_*((QS^o)^{hC_{p^n}}, \mathbb{F}_p)$ to allow the conclusions we want, cf. [BM]. Let me finally note that since K (Coker J) $= 0$, (3.11) does in turn imply assertion (*).

It is natural to compare (3.11) with the étale K-theory of [DF]. In the case at hand,

$$K^{et}(E) = K(E^c)^{hG(E^c/E)}$$

where E^c is the separable closure of E and $G(E^c/E)$ denotes the Galois group. Since the actual fixed set of $K(E^c)$ is $K(E)$ there is a natural inclusion

$$\Gamma : K(E) \to K^{et}(E) \ .$$

For $E = \mathbb{Q}_p$, Dwyer and Friedlander are in the process of determining $K^{et}(E)^\wedge_p$; the answer which evolves is the same as the above for $\mathrm{TC}(\mathbb{Z}_p,p)^\wedge_p$. The Lichtenbaum–Quillen conjecture (in one formulation) asserts that $\Gamma^\wedge_p$ is a homotopy equivalence. So for $E = \mathbb{Q}_p$, Conjecture 3.1 is equivalent to the LQ conjecture.

In this connection it would be of considerable interest to evaluate TC $(\mathbb{Z}_2, 2)$; many of the arguments in [BM] break down for $p = 2$, and one does not expect precisely the answer above. One reason that $p = 2$ is more difficult than the case of odd primes p is that the Adams periodicity map at 2 is a more elaborate construction in homotopy theory than it is at p. It would of course also be very interesting to evaluate TC $(A,p)^\wedge_p$ for integers in general local number fields. One might conjecture that it is always the connected cover of its localization with respect to topological K-theory, when A is torsion free.

Let us finally point out that (3.8) allows us to formulate (3.1) at least for $A = \mathbb{Z}_p$ as a conjectural homotopy Cartesian diagram

$$\begin{array}{ccc} K(\mathbb{Z}_p)^\wedge_p & \to & \mathrm{TC}(\mathbb{Z}_p,p)^\wedge_p \\ \downarrow & & \downarrow \\ K(\mathbb{F}_p)^\wedge_p & \to & \mathrm{TC}(\mathbb{F}_p,p)^\wedge_p \end{array} \tag{3.12}$$

There is a similar formulation in general since TC $(k,p)^\wedge_p \simeq H\mathbb{Z}_p$ for general finite fields of characteristic p, cf. [HM].

4. Topological cyclic homology of $\mathbb{F}_p$

This section outlines as an example the calculation of TC $(\mathbb{F}_p, p)$. The general procedure is the same as the one used in [BM] for determining TC $(\mathbb{Z}_p, p)$, but the details are simpler. The reader is referred to [H1] and [HM] for the proofs of Theorem (3.9) and (3.10) and for a more detailed account of the present outline. First we recall some definitions.

Let T be a G-equivariant spectrum with G a finite group. Following [GM] one defines spectra

$$\begin{aligned} \mathbb{H}_\bullet(G,T) &= (\operatorname{Res} T) \wedge_G EG_+ \\ \mathbb{H}_\bullet(G,T) &= \operatorname{Map}_G(EG_+, \operatorname{Res} T) \\ \hat{\mathbb{H}}(G,T) &= [\tilde{EG} \wedge \operatorname{Map}(EG_+, T)] \end{aligned} \tag{4.1}$$

Here $\operatorname{Res} T$ denotes the weak G-spectrum associated to T, i.e. $\operatorname{Res} T = \{B^n T\}_{n\in\mathbb{N}}$ with the given action of G. The space $\tilde{EG}$ is the (unreduced) suspension of EG, and

$$[\tilde{EG} \wedge \operatorname{Map}(EG,T)]^G = \operatorname{holim}_{\rightarrow} \operatorname{Map}_G(S^V, \tilde{EG} \wedge \operatorname{Map}(EG_+, B^V T)$$

with the limit over all $\mathbb{R}G$-modules. The functor $\mathbb{H}_\bullet(G,T)$ is called the *homotopy orbit* and often denoted T_{hG}; the functor $\mathbb{H}^\bullet(C,T)$ is the homotopy fixed set T^{hG}. In the rest of the paper we use the notation T_{hG} and T^{hG}. The basic tool in our calculations is the following diagram of cofibrations:

$$\begin{array}{ccccc} \text{THH}(F)_{hC_{p^n}} & \xrightarrow{N^h} & \text{THH}(F)^{hC_{p^n}} & \xrightarrow{\Psi} & \hat{\mathbb{H}}(C_{p^n}, \text{THH}(F)) \\ \uparrow \text{id} & & \uparrow \Gamma & & \uparrow \hat{\Gamma} \\ \text{THH}(F)_{hC_{p^n}} & \xrightarrow{N} & \text{THH}(F)^{C_{p^n}} & \xrightarrow{\Phi} & \text{THH}(F)^{C_{p^n}} \end{array} \tag{4.2}$$

(cf. [BM], Theorem 1.10). There are spectral sequences

$$\begin{aligned} \hat{E}^2_{r,s}(C_{p^n}, \text{THH}(F)) &= \hat{H}^{-r}(BC_{p^n}; \pi_s\text{THH}(F)) \Rightarrow \pi_{r+s}\hat{\mathbb{H}}(C_{p^n}, \text{THH}(F)) \\ E^2_{r,s}(C_{p^n}, \text{THH}(F)) &= H^{-r}(BC_{p^n}; \pi_s\text{THH}(F)) \Rightarrow \pi_{r+s}\text{THH}(F)^{hC_{p^n}} \end{aligned} \tag{4.3}$$

Here π_s can be replaced by homotopy groups with coefficients. We first determine the spectral sequence $E_{r,s}$ for THH $(\mathbb{F}_p)$ and homotopy groups with $\mathbb{F}_p$ coefficient.

Recall that the FSP F determines the spectrum F^s, and that there is a "suspension" mapping

$$\sigma : S^1_+ \wedge F^s \to \text{THH}\,(F)$$

When F is the FSP associated to the ring, $\mathbb{F}_p$ then $F^s = H\mathbb{F}_p$, and we have the distinguished element

$$\sigma = \sigma_*(\iota_1 \otimes e_1) \in \pi_2(\text{THH}\,(\mathbb{F}_p)\,;\mathbb{F}_p), e_1 \in \pi_1(H\mathbb{F}_p, \mathbb{F}_p)$$

The proof from [B2] of (3.4) shows that σ is an integral homotopy class and that $\pi_*\text{THH}\,(\mathbb{F}_p) = \mathbb{F}_p\,[\sigma]$. In particular we have for the second spectral sequence in (4.3),

$$E\{u_n\} \otimes \mathbb{F}_p\,[t] \otimes \mathbb{F}_p\,[\sigma] \Rightarrow \pi_*\text{THH}\,(\mathbb{F}_p)^{hC_{p^n}} \tag{4.4}$$

with deg $(u_n) = (-1,0)$, deg $(t) = (-2,0)$ and deg $(\sigma) = (0,2)$. If we instead use homotopy groups with $\mathbb{F}_p$ coefficients we get

$$E\{u_n\} \otimes \mathbb{F}_p\,[t] \otimes E\{e_1\} \otimes \mathbb{F}_p\,[\sigma] \Rightarrow \pi_*(\text{THH}\,(\mathbb{F}_p)^{hC_{p^n}}; \mathbb{F}_p) \tag{4.5}$$

Lemma 4.6 *In* (4.5), $d_2\,(e_1) = t\sigma$ *and* d_2 *maps the other generators to zero .*

Proof. We first prove the corresponding differential in the spectral sequence

$$H^*(BS^1; \pi_*(\text{THH}\,(\mathbb{F}_p); \mathbb{F}_p)) \Rightarrow \pi_*(\text{THH}\,(\mathbb{F}_p)^{hS^1}; \mathbb{F}_p) \tag{*}$$

There is a cofibration of spectra

$$\Sigma^{-2}\text{THH}\,(\mathbb{F}_p) \to \text{Map}_{S^1}(S^3_+, \text{THH}\,(\mathbb{F}_p)) \to \text{THH}\,(\mathbb{F}_p)$$

induced from $S^1_+ \to S^3_+ \to S^1_+ \wedge S^2$. It continues to the right by the map

$$\tilde{A} : \text{THH}\,(\mathbb{F}_p) \to \Sigma^{-1}\text{THH}\,(\mathbb{F}_p)$$

which is adjoint to the S^1-action, $A : S^1_+ \wedge \text{THH}\,(\mathbb{F}_p) \to \text{THH}\,(\mathbb{F}_p)$.

Let $e_1 \in \pi_1(\mathrm{THH}\,(\mathbb{F}_p)\,;\mathbb{F}_p)$ be the image under σ_* of $1 \otimes e_1$. The commutative diagram

$$\begin{array}{ccc} S^1_+ \wedge S^1_+ \wedge H\mathbb{F}_p & \overset{1\wedge\sigma}{\rightarrow} & S^1_+ \wedge \mathrm{THH}\,(\mathbb{F}_p) \\ \downarrow m \wedge \mathrm{id} & & \downarrow A \\ S^1_+ \wedge H\mathbb{F}_p & \overset{\sigma}{\rightarrow} & \mathrm{THH}\,(\mathbb{F}_p) \end{array}$$

shows that $A_*(\iota\otimes e_1) = \sigma$ or equivalently that $\tilde{A}_*(e_1) = \Sigma^{-1}\sigma$. This in turn shows that e_1 does not lift to $\pi_1(\mathrm{Map}_{S^1}(S^3_+, \mathrm{THH}\,(\mathbb{F}_p)); \mathbb{F}_p)$ and thus that there is a non-trivial d_2 on e_1 in the spectral sequence (*). The restriction

$$\mathrm{Res} : \mathrm{THH}\,(\mathbb{F}_p)^{hS^1} \rightarrow \mathrm{THH}\,(\mathbb{F}_p)^{hC_{p^n}}$$

induces a map of spectral sequences which preserves e_1, t and σ on the E^2-level, and the claim follows. □

Given (4.6) one easily calculates the E^∞-terms of the spectral sequences (4.3) for THH $(\mathbb{F}_p)$. The result is

$$\begin{aligned} \pi_*(\hat{\mathbb{H}}(C_{p^n}, \mathrm{THH}\,(\mathbb{F}_p)\,;\mathbb{F}_p) &= E\{u_n\} \otimes \mathbb{F}_p\left[t, t^{-1}\right] \\ \pi_*(\mathrm{THH}\,(\mathbb{F}_p)^{hC_{p^n}}\,;\mathbb{F}_p) &= E\{u_n\} \otimes \mathbb{F}_p\,[t,\sigma]\,/\,(t\sigma) \end{aligned} \qquad (4.7)$$

Lemma 4.8 *The map $\hat{\Gamma}$: THH $(\mathbb{F}_p) \rightarrow \hat{\mathbb{H}}(C_p, \mathrm{THH}\,(\mathbb{F}_p))$ from (4.2) with $n = 1$ induces a homotopy equivalence onto the (-1)-connected cover of the target.*

Proof. The lemma follows from [BM], Lemma 6.4 where the corresponding statement is proved for THH $(\mathbb{Z}_p)$. Indeed, the reduction $\mathbb{Z}_p \rightarrow \mathbb{F}_p$ induces an isomorphism

$$\pi_{2p}(\mathrm{THH}\,(\mathbb{Z}_p)\,;\mathbb{F}_p) \rightarrow \pi_{2p}(\mathrm{THH}\,(\mathbb{F}_p)\,;\mathbb{F}_p)$$

In the diagram

$$\begin{array}{ccc} \pi_{2p}\,(\mathrm{THH}\,(\mathbb{Z}_p)\,;\mathbb{F}_p) & \overset{R_*}{\rightarrow} & \pi_{2p}(\mathrm{THH}\,(\mathbb{F}_p)\,;\mathbb{F}_p) \\ \downarrow \hat{\Gamma}_* & & \downarrow \hat{\Gamma}_* \\ \pi_{2p}(\hat{\mathbb{H}}(C_p; \mathrm{THH}\,(\mathbb{Z}_p)); \mathbb{F}_p) & \overset{\hat{R}_*}{\rightarrow} & \pi_{2p}\left(\hat{\mathbb{H}}\,(C_p, \mathrm{THH}\,(\mathbb{F}_p))\,;\mathbb{F}_p\right) \end{array}$$

$\hat{R}_*\hat{\Gamma}_*$ is non-zero by the cited result from [BM], so $\hat{\Gamma}_*(\sigma^p) = t^{-p}$ and hence $\hat{\Gamma}_*(\sigma) = t^{-1}$. Since $e_1\sigma$ is mapped to σ by the Bockstein operator, $\hat{\Gamma}_*(e_1\sigma) \neq 0$ and thus $\hat{\Gamma}_*(e_1) = u_1 t^{-1}$. □

Proposition 4.9 *The maps* Γ and $\hat{\Gamma}$ *in (4.2) for* THH $(\mathbb{F}_p)$ *induce homotopy equivalences onto the* (-1)*-connected cover of their targets.*

Proof. It suffices to examine the induced maps on modulo p homotopy groups in non-negative degrees. The argument is inductive starting with (4.8). In the model

$$\hat{\mathbb{H}}(C_p, \mathrm{THH}\,(\mathbb{F}_p)) \simeq [\tilde{EC}_{p^n} \wedge \mathrm{Map}\,(EC_{p^n+}, \mathrm{THH}\,(\mathbb{F}_p))]^{C_p}$$

we have an action of $C_{p^{n-1}} = C_{p^n}/C_p$ and

$$\hat{\mathbb{H}}(C_{p^n}, \mathrm{THH}\,(\mathbb{F}_p)) \simeq \hat{\mathbb{H}}(C_p, \mathrm{THH}\,(\mathbb{F}_p))^{C_{p^{n-1}}}$$

Moreover, the mapping from (4.8) (which we now write with non-capitals)

$$\gamma : \mathrm{THH}\,(\mathbb{F}_p) \to \hat{\mathbb{H}}(C_p, \mathrm{THH}\,(\mathbb{F}_p))$$

becomes $C_{p^{n-1}}$-equivariant. The induced mapping of fixed sets is the map from (4.2): $\hat{\gamma}^{C_{p^{n-1}}} \simeq \hat{\Gamma}$. Consider now the following diagram with $n \geq 2$

$$\begin{array}{ccc} T\,(\mathbb{F}_p)^{C_{p^{n-1}}} & \xrightarrow{\Gamma_{n-1}} & T\,(\mathbb{F}_p)^{hC_{p^{n-1}}} \\ \downarrow \hat{\gamma}^{C_{p^{n-1}}} & & \downarrow \hat{\gamma}^{hC_{p^{n-1}}} \\ \hat{\mathbb{H}}(C_p, \mathrm{THH}\,(\mathbb{F}_p))^{C_{p^{n-1}}} & \xrightarrow{G} & \hat{\mathbb{H}}\,(C_p, \mathrm{THH}\,(\mathbb{F}_p))^{hC_{p^{n-1}}} \end{array} \qquad (*)$$

and with Γ_{n-1} and G inclusions of fixed sets into homotopy fixed sets.

The map $\hat{\gamma}^{hC_{p^{n-1}}}$ is a homotopy equivalence (in positive degrees) since $\hat{\gamma}$ is, and since this property is preserved under taking homotopy fixed sets. Also, the source and the target of G have (abstractly) isomorphic homotopy groups by a calculation quite similar to the one giving (4.7). In fact G is a homotopy equivalence. This follows from the diagram

$$\begin{array}{ccc} & \hat{\mathbb{H}}\,(C_p, \mathrm{THH}\,(\mathbb{F}_p))_{hC_{p^{n-1}}} & \\ N \swarrow & & \searrow N^h \\ \hat{\mathbb{H}}\,(C_p, \mathrm{THH}\,(\mathbb{F}_p))^{C_{p^{n-1}}} & \xrightarrow{G} & \hat{\mathbb{H}}\,(C_p, \mathrm{THH}\,(\mathbb{F}_p))^{hC_{p^{n-1}}} \end{array}$$

which exists for any $C_{p^{n-1}}$-equivariant spectrum, and from the calculational fact that

$$\hat{\mathbb{H}}(C_{p^{n-1}}, \hat{\mathbb{H}}(C_p, \mathrm{THH}\ (\mathbb{F}_p))) \simeq O \tag{4.10}$$

To check (4.10) one uses the spectral sequence with E^2-term

$$E\{u_{n-1}\} \otimes \mathbb{F}_p[t, t^{-1}] \otimes E\{e_1\} \otimes \mathbb{F}_p[\sigma, \sigma^{-1}],$$

$d^2\,(e_1) = t\sigma$, and converging to the modulo p homotopy groups of the left-hand side in (4.10). Since $d^2(e_1 t^{-1}\sigma^{-1}) = 1$, the E^3-term vanishes, and (4.10) follows.

We have shown that $\pi_*\,(G, \mathbb{F}_p)$ and $\pi_*(\hat{\gamma}^{hC_{p^{n-1}}}, \mathbb{F}_p)$ are isomorphisms in the diagram (*). Assuming in (*) that $\pi_*(\Gamma_{n-1}, \mathbb{F}_p)$ is an isomorphism for $* \geq 0$ we get $\pi_*(\hat{\gamma}^{C_{p^{n-1}}}; \mathbb{F}_p)$ is an isomorphism for $* \geq 0$ and can then use the exact homotopy sequence of (4.2) to show that $\pi_*(\Gamma_n; \mathbb{F}_p)$ is an isomorphism. Starting with (4.8) we therefore inductively prove (4.9). □

Lemma 4.11 For $i > 0$ $\quad \Phi_* = 0 : \pi_i(\mathrm{THH}\ (\mathbb{F}_p)^{C_{p^n}}; \mathbb{F}_p) \to \pi_i(\mathrm{THH}\ (\mathbb{F}_p)^{C_{p^{n-1}}}; \mathbb{F}_p)$.

Proof. In the spectral sequence

$$\hat{E}^2_{r,s} = \hat{H}^{-r}(BC_{p^n}; \pi_s(\mathrm{THH}\ (\mathbb{F}_p)\,; \mathbb{F}_p)) \Rightarrow \pi_{r+s}(\hat{\mathbb{H}}(C_{p^n}; \mathrm{THH}\ (\mathbb{F}_p)); \mathbb{F}_p)$$

the differentials which cross over the axis $r = 0$ correspond to norm map N^h_*, cf. [BM], Lemma 1.15. Now, in the spectral sequence we have

$$d^2(e_1 u_n t^{-1}) = u_n \sigma,\ d^2(e_1 t^{-1}) = \sigma$$

and hence (cf. 4.7) that

$$N^h_* : \pi_*(\mathrm{THH}\,(\mathbb{F}_p)_{hC_{p^n}}; \mathbb{F}_p) \to \pi_*(\mathrm{THH}\ (\mathbb{F}_p)^{hC_{p^n}}; \mathbb{F}_p)$$

is surjective for $* > 0$. It follows from (4.2) and (4.9) that $\Phi_* = 0$ in positive degrees. □

By definition, TC $(\mathbb{F}_p, p)$ is connected, and

$$\begin{aligned} \pi_o \mathrm{TC}\ (\mathbb{F}_p, p) &= \pi_o \mathrm{ho}\varprojlim \mathrm{THH}\ (\mathbb{F}_p)^{C_{p^n}} \\ &\simeq \varprojlim \pi_o(\mathrm{THH}\ (\mathbb{F}_p)^{C_{p^n}}) \end{aligned} \quad 2$$

Since $\pi_o(\mathrm{THH}\,(\mathbb{F}_p);\mathbb{F}_p) = \mathbb{F}_p$, the integral π_o is cyclic. The bottom cofibration in (4.2) then shows inductively that $\pi_o\mathrm{THH}\,(\mathbb{F}_p)^{C_{p^n}} = \mathbb{Z}/p^{n+1}$. As a consequence we see that

$$\pi_o\mathrm{TC}\,(F_p, p) = \mathbb{Z}_p$$

It follows from (4.11) that $(\Phi\text{-id})_*$ is the identity on $\varprojlim \pi_i(\mathrm{THH}\ (\mathbb{F}_p)^{C_{p^n}}; \mathbb{F}_p)$ for $i > 0$, and hence that $\pi_i(\mathrm{TC}\,(\mathbb{F}_p,p);\mathbb{F}_p) = 0$ for $i > 0$. But then $\pi_i\mathrm{TC}\,(\mathbb{F}_p,p) = 0$ as well. This completes the proof of Theorem 3.8: $\mathrm{TC}\ (\mathbb{F}_p,p)^\wedge_p \simeq H\mathbb{Z}_p$.

Remark 4.12 In the integral spectral sequence

$$\hat{H}^*(BC_{p^n}; \pi_*\mathrm{THH}\ (\mathbb{F}_p)) \Rightarrow \pi_*\hat{\mathbb{H}}(C_{p^n}; \mathrm{THH}\,(\mathbb{F}_p))$$

the only non-trivial differentials are generated multiplicatively from

$$d^{2n+1}\,(u_n) = t^{n+1}\sigma^n$$

Indeed, by the structure of the modulo p spectral sequence it follows that the extensions going from $\hat{E}^\infty$ to the actual homotopy groups are maximally non-trivial. Now, if u_n and hence u_nt^{-1} were permanent cycles, then $\pi_o\hat{\mathbb{H}}(C_{p^n}, \mathrm{THH}\ (\mathbb{F}_p))$ would be equal to $\mathbb{Z}_p$, but it cannot be torsion free. Hence for some k, $d^{2k+1}(u_nt^{-1}) = t^k\sigma^k$. That $k = n$ follows from the Bockstein relation $\beta_n(u_nt^{-1}) = 1$. We have shown that $\pi_{2i}\mathrm{THH}\ (\mathbb{F}_p)^{C_{p^n}} = \mathbb{Z}/p^{n+1}$ and $\pi_{2i-1}\mathrm{THH}\ (\mathbb{F}_p)^{C_{p^n}} = 0$.

When one attempts to calculate topological cyclic homology of other simple rings, e.g. $R = \mathbb{F}_pG, R = \mathbb{F}_p\,[t]\,/\,(t^n)$ then one generally runs into the problem of calculating the homotopy groups of the spectrum $(\mathrm{THH}\ (R) \wedge X)^{C_{p^n}}$ for certain C_{p^n}-spaces X. This requires, except in some very special cases, some basic understanding of the $C_{p^{n-1}}$-equivariant homology theory $\mathrm{THH}\,(R)_*^{C_{p^n}}$, which we do not possess at the time of writing, except for $R = \mathbb{F}_p$, cf. [HM]. Concretely, let me pose the problem of evaluating $\mathrm{THH}\ (R)^{C_{p^n}}\ (SW)$ for all representations W when $R = \mathbb{Z}_p$ (SW = unit sphere of W).

5. The cyclotomic trace in A-theory

This section describes some (partially unpublished) results about $\mathrm{TC}\ (F,p)$ when F is an FSP of the form $F\,(U) = U \wedge \Gamma_+$. We adopt the notation

TC $(B\Gamma, p)$ for TC (F, p), so that the cyclotomic trace becomes

$$\mathrm{Trc} : A(X) \to \mathrm{TC}\,(X, p)\,.$$

Let QY denote the suspension spectrum of Y and $Q_C(Y)$ the C-equivariant suspension spectrum. We write $\Sigma_+(Y) = Y\coprod\{+\}, \Lambda X$ for the free loop space and $\Delta_p : \Lambda X \to \Lambda X$ for the p fold power map $(\Delta_p(\lambda)(z) = \lambda(z^p))$.

Theorem 5.1 ([BHM]) (i) THH $(X) \simeq {}_C Q_C(\Lambda X_+)$ *for each finite cyclic group.*
(ii) *There is a homotopy Cartesian diagram*

$$\begin{array}{ccc} \mathrm{TC}\,(X,p)_p^\wedge & \longrightarrow & Q(\Sigma_+(ES^1 \times_{S^1} \Lambda X))_p^\wedge \\ \downarrow & & \downarrow S^1 - \text{transfer} \\ Q(\Lambda X_+) & \stackrel{\mathrm{id}-\Delta_p}{\longrightarrow} & Q(\Lambda X_+) \end{array}$$

Remark 5.2 It follows from 5.1 (ii) that one *cannot* have a homotopy equivalence

$$S^1_+ \wedge \mathrm{TC}\,(\mathbb{Z}, p)_p^\wedge \stackrel{\simeq}{\to} \mathrm{TC}\,(\mathbb{Z}[v, v^{-1}], p)_p^\wedge$$

Indeed in the range less than $2p-3$ there is no difference between the p-completed sphere spectrum and $H\mathbb{Z}_p$ and therefore also no difference between TC $(*, p)_p^\wedge$ and TC $(\mathbb{Z}, p)_p^\wedge$ or between TC $(S^1, p)_p^\wedge$ and TC $(\mathbb{Z}[v, v^{-1}], p)_p^\wedge$, and one can easily evaluate (5.1), (ii) for $X = S^1$ to get a counter example.

Let G be a p-group. Then

$$\Delta_p : \Lambda BG/BG \longrightarrow \Lambda BG/BG$$

is nilpotent, and there is a homotopy Cartesian diagram

$$\begin{array}{ccc} Q(\Lambda BG) & \stackrel{id-\Delta_p}{\to} & Q(\Lambda BG) \\ \downarrow \mathrm{ev} & & \downarrow \mathrm{ev} \\ Q(BG) & \stackrel{0}{\to} & Q(BG) \end{array}$$

where ev is the map which evaluates a loop at 1. It follows that

$$\mathrm{TC}\,(BG, p)_p^\wedge \simeq Q(BG_+)_p^\wedge \times \mathrm{hofib}\,(Q(\Sigma_+(ES^1 \times_{S^1} \Lambda BG)) \to Q(BG_+))_p^\wedge$$

Since the map from QS^0 to $H\mathbb{Z}$ is $2p-3$-connected at p,

$$\pi_2\mathrm{TC}\,(BG,p)_p^\wedge \simeq \pi_2\mathrm{TC}\,(\mathbb{Z}_pG,p)_p^\wedge$$

when p is odd. Moreover,

$$\pi_2 Q(\Sigma_+(ES^1\times_{S^1}\Lambda BG))_p^\wedge = H_1(ES^1\times_{S^1}\Lambda BG;\mathbb{Z}_p) = HC_1(\mathbb{Z}_pG)$$

so the exact sequence

$$\ldots \to \pi_2(\mathrm{TC}\,(\mathbb{Z}_pG,p))/H_2(G;\mathbb{Z}_p) \to HC_1(\mathbb{Z}_pG) \to H_2(G;\mathbb{Z}_p) \to \ldots \tag{5.3}$$

In the notation from [O] we have that

$$\pi_2\mathrm{TC}\,(\mathbb{Z}_pG,p)/H_2(G;\mathbb{Z}_p) = \mathrm{Wh}_2^Z(\mathbb{Z}_pG),$$

and (5.3) is precisely Conjecture 0.1 from [O], proved for G *abelian* in [O], Theorem 3.9.

The result listed in (5.1), (ii) is closely related to the S^1-fixed set or homotopy fixed set of THH (X) when ΛX has finite type. For example

$$\mathrm{THH}\,(*)^{S^1} \simeq \mathrm{THH}\,(*)^{hS^1} \simeq \prod_1^\infty Q(\Sigma_+BS^1)\times QS^o$$

(after completion).

Theorem 5.4 ([BCCGHM]) *For simply connected spaces,* $\mathrm{Trc}\,:\tilde{A}\,(X)_p^\wedge \xrightarrow{\simeq} \widetilde{\mathrm{TC}}\,(X,p)_p^\wedge$ *is a homotopy equivalence.*

Here $\tilde{A}$ and $\widetilde{\mathrm{TC}}$ denote the reduced theories $A\,(X) = \tilde{\mathrm{A}}\,(X)\times A\,(*)$ etc. The proof of (5.4) is not very difficult, but it is *very* indirect. It is an application of the "Calculus of Functors" from [G1]. The point is that $A\,(X)_p^\wedge$ and $\mathrm{TC}\,(X,p)_p^\wedge$ have the same Goodwillie derivatives. In [G2] it is conjectured for any 1-connected map $\varphi: F_1\to F_2$ of FSP's that

$$\begin{array}{ccc} K(F_1)_p^\wedge & \to & \mathrm{TC}\,(F_1,p)_p^\wedge \\ \downarrow & & \downarrow \\ K(F_2)_p^\wedge & \to & \mathrm{TC}\,(F_2,p)_p^\wedge \end{array}$$

is homotopy Cartesian. In particular one has

Conjecture 5.6 *The diagrams*

$$\begin{array}{ccc} A\,(X)^{\wedge}_{p} & \to & \mathrm{TC}\,(X,p)^{\wedge}_{p} \\ \downarrow & & \downarrow \\ A\,(B\pi_1 X)^{\wedge}_{p} & \to & \mathrm{TC}\,(B\pi_1 X,p)^{\wedge}_{p} \end{array} \qquad \text{(i)}$$

$$\begin{array}{ccc} A\,(B\Gamma)^{\wedge}_{p} & \to & \mathrm{TC}\,(B\Gamma,p)^{\wedge}_{p} \\ \downarrow & & \downarrow \\ K\,(\mathbb{Z}\Gamma)^{\wedge}_{p} & \to & \mathrm{TC}\,(\mathbb{Z}\,[\Gamma]\,,p)^{\wedge}_{p} \end{array} \qquad \text{(ii)}$$

are homotopy Cartesian.

The first conjecture is almost certainly true; one just needs to prove that Trc induces a homotopy equivalence between the two abstractly equivalent derivatives, cf. Remark 2.5 of [BCCGHM]. Moreover, Goodwillie has a convincing outline of a proof that 5.6 (i) implies 5.6 (ii), but some nasty details are involved. In any case it is very generally believed that (5.6) is true.

Let $A\,(X;\mathbb{Z}_p)$ be the A-theory based on completed spheres, or more precisely

$$A(X;\mathbb{Z}_p)^{\wedge}_{p} = K(U \mapsto U^{\wedge}_{p} \wedge \Omega X_{+})^{\wedge}_{p}$$

When $X = B\Gamma$ for a discrete group Γ we have a linearization map $A(B\Gamma;\mathbb{Z}_p)^{\wedge}_{p} \to K(\mathbb{Z}_p\Gamma)^{\wedge}_{p}$, and one can formulate the analogue of Conjecture 5.4 (ii) in this situation. Since one easily shows that

$$\mathrm{TC}\,(R,p)^{\wedge}_{p} \simeq \mathrm{TC}\,(R \otimes \mathbb{Z}_p, p)^{\wedge}_{p}$$

at least when R is a ring which is finitely generated over $\mathbb{Z}$ one expects a homotopy Cartesian diagram

$$\begin{array}{ccc} A\,(B\Gamma)^{\wedge}_{p} & \to & A(B\Gamma;\mathbb{Z}_p)^{\wedge}_{p} \\ \downarrow & & \downarrow \\ K\,(\mathbb{Z}\Gamma)^{\wedge}_{p} & \to & K\,(\mathbb{Z}_p\Gamma)^{\wedge}_{p} \end{array}$$

and this appears reasonable enough. Note that $K\,(\mathbb{Z})^{\wedge}_{p} \to K(\mathbb{Z}_p)^{\wedge}_{p}$ is far from being a homotopy equivalence.

Let us finally briefly consider assembly maps. There is a homotopy commutative diagram of spectra

$$\begin{array}{ccc} X_+ \wedge A(*) & \rightarrow & A(X) \\ \downarrow \mathrm{id} \wedge \mathrm{Trc} & & \downarrow \mathrm{Trc} \\ X_+ \wedge \mathrm{TC}(*, p) & \rightarrow & \mathrm{TC}(X, p) \end{array}$$

where the horizontal arrows are assembly maps. The cofiber of the upper one is Waldhausen's functor $\mathrm{Wh}^{\mathrm{Top}}$. By celebrated theorems of Igusa [I] and Waldhausen [W2], there is a $(\dim M - 7)/3$ connected map.

$$\Omega^2 \mathrm{Wh}^{\mathrm{Top}}(M) \rightarrow \mathrm{Top}(M \times I, M \times 0)$$

at least if M is smoothable. Here Top denotes the space of homeomorphisms.

For closed Riemannian manifolds with negative sectional curvature, Farrell and Jones [FJ] proves that

$$\mathrm{Wh}^{\mathrm{Top}}(M) \simeq \prod \mathrm{Wh}^{\mathrm{Top}}(S^1)$$

where $\prod$ denotes the product over conjugacy classes in $\pi_1 M$. If (5.6), (ii) were true for $\Gamma = \mathbb{Z}$ then using that $S^1_+ \wedge K(\mathbb{Z}) \simeq K(\mathbb{Z}[v, v^{-1}])$ we would get a cofibration of spectra

$$\mathrm{Wh}^{\mathrm{Top}}(S^1)^\wedge_p \rightarrow \overline{\mathrm{TC}}(S^1, p)^\wedge_p \rightarrow \overline{\mathrm{TC}}(\mathbb{Z}[v, v^{-1}], p)^\wedge_p$$

where $\overline{\mathrm{TC}}(\,, p)$ denotes the cofiber of the TC-assembly map. In any case we have the important

Problem 5.7 *Evaluate* $A(S^1)$.

A theorem of Weiss and Williams gives a $(\dim M - 7)/3$-connected map

$$\widetilde{\mathrm{Top}}(M)/\mathrm{Top}(M) \rightarrow \Omega \mathrm{Wh}^{\mathrm{Top}}(M)_{h\mathbb{Z}/2}$$

for a certain involution on $\Omega \mathrm{Wh}^{\mathrm{Top}}(M)$. The space $\widetilde{\mathrm{Top}}(M)$ of block homeomorphisms is contractible by the celebrated rigidity theorem of Farrell and Jones, [FJ] when M is a closed manifold of negative sectional curvature. Thus one needs to examine the $\mathbb{Z}/2$-equivariance properties of the cyclotomic trace. The fixed point set of the relevant involution will be *the topological dihedral homology.* In terms of (5.1), (ii) my guess is that the relevant involution on TC (X, p) (compatible with the involution $A \mapsto (A^*)^{-1}$ on linear K-theory) can be described as follows:

Consider the $O(2)$-action on ΛX which extends the S^1-action by adding the obvious reflection. One gets an induced $\mathbb{Z}/2$-action on $EO\,(2) \times_{S^1} \Lambda X$, and then replaces the upper right-hand corner in (5.1), (ii) by the $\mathbb{Z}/2$-spectrum

$$Q_{\mathbb{Z}/2}(S(\mathbb{R} \oplus \mathbb{R}_-) \wedge (EO\,(2) \times_{S^1} \Lambda X_+))$$

The two other corners would be $Q_{\mathbb{Z}/2}(\Lambda X_+)$ with the reflection $\mathbb{Z}/2$-action on ΛX. In the smooth category a main theorem of Waldhausen asserts that

$$A\,(M) \simeq Q(M_+) \times \mathrm{Wh}^{\mathrm{Diff}}\,(M)$$

Again by [WW] there is a $(\dim M - 7)\,/3$-connected map

$$\widetilde{\mathrm{Diff}}\,(M)\,/\mathrm{Diff}\,(M) \to \Omega \mathrm{Wh}^{\mathrm{Diff}}\,(M)_{h\mathbb{Z}/2}$$

but there is no rigidity result for $\widetilde{\mathrm{Diff}}\,(M)$. One has a homotopy fibration ('surgery theory')

$$\Sigma \mathbb{L}(\pi_1 M) \to F\,(M)\,/\widetilde{\mathrm{Diff}}\,(M) \to \mathrm{Map}\,(M, F/O)$$

but it is hard to make definite calculations, except rationally where there is no distinction between $\widetilde{\mathrm{Top}}$ and $\widetilde{\mathrm{Diff}}$.

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