Algebraic K-theory and traces

Dedicated to Wu Chung Hsiang
on the occasion of his 60'th birthday

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In the invitation to speak at the seminar, S.-T. Yau stated the intent of the lectures and the accompanying publication to be that a graduate student, having heard the lecture and read the manuscript, should be able to start research of his or her own on the subject. He added that it was desirable if the manuscript contained some new original results as well. I do not know if this is possible to achieve in a single paper, but it is a noble goal. The present blend between a traditional expository article and a detailed exposition of the subject is in any case my attempt at this goal.

Contents

1 Introduction

The present paper is an attempt to give an overview of topological cyclic homology and its relation to algebraic K-theory. In the 'classical' setting, algebraic K-theory associates to a ring A a space $K(A)$. The homotopy groups of $K(A)$ are Quillen's higher K-groups. They have proved very difficult to calculate, and are, for example, to this day not even known for the ring of rational integers.

The homology of (a component of) $K(A)$ is the group homology of the group $\mathrm{GL}_\infty(A)$ of invertible matrices of the ring. This was early on used by Quillen and Borel to evaluate K-theory of finite fields and the torsion free part of K-theory of algebraic integers, respectively. Later Suslin evaluated the homotopy groups with finite coefficients of K-theory of algebraically closed fields, or what amounts to the same thing, the profinite completion $K(F)^\wedge$. In particular he showed that $K(\mathbb{C})^\wedge$ is equivalent to the profinite completion of the space which classifies complex vector bundles. Bott periodicity then calculates $\pi_i K(\mathbb{C})^\wedge$ to be a copy of the profinite integers when i is even and zero for i odd.

This development inspired another calculational approach to the K-groups, namely via étale K-theory, introduced by Friedlander.

Given a Galois extension $F \subset E$ with group G, $K(F) \sim K(E)^G$. This is no big calculational help, but if one replaces the actual fixed sets by the socalled homotopy fixed sets, a construction introduced by Sullivan in the sixties in connection with his solution of the Adams conjecture, then calculations become possible. The homotopy fixed set is the function space (spectrum)

$$K(E)^{hG} = \mathrm{Map}_G(EG, K(E))$$

where EG is the contractible free G space, and where Map_G denotes the space of G-mappings. The filtration of EG by its skeleta induces a spectral sequence

$$H^*(G, K_*(E)) \Rightarrow \pi_* K(E)^{hG}$$

which in favorable situations can be determined. The étale K-theory of a field F is, very roughly speaking, the homotopy fixed set of $K(\hat{F})^{hG}$ where $\hat{F}$ is the closure of F. In the characteristic zero situation $K(\hat{F})^\wedge \sim K(\mathbb{C})^\wedge$ by Suslin, so the calculation of étale K-theory of fields is intimately tied to Galois cohomology. There has been a lot of efforts by many people to evaluate étale K-theory, and in particular by W. Dwyer, E. Friedlander, S. Mitchell and Bob Thomasson. But the basic question remains: how close is

$$K(F) \to K^{\mathrm{et}}(F)$$

to be a (profinite) homotopy equivalence? In one formulation, the Lichtenbaum-Quillen conjecture asserts this to be the case (above dimension 1) for number fields.

For small values of i, K_i(fields) have been extensively calculated in work of Merkurjev and Suslin. The reader is referred to Suslin's address at the ICM 1990.

In another direction, Waldhausen generalized Quillen's K-theory of rings to include certain 'rings up to homotopy', such as $\Omega^\infty S^\infty(\Omega X_+)$. The resulting functor $A(X)$ is intimately related to the space of automorphisms (homeomorphisms or diffeomorphisms) of X when it is a (high dimensional) manifold.

The approach to K-theory (of rings or spaces) in this paper is to study a certain trace type invariant

$$\mathrm{trc}\colon K(A) \to \mathrm{TC}(A).$$

The target is a topological version of Connes' cyclic homology; We call it topological cyclic homology but maybe trace homology was a better word.

From a superficial viewpoint the cyclotomic trace records the traces of all powers of matrices, so could also be called the characteristic polynomial invariant. It works equally well for Waldhausen's K-theory of spaces, and was introduced in joint work with M. Bökstedt and W.-C. Hsiang [BHM] in order to solve the K-theory analogue of Novikov's conjecture about homotopy invariance of the higher signatures of manifolds. The construction was inspired by ideas of T. Goodwillie. Here however, I shall be mostly concerned with the situation for rings.

There is a map from $\mathrm{TC}(A)$ to another functor denoted $\mathrm{THH}(A)$, the topological Hochschild homology of R, and trc is a lifting of this topological Dennis trace.

Let me briefly sketch the construction of $\mathrm{THH}(A)$. Consider the simplicial abelian group

$$Z_\bullet(A)\colon\ \cdots A\otimes A\otimes A \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} A\otimes A \rightrightarrows A$$

where the face operators sends $a_1\otimes a_2\otimes a_3$ into $a_1a_2\otimes a_3$, $a_1\otimes a_2a_3$, $a_3a_1\otimes a_2$ etc. The homotopy groups of $Z_\bullet(A)$, or what is the same thing, the homology groups of the associated chain complex, are the Hochschild homology groups $\mathrm{HH}_*(A)$.

The basic idea, suggested by T. Goodwillie, is to replace A by the Eilenberg-MacLane spectrum it generates, and $\otimes$ by smash product of spectra. This was carried out by Bökstedt, and leads to a simplicial space $\mathrm{THH}(A)$. The extra structure in $Z_\bullet(A)$ which comes from the cyclic rotation of the tensor factors is also present after the indicated substitutions, and via Connes' theory of cyclic sets, it implies a circle action on $\mathrm{THH}(A)$.

Connes initially defined cyclic homology by replacing $Z_*(A)$ by the complex $C_*(A)$ whose n'th term is $Z_n(A)/C_{n+1}$, the quotient group by the cyclic rotation of factors. It is crude construction to divide out a non-free group action—usually one gets a better theory by instead taking the Borel quotient. This was done in papers of Loday-Quillen and Feigin-Tsygan who replaced the quotients $A^{\otimes n}/C_n$ by $W^{(n)}\otimes_{C_n} A^{\otimes n}$ where $W^{(n)}$ is the standard free $\mathbb{Z}[C_n]$ resolution of $\mathbb{Z}$. In the topological situation of $\mathrm{THH}(A)$ it is better to take fixed sets $\mathrm{THH}(A)^C$ for the various subgroups of the circle. Had the circle action on $\mathrm{THH}(A)$ been free, the fixed sets would have been the Borel orbits $\mathrm{THH}(A)_{hC} = \mathrm{THH}(A)\wedge_C EC_+$. This is not the case, and the fixed sets $\mathrm{THH}(R)^C$ is a mixture of Borel quotients, one for each strata of the action. In our topological situation it turns out that there is a certain map

$$R\colon \mathrm{THH}(A)^{C_n}\to \mathrm{THH}(A)^{C_m}$$

whenever m divides n, which one does not see in the linear setting. This map mixes the stata. We also have the inclusion of fixed sets

$$F\colon \mathrm{THH}(A)^{C_n}\to \mathrm{THH}(A)^{C_m}.$$

The topological cyclic homology $\mathrm{TC}(A)$ is defined to be the homotopy theoretical limit of $\mathrm{THH}(A)^{C_n}$ over the maps R and F as C_n varies over all cyclic subgroups. The basic theory of $\mathrm{THH}(A)$ and $\mathrm{TC}(A)$ is described in chap. 2 below, where we also recall the construction of

$$\mathrm{trc}\colon K(A) \to \mathrm{TC}(A). \tag{1.1}$$

Actually, both $K(A)$, $\mathrm{TC}(A)$ and $\mathrm{THH}(A)$ are spectra in the sense that there are sequences of spaces $K(A)_{\mathbf{R}^n}$ etc. so that $K(A)$ is equivalent to the n'th based loop space of $K(A)_{\mathbf{R}^n}$ etc., and trc preserves this structure. I write $\mathrm{TH}(A)$ for the spectrum $\{\mathrm{THH}(A)_{\mathbf{R}^n}\}$ but do not introduce special notation for the spectra $K(A)$ and $\mathrm{TC}(A)$.

The following chap. 3 presents results of Dundas, Goodwillie and McCarthy. The following theorem is proved in sect. 3.4.

Theorem 1.2 (McCarthy). *For a surjection of rings $f\colon A \to \bar{A}$ with nilpotent kernel,*

$$\begin{array}{ccc} K(A) & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(\bar{A}) & \xrightarrow{\mathrm{trc}} & \mathrm{TC}(\bar{A}) \end{array}$$

becomes a homotopy Cartesian diagram after profinite completion.

In particular, the relative homotopy groups with finite coefficients of the two vertical maps agree. Earlier results of this nature have appeared in [G4], [G5], [BCCGHM]. The proof is based upon Goodwillie's "calculus of functors"; it is very indirect, and does not in any way produce an explicit inverse from $\mathrm{TC}(A \to \bar{A})$ to $K(A \to \bar{A})$.

The trace (1.1) cannot in general induce an isomorphism of (profinite) homotopy groups. Here is one reason: $\mathrm{TC}(A)$ is constructed out of Eilenberg-MacLane spectra $H(A)$. Now $H(A)^\wedge \sim H(\hat{A})$ since $H(A)$ is characterized by its homotopy groups. This persists to TC,

$$\mathrm{TC}(A)^\wedge \sim \mathrm{TC}(A \otimes \hat{\mathbb{Z}})$$

at least if A is finite over $\mathbb{Z}$. However, it is well known that K-theory does not have this property. Thus (1.1) has little chance of inducing isomorphism on profinite homotopy unless one restricts attention to complete rings.

There is an extension due to B. Dundas of theorem 1.2 to the setting of Waldhausen's functor $A(X)$, namely the following result which is outlined in sect. 3.5.

Theorem 1.3. (Dundas). For any space X the diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\text{trc}} & \mathrm{TC}(X) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}[\pi_1 X]) & \xrightarrow{\text{trc}} & \mathrm{TC}(\mathbb{Z}[\pi_1 X]) \end{array}$$

becomes homotopy Cartesian after profinite completion.

Let k be a finite field of characteristic $p \neq 0$, and let $W(k)$ be its ring of Witt vectors. Chap. 4 outlines the proof of the following joint result with L. Hesselholt

Theorem 1.4 ([HM]) *For finitely generated $W(k)$-algebras,*

$$\text{trc}\colon K(A)_p^\wedge \to \mathrm{TC}(A)_p^\wedge$$

is a homotopy equivalence (in positive dimensions).

Chapter 4 also gives a new (simpler) proof of one of the main results from [BHM], namely that the assembly map

$$K(\mathbb{Z}) \wedge B\Gamma_+ \to K(\mathbb{Z}\Gamma)$$

is a rational equivalence for a large class of big groups, e.g. for the groups Γ which have finitely generated Eilenberg-MacLane cohomology in each dimension. The simplification of the original proof is made possible by theorem 1.3. Chapter 4 further calculates $\mathrm{TC}(X)^\wedge$ in terms of more traditional functors in algebraic topology; these involve the free loop space of X.

The functor $\mathrm{TC}(A)$ is not very easy to calculate, but it does lent itself to analysis by classical methods of algebraic topology. The basic approach, so far, has been to use the following diagram of (co)fibrations (in the category of spectra)

$$\begin{array}{ccccc} \mathrm{TH}(A)_{hC_{p^n}} & \longrightarrow & \mathrm{TH}(A)^{C_{p^n}} & \xrightarrow{R} & \mathrm{TH}(A)^{C_{p^{n-1}}} \\ \| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \hat{\Gamma}} \\ \mathrm{TH}(A)_{hC_{p^n}} & \xrightarrow{N} & \mathrm{TH}(A)^{hC_{p^n}} & \longrightarrow & \hat{\mathbb{H}}(C_{p^n}, \mathrm{TH}(A)) \end{array}$$

The lower cofibration is usually called the norm cofibration. In order to define it one uses that the spectrum $\mathrm{TH}(A)$ can be extended to an S^1-equivariant spectrum $T(A)$. Roughly speaking this means that there are spaces $T(A)_V$, one for each finite dimensional representation V of S^1 such that $\mathrm{TH}(A) \sim_{S^1} \mathrm{Map}(S^V, T(A)_V)$. Here S^V denotes the one-point compactification of V with its induced S^1-action. The construction of the norm cofibration is due to J. Greenlees and J. P. May.

The point of the diagram is firstly that there are spectral sequences which approximates the terms in the bottom sequence, e.g.

$$\hat{H}^s(C_{p^n}; \pi_t \mathrm{TH}(A)) \Rightarrow \pi_{s+t}\hat{\mathbb{H}}(C_{p^n}\mathrm{TH}(A))$$

where $\hat{H}^s$ denotes Tate cohomology. Secondly, it turns out that the maps Γ and $\hat{\Gamma}$ in many situations are homotopy equivalences in non-negative degrees. This is reminisant of the Segal conjecture (which corresponds to $\mathrm{TH}(\Omega^\infty S^\infty)$) where Γ and $\hat{\Gamma}$ are actual homotopy equivalences. In particular one expects Γ, $\hat{\Gamma}$ to give equivalence (in positive degrees) for integers in local number fields with non-zero residue characteristic.

This has been verified in the unramified situation, $A = W(\mathbb{F}_{p^s})$, where the calculation of TC has been carried through. In order to describe the result, let $\mathrm{im}\, J_p$ be the homotopy fiber in

$$\mathrm{im}\, J_p \to (BU \times \mathbb{Z})_p \xrightarrow{\psi^k - 1} BU_p$$

where ψ^k is the Adams operation for an integer k which generates the units in $\mathbb{Z}/p^2\mathbb{Z}$, i.e. a topological generator of the units of the p-adic integers $\mathbb{Z}_p$. The bottom homotopy group of $\mathrm{im}\, J_p$ is a copy of $\mathbb{Z}_p$, and

$$\pi_{2n-1}(\mathrm{im}\, J_p) = \begin{cases} \mathbb{Z}/p^{v_p(n+1)} & \text{if } n \equiv 0\ (p-1) \\ 0 & \text{if not} \end{cases}$$

while $\pi_{2n}(\mathrm{im}\, J_p) = 0$ for $n > 0$. ($v_p(-)$ denotes the p-adic valuation). Let $B\,\mathrm{im}\, J_p$ denote the delooping of $\mathrm{im}\, J_p$ with $\pi_i(B\,\mathrm{im}\, J_p) = \pi_{i-1}(\mathrm{im}\, J_p)$. Then one has:

Theorem 1.5. ([BHM2]). *Let $\mathbb{F}_{p^s}$ be the finite field with p^s elements and $A_s = W(\mathbb{F}_{p^s})$ its Witt-vectors. Then for p odd,*

$$\mathrm{TC}(A_s)^\wedge_p \sim \mathrm{im}\, J_p \times B\,\mathrm{im}\, J_p \times SU^\wedge_p \times U^\wedge_p \times \cdots \times U^\wedge_p \quad (s-1 \textit{ copies of } U)$$

where SU is the special unitary group, and U the unitary group.

The proof of theorem 1.5 is a long and complex calculation which requires a thorough knowledge of homotopy theory. It was in fact the first calculation made of the TC functor applied to rings. The general calculational scheme developed in this case was later exploited in a number of less complicated situations. Theorem 1.5 in conjunction with the Dwyer-Mitchell calculation of $K^{\text{et}}(A_s)$, [DM], verifies the conjecture of Lichtenbaum and Quillen for these rings.

The first three sections of chapter 5 give other examples of TC-calculations in situations where theorem 1.4 applies. Sect. 5.1 studies K-theory of group rings of finite groups. In terms of concrete calculations the main result is:

Theorem 1.6. *Let k be a finite field of characteristic $p > 0$, and let C be a cyclic group of p-power order. Then the p-primary part of K-theory is given by*

$$K_{2n-1}(k[C])_{(p)} \cong K_1(k[C])_{(p)}^{\oplus n}$$

and $K_{2n}(k[C])_{(p)} = 0$ for $n > 0$.

The next two sections 5.2 and 5.3 outline joint work with L. Hesselholt. The main result is the following

Theorem 1.7. *Let k be a perfect field of characteristic $p > 0$. Then*

$$K_{2m-1}\big(k[x]/(x^n)\big)_{(p)} = \mathbf{W}_{nm-1}(k)/V_n\mathbf{W}_{m-1}(k)$$

and $K_{2m}(k[x]/(x^n))_{(p)} = 0$ for $n > 0$.

Here $\mathbf{W}(k)$ denotes the big Witt-vectors, that is, $\mathbf{W}(k) = (1 + k[[t]])^\times$, the multiplicative group of power series beginning with 1, $\mathbf{W}_r(k)$ is the corresponding truncated version

$$\mathbf{W}_r(k) = (1 + k[[t]])^\times/(1 + t^{r+1}k[[t]])^\times,$$

and V_s is the Verschiebung map which takes a power series $f(t)$ to $f(t^s)$. Sect. 5.3 is just an example; it evaluates the groups $\text{Nil}_*(A)$ for the rings of theorem 1.7.

Finally it is in order to point out that $\text{TC}(A)$ only contains information about K-theory at the residue characteristic. The l-primary part of $K(A)$ for $l \neq p$ is however, for the rings under consideration, already known by theorems of Gabber and Suslin et. al.: one may divide out the radical, cf. [Su], which also contains a thorough account on low dimensional calculations.

It is a pleasure to acknowledge the help I have had from M. Bökstedt, B. Dundas and L. Hesselholt in preparing this paper.

2 Topological cyclic homology

This chapter sets the notation to be used in the rest of the paper, reviews the definition of the functors to be discussed and gives the basic constructions.

2.1 Cyclic constructions.

Let G be a topological monoid and E a two sided G space. For technical reasons we assume that the unit $1 \in G$ is a "good" base point, i.e. $\{1\} \subset G$ be a cofibration. The cyclic bar construction of G with coefficients in E is the simplicial space $N_r^{\mathrm{cy}}(G; E)$ with r-simplices

$$N_r^{\mathrm{cy}}(G; E) = E \times G^r \tag{2.1.1}$$

and simplicial structure maps

$$d_i(e, g_1, \dots, g_r) = \begin{cases} (eg_1, g_2, \dots, g_r), & i = 0 \\ (e, g_1, \dots, g_i g_{i+1}, \dots, g_r), & 0 < i < r \\ (g_r e, g_1, \dots, g_{r-1}), & i = r \end{cases}$$
$$s_i(e, g_1, \dots, g_r) = (e, g_1, \dots, g_{i-1}, 1, g_i, \dots, g_r), \quad 0 \le i \le r.$$

Two special cases have particular interest for us, namely $E = *$ and $E = G$ (with its natural two sided G-structure). In these cases we shorten the notation to

$$N_\bullet G = N_\bullet^{\mathrm{cy}}(G; *), \quad N_\bullet^{\mathrm{cy}} G = N_\bullet^{\mathrm{cy}}(G; G).$$

The simplicial space $N_\bullet^{\mathrm{cy}} G$ has extra structure; it is a *cyclic set* in the sense of Connes: one has the cyclic permutation

$$t_r \colon N_r^{\mathrm{cy}} G \to N_r^{\mathrm{cy}} G, \quad t_r(g_0, \dots, g_r) = (g_r, g_0, \dots, g_{r-1})$$

with the following extra relations, as the reader can easily check,

$$\begin{aligned} d_i t_r &= t_{r-1} d_{i-1}, \quad 1 \le i \le r \\ d_0 t_r &= d_r \\ s_i t_r &= t_{r+1} s_{i-1}, \quad 1 \le i \le r \\ s_0 t_r &= t_{r+1}^2 s_r \\ t_r^{r+1} &= 1. \end{aligned} \tag{2.1.2}$$

Let Δ be the usual simplicial category with objects $[r] = \{0, \dots, r\}$ and order preserving maps, so that a simplicial space is a functor from Δ^{op} to $\{\text{spaces}\}$. It is contained in a category Λ with the same objects but with

$$\Lambda([r], [s]) = \Delta([r], [s]) \times C_{r+1}$$

where C_{r+1} is the cyclic group of order $r+1$. A cyclic space is just a functor from Λ^{op} to {spaces}, see e.g. [J] for further information.

For a simplicial space $X_\bullet$ we let $|X_\bullet|$ denote the usual topological realization,

$$|X_\bullet| = \coprod_{r=0}^{\infty} \Delta^r \times X_r / \sim; \quad (d^i u, x) \sim (u, d_i x), \quad (s^i u, x) \sim (u, s_i x),$$

where $d^i \colon \Delta^{r-1} \to \Delta^r$, $s^i \colon \Delta^r \to \Delta^{r-1}$ are the face and degeneracy operators of the standard simplex. The realization of the cyclic r-simplex $\Lambda[r]_\bullet = \Lambda([\bullet], [r])$ can be calculated to be

$$\Lambda^r \cong \mathbb{R}/\mathbb{Z} \times \Delta^r = S^1 \times \Delta^r. \tag{2.1.3}$$

It is a cocyclic space, that is a functor from Λ into {spaces}. There are two good choices of the homeomorphism in (2.1.3). One can either choose it so that

(i) $t^r(\theta, u_0, \dots, u_r) = (\theta - u_0, u_1, \dots, u_r, u_0)$ or so that

(ii) $t^r(\theta, u_0, \dots, u_r) = \big(\theta - 1/(r+1), u_1, \dots, u_r, u_0\big)$.

In case (i), the cosimplicial maps d^i, s^i are $\mathrm{Id}_{S^1} \times d^i$, $\mathrm{Id}_{S^1} \times s^i$ with d^i, s^i being the usual cosimplicial maps on $\Delta^\bullet$; in case (ii) d^i and s^i depends on the circle coordinate. The realization of cyclic spaces comes equipped with a natural action of S^1. Indeed it is easy to see for a cyclic space $Z_\bullet$ that

$$|Z_\bullet| \cong \coprod_{r=0}^{\infty} \Lambda^r \times Z_r / \approx \tag{2.1.4}$$

where the identifications $\approx$ are

$$(d^i u, z) \approx (u, d_i z), \quad (s^i u, z) \approx (u, s_i z), \quad (t^r u, z) \approx (u, t_r z).$$

The S^1-action on the circle factor of Λ^r descents to the claimed S^1-action on $|Z_\bullet|$. For further information on cyclic spaces we refer the reader to [C], [J], [DHK].

The homotopy theory of spaces X equipped with an action of a group G is governed by the homotopy theory of its fixed sets X^H, $H \subseteq G$ (H closed if G is Lie). In particular a G-map $f \colon X \to Y$ is a weak homotopy equivalence if and only if its induced map on H-fixed set is for all (closed) $H \subseteq G$. Thus it is important to be able to calculate fixed set $|Z_\bullet|^C$ for the realization of a cyclic set, where C is finite cyclic or $C = S^1$. It is not hard to see from (2.1.4) that

$$|Z_\bullet|^{S^1} = \{z \in Z_0 \mid s_0 z = t_1 s_0 z\}$$

but it is harder to use (2.1.4) to get information about $|Z_\bullet|^C$ when $C \subset S^1$ is finite.

There is however a simple devise, *edgewise subdivision*, which can be used to effectively calculate $|Z_\bullet|^C$. Let $X_\bullet$ be any simplicial space, and C a cyclic group of order c. We consider $X_\bullet \colon \Delta^{\mathrm{op}} \to \{\text{spaces}\}$ and define

$$\begin{aligned} &sd_C \colon \Delta \to \Delta \\ &sd_C[r] = [r] \amalg \cdots \amalg [r], \quad c = \text{summands} \\ &sd_C\phi = \phi \amalg \cdots \amalg \phi, \quad \phi \in \Delta([r],[s]). \end{aligned} \tag{2.1.5}$$

The composition $X_\bullet \circ sd_C \colon \Delta^{\mathrm{op}} \to \{\text{spaces}\}$ is the subdivided simplicial space, denoted $sd_C X_\bullet$. Its space of r-simplices is equal to $X_{c(r+1)-1}$.

The diagonal inclusion of Δ^r into the c fold join $\Delta^r * \cdots * \Delta^r$ induces a (non-simplicial) map D from the realization of $sd_C X_\bullet$ into the realization of $X_\bullet$. If $X_\bullet$ is a cyclic space then $|sd_C X_\bullet|$ has a natural $\mathbb{R}/c\mathbb{Z}$ action, which restricts to a simplicial $\mathbb{Z}/c\mathbb{Z}$ action. Indeed $t^{r+1}_{c(r+1)-1}$ acts simplicially on the r-simplices of $sd_C X_\bullet$. From [BHM], sect. 1 we have:

Lemma 2.1.6. *The map $D \colon |sd_C X_\bullet| \to |X_\bullet|$ is a homeomorphism. Moreover, if $X_\bullet$ is cyclic then D is S^1-equivariant when $\mathbb{R}/c\mathbb{Z}$ is identified with the circle in the usual way.* □

For a cyclic space $Z_\bullet$, the action of the $(r+1)$'st power $t^{r+1}_{c(r+1)-1}$ is a simplicial map of $sd_C Z_\bullet$ of order c, so induces a simplicial C-action on $|sd_C Z_\bullet|$, and hence via D an action of C on $|Z_\bullet|$. For example it is not hard to see that

$$sd_C N^{\mathrm{cy}}_\bullet(G) \cong N^{\mathrm{cy}}_\bullet(E, G^c)$$

where $E = G^c$ (c fold Cartesian product) with its componentwise left G^c-action, and right G^c-action given by

$$(g_1, \dots, g_c)(e_1, \dots, e_c) = (g_c e_1, g_1 e_2, \dots, g_{c-1} e_c).$$

The action of C on $sd_C N^{\mathrm{cy}}(G)$ corresponds under the above identification to the cyclic permutation action on E and G^c, so there is a homeomorphism

$$\Delta_C \colon N^{\mathrm{cy}}_\bullet(G) \xrightarrow{\cong} sd_C N^{\mathrm{cy}}_\bullet(G)^C \tag{2.1.7}$$

with Δ_C induced from the diagonal map $G \to G^c$.

We now suppose that our topological monoid is group-like, that is, $\pi_0 G$ is a group. In this case

$$BG = |N_\bullet G|$$

and the canonical map $G \to \Omega BG$ is a weak homotopy equivalence, in sign: $\Omega BG \sim G$. Moreover,

$$B^{\mathrm{cy}}G = |N_\bullet^{\mathrm{cy}}G|$$

is equivalent to the free loop space ΛBG of BG. Indeed, the projection

$$N_\bullet^{\mathrm{cy}}(G;G) \to N_\bullet^{\mathrm{cy}}(G;*)$$

induces a map from $B^{\mathrm{cy}}G$ to BG and the adjoint of the map

$$S^1 \times B^{\mathrm{cy}}G \to B^{\mathrm{cy}}G \to BG$$

defines the equivalence (cf. [G1],[BF])

$$q\colon B^{\mathrm{cy}}G \xrightarrow{\sim} \Lambda BG. \tag{2.1.8}$$

This is not a (weakly homotopy) equivalence of S^1-spaces since the S^1-fixed sets do not agree. However, for each finite subgroup $C \subset S^1$,

$$q^C\colon (B^{\mathrm{cy}}G)^C \xrightarrow{\sim} (\Lambda BG)^C \tag{2.1.9}$$

is an equivalence. This follows easily upon using (2.1.7) and the obvious homeomorphism

$$\Delta_c\colon \Lambda BG \xrightarrow{\cong} (\Lambda BG)^C, \quad \Delta(\lambda)(z) = \lambda(z^c).$$

Indeed, q^C identifies with q under the identifications induced from Δ_C and Δ_c (cf. [BHM], proposition 2.6). Let me give the proof of (2.1.8) when G is a group, and refer to [BF], [G1] for the group-like case. One starts with a rewriting of $N^{\mathrm{cy}}(G)$, namely via the bijection

$$f\colon N_\bullet^{\mathrm{cy}}(G) \xrightarrow{\cong} \mathrm{Ad}G \times_G E_\bullet G,$$

where $\mathrm{Ad}G$ denotes G with conjugation action, and $E_\bullet G$ is the left acyclic bar construction whose k-simplices are $g[g_1|\cdots|g_k]$; the map is defined as

$$f(g_0, \dots g_k) = g_k \cdots g_1 g_0 [g_1|\cdots|g_{k-1}].$$

The topological realization of $E_\bullet G$ is the free contractible G-space, and $\mathrm{Ad}G \times_G EG \sim \Lambda BG$. Indeed a loop $\lambda(t) \in \Lambda BG$ is mapped into $(g_\lambda, \tilde{\lambda}(1))$ where $\tilde{\lambda}(t)$ is a lift of $\lambda\colon [0,1] \to BG$ and g_λ is the holonomy: $g_\lambda \cdot \tilde{\lambda}(0) = \tilde{\lambda}(1)$. When G is compact Lie one needs a connection in $EG \to BG$, and $\tilde{\lambda}$ will be a parallel curve in EG.

The above have generalizations to the nerve and cyclic nerve of a category $\mathcal{C}$. The nerve $N_\bullet \mathcal{C}$ is the simplicial set with

$$N_r\mathcal{C} = \{c_r \xrightarrow{f_r} c_{r-1} \longrightarrow \cdots \xrightarrow{f_1} c_0\},$$

the set of r composable maps. Similarly $N_\bullet^{\mathrm{cy}}(\mathcal{C})$ has r-simplices

$$N_r^{\mathrm{cy}}\mathcal{C} = \{c_r \xrightarrow{f_r} c_{r-1} \longrightarrow \cdots \xrightarrow{f_1} c_0 \xrightarrow{f_0} c_r\}$$

and boundary maps similar to (2.1.1). For categories with only one object, monoids, this agrees with the above constructions. If we restrict the morphisms of $\mathcal{C}$ to be isomorphisms we obtain a subcategory $i\mathcal{C}$, and (2.1.8) generalizes to

$$|N_\bullet^{\mathrm{cy}}(i\mathcal{C})| \sim \Lambda|N_\bullet(i\mathcal{C})|.$$

These more general concepts will be used in the next chapter.

We close this section with a rewriting of $N_\bullet G$, due to Waldhausen, [W2], in the special case where G is a semi-direct product. Let $(V, +)$ be an abelian monoid equipped with a two-sided action of the monoid Γ. Denote by $G = V \rtimes \Gamma$ the semi-direct product with multiplication

$$(v_1, g_1)(v_2, g_2) = (v_1 g_2 + g_1 v_2, g_1 g_2).$$

Let $N_\bullet V$ be the bar construction of $(V, +)$. It inherits a simplicial two sided action of Γ, and we can form the bisimplicial set

$$[r], [s] \mapsto N_r^{\mathrm{cy}}(\Gamma; N_s V).$$

Its diagonal simplicial set with r-simplices $N_r^{\mathrm{cy}}(\Gamma; N_r V)$ is denoted $\delta N_\bullet^{\mathrm{cy}}(\Gamma; N_\bullet V)$. Consider the simplicial map

$$u\colon \delta N_\bullet^{\mathrm{cy}}(\Gamma, N_\bullet V) \to N_\bullet(V \rtimes \Gamma)$$

given on r-simplices by

$$\begin{aligned} &u(v_1, \ldots, v_r, \gamma_1, \ldots, \gamma_r) = \\ &((\gamma_1 \cdots \gamma_r v_1 \gamma_1, \gamma_1), (\gamma_2 \cdots \gamma_r v_2 \gamma_1 \gamma_2, \gamma_2), \ldots, (\gamma_r v_r \gamma_1 \cdots \gamma_r, \gamma_r)). \end{aligned}$$

The map u can be understood as the composition of two maps: one starts with a rewriting of the left hand side, similar to the above f, and then rearranges factors upon using the semi-direct product. When Γ is a group u is a bijection. In general one has from [W2], lemma 2.3.1:

Lemma 2.1.10. *If Γ is a group-like monoid then u induces a weak homotopy equivalence of topological realizations.* □

The map u will be used in the next chapter for $\Gamma = \mathrm{GL}_k(R)$, $V = \mathrm{M}_k(V)$, where R is a unital ring and V is an R-bimodule. In this case the semi-direct product ring $V \rtimes R$ has

$$\mathrm{GL}_k(V \rtimes R) = \mathrm{M}_k(V) \rtimes \mathrm{GL}_k(R).$$

2.2 Simplicial spaces.

We have already in sect. 2.1 used simplicial sets and spaces. This will continue even more extensively in later sections, and it is in order to collect some of the relevant properties of simplicial sets and spaces.

Let us first point out that we use the word *space* to mean a based compactly generated Hausdorff space, and that all constructions are to be taken in this category.

A map $f\colon X \to Y$ is called k-connected if it induces an isomorphism on homotopy groups in degrees less than k and an epimorphism in degree k, i.e. if the homotopy fiber is $(k-1)$-connected. The convention is that every space is (-2)-connected, non-empty spaces are (-1)-connected and path connected spaces are 0-connected. It is an equivalence if it is k-connected for all k, and in general two spaces X and Y are called equivalent ($X \sim Y$) if they can be connected by a string of equivalences. In almost all cases to be considered, our spaces will have the homotopy type of CW complexes, and in this case $X \sim Y$ iff they are homotopy equivalent in the ordinary sense. The homotopy groups of a simplicial space (or set) $X_\bullet$ will mean the homotopy groups of the topological realization $\|X_\bullet\|$ below.

This is the bigger realization, sometimes called the fat realization, which only depends on the face operators in $X_\bullet$, i.e. on the functor

$$X_\bullet\colon \Delta_m^{\mathrm{op}} \to \{\text{spaces}\}$$

where $\Delta_m \subset \Delta$ is the subcategory of injective maps in Δ. Such a functor is called a Δ-space, [RS], and a presimplicial space in [DM2]. Its realization

$$\|X_\bullet\| = \coprod_{r=0}^{\infty} \Delta^r \times X_r/(d^i u, x) \sim (u, d_i x) \tag{2.2.1}$$

has $|X_\bullet|$ as a quotient when $X_\bullet$ is simplicial.

For simplicial sets,

$$\|X_\bullet\| \to |X_\bullet|$$

is an equivalence, but this is not always true for simplicial spaces.

A simplicial space $X_\bullet\colon \Delta^{\mathrm{op}} \to \{\text{spaces}\}$ is called "good" (or "proper"), [Se1] (or [May1]) if the inclusion of its degenerate simplices

$$\bigcup_{i=0}^{r-1} s_i(X_{r-1}) \subset X_r$$

is a cofibration (an NDR-pair). For such, the two realizations $|X_\bullet|$ and $\|X_\bullet\|$ are equivalent, cf. [Se1], appendix.

Any bisimplicial set (functor from $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ into sets) give rise to two "good" simplicial spaces

$$[r] \to |X_{r,\bullet}|, \quad [s] \to |X_{\bullet,s}|.$$

Their realizations are each homeomorphic to the realization of the diagonal simplicial set $\delta X_{\bullet,\bullet}$, and similarly for multisimplicial sets.

The homotopy fiber of a map $f\colon X \to Y$ of spaces with respect to $* \in Y$ is the space

$$\mathrm{hF}(f) = \{(x, \lambda) \in X \times Y^I \mid f(x) = \lambda(0), * = \lambda(1)\}$$

and there is a long exact sequence of homotopy groups

$$\cdots \to \pi_i X \to \pi_i Y \to \pi_{i-1}(\mathrm{hF}(f)) \to \pi_{i-1} X \to \cdots$$

so f is k-connected precisely if $\mathrm{hF}(f)$ is $(k-1)$-connected (for each choice of $*$).

Given a map of simplicial spaces, $f_\bullet\colon X_\bullet \to Y_\bullet$ and a base point $*_\bullet \in Y_\bullet$ there is a natural map

$$\big|[r] \to \mathrm{hF}(X_r \to Y_r)\big| \to \mathrm{hF}\big(|X_\bullet| \to |Y_\bullet|\big). \tag{2.2.2}$$

This is an equivalence if each Y_r is 0-connected, provided $X_\bullet$ and $Y_\bullet$ both are "good". In particular $|f_\bullet|$ is an equivalence when each $f_r\colon X_r \to Y_r$ is an equivalence. The associated fat realizations are equivalent without the goodness assumption.

The homotopy fiber and the dual notion of homotopy cofiber,

$$\mathrm{cof}(f) = (Y \times I) \amalg X / \big\langle f(x) \sim (y,1), * \sim (y,0)\big\rangle$$

are special cases of the homotopy limit and the homotopy colimit functor from a small category into spaces, cf. [BK], [G4].

Let us next consider function spaces between pointed spaces X and Y. Denote by $F(X,Y)$ the function space (in the compact open topology) of pointed maps from X to Y.

Suppose X is a pointed CW complex, e.g. the realization of a simplicial set, and that $\dim X \le n$. There is a natural map

$$\phi\colon \big|[r] \to F(X, Y_r)\big| \to F(X, |Y_\bullet|). \tag{2.2.3}$$

If $Y_\bullet$ is a "good" simplicial space and each Y_r is $(\dim X - 1)$-connected, then ϕ is an equivalence. In particular, the loop space of a "good" simplicial space $Y_\bullet$ with each Y_r 0-connected can be computed degreewise:

$$\Omega|Y_\bullet| \sim \big|[r] \to \Omega Y_r\big|$$

cf. [May1], sect. 12.

Given based simplicial sets there is a simplicial version $F_\bullet(X_\bullet, Y_\bullet)$ of the mapping space which we shall occasionally use. Its r-simplices consists of the simplicial maps

$$\delta(\Delta[r]_\bullet \times X_\bullet) \to Y_\bullet$$

which maps $\Delta[r]_\bullet \times *_\bullet$ to the base point of $Y_\bullet$. Here $\Delta[r]_\bullet$ is the simplicial r-simplex with $\Delta[r]_s = \Delta\big([s],[r]\big)$. More generally, for each based $K_\bullet$,

$$\mathrm{Map}(K_\bullet, F_\bullet(X_\bullet, Y_\bullet)) = \mathrm{Map}(\delta(K_\bullet \wedge X_\bullet), Y_\bullet)$$

where Map denotes the set of based simplicial maps. In particular, we see for $Y_\bullet = \sin_\bullet Y$, the singular complex of Y, that

$$\sin_\bullet F(|X_\bullet|, Y) \cong F_\bullet(X_\bullet, \sin_\bullet Y)$$

(take $K_\bullet = \Delta[r]_\bullet$). Since $Y_\bullet \sim \sin_\bullet |Y_\bullet|$ when $Y_\bullet$ is fibrant (Kan complex) we see in this case that

$$F(|X_\bullet|, |Y_\bullet|) \sim |F_\bullet(X_\bullet, Y_\bullet)|.$$

Let us finally remind the reader that a simplicial group, $X_\bullet \colon \Delta^{\mathrm{op}} \to \{\text{groups}\}$, is always a Kan complex. For simplicial abelian groups $A_\bullet$ and $B_\bullet$, the function complex $\mathrm{s_\bullet Ab}(A_\bullet, B_\bullet)$ has the property that

$$\mathrm{s_\bullet Ab}(A_\bullet, B_\bullet) \cong \mathrm{s_\bullet Ab}\big(\delta A_\bullet \otimes \tilde{\mathbb{Z}}(S^n_\bullet), \delta A_\bullet \otimes \tilde{\mathbb{Z}}(S^n_\bullet)\big).$$

In particular, $\delta A_\bullet \otimes \tilde{\mathbb{Z}}(S^n_\bullet)$ is a deloop of $A_\bullet$. Here $\tilde{\mathbb{Z}}(S^n_\bullet)$ is the free abelian group of the simplicial n-sphere modulo the relations $\lambda *_\bullet = 0$ and $0 \cdot x = 0$. The reader is referred to [Q1] and [May2] for further details on simplicial sets.

Many constructions later in the paper are functors of fixed sets of the topological realizations of cyclic sets and spaces. Examples have already appeared in sect. 2.1. A map of cyclic sets (spaces) $f_\bullet \colon X_\bullet \to Y_\bullet$ induces an S^1-equivariant map. It is an S^1-equivariant homotopy equivalence if $f^C_\bullet \colon X^C_\bullet \to Y^C_\bullet$ induces an equivalence for all closed subgroups of S^1. This includes S^1 itself. But for some purposes of the paper, the S^1 fixed set is exceptional, and only the C-fixed sets for finite C matters. We therefore introduce the notions $X \sim_{C_\infty} Y$ resp. $X \sim_{C_{p^\infty}} Y$ to mean that X and Y can be connected with a sequence of S^1-maps which induce equivalences on all C_n fixed sets, resp. C_{p^n} fixed sets.

In the rest of the paper we shall tacitly assume that our simplicial spaces are "good". This will sometimes have to be verified, but we shall not go into such details below.

2.3 Topological Hochschild homology.

Given a unital ring A and an A-bimodule V we can form its *cyclic construction* $Z_\bullet(A;V)$. It is a simplicial abelian group with r-simplices

$$Z_r(A;V) = V \otimes A^{\otimes r} \tag{2.3.1}$$

and face and degeneracy operators

$$d_i(v \otimes a_1 \otimes \cdots \otimes a_r) = \begin{cases} va_1 \otimes a_2 \otimes \cdots \otimes a_r, & i = 0 \\ v \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_r, & 0 < i < r \\ a_r v \otimes a_1 \otimes \cdots \otimes a_{r-1}, & i = r \end{cases}$$
$$s_i(v \otimes a_1 \otimes \cdots \otimes a_r) = v \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes \cdots \otimes a_r, \quad 0 \le i \le r$$

cf. (2.1.1). When $V = A$ this becomes a cyclic set upon defining

$$t_r(a_0 \otimes \cdots \otimes a_r) = a_r \otimes a_0 \otimes \cdots \otimes a_{r-1}.$$

The topological realization of $Z_\bullet(A;V)$ is denoted $\mathrm{HH}(A;V)$ or when $V = A$ just $\mathrm{HH}(A)$. Its homotopy groups are the Hochschild homology groups,

$$\mathrm{HH}_i(A;V) = \pi_i \mathrm{HH}(A;V).$$

Indeed for any simplicial abelian group $Z_\bullet$ the homotopy groups of $|Z_\bullet|$ can be calculated as the homology of the associated chain complex Z_* with

$$d\colon Z_i \to Z_{i-1}, \quad dz = \sum_{\nu=0}^{i} (-1)^i d_\nu(z)$$

and $Z_*(A;V)$ is the standard Hochschild complex. The space $\mathrm{HH}(A)$ is the topological realization of a cyclic set, so comes equipped with a natural action of S^1, which keeps the base point invariant. Hence it gives a map

$$A\colon S^1_+ \wedge \mathrm{HH}(A) \to \mathrm{HH}(A)$$

which is the identity on the subspace $\mathrm{HH}(A)$. Exterior product with the generator $[S^1] \in \pi_1 S^1$ induces a map from $\mathrm{HH}_r(A)$ to $\mathrm{HH}_{r+1}(A)$. This is Connes' B-operator, cf. [H1].

T. Goodwillie suggested a decade ago to define the topological Hochschild homology analogously by replacing A with the Eilenberg-MacLane spectrum it generates, and the tensor product by smash product of spectra. Some care is needed in order to make these substitutions because smash products of spectra are not easily made strictly associative. M. Bökstedt in [B1] got around this difficulty in a way we now describe; see also [Br], appendix.

Let Top_* denote the category of based spaces and continuous maps, and let $L\colon \mathrm{Top}_* \to \mathrm{Top}_*$ be a continuous functor such that $L(*) = *$, that is, the function

$$F(X,Y) \xrightarrow{L} F(L(X), L(Y))$$

is continuous and maps the constant map to the constant map. Given $X, Y \in \mathrm{Top}_*$, we have maps

$$L(Y) \to L(X \wedge Y)$$

for each $x \in X$, induced from the corresponding inclusion of Y in $X \wedge Y$, so altogether a function

$$\sigma_{X,Y}\colon X \wedge L(Y) \to L(X \wedge Y)$$

and the assumptions on L implies that this is continuous; σ is called the *assembly map.*

Definition 2.3.2. (Bökstedt). A functor with smash product (FSP) is a functor L with an assembly map together with natural transformations

$$\begin{aligned} 1_X\colon X &\to L(X) \\ \mu_{X,Y}\colon L(X) \wedge L(Y) &\to L(X \wedge Y) \end{aligned}$$

such that

$$\begin{aligned} &\text{(i)} && \mu_{X,Y} \circ (1_X \wedge \mathrm{id}_{L(Y)}) = \sigma_{X,Y} \\ &&& \mu_{X,Y} \circ (\mathrm{id}_{L(X)} \wedge 1_Y) = L(\pi) \circ \sigma_{Y,X} \circ \pi \\ &\text{(ii)} && \mu_{X\wedge Y,Z} \circ (\mu_{X,Y} \wedge \mathrm{id}_{L(Z)}) = \mu_{X,Y\wedge Z} \circ (\mathrm{id}_{L(X)} \wedge \mu_{Y,Z}) \end{aligned}$$

where π switches factors.

The FSP is called 0-connected if it maps n-connected spaces into n-connected spaces and if

$$\sigma_X\colon S^1 \wedge L(X) \to L(S^1) \wedge L(X) \to L(S^1 \wedge X)$$

is $2n - c$ connected whenever X is n-connected (c independent of n).

Any unital ring A induces a 0-connected FSP which we denote $\tilde{A}$. It takes a based space X into the Dold-Thom construction: the configuration space of particles in X with labels in A:

$$\tilde{A}(X) = \{\Sigma a_i x_i \mid x_i \in X, a_i \in A\}/\langle a \cdot * = 0, 0 \cdot x = *\rangle. \tag{2.3.3}$$

It is a 0-connected FSP and $\tilde{A}(S^n)$ is the Eilenberg-MacLane space of type (A, n) as

$$\pi_i \tilde{A}(X) = \tilde{H}_i(X; A).$$

A topological monoid G induces a 0-connected FSP $\tilde{G}$, namely

$$\tilde{G}(X) = X \wedge G_+ \tag{2.3.4}$$

with the obvious 1_X and $\mu_{X,Y}$.

Definition 2.3.5. A functor with smash product is called commutative if

$$\mu_{X,Y} \circ \pi = L(\pi) \circ \mu_{Y,X}$$

where π switches factors.

The FSP's $\tilde{A}$ and $\tilde{G}$ above are commutative when A and G are commutative. Let I denote the category of finite sets and injective maps. Its objects are the sets $\mathbf{n} = \{1, \dots, n\}$ with $\mathbf{0} = \emptyset$. A morphism $f \in I(\mathbf{n}, \mathbf{m})$ can be written as $\sigma \circ i$ where $\sigma \in \Sigma_m$ and i is the standard inclusion. The Cartesian products I^{r+1} form a cyclic category in that there are structure maps

$$d_i \colon I^{r+1} \to I^r, \quad s_i \colon I^r \to I^{r+1}, \quad t_r \colon I^r \to I^r$$

given by

$$\begin{aligned}
d_i(x_0, \dots, x_r) &= \begin{cases} (x_0, \dots, x_i \amalg x_{i+1}, \dots, x_r) & 0 \le i < n \\ (x_n \amalg x_0, x_1, \dots, x_r) & i = n \end{cases} \\
s_i(x_0, \dots, x_r) &= (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_r) \\
t_i(x_0, \dots, x_r) &= (x_r, x_0, \dots, x_{r-1}).
\end{aligned}$$

For $x \in I$ we let S^x be the one point compactification of $\mathbb{R}^x$. For a based space X, consider the functor

$$G_r^X(L) \colon I^{r+1} \to \{\text{spaces}\}$$

given by

$$G_r^X(L, x_0, \dots, x_r) = F\big(S^{x_0} \wedge \dots \wedge S^{x_r}, L(S^{x_0}) \wedge \dots \wedge L(S^{x_r}) \wedge X\big),$$

where F denotes the pointed function space. Using the properties of L we find maps

$$\begin{aligned}
d_i &\colon G_r^X(L, x_0, \dots, x_r) \to G_{r-1}^X(L, d_i(x_0, \dots, x_r)) \\
s_i &\colon G_{r-1}^X(L, x_0, \dots, x_{r-1}) \to G_r^X(L, s_i(x_0, \dots, x_{r-1})) \\
t_i &\colon G_r^X(L, x_0, \dots, x_r) \to G_r^X(L, t_i(x_0, \dots, x_r))
\end{aligned}$$

similar to the maps of sect. 2.1, and we can define a cyclic space $\mathrm{THH}^X_\bullet(L)$ by setting

$$\mathrm{THH}^X_r(L) = \underset{I^{r+1}}{\underrightarrow{\mathrm{holim}}} G^X_r(L). \tag{2.3.6}$$

The realization of $\mathrm{THH}^X_\bullet(L)$ is denoted $\mathrm{THH}^X(L)$; if $X = S^0$ we just write $\mathrm{THH}(L)$.

Lemma 2.3.7. ([B1]) *For a 0-connected* FSP L *and given integer* i,

$$\pi_i \mathrm{THH}^X_r(L) = \pi_i G^X_r(L, x_0, \dots, x_r)$$

provided $x_0, \dots, x_r$ *are sufficiently large.*

Proof. Here is Bökstedt's argument. The category I of finite sets and injective maps has the following structure:

an associative product $\mu\colon I \times I \to I$ (1)

natural transformations between μ and the two projections (2)

a decreasing filtration $F_i I$ with $\mu(F_i I, F_j I) \subset F_{i+j} I$ (3)

Indeed, $\mu(n, m) = n + m$ and $F_i I = \{n \mid n \geq i\}$. Such a category is called a good limit category. These are preserved under Cartesian product, so I^{r+1} is also a good index category.

For $\mathbf{x} = (x_0, \dots, x_r)$, write $G_r(\mathbf{x})$ instead of $G^X_r(L, \mathbf{x})$. To each $\lambda \geq 0$ there exists an i so that $G_r(\mathbf{x}) \to G_r(\mathbf{y})$ is λ connected for each $\mathbf{x} \to \mathbf{y}$ in $F_i = F_i I^{r+1}$. Now it suffices to prove that the following two maps are λ-connected:

$$\underset{\mathbf{x} \in F_i}{\underrightarrow{\mathrm{holim}}}\, G_r(\mathbf{x}) \to \underset{\mathbf{x} \in I^{r+1}}{\underrightarrow{\mathrm{holim}}}\, G_r(\mathbf{x}) \tag{a}$$

$$G_r(\mathbf{y}) \to \underset{\mathbf{x} \in F_i}{\underrightarrow{\mathrm{holim}}}\, G_r(\mathbf{x}), \qquad \mathbf{y} \in F_i \tag{b}$$

The map in (a) is an equivalence; an inverse is induced from $\mu(\mathbf{y}, -)\colon I^{r+1} \to F_i$ for some fixed $\mathbf{y} \in F_i$. This uses property (2) above. To show that (b) is a λ-equivalence, one first argues that the space BF_i (=realization of the nerve) is contractible. Indeed, μ induces a product on BF_i, and by (2) it has a homotopy unit. Condition (2) also yields that $\pi_0 BF_i = 0$. But a connected H-space has a homotopy antipode:

$$-\mathrm{Id}\colon BF_i \to BF_i,$$

with

$$BF_i \xrightarrow{\Delta} BF_i \times BF_i \xrightarrow{-\mathrm{Id}\times\mathrm{Id}} BF_i \times BF_i \xrightarrow{\mu} BF_i$$

homotopy to a constant. Since by (2),

$$\mu \sim pr_2 \colon BF_i \times BF_i \to BF_i$$

we conclude that BF_i is contractible. Finally the projection

$$p\colon \underset{\mathbf{x}\in F_i}{\underrightarrow{\mathrm{holim}}}\, G_r(\mathbf{x}) \to BF_i$$

is a λ-quasifibration in the sense that $G_r(\mathbf{y}) = p^{-1}(\mathbf{y})$ is λ-equivalent to the homotopy fiber. This follows from the last lemma in sect. 1 of [Q2] upon passing to the λ-coskeleton of $G_r(\mathbf{x})$. □

An FSP L induces a ring (pre)spectrum L^S whose n'th term is $L(S^n)$. We note that $\mathrm{THH}^X(L)$ only depends on L^S in the sense that if $L_1 \to L_2$ is a map of FSP's so that $L_1^S \to L_2^S$ is a homotopy equivalence of (pre)spectra then $\mathrm{THH}^X(L_1) \sim \mathrm{THH}^X(L_2)$.

A 0-connected FSP L gives rise to a ring $\pi_0 L$ by linearization, namely

$$\pi_0 L = \underset{n}{\underrightarrow{\lim}}\, \pi_n(L(S^n))$$

($\pi_0 \tilde{A} = A$), and the map

$$G_r(L, x_0, \dots, x_r) \to \pi_0 G_r(L, x_0, \dots, x_r)$$

induces a map $\mathrm{THH}(L) \to \mathrm{HH}(\pi_0 L)$ since

$$\begin{aligned}\pi_{x_0+\cdots+x_r} G(L; x_0, \dots, x_r) &= \tilde{H}_{x_0+\cdots+x_r}(L(S^{x_0}\wedge) \cdots \wedge L(S^{x_r})) \\ &= \pi_0(L) \otimes \cdots \otimes \pi_0(L).\end{aligned}$$

As $\mathrm{THH}^X_\bullet(L)$ is a cyclic space, the realization $\mathrm{THH}^X(L)$ inherits a continuous action of S^1, sect. 2.1, which will be of fundamental importance later in the paper.

There are a number of variations of the construction. First, we may define $\mathrm{THH}(L, M)$ when M is an L-bimodule. This is a functor from pointed spaces to itself with an assembly map

$$\sigma_{X,Y} \colon X \wedge M(Y) \to M(X \wedge Y)$$

and structure maps

$$\begin{aligned} l_{X,Y} &\colon L(X) \wedge M(Y) \to M(X \wedge Y) \\ r_{X,Y} &\colon M(X) \wedge L(Y) \to M(X \wedge Y),\end{aligned}$$

satisfying the obvious compatibility relations which we leave for the reader to explicate. One defines

$$\mathrm{THH}^X_r(L;M) = \underset{\overrightarrow{I^{r+1}}}{\mathrm{holim}}\, F\big(S^{x_0}\wedge\cdots\wedge S^{x_r}, M(S^{x_0})\wedge L(S^{x_1})\wedge\cdots\wedge L(S^{x_r})\wedge X\big)$$

and gets a simplicial space with realization $\mathrm{THH}^X(L;M)$ and a linearization map

$$\mathrm{THH}(L;M)\to \mathrm{HH}(\pi_0 L, \pi_0 M). \tag{2.3.8}$$

This is a rational equivalence when $L=\tilde{A}$, $M=\tilde{V}$.

Second, we may vary the concept of FSP to the simplicial setting and consider simplicial endo-functors of based simplicial sets

$$L_\bullet\colon \mathrm{s_\bullet sets_*}\to \mathrm{s_\bullet sets_*}$$

with properties analogous to the ones given in definition 2.3.2. In this setting the FSP $\tilde{A}$ associated to a ring A is simply

$$\tilde{A}(X_\bullet) = A[X_\bullet]/A\cdot *_\bullet = 0$$

where $A[X_\bullet]$ denotes the simplicial abelian group whose k-simplices is the free A-module with basis X_k. One defines $\mathrm{THH}_\bullet(L_\bullet)$ by using the simplicial function space, assuming $L_\bullet(S^n_\bullet)$ be fibrant, or one can follow $L_\bullet$ by realization, and use the above construction.

Third, there is a variation of $\mathrm{THH}(L)$ which defines $\mathrm{THH}(\mathcal{C})$ of an additive category, cf. [DM2]. The definition is as follows. Consider the functor

$$\tilde{\mathcal{C}}_\bullet\colon \mathrm{s_\bullet sets_*}\to (\mathrm{s_\bullet Ab})^{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}$$

which to a simplicial set $X_\bullet$ associates the functor from $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}$ to $\mathrm{s_\bullet Ab}$

$$\tilde{\mathcal{C}}_\bullet(X_\bullet)(c_1,c_2) = \mathrm{Hom}_{\mathcal{C}}(c_1,c_2)\otimes\tilde{\mathbb{Z}}(X_\bullet)$$

Write $\mathbf{x}=(x_0,\dots,x_r)$,

$$V_\bullet(\mathcal{C},\mathbf{x}) = \bigvee_{(c_0,\dots,c_r)\in\mathcal{C}^{r+1}} \tilde{\mathcal{C}}_\bullet(S^{x_0}_\bullet)(c_0,c_r)\wedge\cdots\wedge\tilde{\mathcal{C}}_\bullet(S^{x_r})(c_r,c_{r-1}),$$

and

$$G_r(\mathcal{C},\mathbf{x}) = F_\bullet\big(S^{x_0}_\bullet\wedge\cdots\wedge S^{x_r}_\bullet, V_\bullet(\mathcal{C},\mathbf{x})\big).$$

Here $S^{x_0}_\bullet$ is, say the x_0-fold smash product of the simplicial circle $S^1_\bullet = \Delta_\bullet[1]/\partial\Delta_\bullet[1]$, and $F_\bullet$ is the simplicial mapping space. Then, as before,

$$\mathrm{THH}_\bullet(\mathcal{C}) = \underset{\overrightarrow{I^{\bullet+1}}}{\mathrm{holim}}\, G_\bullet(\mathcal{C};\mathbf{x}) \tag{2.3.9}$$

with realization $\mathrm{THH}(\mathcal{C})$.

If $\mathcal{C}$ is the category of free A-modules of a given rank n, then $\mathrm{THH}(\mathcal{C})$ is obviously equal to $\mathrm{THH}(\mathrm{M}_n(A))$. By Morita invariance, cf. proposition 2.6.5 below, this is equivalent to $\mathrm{THH}(A)$. If $\mathcal{C} = \mathcal{F}_A$, the category of free A-modules, then the k-simplices of $\mathrm{THH}(\mathcal{C})$ consists of matrices of varying sizes. By adding zeros in a suitable way to get them to have equal size, one does not change anything up to homotopy, so $\mathrm{THH}(\mathcal{C}) \sim \mathrm{THH}(A)$ also in this case. (The reader can easily supply the argument by constructing the required homotopies, step by step). More generally, if $\mathcal{C} = \mathcal{P}_A$ is the category of finitely generated projective modules, then by adding complements to modules and zero homomorphisms, one gets (as pointed out in [DM2]):

Lemma 2.3.10. *For the category of finitely generated projective modules* $\mathrm{THH}(\mathcal{P}_A) \sim \mathrm{THH}(A)$. □

The construction $\mathrm{THH}_\bullet(\mathcal{C})$ is clearly a cyclic set, so $\mathrm{THH}(\mathcal{C})$ has an S^1-action. The equivalence in the above lemma is actually a C_∞ equivalence. This can be seen upon using subdivisions and lemmas 3.10–12 of [BHM].

In general one may associate to $\mathcal{C}$ the simplicial FSP:

$$L_{\mathcal{C}}(X) = \prod_{c_1 \in \mathcal{C}} \bigvee_{c_2 \in \mathcal{C}} \tilde{\mathcal{C}}_\bullet(X)(c_1, c_2).$$

Then $\mathrm{THH}(\mathcal{C}) \simeq \mathrm{THH}(L_{\mathcal{C}})$ cf. [DM], lemma 1.6.22, so (2.3.9) is not really a generalization. It is however a *very* convenient formulation, as we shall see in the next chapter, and $L_{\mathcal{C}}(X)$ is not functorial in $\mathcal{C}$.

Remark 2.3.11. The ring (pre)spectrum L^S associated to an FSP is very special: it has a strictly associative multiplication, and for commutative L it is strictly commutative. Most of the (pre)spectra which otherwise appear in algebraic topology do not have such a "strict" structure—they are merely "homotopy everything associative" (A_∞-spectra) or "homotopy everything commutative" (E_∞-spectra). Recently Elmendorf, Kriz, Mandell and May have recast the category of A_∞ and E_∞-spectra into what they call $\mathcal{L}$-rings and commutative $\mathcal{L}$-rings, [EKMM]. Such an animal E has an associative product $\mu_{\mathcal{L}}\colon E \wedge_{\mathcal{L}} E \to E$. There is no (strict) unit for $\mu_{\mathcal{L}}$, but one may still define $\mathrm{THH}(E)$ by imitating the algebraic construction $Z_\bullet(A)$ of (2.3.1), forgetting the degeneracy operators, cf. (2.2.1). ([EKMM] also introduces S-rings and product $\mu_S\colon E \wedge_S E \to E$ with a unit, and show that the two categories are equivalent, so for S-rings one has $\mathrm{THH}_\bullet(E)$ with degeneracies). More importantly for this paper, Jeff Smith has pointed out that each $\mathcal{L}$-ring

E gives rise to an FSP. Thus FSP's are rich in supply also from the point of view of A_∞ and E_∞-spectra.

In the rest of the paper all FSP's will be assumed to be 0-connected.

2.4 Cyclotomic spectra.

This section constructs from $\mathrm{THH}(L)$ an equivariant S^1-spectrum with extra structure, a socalled cyclotomic spectrum.

Let G be a compact Lie group. For any finite dimensional G-representation space V we write S^V for its one point compactification, and if X is a G-space, $\Omega^V X$ for the (based) mapping space $F(S^V, X)$ with its conjugate G-action.

Roughly speaking a G-spectrum T is a G-space T with a specific delooping $T(V)$ for each G-representation, so that T and $\Omega^V T(V)$ are G-equivalent (or even G-homeomorphic). However, due to the many G-automorphisms of V, some real care is needed to make consistent definitions. (For example, the signs which show up for spectra when $G = 1$ blow up to become elements in the Burnside ring of G).

We shall here follow the approach to G-spectra given in [LMS], and we give a brief account before introducing the concept of cyclotomic spectra. Let G be a compact Lie group and $\mathcal{U}$ a "complete G-universe", i.e. an infinite dimensional G-vector space with a G-invariant inner product which contains each finite dimensional representation of G.

A G-*prespectrum* indexed on $\mathcal{U}$ is a collection of G-spaces $t(V)$, one for each finite dimensional G-space $V \subset \mathcal{U}$ together with a transitive system of G-maps

$$\sigma\colon t(V) \to \Omega^{W-V} t(W)$$

Here $W - V$ denotes the orthogonal complement of V in W. It is a G-*spectrum* if the structure maps σ are all homeomorphisms. A map $f\colon t \to t'$ of G-prespectra consists of G-maps $f(V)\colon t(V) \to t'(V)$ which commute strictly with the structure maps. The category of G-prespectra indexed on $\mathcal{U}$ is denoted $GP\mathcal{U}$ and $GS\mathcal{U}$ denotes the full subcategory of G-spectra. The forgetful functor $l\colon GS\mathcal{U} \to GP\mathcal{U}$ has a left adjoint $\mathbf{L}$. It is given by the colimit over the structure maps

$$\mathbf{L}t(V) \cong \varinjlim_{W \subset \mathcal{U}} \Omega^{W-V} t(W),$$

provided that each σ is an inclusion, i.e. induces a homeomorphism onto its image. (This can always be arranged by thickening up t, to such a prespectrum t^τ, cf. [HM], appendix A).

Suppose C is a closed subgroup in G with quotient J and $T \in G\mathcal{SU}$. There are two possible notions of an associated fixed point spectrum in $J\mathcal{SU}^C$, in [A], [LMS] denoted T^C and $\Phi^C T$ respectively. Their V'th spaces are

$$T^C(V) = T(V)^C, \quad \Phi^C T(V) = \varinjlim_{W \subset \mathcal{U}} \Omega^{W^C - V} T(W)^C, \quad V = V^C \tag{2.4.1}$$

and the structure maps are the evident ones. Since $T(V) \cong \Omega^{W-V} T(W)$ when $V \subset W$ the replacement of a C-equivariant map from S^{W-V} to $T(W)$ with its induced map on C-fixed sets induces a map $s_C\colon T^C \to \Phi^C T$. If $T = \mathbf{L}t$, then $\Phi^C T(V) = \lim \Omega^{W^C - V} t(W)^C$, see e.g. [HM], lemma 1.1.

In the case $G = 1$, the concept of prespectra differ from the usual one in that it is indexed on finite dimensional vector spaces, rather than on just the positive integers n (or $\mathbb{R}^n$). But the two categories are equivalent; the relationship is similar to the relation between a category and its skeleton category. The category of spectra is similar to what used to be called Ω-spectra, where one just demanded that σ be a homotopy equivalence. The functor $T \mapsto \mathbf{L}T^t$ brings us from Ω-spectra to spectra.

We need a few further results. It can all be found in [LMS], chap. 1–2, but the reader which is not accustomed with spectra should first consult [A] to get oriented in the subject.

Let $G \subset H$ be a closed subgroup. There is a pair of adjoint functors

$$i^*\colon G\mathcal{SU} \to G\mathcal{SU}^H, \qquad i_*\colon G\mathcal{SU}^H \to G\mathcal{SU}$$

with i^* the obvious restriction, and

$$i_*(T)(W) = \varinjlim \Omega^{V-W}\big(T(V^H) \wedge S^{V-V^H}\big).$$

Here V runs over the finite dimensional G subspaces which contain W. Given a based G-space X, $\Sigma^\infty(X) \in G\mathcal{SU}^G$ denotes its suspension spectrum, i.e. the spectrum associated with the prespectrum $V \mapsto S^V \wedge X$ for $V \subset \mathcal{U}^G$. Then $i_*(\Sigma^\infty X) = \Sigma^\infty_G X$ is the corresponding equivariant spectrum in $G\mathcal{SU}$.

Maybe the most important construction in the category of spectra is the transfer. Given $T \in G\mathcal{SU}$ and a free G-space E, the transfer is a map

$$\tau\colon j^*T \wedge_G E_+ \to j^*(\Sigma^{-\mathrm{Ad}(G)} T \wedge E_+)^G.$$

Here $j\colon \mathcal{U}^G \to \mathcal{U}$, $\mathrm{Ad}(G)$ the adjoint representation and $\Sigma^{-A}T$ is the function spectrum $F(S^A, T)$ or the equivalently internal delooping $(\Sigma^{-A}T)(V) = T(A \oplus V)$. It follows from [LMS], theorem II.7.1 that τ is a homotopy equivalence. Indeed, II.7.1 proves the result when $T = j_* T_0$, $T_0 \in G\mathcal{SU}^G$. The general case follows because the natural G-map

$$j_* j^* T \wedge E_+ \to T \wedge E_+$$

is a non-equivariant homotopy equivalence, hence as E is G-free a G-equivariant one.

The second result we need is that "induction" and "coinduction" agree, cf. [LMS], theorem II.6.2. Let $T \in GS\mathcal{U}^H$, and let $L = T_{\{H\}}(G/H)$ be the tangent space at the base point $\{H\}$, with its H-action. There is a G-equivalence

$$\omega\colon F(G_+, \Sigma^L T)^H \xrightarrow{\sim} G/H_+ \wedge T.$$

See also [HM], sect. 7. In our applications $G = S^1$ or is finite, and $H \subset G$ is finite. In this case we get the equivalences

$$\begin{aligned} \tau\colon \Sigma^A j^* T \wedge_G E_+ &\xrightarrow{\sim} j^*(T \wedge E_+)^G, \\ \omega\colon \Sigma^A F(G/H_+, T) &\xrightarrow{\sim} G/H_+ \wedge T \end{aligned} \tag{2.4.2}$$

with $A = \mathbb{R}$ if $G = S^1$, and $A = 0$ if G is finite.

The smash products above are to be taken in the category of spectra: if X is a G-space and t a G-prespectrum then $t \wedge X_+$ is the prespectrum whose V'th term is $t(V) \wedge X_+$. If T is a G-spectrum then $T \wedge X_+ := \mathbf{L}(X_+ \wedge lT)$. It is worth pointing out that

$$\Phi^C(T \wedge X_+) \sim_G \Phi^C T \wedge X_+^C.$$

This follows from the equivalence $\Phi^C T \sim_G \Phi^C t$, mentioned above.

We will now fix G to be the circle group S^1. Write $\mathbb{C}(n)$ for the one-dimensional representation where $z \in S^1$ acts as multiplication with z^n, and take

$$\mathcal{U} = \bigoplus_{n \in \mathbf{Z}, \alpha \in \mathbf{N}} \mathbb{C}(n)_\alpha.$$

If $C \subset S^1$ is cyclic of order c then $\mathcal{U}^C \subset \mathcal{U}$ is precisely the summands with $n \in c\mathbb{Z}$.

Next we consider the homotopy fiber of $s_C\colon T^C \to \Phi^C T$ when C is a cyclic p-group. Let $j\colon \mathcal{U}^G \to \mathcal{U}^C$ be the inclusion of the G-trivial universe and let D be a J-spectrum. We call j^*T with its J-action forgotten for the underlying non-equivariant spectrum D. The following is proved in [HM], sect. 1 or in [BHM]:

Proposition 2.4.3. *Suppose C is a cyclic p-group. For any S^1-spectrum T there is a cofibration sequence of non-equivariant spectra*

$$T_{hC} \xrightarrow{N} T^C \xrightarrow{s_{C_p}^{C/C_p}} (\Phi^{C_p} T)^{C/C_p}.$$

Here $T_{hC} = EC_+ \wedge_C j^*T$ is the homotopy orbit spectrum. □

The cofibration sequence of proposition 2.4.3 is the C fixed point of $T \wedge (EC_+ \to S^0 \to \widetilde{EC})$. One identifies the terms by use of (2.4.2) and the easy fact that $(T \wedge \widetilde{EC})^{C_p} \xrightarrow{\sim} \Phi^{C_p}T$, cf. lemma 4.1.2 below.

The circle $G = S^1$ has the nice property that any S^1/C-space X can be viewed as an S^1-space by identifying S^1 with S^1/C via the $|C|$'th root map $\rho_C\colon S^1 \xrightarrow{\cong} S^1/C$. We call this S^1-space $\rho_C^*(X)$. We can also use ρ_C to view S^1/C-spectra as S^1-spectra. Indeed, given an S^1/C-spectrum D indexed on $\mathcal{U}^C$ we have the S^1-spectrum $\rho_C^\# D$ indexed on $\rho_C^*\mathcal{U}^C$, with

$$\rho_C^\# D(V) = \rho_C^* D((\rho_C^{-1})^*(V)).$$

In our case

$$\rho_C^*\mathcal{U}^C = \bigoplus_{\alpha\in\mathbf{N}, n\in c\mathbf{Z}} \mathbb{C}(n/c) = \mathcal{U}$$

so $\rho_C^\# D$ becomes an S^1-spectrum, again indexed on $\mathcal{U}$.

Definition 2.4.4. A *cyclotomic spectrum* is an S^1-spectrum indexed on $\mathcal{U}$ together with an S^1-equivalence

$$r_C\colon \rho_C^\# \Phi^C T \to T$$

for every finite $C \subset S^1$, such that for any pair of finite subgroups the diagram

$$\begin{array}{ccc}
\rho_{C_r}^\# \Phi^{C_r} \rho_{C_s}^\# \Phi^{C_s} T & = & \rho_{C_{rs}}^\# \Phi^{C_{rs}} T \\
{\scriptstyle \rho_{C_r}^\# \Phi^{C_r} r_{C_s}} \downarrow & & \downarrow {\scriptstyle r_{C_{rs}}} \\
\rho_{C_r}^\# \Phi^{C_r} T & \xrightarrow{r_{C_r}} & T
\end{array}$$

commutes.

The cyclotomic condition is analogous to the property of free loop spaces: $(\Lambda X)^C \cong \Lambda X$, and indeed the S^1-suspension spectrum $\Sigma_{S^1}^\infty(\Lambda X)$ is easily seen to be cyclotomic.

More generally, $\mathrm{THH}^X(L)$ induces a cyclotomic spectrum for every FSP. We proceed to explain this. Let us write $\mathrm{THH}(L;V)$ instead of $\mathrm{THH}^{S^V}(L)$. It is the realization of a cyclic space, so gets an S^1-action from this structure. On the other hand, being a functor in V (or S^V) it has a second S^1-action, and altogether an $S^1 \times S^1$-action. We write $t(L)(V) = \mathrm{THH}(L;V)$ equipped with the diagonal S^1-action. This defines an S^1-prespectrum and we let $T(L)$ be the associated equivariant spectrum, $T(L) = \mathbf{L}t(L)$.

Actually, it is not very hard to see that the adjoint of the natural map

$$S^V \wedge \mathrm{THH}(L;W) \to \mathrm{THH}(L; V \oplus W)$$

is a C-equivalence for each finite subgroup of the circle, so that $T(V) \sim_C \mathrm{THH}(L;V)$, cf. [HM], proposition 1.4.

In order to describe the cyclotomic structure maps r_C we use the subdivision operator sd_C introduced in sect. 2.1. For a cyclic space $Z_\bullet$, $sd_C Z_\bullet$ has a simplicial C-action, and its realization $|sd_C Z_\bullet|$ an $\mathbb{R}/c\mathbb{Z}$ action which extends the $C = \mathbb{Z}/c\mathbb{Z}$ action. The homeomorphism

$$D\colon |sd_C Z_\bullet| \to |Z_\bullet|$$

becomes an S^1-map when $\mathbb{R}/c\mathbb{Z}$ is identified with $\mathbb{R}/\mathbb{Z} = S^1$ by division with c.

We now define a simplicial map

$$r'_C\colon sd_C \mathrm{THH}_\bullet(L;V)^C \to \mathrm{THH}_\bullet(L;V^C)$$

for each cyclic subgroup $C \subset S^1$. Let $c = |C|$. With the notation of sect. 2.3, the r-simplices of $sd_C\mathrm{THH}(L;V)$ is the homotopy colimit

$$sd_C \mathrm{THH}_r(L;V) = \underset{\mathbf{x}\in I^{(r+1)c}}{\underrightarrow{\mathrm{holim}}}\ G^V_{(r+1)c-1}(L;\mathbf{x})$$

where

$$G^V_s(L;\mathbf{x}) = F\big(S^{x_0} \wedge \cdots \wedge S^{x_s}, L(S^{x_0}) \wedge \cdots \wedge L(S^{x_s}) \wedge S^V\big).$$

The c-fold diagonal $\Delta_c\colon I^{r+1} \to I^{(r+1)c}$ gives a C-equivariant inclusion

$$\underset{I^{r+1}}{\underrightarrow{\mathrm{holim}}}\, G^V_{c(r+1)-1}(L) \circ \Delta_C \to \underset{I^{(r+1)c}}{\underrightarrow{\mathrm{holim}}}\, G^V_{c(r+1)-1}(L)$$

which induces a homeomorphism of C-fixed sets, and for $\mathbf{x} \in I^{r+1}$,

$$\begin{aligned} &G^V_{c(r+1)-1}(L)(\Delta_C(\mathbf{x})) \\ &\qquad = F((S^{x_0})^{(c)} \wedge \cdots \wedge (S^{x_r})^{(c)}, L(S^{x_0})^{(c)} \wedge \cdots \wedge L(S^{x_r})^{(c)} \wedge S^V) \end{aligned}$$

where $Y^{(c)}$ is the c-fold smash power, and the action of C is by cyclic permutation of factors; then $(Y^{(c)})^C$ is the diagonal copy of Y.

The above formula is quite similar to the identification of $sd_C N^{\mathrm{cy}}_\bullet(G) = N^{\mathrm{cy}}_\bullet(E;G^c)$ explained in sect. 2.1, but this time there is no diagonal homeomorphism

$$\Delta_C\colon N^{\mathrm{cy}}_\bullet(G) \xrightarrow{\cong} (sd_C N^{\mathrm{cy}}_\bullet(G))^C.$$

Even in the linear case of $Z_\bullet(R)$ we do not have such a map since $\Delta(r) = r\otimes\cdots\otimes r$ is not linear. However there is a map in the other direction. Indeed, given any two pointed C-spaces Y_1 and Y_2 one has the obvious map

$$r'_C\colon F(Y_1,Y_2)^C \to F(Y_1^C, Y_2^C)$$

which restricts a C-equivariant map to the induced map on the C-fixed points. This gives a map

$$r'_C\colon G^V_{(r+1)c-1}(L,\Delta_C(\mathbf{x}))\to G^{V^C}_r(L,\mathbf{x}),\quad \mathbf{x}\in I^{r+1}$$

and induces a simplicial map

$$r'_C\colon sd_C\mathrm{THH}_\bullet(L;V)^C\to \mathrm{THH}_\bullet(L;V^C).$$

Taking realization and composing with the inverse of the homeomorphism D we have obtained

$$r_C\colon \mathrm{THH}(L;V)^C\to \mathrm{THH}(L;V^C).$$

This is S^1-equivariant, when one identifies the S^1/C-action in the domain with the S^1-action via ρ_C, so induces an S^1-map from

$$\rho_C^*\Phi^C t(L)(W)=\varinjlim_{V\subset\mathcal{U}}\Omega^{V^C-W}\mathrm{THH}(L;V)^C$$

into

$$T(L)(W)=\varinjlim_{V\in\mathcal{U}}\Omega^{V^C-W}\mathrm{THH}(L;V^C).$$

Since $\Phi^C T(L)=\Phi^C t(L)$, we do get a map

$$r_C\colon \rho_C^{\#}\Phi^C T(L)\to T(L)$$

of S^1-spectra. This is an S^1-equivalence by [HM], proposition 1.5, so we have

Theorem 2.4.5. *For every FSP the S^1-spectrum $T(L)$, induced from the prespectrum* $\mathrm{THH}(L;V)$, *is cyclotomic.* □

The essential point in this and the next chapter is the spectrum $T(L)$, but only considered as a spectrum in the usual sense equipped with an action of S^1. To separate out this, let me introduce the notation $\mathrm{TH}(L)$ for this weakened form,

$$\mathrm{TH}(L)=T(L)\mid \mathcal{U}^{S^1}=j^*T(L),\quad j\colon \mathcal{U}^{S^1}\to\mathcal{U}.$$

The reason for introducing the extra notation is to underline the fact that $\mathrm{TH}(L)\wedge E_+$ and $j^*(T(L)\wedge E_+)$, $j\colon\mathcal{U}^C\to\mathcal{U}$ are quite different. If for example E is S^1-free then $(\mathrm{TH}(L)\wedge E_+)^C\sim 0$ whereas $(T(L)\wedge E_+)^C\sim \mathrm{TH}(L)\wedge_C E_+$ by (2.4.2).

We shall continuously use the following special case of proposition 2.4.3; we call it the *fundamental cofibration sequence*

$$\mathrm{TH}(L)_{hC_{p^n}}\longrightarrow \mathrm{TH}(L)^{C_{p^n}}\xrightarrow{R_p}\mathrm{TH}(L)^{C_{p^{n-1}}}. \tag{2.4.6}$$

2.5 Cyclic homology of cyclotomic spectra.

Given any FSP we saw in the last section that the S^1-spectrum $T(L)$ associated to the prespectrum $\mathrm{THH}(L;S^V)$ comes equipped with an S^1-equivalence

$$r_C\colon \rho_C^{\#}\Phi^C T(L) \xrightarrow{\sim} T(L).$$

We now use this structure to define a new functor $\mathrm{TC}(L)$, the topological cyclic homology of L, initially defined in [BHM].

Let $\mathbb{I}$ be the category where objects are the natural numbers, $\mathrm{ob}\,\mathbb{I} = \{1,2,3,\dots\}$, and with two morphisms $R_r, F_r\colon n \to m$, whenever $n = rm$, subject to the relations

$$\begin{aligned} &R_1 = F_1 = \mathrm{id}_n \\ &R_r R_s = R_{rs}, \quad F_r F_s = F_{rs} \\ &R_r F_s = F_s R_r. \end{aligned} \tag{2.5.1}$$

For a prime p, we let $\mathbb{I}_p$ be the full subcategory with $ob\mathbb{I}_p = \{1,p,p^2,\dots\}$. A cyclotomic spectrum T defines a functor from $\mathbb{I}$ to the category of non-equivariant spectra. Indeed when $n = lm$ we have two commuting maps

$$R_l, F_l\colon T^{C_n} \to T^{C_m}.$$

Here T^{C_n} and T^{C_m} are considered as ordinary (non-equivariant) spectra. The map F_l, called the *Frobenius* map, is simply the inclusion of fixed points $(C_m \subset C_n)$. The map R_l, called the *restriction* map, is the composite

$$R_l\colon T^{C_n} = (\rho_C^{\#}T^C)^{C_m} \xrightarrow{s_C} (\rho_C^{\#}\Phi^C T)^{C_m} \xrightarrow{r_C} T^{C_m}$$

where $C = C_l$ and $s_C\colon T^C \to \Phi^C T$ is the map from (2.4.10), and where r_C is the cyclotomic structure map.

Definition 2.5.2. If T is a cyclotomic spectrum, then

$$\mathrm{TC}(T;p) = \operatorname*{holim}_{\overleftarrow{\mathbb{I}_p}} T^{C_{p^s}}, \quad \mathrm{TC}(T) = \operatorname*{holim}_{\overleftarrow{\mathbb{I}}} T^{C_n}.$$

For a functor with smash product L, we write $\mathrm{TC}(L) = \mathrm{TC}(T(L))$ and similarly for $\mathrm{TC}(L;p)$.

The homotopy limit which defines $\mathrm{TC}(T;p)$ may be formed in two steps. First we can take the homotopy limit over F_p (resp. R_p). Since R_p and F_p commute, R_p (resp. F_p) induces a self-map of this homotopy limit, and we may take the homotopy fixed points. More precisely, let

$$\mathrm{TR}(T;p) = \operatorname*{holim}_{\overleftarrow{R_p}} T^{C_{p^s}}, \quad \mathrm{TF}(T;p) = \operatorname*{holim}_{\overleftarrow{F_p}} T^{C_{p^s}}. \tag{2.5.3}$$

Then F_p induces an endomorphism of $\mathrm{TR}(T;p)$ and R_p an endomorphism of $\mathrm{TF}(T;p)$, and

$$\mathrm{TC}(T;p) \cong \mathrm{TR}(T;p)^{h\langle F_p\rangle} \cong \mathrm{TF}(T;p)^{h\langle R_p\rangle}.$$

The homotopy inverse limit of a string of maps $\cdots \to X_n \to X_{n-1} \to \cdots$ is a homotopy equivalent to the categorical limit provided each map is a fibration. Here $\langle F_p\rangle$ is the free monoid on F_p and $X^{h\langle F_p\rangle}$ denotes the $\langle F_p\rangle$-homotopy fixed points of X, or in other words, the homotopy fiber of $\mathrm{id} - F_p$. This was the definition used for $\mathrm{TC}(T;p)$ in [BHM].

There is a similar description of $\mathrm{TC}(L)$. Let

$$\mathrm{TR}(T) = \operatorname*{holim}_{\substack{\longleftarrow \\ R}} T^{C_n}, \quad \mathrm{TF}(T) = \operatorname*{holim}_{\substack{\longleftarrow \\ F}} T^{C_n}, \tag{2.5.4}$$

then

$$\mathrm{TC}(T) = \mathrm{TR}(T)^{hF} = \mathrm{TF}(T)^{hR},$$

where hF denotes the homotopy fixed set of the multiplicative monoid of natural numbers acting of $\mathrm{TR}(T)$ through the maps F_s, $s \geq 1$. The inclusions $\{1\} \subset \mathbb{I}_p \subset \mathbb{I}$ induce maps

$$\mathrm{TC}(T) \to \mathrm{TC}(T,p) \to T.$$

The following theorem, basically due to Goodwillie, cf. [HM], sect. 3, tells us that $\mathrm{TC}(L)$ is not really a stronger functor than the collection $\mathrm{TC}(L,p)$ for all primes p.

Theorem 2.5.5. *The projections* $\mathrm{TC}(T) \to \mathrm{TC}(T;p)$ *induce an equivalence of* $\mathrm{TC}(T)$ *with the fiber product of the* $\mathrm{TC}(T;p)$*'s over* T. *Moreover, the functors agree after p-adic completion,* $\mathrm{TC}(T)^\wedge_p \simeq \mathrm{TC}(T;p)^\wedge_p$. □

Remark 2.5.6. T. Goodwillie has introduced the following alternative definition of $\mathrm{TC}(L)$ which has the advantage of allowing an integral description of Waldhausen's reduced A-theory, cf. [G5].

The fixed set $\mathrm{TH}(L)^{C_n}$ has the natural S^1/C_n-action so each $\rho^\#_{C_n}\mathrm{TH}(L)^{C_n}$ is a spectrum with an S^1-action (an S^1-spectrum indexed on $\mathcal{U}^{S^1}$). If $n = rm$ then

$$F_r \colon \rho^\#_{C_n}\mathrm{TH}(L)^{C_n} \to \rho^\#_{C_m}\mathrm{TH}(L)^{C_m}$$

satisfies

$$F_r(\theta^r x) = \theta F_r(x), \quad \theta \in S^1.$$

Let M be the semi-direct product

$$M = \{(r,\theta) \mid r \in \mathbb{N},\ \theta \in S^1,\ \theta r = r\theta^r\}.$$

It acts on

$$\mathrm{TR}(L) = \operatorname*{holim}_{\overleftarrow{R}} \rho^{\#}_{C_n} \mathrm{TH}(L)^{C_n}.$$

Goodwillie defines:

$$\mathcal{TC}(L) = \mathrm{TR}(L)^{hM} = (\mathrm{TR}(L)^{hS^1})^{h\mathbb{N}},$$

and shows that its p-adic completion is equivalent to $\mathrm{TC}(L,p)^{\wedge}_p$.

In later chapters we shall be concerned with the calculation of $\mathrm{TC}(L)$ primarily for the FSP $\tilde{A}$ associated to a ring, cf. (2.3.3). In this case we write $T(A)$ and $\mathrm{TC}(A)$ etc. instead of $T(\tilde{A})$ and $\mathrm{TC}(\tilde{A})$.

Since $T(L)$ and its fixed points are (-1)-connected spectra, $\mathrm{TC}(L)$ is always (-2)-connected. In [HM], sect. 2 we calculated the component groups $\pi_0 T(A)^{C_{p^n}}$, and in particular:

Theorem 2.5.7. *For a commutative ring A, there is a natural isomorphism*

$$I\colon W(A,p) \to \pi_0 \mathrm{TR}(A,p)$$

where $W(A,p)$ denotes the p-typical Witt vectors. Moreover, the self map F on $\mathrm{TR}(A,p)$ corresponds to the Frobenius map of Witt-vectors.

It follows that we have the exact sequence

$$\mathrm{TC}_0(A,p) \longrightarrow W(A,p) \xrightarrow{1-F} W(A,p) \longrightarrow \mathrm{TC}_{-1}(A,p) \longrightarrow 0$$

for the two lowest dimensional homotopy groups of $\mathrm{TC}(A,p)$. The left hand arrow is often injective, but not always.

Addendum 2.5.8. For finite subgroups $H \subset K$ of the circle, there is a map $\Sigma^{\infty}_{S^1}(S^1/K_+) \to \Sigma^{\infty}_{S^1}(S^1/H_+)$, namely the Thom collapse map of an equivariant embedding $G/H \subset G/K \times V$. It induces a map of spectra

$$F\big(\Sigma^{\infty}_{S^1}(S^1/H_+), T(A)\big) \to F\big(\Sigma^{\infty}_{S^1}(S^1/K_+), T(A)\big)^{S^1},$$

that is a map

$$V\colon T(A)^H \to T(A)^K,$$

well-defined up to homotopy. In particular we get

$$V\colon \pi_0 T(A)^{C_{p^n}} \to \pi_0 T(A)^{C_{p^{n+1}}}.$$

Theorem 2.5.7 extends to the statement that there is an isomorphism

$$I\colon \pi_0 T(A)^{C_{p^n}} \xrightarrow{\cong} W_n(A,p)$$

into the p-typical Witt vectors of length $n+1$ with $\pi_0 F$, $\pi_0 R$ and $\pi_0 V$ corresponding to Frobenius, Restriction and Verschiebung, cf. [HM], theorem 2.3.

We close with two remarks of homotopy theoretic nature.

Remark 2.5.9. Given an $\mathcal{L}$ or S-ring E (cf. 2.3.11), the direct construction THH(E) from [EKMM] is *not* cyclotomic. The price one pays for making the spectrum multiplication $\mu\colon E \wedge_S E \to E$ associative is that there are no "diagonal fixed points" under the cyclic group action on the S-smash powers, and this prevents the cyclotomic property. Passing to Jeff Smith's associated FSP $\tilde{E}$ is one way around this. There might be other ways.

Remark 2.5.10. For a commutative FSP L, one can iterate the construction TC(L) to obtain $\mathrm{TC}^{(n)}(L)$ for each $n \geq 1$, cf. [HM], sect. 3.6. In view of the calculational results of sect. 4 below it is an interesting challenge in homotopy theory to study $\mathrm{TC}^{(n)}(\mathbb{F}_p)$ and $\mathrm{TC}^{(n)}(\mathbb{Z}_p)$.

2.6 The cyclotomic trace.

We begin by defining the K-theory of an FSP. Given L we can consider the associated infinite loop space

$$QL = \varinjlim \Omega^x L(S^x).$$

The components

$$\pi_0 QL = \varinjlim \pi_x L(S^x)$$

is a ring, and we denote by $\widehat{\mathrm{GL}}_1(L) \subset QL$ the subspace of invertible components. This is by definition a group-like monoid.

We let $\mathrm{M}_n(L)$ be the FSP of $n \times n$ matrices over L defined as

$$\mathrm{M}_n(L)(X) = F(\mathbf{n}_+, \mathbf{n}_+ \wedge L(X)), \qquad \mathbf{n} = \{1, \dots, n\}$$

and set $\widehat{\mathrm{GL}}_n(L) = \widehat{\mathrm{GL}}_1(\mathrm{M}_n(L))$, again a group-like monoid. Direct sum of matrices give maps

$$\widehat{\mathrm{GL}}_n(L) \times \widehat{\mathrm{GL}}_m(L) \to \widehat{\mathrm{GL}}_{n+m}(L)$$

which induces a monoid structure on the disjoint union of the $B\widehat{\mathrm{GL}}_n(L)$. Its group-completion is $K(L)$, that is:

$$K(L) = \Omega B\Big(\coprod_{n=0}^{\infty} B\widehat{\mathrm{GL}}_n(L)\Big) \sim B\widehat{\mathrm{GL}}_\infty(L)^+ \times \mathbb{Z}, \qquad (2.6.1)$$

where the superscript $+$ is Quillen's plus construction.

If A is a unital ring, and $\tilde{A}$ the associated FSP, cf. (2.3.3), then $\pi_0 Q(\tilde{A}) = A$, and the natural map $Q\tilde{A} \to A$ is an equivalence. It follows that

$$B\widehat{\mathrm{GL}}_n(\tilde{A}) \to B\widehat{\mathrm{GL}}_n(A)$$

is an equivalence, and in turn that

$$K(\tilde{A}) \to B\mathrm{GL}(A)^+ \times \mathbb{Z} = K(A)$$

is an equivalence. Thus $K(\tilde{A})$ is just another model for Quillen's $K(A)$-space (the version where $K_0(A) = \mathbb{Z}$, rather than the projective class group).

If L is the FSP $\tilde{G}$ of (2.3.4) associated to a topological group-like monoid G, then $K(\tilde{G})$ is a model for Waldhausen's $A(BG)$, again the version with $\pi_0 A(BG) = \mathbb{Z}$.

The space $K(L)$ is an infinite loop space, that is, it is the zero'th space of a connective spectrum which again will be denoted $K(L)$. The deloopings are not as concrete as the deloopings of $\mathrm{TH}(L)$ and $\mathrm{TC}(L)$ above. One has to use the abstract machinery of Segal's Γ-spaces or the equivalent machinery May's operads, or the original approach of Boardman-Vogt.

The *cyclotomic trace* from [BHM] is a spectrum map

$$\mathrm{trc}\colon K(L) \to \mathrm{TC}(L).$$

It is highly technical to construct, so I shall here only give a rough outline of the ideas involved to the extend it throws light on the definition of $\mathrm{TC}(L)$. The interested reader can consult the original source, and [HM], sect. 1.6 for the equivalence of the abstract Γ-space delooping of $\mathrm{TC}(L)$ and the concrete one above.

I begin by recalling K. Dennis' trace map in the linear situation,

$$\mathrm{Tr}\colon K(A) \to \mathrm{HH}(A) \qquad (2.6.2)$$

Remember here that $\mathrm{HH}(A)$ denotes the topological realization of the standard cyclic abelian group $Z_\bullet(A)$. We proceed simplicially, and consider

$$N_\bullet \mathrm{GL}_n(A) \xrightarrow{I_\bullet} N_\bullet^{\mathrm{cy}}(\mathrm{GL}_n(A)) \xrightarrow{S_\bullet} Z_\bullet(\mathrm{M}_n(A)) \qquad (2.6.3)$$

with

$$I(g_1, \cdots, g_r) = (g_0, g_1, \ldots, g_r), \quad g_0 = (g_1 \cdots g_r)^{-1}$$
$$S(g_0, \ldots, g_r) = g_0 \otimes \cdots \otimes g_n.$$

We have the simplicial map

$$\begin{aligned} &\mathrm{Tr}_\bullet^{(n)} \colon Z_\bullet(\mathrm{M}_n(A)) \to Z_\bullet(A), \\ &\mathrm{Tr}_r^{(n)}(X_0 \otimes \cdots \otimes X_r) = \sum X_0(i_0, i_1) \otimes \cdots \otimes X_r(i_r, i_0). \end{aligned} \tag{2.6.4}$$

It induces a homotopy equivalence

$$\mathrm{Tr}^{(n)} \colon \mathrm{HH}(\mathrm{M}_n(A)) \xrightarrow{\sim} \mathrm{HH}(A). \quad \text{(Morita invariance)}$$

Indeed, if $i \colon A \to \mathrm{M}_n(A)$ is the inclusion which maps $a \in A$ into the matrix with a on the $(1,1)$ entry and zero elsewhere, then the simplicial map

$$Z_\bullet(i) \colon Z_\bullet(A) \to Z_\bullet(\mathrm{M}_n(A))$$

induces a map from $\mathrm{HH}(A)$ to $\mathrm{HH}(\mathrm{M}_n(A))$ which is an inverse to $\mathrm{Tr}^{(n)}$. Consider the composition of (2.6.3) and (2.6.4):

$$\begin{aligned} &\mathrm{Tr}_\bullet \colon N_\bullet(\mathrm{GL}_n(A)) \to \mathbb{Z}_\bullet(A) \\ &\mathrm{Tr}_r = \mathrm{Tr}_r^{(n)} \circ S_r \circ I_r - s_0^{(r-1)}(n) \end{aligned}$$

where s_0 is the degeneracy operator in $Z_\bullet(A)$. It is easy to check that

$$\begin{tikzcd} N_\bullet(\mathrm{GL}_n(A)) \arrow[rr] \arrow[dr, "\mathrm{Tr}_\bullet"'] & & N_\bullet(\mathrm{GL}_{n+1}(A)) \arrow[dl, "\mathrm{Tr}_\bullet"] \\ & Z_\bullet(A) & \end{tikzcd}$$

is commutative, so the topological realization of $\mathrm{Tr}_\bullet$ induces the map in (2.6.2).

The above linear trace map can be generalized to give

$$\mathrm{tr} \colon K(L) \to \mathrm{THH}(L)$$

for each FSP, but two issues have to be addressed: $\widehat{\mathrm{GL}}_n(L)$ has no strict inverses and (2.6.3) does note make sense a priori in $\mathrm{THH}_\bullet(L)$.

There is a standard way to get around the lack of strict inverses by group completing the monoid, see below. For now we simply use (2.1.8):

$$|N^{\mathrm{cy}} B\widehat{\mathrm{GL}}_n(L)| \sim \Lambda B\widehat{\mathrm{GL}}_n(L)$$

and replace $|I_\bullet|$ by the inclusion of $B\widehat{\mathrm{GL}}_n(L)$ into the free loop space as the constant loops. The second map

$$S_\bullet \colon N_\bullet^{\mathrm{cy}}\widehat{\mathrm{GL}}_n(L) \to \mathrm{THH}_\bullet(L)$$

maps a string $(g_0, \dots, g_r)$ into the smash product $g_0 \wedge \cdots \wedge g_r \in \mathrm{THH}_r(L)$ upon thinking of each g_i as a limit of maps $S^{x_i} \to \mathrm{M}_n(L)(S^{x_i})$. Finally we have Morita invariance:

Recall our convention that two S^1-spaces are called C_∞-equivariant if they are connected by a string of S^1-maps which induce equivalences of C-fixed sets for every finite subgroup of S^1.

Proposition 2.6.5. *For every FSP there is a C_∞-equivalence*

$$\mathrm{THH}^X(\mathrm{M}_n(L)) \sim \mathrm{THH}^X(L)$$

which defines a C_∞-equivalence of the associated equivariant spectra $T(\mathrm{M}_n(L))$ and $T(L)$.

Proof. I briefly sketch a proof, modelled upon the linear case treated above. This approach is different from the one of [BHM]. Details can be found in a forthcoming paper by C. Schlichtkrull, [Sch]. See also [DM2]. We can rewrite

$$\mathrm{M}_n(L)(X) = \prod^n \bigvee^n L(X)$$

and have the subfunctor

$$W_n(L)(X) = \bigvee^n \bigvee^n L(X).$$

It is an "FSP without unit". We can restrict the simplicial space

$$\mathrm{THH}_\bullet(L;V) \colon \Delta^{\mathrm{op}} \to \text{spaces} \quad (\mathrm{THH}_\bullet(L;V) = \mathrm{THH}_\bullet^{S^V}(L))$$

to the subcategory of injective maps in Δ^{op}, i.e. we forget degeneracy operators and consider $\mathrm{THH}_\bullet(L;V)$ only as a Δ-space (presimplicial space) in the sense if [RS]. Then $\mathrm{THH}_\bullet(W_n(L);V)$ is defined, and the inclusion of Δ-sets

$$\mathrm{THH}_\bullet(W_n(L);V) \to \mathrm{THH}_\bullet(\mathrm{M}_n(L);V)$$

induces an equivalence upon applying the realization functor $\|\cdot\|$ of Δ-sets. On the other hand, the projection

$$\|\mathrm{THH}_\bullet(\mathrm{M}_n(L);V)\| \to |\mathrm{THH}_\bullet(\mathrm{M}_n(L);V)|$$

is a C-equivalence.

Second, suitable evaluation defines a map from $W_n(L)$ to L, analogous to the linear situation

$$\mathrm{ev}\colon \mathrm{Hom}_A(A^{\oplus n}, A) \otimes A^{\oplus n} \to A,$$

and we can imitate the map of (2.6.4) to get

$$\mathrm{tr}^{(n)}\colon \mathrm{THH}_\bullet(W_n(L); V) \to \mathrm{THH}_\bullet(L; V),$$

This induces the required equivalence. □

The resulting trace map, valid for any FSP, tr: $K(L) \to \mathrm{THH}(L)$, is Bökstedt's topological version of Dennis' trace map. It is far from obvious, however, that tr is a map of spectra. See the final paragraph of this section.

It is time to explain how to lift the topological Dennis trace into the fixed sets $\mathrm{TH}(L)^C$ of the finite subgroups $C \subset S^1$. Suppose first that G is a (topological) group.

The simplicial map (cf. 2.1.7)

$$\delta_C\colon N_\bullet(G) \xrightarrow{I_\bullet} N_\bullet^{\mathrm{cy}}(G) \xrightarrow{\Delta_C} sd_C N_\bullet^{\mathrm{cy}}(G)^C$$

has topological realization homotopic to

$$\delta_c\colon BG \xrightarrow{I} \Lambda BG \xrightarrow{\Delta_c} (\Lambda BG)^C, \quad \Delta_c(\lambda(\theta)) = \lambda(\theta^c)$$

where $c = |C|$ and I is the inclusion into the constant loops.

For a subgroup $C_0 \subset C$, the composition of δ_c with the inclusion of $(\Lambda BG)^C$ into $(\Lambda BG)^{C_0}$ is equal to δ_{c_0} since Δ_c leaves constant loops invariant. On the simplicial level it is therefore not surprising that there is a natural homotopy between δ_{C_0} and the composition

$$|N_\bullet(G)| \xrightarrow{\delta_C} |sd_C N_\bullet^{\mathrm{cy}}(G)|^C \xrightarrow{\mathrm{incl}} |sd_C N_\bullet^{\mathrm{cy}}(G)|^{C_0} \xrightarrow{D} |sd_{C_0} N_\bullet^{\mathrm{cy}}(G)|^{C_0}$$

where D is the subdivision homeomorphism of lemma 2.1.6. Thus if we write

$$F_{C/C_0}\colon |sd_C N_\bullet^{\mathrm{cy}}(G)|^C \longrightarrow |sd_{C_0} N_\bullet^{\mathrm{cy}}(G)|^{C_0}$$
$$R_{C/C_0}\colon |sd_C N_\bullet^{\mathrm{cy}}(G)|^C \xrightarrow{\cong} |sd_{C_0} N_\bullet^{\mathrm{cy}}(G)|^{C_0}, \quad R_{C/C_0} = \Delta_{C/C_0}^{-1}$$

we have

$$F_{C/C_0} \circ \delta_C \sim \delta_{C_0}, \quad R_{C/C_0} \circ \delta_C = \delta_{C_0} \tag{2.6.6}$$

with a specified homotopy in the first relation. There results a diagram

$$\begin{tikzcd} & \varprojlim_R |sd_C N^{\mathrm{cy}}_\bullet(G)|^C \arrow[d, "F"] \\ |N_\bullet(G)| \arrow[ur, "\delta_C"] \arrow[dr, "\delta_C"'] & \\ & \varprojlim_R |sd_C N^{\mathrm{cy}}_\bullet(G)|^C \end{tikzcd}$$

which is homotopy commutative via a specified homotopy equivalence and thus a map

$$\delta \colon |N_\bullet(G)| \to \left(\varprojlim_R |sd_C N^{\mathrm{cy}}_\bullet(G)|^C\right)^{hF} \qquad (2.6.7)$$

into the homotopy fiber of $F - \mathrm{id}$.

We want to apply (2.6.7) to $G = \widehat{\mathrm{GL}}_k(L)$, so must generalize to group-like topological monoids where $I_\bullet$ a priori does not exist.

The standard way to overcome the lack of strict inverses is to group complete the topological monoid: there are functors $G \mapsto G^\vee$ and $G \mapsto G^\wedge$, and natural transformations $G \leftarrow G^\vee \to G^\wedge$ which induces equivalences of the constructions $N_\bullet(\ )$ and $N^{\mathrm{cy}}_\bullet(\ )$ when G is group-like. Here $G^\vee$ is a free monoid and $G^\wedge$ is a topological group, cf. [BF, p. 331] or [G2], sect. I.1.8. Consider the homotopy pull-back

$$\begin{tikzcd} B'G \arrow[r] \arrow[d, "N"] & \operatorname{holim}_{\overleftarrow{R}} |sd_C N^{\mathrm{cy}}_\bullet(G^\vee)|^C \arrow[d, "\sim"] \\ |N_\bullet(G^\vee)| \arrow[r, "\delta^h"] & \operatorname{holim}_{\overleftarrow{R}} |sd_C N^{\mathrm{cy}}_\bullet(G^\wedge)|^C \end{tikzcd}$$

where δ^h is the composition of δ with the inclusion of $\varprojlim_R$ into $\operatorname{holim}_{\overleftarrow{R}}$.

When $G = \widehat{\mathrm{GL}}_k(L)$, the simplicial map

$$S_\bullet \colon N^{\mathrm{cy}}_\bullet(\widehat{\mathrm{GL}}_k(L)) \to \mathrm{THH}_\bullet(\mathrm{M}_k(L))$$

is cyclic in the sense of Connes, and the induced maps on C-fixed sets commute with the F and R-maps. One gets a map

$$\operatorname{holim}_{\overleftarrow{R}} \left| sd_C N^{\mathrm{cy}}_\bullet(\widehat{\mathrm{GL}}_k(L))^C \right|^{hF} \to \operatorname{holim}_{\overleftarrow{R}} \left| sd_C \mathrm{THH}_\bullet(\mathrm{M}_k(L))^C \right|^{hF}.$$

The target is $\mathrm{TC}(\mathrm{M}_k(L))$. It is by (2.6.5) and (2.4.6) equivalent to $\mathrm{TC}(L)$. Thus we have for each k a string of maps

$$\mathrm{trc} \colon B\widehat{\mathrm{GL}}_k \xleftarrow{\sim} B'\widehat{\mathrm{GL}}_k(L) \longrightarrow \mathrm{TC}(\mathrm{M}_k(L)) \xrightarrow{\sim} \mathrm{TC}(L)$$

which in turn induces a map from $K(L)$ to $\mathrm{TC}(L)$, the cyclotomic trace.

In order to see that trc is in fact a map of spectra, one uses e.g. Segal's Γ-structure on $\amalg B\widehat{\mathrm{GL}}_k(L)$ and a corresponding structure on $\amalg \mathrm{THH}(\mathrm{M}_k(L))$, cf. [BHM], sect. 4. Finally, the associated abstract delooping of $\mathrm{TC}(L)$ and the concrete one from sect. 2.5 agree by [HM], sect. 1.6. I return to a different solution to this in the next chapter, but it is in order to mention that the Γ-space approach is based upon the following

Proposition 2.6.8 ([BHM]). *For a product of* FSP*'s there is a* C_∞-*equivalence*

$$\mathrm{TH}(L_1 \times L_2) \sim \mathrm{TH}(L_1) \times \mathrm{TH}(L_2). \qquad \square$$

3 The relative theorems

The end result of this chapter is a proof of the following conjecture from [G5]: Let $f\colon L_1 \to L_2$ be a map of FSP's such that $\pi_0 L_1 \to \pi_0 L_2$ is a surjection of rings with nilpotent kernel. Then

$$\begin{array}{ccc} K(L_1) & \longrightarrow & \mathrm{TC}(L_1) \\ \downarrow & & \downarrow \\ K(L_2) & \longrightarrow & \mathrm{TC}(L_2) \end{array}$$

becomes homotopy Cartesian after profinite completion.

The proof proceeds in three steps, due to Dundas-McCarthy [DM1], McCarthy [Mc] and Dundas [D], respectively, and uses Goodwillie's black magic: *calculus of functors*, [G3], [G4]. The exposition is based on these papers and on [DM2]. I have had invaluable help from B. Dundas with some of the details below.

3.1 Calculus of functors.

Calculus of functors is a general procedure, devised by Goodwillie, for proving relative theorems as above. The reader is referred to [G3], [G4] for more details.

We shall consider certain functors

$$F\colon \mathrm{s_\bullet sets} \to \{\text{prespectra}\}$$

from the category of simplicial sets (or spaces) to the category of prespectra. I here use prespectra indexed only on $\mathbb{R}^n$, not the coordinate free ones of May.

The functors we consider are supposed to satisfy the following two axioms:

(i) A homotopy $f_t\colon X_1 \to X_2$ induces a natural homotopy $F(f_t)\colon F(X_1) \to F(X_2)$.

(ii) For each $X \in \mathrm{s_\bullet sets}$ and each prime p, the mod p homotopy groups satisfy

$$\pi_i(F(X); \mathbb{F}_p) = \varinjlim \pi_i F(X^{(\alpha)}; \mathbb{F}_p)$$

where $X^{(\alpha)}$ runs over the finite subcomplexes of X.

Condition (i) implies that F is a homotopy functor; (ii) is called the p-limit axiom.

Given such an F and a fixed $(X, x) \in \mathrm{s_\bullet sets_*}$ there is a new functor on $\mathrm{s_\bullet sets_*}$, namely

$$\Phi(Y) = \mathrm{fib}(F(X \vee Y) \to F(X)).$$

Consider the commutative diagram

$$\begin{array}{ccc} \Phi(Y) & \longrightarrow & \Phi(C_+Y) \\ \downarrow & & \downarrow \\ \Phi(C_-Y) & \longrightarrow & \Phi(S^1 \wedge Y) \end{array}$$

where $C_\pm Y$ are the two cones in the reduced suspension $S^1 \wedge Y$. The standard retractions of the cones induce retractions of the two off diagonal terms, and in turn a map

$$\Phi(Y) \to \Omega\Phi(S^1 \wedge Y). \tag{3.1.1}$$

The homotopy colimit of these maps is called the *differential* of F at (X, x). More importantly for our purpose we have

Definition 3.1.2. The derivative of F at (X, x) is the prespectrum whose n'th term is

$$\partial_x F(X)(\mathbb{R}^n) = \Phi(S^n)$$

and with structure maps

$$S^1 \wedge \partial_x F(X)(\mathbb{R}^n) \to \partial_x F(X)(\mathbb{R}^{n+1})$$

being the adjoints of (3.1.1).

For example the derivative of the functor

$$F(X) = \Sigma^\infty(X_+^n)$$

of the suspension spectrum of the n-fold Cartesian power of X is

$$\partial_x F(X) = \bigvee^n \Sigma^\infty(X_+^{n-1})$$

We next define Goodwillie's concept of analytic functors. The simplest ones are the linear functors. They are the homotopy functors which map a homotopy coCartesian square

$$\begin{array}{ccc} Y_\emptyset & \longrightarrow & Y_{\{2\}} \\ \downarrow & & \downarrow \\ Y_{\{1\}} & \longrightarrow & Y_{\{1,2\}} \end{array}$$

into a homotopy Cartesian square

$$\begin{array}{ccc} L(Y_\emptyset) & \longrightarrow & L(Y_{\{2\}}) \\ \downarrow & & \downarrow \\ L(Y_{\{1\}}) & \longrightarrow & L(Y_{\{1,2\}}), \end{array}$$

and has $F(*) \sim *$.

Here homotopy Cartesian and homotopy coCartesian means that the canonical maps

$$Y_\emptyset \xrightarrow{a} \underleftarrow{\operatorname{holim}}(Y_{\{1\}} \to Y_{\{1,2\}} \leftarrow Y_{\{2\}})$$

$$Y_{\{1,2\}} \xleftarrow{b} \underrightarrow{\operatorname{holim}}(Y_{\{1\}} \leftarrow Y_\emptyset \to Y_{\{2\}})$$

are equivalences.

To define the concept of *analytic functors*, one needs to consider n-dimensional cubes of spaces and spectra, i.e. functors

$$\mathfrak{X}\colon \mathcal{P}(S) \to \mathcal{C}, \quad \mathcal{C} = \mathrm{s_\bullet sets}, \{\mathrm{spectra}\}$$

from the category of posets of the finite set S. If $S = \mathbf{n}$, then $\mathfrak{X}$ is called an n-cube. Generalizing the above, $\mathfrak{X}$ is called k-Cartesian or k-coCartesian if

$$\mathfrak{X}(\emptyset) \xrightarrow{a} \underleftarrow{\operatorname{holim}}_{\mathcal{P}_0(S)} \mathfrak{X}, \quad \mathcal{P}_0 = \mathcal{P}(S) - \{\emptyset\}$$

$$\mathfrak{X}(S) \xleftarrow{b} \underrightarrow{\operatorname{holim}}_{\mathcal{P}_1(S)} \mathfrak{X}, \quad \mathcal{P}_1 = \mathcal{P}(S) - \{S\}$$

are k-equivalences.

Given $U \subset T \subset S$ the face $\partial_U^T \mathfrak{X}$ is the $T - U$ cube given by

$$\partial_U^T \mathfrak{X}(V)\mathfrak{X}(V \cup U).$$

We shall consider ***strongly coCartesian*** cubes, that is, cubes $\mathfrak{X}$ where each 2-dimensional face $\partial_U^T \mathfrak{X}$ is k-coCartesian for all k. This implies in particular that the total cube is homotopy coCartesian.

Definition 3.1.3. A functor F: $\mathrm{s_\bullet sets} \to \{\text{spectra}\}$ is called stably n-excisive if the following statement $E_n(c, \kappa)$ is true for some numbers c and κ:

$E_n(c, \kappa)$: Given any strongly coCartesian $(n+1)$-cube $\mathfrak{X}$ with $\mathfrak{X}(\emptyset) \to \mathfrak{X}(\{s\})$ k_s-connected and $k_s \geq \kappa$, then the $(n+1)$-cube $F(\mathfrak{X})$ is $(-c + \Sigma k_s)$-Cartesian.

Definition 3.1.4. A homotopy functor F is called ρ-analytic if for some q, independent of n, F satisfies $E_n(n\rho - q, \rho + 1)$ for all n.

Let (A, P) be a pair of a unitary ring and an A bimodule P. For each based simplicial set $Y_\bullet \in \mathrm{s_\bullet sets_*}$ we have the simplicial ring

$$(A \ltimes P)(Y_\bullet) = A \oplus \tilde{P}(Y_\bullet), \quad \tilde{P}(Y_\bullet) = P[Y_\bullet]/P[*_\bullet]$$

with multiplication

$$(a_1, p_1)(a_2, p_2) = (a_1 a_2, a_1 p_2 + p_1 a_2).$$

We shall see in sect. 3.3 below that the realization of the simplicial functors

$$\begin{aligned} &[r] \to \mathrm{hF}\big(K(A \oplus \tilde{P}(Y_r)) \to K(A)\big) \\ &[r] \to \mathrm{hF}\big(\mathrm{TC}(A \oplus \tilde{P}(Y_r)) \to \mathrm{TC}(A)\big) \end{aligned} \tag{3.1.5}$$

both satisfies $E_n(-2 - n, 0)$ for all n; thus they are (-1)-analytic; hF = homotopy fiber.

The main theorem of Goodwillie's about analytic functors is the following

Theorem 3.1.6. *Suppose $\theta\colon F \to G$ is a natural transformation between ρ-analytic functors such that $\partial_x\theta(X)\colon \partial_x F(X) \to \partial_x G(X)$ is an equivalence of prespectra. Then for every $(\rho+1)$-connected map $Y \to K$ in $\mathrm{s_\bullet Sets_*}$, the diagram*

$$\begin{array}{ccc} F(Y) & \longrightarrow & G(Y) \\ \downarrow & & \downarrow \\ F(K) & \longrightarrow & G(K) \end{array}$$

is homotopy Cartesian.

The cyclotomic trace of sect. 2.6 defines a natural transformation between the two functors in (3.1.5), which turns out to satisfy the conditions of the above theorem after profinite completion, cf. sect. 3.2 and sect. 3.3 below, so theorem 3.1.6 implies that

$$\begin{array}{ccc} K(A\oplus \tilde{P}(Y_\bullet))^\wedge & \longrightarrow & \mathrm{TC}(A\oplus \tilde{P}_\bullet(Y))^\wedge \\ \downarrow & & \downarrow \\ K(A)^\wedge & \longrightarrow & \mathrm{TC}(A)^\wedge \end{array} \tag{3.1.7}$$

is homotopy Cartesian, where the upper horizontal line is calculated degree-wise. Indeed, the homotopy fibers of the vertical arrows are the relative theories of (3.1.5), and they vanish for $Y_\bullet = *_\bullet$, so agree by the theorem.

3.2 K- and THH of additive split exact categories.

In this section $\mathcal{C}$ is an additive split exact category, e.g. the category of projective modules $\mathcal{P}_A$ over a ring, or its subcategory $\mathcal{F}_A$ of free modules.

Recall that Waldhausen in [W3] associated to $\mathcal{C}$ a simplicial set (in fact a simplicial category) $S_\bullet\mathcal{C}$. The r-simplices of objects in $S_r\mathcal{C}$ is the set of diagrams

$$\begin{array}{ccccccc} C_1 \rightarrowtail & C_2 & \rightarrowtail & C_3 & \rightarrowtail \cdots \rightarrowtail & C_r \\ & \downarrow & & \downarrow & & \downarrow \\ & C_{12} & \rightarrowtail & C_{13} & \rightarrowtail \cdots \rightarrowtail & C_{1r} \\ & & & \downarrow & & \downarrow \\ & & & C_{23} & \rightarrowtail \cdots \rightarrowtail & C_{2r} \\ & & & & & \downarrow \\ & & & & & \vdots \\ & & & & & \downarrow \\ & & & & & C_{r-1,r} \end{array} \tag{3.2.1}$$

with

$$0 \to C_{ij} \to C_{ik} \to C_{jk} \to 0$$

a (split) exact sequence. Thus $S_r\mathcal{C} = 0$ for $r = 0$, $S_1\mathcal{C} = \mathcal{C}$, and in general $S_r\mathcal{C}$ is the category of flags involving r objects with choice of quotients.

The objects of $S_\bullet\mathcal{C}$ form a simplicial set where d_0 forgets the first row (divides out C_1) and where d_i contracts the flag by forgetting C_i and the row $C_{i,\bullet}$. The degeneracy operator s_i inserts an extra C_i, so for example s_0 and s_1 from $S_1\mathcal{C} = \mathcal{C}$ to $S_2\mathcal{C}$ sends C to $0 \rightarrowtail C$ and $C \rightarrowtail C$, respectively.

The nerve of the isomorphism category $iS_\bullet(\mathcal{C})$ of flags defines a bisimplicial space

$$[s],[r] \to N_s(iS_r\mathcal{C}).$$

The loop space of its realization is Waldhausen's definition of $K(\mathcal{C})$,

$$K(\mathcal{C}) = \Omega|N_\bullet(iS_\bullet\mathcal{C})| \tag{3.2.2}$$

(of course, Waldhausen's definition applies to much more general situations). In order to relate this to the previous definition of K-theory, recall from sect. 2.2 that we can realize the double complex in two steps. Let us first realize the r-direction. There is an obvious map

$$\Delta^1 \times N_s(iS_1(\mathcal{C})) \to |N_s(iS_\bullet\mathcal{C})|$$

(the inclusion of the 1-skeleton), and since $S_0\mathcal{C} = \{0\}$, it factors over

$$\sigma\colon S^1 \wedge N_s(iS_1\mathcal{C}) \to |N_s(iS_\bullet\mathcal{C})|.$$

Realizing the s-direction and adjoining σ we get a map

$$|N_\bullet(\mathcal{C})| \xrightarrow{\mathrm{ad}(\sigma)} \Omega|N_\bullet(iS_\bullet\mathcal{C})| \tag{3.2.3}$$

which turns out to be a *group completion*, cf. [W3], sect. 1.6. When $\mathcal{C} = \mathcal{F}_A$ then

$$|N_\bullet(\mathcal{F}_A)| = \coprod_{n=0}^{\infty} B\mathrm{GL}_n(A)$$

so the above definition of K-theory agrees with the earlier one from sect. 2.6 in this case.

The iterated degeneracy operator in the s-direction defines a map

$$s\colon N_0(iS_\bullet\mathcal{C}) \to N_s(iS_\bullet\mathcal{C})$$

with a one-sided inverse d_0^s, and gives a map

$$s\colon |N_0(iS_\bullet\mathcal{C})| \xrightarrow{\sim} |N_\bullet(iS_\bullet\mathcal{C})|. \tag{3.2.4}$$

Corollary 1.4.1 of [W3] states that (3.2.4) is an equivalence. Thus one can recast (3.2.2) as

$$K(\mathcal{C}) \cong \Omega|\operatorname{ob} S_\bullet\mathcal{C}| = \Omega\big|[r] \to S_r\mathcal{C}\big|. \tag{3.2.5}$$

When $\mathcal{C} = \mathcal{P}_A$, the projective modules, then (3.2.4), and (3.2.6) below, implies that

$$K_0(\mathcal{P}_A) = \pi_1|\operatorname{ob} S_\bullet\mathcal{P}_A| \cong H_1(\operatorname{ob} S_\bullet\mathcal{P}_A) = K_0(A),$$

the projective class group of the ring A.

The $S_\bullet$ construction can be iterated, and defines a (-1)-connected spectrum whose $(n-1)$'st term is $\Omega|\operatorname{ob} S_\bullet^{(n)}\mathcal{C}|$. The natural maps

$$|\operatorname{ob} S_\bullet^{(n-1)}\mathcal{C}| \to \Omega|\operatorname{ob} S_\bullet^{(n)}\mathcal{C}| \tag{3.2.6}$$

are equivalences for $n > 1$, cf. [W3], proposition 1.5.3, so the $S_\bullet$-construction deloops $K(\mathcal{C})$ beyond the first step

$$K(\mathcal{C}) \simeq \Omega^n|\operatorname{ob} S_\bullet^{(n)}\mathcal{C}|, \quad n \geq 1.$$

We now turn to $\mathrm{THH}(\mathcal{C})$, following [DM2]. We have already presented the definition in (2.3.9), and can try to imitate the two key results above, (3.2.3) and (3.2.4), for $N_\bullet(-)$ replaced by $\mathrm{THH}_\bullet(-)$. In fact, since $\mathrm{THH}(-)$ is already a spectrum, one expects that (3.2.3) be an equivalence, and this indeed happens. Here are some details.

We can think of (3.2.2) as

$$K(\mathcal{C}) = \Omega\big|[r] \to |N_\bullet(iS_r\mathcal{C})|\big|$$

and can similarly consider the simplicial space

$$[r] \to \mathrm{THH}(S_r\mathcal{C}) = |\mathrm{THH}_\bullet(S_r\mathcal{C})|$$

which we denote for short $\mathrm{THH}(S_\bullet\mathcal{C})$. There are maps

$$\sigma\colon S^1 \wedge \mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(S_\bullet\mathcal{C})$$
$$s\colon \mathrm{THH}_0(S_r\mathcal{C}) \to \mathrm{THH}_\bullet(S_r\mathcal{C})$$

defined as above.

Theorem 3.2.7. ([DM2]) *The maps σ and s define equivalences*

(i) $\quad \mathrm{THH}(\mathcal{C}) \xrightarrow{\simeq} \Omega|\mathrm{THH}(S_\bullet\mathcal{C})|$

(ii) $\quad \varinjlim \Omega^n|\mathrm{THH}_0(S_\bullet^{(n)}\mathcal{C})| \xrightarrow{\simeq} \varinjlim \Omega^n|\mathrm{THH}(S_\bullet^{(n)}\mathcal{C})|$

Proof of (i) (sketch). The proof is modelled upon [W3], proposition 1.53. Consider the functor $S_n\mathcal{C} \to \mathcal{C}^n$ which to the flag (3.2.1) associates the n-tuple $(C_1, C_{12}, \dots, C_{n-1,n})$. It induces an equivalence

$$\mathrm{THH}(S_n\mathcal{C}) \xrightarrow{\simeq} \mathrm{THH}(\mathcal{C})^n.$$

This is an application of Morita invariance and (2.6.8): the trace of a triangular matrix only depends on the diagonal entries.

Now recall for any simplicial space $X_\bullet$ the simplicial path space construction $P_\bullet X_\bullet$. It has n-simplices $P_n X_\bullet = X_{n+1}$ and face and degeneracy operators are shifted up by 1. The extra degeneracy $s_0\colon X_n \to X_{n+1}$ not used in $P_\bullet X_\bullet$ gives an equivalence $|P_\bullet X_\bullet| \sim X_0$, so $|P_\bullet X_\bullet|$ is contractible when X_0 consists of a single point. Moreover we have a sequence

$$X_1 \xrightarrow{s_0^\bullet} P_\bullet X_\bullet \xrightarrow{d_0} X_\bullet$$

of simplicial sets upon considering X_1 the constant simplicial space. We now have for each r the diagram

$$\begin{tikzcd} \mathrm{THH}(\mathcal{C}) \arrow[r] \arrow[d, equal] & \mathrm{THH}(P_r S_\bullet \mathcal{C}) \arrow[r] \arrow[d, "\sim"] & \mathrm{THH}(S_r \mathcal{C}) \arrow[d, "\sim"] \\ \mathrm{THH}(\mathcal{C}) \arrow[r] & \mathrm{THH}(\mathcal{C})^{r+1} \arrow[r] & \mathrm{THH}(\mathcal{C})^r \end{tikzcd}$$

so the sequence

$$\mathrm{THH}(\mathcal{C}) \to \mathrm{THH}(P_\bullet S_\bullet \mathcal{C}) \to \mathrm{THH}(S_\bullet \mathcal{C})$$

is a degreewise homotopy fibration, and hence becomes a homotopy fibration after realization, since THH$(-)$ is equivalent to an abelian group complex, see (3.2.8) below. Finally $|\mathrm{THH}(P_\bullet S_\bullet \mathcal{C})| \sim *$. □

The proof of (ii) is more delicate and requires some rewritings of THH$(\mathcal{C})$ which we now present. We have for each number x the simplicial abelian group

$$\mathcal{C}^x(c_0, c_1) = \mathrm{Hom}_{\mathcal{C}}(c_0, c_1) \otimes \tilde{\mathbb{Z}}(S^x_\bullet) = \mathcal{C}(x_0, x_1) \otimes \tilde{\mathbb{Z}}(S^x_\bullet)$$

and associated simplicial sets, one for each r,

$$V_{r,\bullet}(\mathcal{C}, \mathbf{x}) = \bigvee_{c_0, \dots, c_r \in \mathcal{C}} \delta\big(\mathcal{C}^{x_0}(c_0, c_r) \wedge \mathcal{C}^{x_1}(c_1, c_0) \wedge \cdots \wedge \mathcal{C}^{x_r}(c_r, c_{r-1})\big)$$

where δ denotes the diagonal in the stated multisimplicial set. There are simplicial maps

$$V_{r,\bullet}(\mathcal{C}, \mathbf{x}) \xrightarrow{d_i} V_{r-1,\bullet}(\mathcal{C}, d_i\mathbf{x})$$
$$V_{r,\bullet}(\mathcal{C}, \mathbf{x}) \xrightarrow{s_i} V_{r+1,\bullet}(\mathcal{C}, s_i\mathbf{x})$$

and we let

$$\mathrm{THH}_{r,\bullet}(\mathcal{C}) = \underset{\mathbf{x} \in I^{r+1}}{\underrightarrow{\mathrm{holim}}}\, s_\bullet \mathcal{C}\big(\delta S^{x_0}_\bullet \wedge \cdots \wedge S^{x_r}_\bullet, V_{r,\bullet}(\mathcal{C}, \mathbf{x})\big).$$

Here $s_\bullet \mathcal{C}$ is the simplicial mapping space. This gives a bisimplicial set $\mathrm{THH}_{r,s}(\mathcal{C})$ whose realization is the THH$(\mathcal{C})$ defined in sect. 2.3.

We now vary the definition by replacing $V_{r,\bullet}(\mathcal{C},\mathbf{x})$ by

$$\begin{aligned} & V^{\oplus}_{r,\bullet}(\mathcal{C},\mathbf{x}) \\ = & \bigoplus_{c_0,\dots,c_r\in\mathcal{C}} \mathcal{C}^{x_0}(c_0,c_r)\otimes\tilde{\mathbb{Z}}(\mathcal{C}^{x_1}(c_1,c_0)\wedge\cdots\wedge\mathcal{C}^{x_r}(c_r,c_{r-1})) \\ = & \bigoplus \mathcal{C}^{x_0}(c_0,c_r)\otimes\tilde{\mathbb{Z}}(\mathcal{C}^{x_1}(c_1,c_0))\otimes\cdots\otimes\tilde{\mathbb{Z}}(\mathcal{C}^{c_r}(c_r,c_{r-1})) \end{aligned} \tag{3.2.8}$$

and write $\mathrm{THH}^{\oplus}_{\bullet,\bullet}(\mathcal{C})$ for the corresponding bisimplicial abelian group. The inclusion of $V_{\bullet,\bullet}(\mathcal{C},\mathbf{x})$ in $V^{\oplus}_{\bullet,\bullet}(\mathcal{C},\mathbf{x})$ induces a simplicial map

$$\theta_\bullet\colon \mathrm{THH}_\bullet(\mathcal{C})\to\mathrm{THH}^{\oplus}_\bullet(\mathcal{C}) \tag{3.2.9}$$

which is an equivalence. This follows from lemma 2.3.7, the well-known isomorphisms

$$\pi_i\tilde{M}(Y)\cong H_i(Y;M)\cong\pi_{i+x}(Y\wedge\tilde{M}(S^x)),\quad i<x,$$

and because the inclusion of the wedge in the product (direct sum) is $2\Sigma x_\nu-1$ connected.

Recall that $S_2\mathcal{C}$ is the category of (split) exact sequences in $\mathcal{C}$. The morphisms are commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & C_0' & \longrightarrow & C_1' & \longrightarrow & C_2' & \longrightarrow & 0 \end{array}$$

We use the notation (f_0,f_1,f_2) for this morphism. The simplicial functors

$$d_0,d_1,d_2\colon S_2\mathcal{C}\to\mathcal{C}=S_1\mathcal{C}$$

induce simplicial maps

$$\bar{d}_0,\bar{d}_1,\bar{d}_2\colon \mathrm{THH}^{\oplus}_\bullet(S_2\mathcal{C})\to\mathrm{THH}^{\oplus}_\bullet(\mathcal{C})$$

and we have (in preparation for the proof of theorem 3.2.7 (ii))

Lemma 3.2.10. *For each r, there are natural transformations*

$$T^{(\nu)}_r\colon \mathrm{THH}^{\oplus}_r(\mathcal{C})\to\mathrm{THH}^{\oplus}_r(S_2\mathcal{C}),\quad \nu=1,2$$

such that

$$\bar{d}_0T^{(1)}_r=\mathrm{id},\quad \bar{d}_2T^{(1)}_r=0=\bar{d}_0T^{(2)}_r,\quad \bar{d}_1T^{(1)}_r=\bar{d}_1T^{(2)}_r,\quad \bar{d}_2T^{(2)}_r=s^r_0\circ d^r_0.$$

Proof. Given objects $\mathbf{c} = (c_0, \dots, c_r) \in \mathcal{C}^{r+1}$ and morphisms $\alpha_0 \in \mathcal{C}(c_0, c_r)$, $\alpha_k \in \mathcal{C}(c_k, c_{k-1})$ for $k = 1, \dots, r$ we define objects $\Delta_k^{(\nu)} = \Delta_k^{(\nu)}(\mathbf{c}, \boldsymbol{\alpha})$ of $S_2\mathcal{C}$

$$\Delta_k^{(1)}\colon 0 \longrightarrow C_r \xrightarrow{i_1} C_r \oplus C_k \xrightarrow{\pi_2} C_k \longrightarrow 0$$
$$\Delta_k^{(2)}\colon 0 \longrightarrow C_r \xrightarrow{\left(\begin{smallmatrix}1\\ \beta_k\end{smallmatrix}\right)} C_r \oplus C_k \xrightarrow{(\beta_k, -1)} C_k \longrightarrow 0$$

where $\beta_k = \alpha_{k+1} \cdots \alpha_r$ for $0 \le k < r$ and $\beta_r = 1$.

With these notions we define $t_\bullet^{(\nu)}$

$$t_r^{(1)}\colon s_\bullet\mathcal{C}(S_\bullet^{\mathbf{x}}, V_{r,\bullet}^{\oplus}(\mathcal{C}, \mathbf{x})) \to s_\bullet\mathcal{C}(S_\bullet^{\mathbf{x}}, V_{r,\bullet}^{\oplus}(S_2\mathcal{C}; \mathbf{x}))$$
$$t_r^{(2)}\colon s_\bullet\mathcal{C}(S_\bullet^{\mathbf{x}}, V_{r,\bullet}^{\oplus}(\mathcal{C}; \mathbf{x})) \to s_\bullet\mathcal{C}(S_\bullet^{\mathbf{y}}, V_{r,\bullet}^{\oplus}(S_2\mathcal{C}; \mathbf{y}))$$

where $\mathbf{y} = (x_0 + \cdots + x_r, x_1 + \cdots + x_r, \dots, x_r)$ and $S_\bullet^{\mathbf{x}} = S_\bullet^{x_0} \wedge \cdots \wedge S_\bullet^{x_r}$. If $(x_0, \dots, x_r) = (0, \dots, 0)$ the formulas are:

$$\begin{aligned}
&t_r^{(1)}(\alpha_0 \otimes \cdots \otimes \alpha_r) \\
&\qquad = \left(1, \left(\begin{smallmatrix}0 & 0\\ \alpha_0 & \alpha_0\end{smallmatrix}\right), \alpha_0\right) \otimes \left(1, \left(\begin{smallmatrix}1 & 0\\ 0 & \alpha_1\end{smallmatrix}\right), \alpha_1\right) \otimes \cdots \otimes \left(1, \left(\begin{smallmatrix}1 & 0\\ 0 & \alpha_r\end{smallmatrix}\right), \alpha_r\right) \\
&t_r^{(2)}(\alpha_0 \otimes \cdots \otimes \alpha_r) \\
&\qquad = \left(\alpha_0 \cdots \alpha_r, \left(\begin{smallmatrix}0 & 0\\ \alpha_0 & \alpha_0\end{smallmatrix}\right), 0\right) \otimes \left(1, \left(\begin{smallmatrix}1 & 0\\ 0 & \alpha_1\end{smallmatrix}\right), \alpha_1\right) \otimes \cdots \otimes \left(1, \left(\begin{smallmatrix}1 & 0\\ 0 & \alpha_r\end{smallmatrix}\right), \alpha_r\right).
\end{aligned}$$

For general $\mathbf{x}$, one needs to replace $\mathcal{C}^x(c, d)$ by the equivalent $s_\bullet\mathcal{C}(c, d \otimes \tilde{\mathbb{Z}}(S^x))$ and one must use suitable suspension maps

$$s_\bullet\mathcal{C}(c, d) \xrightarrow{\sim} s_\bullet\mathcal{C}(c \otimes \tilde{\mathbb{Z}}(S^y), d \otimes \tilde{\mathbb{Z}}(S^y))$$

in order to define both $\Delta_k^{(2)}(\mathbf{c}, \boldsymbol{\alpha})$ and $t_r^{(2)}$. Details are left for the reader to carry out, who can also consult [DM2]. We set

$$T_r^{(\nu)} = \underrightarrow{\operatorname{holim}}_{I^{r+1}}\, t_r^{(\nu)}\colon \mathrm{THH}_r^{\oplus}(\mathcal{C}) \to \mathrm{THH}_r^{\oplus}(S_2\mathcal{C}).$$

These are the required maps, and the required relations are obvious to check. □

We assumed $\mathcal{C}$ to be an additive split exact category, so $S_2\mathcal{C}$ is equivalent to $\mathcal{C} \times \mathcal{C}$: there are functors both ways whose composites are naturally isomorphic to the identity. Indeed,

$$S_2\mathcal{C} \xrightarrow{(d_0, d_2)} \mathcal{C} \times \mathcal{C}, \quad \mathcal{C} \times \mathcal{C} \xrightarrow{s_0 \oplus s_1} S_2\mathcal{C}$$

are the two functors. One composite is the identity; the other sends each object to an isomorphic object, and one may easily construct the required *natural* isomorphism.

Functors such as $\mathrm{THH}_r^{\oplus}(\mathcal{C})$ does not map equivalent categories into homotopy equivalent spaces (check e.g. $r = 0$). However the composite functor $\mathrm{THH}_r^{\oplus}(S_\bullet\mathcal{C})$ does have this property.

Lemma 3.2.11. *Let $g_0, g_1 : \mathcal{C} \to \mathcal{D}$ be naturally isomorphic functors between exact categories. Then there is a simplicial functor*

$$G\colon \Delta[1]_\bullet \times S_\bullet\mathcal{C} \to S_\bullet\mathcal{D}$$

which restricts to $S_\bullet f_0$ and $S_\bullet f_1$ at the two ends. Here $\Delta[1]_\bullet$ is the simplicial 1-simplex considered as a discrete category. □

The lemma is proved in [W3], although only stated on objects. Since simplicial homotopies are preserved by functors, the induced maps

$$\mathrm{THH}_r^{\oplus}(S_\bullet\mathcal{C}) \rightrightarrows \mathrm{THH}_r^{\oplus}(S_\bullet\mathcal{D})$$

are homotopic. In particular

$$\mathrm{THH}_r^{\oplus}(S_\bullet S_2\mathcal{C}) \sim \mathrm{THH}_r^{\oplus}(S_\bullet\mathcal{C} \times S_\bullet\mathcal{C}).$$

Consider a functor X from additive exact categories to simplicial or topological groups with $X(0) = 0$ and $X(\mathcal{C} \times \mathcal{D}) \xrightarrow{\sim} X(\mathcal{C}) \times X(\mathcal{D})$. Let $Y(\mathcal{C}) = Y(S_\bullet\mathcal{C})$, a bisimplicial abelian group. Then

$$\overline{d}_1 \sim \overline{d}_0 + \overline{d}_2 = Y(S_2\mathcal{C}) \to Y(\mathcal{C}) \tag{3.2.12}$$

where $\overline{d}_i = Y(d_i)$. This follows from the homotopy commutative diagram

$$\begin{tikzcd} Y(S_2\mathcal{C}) \arrow[r, "Y(d_0\times d_1)"] \arrow[dr, "\overline{d}_0\times\overline{d}_2"'] & Y(\mathcal{C}\times\mathcal{C}) \arrow[r, "Y(s_0\oplus s_2)"] & Y(S_2\mathcal{C}) \\ & Y(\mathcal{C})\times Y(\mathcal{C}) \arrow[u, "\overline{i}_1+\overline{i}_2"] \arrow[ur, "\overline{s}_0+\overline{s}_1"'] & \end{tikzcd}$$

The right hand triangle homotopy commutes because it does so after composing with the equivalence $\overline{d}_0 \times \overline{d}_2$, $d_2 s_0 = 0 = d_0 s_1$. The left hand triangle commutes because $\overline{pr}_1 \times \overline{pr}_2$ is a homotopy inverse to $\overline{i}_1 + \overline{i}_2$. Finally the horizontal composite is homotopic to the identity by lemma 3.2.11, and $\overline{d}_1(\overline{s}_0 + \overline{s}_1)$ is equal to addition.

The functor $\mathrm{THH}_r^{\oplus}(\mathcal{C})$ does not preserve product, but the functor

$$X_r(\mathcal{C}) = \varinjlim \Omega^k \mathrm{THH}_r^{\oplus}(S_\bullet^{(k)}\mathcal{C}) \tag{3.2.13}$$

does. This is formal and true for any functor Z with $Z(0) = 0$ as the map of multisimplicial sets

$$Z(S_\bullet^{(k)}\mathcal{C} \times S_\bullet^{(k)}\mathcal{C}) \to Z(S_\bullet^{(k)}\mathcal{C}) \times Z(S_\bullet^{(k)}\mathcal{C})$$

is an isomorphism when the sum of the $2k$ simplicial degrees is less than $2k$ (because $S_0\mathcal{C} = 0$). In particular the map is $2k$-connected.

Proof of theorem 3.2.7 (ii). With the notation from (3.2.13) have from (3.2.12)

$$\overline{d}_1 \sim \overline{d}_0 + \overline{d}_2 \colon X_r(S_\bullet S_2\mathcal{C}) \to X_r(S_\bullet\mathcal{C}).$$

Lemma 3.2.10 can be applied to X_r as well as to $\mathrm{THH}_r^\oplus$, and shows that the composition

$$X_r(\mathcal{C}) \xrightarrow{d_0^r} X_0(\mathcal{C}) \xrightarrow{s_0^r} X_r(\mathcal{C})$$

is homotopic to the identity. Indeed

$$\mathrm{id} = d_0\mathrm{Tr}^{(1)} + \overline{d}_2\mathrm{Tr}^{(1)} = \overline{d}_1\mathrm{Tr}^{(2)} \sim \overline{d}_0\mathrm{Tr}^{(2)} + \overline{d}_2\mathrm{Tr}^{(2)} = d_0^r s_0^r.$$

The other composition is obviously the identity.

Thus $X_\bullet(\mathcal{C})$ is a simplicial space in which the simplicial structure maps are all homotopy equivalences; for such $X_0(\mathcal{C}) \sim |X_\bullet(\mathcal{C})|$ □

Theorem 3.2.7 allows a slick definition of the topological Dennis trace

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{THH}(\mathcal{C}),$$

namely as the composite

$$\Omega|S_\bullet\mathcal{C}| \to \Omega|\mathrm{THH}_0(S_\bullet\mathcal{C})| \to \Omega|\mathrm{THH}(S_\bullet\mathcal{C})| \sim \mathrm{THH}(\mathcal{C}) \tag{3.2.14}$$

where the first map is induced from sending an object $\mathbf{C} \in S_r\mathcal{C}$ into $\mathrm{id}_{\mathbf{C}} \in \mathrm{Hom}_{S_\bullet\mathcal{C}}(\mathbf{C}, \mathbf{C})$.

We can introduce the spectrum $\mathrm{TH}(\mathcal{C})$ either by iterating the $S_\bullet$-construction or by introducing a dummy variable similar to what we did in the case of $\mathrm{THH}(L)$. The corresponding deloops (spectra) are equivalent by the standard argument which makes use of both deloops:

$$\mathrm{THH}(S_\bullet^{(n)}\mathcal{C}) \sim \Omega^n\mathrm{THH}^{S^n}(S_\bullet^{(n)}\mathcal{C}) \sim \mathrm{THH}^{S^n}(\mathcal{C})$$

(cf. [BM] sect. 1).

If we use the iteration of the $S_\bullet$-construction to define the spectrum, then it is obvious that the map in (3.2.14) is a map of spectra.

Later in the chapter we shall consider $\mathrm{THH}^{\oplus}(\mathcal{C};M)$ where $M\colon \mathcal{C}^0\times\mathcal{C}\to \mathrm{Ab}$. It is defined by replacing $V^{\oplus}_{r,\bullet}(\mathcal{C},\mathbf{x})$ by

$$V^{\oplus}_{r,\bullet}(\mathcal{C};M,\mathbf{x})=\bigoplus M^{x_0}(c_0,c_r)\otimes\tilde{\mathbb{Z}}\mathcal{C}^{x_1}(c_1,c_0)\otimes\cdots\otimes\tilde{\mathbb{Z}}\mathcal{C}^{x_r}(c_r,c_{r-1}).\quad(3.2.15)$$

If $\mathcal{C}$ is the catetegory of projective or free modules and M is an A-bimodule then

$$M^{x_0}(c_0,c_r)=\mathrm{Hom}_A(c_0,c_r\otimes_A M)$$

extends to a functor on $S_\bullet\mathcal{C}$, and the proof of theorem 3.2.7 extends word for word to give

$$\mathrm{THH}^{\oplus}(A,M)\sim\varinjlim_{p}\Omega^p\big(|\mathrm{THH}^{\oplus}_0(S^{(p)}_\bullet\mathcal{P}_A,M)|\big).$$

Moreover, in this linear situation, one can omit the homotopy colimit over x_0 in the definition of $\mathrm{THH}^{\oplus}_0$. Indeed, for any number x

$$\begin{aligned}\mathrm{Hom}_A(a,b)&\xrightarrow{\sim}s_\bullet\mathcal{P}_A\big(a\otimes\tilde{\mathbb{Z}}(S^x_\bullet),b\otimes\tilde{\mathbb{Z}}(S^x_\bullet)\big)\\&\xrightarrow{\cong}s_\bullet\mathrm{Sets}_*\big(S^x_\bullet,s_\bullet\mathcal{P}_A(a,b\otimes\tilde{\mathbb{Z}}(S^x_\bullet))\big)\end{aligned}$$

where $S^x_\bullet=\Delta[x]_\bullet/\partial$ is the simplicial x-sphere, and $\mathrm{Hom}_A(a,b)$ is considered the constant simplicial group, cf. [Q1]. We have proved

Corollary 3.2.16. *For an A-bimodule M,*

$$\mathrm{THH}(A,M)\sim\varinjlim_{p}\Omega^p\left|\bigoplus_{\mathbf{c}\in S^{(p)}_\bullet\mathcal{P}_A}\mathrm{Hom}(\mathbf{c},\mathbf{c}\otimes M)\right|.\qquad\square$$

Remark 3.2.17. If we let $\mathbf{x}=\mathbf{0}$ in (3.2.15) we obtain a bisimplicial abelian group $V^{\oplus}_{r,s}(\mathcal{C},M,\mathbf{0})$ which is constant in the s-direction. Following [DM1] we write

$$F_r(\mathcal{C},M)=V_{r,0}(\mathcal{C},M,\mathbf{0})\cong\bigoplus_{c_r\to\cdots\to c_0\in N_r\mathcal{C}}M(c_0,c_r).$$

The homotopy groups of $|F_\bullet(\mathcal{C},M)|$, or equivalently the homology groups of the associated chain complex $F_*(\mathcal{C},M)$, is usually denoted $H_*(\mathcal{C};M)$ and is called the (non-additive) homology of $\mathcal{C}$ with coefficients in M. Dundas and McCarthy proves theorem 3.2.7 for this functor by an argument almost identical to the above. The diagram

$$\begin{array}{ccc}\Omega^\infty|F_\bullet(S^{(\infty)}_\bullet\mathcal{P}_A;M)| & \longrightarrow & \Omega^\infty|\mathrm{THH}^{\oplus}(S^{(\infty)}_\bullet\mathcal{P}_A;M)|\\ \uparrow\sim & & \uparrow\sim\\ \Omega^\infty|F_0(S^{\infty}_\bullet\mathcal{P}_A;M)| & \xrightarrow{\sim} & \Omega^\infty|\mathrm{THH}^{\oplus}_0(S^{(\infty)}_\bullet\mathcal{P}_A;M)|\end{array}$$

then shows that $\pi_*\mathrm{THH}(A;M) \cong H_*(\mathcal{P}_A, M)$. This is a special case of a theorem due to Pirashvili and Waldhausen, [PW].

3.3 Stable K- and TC-theory.

Let A be a ring, V an A-bimodule and $A \ltimes V$ the semiproduct ring. We may replace V by the $(n-1)$-connected simplicial A-bimodule $\tilde{V}(S^n_\bullet)$ and consider the simplicial ring $A \ltimes \tilde{V}(S^n_\bullet)$. This can be thought of as a small deformation of A. We want to measure the difference between $K(A)$ and $K(A \ltimes \tilde{V}(S^n_\bullet))$.

Recall from [W1] that K-theory of a simplicial ring $R_\bullet$ is defined as

$$K(R_\bullet) = \Omega B\left(\coprod_n B\widehat{\mathrm{GL}}_n(R_\bullet)\right) = \mathbb{Z} \times B\widehat{\mathrm{GL}}_\infty(R_\bullet)^+ \tag{3.3.1}$$

where $\widehat{\mathrm{GL}}_n(R_\bullet) \subset \mathrm{M}_n(R_\bullet)$ is the group like simplicial monoid of matrices which map to invertible matrices in $\mathrm{M}_n(\pi_0 R_\bullet)$, and $B\widehat{\mathrm{GL}}_n(R_\bullet) = |N_\bullet(\widehat{\mathrm{GL}}_n(R_\bullet))|$. Alternatively we can use (2.6.1) for the FSP

$$\tilde{R}_\bullet(X) = \big|[p] \to \tilde{R}_p(X)\big|$$

Indeed $K(R_\bullet) \simeq K(\tilde{R}_\bullet)$. There is another, more straightforward possibility, namely to consider the simplicial monoid $\mathrm{GL}(R_\bullet)$ with p-simplices $\mathrm{GL}(R_p)$. This leads to degreewise K-theory, $\big|[p] \to K(R_p)\big|$, which however is not a homotopy invariant of $R_\bullet$, and does not agree with (3.3.1) in general.

For a map of (simplicial) rings $R_\bullet \to S_\bullet$ we write

$$K(R_\bullet \to S_\bullet) = \mathrm{hF}\big(K(R_\bullet) \to K(S_\bullet)\big).$$

Lemma 3.3.2 ([G2]). *Let $R_\bullet$ be a simplicial ring and $I_\bullet \subset R_\bullet$ a (degreewise) square zero ideal. Then*

$$K(R_\bullet \to R_\bullet/I_\bullet) \sim \big|[p] \to K\big(R_p \to R_p/I_p\big)\big|.$$

(There is a little gap in the argument from [G2], lemma I.2.2 where it was used without proof that the diagram

$$\begin{array}{ccc} B\widehat{\mathrm{GL}}(R_\bullet) & \longrightarrow & B\mathrm{GL}(R_\bullet) \\ \downarrow & & \downarrow \\ B\widehat{\mathrm{GL}}(R_\bullet)^+ & \longrightarrow & B\mathrm{GL}(R_\bullet)^+ \end{array} \quad \text{(Quillen's plus)}$$

is homotopy Cartesian. This was repaired in [FOV]).

Definition 3.3.3 ([W2]). The stable K-theory $K^s(A;V)$ is the functor

$$K^s(A;V) = \varinjlim_n \Omega^{n+1} K\big(A \ltimes \tilde{V}(S^n_\bullet) \to A\big)$$

The limit system in the definition, i.e. the maps from $K\big(A \oplus \tilde{V}(S^n_\bullet) \to A\big)$ to $\Omega K\big(A \oplus \tilde{V}(S^{n+1}_\bullet) \to A\big)$, are the ones given in (3.1.1). $K^s(A;V)$ is a spectrum whose k'th space may be given by replacing the $(n+1)$'st loop space in the definition by the $(n+1-k)$'th loop space.

The lemma above shows that we might as well have defined the stable K-theory degreewise as

$$K^s(A,V) = \varinjlim_n \Omega^{n+1} \big|[r] \to K\big(A \ltimes \tilde{V}(S^n_r) \to A\big)\big| \tag{3.3.4}$$

which is the point of view to be used below.

The reader can note the resemblance of K^s with the algebraic "tangent space" of K-theory:

$$TK(A,V) = K(A \ltimes V \to A).$$

In $K^s(A,V)$ one has further made V "small" by passing to the simplicial setting, where one can make V "close to the 0-module" upon replacing it with $\tilde{V}(S^n_\bullet)$, which "approaches 0" in the homotopy sense as $n \to \infty$. Further details on stable K-theory can be found in [K].

The above can be generalized to the setting of FSP's. Indeed, let L be an FSP and M a module over L as in sect. 2.3. One defines

$$(L \ltimes M[n])(X) = L(X) \vee (S^n \wedge M(X))$$

(one could also use $L(X) \vee M(S^n \wedge X)$ as the two definitions give stably equivalent FSP's).

$$\begin{aligned} K^s(L;M) &= \varinjlim \Omega^{n+1} K\big(L \ltimes M[n] \to L\big) \\ \mathrm{TC}^s(L;M) &= \varinjlim \Omega^{n+1} \mathrm{TC}\big(L \ltimes M[n] \to L\big). \end{aligned} \tag{3.3.5}$$

The topological Dennis trace

$$\mathrm{tr}\colon K(L,M) \to \mathrm{TH}(L,M)$$

factors over $K^s(L,M)$ and long ago, Waldhausen conjectured that the resulting map

$$K^s(L;M) \xrightarrow{\mathrm{tr}} \mathrm{TH}(L;M) \tag{3.3.6}$$

is an equivalence.

The rest of the section is a presentation of the Dundas-McCarthy proof of (3.3.6) in the linear situation, corresponding to $L = \tilde{A}$, $M = \tilde{V}$, the FSP's associated with a ring and a bimodule, and of Hesselholt's corresponding result for TC.

Consider the category $\mathcal{P}(A, V)$ of pairs (P, α) of a projective A-module P and an A-linear homomorphism $\alpha\colon P \to P \otimes_A V$. The morphisms from (P, α) to (P', α') are maps $f\colon P \to P'$ such that

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P \otimes_A V \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes 1} \\ P' & \xrightarrow{\alpha'} & P' \otimes_A V \end{array}$$

are commutative.

The K-theory of $\mathcal{P}(A, V)$ will be denoted $K^{\mathrm{cy}}(A; V)$; in the simplicial setting we make the following

Definition 3.3.7. For a simplicial A-bimodule $V_\bullet$,

$$K^{\mathrm{cy}}(A; V_\bullet) = \big|[r] \to K(\mathcal{P}(A; V_r))\big|.$$

Clearly, $K^{\mathrm{cy}}(A; 0) = K(A)$ and we set $\tilde{K}^{\mathrm{cy}}(A; V_\bullet) = \mathrm{hF}\big(K(A; V_\bullet) \to K(A)\big)$. **Lemma 3.3.8.** *There are homotopy equivalences*

(i) $K\big(A \ltimes V_\bullet \to A\big) \sim \tilde{K}^{\mathrm{cy}}(A; \delta\tilde{V}_\bullet(S^1_\bullet))$

(ii) $K^s(A; V) \sim \underset{\overrightarrow{n}}{\mathrm{holim}}\,\Omega^{n+1}\tilde{K}^{\mathrm{cy}}(A, \tilde{V}(S^{n+1}_\bullet))$.

Proof. The second statement follows from the first since $N_\bullet\tilde{V}(X_\bullet) = \tilde{V}(S^1_\bullet \wedge X_\bullet)$, so we have left to prove (i).

Since we are considering the relative groups, we may replace $\mathcal{P}(A, V)$ by $\mathcal{F}(A, V)$ in the definitions. But

$$N_\bullet(i\mathcal{F}_V) \sim \coprod_{k=1}^{\infty} N_\bullet(im_k\mathcal{F}_V) \tag{3.3.9}$$

where $m_k\mathcal{F}_V$ is the full subcategories of pairs (A^k, α), $\alpha \in \mathrm{M}_k(V)$ and where i indicates that we are only considering isomorphisms. An r-simplex of

$N_\bullet(im_k\mathcal{F}(A,V))$ is determined by a string $(\alpha_0; f_1, \dots, f_k)$ with $f_i \in \mathrm{GL}_k(A)$ and $\alpha_0 \in \mathrm{M}_k(V)$. Thus

$$N_\bullet(im_k\mathcal{F}(A,V)) \cong N_\bullet^{\mathrm{cy}}(\mathrm{GL}_k(A); \mathrm{M}_k(V)) \tag{3.3.10}$$

upon sending $(\alpha_0; f_1, \dots, f_k)$ into $((f_1 \cdots f_k)^{-1}; f_1, \dots, f_k)$, cf. (2.1.1). From (2.1.10) we have

$$\delta N_\bullet^{\mathrm{cy}}(\mathrm{GL}_k(A), N_\bullet \mathrm{M}_k(V)) \cong N_\bullet(\mathrm{GL}_k(A \ltimes V)) \tag{3.3.11}$$

so, extending (degreewise) to simplicial modules $V_\bullet$, and taking group completions, the result follows. $\square$

Theorem 3.3.12 ([DM1]). *For any A-bimodule M, the trace defines an equivalence*

$$K^s(A, M) \xrightarrow{\sim} \mathrm{THH}(A, M).$$

Proof. We use the model $\mathrm{THH}^\oplus$ for THH. Indeed corollary 3.2.16 gives

$$\mathrm{THH}(A,M) \sim \varinjlim \Omega^p \left| \left(\bigoplus_{\mathbf{C} \in S_\bullet^{(p)} \mathcal{P}_A} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A M) \right) \right|.$$

By definition

$$K^{\mathrm{cy}}(A; M) = \Omega^p |K(S_\bullet^{(p)} \mathcal{P}(A,M))| = \Omega^p \left| \left(\coprod_{\mathbf{C} \in S_\bullet^{(p)} \mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A M) \right) \right|,$$

with $\mathcal{P} = \mathcal{P}_A$. We shall compare these definitions when M is replaced by the simplicial bimodule $W_\bullet = \tilde{M}(S_\bullet^n)$, M applied to the simplicial n-sphere. Both functors are defined degreewise

$$\begin{aligned} \mathrm{THH}(A, W_\bullet) &= |[r] \to \mathrm{THH}(A; W_r)| \\ K^{\mathrm{cy}}(A, W_\bullet) &= |[r] \to K^{\mathrm{cy}}(A; W_r)|\,. \end{aligned}$$

Actually, we are interested in the relative functor $\tilde{K}^{\mathrm{cy}}(A, W_\bullet)$. Consider the coCartesian diagram

$$\begin{array}{ccc} \left| \coprod_{\mathbf{C} \in S_\bullet^{(p)} \mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A W_\bullet) \right| & \longrightarrow & |S_\bullet^{(p)} \mathcal{P}| \\ \Big\downarrow {\scriptstyle -/|S_\bullet^{(p)}\mathcal{P}|} & & \Big\downarrow {\scriptstyle -/|S_\bullet^{(p)}\mathcal{P}|} \\ \left| \bigvee_{\mathbf{C} \in S_\bullet^{(p)} \mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A W_\bullet) \right| & \longrightarrow & * \end{array}$$

Each of the spaces are at least $(p-1)$-connected, since the $S_\bullet$-construction applied to any category adds one to the connectivity. It follows the vertical homotopy fiber is $(2p-2)$-equivalent to the space $|S_\bullet^{(p)}\mathcal{P}|$ which was divided out, and hence that the vertical homotopy fibers agree in the same range. Since $p \gg 2n$, it follows that

$$\tilde{K}^{\mathrm{cy}}(A; W_\bullet) \sim_{2n} \Omega^p \left| \bigvee_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C}\otimes_A W_\bullet) \right|.$$

It is clear from the definition of trace given in (3.2.11) that it (under the equivalences above) corresponds to the natural inclusion

$$\bigvee_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C}\otimes_A W_\bullet) \to \bigoplus_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C}\otimes_A W_\bullet).$$

This map is $(p+2n)$-connected. Indeed, the inclusion of a wedge of n-connected spaces into the product is $2n$-connected, so the corresponding map indexed over $S_r^{(p)}\mathcal{P}$ is $(2n+1)$-connected. Thus the homotopy fiber of the map in question is a bisimplicial set $F_{\bullet,\bullet}$ with $|F_{r,\bullet}|$ $(2n+1)$-connected for $r \geq p$ and $|X_{r,\bullet}|$ contractible for $r < p$ (since $|S_\bullet^{(p)}|$ is $(p-1)$-connected. The standard spectral sequence

$$H_r(H_s(F_{\bullet,\bullet})) \Rightarrow H_{r+s}(\delta F_{\bullet,\bullet})$$

is zero for $r < p$ and $s \leq 2n+1$, so gives the connectivity conclusion. The theorem now follows from the equivalences

$$K^s(A; M) = \underrightarrow{\mathrm{holim}}\, \Omega^{n+1} K^{\mathrm{cy}}(A; \tilde{M}(S_\bullet^{n+1}))$$
$$\mathrm{TH}(A; M) = \underrightarrow{\mathrm{holim}}\, \Omega^{n+1}\mathrm{TH}(A; \tilde{M}(S_\bullet^{n+1})). \qquad \square$$

We remark that the above proof also contains a proof of

Addendum 3.3.13. *For a simplicial A bimodule $V_\bullet$,*

(i) $$\tilde{K}^{\mathrm{cy}}(A; V_\bullet) = \varinjlim_p \Omega^p \left| \bigvee_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C}\otimes_A V_\bullet) \right|$$

(ii) $$\tilde{K}^{\mathrm{cy}}(A; \tilde{V}(X_\bullet)) = \varinjlim_p \Omega^p \left| \bigvee_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}} \mathrm{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C}\otimes_A V_\bullet)^{\sim}(X_\bullet) \right|.$$

Theorem 3.3.14 ([H1]). *For any* FSP L *and* L*-bimodule* M*, the profinite completions of* $\mathrm{TC}^s(L;M)$ *and* $\mathrm{TH}(L;M)$ *are equivalent.*

Proof (sketch). Recall from (3.3.5) that

$$\mathrm{TC}^s(L;M) = \varinjlim \Omega^{n+1}\mathrm{TC}\,(L \ltimes M[n] \to L)$$

where $L \ltimes M[n]$ is the FSP

$$(L \ltimes M[n])(X) = L(X) \vee (S^n \wedge M(X)).$$

We may decompose

$$(L(S^{x_0}) \vee M[n](S^{x_0})) \wedge \cdots \wedge (L(S^{x_r}) \vee M[n](S^{x_r}))$$

into a wedge, and collect the factors which contain a given number of copies of $M[n](S^x)$. This gives a decomposition of cyclic spaces

$$\mathrm{TH}(L \ltimes M[n]) = \bigvee_{a=0}^{\infty} T_a(L;M[n])$$

with $T_0(L;M[n]) = \mathrm{TH}(L)$. Moreover $T_1(L;M[n])$ is a simplicial spectrum whose k-simplices has exactly one copy of $M[n]$, but sitting at any of the $(k+1)$ positions available, i.e.

$$T_1(L;M[n])_k = C_{k+1\,+} \wedge \mathrm{TH}(L;M[n]).$$

The realization of this cyclic space is $S^1_+ \wedge \mathrm{TH}(L;M[n])$ with its natural action of S^1 (in the first factor), so

$$T_1(L;M[n]) = S^1_+ \wedge \mathrm{TH}(L;M[n]).$$

The cyclotomic structure map R_p maps

$$R_p\colon T_a(L;M[n])^{C_{p^r}} \to T_{a/p}(L;M[n])^{C_{p^{r-1}}}$$

if $p|a$ and trivially otherwise. By (2.4.6) this map is $(na-1)$-connected. Hence if $(k,p)=1$

$$T_{p^s k}(L;M[n])^{C_{p^r}} \sim_{kpn-1} T_k(L;M[n])^{C_{p^{r-s}}},$$

and again by (2.4.6), $T_k(L;M[n])^{C_{p^{r-s}}} \sim T_k(L;M[n])_{hC_{p^{r-s}}}$, which is $(kn-1)$-connected (as T_k contains k copies of the $(n-1)$-connected $M[n]$).

We are only interested in the range $< 2n$, so $T_{kp^s}(L;M[n])^{C_{p^r}}$ can be disregarded when $k>1$. Thus by theorem 2.5.5,

$$\mathrm{TC}\,(L \ltimes M[n] \to L)^\wedge_p \sim_{2n-1} \left(\underset{\mathrm{I}_p}{\operatorname{holim}} \left(\bigvee_{s=0}^{\infty} T_{p^s}(L;M[n])\right)^{C_{p^r}}\right)^\wedge_p.$$

Moreover,

$$R_p^{(s)}\colon T_{p^s}(L;M[n])^{C_{p^r}} \sim_{2n-1} T_1(L;M[n])^{C_p^{r-s}} \quad \text{for } r \geq s$$

and $T_{p^s}(L;M[n])^{C_{p^r}} \sim_{2n-1} 0$ if $r < s$. Hence

$$\left(\bigvee_{s=0}^{\infty} T_{p^s}(L;M[n])\right)^{C_{p^r}} \sim_{2n-1} \bigvee_{t=0}^{r} T_1(L;M[n])^{C_{p^t}} = \prod_{t=0}^{r} T_1(L;M[n])^{C_{p^t}}$$

(as we work with spectra, there is no difference between finite wedges and finite products). The R_p-map corresponds to projection on the first r factors, so

$$\underset{\overleftarrow{R_p}}{\operatorname{holim}} \left(\bigvee_{s=0}^{\infty} T_{p^s}(L;M[n])\right)^{C_{p^r}} = \prod_{t=0}^{\infty} T_1(L;M[n])^{C_{p^t}}$$

and by (2.5.4) one concludes that

$$\mathrm{TC}\,(L \ltimes M[n] \to L)_p^{\wedge} \sim \left(\underset{\overleftarrow{F_p}}{\operatorname{holim}}\, T_1(L;M[n])^{C_{p^t}}\right)_p^{\wedge}.$$

The action of S^1 (and hence C_{p^t}) on

$$T_1(L;M[n]) = S^1_+ \wedge \mathrm{TH}(L;M[n])$$

is free, and in this case the action can be divided out, so

$$\underset{\overleftarrow{F_p}}{\operatorname{holim}}\, T_1(L;M[n])^{C_{p^t}} \sim \underset{\longleftarrow}{\operatorname{holim}}\, (S^1/C_{p^t+} \wedge \mathrm{TH}(L;M[n]))$$

where the limit on the right is via transfers (in the suspension spectrum $\Sigma^\infty(S/C_{p^t+})$). If we identify $S^1/C_{p^t} = S^1$ then we obtain a (co)fibration of limit systems:

$$\begin{array}{ccccc}
\mathrm{TH}(L;M[n]) & \longrightarrow & S^1_+ \wedge \mathrm{TH}(L;M[n]) & \longrightarrow & S^1 \wedge \mathrm{TH}(L;M[n]) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle F_p} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{TH}(L;M[n]) & \longrightarrow & S^1_+ \wedge \mathrm{TH}(L;M[n]) & \longrightarrow & S^1 \wedge \mathrm{TH}(L;M[n])
\end{array}$$

This implies a cofibration in the limit. Since

$$\underset{\longleftarrow}{\operatorname{holim}}(\mathrm{TH}(L;M[n]),p)_p^{\wedge} \sim 0$$

we are finished. □

3.4 McCarthy's theorem.

The presentation in this section is my writeup of lectures given by McCarthy in Aarhus, July 1994.

Theorem 3.4.1 (McCarthy). *Let $R \to S$ be a surjection of rings with nilpotent kernel. Then the diagram*

$$\begin{array}{ccc} K(R)^{\wedge} & \longrightarrow & \mathrm{TC}(R)^{\wedge} \\ \downarrow & & \downarrow \\ K(S)^{\wedge} & \longrightarrow & \mathrm{TC}(S)^{\wedge} \end{array}$$

of profinitely completed spectra is homotopy Cartesian. In particular

$$K(R \to S)^{\wedge} \sim \mathrm{TC}(R \to S)^{\wedge}.$$

The obvious induction shows that it suffices to prove the theorem when the kernel is a square zero ideal; this will be assumed in the rest of the section.

Associated to a simplicial ring $R_\bullet$ we have the FSP

$$\tilde{R}_\bullet(X) = \left|[s] \to \tilde{R}_s(X)\right|.$$

We write $\mathrm{TC}(R_\bullet)$ instead of $\mathrm{TC}(\tilde{R}_\bullet)$. If $R_\bullet \to R'_\bullet$ is a simplicial equivalence (i.e. $|R_\bullet| \to |R'_\bullet|$ a homotopy equivalence) then the induced map of FSP's $\tilde{R}_\bullet \to \tilde{R}'_\bullet$ is a stable equivalence in the sense that

$$\varinjlim_n \Omega^n(\tilde{R}_\bullet(S^n)) \to \varinjlim_n \Omega^n(\tilde{R}'_\bullet(S^n))$$

is an equivalence, and in this case

$$\mathrm{TC}(R_\bullet) \xrightarrow{\sim} \mathrm{TC}(R'_\bullet)$$

cf. sect. 2.6, so $\mathrm{TC}(R_\bullet)$ only depends on the homotopy type of $R_\bullet$. On the other hand, we have the possibility of calculating TC degreewise. In contrast to K-theory where the two definitions do not agree in general we have

Proposition 3.4.2. $\mathrm{TC}(R_\bullet) \sim |[s] \to \mathrm{TC}(R_s)|$.

Proof. Since

$$\Omega^{n_0+\cdots+n_k}\left|[s] \to \tilde{R}_s(S^{n_0}) \wedge \cdots \wedge \tilde{R}_s(S^{n_k})\right| \sim$$
$$\left|[s] \to \Omega^{n_0+\cdots+n_k}(\tilde{R}_s(S^{n_0}) \wedge \cdots \wedge \tilde{R}_s(S^{n_k}))\right|$$

we see that the topological Hochschild spectrum $\mathrm{TH}(\tilde{R}_\bullet)$ can be calculated degreewise:

$$\mathrm{TH}(\tilde{R}_\bullet) \sim \Big|[s] \to \mathrm{TH}(\tilde{R}_s)\Big|.$$

The fundamental cofibration sequence of proposition 2.4.3 then shows that the same assertion is true for fixed sets

$$\mathrm{TH}(\tilde{R}_\bullet)^{C_{p^n}} \sim \Big|[s] \to \mathrm{TH}(\tilde{R}_s)^{C_{p^n}}\Big|$$

and upon taking inverse limit

$$\mathrm{TF}(\tilde{R}_\bullet, p) \sim \Big|[s] \to \mathrm{TF}(\tilde{R}_s, p)\Big|$$

cf. (2.5.3) for notation. There is a salient point here: realization does not in general commute with homotopy inverse limits; however in the above situation it does as $\mathrm{TH}(\tilde{R}_s) \sim \Omega\mathrm{THH}(\tilde{R}_s; S^1)$, so $\mathrm{TH}(\tilde{R}_s)$ is equivalent to a Kan simplicial set. For such, realization do commute with homotopy inverse limits.

Finally the homotopy fibrations

$$\mathrm{TC}(\tilde{R}_\bullet, p) \longrightarrow \mathrm{TF}(\tilde{R}_\bullet, p) \xrightarrow{R_p - \mathrm{id}} \mathrm{TF}(\tilde{R}_\bullet, p)$$
$$\mathrm{TC}(\tilde{R}_s, p) \longrightarrow \mathrm{TF}(\tilde{R}_s, p) \xrightarrow{R_p - \mathrm{id}} \mathrm{TF}(\tilde{R}_s, p)$$

show that $\mathrm{TC}(\tilde{R}_\bullet, p)$ can be calculated degreewise. Now apply theorem 2.5.5 to obtain the result for $\mathrm{TC}(\tilde{R}_\bullet)$. □

Lemma 3.4.3 ([G2]). *If the theorem is true in the special case where R is a semi-direct product ring $R = A \ltimes M$ and $S = A$ then it is true in general.*

Proof. Goodwillie associates to S a simplicial ring $\Phi_\bullet(S)$ with a simplicial map $\Phi_\bullet(S) \to S$ (when S is regarded as the constant simplicial ring) such that

(i) $\Phi_r(S)$ is free associative for each r

(ii) $|\Phi_\bullet(S)| \xrightarrow{\phi} S$ is an equivalence.

Indeed, $\Phi_\bullet(S)$ is the simplicial ring with $\Phi_r(S) = (FG)^{r+1}(S)$ where G is the forgetful functor from rings to sets and F its left adjoint free functor: $\Phi_\bullet(S)$ is the "bar-construction", cf. [G2], sect. I.1.6. Write $A_\bullet = \Phi_\bullet(S)$ and consider the (degreewise) pull-back

$$\begin{array}{ccc} B_\bullet & \longrightarrow & A_\bullet \\ \downarrow{\scriptstyle \bar{\phi}} & & \downarrow{\scriptstyle \phi} \\ R & \longrightarrow & S \end{array}$$

Then $M = \ker(B_\bullet \to A_\bullet)$ is the constant ideal $M = \ker(R \to S)$. Since ϕ, and hence $\bar{\phi}$, is an equivalence

$$K(R \to S) \sim K(B_\bullet \to A_\bullet).$$

The latter can be calculated degreewise by lemma 3.3.2,

$$K(B_\bullet \to A_\bullet) \sim |[r] \to K(B_r \to A_r)|\,.$$

Now A_r is free, so $B_r \to A_r$ is a split surjection, and hence $B_r = M_r \ltimes A_r$. With the assumption,

$$K(B_r \to A_r)^\wedge \sim \mathrm{TC}(B_r \to A_r)^\wedge$$

so in conclusion

$$K(B \to A)^\wedge \sim |[r] \to \mathrm{TC}(B_r \to A_r)^\wedge| \sim \mathrm{TC}(B \to A)^\wedge$$

by the previous proposition. □

The idea behind the proof of theorem 3.4.1 is to use calculus of functors on the cyclotomic trace

$$\mathrm{trc}\colon K\left(A \ltimes \tilde{M}(X_\bullet) \to A\right) \to \mathrm{TC}\left(A \ltimes \tilde{M}(X_\bullet) \to A\right)$$

cf. sect. 3.1. First we need:

Proposition 3.4.4. *For any ring A and bimodule M,*

$$\begin{array}{ll} \text{(i)} & X_\bullet \to K(A \ltimes \tilde{M}(X_\bullet)) \\ \text{(ii)} & X_\bullet \to \mathrm{TC}(A \ltimes \tilde{M}(X_\bullet)) \end{array}$$

are (-1)-analytic as functors from based simplicial sets to spectra.

Proof. For K-theory we can use the equivalence of lemma 3.3.8(i),

$$K\left(A \ltimes \tilde{M}(X_\bullet) \to A\right) \sim \tilde{K}^{\mathrm{cy}}(A, \tilde{M}(X_\bullet \wedge S^1_\bullet))$$

and the general fact that a functor

$$F\colon \mathrm{s_\bullet sets_*} \to \{\text{spectra}\}$$

is ρ-analytic if (and only if) $F((-)\wedge S^1_\bullet)$ is $(\rho-1)$-analytic. The latter follows directly from the definition of analyticity. Indeed if F is say 0-analytic, and $\mathfrak{X}$ is a strictly coCartesian $(n+1)$-cube with $\mathfrak{X}(\emptyset) \to \mathfrak{X}(s)$ k_s-connected ($k_s \geq 0$)

then the suspended cube has $\mathfrak{X}(\emptyset) \wedge S^1 \to \mathfrak{X}(s) \wedge S^1$ (k_s+1)-connected, so by assumption

$$a\colon F(\mathfrak{X}(\emptyset) \wedge S^1) \to \underset{S\neq\emptyset}{\underleftarrow{\operatorname{holim}}}\, F(\mathfrak{X}(S) \wedge S^1)$$

is $(q + \Sigma(k_s+1))$-connected. Hence if F satisfies the condition $E_n(-q,1)$ then $F((-)\wedge S^1)$ satisfies $E_n(-n-q-1,0)$, so is (-1)-analytic.

To see that $\tilde{K}^{\mathrm{cy}}(A, \tilde{M}(X_\bullet))$ is 0-analytic we use the description of addendum 3.3.13:

$$\tilde{K}^{\mathrm{cy}}(A, \tilde{M}(X_\bullet)) \sim \operatorname{holim} \Omega^p \left(\bigvee_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}} \operatorname{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A M)^{\sim}(X_\bullet) \right).$$

Given a strongly coCartesian $(n+1)$-cube $\mathfrak{X}$. For given $\mathbf{C} \in S_\bullet^{(p)}$, the cube

$$\operatorname{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A M)^{\sim}(\mathfrak{X})$$

is homotopy Cartesian for each p: this is true for $\tilde{M}(\mathfrak{X})$ for any abelian M.

It follows from the dual Blaker-Massey theorem, [G4], theorem 2.6 that the above strongly Cartesian cube is also $n+\Sigma k_s$ coCartesian. Taking wedge over $\mathbf{C} \in S_\bullet^{(p)}$ we obtain an $(n+p+\Sigma k_s)$-coCartesian cube. (The extra p appears because $S_\bullet^{(p)}$ is $(p-1)$-connected, cf. the last part of the proof for theorem 3.3.12). By [G4], theorem 2.5, the cube

$$\bigvee_{\mathbf{C}\in S_\bullet^{(p)}\mathcal{P}_A} \operatorname{Hom}_{S_\bullet^{(p)}}(\mathbf{C}, \mathbf{C} \otimes_A M_\bullet)^{\sim}(\mathfrak{X})$$

is $(p+\Sigma k_s)$-Cartesian, and looping down p times there results a (Σk_s)-Cartesian cube. This proves (i).

The FSP associated to the simplicial ring $A \ltimes \tilde{M}(X_\bullet)$ is equivalent to the FSP which sends $Y_\bullet$ to $\tilde{A}(Y_\bullet) \vee \tilde{M}(X_\bullet \wedge Y_\bullet)$. Thus we have the decomposition of spectra

$$\mathrm{TH}\left(A \ltimes \tilde{M}(X_\bullet)\right) \sim \bigvee_{n=0}^{\infty} T_n\left(A; \tilde{M}(X_\bullet)\right)$$

also used in the proof of theorem 3.3.14.

One now first shows that the functor $M^{(n)}(X_\bullet) = \tilde{M}(X_\bullet) \wedge \cdots \wedge \tilde{M}(X_\bullet)$ is (-1)-analytic. This is a non-trivial task. The functor $T_n\left(A, \tilde{M}(X_\bullet)\right)$ involves n smash copies of $\tilde{M}(X_\bullet)$ in each degree, and is thus (-1)-analytic as well. Hence $\mathrm{TH}\left(A \ltimes \tilde{M}(X_\bullet)\right)$ is (-1)-analytic. The cofibrations of spectra (2.4.6)

$$\mathrm{TH}(A \ltimes \tilde{M}(X_\bullet))_{hC_{p^n}} \to \mathrm{TH}(A \ltimes \tilde{M}(X_\bullet))^{C_{p^n}} \to \mathrm{TH}(A \ltimes \tilde{M}(X_\bullet))^{C_{p^{n-1}}}$$

then give (inductively) that each of the fixed sets is (-1)-analytic. Taking inverse limit we see that

$$X_\bullet \mapsto \mathrm{TF}(A \ltimes \tilde{M}(X_\bullet), p)$$

is (-1)-analytic, and then that $\mathrm{TC}(A \ltimes \tilde{M}(X_\bullet), p)$ has the same property. Apply theorem 2.5.5 to complete the proof. □

Lemma 3.4.5. *The functors*

$$\begin{aligned} X_\bullet &\to K(A \ltimes \tilde{M}(X_\bullet)) \\ X_\bullet &\to \mathrm{TC}(A \ltimes \tilde{M}(X_\bullet)) \end{aligned}$$

satisfies the p-limit axiom (ii) *of sect. 3.1 for each prime p.* □

This is well-known for K-theory. The proof for TC follows the scheme of the previous lemma: first do TH and then induct over the fundamental cofibrations, (2.4.6).

We next evaluate the differential $\partial_x F$ of the two functors in question, cf. definition 3.1.2.

Lemma 3.4.6. *The functors* $K(A \ltimes \tilde{M}(X_\bullet))$ *and* $\mathrm{TC}(A \ltimes \tilde{M}(X_\bullet))_p^\wedge$ *have as differentials the spectrum* $|[p] \to \mathrm{TH}(A \times M(X_p); M)|$ *and its p-completion, respectively.*

Proof. This is really a consequence of results in the previous section, namely theorems 3.3.12 and 3.3.14.

$$\partial_x K(A \ltimes \tilde{M}(X_\bullet)) = \varinjlim \Omega^{n+1} K\left(A \ltimes \tilde{M}(X_\bullet \vee S^n_\bullet) \to A \ltimes \tilde{M}(X_\bullet)\right).$$

But $\tilde{M}(X_\bullet \vee S^n_\bullet) = \tilde{M}(X_\bullet) \oplus \tilde{M}(S^n_\bullet)$ and thus

$$A \ltimes \tilde{M}(X_\bullet \vee S^n_\bullet) = (A \ltimes \tilde{M}(X_\bullet)) \ltimes \tilde{M}(S^n_\bullet)$$

where on the right hand side the action is through the projection $A \ltimes \tilde{M}(X_\bullet) \to A$. Write $B_\bullet = A \ltimes \tilde{M}(X_\bullet)$. The analogue of lemma 3.3.2 for bisimplicial rings shows that

$$K\left(B_\bullet \ltimes \tilde{M}(S^n_\bullet) \to B_\bullet\right) \sim \left|[p] \to K\left(B_p \ltimes \tilde{M}(S^n_\bullet) \to B_p\right)\right|$$

and by 3.3.8(i) and 3.3.12

$$\begin{aligned} &\varinjlim \Omega^{n+1} K\left(B_p \ltimes \tilde{M}(S^n_\bullet) \to B_p\right) \sim \\ &\qquad \varinjlim \Omega^{n+1} \tilde{K}^{\mathrm{cy}}(B_p, \tilde{M}(S^{n+1}_\bullet)) \sim \mathrm{TH}(B_p; M). \end{aligned}$$

Similarly,

$$\varinjlim \Omega^{n+1}\mathrm{TC}(B_p \ltimes \tilde{M}(S^n_\bullet))^\wedge_p \sim \mathrm{TH}(B_p; M)^\wedge_p$$

by theorem 3.3.14, and proposition 3.4.2 supplies the conclusion. $\square$

Finally, one must check that the p-completion of $\partial_x\mathrm{trc}$ induces the equivalence. This follows from the following homotopy commutative diagram of spectra, where $M_\bullet = \tilde{M}(S^m_\bullet)$ for an B-bimodule M:

$$\begin{tikzcd}
 & \widetilde{\mathrm{TC}}(B\oplus M_\bullet)^\wedge_p \arrow[d] & \operatorname{holim}_{\leftarrow} S^1/C_{p^r+}\wedge \mathrm{TH}(R,M_\bullet)^\wedge_p \arrow[l, "\sim 2m"'] \arrow[d] \\
\tilde{K}(B\oplus M_\bullet)^\wedge_p \arrow[ur, "\mathrm{trc}"] \arrow[r, "\mathrm{tr}"] \arrow[dr, "\mathrm{tr}", "\sim 2m+2"'] & \widetilde{\mathrm{TH}}(B\oplus M_\bullet)^\wedge_p & S^1_+\wedge \mathrm{TH}(R,M_\bullet)^\wedge_p \arrow[l, "\sim 2m"'] \arrow[d] \\
 & \widetilde{\mathrm{TH}}(B;\tilde{M}_\bullet(S^1))^\wedge_p & S^1\wedge \mathrm{TH}(R,M)^\wedge_p \arrow[l, "\sim 2m"]
\end{tikzcd}$$

The two upper vertical maps are the natural ones which map a homotopy inverse limit into its initial term. The right-hand vertical composition is an equivalence (cf. the proof of theorem 3.3.14), and the notation is

$$\tilde{K}(B\oplus M_\bullet) = K\,(B\oplus M_\bullet \to B)$$

etc. This completes McCarthy's proof of theorem 3.4.1, as I have understood his Aarhus lectures.

Addendum 3.4.7. (McCarthy) *Suppose $f_\bullet\colon R_\bullet \to S_\bullet$ is a map of simplicial rings and that $\pi_0(|f_\bullet|)$ is surjective and has nilpotent kernel. Then*

$$\begin{tikzcd}
K(R_\bullet)^\wedge \arrow[r] \arrow[d] & \mathrm{TC}(R_\bullet)^\wedge \arrow[d] \\
K(S_\bullet)^\wedge \arrow[r] & \mathrm{TC}(S_\bullet)^\wedge
\end{tikzcd}$$

is homotopy Cartesian.

The proof is the same with the exception of lemma 3.4.3 where one has to add an extra step, passing from nilpotency on the π_0-level to nilpotency on the simplicial ring level, cf. [G2], lemma I.3.3.

3.5 Dundas' theorem.

This section gives a brief outline of the proof from [D] of Goodwillie's conjecture:

Theorem 3.5.1. (Dundas). *Let $f\colon L_1 \to L_2$ be a map of FSP's with $\pi_0(f)$ surjective and* $\ker \pi_0(f)$ *nilpotent. Then the diagram*

$$\begin{tikzcd} K(L_1)^\wedge \arrow[r] \arrow[d] & \mathrm{TC}(L_1)^\wedge \arrow[d] \\ K(L_2)^\wedge \arrow[r] & \mathrm{TC}(L_2)^\wedge \end{tikzcd}$$

is homotopy Cartesian.

The general idea is to approximate the FSP's L_i by FSP's coming from simplicial rings, and then use McCarthy's theorem 3.4.7 to derive the conclusion. This is similar in spirit to the cosimplicial resolution of a space (simplicial set) by Eilenberg-MacLane spaces.

Let X be a $(k-1)$-connected space (simplicial set) with $k > 1$. By the Hurewicz theorem, $\pi_k X \xrightarrow{\cong} H_k X$ and $\pi_{k+1}X \to H_{k+1}X$ is surjective. In other words, the linearization map $X \xrightarrow{h} \tilde{\mathbb{Z}}X$ is $(k+1)$-connected. The relative version of this is as follows. Suppose $f\colon X \to Y$ is a $(k+1)$-connected map and X is $(k-1)$-connected. Then the 2-cube

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "h"] \\ \tilde{\mathbb{Z}}X \arrow[r, "\tilde{\mathbb{Z}}f"] & \tilde{\mathbb{Z}}Y \end{tikzcd} \tag{3.5.2}$$

is $(k+2)$-Cartesian in the sense of sect. 3.1.

Indeed, let C be the (homotopy) cofiber of f, and let F be the homotopy fiber. Then F is k-connected and C is $(k+1)$-connected, and the left hand vertical map is the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ \Omega C \arrow[r] & * \arrow[r] & C \end{tikzcd}$$

is $(k+2)$-connected. (This follows for example from the Serre spectral sequence of the involved homotopy fibrations). On the other hand, $\tilde{\mathbb{Z}}(-)$ sends a cofibration into a fibration, so $\mathrm{hF}(\tilde{\mathbb{Z}}f) \sim \Omega\tilde{\mathbb{Z}}C$: apply $\tilde{\mathbb{Z}}$ the the right hand coCartesian square above. Since C is $(k+1)$-connected, $\Omega C \to \Omega\tilde{\mathbb{Z}}C$ is $(k+2)$-connected. Thus $F \to \mathrm{hF}(\tilde{\mathbb{Z}}f)$ is $(k+2)$-connected; its homotopy fiber is equal to the homotopy fiber of

$$a\colon X \to \underset{\longleftarrow}{\mathrm{holim}}\left(\tilde{\mathbb{Z}}X \xrightarrow{\tilde{\mathbb{Z}}f} \tilde{\mathbb{Z}}Y \longleftarrow Y\right)$$

so (3.5.2) is $(k+2)$-Cartesian. Roughly the same argument proves

Lemma 3.5.3. ([D]). *Let $\mathfrak{X}$ be an $(n+k)$-Cartesian n-cube, $k > 1$ such that each sub m-cube is $(m+k)$-Cartesian. Then the $(n+1)$-cube $\mathfrak{X} \to \tilde{\mathbb{Z}}\mathfrak{X}$ is $(n+1+k)$-Cartesian.* □

Starting now with a $(k-1)$-connected space, one can inductively define n-cubes $\mathfrak{Z}_n(X)$ as follows:

$$\mathfrak{Z}_1(x) = \left\{X \to \tilde{\mathbb{Z}}X\right\}, \qquad \mathfrak{Z}_2(X) = \left\{\begin{array}{ccc} X & \xrightarrow{h_X} & \tilde{\mathbb{Z}}X \\ \downarrow{\scriptstyle h_X} & & {\scriptstyle h_{\tilde{\mathbb{Z}}X}}\downarrow \\ \tilde{\mathbb{Z}}X & \xrightarrow{\tilde{\mathbb{Z}}h_X} & \tilde{\mathbb{Z}}\tilde{\mathbb{Z}}X \end{array}\right\},$$

and in general

$$\mathfrak{Z}_n(X) = \left\{\mathfrak{Z}_{n-1}(X) \to \tilde{\mathbb{Z}}(\mathfrak{Z}_{n-1}(X))\right\}.$$

The lemma tells us that $\mathfrak{Z}_n(X)$ is $(n+k)$-Cartesian. For an FSP L, each vertex $\mathfrak{Z}_n(L(X))_S$ defines a new FSP $\mathfrak{Z}_n(L)_S$ with $L = \mathfrak{Z}_n(L)_\emptyset$, and with

$$a_L(X)\colon L(X) \to \underset{\substack{\longleftarrow \\ S\neq\emptyset}}{\mathrm{holim}}\, \mathfrak{Z}_n(L)_S(X)$$

$(n+k)$-connected when X (hence $L(X)$) is $(k-1)$-connected.

One could similarly start with the functor $\tilde{\mathbb{Z}}^q = \tilde{\mathbb{Z}} \circ \cdots \circ \tilde{\mathbb{Z}}$ instead of $\tilde{\mathbb{Z}}$. It is still true that $X \to \tilde{\mathbb{Z}}^q X$ is $(k+1)$-connected for a $(k-1)$-connected X, and one obtains corresponding cubes $\mathfrak{Z}_n^q(L)$ with $a_L^q(X)$ $(n+k)$-connected.

Proposition 3.5.4. *The map a_L induces a map*

$$\mathrm{TC}(L)_p^\wedge \to \underset{\substack{\longleftarrow \\ S\neq\emptyset}}{\mathrm{holim}}\, \mathrm{TC}(\mathfrak{Z}_n(L)_S)_p^\wedge$$

which is $(n-1)$-connected.

Proof. Here is Dundas' argument. It is enough to show that

$$\mathrm{TH}(L) \to \underset{\substack{\longleftarrow \\ S\neq\emptyset}}{\mathrm{holim}}\, \mathrm{TH}(\mathfrak{Z}_n(L)_S)$$

is n-connected, since inductive use of the fundamental cofibration then gives the same conclusion for all C_{p^n}-fixed sets, hence for $\mathrm{TF}(L,p)$, and finally for $\mathrm{TC}(L,p)$ with n replaced by $n-1$.

Now $\mathrm{TH}(L)$ is the prespectrum $\{|\mathrm{THH}_\bullet(L;S^m)|\}_m$, and it suffices to argue that

$$\mathrm{THH}_r(L;S^m) \to \underset{S\neq\emptyset}{\underleftarrow{\mathrm{holim}}}\,\mathrm{THH}_r(\mathfrak{Z}_n(L)_S;S^m)$$

is $(n+m)$-connected for all r. This is lemma 3.5.2 when $r=0$. In general,

$$\mathrm{THH}_r\left(\mathfrak{Z}_n(L)_S;S^m\right) \sim \mathfrak{Z}_n^{r+1}\left(\mathrm{THH}_r(L;S^m)\right)_S. \tag{2}$$

The map is induced from the natural map

$$\sigma\colon \tilde{\mathbb{Z}}^q L(S^{x_0})\wedge\cdots\wedge\tilde{\mathbb{Z}}^q L(S^{x_r}) \to \tilde{\mathbb{Z}}^{q(r+1)}\left(L(S^{x_0})\wedge\cdots\wedge L(S^{x_r})\right).$$

In turn, σ is constructed from iterated use of the assembly map $X\wedge\tilde{\mathbb{Z}}Y\to\tilde{\mathbb{Z}}(X\wedge Y)$. For example, $\tilde{\mathbb{Z}}X\wedge\tilde{\mathbb{Z}}Y\xrightarrow{\sigma}\tilde{\mathbb{Z}}(X\wedge\tilde{\mathbb{Z}}Y)\xrightarrow{\tilde{\mathbb{Z}}\sigma}\tilde{\mathbb{Z}}\left(\tilde{\mathbb{Z}}(X\wedge Y)\right)$. The equivalence statement (2) amounts to the easy fact that $X\wedge\tilde{\mathbb{Z}}(S^n)\to\tilde{\mathbb{Z}}(X\wedge S^n)$ is $(2n-1)$-connected. To finish the proof one applies (3.5.3) with a_L replaced by a_L^q. □

The next result is of similar complexity but I refrain from giving the proof, and refer the reader to [D].

Proposition 3.5.5. *The map*

$$K(L)\to\underset{S\neq\emptyset}{\underleftarrow{\mathrm{holim}}}\,K(\mathfrak{Z}_n(L)_S)$$

is $(n+1)$-connected. □

For $S\neq\emptyset$, $\mathfrak{Z}_n(L)_S$ is equivalent to an FSP associated to a simplicial ring, namely to a simplicial version of $\underset{k}{\underrightarrow{\lim}}\,\Omega^k\mathfrak{Z}_n(L)_S(S^k)$ and $\pi_0\mathfrak{Z}(L)_S=\pi_0 L$, so theorem 3.4.7 applies to show that

$$K\left(\mathfrak{Z}_n(L_1)_S\to\mathfrak{Z}_n(L_2)_S\right)^\wedge\sim\mathrm{TC}\left(\mathfrak{Z}_n(L_1)_S\to\mathfrak{Z}_n(L_2)_S\right)^\wedge$$

when $S\neq\emptyset$. The two previous propositions combine to give the same for $S=\emptyset$. This completes my outline of theorem 3.5.1.

Let G be a topological (or simplicial) monoid homotopy equivalent to ΩX, and $\tilde{G}$ the corresponding FSP, so that $K(\tilde{G})$ is Waldhausen's $A(X)$.

The theorem applies to $\tilde{G} \to \pi_0\tilde{G} = \widetilde{\pi_1 X}$, and to $\widetilde{\pi_1 X} \to \mathbb{Z}[\pi_1 X]^\sim$, so gives a homotopy Cartesian diagram

$$\begin{array}{ccc} A(X)^\wedge & \longrightarrow & \mathrm{TC}(X)^\wedge \\ \downarrow & & \downarrow \\ K(\mathbb{Z}[\pi_1 X])^\wedge & \longrightarrow & \mathrm{TC}(\mathbb{Z}[\pi_1 X])^\wedge \end{array} \tag{3.5.6}$$

The terms on the right-hand side is examined in the next two chapters, and a lot is known. Thus theorem 3.5.1 to some extend reduces the calculation of $A(X)$ to linear K-theory.

4 The absolute theorems

This chapter outlines the proof of the theorem from [HM] that $K(A)$ and $\mathrm{TC}(A)$ agrees after p-adic completion for a large class of p-complete rings, namely for the rings which are finitely generated modules over Witt vectors of perfect fields k of positive characteristic p. It also calculates TC for the FSP's associated with a group like monoid, and gives the relation to Waldhausen's A-functor.

4.1 General approach to TC calculations.

Since $\mathrm{TC}(L)$ is build out of the fixed sets $\mathrm{TH}(L)^C$ the basic calculational problem is to get a hold of $\pi_*\mathrm{TH}(L)^C$ for the cyclic subgroups of the circle. It suffices by theorem 2.5.5 to let C run over the cyclic p-groups, where we have the fundamental cofibration of sect. 2.4

$$\mathrm{TH}(L)_{hC_{p^n}} \longrightarrow \mathrm{TH}(L)^{C_{p^n}} \xrightarrow{R} \mathrm{TH}(L)^{C_{p^{n-1}}}$$

to ease calculations.

Recall that $\mathrm{TH}(L)$ is the restriction of an S^1-invariant spectrum $T(L)$. In the notation of sect. 2.4, $\mathrm{TH}(L) = j^*T(L)$ where $j\colon \mathcal{U}^{S^1} \to \mathcal{U}$. Moreover, the "geometric fix point" spectrum $\Phi^{C_p}T(L)$ of (2.4.1) is equivalent to $T(L)$ by theorem 2.4.5,

$$\rho^{\#}_{C_p}\Phi^{C_p}T(L) \sim_{S^1} T(L).$$

The general approach to the calculation of $\pi_*T(L)^C$ is to replace $T(L)$ by the function spectrum $F(ES^1_+, T(L))$, and to use spectral sequences for calculating the C_{p^n}-fixed points of the function spectrum. This leaves us then for each FSP L with the problem of how close the natural map

$$\pi_*T(L)^{C_{p^n}} \to \pi_*F(ES^1_+, T(L))^{C_{p^n}}$$

is to be an isomorphism. Here ES^1 is the free contractible S^1-space

$$ES^1 = \bigcup_{n=0}^{\infty} S(\mathbb{C}^{n+1}) = \bigcup_{n=0}^{\infty} S^{2n+1}$$

with its standard S^1-action (orbit space $\mathbb{C}P^\infty$), and $F(ES^1_+, T(L))$ is the equivariant S^1-spectrum whose V'th term is $F(ES^1_+, T(L)(V))$, the space of based maps from $ES^1_+ = ES^1 \cup \{+\}$ into the V'th space of $T(L)$, with S^1 acting by conjugation.

Following [GM] we define for each finite p-group C_{p^n},

$$\hat{\mathbb{H}}(C_{p^n}, T(L)) = \left(F(ES^1_+, T(L)) \wedge \tilde{E}S^1\right)^{C_{p^n}} \tag{4.1.1}$$

and call it the C_{p^n}-Tate spectrum of $T(L)$. It is an S^1/C_{p^n}-equivariant spectrum indexed on $\mathcal{U}^{C_{p^n}}$. The space

$$\widetilde{E}S^1 = \bigcup_{n=0}^{\infty} S(\mathbb{C}^n \oplus \mathbb{R}) = \bigcup_{n=0}^{\infty} S^{2n},$$

with S^1-action induced from complex multiplication in C^n, is contractible but not equivariantly: $(\widetilde{E}S^1)^C = S(\mathbb{R}) = S^0$ for each $C \subsetneq S^1$.

Lemma 4.1.2. *For any two based C_{p^n}-spaces X and Y, the restriction to C_p-fixed sets induces a weak C_{p^n}/C_p-homotopy equivalence*

$$F(X, Y \wedge \widetilde{E}S^1)^{C_p} \xrightarrow{\sim} F(X^{C_p}, Y^{C_p}).$$

Proof. We may assume X and Y are C_{p^n}-equivariant CW complexes, e.g. by replacing them with the realization of their singular complexes. The singular set of the C_{p^n} space X is X^{C_p}, so $X - X^{C_p}$ has a free C_{p^n}-action,

$$X = X^{C_p} \cup_\partial (\amalg C_{p^n +} \wedge D^{k_i}).$$

Given ϕ: $X^{C_p} \to Y^{C_p} = (Y \wedge \widetilde{E}S^1)^{C_p}$, one can extend ϕ cell by cell to a C_{p^n}-equivariant map from X to $Y \wedge \widetilde{E}S^1$. Indeed the obstructions to extend lie in

$$\pi_0 F(C_{p^n +} \wedge S^{k_i}, Y \wedge \widetilde{E}S^1)^{C_{p^n}} = \pi_0 F(S^{k_i}, Y \wedge \widetilde{E}S^1) = 0.$$

This proves that the map is surjective on π_0, and hence on π_n by replacing X by $X \wedge S^n$. Injectivity is similar. $\square$

Recall that the smash product of $T(L) \in S^1 S\mathcal{U}$ and a based S^1-space X is the spectrification of the obvious prespectrum, or concretely

$$(T(L) \wedge X)(V) = \varinjlim_{W \supset V} \Omega^{W-V} (T(L)(W) \wedge X). \tag{4.1.3}$$

It follows from lemma 4.1.2 that

$$\Phi^{C_p} T(L) \sim_{C_{p^n}} \left(T(L) \wedge \widetilde{E}S^1\right)^{C_p}$$

and in particular that

$$\hat{\mathbb{H}}(C_{p^n}, T(L)) \sim \Phi^{C_p} F\left(ES^1_+, T(L)\right)^{C_{p^n}/C_p} \tag{4.1.4}$$

Let $C \subseteq S^1$ be any subgroup. We have the pair of adjoint functors j_* and j^* of sect. 2.4 where $j: \mathcal{U}^C \to \mathcal{U}$ is the inclusion, and the maps from (2.4.2),

$$\begin{aligned} &\tau_C : j^*T \wedge_C ES^1_+ \to (T \wedge ES^1_+)^C, \quad C \text{ finite} \\ &\tau_{S^1} : \Sigma j^*T \wedge_{S^1} ES^1_+ \to (T \wedge ES^1_+)^{S^1}, \quad C = S^1. \end{aligned} \tag{4.1.5}$$

The maps fit together with the non-equivariant transfer maps

$$\begin{aligned} &\mathrm{trf}_C^D : j^*T \wedge_D ES^1_+ \to j^*T \wedge_C ES^1_+, \quad D \supset C \\ &\mathrm{trf}_C^{S^1} : \Sigma j^*T \wedge_{S^1} ES^1_+ \to j^*T \wedge_C ES^1_+ \end{aligned}$$

in homotopy commutative diagrams, namely

$$\tau_C \circ \mathrm{trf}_C^D \sim F \circ \tau_D, \quad \tau_C \circ \mathrm{trf}_C^{S^1} \sim F \circ \tau_{S^1} \tag{4.1.6}$$

where F denotes inclusion of fixed sets as usual, cf. [A], [LMS].

Since $\widetilde{E}S^1 = ES^1 * S^0$, the unreduced suspension of ES^1, there is an S^1-equivariant cofibration sequence

$$ES^1_+ \to S^0 \to \widetilde{E}S^1 \to \Sigma(ES^1_+) \to \cdots$$

which induces a cofibration of equivariant spectra upon smashing it with the S^1-equivariant function spectrum $F\left(ES^1_+, T(L)\right)$. We take C_{p^n}-fixed sets and apply (4.1.4) and (4.1.5) to get the norm cofibration of [GM]:

$$\mathrm{TH}(L) \wedge_{C_{p^n}} ES^1_+ \to F(ES^1_+, \mathrm{TH}(L))^{C_{p^n}} \to \hat{\mathbb{H}}(C_{p^n}, T(L)). \tag{4.1.7}$$

By definition it appears that $\hat{\mathbb{H}}(C_{p^n}, T(L))$ depends on the full equivariant structure of $T(L)$, and not only on $\mathrm{TH}(L)$, but this is not really the case. The adjunction $j_*\mathrm{TH}(L) \to T(L)$ induces a map

$$\hat{\mathbb{H}}(C_{p^n}, j_*\mathrm{TH}(L)) \to \hat{\mathbb{H}}(C_{p^n}, T(L))$$

which also fits into the cofibration sequence above; it must be an equivalence by a 5-lemma argument. Thus we shall often write $\hat{\mathbb{H}}(C_{p^n}, \mathrm{TH}(L))$ instead of $\hat{\mathbb{H}}(C_{p^n}, T(L))$. We shall also use the costumary abbreviations

$$\begin{aligned}\mathrm{TH}(L)_{hC_{p^n}} &= \mathrm{TH}(L) \wedge_{C_{p^n}} ES^1_+\\ \mathrm{TH}(L)^{hC_{p^n}} &= F(ES^1_+, \mathrm{TH}(L))^{C_{p^n}}.\end{aligned}$$

With these notions we have

Proposition 4.1.8. *There is a homotopy commutative diagram of cofibrations (of non-equivariant spectra)*

$$\begin{tikzcd}
\mathrm{TH}(L)_{hC_{p^n}} \arrow[r, "N"] \arrow[d, "\mathrm{id}"] & \mathrm{TH}(L)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma"] & \mathrm{TH}(L)^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}"] \\
\mathrm{TH}(L)_{hC_{p^n}} \arrow[r, "N^h"] & \mathrm{TH}(L)^{hC_{p^n}} \arrow[r, "R^h"] & \hat{\mathbb{H}}(C_{p^n}, \mathrm{TH}(L))
\end{tikzcd}$$

Remark 4.1.9. The S^1-fixed set of $\mathrm{TH}(L)$ is contained in $\mathrm{THH}_0(L)$, cf. sect. 2.1, and is of no relevance. In particular the upper horizontal sequence in (4.1.7) has no analogue for S^1 fixed sets. But the lower sequence does have an S^1-version, namely

$$\Sigma\mathrm{TH}(L)_{hS^1} \to \mathrm{TH}(L)^{hS^1} \to \hat{\mathbb{H}}(S^1, \mathrm{TH}(L))$$

with the right-hand term defined by (4.1.1) upon replacing the C_{p^n} fixed set by the S^1 fixed set, cf. [GM].

Example 4.1.10. In the special case of the identity FSP, $L(X) = X$, $T(L)$ is the equivariant sphere spectrum,

$$T(L)(W) \sim_{C_\infty} \varinjlim_{V} \Omega^{V-W} S^W, \quad V \subset \mathcal{U}^C$$

cf. lemma 4.4.4 below. In this case the diagram of proposition 4.1.7 is completely known. Listing only the 0'th terms of the spectra we have

$$\begin{aligned}\mathrm{TH}(\mathrm{Id})^{C_{p^n}} &\sim \Omega^\infty S^\infty(BC_{p^n+}) \times \cdots \times \Omega^\infty S^\infty(BC_{p+}) \times \Omega^\infty S^\infty(S^0)\\ \mathrm{TH}(\mathrm{Id})_{hC_{p^n}} &\sim \Omega^\infty S^\infty(BC_{p^n+})\end{aligned}$$

where $\Omega^\infty S^\infty(X_+) = \varinjlim \Omega^k(S^k \wedge X_+)$. The map R is the projection onto the last n factors. Moreover, the affirmed Segal conjecture tells us that the profinite completions of Γ_n and $\hat{\Gamma}_n$ are equivalences for all n.

One may get information about the homotopy groups of the terms in the norm cofibration by spectral sequences. Let M be a coefficient group (usually $M = \mathbb{Z}_p$ or $M = \mathbb{F}_p$). To ensure convergence of the spectral sequences I will assume that $\pi_*(T(L); M)$ is a finitely generated $\mathbb{Z}_p$-module in each degree.

The spectral sequences were set up in [BM], sect. 2 and in [GM], sect. 10; in [BM] by using a topological version (due to Greenlees) of the complete resolution in usual Tate cohomology of groups and in [GM] by the dual viewpoint where one uses the equivariant Postnikov tower of the spectrum. In our case, the spectral sequences takes the form

$$\text{(i)}\quad E^2_{s,t}\left(T(L)_{hC_{p^n}}; M\right) = H_s\left(C_{p^n}; \pi_t(T(L); M)\right) \Rightarrow \pi_{s+t}\left(T(L)_{hC_{p^n}}; M\right)$$

$$\text{(ii)}\quad E^2_{-s,t}\left(T(L)^{hC_{p^n}}; M\right) = H^s\left(C_{p^n}; \pi_t(T(L); M)\right) \Rightarrow \pi_{t-s}\left(T(L)^{hC_{p^n}}; M\right)$$

$$\text{(iii)}\quad E^2_{-s,t}\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right) = \hat{H}^s\left(C_{p^n}; \pi_t(T(L); M)\right) \Rightarrow \pi_{t-s}\left(\hat{\mathbb{H}}(C_{p^n}; T(L)); M\right)$$

The spectral sequences are concentrated in the upper half plane, the differentials take $E^r_{s,t}$ to $E^r_{s-r,t+r-1}$, and for commutative L the last two spectral sequences have ring structure (with the differentials being derivations) when M is a p-adic ring with p odd. Since the C_{p^n}-action comes from an S^1-action $\pi_*(T(L); M)$ has trivial C_{p^n}-action. Thus for p odd:

$$\begin{aligned} E^2\left(T(L)^{hC_{p^n}}; \mathbb{F}_p\right) &= E\{u_n\} \otimes S\{t\} \otimes \pi_*(T(L); \mathbb{F}_p) \\ E^2\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); \mathbb{F}_p\right) &= E\{u_n\} \otimes S\{t, t^{-1}\} \otimes \pi_*(T(L); \mathbb{F}_p) \end{aligned} \tag{4.1.11}$$

with $\deg(u_n) = (-1, 0)$, $\deg(t) = (-2, 0)$ and $\pi_t(T(L); \mathbb{F}_p)$ sitting in degree $(0, t)$.

In the above H_s, H^s and $\hat{H}^s$ denotes group homology, group cohomology and group Tate cohomology. They are related by the formulas:

$$\hat{H}^{-s}(G; A) = \begin{cases} H^{-s}(G; A), & s \leq 0 \\ H_{s-1}(G; A), & s > -1 \\ \ker\left(\text{Norm}\colon H_0(G; A) \to H^0(G; A)\right), & s = -1 \\ \operatorname{coker}\left(\text{Norm}\colon H_0(G; A) \to H^0(G; A)\right), & s = 0 \end{cases}$$

When $G = C_{p^n}$ and $pA = 0$ then Norm $= 0$, so we see that

$$E^2_{-s,t}\left(\hat{\mathbb{H}}(C_{p^n}; T(L)); \mathbb{F}_p\right) = \begin{cases} E^2_{-s,t}\left(T(L)^{hC_{p^n}}; \mathbb{F}_p\right), & s \geq 0 \\ E^2_{-s-1,t}\left(T(L)_{hC_{p^n}}; \mathbb{F}_p\right), & s < 0 \end{cases}$$

It is important for calculation of $\pi_*(\mathrm{TC}(L); \mathbb{F}_p)$ to identify the R-map, or in the setting of the norm cofibration to identify $\pi_*(R^h)$. This is connected with the differentials in the spectral sequence for $\hat{\mathbb{H}}(C_{p^n}, T(L))$ which cross over the line $-s = 1/2$ in $E^r_{-s,t}$. Indeed, the maps in

$$T(L)^{hC_{p^n}} \xrightarrow{R^h} \hat{\mathbb{H}}(C_{p^n}, T(L)) \xrightarrow{\partial} \Sigma T(L)_{hC_{p^n}}$$

induce homomorphisms of spectral sequences

$$\begin{aligned} E^r(R^h)\colon E^r_{-s,t}\left(T(L)^{hC_{p^n}}; M\right) &\to E^r_{-s,t}\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right) \\ E^r(\partial)\colon E^r_{-s,t}\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right) &\to E^r_{-s-1,t}\left(T(L)_{hC_{p^n}}; M\right) \end{aligned} \tag{4.1.12}$$

with $E^r(R^h)$ surjective for $s \geq 0$, and $E^r(\partial)$ injective for $s < 0$. In a situation where one can calculate the spectral sequences one will also know $E^\infty(R^h)$ and $E^\infty(\partial)$, and hence since the spectral sequences converge,

$$\begin{aligned} E^0\pi_* R^h\colon E^0\pi_*\left(T(L)^{hC_{p^n}}; M\right) &\to E^0\pi_*\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right) \\ E^0(\pi_*\partial)\colon E^0\pi_*\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right) &\to E^0\pi_{*-1}\left(T(L)_{hC_{p^n}}; M\right) \end{aligned}$$

In general this is of course not sufficient to give, say $\pi_* R^h$; there might be filtration shifts. The following lemma goes a long way to overcome this difficulty.

Lemma 4.1.13. *If $\alpha \in E^0\pi_{s+t}\left(T(L)^{hC_{p^n}}; M\right)$ is in the kernel of $E^0(\pi_* R^h)$ then there exists an element $\beta \in \pi_{s+t}\left(T(L)_{hC_{p^n}}; M\right)$ with $E^0\pi_*(N^h)(\beta) = \alpha$.*

Proof. This is a special case of [BM], theorem 2.15. The argument can be outlined as follows. By assumption $E^\infty(R^h)(\alpha) = 0$. The reason must be that there exists an $r > s$ such that α belongs to the image of

$$d^r\colon E^r_{r-s,t-r+1}\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right) \to E^r_{-s,t}\left(\hat{\mathbb{H}}(C_{p^n}, T(L)); M\right)$$

say $\alpha = d^r(\gamma)$. Now $\gamma = E^r(\partial)(\overline{\beta})$ and $\overline{\beta}$ will be an infinite cycle in $E^r\left(T(L)_{hC_{p^n}}; M\right)$. Thus $\overline{\beta}$ represents an element of $E^\infty\left(T(L)_{hC_{p^n}}; M\right)$, and one can pick a suitable representative. More details can be found in [BM], sect. 2. □

The d^2-differential in the spectral sequences is connected to the action

$$A\colon S^1_+ \wedge \mathrm{TH}(L) \to \mathrm{TH}(L)$$

as follows. The stable homotopy $\pi_1^s(S^1_+) \cong \pi_4(\Sigma^3(S^1_+)) \xrightarrow{\cong} \pi_4(S^4) \oplus \pi_4(S^3)$ is $\mathbb{Z} \oplus \mathbb{Z}/2$, generated by the σ = id and the Hopf map η. Thus we get operators

$$[S^1], \eta\colon \pi_i(\mathrm{TH}(L)) \longrightarrow \pi_{i+1}(S^1_+ \wedge \mathrm{TH}(L)) \xrightarrow{A_*} \pi_{i+1}\mathrm{TH}(L)$$

where the first map is exterior product with σ and η, respectively. There are induced operations

$$\hat{H}^s\left(C_{p^n}; \pi_t(\mathrm{TH}(L); M)\right) \to \hat{H}^s\left(C_{p^n}; \pi_{t+1}(\mathrm{TH}(L); M)\right)$$

which we can compose with the periodicity isomorphism

$$\hat{H}^s\left(C_{p^n}; \pi_{t+1}(\mathrm{TH}(L); M)\right) \xrightarrow{\cong} \hat{H}^{s+2}\left(C_{p^n}; \pi_{t+1}(\mathrm{TH}(L); M)\right)$$

to get maps $[S^1]_\#$, $\eta_\#$.

Proposition 4.1.14. *In the spectral sequence $E^r_{*,*}\left(\hat{\mathbb{H}}(C_{p^n}, \mathrm{TH}(L)); M\right)$, the d^2-differential*

$$d^2\colon \hat{H}^s\left(C_{p^n}; \pi_t(\mathrm{TH}(L); M)\right) \to \hat{H}^{s+2}\left(C_{p^n}; \pi_t(\mathrm{TH}(L); M)\right)$$

is equal to $[S^1]_\#$, provided η acts trivially on $\pi_*(\mathrm{TH}(L); M)$. □

This is proved in [H2] when C_{p^n} is replaced by S^1, and the above can be deduced from this case. The assumption that $\eta_\#$ be zero is satisfied for the linear FSP's $L = \tilde{A}$ associated with a ring because $\mathrm{TH}(A) \sim \mathrm{TH}^{\oplus}(A)$ is a product of Eilenberg-MacLane spectra.

We have left to consider the *homotopy limit problem*, i.e. the homotopical behavior of

$$\hat{\Gamma}_n\colon \mathrm{TH}(L)^{C_{p^{n-1}}} \to \hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(L)\right).$$

In the special case of L = Id it is a homotopy equivalence, but this is too much to expect in general. The domain is a (-1)-connected spectrum, but this is often false for the right hand side, e.g. when $L = \tilde{\mathbb{F}}_p$ as we shall see in sect. 4.2 below. The best one could hope for would be that $\pi_i(\hat{\Gamma}_n)$, and hence also $\pi_i(\Gamma_n)$, be isomorphisms for $i \geq 0$. This unfortunately is also not true. For the FSP $\tilde{A}$ associated with truncated polynomial algebras $A = k[t]/(t^n)$, the two sides have different homotopy groups in all even dimensions; this is an easy consequence of sect. 5.2. The only completely general theorem is the following result of S. Tsalidis:

Theorem 4.1.15. ([T]) *Suppose*

$$\pi_i(\hat{\Gamma}_1)\colon \pi_i(\mathrm{TH}(L); \mathbb{F}_p) \to \pi_i\left(\hat{\mathbb{H}}(C_p, \mathrm{TH}(L)); \mathbb{F}_p\right)$$

is an isomorphism for $i \geq i_0$. Then the same is true for $\pi_i(\hat{\Gamma}_n)$ for all $n \geq 1$. □

Tsalidis' proof is similar to the induction step from C_p to C_{p^n} in the proof of the affirmed Segal conjecture.

Calculations from [H2] show that if $\pi_i(\hat{\Gamma}_1; \mathbb{F}_p)$ is an isomorphism in non-negative degrees for a ring A then the same is the case for the polynomial algebra $A[t]$ and more generally for any smooth A-algebra. In [BM1] and in sect. 5.4 below the assumption of theorem 4.1.15 is established for $A = W(\mathbb{F}_{p^s})$, with $i_0 = 0$. Optimistically one would hope for

Conjecture 4.1.16. For a regular ring A,

$$\pi_i(\hat{\Gamma}_1; \mathbb{F}_p)\colon \pi_i(\mathrm{TH}(A); \mathbb{F}_p) \to \pi_i\left(\hat{\mathbb{H}}(C_p, \mathrm{TH}(A)); \mathbb{F}_p\right)$$

is an isomorphism when $i \geq 0$.

Note that the statement is equivalent to the assertion that

$$\hat{\Gamma}_n\colon \mathrm{TH}(A)^{C_{p^n}} \to \hat{\mathbb{H}}\left(C_{p^{n+1}}, \mathrm{TH}(A)\right)[0, \infty)$$

becomes a homotopy equivalence after p-adic completion, with $[0, \infty)$ indicating (-1)-connected cover.

4.2 The spectrum $\mathrm{TC}(\mathbb{F}_p)$.

This section illustrates sect. 4.1 by completely determining the spectra $\mathrm{TH}(\mathbb{F}_p)^{C_{p^n}}$ and $\mathrm{TC}(\mathbb{F}_p)$. The calculation was originally carried out in [M], but [HM], sect. 4.1–3 is a better place to look for additional details.

For any ring, $\mathrm{THH}(A)$ is the realization of a simplicial abelian group, cf. sect. 3.2, so its homotopy type is determined by its homotopy groups:

$$\mathrm{TH}(A) \sim \bigvee_{n=0}^{\infty} \Sigma^n H(\pi_n \mathrm{TH}(A)) \sim \prod_{n=0}^{\infty} \Sigma^n H(\pi_n \mathrm{TH}(A)) \tag{4.2.1}$$

where $H(-)$ is the Eilenberg-MacLane spectrum with $\pi_0 H(B) = B$ and $\pi_i H(B) = 0$ for $i \neq 0$, and Σ^n is the suspension functor.

One may filter $\mathrm{TH}(A)$ by skeletons, since it is the realization of a simplicial construction. This leads to a spectral sequence,

$$E^2(A) = \mathrm{HH}_*(\mathcal{A}_A) \Rightarrow \pi_*(\mathrm{TH}(A); \mathbb{F}_p) \tag{4.2.2}$$

with $\mathcal{A}_A = H_*(H(A);\mathbb{F}_p)$. This spectral sequence was used by Bökstedt to calculate $\mathrm{TH}(\mathbb{F}_p)$. I refer the reader to [B1] or [HM], sect. 4.2 for details. Different calculational methods can be found in [Br] or [FLS].

The 0-skeleton of $\mathrm{TH}(A)$ is the Eilenberg-MacLane spectrum $H(A)$, and one may use the S^1-action to get the map

$$\sigma\colon S^1_+ \wedge HA \to S^1_+ \wedge \mathrm{TH}(A) \to \mathrm{TH}(A). \tag{4.2.3}$$

For $A = \mathbb{F}_p$ we have $\tau_0 \in \pi_1(H\mathbb{F}_p;\mathbb{F}_p)$ and can consider $\sigma_*([S^1]\wedge\tau_0) \in \pi_2(\mathrm{TH}(\mathbb{F}_p);\mathbb{F}_p)$, where $[S^1] \in \pi_1^S(S^1_+)$ was defined in the previous section.

Theorem 4.2.4. ([B1], [Br]). *The reduction*

$$\mathrm{red}_p\colon \pi_2\mathrm{TH}(\mathbb{F}_p) \to \pi_2(\mathrm{TH}(\mathbb{F}_p);\mathbb{F}_p)$$

is an isomorphism, and

$$\pi_*\mathrm{TH}(\mathbb{F}_p) = S_{\mathbb{F}_p}\{\sigma\},$$

the polynomial algebra on σ of degree 2 with $\mathrm{red}_p(\sigma) = \sigma_*([S^1]\wedge\tau_0)$. □

Combined with (4.1.11) we can explicate the E^2-terms of the spectral sequence $\hat{E}^r(C_{p^n};M) = E^r\left(\hat{\mathbb{H}}(C_{p^n},T(\mathbb{F}_p));M\right)$ for $M = \mathbb{F}_p$, $\mathbb{Z}_p$ to be

$$\begin{aligned}
\hat{E}^2(C_{p^n};\mathbb{F}_p) &= E_{\mathbb{F}_p}\{u_n\}\otimes S_{\mathbb{F}_p}\{t,t^{-1}\}\otimes E_{\mathbb{F}_p}\{e_1\}\otimes S_{\mathbb{F}_p}\{\sigma\}\\
\hat{E}^2(C_{p^n};\mathbb{Z}_p) &= E_{\mathbb{F}_p}\{u_n\}\otimes S_{\mathbb{F}_p}\{t,t^{-1}\}\otimes S_{\mathbb{F}_p}\{\sigma\}
\end{aligned}$$

except if $p = 2$ and $n = 1$ where the first two terms are replaced by $S\{u_1,u_1^{-1}\}$. The mod p Bockstein operator maps $e_1\sigma^l$ to σ^l for $l \geq 0$. For p odd, $\hat{E}^r(C_{p^n};\mathbb{F}_p)$ is a spectral sequence of algebras. If $p = 2$ there is the usual trouble with products in $\pi_*(T;\mathbb{F}_2)$ but in all cases, $\hat{E}^r(C_{p^n};\mathbb{F}_p)$ is an algebra over $\hat{E}^r(C_{p^n};\mathbb{Z}_p)$.

Lemma 4.2.5. *The non-zero differentials in $\hat{E}^r(C_{p^n};\mathbb{F}_p)$ are generated from $d^2e_1 = t\sigma$ in the module structure over $\hat{E}^r(C_{p^n};\mathbb{Z}_p)$. In particular*

$$\begin{aligned}
&\pi_*\left(\hat{\mathbb{H}}(C_{p^n},\mathrm{TH}(\mathbb{F}_p));\mathbb{F}_p\right) \cong E_{\mathbb{F}_p}\{u_n\}\otimes S_{\mathbb{F}_p}\{t,t^{-1}\}, \quad p \text{ odd or } n>1\\
&\pi_*\left(\hat{\mathbb{H}}(C_2,\mathrm{TH}(\mathbb{F}_2));\mathbb{F}_2\right) \cong S_{\mathbb{F}_2}\{u_1,u_1^{-1}\}
\end{aligned}$$

with $\deg(t) = -2$, $\deg u_n = -1$.

Proof. Since $e_1 = \sigma_*(\tau_0)$, $\tau_0 \in \pi_1(H\mathbb{F}_p;\mathbb{F}_p)$ and $\mathrm{red}_p(\sigma) = \sigma_*([S^1] \wedge \tau_0)$ we have in the notation of proposition 4.1.14,

$$[S^1]_\#(e_1) = \sigma, \qquad [S^1]_\#(1) = 0,$$

and hence $[S^1]_\#(e_1\sigma^l) = \sigma^{l+1}$. The d^2-differential then follow from (4.1.14), and a routine cohomology calculation gives

$$\hat{E}^3(C_{p^n};\mathbb{F}_p) = E_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\}$$

(with $u_1^2 = t$ if $p = 2$ and $n = 1$). For degree reasons there can be no further differentials. For p odd (and $p = 2$, $n = 1$) this is a free commutative algebra in the graded sense, and the stated value of the mod p homotopy is immediate. If $p - 2$ and $n > 1$ one uses that the mod p Bockstein on u_n is trivial. □

For $n = 1$, the mod p Bockstein relation $\beta(u_1) = t$ gives that

$$\pi_*\hat{\mathbb{H}}(C_p, \mathrm{TH}(\mathbb{F}_p)) = S_{\mathbb{F}_p}\{t, t^{-1}\}$$

(with $t = u_1^2$ if $p = 2$). We next check the assumption of theorem 4.1.15.

Lemma 4.2.6. *The homomorphism*

$$\pi_i(\hat{\Gamma}_1;\mathbb{F}_p)\colon \pi_i(\mathrm{TH}(\mathbb{F}_p);\mathbb{F}_p) \to \pi_i\left(\hat{\mathbb{H}}(C_p, \mathrm{TH}(\mathbb{F}_p));\mathbb{F}_p\right)$$

is an isomorphism when $i \geq 0$.

Proof. Since $\hat{\Gamma}_1\colon \mathrm{TH}(\mathbb{F}_p) \to \hat{\mathbb{H}}(C_p, \mathrm{TH}(\mathbb{F}_p))$ is multiplicative, it suffices to see that $\pi_2(\hat{\Gamma}_1;\mathbb{F}_p)$ is an isomorphism.

Continuing the cofibration diagram of (4.1.8), $n = 1$, to the right, gives a homotopy commutative square of S^1-spectra

$$\begin{tikzcd}
\mathrm{TH}(\mathbb{F}_p) \arrow[r, "\partial"] \arrow[d, "\hat{\Gamma}_1"] & \Sigma\rho^{\#}_{C_p}\left(\mathrm{TH}(\mathbb{F}_p)_{hC_p}\right) \arrow[r, "\Sigma N"] \arrow[d, "\mathrm{id}"] & \Sigma\rho^{\#}_{C_p}\mathrm{TH}(\mathbb{F}_p)^{C_p} \arrow[d, "\Gamma_1"] \\
\rho^{\#}_{C_p}\hat{\mathbb{H}}(C_p, \mathrm{TH}(\mathbb{F}_p)) \arrow[r, "\partial^h"] & \Sigma\rho^{\#}_{C_p}\left(\mathrm{TH}(\mathbb{F}_p)_{hC_p}\right) \arrow[r, "\Sigma N^h"] & \Sigma\rho^{\#}_{C_p}\mathrm{TH}(\mathbb{F}_p)^{hC_p}
\end{tikzcd}$$

Here as usual $\rho^{\#}_{C_p}$ indicates that the S^1/C_p-spectra are to be considered as S^1-spectra under the p'th root isomorphism $S^1 \to S^1/C_p$.

Now $\sigma = [S^1]_\#(\tau_0)$, so we are done if we can show that $e_0 = \pi_i(\partial;\mathbb{F}_p)(\tau_0)$ is non-zero in $\pi_0\left(\mathrm{TH}(\mathbb{F}_p)_{hC_p};\mathbb{F}_p\right)$, and $[S^1]_\#(e_0) \neq 0$.

The spectral sequence $E^r(\mathrm{TH}(\mathbb{F}_p);\mathbb{Z}_p)$ gives $\pi_0\mathrm{TH}(\mathbb{F}_p) = \mathbb{Z}/p$, and by (2.5.8) $\pi_0\mathrm{TH}(\mathbb{F}_p)^{C_p} = \mathbb{Z}/p^2$. The fundamental cofibration thus induces the exact non-split sequence

$$0 \longrightarrow \pi_0\mathrm{TH}(\mathbb{F}_p)_{hC_p} \xrightarrow{N} \pi_0\mathrm{TH}(\mathbb{F}_p)^{C_p} \xrightarrow{R} \pi_0\mathrm{TH}(\mathbb{F}_p) \longrightarrow 0$$

so $\pi_0(N;\mathbb{F}_p) = 0$, and $\pi_1(\partial;\mathbb{F}_p)$ must be surjective. Finally, the inclusion

$$T(\mathbb{F}_p) \wedge_{C_p} S^1_+ \to T(\mathbb{F}_p) \wedge_{C_p} ES^1_+$$

coming from $S^1 \subset ES^1$ induces a monomorphism on $\pi_i(-;\mathbb{F}_p)$ for $i = 0, 1$. The homeomorphism

$$\rho^{\#}_{C_p}\left(\mathrm{TH}(\mathbb{F}_p) \wedge_{C_p} S^1_+\right) \to \mathrm{TH}(\mathbb{F}_p) \wedge S^1_+, \quad (x,\theta) \mapsto (\theta^{-1}x, \theta)$$

map the diagonal S^1-structure in the domain to the extended S^1-structure in the range. Hence

$$[S^1]_{\#}\colon \pi_0\left(\rho^{\#}_{C_p}\mathrm{TH}(\mathbb{F}_p) \wedge_{C_p} S^1_+;\mathbb{F}_p\right) \to \pi_1\left(\rho^{\#}_{C_p}\mathrm{TH}(\mathbb{F}_p) \wedge_{C_p} S^1_+;\mathbb{F}_p\right)$$

must be injective. □

The spectrum $\mathrm{TH}(\mathbb{F}_p)$ is p-complete, and inductive use of the fundamental cofibration (2.4.6) implies the same for $\mathrm{TH}(\mathbb{F}_p)^{C_{p^n}}$ for each n. Thus

$$\pi_*\mathrm{TH}(\mathbb{F}_p)^{C_{p^n}} = \pi_*\left(\mathrm{TH}(\mathbb{F}_p)^{C_{p^n}};\mathbb{Z}_p\right).$$

Proposition 4.2.7. *For $n \geq 1$,*

$$\pi_*T(\mathbb{F}_p)^{C_{p^n}} = S_{\mathbb{Z}/p^{n+1}}\{\sigma_n\}$$

with $\deg\sigma_n = 2$. Moreover, $F(\sigma_n) = \sigma_{n-1}$ and $R(\sigma_n) = \lambda_n p\sigma_{n-1}$ with $\lambda_n \in \mathbb{Z}/p^n$ a unit.

Proof. Theorem 4.1.15 shows that

$$\hat{\Gamma}\colon \pi_*\mathrm{TH}(\mathbb{F}_p)^{C_{p^n}} \to \pi_*\hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)\right)$$

is an isomorphism in non-negative degrees. For the target, the integral spectral sequence $\hat{E}^r(C_{p^n};\mathbb{Z}_p)$ has

$$\hat{E}^2 = E_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}\{\sigma\}.$$

The elements t and σ are infinite cycles. Indeed the inclusion of S^1 fixed sets into C_{p^n} fixed sets gives a map

$$\hat{\mathbb{H}}\left(S^1, \mathrm{TH}(\mathbb{F}_p)\right) \to \hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)\right)$$

cf. (4.1.9), and an induced map of spectral sequence. The E^2-term of the range is

$$E^2\left(\hat{\mathbb{H}}(S^1, \mathrm{TH}(\mathbb{F}_p)); \mathbb{Z}_p\right) = S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}\{\sigma\},$$

so is concentrated in even total degrees. Thus $E^2 = E^\infty$. On the other hand it injects into the $\hat{E}^2$ above. Thus $t^k\sigma^l$ are all infinite cycles.

We claim that u_n survives to $\hat{E}^{2n+1}(C_{p^n}; \mathbb{Z}_p)$ and that $d^{2n+1}(u_n) = t^{n+1}\sigma^n$. Indeed, the first non-trivial differential on u_n must be of the form

$$d^{2r+1}(u_n) = t^{r+1}\sigma^r$$

for some r. Given this it is easy to solve the spectral sequence. In particular

$$E^0\pi_0\hat{\mathbb{H}}\left(C_{p^n}; \mathrm{TH}(\mathbb{F}_p)\right) = \mathbb{F}_p^{\oplus r}$$

generated by $1, t\sigma, \ldots, (t\sigma)^{r-1}$, $(d^{2r+1}(u_n t^{-1}) = (t\sigma)^r)$. Since $\pi_0\hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)\right) \cong \pi_0\mathrm{TH}(\mathbb{F}_p)^{C_{p^{n-1}}}$ is $\mathbb{Z}/p^n$ by (2.5.8), we conclude that $r = n$. Moreover,

$$E^0\pi_{2k}\hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)\right) = \mathbb{F}_p^{\oplus n}$$

generated by $\sigma^k, \sigma^{k+1}t, \ldots, \sigma^{k+n}t^n$, and $\pi_{2k+1}\hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)\right) = 0$. Since in addition $\pi_{2k}\left(\hat{\mathbb{H}}(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)); \mathbb{F}_p\right)$ is a single copy of $\mathbb{F}_p$ we must have

$$\pi_{2k}\hat{\mathbb{H}}\left(C_{p^n}; \mathrm{TH}(\mathbb{F}_p)\right) = \mathbb{Z}/p^n$$

for all $k \geq 0$. One more application of theorem 4.1.15 gives the stated homotopy groups. The inclusion F corresponds under $\hat{\Gamma}$ to the inclusion

$$\hat{\mathbb{H}}\left(C_{p^{n+1}}, \mathrm{TH}(\mathbb{F}_p)\right) \xrightarrow{F^h} \hat{\mathbb{H}}\left(C_{p^n}, \mathrm{TH}(\mathbb{F}_p)\right)$$

so $\pi_{2k}(F^h)$ must be surjective, and we can pick the generator to satisfy $F(\sigma_n) = \sigma_{n-1}$.

Finally the exact sequence

$$\pi_2 T(\mathbb{F}_p)^{C_{p^n}} \xrightarrow{R} \pi_2 T(\mathbb{F}_p)^{C_{p^{n-1}}} \xrightarrow{\partial_*} \pi_1 T(\mathbb{F}_p)_{hC_{p^n}} \xrightarrow{N} \pi_1 T(\mathbb{F}_p)^{C_{p^n}},$$

with $\pi_1 T(\mathbb{F}_p)^{C_{p^n}} = 0$ and $\pi_1 T(\mathbb{F}_p)_{hC_{p^n}} = \mathbb{Z}/p$, yields the stated value of R. $\square$

Corollary 4.2.8. $\mathrm{TC}(\mathbb{F}_p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p$

Proof. We use the cofibration sequence of sect. 2.5,

$$\mathrm{TC}(\mathbb{F}_p,p) \longrightarrow \mathrm{TR}(\mathbb{F}_p,p) \xrightarrow{F-1} \mathrm{TR}(\mathbb{F}_p,p).$$

The previous proposition yields

$$\pi_k \mathrm{TR}(\mathbb{F}_p,p) = \varprojlim_R \pi_k \mathrm{TH}(\mathbb{F}_p)^{C_{p^n}} = \begin{cases} 0 & \text{for } k > 0 \\ \mathbb{Z}_p & \text{for } k = 0 \end{cases}$$

so that

$$\pi_0 \mathrm{TC}(\mathbb{F}_p,p) = \mathbb{Z}_p, \quad \pi_{-1}\mathrm{TC}(\mathbb{F}_p,p) = \mathbb{Z}_p$$

and $\pi_k \mathrm{TC}(\mathbb{F}_p,p) = 0$ otherwise. Finally, $\mathrm{TC}(\mathbb{F}_p)$ is p-complete and by theorem 2.5.5 equal to $\mathrm{TC}(\mathbb{F}_p,p)$. □

4.3 The absolute theorem: linear case.

This section sketches the proof of theorem 1.3 of the introduction. It is joint work with L. Hesselholt, and further details can be found in [HM], sect. 4.5, 5.1, 5.2 and [HM], appendix B.

We fix a perfect field k of positive characteristic p, and consider algebras A over the (p-typical) Witt vectors $W(k)$ which are finitely generated as modules; for short: *finite* $W(k)$-algebras. If k is finite the assumption is that A be a finite $\mathbb{Z}_p$-algebra. We use the notation

$$K_i(A;\mathbb{Z}_p) = \pi_i(K(A)^\wedge_p)$$
$$\mathrm{TC}_i(A;\mathbb{Z}_p) = \pi_i(\mathrm{TC}(A)^\wedge_p)$$

and want to prove

Theorem 4.3.1. *For finite $W(k)$-algebras, the cyclotomic trace*

$$\mathrm{trc}\colon K_i(A;\mathbb{Z}_p) \to \mathrm{TC}_i(A;\mathbb{Z}_p)$$

is an isomorphism, for $i \geq 0$.

The ring of Witt vectors $W(k)$ is a P.I.D and is p-adically complete. Since A is finite over $W(k)$,

$$A = \varprojlim A/p^n A,$$

and we can introduce the continuous version of the functors:

$$K^{\mathrm{top}}(A) = \operatorname{holim}_{\longleftarrow} K(A/p^n A), \quad \mathrm{TC}^{\mathrm{top}}(A) = \operatorname{holim}_{\longleftarrow} \mathrm{TC}(A/p^n A).$$

There are exact sequences

$$0 \to \varprojlim{}^{(1)} K_{i+1}(A/p^nA;\mathbb{Z}_p) \to K_i^{\mathrm{top}}(A;\mathbb{Z}_p) \to \varprojlim K_i(A/p^nA;\mathbb{Z}_p) \to 0$$
$$0 \to \varprojlim{}^{(1)} \mathrm{TC}_{i+1}(A/p^nA;\mathbb{Z}_p) \to \mathrm{TC}_i^{\mathrm{top}}(A;\mathbb{Z}_p) \to \varprojlim \mathrm{TC}_i(A/p^nA;\mathbb{Z}_p) \to 0$$

cf. [BK], p. 249 and p. 299.

The proof of theorem 4.3.1 is broken down into three statements to be considered separately below:

$$\begin{array}{lll} \text{(i)} & K_i(A/pA;\mathbb{Z}_p) \xrightarrow{\cong} \mathrm{TC}_i(A/pA;\mathbb{Z}_p), & i \geq 0 \\ \text{(ii)} & \mathrm{TC}_i(A;\mathbb{Z}_p) \xrightarrow{\cong} \mathrm{TC}_i^{\mathrm{top}}(A;\mathbb{Z}_p), & i \geq 0 \\ \text{(iii)} & K_i(A;\mathbb{Z}_p) \xrightarrow{\cong} K_i^{\mathrm{top}}(A;\mathbb{Z}_p), & i \geq 0 \end{array}$$

Indeed, given (i), McCarthy's theorem 3.4.12 show that

$$\mathrm{trc}\colon K_i(A/p^nA;\mathbb{Z}_p) \to \mathrm{TC}_i(A/p^nA;\mathbb{Z}_p)$$

is an isomorphism for all $i \geq 0$, and hence by the short exact sequences above that

$$\mathrm{trc}\colon K_i^{\mathrm{top}}(A;\mathbb{Z}_p) \xrightarrow{\cong} \mathrm{TC}_i^{\mathrm{top}}(A;Z_p), \quad i \geq 0.$$

Use of (ii) and (iii) completes the proof.

I begin with (i). For $A = W(k)$, $A/pA = k$. If k is finite then $K(k)_p^\wedge \simeq H\mathbb{Z}_p$ by [Q3]. For general perfect fields the same holds by [Kr]. We must therefore first extend Corollary 4.2.8 to general perfect fields. The result we need is

Theorem 4.3.2. *For a perfect field of characteristic $p > 0$, there is a homotopy equivalence* $\mathrm{TR}(k,p) \sim HW(k)$

Given this, we can calculate $\mathrm{TC}(k,p)$ from the cofibration

$$\mathrm{TC}(k,p) \longrightarrow \mathrm{TR}(k,p) \xrightarrow{1-F} \mathrm{TR}(k,p),$$

since by theorem 2.5.7 we know that

$$\pi_0 F\colon \pi_0(\mathrm{TR}(k,p);\mathbb{F}_p) \to \pi_0(\mathrm{TR}(k,p);\mathbb{Z}_p)$$

induces the Frobenius homomorphism of Witt vectors. Moreover

$$\ker\left(F - \mathrm{id}\colon W(k) \to W(k)\right) = W(k^F)$$

and $k^F = \mathbb{F}_p$ so $W(k^F) = \mathbb{Z}_p$. Thus theorem 4.3.2 gives

$$\mathrm{TC}_i(k;\mathbb{Z}_p) = \begin{cases} 0, & i \geq 0 \\ \mathbb{Z}_p, & i = 0 \\ \operatorname{cok}(F - \mathrm{id}\colon W(k) \to W(k)), & i = -1 \end{cases} \tag{4.3.3}$$

and hence $K_i(k;\mathbb{Z}_p) \cong \mathrm{TC}_i(k;\mathbb{Z}_p)$ for $i \geq 0$.

Proof of 4.3.2. For a perfect field of positive characteristic the usual Hochschild homology groups $\mathrm{HH}_*(k)$ vanish in higher degrees, and $\mathrm{HH}_0(k) = k$. It then follows from the spectral sequence (4.2.2) that

$$\pi_*(\mathrm{TH}(k)) = k \otimes \pi_*\mathrm{TH}(\mathbb{F}_p).$$

The cofibration sequence

$$\mathrm{TH}(k)_{hC_{p^n}} \xrightarrow{N} \mathrm{TH}(k)^{C_{p^n}} \xrightarrow{R} \mathrm{TH}(k)^{C_{p^{n-1}}}$$

was derived from taking C_{p^n} fixed points, so $\mathrm{TH}(k)^{C_{p^n}}$ acts on it. In particular, the homotopy groups are $\pi_0\mathrm{TH}(k)^{C_{p^n}}$-modules, and by (2.5.8) $W_{n+1}(k)$-modules. The inclusion $\mathbb{F}_p \subset k$ induces $W_{n+1}(k)$-homomorphisms:

$$\begin{array}{ll} \text{(i)} & W_{n+1}(k) \otimes \pi_i\mathrm{TH}(\mathbb{F}_p)_{hC_{p^n}} \to \pi_i\mathrm{TH}(k)_{hC_{p^n}} \\ \text{(ii)} & W_{n+1}(k) \otimes \pi_i\mathrm{TH}(\mathbb{F}_p)^{C_{p^n}} \to \pi_i\mathrm{TH}(k)^{C_{p^n}} \\ \text{(iii)} & W_{n+1}(k) \otimes \pi_i\mathrm{TH}(\mathbb{F}_p)^{C_{p^{n-1}}} \to \pi_i\mathrm{TH}(k)^{C_{p^{n-1}}} \end{array}$$

Now $\pi_i\mathrm{TH}(\mathbb{F}_p)^{C_{p^{n-1}}} = \mathbb{Z}/p^n$ and $W_{n+1}(k) \otimes \mathbb{Z}/p^n = W_n(k)$, so the domain of (iii) is $W_n(k) \otimes \pi_i\mathrm{TH}(\mathbb{F}_p)^{C_{p^{n-1}}}$. We may inductively assume the third arrow to be an isomorphism. Thus we are done by the 5-lemma, if we can show that (i) is an isomorphism. This follows from the spectral sequence

$$H_*\left(C_{p^n};\pi_*\mathrm{TH}(k)\right) \Rightarrow \pi_*\mathrm{TH}(k)_{hC_{p^n}}.$$

Indeed, it is a spectral sequence of $W_{n+1}(k)$-modules when the $W_{n+1}(k)$-structure on the E^2-term is via $F^n\colon W_{n+1}(k) \to W_1(k) = k$ and

$$W_{n+1}(k) \otimes (F^n)^{\#}\pi_i\mathrm{TH}(\mathbb{F}_p) \cong (F^n)^{\#}\pi_*(\mathrm{TH}(k)).$$

We conclude that the homomorphisms in (i), (ii) and (iii) are isomorphisms. Now (4.2.8) gives

$$\pi_*\mathrm{TH}(k)^{C_{p^n}} = S_{W_{n+1}(k)}\{\sigma_n\} \tag{4.3.4}$$

with $R(\sigma_n) = \lambda_n p\sigma_{n-1}$ with $\lambda_n \in W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$ a unit. In conclusion,

$$\varprojlim \pi_*\mathrm{TH}(k)^{C_{p^n}} = \begin{cases} 0 & \text{for } * > 0 \\ W(k) & \text{for } * = 0 \end{cases}$$

$$\varprojlim{}^{(1)}\pi_*\mathrm{TH}(k)^{C_{p^n}} = 0$$

as the limit system is obviously Mittag-Leffler, cf. [BK], p. 256. □

Theorem 4.3.5. *If A is a semi-simple k-algebra then* $K_i(A; Z_p) \cong \mathrm{TC}_i(A; \mathbb{Z}_p)$ *for* $i \geq 0$.

Proof. Both functors preserve products so it suffices to do the case of a simple algebra. If $A = \mathrm{M}_n(k)$ then we are done by Morita invariance:

$$K_i(\mathrm{M}_n(k)) = K_i(k), \quad \mathrm{TC}_i(\mathrm{M}_n(k)) = \mathrm{TC}_i(k)$$

and theorem 4.3.2. In general, we only know that

$$A \otimes_k k' \cong \mathrm{M}_n(k')$$

for a Galois extension k' with $|k' : k|$ prime to p. (The existence of such a k' is a consequence of the lack of p-torsion in the Brauer group $\mathrm{Br}(k)$). Finally, the horizontal compositions in the diagram

$$\begin{array}{ccccc} K_i(A; \mathbb{Z}_p) & \xrightarrow{i_*} & K_i(A \otimes_k k'; \mathbb{Z}_p) & \xrightarrow{i^*} & K_i(A; \mathbb{Z}_p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TC}_i(A; \mathbb{Z}_p) & \xrightarrow{i_*} & \mathrm{TC}_i(A \otimes'_k; \mathbb{Z}_p) & \xrightarrow{i^*} & \mathrm{TC}_i(A; \mathbb{Z}_p) \end{array}$$

are isomorphisms since $|k' : k|$ is a unit of $\mathbb{Z}_p$, and the middle arrow is an isomorphism. (Here i^* is the composition of the functors applied to $A \otimes_k k' \to \mathrm{End}_A(A \otimes_k k')$ and Morita invariance). □

Corollary 4.3.6. *If A satisfies the assumption of theorem* 4.3.1, *then* $\mathrm{trc}\colon K_i^{\mathrm{top}}(A; \mathbb{Z}_p) \to \mathrm{TC}_i^{\mathrm{top}}(A; \mathbb{Z}_p)$ *is an isomorphism for* $i \geq 0$.

Proof. We are reduced to check that

$$\mathrm{trc}\colon K_i(A/pA; \mathbb{Z}_p) \to \mathrm{TC}_i(A/pA; \mathbb{Z}_p)$$

is an isomorphism. But A/pA is artenian, so its radical J is nilpotent. Thus by theorem 3.4.1 it is enough that the cyclotomic trace induce isomorphism for the algebra $(A/pA)/J$, which is semi-simple. Apply theorem 4.3.5. □

Theorem 4.3.7. *In the situation of theorem 4.3.1, the natural map*

$$\mathrm{TC}_i(A; \mathbb{Z}_p) \to \mathrm{TC}_i^{\mathrm{top}}(A; \mathbb{Z}_p)$$

is an isomorphism.

Proof. It is enough to prove the statement with $\mathbb{F}_p$ coefficients: a map of p-complete spaces is a homotopy equivalence if the induced homomorphism on $\bmod p$ homotopy groups is an isomorphism.

The functor which to A associates the Eilenberg-MacLane spectrum HA is continuous, $\pi_i HA \cong \varprojlim \pi_i HA/p^n A$ when $A = \varprojlim A/p^n A$. The same is true for the r fold smash product, $HA^{(r)} = HA \wedge \cdots \wedge HA$,

$$\pi_*(HA^{(r)}; \mathbb{F}_p) \xrightarrow{\cong} \pi_* \left(\operatorname{holim}_{\overleftarrow{n}} H(A/p^n A)^{(r)}; \mathbb{F}_p \right).$$

This is an easy calculation based on the isomorphism

$$\pi_*(HA^{(r)}; \mathbb{F}_p) \cong H_*(HA^{(r-1)}; k) \oplus H_{*-1}(HA^{(r-1)}; k)$$

cf. [HM], lemma 5.1. It implies that the k-simplices

$$\mathrm{THH}_k(A)_p^\wedge \simeq \operatorname{holim}_{\leftarrow} \mathrm{THH}_k(A)_p^\wedge.$$

The simplicial group model $\mathrm{THH}_\bullet^\oplus$ for $\mathrm{THH}_\bullet$, cf. sect. 2.4, is a Kan complex, and for such homotopy inverse limits commutes with realizations, so we get

$$\mathrm{THH}(A)_p^\wedge \sim \operatorname{holim}_{\leftarrow} \mathrm{THH}(A/p^n A)_p^\wedge.$$

The same relation the holds for the spectra $\mathrm{TH}(A)$ and $\mathrm{TH}(A/p^n A)$.

Finally inductive use of the fundamental cofibration sequence shows that the fixed sets $(\mathrm{TH}(A)^{C_{p^n}})_p^\wedge$ are continuous, and since $\mathrm{TC}(A)_p^\wedge$ is a homotopy inverse limit construction, $\mathrm{TC}(A)_p^\wedge$ must be continuous. □

Theorem 4.3.8. *For the rings in theorem 4.3.1,*

$$K_i(A; \mathbb{Z}_p) \xrightarrow{\cong} K_i^{\mathrm{top}}(A; \mathbb{Z}_p)$$

Proof. Let F be the field of fractions of $W(k)$, and let $E = A \otimes_{W(k)} F$ with radical $J(E)$. Then $J = A \cap J(E)$ is a nilpotent ideal of A and it suffices, again the theorem 3.4.1, to show the theorem for A/J. But

$$A/J \otimes_{W(k)} F = E/J(E)$$

is semi-simple, and for such algebras results of Gabber, Suslin and Suslin-Yufryakov give the result, cf. [HM], appendix B for more details. □

Theorem 4.3.1 is probably the optimal result for K-theory calculations by traces. One would have liked to have a similar isomorphism for other rings, and in particular for the ring of rational integers. But

$$\mathrm{TC}_i(A; \mathbb{Z}_p) \xrightarrow{\cong} \mathrm{TC}_i(\varprojlim A/p^n A; \mathbb{Z}_p)$$

at least when A is finite over $\mathbb{Z}$. Indeed, this holds for the functor $A \mapsto (HA)^\wedge_p$ and hence adapting the argument of theorem 4.3.7 also for $\mathrm{TC}(A)^\wedge_p$. But K-theory does not have this property. One would also like to drop the finiteness assumption on A, and could wonder what would happen for $A = k[[X]]$. For such a ring the arguments proving theorem 4.3.7 and theorem 4.3.8 break down. In the first case for the simple reason that the r fold tensor power of A is not $k[[X_1, \dots, X_r]]$ - one needs completed tensor products.

4.4 The absolute theorem: group-like case.

This section examines $\mathrm{TC}(L)$ for a certain class of FSP's which include the $\tilde{G}$ of (2.3.4). The results are mostly a reformulation of parts of [BHM].

Definition 4.4.1. An FSP L is called group-like if the associated cyclotomic spectrum $T(L)$ satisfies the following condition:
For each finite cyclic group C there is an equivariant map of spectra

$$\sigma_C \colon \Phi^C T(L) \to T(L)^C,$$

natural with respect to inclusions $C_1 \subset C_2$, such that σ_C splits the natural map $s_C \colon T(L)^C \to \Phi^C T(L)$, $s_C \circ \sigma_C = \mathrm{id}$.

For group-like L, the fundamental cofibration

$$\mathrm{TH}(L)_{hC_{p^n}} \longrightarrow \mathrm{TH}(L)^{C_{p^n}} \xrightarrow{R} \mathrm{TH}(L)^{C_{p^{N-1}}}$$

is split by the map

$$S_{n-1} \colon \mathrm{TH}(L)^{C_{p^{n-1}}} \to \mathrm{TH}(L)^{C_{p^n}}$$

coming from the identification of $\rho^\#_{C_p} \Phi^{C_p} T(L)$ with $T(L)$, and

$$RS_{n-1} = \mathrm{id}, \quad FS_{n-1} \sim S_{n-2}F \quad (n \geq 2). \tag{4.4.2}$$

We recall from (4.1.5) that the fiber of R was identified as $\mathrm{TH}(L)_{hC_{p^n}}$ by the transfer map

$$\tau_{C_{p^n}} \colon \mathrm{TH}(L)_{hC_{p^n}} = \mathrm{TH}(L) \wedge_{C_{p^n}} ES^1_+ \xrightarrow{\sim} (T(L) \wedge ES^1_+)^{C_{p^n}}.$$

Naturality of transfers shows that

$$\begin{array}{ccc}
\mathrm{TH}(L)_{hC_{p^n}} & \xrightarrow{\sim} & T(L)^{C_{p^n}} \\
\downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle F} \\
TH(L)_{hC_{p^{n-1}}} & \xrightarrow{\sim} & T(L)^{C_{p^{n-1}}}
\end{array}$$

is homotopy commutative with τ_n being a suitable transfer map.

Proposition 4.4.3. *For a group-like FSP there is a homotopy Cartesian diagram*

$$\begin{tikzcd}
\mathrm{TC}(L)^\wedge_p \arrow[r] \arrow[d] & \left(\underset{\overleftarrow{\tau_n}}{\mathrm{holim}}\,\mathrm{TH}(L)_{hC_{p^n}}\right)^\wedge_p \arrow[d, "\mathrm{proj}"] \\
\mathrm{TH}(L)^\wedge_p \arrow[r, "FS_0-1"] & \mathrm{TH}(L)^\wedge_p
\end{tikzcd}$$

Proof. The splittings of (4.4.2) give equivalences

$$\bigvee_{i=0}^{n} \mathrm{TH}(L)_{hC_{p^i}} \xrightarrow{\sim} \mathrm{TH}(L)^{C_{p^n}}$$

such that on the left hand side R corresponds to projection. Hence

$$\mathrm{TR}(L,p) \sim \prod_{i=0}^{\infty} \mathrm{TH}(L)_{hC_{p^i}}.$$

Under this equivalence $F(x_0, x_1, \ldots) = (Fx_1 + FSx_0, Fx_2, \ldots)$, and the diagram

$$\begin{tikzcd}
\mathrm{TH}(L) \arrow[r] \arrow[d, "FS_0-\mathrm{id}"] & \prod_{i=0}^{\infty} \mathrm{TH}(L)_{hC_{p^i}} \arrow[r] \arrow[d, "F-\mathrm{id}"] & \prod_{i=1}^{\infty} \mathrm{TH}(L)_{hC_{p^i}} \arrow[d, "F-\mathrm{id}"] \\
\mathrm{TH}(L) \arrow[r] & \prod_{i=0}^{\infty} \mathrm{TH}(L)_{hC_{p^i}} \arrow[r] & \prod_{i=1}^{\infty} \mathrm{TH}(L)_{hC_{p^i}}
\end{tikzcd}$$

gives the cofibration

$$\mathrm{hF}(FS_0 - \mathrm{id})^\wedge_p \to \mathrm{TC}(L,p)^\wedge_p \to \underset{\overleftarrow{\tau_n}}{\mathrm{holim}} \left(\mathrm{TH}(L)_{hC_{p^i}}\right)^\wedge_p$$

upon taking vertical homotopy fibers. Apply theorem 2.5.5. □

Lemma 4.4.4. *For the identity FSP, $T(\mathrm{Id}) \sim_{C_\infty} \Sigma^\infty_{S^1}(S^0)$, where the right-hand side is the equivariant sphere spectrum.*

Proof. Recall from sect. 2.1 the subdivision S^1-homeomorphism

$$|sd_C \mathrm{THH}_\bullet(L;V)| \xrightarrow{D} |\mathrm{THH}_\bullet(L;V)|,$$

where C is a finite cyclic group of order c.

The space of 0-simplices in $sd_C\mathrm{THH}_\bullet(\mathrm{Id};V)$ is equal to $\mathrm{THH}_{c-1}(\mathrm{Id};V)$ and there is a natural C-map

$$i_C\colon \varinjlim \Omega^{m\mathbf{R}C}(S^{m\mathbf{R}C}\wedge S^V)\to |sd_C\mathrm{THH}_\bullet(\mathrm{Id};V)|$$

which is a C-homotopy equivalence onto the space of 0-simplices. The simplicial structure maps are C-homotopy equivalences, so the topological realization is C-homotopy equivalent to the space of 0-simplices, cf. sect. 2.2. Hence i_C is a C-homotopy equivalence. The diagram

$$\begin{tikzcd} {|sd_C\mathrm{THH}_\bullet(L;V)|} \arrow[rr, "D"] & & {|\mathrm{THH}_\bullet(L;V)|} \\ & S^V \arrow[ul, "i_C"] \arrow[ur, "i"'] & \end{tikzcd}$$

is commutative. It follows that the S^1-map

$$\Sigma^\infty_{S^1}(S^V)\to T(\mathrm{Id})(V)$$

induced by i is a C-homotopy equivalence for each finite C. □

For any FSP L and monoid G we may define a new FSP by

$$L[G](X)=L(X)\wedge G_+. \tag{4.4.5}$$

If $L=\mathrm{Id}$ this is precisely $\tilde{G}$ of (2.3.4). If $L=\tilde{A}$ for a commutative ring A, the map $\tilde{A}[G]\to\widetilde{A[G]}$ is a stable equivalence, so there are equivalences

$$K(\tilde{A}[G])\sim K(A[G]),\quad \mathrm{TC}(\tilde{A}[G])\sim\mathrm{TC}(A[G])$$

for every discrete group. When G is a group-like topological monoid, the cyclic classifying space $B^{\mathrm{cy}}G=|N^{\mathrm{cy}}_\bullet(G)|$ was identified in sect. 2.1 to be the free loop space ΛBG of the ordinary classifying space BG. Moreover, if δ_C is the composite homeomorphism

$$\delta_C\colon |N^{\mathrm{cy}}_\bullet(G)|\xrightarrow{\Delta_C}|sd_CN^{\mathrm{cy}}_\bullet(G)|^C\xrightarrow{D}|N^{\mathrm{cy}}_\bullet(G)|$$

then there is a commutative diagram ([BHM], proposition 2.5)

$$\begin{tikzcd} B^{\mathrm{cy}}G \arrow[r, "\delta_C"] \arrow[d] & (B^{\mathrm{cy}}G)^C \arrow[d] \\ \Lambda BG \arrow[r, "\Delta_c"] & (\Lambda BG)^C, \end{tikzcd}\quad \Delta_c(\lambda)(z)=\lambda(z^c) \tag{4.4.6}$$

Given any cyclotomic spectrum T and any space X, the spectrum smash product $T \wedge \Lambda X_+$ is again cyclotomic. Indeed, there is a canonical map from right to left:

$$\Phi^C(T \wedge \Lambda X_+) \sim \Phi^C T \wedge (\Lambda X)_+^C$$

which is an S^1/C-equivalence, and

$$r_C \wedge \Delta_c^{-1} \colon \rho_C^\# \Phi^C T \wedge \rho_C^\# \Lambda X_+^C \to T \wedge \Lambda X_+$$

defines the required equivalence, cf. sect. 2.4.

Lemma 4.4.7. *There is an S^1-equivalence of cyclotomic spectra, $T(L[G]) \sim_{S^1} T(L) \wedge \Lambda BG_+$, provided G is group-like.*

Proof. Consider the bi-simplicial space $X_{\bullet,\bullet}(G;V)$ with

$$X_{k,l}(G;V) = \underset{\mathbf{x}\in I^{k+1}}{\underrightarrow{\operatorname{holim}}}\, F\left(S^{x_0} \wedge \cdots \wedge S^{x_k}, F(S^{x_0}) \wedge \cdots \wedge F(S^{x_k}) \wedge G_+^l \wedge S^V\right).$$

Cyclic permutation of factors make it a bi-cyclic space. The map

$$X_{k,l}(1;V) \wedge G_+^l \to X_{k,l}(G;V) \tag{1}$$

becomes highly connected as an equivariant map as V runs through the S^1-universe $\mathcal{U}$ (one needs $\dim V^C \to \infty$ for all $C \subseteq S^1$).

The diagonal complex $\delta X_{\bullet,\bullet}(G;V)$ is precisely $\mathrm{THH}_\bullet(L[G];V)$ with realization $\mathrm{THH}(L[G];V)$. On the other hand, if we instead first realize the l-direction and then the k-direction and use (1), then we get a highly connected S^1-map

$$\mathrm{THH}(L;V) \wedge \Lambda BG_+ \to \mathrm{THH}(L[G];V).$$

Use of subdivision and (4.4.6) shows that the corresponding map on C-fixed sets become highly connected when V runs over $\mathcal{U}$, so the two prespectra are equivalent. Moreover, the corresponding cyclotomic structure maps agree. Apply spectrification. $\square$

Corollary 4.4.8. *The FSP $\tilde{G}$ is group-like if G is.*

Proof. The previous result tells us that $T(\tilde{G}) \sim_{C_\infty} \Sigma_{S^1}^\infty(\Lambda BG_+)$. But the suspension spectrum satisfies the requirement of (4.4.1). This is a consequence of the tom Dieck-Segal splitting, valid for any based S^1-space X:

$$\Sigma_{S^1}^\infty(X)^C \sim_{S^1/C} \bigvee_{H\subseteq C} \Sigma_{S^1/C}^\infty E_{S^1/C}(C/H)_+ \wedge_{C/H} X^H$$

$$\Phi^C(\Sigma_{S^1}^\infty(X)) \sim \Sigma_{S^1/C}^\infty(X^C)$$

Here $E_G(\Gamma)$ is the G-equivariant model of $E\Gamma$. The map s_C is the projection onto the factor $C = H$ and σ_C is the obvious inclusion, cf. [tD], [LMS]. $\square$

The next theorem is similar to lemma 5.15 of [BHM], but avoids the assumption that T has finite p-type. It contradicts the "counter-example" presented in [BHM], p. 498–499, which is wrong. The mistake occurs in the identification of $(t_m^{m-1})_r$ on p. 499. The mistake was pointed out by T. Goodwillie, and the proof below is due to him.

Lemma 4.4.9. *For any equivariant S^1-spectrum T, the S^1-transfer induces an isomorphism*

$$\pi_*(\Sigma T_{hS^1};\mathbb{F}_p) \to \pi_*(\operatorname{holim}_{\longleftarrow} T_{hC_{p^n}};\mathbb{F}_p).$$

Proof. The skeletons of ES^1 are the spheres $S^{2k-1} \in \mathbb{C}^k$ with the standard action of S^1. There is the cofibration diagram

$$\begin{tikzcd}
S^1 \wedge (S^{2k-1}_+ \wedge_{S^1} T) \arrow[r] \arrow[d, "\mathrm{trf}"] & S^1 \wedge (S^{2k+1}_+ \wedge_{S^1} T) \arrow[r] \arrow[d, "\mathrm{trf}"] & S^1 \wedge (S^{2k+1}/S^{2k-1} \wedge_{S^1} T) \arrow[d, "\mathrm{trf}"] \\
S^{2k-1}_+ \wedge_{C_{p^n}} T \arrow[r] & S^{2k+1}_+ \wedge_{C_{p^n}} T \arrow[r] & S^{2k+1}/S^{2k-1} \wedge_{C_{p^n}} T
\end{tikzcd} \tag{1}$$

Now $S^{2k+1}/S^{2k-1} \simeq_{S^1} S^1_+ \wedge S^{2k}$; the S^1-action on the right hand side is the diagonal action with $S^{2k} = \Sigma(S^{2k-1})$. However for any S^1-space or spectrum X,

$$S^1_+ \wedge X \cong_{S^1} S^1_+ \wedge |X|, \quad (z,x) \mapsto (z, z^{-1}x)$$

where the bars indicate X with no S^1-action. In particular,

$$S^1_+ \wedge S^{2k} \wedge T \simeq_{S^1} S^1_+ \wedge |S^{2k} \wedge T|$$

and the upper right hand term in (1) may be identified as

$$S^1 \wedge S^{2k+1}/S^{2k-1} \wedge_{S^1} T \simeq S^1 \wedge |S^{2k} \wedge T|.$$

Moreover, the right-hand vertical map in (1) can be identified as the smash product of the transfer

$$\tau\colon S^1 \wedge \Sigma^\infty(S^1_+/S^1) \to \Sigma^\infty(S^1_+/C_{p^n}) \tag{2}$$

with $|S^{2k} \wedge T|$. The transfers

$$\tau_n\colon \Sigma^\infty(S^1_+/C_{p^{n-1}}) \to \Sigma^\infty(S^1_+/C_{p^n})$$

of the C_p-covering $S^1/C_{p^{n-1}} \to S^1/C_{p^n}$ are known as follows. If we identify S^1/C_{p^n} with S^1 (via $\rho_{C_{p^n}}$), and use the splitting

$$\Sigma^\infty(S^1_+) = \Sigma^\infty(S^1) \vee \Sigma^\infty(S^0)$$

induced by the projections, then τ_n becomes the matrix

$$\tau_n = \begin{pmatrix} \mathrm{id} & 0 \\ \eta & p \end{pmatrix} \tag{3}$$

with $\eta \in \pi_1(\Sigma^\infty(S^0)) = \mathbb{Z}/2$ the non-trivial element. This can be seen for example by using ω of (2.4.2). Since the transfers in the limit system

$$\mathrm{trf}_n\colon S^{2k+1} \,/\, S^{2k-1} \wedge_{C_{p^{n-1}}} T \to S^{2k+1}/S^{2k-1} \wedge_{C_{p^n}} T$$

can be identified with $\tau_n \wedge |S^{2k} \wedge T|$,

$$\operatorname*{holim}_{\substack{\longleftarrow \\ \mathrm{trf}_n}} S^{2k+1}/S^{2k-1} \wedge_{C_{p^{n-1}}} T \simeq \operatorname*{holim}_{\substack{\longleftarrow \\ \tau_n}} \Sigma^\infty(S^1_+) \wedge |S^{2k} \wedge T|,$$

and we obtain from (3) a cofibration

$$S^1 \wedge S^{2k} \wedge T \to \operatorname*{holim}_{\substack{\longleftarrow \\ \tau_n}} \Sigma^\infty(S^1_+) \wedge |S^{2k} \wedge T| \to \operatorname*{holim}_{\substack{\longleftarrow \\ p}} S^{2k} \wedge T.$$

We can calculate the mod p homotopy groups of the right hand term by the exact sequence

$$0 \to \varprojlim{}^{(1)} \pi_{i-1}(S^{2k} \wedge T; \mathbb{F}_p) \to \pi_i(\operatorname{holim}_{\leftarrow}(S^{2k} \wedge T); \mathbb{F}_p) \to \lim_{\leftarrow} \pi_i(S^{2k} \wedge T; \mathbb{F}_p) \to 0.$$

The outer terms vanish, so in conclusion

$$\pi_i(S^1 \wedge S^{2k} \wedge T; \mathbb{F}_p) \cong \pi_i(\operatorname*{holim}_{\substack{\longleftarrow \\ \mathrm{trf}_n}} S^{2k+1}/S^{2k-1} \wedge_{C_{p^n}} T; \mathbb{F}_p),$$

and comparing with (2) it follows that the right-hand vertical maps in (1) induces an isomorphism

$$\pi_i(\Sigma(S^{2k+1}/S^{2k-1} \wedge_{S^1} T); \mathbb{F}_p) \xrightarrow{\cong} \pi_i(\operatorname*{holim}_{\longleftarrow} S^{2k+1}/S^{2k-1} \wedge_{C_{p^n}} T; \mathbb{F}_p).$$

We can finally make the obvious induction over k. □

Remark 4.4.10. The lemma can be restated as a homotopy equivalence of p-completed spaces,

$$(S^1 \wedge T_{hS^1})^\wedge_p \simeq (\operatorname{holim}_{\leftarrow} T_{hC_{p^n}})^\wedge_p.$$

Corollary 4.4.11. *For a group-like FSP, there is a homotopy Cartesian diagram of (non-equivariant) spectra*

$$\begin{tikzcd} \mathrm{TC}(L)_p^\wedge \arrow[r] \arrow[d] & (\Sigma \mathrm{TH}(L)_{hS^1})_p^\wedge \arrow[d, "\mathrm{trf}_{S^1}"] \\ \mathrm{TH}(L)_p^\wedge \arrow[r, "FS_0-\mathrm{id}"] & \mathrm{TH}(L)_p^\wedge \end{tikzcd}$$

Moreover, if $L = \tilde{G}$ for a group-like monoid, then $\mathrm{TH}(L) = \Sigma^\infty(\Lambda BG_+)$ and $FS_0 = \Sigma^\infty(\Delta_{p+})$ where $\Delta_p(\lambda)(z) = \lambda(z^p)$.

Proof. Only the last point need any explanation. It comes from the Segal-tom Dieck splitting used in the proof of corollary 4.4.8:

$$\begin{aligned} \Sigma^\infty_{S^1}(\Lambda BG_+)^{C_p} &\sim \Sigma^\infty(\Lambda BG_{hC_p+}) \vee \Sigma^\infty(\Lambda BG_+^{C_p}) \\ &\cong \Sigma^\infty(\Lambda BG_{hC_p+}) \vee \Sigma^\infty(\Lambda BG_+) \end{aligned}$$

where the last homeomorphism is $\mathrm{id} \vee \Sigma^\infty_+(\Delta_p^{-1})$. The map F becomes the sum of the transfer

$$\Sigma^\infty_+(\Lambda BG_{hC_p}) \to \Sigma^\infty(\Lambda BG_+)$$

and the inclusion

$$\Sigma^\infty(\Lambda BG_+^{C_p}) \to \Sigma^\infty(\Lambda BG_+)$$

and

$$S_0 \colon \Sigma^\infty(\Lambda BG_+) \to \Sigma^\infty_+(\Lambda BG_{hC_p}) \vee \Sigma^\infty(\Delta BG_+^{C_p})$$

is the inclusion in the second factor via $\Sigma^\infty_+(\Delta_p)$ □

Recall for an FSP L that we write $\pi_0 L$ for the associated ring $\pi_0 L = \varinjlim \pi_n L(S^n)$.

Theorem 4.4.12. *Suppose L is an FSP so that $\pi_0 L$ is a finite $W(k)$-algebra for some perfect field k of characteristic p. Then*

$$\mathrm{trc} \colon K(L)_p^\wedge \to \mathrm{TC}(L)_p^\wedge$$

is a homotopy equivalence.

Proof. Dundas' theorem 3.5.1 gives the homotopy Cartesian square

$$\begin{tikzcd} K(L)_p^\wedge \arrow[r] \arrow[d] & \mathrm{TC}(L)_p^\wedge \arrow[d] \\ K(\pi_0 L)_p^\wedge \arrow[r] & \mathrm{TC}(\pi_0 L)_p^\wedge \end{tikzcd}$$

and the bottom arrow is a homotopy equivalence by theorem 4.3.1. □

4.5 The K-theory assembly map.

For a discrete group G and a commutative ring R, $\mathrm{GL}_n(R[G])$ contains $\mathrm{GL}_n(R) \times G$ as a subgroup, namely as the tensor product of $(n \times n)$-matrices over R and elements $g \in G$ considered as (1×1)-matrices over $R[G]$. Taking classifying spaces gives a map

$$B\mathrm{GL}_n(R) \times BG \to B\mathrm{GL}_n(R[G]).$$

This induces a map of spectra

$$a_K \colon K(R) \wedge BG_+ \to K(R[G])$$

usually called the assembly map. Indeed, one may either use Segal's Γ-space definition, May's operad version or Waldhausen's definition of $K(A)$ to do the details, or one can use the device of ring suspensions as in the original source, [L1].

The study of a_K has long been promoted by W. C. Hsiang, who e.g. in [Hs], conjectured that a_K is a rational injection, provided R is regular and BG is a finite complex. The conjecture is often called the K-theory Novikov conjecture. The reason is that there is a similar assembly map in L-theory, initially constructed by F. Quinn,

$$a_L \colon L(R) \wedge BG_+ \to L(R[G])$$

and (rational) injectivity of a_L (for $R = \mathbb{Z}$ and BG a manifold) translates via the surgery exact sequence to Novikov's original conjecture about the homotopy invariance of the higher signatures.

The definition of a_K extends to the case of FSP's to give a map of spectra

$$a_K \colon K(L) \wedge BG_+ \to K(L[G]).$$

(Here G could be any group-like monoid, and thus BG any space. For $L = \mathrm{Id}$ the above becomes Waldhausen's assembly map $A(*) \wedge X_+ \to A(X)$). The study of the assembly map when $L = \mathrm{Id}$ was the main motivation behind [BHM]. We can now present a somewhat easier proof of the main result from [BHM], thanks to Dundas' relative theorem 3.5.1.

There is an obvious assembly map

$$\mathrm{THH}(L; V) \wedge \Lambda BG_+ \to \mathrm{THH}(L[G]; V)$$

(cf. lemma 4.4.7) and hence via the inclusion

$$BG \to \Lambda BG$$

an assembly map

$$\mathrm{THH}(L;V)\wedge BG_+ \to \mathrm{THH}(L[G];V).$$

This passes to an assembly map of cyclotomic spectra and induces

$$a_{\mathrm{TC}}\colon \mathrm{TC}(L)\wedge BG_+ \to \mathrm{TC}(L[G])$$

so that the diagram

$$\begin{array}{ccc} K(L)\wedge BG_+ & \xrightarrow{a_K} & K(L[G]) \\ \downarrow{\scriptstyle \mathrm{trc}\wedge\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{trc}} \\ \mathrm{TC}(L)\wedge BG_+ & \xrightarrow{a_{\mathrm{TC}}} & \mathrm{TC}(L[G]) \end{array} \tag{4.5.1}$$

is commutative.

For each FSP L, we can from its p-adic completion L_p, $L_p(S) = L(S)^\wedge_p$. (It should be remembered that $X^\wedge_p \wedge Y^\wedge_p$ is not p-complete; but this causes no problems because we are always completing the functors on the outside, so there are no unpleasant surprises in $\mathrm{THH}(L_p)^\wedge_p$ etc.)

Theorem 4.5.2. *For a discrete group G, the assembly map*

$$a_K\colon K(\mathrm{Id}_p)\wedge BG_+ \to K(\mathrm{Id}_p[G])$$

becomes split injective after p-adic completion.

Proof. We compose with the cyclotomic trace and consider

$$\begin{array}{ccc} (K(\mathrm{Id}_p)\wedge BG_+)^\wedge_p & \xrightarrow{a_K} & K(\mathrm{Id}_p[G])^\wedge_p \\ \downarrow & & \downarrow \\ (\mathrm{TC}(\mathrm{Id}_p)\wedge BG_+)^\wedge_p & \xrightarrow{a_{\mathrm{TC}}} & \mathrm{TC}(\mathrm{Id}_p[G])^\wedge_p \end{array} \tag{1}$$

Now corollary 4.4.11 gives the homotopy Cartesian diagram

$$\begin{array}{ccc} \mathrm{TC}(\mathrm{Id}_p[G])^\wedge_p & \longrightarrow & \left(\Sigma(\Sigma^\infty_{S^1}(\Lambda BG_+))_{hS^1}\right)^\wedge_p \\ \downarrow & & \downarrow \\ \Sigma^\infty(\Lambda BG_+)^\wedge_p & \xrightarrow{\Sigma^\infty(\Delta_p)-\mathrm{id}} & \Sigma^\infty(\Lambda BG_+)^\wedge_p \end{array} \tag{2}$$

upon using the obvious equivalence between $\mathrm{TH}(\mathrm{Id}_p)^\wedge_p$ and $\mathrm{TH}(\mathrm{Id})^\wedge_p$ together with lemma 4.4.4 and lemma 4.4.7.

The component group $\pi_0(\Lambda BG)$ is the set of free homotopy classes of maps from the circle into BG, and hence equal to the conjugary classes of elements in G. Let $\Lambda_{[1]}BG$ be the component of the identity element. There are S^1-equivariant maps

$$\Lambda BG_+ \xrightarrow{\text{proj}} \Lambda_{[1]}BG_+ \xleftarrow{\text{incl}} BG_+. \tag{3}$$

The inclusion is a homotopy equivalence, but not an equivariant one. Anyway, the weak statement is enough to ensure that

$$ES^1 \times_{S^1} BG \longrightarrow ES^1 \times_{S^1} \Lambda_{[1]}BG$$

is a homotopy equivalence, and since

$$\Sigma^\infty_{S^1}(\Lambda BG_+)_{hS^1} = \Sigma^\infty(ES^1 \times_{S^1} \Lambda BG_+)$$

diagram (1) projects to the homotopy Cartesian diagram

$$\begin{array}{ccc} (\mathrm{TC}(\mathrm{Id}_p) \wedge BG_+)^\wedge_p & \longrightarrow & \Sigma(\Sigma^\infty BS^1_+ \wedge BG_+)^\wedge_p \\ \downarrow & & \downarrow \\ \Sigma^\infty(BG_+)^\wedge_p & \longrightarrow & \Sigma^\infty(BG_+)^\wedge_p \end{array}$$

Moreover,

$$(\mathrm{TC}(\mathrm{Id}_p) \wedge BG_+)^\wedge_p \xrightarrow{a_{\mathrm{TC}}} \mathrm{TC}(\mathrm{Id}_p[G])^\wedge_p \xrightarrow{\text{proj}} (\mathrm{TC}(\mathrm{Id}_p) \wedge BG_+)^\wedge_p$$

is the identity, and thus a_{TC} is split injective after p-adic completion. Now apply theorem 4.4.12 and diagram (1) to conclude the proof. □

Soulé proved in [Sou] that

$$\pi_{4n+1}(K(\mathbb{Z}); \mathbb{Q}_p) \to \pi_{4n+1}(K(\mathbb{Z}_p); \mathbb{Q}_p) \tag{4.5.3}$$

is an isomorphism provided the p-adic L-function $L_p(1+2n, \omega^{-2n}) \neq 0$ (both groups are equal to $\mathbb{Q}_p$). This is certainly the case for regular primes and maybe always. Soulé proved (4.5.3) by using the étale cohomology invariant. It was reproved in [BHM] by cyclotomic trace considerations. One can use (4.5.3) to translate theorem 4.5.2 into a rational statement, namely

Theorem 4.5.4. ([BHM]). *If G is a discrete group for which each $H_i(BG; \mathbb{Z})$ is finitely generated, then the K-theory assembly map*

$$a_K \colon K(\mathbb{Z}) \wedge BG_+ \longrightarrow K(\mathbb{Z}G)$$

induce an injection on rational homotopy groups.

Proof. The linearization maps

$$K(\mathrm{Id}) \to K(\mathbb{Z}), \quad K(\mathrm{Id}[G]) \to K(\mathbb{Z}G)$$

are rational equivalences, essentially because the homotopy groups of $\Sigma^\infty(S^0)$ are finite in positive degrees, cf. [W1]. Thus it suffices to show the statement for

$$a_K\colon K(\mathrm{Id}) \wedge BG_+ \to K(\mathrm{Id}[G]).$$

We have

$$K(\mathrm{Id}[G]) \to \mathrm{TC}(\mathrm{Id}[G]) \to (\mathrm{TC}(\mathrm{Id}) \wedge BG_p)_p^\wedge$$

and must show

$$\mathrm{trc} \wedge \mathrm{id}_{BG}\colon K(\mathrm{Id}) \wedge BG_+ \to (\mathrm{TC}(\mathrm{Id}) \wedge BG_+)_p^\wedge$$

is rational injective. This is the case because

$$K(\mathrm{Id})_p^\wedge \to K(\mathrm{Id}_p)_p^\wedge \simeq \mathrm{TC}(\mathrm{Id}_p)_p^\wedge$$

is rationally the same as $K(\mathbb{Z})_p^\wedge \to K(\mathbb{Z}_p)_p^\wedge$, and because we can choose p to be a regular prime and apply (4.5.3). □

Remark 4.5.5. It would be nice if the above argument could be extended to L-theory, and thus proving the original Novikov conjecture for the groups with finitely generated Eilenberg-MacLane homology. There is a variant of $\mathrm{TC}(R)$, namely the topological Dihedral homology $\mathrm{TD}(R)$, which imitates the linear construction of [L2]. It is the fixed set of a suitable involution on $\mathrm{TC}(R)$, $\mathrm{TD}(R) = \mathrm{TC}(R)^{\mathbb{Z}/2}$, and there is a map from Hermitian K-theory into $\mathrm{TD}(R)$, at least when $1/2 \in R$. The basic problem with this approach however, is that $\mathrm{TD}(R)_p^\wedge \to \mathrm{TD}(R \otimes \mathbb{Z}_p)_p^\wedge$ is again an equivalence (under suitable finiteness conditions on R). But in contrast to (4.5.3), $L(\mathbb{Z}) \to L(\mathbb{Z}_p)$ is rationally trivial for all primes, so one cannot extend the K-theory proof directly.

There might be a chance of proceding indirectly as follows. Let E be the maximal abelian extension of $\mathbb{Q}_p$, and let A be the integers of E. If one could produce a signature type rationally injective map from $L(\mathbb{Z}[g])$ to $K(A[G])$, or maybe into some completion $\mathcal{A}(G)$ of $A[G]$, like the C^*-algebra associated with $\mathbb{C}[G]$, then one could study the K-theory assembly map on $A[G]$ (or $\mathcal{A}(G)$) using the techniques above.

In this connection one should remember the theorems of Suslin that $K(\hat{E})_p^\wedge \simeq K(\mathbb{C})_p^\wedge$ for the algebraic closure of E and that $K(\mathbb{C})_p^\wedge \simeq BU_p^\wedge$. The latter equivalence comes from the roots of unity: the map $BS^1 \to B\mathrm{GL}_1(\mathbb{C}) \to K(\mathbb{C})$ extends to $\Omega^\infty S^\infty(BS^1) \to K(\mathbb{C})$, and gives via the splitting $\Omega^\infty S^\infty(BS^1) \sim BY \times X$ the required map from BU to $K(\mathbb{C})$, I believe.

The same procedure gives a map from $BU_p^\wedge \to K(A)_p^\wedge$ because $\mu(A) = \mathbb{Q}/\mathbb{Z}$ and $B(\mathbb{Q}/\mathbb{Z})_p^\wedge \sim (BS^1)_p^\wedge$.

This remark represents years of discussions with W. C. Hsiang.

The main interest in the assembly map a_K lies in its relationship to automorphism groups of manifolds. For a group-like monoid, such as $G = \Omega X$, $K(\mathrm{Id}[G])$ is Waldhausen's $A(X)$ and in particular $K(\mathrm{Id}) = A(*)$, so that the assembly map takes the form

$$a_A\colon A(*) \wedge X_+ \to A(X).$$

Waldhausen defined the spectrum $\mathrm{Wh}^{\mathrm{top}}(X)$ to be the cofiber of a_A.

For a manifold M, the space of topological pseudo isotopies $\mathcal{P}^{\mathrm{top}}(M)$ is defined as the space of homeomorphisms of $\mathrm{M}^n \times I$ which is the identity on $\mathrm{M}^n \times 0 \cup \partial M \times I$. A celebrated result of Waldhausen [W4] states that

$$\Omega^2 \mathrm{Wh}^{\mathrm{top}}(M) \sim \underset{\underset{k}{\longrightarrow}}{\mathrm{holim}}\, \mathcal{P}^{\mathrm{top}}(M \times D^k). \tag{4.5.6}$$

Moreover, the stability theorem of K. Igsa, [I] asserts that the map

$$\mathcal{P}^{\mathrm{top}}(M) \to \underset{\longrightarrow}{\mathrm{holim}}\, \mathcal{P}^{\mathrm{top}}(M \times D^k)$$

is $(\dim M - 7)/3$-connected, at least if M is smoothable

Farell and Jones has in [FJ] shown that for a negatively curved manifold M, $\mathrm{Wh}^{\mathrm{top}}(S^1)$ determines $\mathrm{Wh}^{\mathrm{top}}(M)$. Thus it would be of considerable interest to determine $\mathrm{Wh}^{\mathrm{top}}(S^1)$. Theorem 3.5.1, proposition 4.4.3 and corollary 4.4.11 reduces this to the problem of studying the linearization map

$$L^{\langle 1\rangle}\colon \mathrm{TC}^{\langle 1\rangle}(S^1,p)_p^\wedge \longrightarrow \mathrm{TC}^{\langle 1\rangle}(\mathbb{Z}[t,t^{-1}],p)_p^\wedge$$

where $\mathrm{TC}^{\langle 1\rangle}(-)$ is the cofiber of a_{TC}. Indeed, the K-theory assembly map $S^1_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[t,t^{-1}])$ is an equivalence, so the fiber of $L^{\langle 1\rangle}$ is $\mathrm{Wh}^{\mathrm{top}}(S^1)_p^\wedge$. See also remark 5.4.8 below. See [M] for more details.

5 Calculations in K-theory

This chapter evaluates the higher K-groups $K_i(R;\mathbb{Z}_p)$ with p-adic coefficients in a number of cases where the K-groups were not previously known. The rings we consider are all of the type where the absolute theorem of the previous chapter applies, and the functor we actually calculate is $\mathrm{TC}(R)_p^\wedge$.

5.1 On the K-theory of group rings.

Let A be a finite algebra over $W(k)$, the Witt vectors of a finite field k of characteristic p. For a finite group G, the group ring $A[G]$ is again finite, so

$$K_i(A[G];\mathbb{Z}_p) \cong \mathrm{TC}_i(A[G];\mathbb{Z}_p).$$

By general induction theory, cf. [O1]

$$K(A[G])^{\wedge}_p \simeq \underleftarrow{\operatorname{holim}}\, K(A[\Gamma])^{\wedge}_p$$

where Γ runs over the hyper-elementary subgroups of G, that is, the subgroups of the form $\Gamma = C_N \ltimes P$ where P a p-group and $(N,p) = 1$. It follows that $A[\Gamma]$ decomposes into a product of twisted group rings $B^t[P]$ for unramified extensions B/A.

We here study the case of an untwisted group-ring $A[P]$. In terms of explicit values our main result is

Theorem 5.1.1. *For a perfect field k of characteristic $p > 0$, $K_{2n-1}(k[C_{p^N}];\mathbb{Z}_p) = K_1(k[C_{p^N}];\mathbb{Z}_p)^{\oplus n}$ and $K_{2n}(k[C_{p^N}];\mathbb{Z}_p) = 0$ when $n > 0$.*

The K_1-group on the left is the p-part of the units $k[C_{p^N}]^\times$ which is easily calculated, cf. theorem 5.1.16 below. Note also that $k[C_{p^N}]/\mathrm{rad} = k$, so that $K_i(k[C_{p^N}];\mathbb{Z}_l) = K_i(k;\mathbb{Z}_l)$ for $(l,p) = 1$.

Our starting point is lemma 4.4.7,

$$T(A[P]) \sim_{S^1} T(A) \wedge \Lambda BP_+.$$

Let $X(P)$ denote the conjugacy classes of elements in P. Then $\pi_0(\Lambda BP) = X(P)$, and the p'th power map $\Delta\colon X(P) \to X(P)$ has $\Delta^N X(P) = 1$ when P has exponent p^N. Define a filtration of $X(P)$,

$$\{1\} = X_0(P) \subset X_1(P) \subset \cdots \subset X_N(P) = X(P), \quad X_k(P) = \{g \mid g^{p^k} = 1\}$$

and a corresponding filtration of $\Lambda = \Lambda BP$

$$\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_N = \Lambda \tag{5.1.2}$$

where $\Lambda_k = \coprod_{\gamma \in X_k(P)} \Lambda_\gamma BP$ is the set of components corresponding to the listed conjugacy classes. We note (from [BHM], sect. 7) that

$$\Lambda_\gamma BP \sim BC_P(\gamma)$$

the classifying space of the centralizer of γ.

We are interested in the C_{p^n}-action on $T(A[P])$. The p'th power map $\Delta\colon \Lambda \xrightarrow{\cong} \Lambda^{C_p} \subset \Lambda$ maps Λ_k homeomorphically into $\Lambda_{k-1}^{C_p}$, so in (5.1.2), $\Lambda_N - \Lambda_{N-1}$ is the free stratum, and

$$\Delta\colon \Lambda_k - \Lambda_{k-1} \to (\Lambda_{k-1} - \Lambda_{k-2})^{C_p}$$

is a homeomorphism for $1 < k \leq N$. Let $\mathrm{TC}^{\langle 1\rangle}(A[G],p)$ denote the cofiber of the assembly map from sect. 4.5,

$$\mathrm{TC}(A,p) \wedge BP_+ \xrightarrow{\alpha_{\mathrm{TC}}} \mathrm{TC}(A[P],p) \longrightarrow \mathrm{TC}^{\langle 1\rangle}(A[P],p), \tag{5.1.3}$$

and write $T \ltimes X = T \wedge X_+$.

Proposition 5.1.4. *One has*

$$\mathrm{TC}^{\langle 1\rangle}(A[P],p) \simeq \operatorname*{\underleftarrow{holim}}_{F} (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^n}},$$

where the limit runs over inclusions of fixed sets.

Proof. In the proof we write $B = BP$. The inclusion $i\colon B \to \Lambda$ of B into the constant loops induces a cofibration sequence of cyclotomic spectra

$$T(A) \ltimes B \to T(A) \ltimes \Lambda \to T(A) \wedge \Lambda/B$$

This gives a cofibration sequence of fixed sets, and hence the cofibration sequence

$$\operatorname*{\underleftarrow{holim}}_{F,R}(T(A) \ltimes B)^{C_{p^n}} \to \mathrm{TC}(A[G],p) \to \operatorname*{\underleftarrow{holim}}_{F,R}(T(A) \wedge \Lambda/B)^{C_{p^n}}.$$

Now $\Delta = \mathrm{id}$ on B, and since B has trivial S^1-action,

$$\operatorname*{\underleftarrow{holim}}_{F,R}(T(A) \ltimes B)^{C_{p^n}} = (\operatorname*{\underleftarrow{holim}}_{F,R} T(A)^{C_{p^n}}) \ltimes B = \mathrm{TC}(A,p) \ltimes B.$$

It follows that

$$\mathrm{TC}^{\langle 1\rangle}(A[G],p) = \operatorname*{\underleftarrow{holim}}_{F,R}(T(A) \wedge \Lambda/B)^{C_{p^n}}. \tag{1}$$

We examine the right-hand side in two steps. First we evaluate the homotopy limit over R and then we use the cofibration

$$\operatorname*{\underleftarrow{holim}}_{F,R}(T(A)\wedge\Lambda/B)^{C_{p^n}} \longrightarrow \operatorname*{\underleftarrow{holim}}_{R}(T(A)\wedge\Lambda/B)^{C_{p^n}} \xrightarrow{F-1} \operatorname*{\underleftarrow{holim}}_{R}(T(A)\wedge\Lambda/B)^{C_{p^n}}. \tag{2}$$

We use the decomposition

$$\Lambda/B = \Lambda_0/B \vee (\Lambda_1 - \Lambda_0)_+ \vee \cdots \vee (\Lambda_N - \Lambda_{N-1})_+$$

and the corresponding decomposition

$$(T(A) \wedge \Lambda/B)^{C_{p^n}} = (T(A) \wedge \Lambda_0/B)^{C_{p^n}} \vee \bigvee_{k=1}^{N} (T(A) \ltimes (\Lambda_k - \Lambda_{k-1}))^{C_{p^n}}.$$

There are the following easy consequences of the cyclotomic structure on $T(A) \ltimes \Lambda$, cf. lemma 4.4.7:

(3)

(i) $\Lambda_0^{C_p}/B \overset{\Delta}{\cong} \Lambda_1/B = \Lambda_0/B \vee (\Lambda_1 - \Lambda_0)_+$

(ii) $R\colon (T(A) \wedge \Lambda_0/B)^{C_{p^n}} \xrightarrow{\simeq} (T(A) \wedge \Lambda_0/B)^{C_{p^{n-1}}} \vee (T(L) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^{n-1}}}$

(iii) $R\colon (T(A) \ltimes (\Lambda_k - \Lambda_{k-1}))^{C_{p^n}} \to (T(L) \ltimes (\Lambda_{k+1} - \Lambda_k))^{C_{p^{n-1}}}, \quad 1 \le k < N$

(iv) $R\colon (T(A) \ltimes (\Lambda_N - \Lambda_{N-1}))^{C_{p^n}} \to 0$

The fundamental cofibration applied to $T = T(A) \wedge \Lambda_0/B$ shows that (3,ii) is a homotopy equivalence. Indeed $(T(A) \wedge \Lambda_0/B)_{hC_{p^n}} \sim 0$ since the inclusion of B in Λ_0 is a non-equivariant homotopy equivalence. If we write

$$X_n = \bigvee_{k=2}^{N} (T(A) \ltimes (\Lambda_k - \Lambda_{k-1}))^{C_{p^n}}$$
$$Y_n = (T(A) \wedge \Lambda_0/B)^{C_{p^n}} \vee (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^n}}$$

and consider the cofibration sequence of limit systems

$$(X_n, R) \to \big((T(A) \wedge \Lambda/B)^{C_{p^n}}, R\big) \to (Y_n, R)$$

it follows from (3,iii–iv) that $R^{N-1}\colon X_n \to X_{n-N+1}$ is null-homotopic. Hence $\underleftarrow{\operatorname{holim}} X_n \simeq 0$, and

$$\underleftarrow{\operatorname{holim}} (T(A) \wedge \Lambda/B)^{C_{p^n}} \simeq \underleftarrow{\operatorname{holim}}\, Y_n.$$

Inductive use of (3,ii) yields

$$Y_n \simeq \bigvee_{i=0}^{n-1} (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^i}}$$

and that $R\colon Y_n \to Y_{n-1}$ corresponds to the obvious projection. Therefore

$$\underset{R}{\underleftarrow{\operatorname{holim}}}\, Y_n \simeq \prod_{i=0}^{\infty} (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^i}}$$

Now it is easy to see that

$$F\colon \prod_{i=0}^{\infty} (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^i}} \to \prod_{i=0}^{\infty} (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^i}}$$

sends $(t_0, t_1, \dots)$ to $(Ft_1, Ft_2, \dots)$ where on the right-hand side

$$F\colon\ (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^k}} \to (T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^{k-1}}}$$

is just inclusion of fixed sets. Thus by (2),

$$\underset{F,R}{\underleftarrow{\operatorname{holim}}}(T(A) \wedge \Lambda/B)^{C_{p^n}} \simeq \underset{F}{\underleftarrow{\operatorname{holim}}}\,(T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^n}} \qquad \square$$

If P has exponent p then $\Lambda_1 - \Lambda_0$ is a free C_{p^n} space, so

$$(T(A) \ltimes (\Lambda_1 - \Lambda_0))^{C_{p^n}} \sim T(A) \ltimes_{C_{p^n}} (\Lambda_1 - \Lambda_0) \sim (T(A) \ltimes (\Lambda_1 - \Lambda_0))_{hC_{p^n}}$$

and lemma 4.4.9 gives

$$\mathrm{TC}^{\langle 1\rangle}(A[P])_p^\wedge \sim (\Sigma(\mathrm{TH}(A) \ltimes (\Lambda_1 - \Lambda_0))_{hS^1})_p^\wedge\,. \tag{5.1.5}$$

For more general P, there is a spectral sequence

$$E^1_{k,l} = \pi_{k+l-1}\left((\mathrm{TH}(A) \ltimes (\Lambda_k - \Lambda_{k-1}))_{hS^1}; \mathbb{Z}_p\right) \Rightarrow \pi_*\left(\mathrm{TC}^{\langle 1\rangle}(A[G]; \mathbb{Z}_p)\right)$$

which might be of use in some situations. In this connection, I note from [J], theorem B that the homology of the homotopy S^1 orbit is closely related to cyclic homology, namely

$$HC_n(C_*(G)) = H_n(\Lambda BG_{hS^1})$$

where $C_*(G)$ denotes the singular chain complex; for discrete G this is equivalent to the group ring. Thus the E^1-term above is a twisted version of certain subgroups of cyclic homology groups associated with the filtration (5.1.2). If one takes a Postnikov decomposition of $\mathrm{TH}(A)$ one obtains a second spectral sequence which converges to the E^1-term and starts out with cyclic homology.

For $A = \mathbb{Z}_p$ with p odd one can in a range instead use theorem 4.4.11 with $L = \mathrm{Id}_p[P]$. Indeed,

$$\mathrm{TH}(\mathrm{Id}_p[P]) \to \mathrm{TH}(\mathbb{Z}_p[P])$$

is $(2p-3)$-connected. The same is then the case when one replaces $\mathrm{TH}(-)$ by $\mathrm{TR}(-)$, and it follows that

$$\mathrm{TC}(\mathrm{Id}_p[P]) \to \mathrm{TC}(\mathbb{Z}_p[P])$$

is $(2p-4)$-connected. One the other hand for a p-group

$$\Delta_p\colon \Lambda BP/BP \to \Lambda BP/BP$$

is nilpotent, so theorem 4.4.11 yields the homotopy Cartesian square

$$\begin{tikzcd}
\mathrm{TC}(\mathrm{Id}_p[P]) \arrow[r] \arrow[d, Leftrightarrow] & \Sigma^\infty(\Sigma_+(\Lambda BP_{hS^1})) \arrow[d, "\mathrm{trf}_{S^1}"] \\
\Sigma^\infty(BP_+) \arrow[r, "0"] & \Sigma^\infty(BP_+)
\end{tikzcd}$$

This gives the exact sequence

$$K_n(\mathbb{Z}_pP;\mathbb{Z}_p)/H_n(P;\mathbb{Z}_p) \to \mathrm{TC}_{n-1}(\mathbb{Z}_pP) \to H_n(P;\mathbb{Z}_p) \to \cdots \tag{5.1.6}$$

exact for $n \le 2p-4$, cf. conjecture 0.1 from [O2]. I leave for the reader to wonder about $p=2$.

I now specialize to $P = C_{p^N}$, the cyclic group of order N, where one can be more explicit.

The components of $\Lambda = \Lambda BC_{p^N}$ are indexed by C_{p^N}, and are denoted Λ_g, $g \in C_{p^N}$. Two elements g_1, g_2 of the same order have S^1-homeomorphic components since there is an automorphism $\phi \in \mathrm{Aut}(C_{p^N})$ with $\phi(g_1) = g_2$ which induces $\phi\colon \Lambda_{g_1} \xrightarrow{\cong} \Lambda_{g_2}$. Moreover, for each component corresponding to a non-generator, one has the S^1-homeomorphism

$$\rho^\#_{C_p}\Lambda_g^{C_p} \cong \coprod\{\Lambda_h \mid h^p = g\} \tag{5.1.7}$$

induced by the p'th power map $\Delta\colon \Lambda_h \longrightarrow \Lambda_g^{C_p}$.

Lemma 5.1.8. *For any cyclotomic spectrum T and $k \ge l$ there is a cofibration sequence of spectra*

$$(\rho^\#_{C_{p^l}}T^{C_{p^l}} \ltimes \Lambda_g)^{C_{p^{k-l}}} \to (T \ltimes \Lambda_{g^{p^l}})^{C_{p^k}} \to \bigvee^{p-1}\bigvee_{j=1}^{l-1}(T \ltimes \Lambda_{g^{p^j}})^{C_{p^{k-l+j}}}.$$

Proof. The l'th iterate $\Delta^l\colon \Lambda_g \to \Lambda_{g^{p^l}}$ embeds Λ_g into one component of $\Lambda_{g^{p^l}}^{C_{p^l}}$, and $\Delta^l(\Lambda_g)$ is (non-equivariantly) equivalent to the ambient space $\Lambda_{g^{p^l}}$. The cofibration of the lemma is induced from

$$\Delta^l(\Lambda_g)_+ \to \Lambda_{g^{p^l}+} \to \Lambda_{g^{p^l}}/\Delta^l(\Lambda_g)$$

upon applying the functor $\rho^{\#}_{C_{p^k}}(T\wedge(-))^{C_{p^k}}$. Since $\Delta^l(\Lambda_g)$ is fixed under C_{p^l},

$$\rho^{\#}_{C_{p^l}}\left(T\ltimes\Delta^l(\Lambda_g)\right)^{C_{p^l}}\sim_{S^1}\rho^{\#}_{C_{p^l}}T^{C_{p^l}}\ltimes\Lambda_g.$$

We use (2.4.3) to calculate the cofiber. Indeed, $\left(T\wedge\Lambda_{g^{p^l}}/\Delta^l(\Lambda_g)\right)_{hC_{p^k}}\sim 0$ so that

$$\left(T\wedge\Lambda_{g^{p^l}}/\Delta^l(\Lambda_g)\right)^{C_{p^k}}\sim\rho^{\#}_{C_p}\Phi^{C_p}\left(T\wedge\Lambda_{g^{p^l}}/\Delta^l(\Lambda_g)\right)^{C_{p^{k-1}}}$$
$$\sim\left(T\wedge\rho^{\#}_{C_p}\Delta(\Lambda_{g^{p^{l-1}}})/\Delta^l(\Lambda_g)\right)^{C_{p^{k-1}}}\vee\bigvee_{j=1}^{p-1}\left(T\ltimes\Lambda_{g^{p^{l-1}}+jp^{N-1}}\right)^{C_{p^{k-1}}}.$$

Each of the $p-1$ wedge terms are equivalent to $(T\ltimes\Lambda_{g^{p^{l-1}}})^{C_{p^{k-1}}}$, and we can iterate. □

The point of the lemma is that the component $\Lambda_{g^{p^l}}$ has been replaced by the simpler components $\Lambda_g,\dots,\Lambda_{g^{p^{l-1}}}$, simpler w.r.t. the C_{p^k}-action. For example, the action of C_{p^k} on Λ_g is free when g is a generator of C_{p^N}. For every equivariant S^1-spectrum T,

$$\begin{aligned}&\text{(i)}\quad (\Sigma T_{hS^1})^\wedge_p\sim(\underleftarrow{\operatorname{holim}}\,T_{hC_{p^n}})^\wedge_p\\ &\text{(ii)}\quad (T^{hS^1})^\wedge_p\sim(\underleftarrow{\operatorname{holim}}\,T^{hC_{p^n}})^\wedge_p\\ &\text{(iii)}\quad \hat{\mathbb{H}}(S^1,T)^\wedge_p\sim\underleftarrow{\operatorname{holim}}\,\hat{\mathbb{H}}(C_{p^n},T)^\wedge_p\end{aligned}\tag{5.1.9}$$

The first equivalence is lemma 4.4.9, the second is an easy consequence of the definitions, and is just an equivariant version of the relation $\underrightarrow{\operatorname{holim}}BC_{p^n}\sim(\mathbb{C}P^\infty)^\wedge_p$. The third equivalence follows by comparing the norm fibration for C_{p^n} and S^1, cf. remark 4.1.9. We consider the convergent sequences with

$$E^2(T^{hS^1};\mathbb{Z}_p)=S_{\mathbb{Z}_p}\{t\}\otimes\pi_*(T;\mathbb{Z}_p)$$
$$E^2(\hat{\mathbb{H}}(S^1,T);\mathbb{Z}_p)=S_{\mathbb{Z}_p}\{t,t^{-1}\}\otimes\pi_*(T;\mathbb{Z}_p)$$

cf. [HM1], [GM] for convergence.

Proposition 5.1.10. *If $g\in C_{p^N}$ is a generator, then the Tate spectrum*

$$\hat{\mathbb{H}}(S^1,\rho^{\#}_{C_{p^l}}T(k)^{C_{p^l}}\ltimes\Lambda_g)^\wedge_p\sim *$$

Proof. We use $\mathbb{Z}_p$ coefficients and have

$$\hat{E}^2_{*,*}=S_{\mathbb{Z}_p}\{t,t^{-1}\}\otimes S_{W_{l+1}(k)}\{\sigma\}\otimes H_*(\Lambda_g;\mathbb{Z}/p^{l+1})$$

with $t \in \hat{E}^2_{-2,0}$, $\sigma \in \hat{E}_{0,2}$ and $H_*(\Lambda_g; \mathbb{Z}/p^{l+1}) \subset \hat{E}^2_{0,*}$, cf. (4.3.4).

The spectrum $T(k)^{C_{p^l}}$ is a product of Eilenberg MacLane spectra, since it is a module over $\mathrm{TR}(k) \sim HW(k)$, and the $\hat{d}^2$-differential is this given by

$$[S^1]_\#\colon H_t(\Lambda_g; \mathbb{Z}/p^{l+1}) \to H_{t+1}(\Lambda_g; \mathbb{Z}/p^{l+1})$$

induced from the action

$$S^1 \times \Lambda_g \to \Lambda_g$$

cf. proposition 4.1.14. The evaluation of loops at 1 gives a non-equivariant homotopy equivalence $\Lambda_g \to BC_{p^N}$, so

$$H_*(\Lambda_g; \mathbb{Z}/p^{l+1}) = E_{\mathbb{Z}/p^{l+1}}\{y_1\} \otimes \Gamma_{\mathbb{Z}/p^{l+1}}\{x_2\}$$

with $\deg(y_1) = 1$, $\deg x_2 = 2$ and with $\Gamma\{x_2\}$ being the divided polynomial algebra. We show in lemma 5.1.12 below that $[S^1]_\#$ multiplies by y_1. Hence

$$\hat{d}^2\,(t^s\gamma_n(x_2)\sigma^r) = t^{s+1}\gamma_n(x_2)y_1\sigma^r, \quad s \in \mathbb{Z}, \quad n \geq 0,$$

and $\hat{E}^3$=0. $\square$

Proposition 5.1.11. *For a generator $g \in C_{p^N}$,*

$$\pi_*\left((\rho^\#_{C_{p^l}} T(k)^{C_{p^l}} \ltimes \Lambda_g)^{hS^1}; \mathbb{Z}_p\right) = S_{W_{l+1}(k)}\{\sigma\} \otimes \tilde{H}_*(BC_{p^l}; \mathbb{Z}_p).$$

Proof. The spectral sequence for the homotopy S^1 fixed set has E^2-term

$$E^2_{*,*} = S_{\mathbb{Z}_p}\{t\} \otimes S_{W_{l+1}(k)}\{\sigma\} \otimes H_*(\Lambda_g; \mathbb{Z}/p^{l+1})$$

with differentials as above. This time, however t^{-1} is not present, so there is no differential to kill the classes $\gamma_n(x_2)y_1\sigma^r$. Thus

$$E^3_{*,*} = S_{W_{l+1}(k)}\{\sigma\} \otimes y_1\Gamma_{\mathbb{Z}/p^{l+1}}\{x_2\},$$

all concentrated on one vertical line, and $E^3_{*,*} = E^\infty_{*,*}$. $\square$

Lemma 5.1.12. *If $g \in C_{p^N}$ is a generator, then the action $S^1 \times \Lambda_g \to \Lambda_g$ induces multiplication by $y_1 \in H_1(\Lambda_g; \mathbb{Z}_p)$ on $H_*(\Lambda_g; \mathbb{Z}/p^{l+1})$.*

Proof. Let $\tilde{g}\colon S^1 \to BC_{p^N}$ represent the homotopy class corresponding to $g \in C_{p^N}$. Consider $\tilde{g}$ as an element of ΛBC_{p^N}. Since C_{p^N} is abelian, BC_{p^N} is an abelian topological group. The map $f\colon BC_{p^N} \to \Lambda BC_{p^N}$ with $f(b)(z) = b\tilde{g}(z)$ lands in Λ_g since we may connect b with a path to $1 \in BC_{p^N}$. Moreover,

f is a homotopy equivalence, since its composition with the evaluation map is homotopic to the identity. The lemma now follows from the homotopy commutative diagram

$$\begin{tikzcd} S^1 \times B \arrow[r, "1\times f"] \arrow[d, "g\times 1"] & S^1 \times \Lambda B \arrow[r, "\text{action}"] & \Lambda B \arrow[d, "\text{ev}"] \\ B \times B \arrow[rr, "\text{mult}"] & & B \end{tikzcd}$$

$\square$

We return to the calculation of the p-adic homotopy groups of $\mathrm{TC}^{\langle 1\rangle}(kC_{p^N})$. They are by proposition 5.1.4 equivalent to

$$\pi_*\left(\bigvee^{p-1} \underset{\overleftarrow{F}}{\mathrm{holim}} \left(T(k) \ltimes \Lambda_{g p^{N-1}}\right)^{C_{p^n}} ; \mathbb{Z}_p\right)$$
$$= \bigoplus^{p-1} \underleftarrow{\lim}\, \pi_* \left(\left(T(k) \ltimes \Lambda_{g p^{N-1}}\right)^{C_{p^n}} ; \mathbb{Z}_p\right),$$

where g generates C_{p^n}. The idea is to use the cofibration sequence of lemma 5.1.8 inductively for $l = 1, \dots, N-1$. One has

$$\underset{\overleftarrow{F}}{\mathrm{holim}}\, (T(k) \ltimes \Lambda_g)^{C_{p^n}} \sim \underset{\overleftarrow{F}}{\mathrm{holim}}\, (T(k) \ltimes \Lambda_g)_{hC_{p^n}} \sim (T(k) \ltimes \Lambda_g)^{hS^1}$$

after p-completion. This follows from (5.1.9,i) and proposition 5.1.10. Proposition 5.1.11 shows inductively that all p-adic homotopy is concentrated in odd degrees. In particular we get, for each l, short exact sequences

$$0 \to \pi_*\left(\rho^{\#}_{C_{p^l}} T(k)^{C_{p^l}} \ltimes \Lambda_g\right)^{hS^1} \to \pi_* \underset{\overleftarrow{F}}{\mathrm{holim}} \left(T(k) \ltimes \Lambda_{g p^l}\right)^{C_{p^n}}$$
$$\to \bigoplus^{p-1} \bigoplus_{j=1}^{l-1} \pi_* \underset{\overleftarrow{F}}{\mathrm{holim}} \left(T(k) \ltimes \Lambda_{g p^j}\right)^{C_{p^n}} \to 0$$

of homotopy groups with $\mathbb{Z}_p$ coefficients. These sequences are also split exact. Indeed the left hand term consists of a sum of groups $W_{l+1}(k) = W(k)/p^{l+1}$, so it suffices to check that

$$p^{l+1}\pi_*\left(\underset{\overleftarrow{F}}{\mathrm{holim}}\, T(k) \ltimes \Lambda_{g p^l}\right)^{C_{p^n}} = 0. \tag{5.1.13}$$

This on the other hand is a consequence of induction theory, upon using a result of C. Schlichtkrull, [Sch], which I now describe.

Let L be an FSP and consider the functor

$$\mathrm{TF}(L[G],p) = \underset{\overleftarrow{F}}{\mathrm{holim}}\, T(L[G])^{C_{p^n}}.$$

For $\Gamma \subset G$ of finite index we have the map

$$\mathrm{Ind}_G^{\Gamma}\colon \mathrm{TF}(L[G],p) \to \mathrm{TF}(L[\Gamma],p)$$

given as the composition of the functor applied to $L[G] \to \mathrm{End}_{\mathrm{L}[\Gamma]}(\mathrm{L}[\mathrm{G}])$ with Morita equivalence. Now

$$\mathrm{TF}(L[G],p) \sim \underset{\overleftarrow{F}}{\mathrm{holim}}\, (T(L) \ltimes \Lambda BG)^{C_{p^n}}$$

decomposes into components,

$$\mathrm{TF}(L[G],p) \sim \bigvee_{[g]\in X(G)} \underset{\overleftarrow{F}}{\mathrm{holim}}\, \left(T(L) \ltimes \Lambda_{[g]}BG\right)^{C_{p^n}}$$

with $\Lambda_{[g]}BG = B_{S^1}C_G(g)$, the classifying space of the centralizer with some action of S^1. It follows that Ind_G^{Γ} decomposes into components,

$$\mathrm{Ind}_G^{\Gamma}([g],[\gamma])\colon \underset{\overleftarrow{F}}{\mathrm{holim}}\, \left(T(L) \ltimes \Lambda_{[g]}BG\right)^{C_{p^n}} \to \underset{\overleftarrow{F}}{\mathrm{holim}}\, \left(T(L) \ltimes \Lambda_{[\gamma]}B\Gamma\right)^{C_{p^n}}.$$

Theorem 5.1.14. ([Sch]) (i) $\mathrm{Ind}_G^{\Gamma}([g],[\gamma]) = 0$ *if* $\gamma \notin [g]$. (ii) *If* $\gamma \in [g]$ *then* Ind_G^{Γ} *is induced from the* S^1*-equivariant covering* $\Lambda_\gamma B\Gamma \to \Lambda_g BG$. □

(The theorem verifies in particular conjecture 7.14 of [BHM]; it undoubtly generalizes to simplicial groups, and should be of help in the study of transfers in Waldhausen's A-theory).

Corollary 5.1.15. *In the limit over k, the cofibration sequences of lemma 5.1.8 become split, for $T = T(k)$.*

Proof. The terms in the limit sequence are modules over $K(k)_p^\wedge = HW(k)$ via the cyclotomic trace, so it suffices to check that the homotopy exact sequence is split. This was above reduced to the statement (5.1.13). We use theorem 5.1.14(i) with $G = C_{p^n}$, $\Gamma = C_{p^{N-l-1}}$ to conclude that

$$\mathrm{Res}^{C_{p^N}}_{C_{p^{N-l-1}}} \circ \mathrm{Ind}^{C_{p^{N-l-1}}}_{C_{p^N}} : \mathrm{TF}(k[C_{p^N}],p) \to \mathrm{TF}(k[C_{p^N}],p)$$

is trivial on $\underleftarrow{\mathrm{holim}}\left(T(k)\ltimes\Lambda_{g^{p^l}}\right)^{C_{p^n}}$. On the other hand the composition induces multiplication by the index p^{l+1} on homotopy. □

Theorem 5.1.16. *For a perfect field of characteristic $p>0$,*

$$\begin{aligned}\pi_{2n-1}\mathrm{TC}^{\langle 1\rangle}(k[C_{p^N}]) &= \pi_1\left(\mathrm{TC}^{\langle 1\rangle}(k[C_{p^N}])\right)^{\oplus n},\\ \pi_{2n}\mathrm{TC}^{\langle 1\rangle}(k[C_{p^N}]) &= 0,\ n\geq 0\end{aligned}$$

Moreover,

$$\pi_1\mathrm{TC}^{\langle 1\rangle}(kC_{p^N}) = \left(W(k)/p^N\right)^{\oplus(p-1)}\oplus\bigoplus_{j=1}^{N-1}\left(W(k)/p^{N-j}\right)^{\oplus(p-1)(p^j-p^{j-1})}.$$

Proof. This follows from corollary 5.1.15 and proposition 5.1.11 upon collecting terms. □

We have left to determine the exact homotopy sequence of

$$\mathrm{TC}(k)\ltimes BC_{p^N}\to\mathrm{TC}(k[C_{p^N}])\to\mathrm{TC}^{\langle 1\rangle}(k[C_{p^N}]).\tag{5.1.17}$$

From (4.3.3) we have

$$\mathrm{TC}(k)\sim H\mathbb{Z}_p\vee\Sigma^{-1}H(\mathbb{Z}_p).\tag{5.1.18}$$

when k is finite. Thus

$$\pi_i\left(\mathrm{TC}(k)\ltimes BC_{p^N}\right) = H_i(BC_{p^N};\mathbb{Z}_p)\oplus H_{i+1}(BC_{p^N};\mathbb{Z}_p),$$

with one copy of $\mathbb{Z}/p^N$ in each degree.

Lemma 5.1.19. *The homotopy exact sequence of (5.1.17) reduces to the exact sequence*

$$\begin{aligned}0&\longrightarrow H_{2n-1}(BC_{p^N};\mathbb{Z}_p)\longrightarrow\mathrm{TC}_{2n-1}(k[C_{p^N}])\\ &\longrightarrow\mathrm{TC}^{\langle 1\rangle}_{2n-1}(k[C_{p^N}])\xrightarrow{\partial_*}H_{2n-1}(BC_{p^N};\mathbb{Z}_p)\longrightarrow 0.\end{aligned}$$

Proof. We must argue that ∂_* is surjective. This is true for $n=1$ because $\mathrm{TC}^{\langle 1\rangle}_2(k[C_{p^N}])=0$ and because the K-theory assembly map is clearly injective in dimension zero.

For $n > 1$ we use that (5.1.17) is a module over $\mathrm{TR}(k) \ltimes BC_{p^N}$, and hence over $\mathrm{TR}(\mathbb{F}_p) \ltimes BC_{p^N}$. Thus

$$\partial\colon \mathrm{TC}^{\langle 1\rangle}(k[C_{p^N}]) \to \Sigma\mathrm{TC}(k) \ltimes BC_{p^N} \to H\mathbb{Z}_p \ltimes BC_{p^N}$$

commutes with the resulting actions

$$\mathrm{TC}_*^{\langle 1\rangle}(k[C_{p^N}]) \otimes H_*(BC_{p^N};\mathbb{Z}/p^N) \to \pi_*\left(\mathrm{TC}^{\langle 1\rangle}(k[C_{p^N}]);\mathbb{Z}/p^N\right)$$
$$H_*(BC_{p^N};\mathbb{Z}_p) \otimes H_*(BC_{p^N};\mathbb{Z}/p^N) \to H_*(BC_{p^N};\mathbb{Z}/p^N)$$

The second map has the property

$$H_1(BC_{p^N};\mathbb{Z}_p)\cdot H_{2n}(BC_{p^N};\mathbb{Z}/p^N) = H_{2n+1}(BC_{p^N};\mathbb{Z}/p^N),$$

and since $H_{2n+1}(BC_{p^N};\mathbb{Z}_p) = H_{2n+1}(BC_{p^N};\mathbb{Z}/p^N)$, surjectivity of ∂_* in dimension 1 gives surjectivity in general. □

Since $\mathrm{TC}_i(k[C_{p^N}]) = K_i(k[C_{p^N}];\mathbb{Z}_p)$ has exponent p^N, lemma 5.1.19 yields the abstract isomorphism

$$\mathrm{TC}_{2n-1}(k[C_{p^N}]) \cong \mathrm{TC}_{2n-1}^{\langle 1\rangle}(k[C_{p^N}]).$$

This proves theorem 5.1.1.

It seems clear that one should be able to calculate $K_*(k[P])$ for more complicated p-groups. It is also natural to attack $K_*(A[P])$ for other base rings, and in particular for $A = \mathbb{Z}_p$ cf. sect. 5.4 below.

I conclude with some remarks about the twisted group ring case, inspired by [O1], ch. 12. Let E be any finite extension of $\mathbb{Q}_p$ and $A \subset E$ the ring of integers. Given a p-group P and any homomorphism $t\colon P \to \mathrm{Gal}(E/\mathbb{Q}_p)$ we have the twisted group ring $A^t[P]$. It contains the untwisted group ring $A[P_0]$, $P_0 = \mathrm{Ker}(t)$. Theorem 12.3 of [O1] states that the inclusion induces an isomorphism

$$K_1(A[P_0])_{P/P_0} \xrightarrow{\cong} K_1(A^t[P]), \tag{5.1.20}$$

where the left hand side denotes the coinvariants of the action induced from $P/P_0 \to \mathrm{Aut}(A) \times \mathrm{Out}(P_0)$. Olivers argument is based upon the integral p-adic logarithm, close in spirit to $\pi_1(\mathrm{trc})$; one may wonder if (5.1.20) generalizes to the statement

$$\mathrm{TC}(A^t[P]) \sim \mathrm{TC}(A[P_0])_{hP/P_0} \quad ?$$

5.2 K-theory of $k[x]/(x^n)$.

This section outlines joint work with Lars Hesselholt. The main result is Theorem 5.2.8 below. A detailed account can be found in [HM], sect. 6–8, when $n = 2$ and will appear in [HM2] when $n > 2$.

Let $\Pi_n = \{0, 1, x, \dots, x^{n-1}\}$, considered as a pointed monoid with 0 as base point and with $x^i = 0$ for $i \geq n$. We form the cyclic construction $N_\bullet^{\mathrm{cy}}(\Pi_n)$. Its set of k-simplices is the $(k+1)$-fold smash power of Π_n, so consists of $k+1$ tuples $(x^{i_0}, \dots, x^{i_k})$ with $(x^{i_0}, \dots, x^{i_k}) = 0$ if some $i_\nu \geq n$; $N_\bullet^{\mathrm{cy}}(\Pi_n)$ becomes a cyclic set when we give it the structure maps of sect. 2.1.

The argument of lemma 4.4.7 gives for any ring A (or even FSP) the equivalence of equivariant spectra

$$T(A[x]/(X^n)) \sim_{S^1} T(A) \wedge |N_\bullet^{\mathrm{cy}}(\Pi_n)|, \tag{5.2.1}$$

There is an analogue of the component decomposition of $N_\bullet^{\mathrm{cy}}(G) = \Lambda BG$, namely

$$N_\bullet^{\mathrm{cy}}(\Pi_n) = \bigvee_{s=0}^{\infty} N_\bullet^{\mathrm{cy}}(\Pi_n; s)$$

where $N_k^{\mathrm{cy}}(\Pi_n; s)$ consists of simplices $(x^{i_0}, \dots, x^{i_k})$ with $\Sigma i_\nu = s$, and $0 \in N_k^{\mathrm{cy}}(\Pi_n; s)$ for all s. The simplex $(x^{(s)}) = (x, \dots, x) \in N_{s-1}^{\mathrm{cy}}(\Pi_n; s)$ is represented by a cyclic map

$$i_{s,\bullet} \colon \Lambda[s-1]_\bullet \to N_\bullet(\Pi_n; s)$$

of the standard cyclic $(s-1)$-simplex. Its realization becomes a map

$$i \colon S^1 \times \Delta^{s-1} \to |N_\bullet(\Pi_n; s)|,$$

cf. (2.1.3). Since $(x^n, x, \dots, x) \in N_{s-n}(\Pi_n; s)$ is the base point, the composite of $i_{s,\bullet}$ with the iterated face operator $d_{s-n+1} \circ \cdots \circ d_s$, maps the corresponding face $S^1 \times \Delta^{s-n}$ to zero. Moreover, as $(x^{(s)})$ is invariant under cyclic permutations, i_s maps the orbit $S^1 \times C_s \cdot \Delta^{s-1} = C_s \cdot (S^1 \times \Delta^{s-1})$ to zero. All in all we obtain a map

$$\hat{i}_s \colon S^1 \times_{C_s} \Delta^{s-1} / S^1 \times_{C_s} C_s \cdot \Delta^{s-n} \to |N_\bullet^{\mathrm{cy}}(\Pi_n; s)|$$

and it is not hard to prove:

Lemma 5.2.2. *The map $\hat{i}_s$ is an S^1-equivariant homeomorphism.* □

For $n = 2$, the domain of $\hat{i}_s$ is $S^1 \times_{C_s} \Delta^{s-1} / \partial(S^1 \times_{C_s} \Delta^{s-1})$. We consider $\Delta^{s-1} \subset \mathbb{R}C_s$ to be the simplex spanned by the group elements $g^i \in \mathbb{R}C_s$.

It projects homeomorphically to the reduced regular representation $\mathbb{R}C_s - \mathbb{R}$ with $\mathbb{R} \subset \mathbb{R}C_s$ the invariant line through $\sum_{i=0}^{s-1} g^i$. Hence we have:

$$\begin{aligned} S^1 \times_{C_s} \Delta^{s-1}/\partial(S^1 \times_{C_s} \Delta^{s-1}) & \\ &\cong S^1 \times_{C_s} D(\mathbb{R}C_s - \mathbb{R})/\partial(S^1 \times_{C_s} D(\mathbb{R}C_s - \mathbb{R})) \\ &\cong S^1_+ \wedge_{C_s} S^{\mathbb{R}C_s - \mathbb{R}}. \end{aligned}$$

If s is odd then $\mathbb{R}C_s - \mathbb{R}$ is a complex representation and

$$S^1_+ \wedge_{C_s} S^{\mathbb{R}C_s - \mathbb{R}} \cong_{S^1} S^1_+/C_s \wedge S^{\mathbb{R}C_s - \mathbb{R}}$$

with diagonal S^1-action on the right hand side. If s is even then $\mathbb{R}C_s - \mathbb{R} = \mathbb{R}_- \oplus V_s$ with V_s complex, and

$$S^1_+ \wedge_{C_s} S^{\mathbb{R}C_s - \mathbb{R}} \cong \operatorname{cof}\left(S^1_+/C_{s/2} \xrightarrow{\Delta} S^1_+/C_s\right) \wedge S^{V_s}$$

with Δ the natural projection.

The above description of $|N^{\text{cy}}_\bullet(\Pi_2; s)|$ has the following generalization when $n > 2$. We use $\mathbb{C}(n)$ to denote the complex S^1-representation where the action of $z \in S^1$ is multiplication with z^n. Suppose $dn < s \leq (d+1)n$, and write

$$V_s = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \cdots \oplus \mathbb{C}(d). \tag{5.2.3}$$

It is an S^1-module and hence by restriction to $C_s \subset S^1$ also a C_s-module.

Theorem 5.2.4. ([HM2]). *Suppose $n \geq 2$ and $dn < s \leq (d+1)n$. Then*

$$S^1 \times_{C_s} \Delta^{s-1}/S^1 \times_{C_s} C_s \cdot \Delta^{s-n} \sim_{S^1} \begin{cases} S^1_+/C_s \wedge S^{V_s}, & s < (d+1)n \\ \operatorname{cofib}\left(S^1_+/C_{d+1} \to S^1_+/C_s\right) \wedge S^{V_s}, & s = (d+1)n \end{cases}$$

Proof. (Outline). The proof is based upon the concept of regular cyclic polytopes of D. Gale, [G]. Let $\pi_d(g) = (\xi_s, \xi_s^2, \dots, \xi_s^d)$, $\xi_s = e^{2\pi i/s}$. The image $P_{s,d} = \pi_d(\Delta^{s-1}) \subset V_s$ is a regular cyclic polytope. Its structure of facets (=codim 1 faces) is completely described in [G]. Using this we prove in [HM2] that

$$\pi_d(\Delta^{s-1}/C_s \cdot \Delta^{s-n}) \sim P_{s,d}/Q_{s,d} \sim S^{V_s}$$

for $dn < s < (d+1)n$ where $Q_{s,d} = \pi_d(C_s \cdot \Delta^{s-n})$. Next, the socalled Buenos Aires formula, [BAG], gives explicit generators of the homology

$$H_*\left(N^{\text{cy}}_\bullet(\Pi_n); \mathbb{Z}\right) = \mathrm{HH}_*\left(\mathbb{Z}[x]/(x^n); \mathbb{Z}\right)$$

in terms of the simplices of $N^{\text{cy}}_\bullet(\Pi_n; s)$. This is used to show that

$$S^1_+ \wedge_{C_s} \Delta^{s-1}/S^1_+ \wedge_{C_s} C_s \cdot \Delta^{s-n} \sim S^1_+ \wedge_{C_s} S^{V_s}$$

when $dn < s < (d+1)n$. The case $s = (d+1)n$ is somewhat more complicated, and will not be outlined here. $\square$

We also need to know the cyclotomic structure of $T\left(A[x]/(x^n)\right)$, similar to lemma 4.4.7, and must calculate the geometric fixed points:

$$\begin{aligned}\rho_{C_p}^{\#}\Phi^{C_p}T\left(A[x]/(x^n)\right) &\sim_{C_{p^\infty}} \rho_{C_p}^{\#}\Phi^{C_p}T(A)\wedge\rho_{C_p}^{\#}|N_\bullet^{\mathrm{cy}}(\Pi_n)|^{C_p}\\ &\sim_{C_{p^\infty}} T(A)\wedge\rho_{C_p}^{\#}|sd_{C_p}N_\bullet^{\mathrm{cy}}(\Pi_n)|^{C_p}.\end{aligned}$$

Comparing with (2.1.7), the isomorphism

$$\Delta_{C_p}\colon N_\bullet^{\mathrm{cy}}(\Pi_n,s)\xrightarrow{\cong} sd_{C_p}N_\bullet^{\mathrm{cy}}(\Pi_n;sp)^{C_p}$$

gives an S^1-map

$$\Delta^{-1}\colon \rho_{C_p}^{\#}sd_{C_p}N_\bullet^{\mathrm{cy}}(\Pi_n;sp)^{C_p}\xrightarrow{\cong} N_\bullet^{\mathrm{cy}}(\Pi_n;s)$$

which when composed with the above gives the required C_{p^∞}-equivalence

$$\rho_{C_p}^{\#}\Phi^{C_p}T\left(A[x]/(x^n)\right)\sim_{C_{p^\infty}} T\left(A[x]/(x^n)\right).$$

It is clear from the definition of V_s that $\rho_{C_p}^{\#}V_{sp}^{C_p}\cong_{S^1} V_s$, and we also have $S^1/C_s\cong_{S^1}\rho_{C_p}^{\#}(S^1/C_{ps})$. This yields a C_{p^∞}-equivalence

$$r_{C_p}(s)\colon \rho_{C_p}^{\#}\Phi^{C_p}\left(T(A)\wedge S^1_+/C_{ps}\wedge S^{V_{ps}}\right)\to T(A)\wedge S^1_+/C_s\wedge S^{V_s}.$$

The proof of theorem 5.2.4 contains the following

Addendum 5.2.5. *The cyclotomic structure of $T(A[x]/(x^n))$ is given by $\bigvee_{s=0}^{\infty} r_{C_p}(s)$.* $\square$

Since we are working in the category of equivariant spectra, $T(A)\wedge S^{V_s}$ is equal to the V_s'th deloop $T(A)(V_s)$ of $T(A)$. With this interpretation $r_{C_p}(s)$ induces

$$R\colon T(A)(V_{ps})^{C_{p^m}}\to T(A)(V_s)^{C_{p^{m-1}}}$$

and we can form the homotopy inverse limit over these maps

Denote by $\widetilde{\mathrm{TC}}(A[x]/(x^n))$ the reduced space, i.e. the homotopy fiber of $\mathrm{TC}(A[x]/(x^n))\to\mathrm{TC}(A)$. Then

$$\mathrm{TC}\left(A[x]/(x^n)\right)\sim\mathrm{TC}(A)\times\widetilde{\mathrm{TC}}\left(A[x]/(x^n)\right).$$

For any S^1-equivariant spectrum $T\in S^1\mathcal{SU}$ and any finite dimensional S^1-module W in the Universe we have the map

$$V_n\colon T(W)^{C_{p^r}}\to T(W)^{C_{np^r}}$$

constructed from equivariant transfers, cf. (2.5.8). If $n = p^{v_p(n)}n'$ with $(p, n') = 1$ we write

$$V_n^{(p)}: T(W)^{C_{p^r}} \to T(W)^{C_{p^{r+v_p(n)}}}$$

instead of $V_{p^{v_p(n)}}$. Then we have the following analogue of [HM], addendum 7.2:

Theorem 5.2.6. *The spectrum $\widetilde{\mathrm{TC}}(A[x]/(x^n))_p^\wedge$ is equivalent to the product of the p-adic completions of*

$$\Pi\left\{\Sigma \operatorname*{holim}_{\overleftarrow{R}} T(A)(V_{p^i l})^{C_{p^i}} \mid (l,p) = 1,\ n' \nmid l\right\}$$

and

$$\Pi\left\{\operatorname{cof}\left(\Sigma \operatorname*{holim}_{\overleftarrow{R}} T(A)(V_{p^i l})^{C_{p^{i-v_p(n)}}} \xrightarrow{V_n^{(p)}} \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(A)(V_{p^i l})^{C_{p^i}}\right) \,\Big|\, (l,p) = 1,\ n' \mid l\right\},$$

where in the second factor $T(A)(V_{p^i l})^{C_{p^{i-v_p(n)}}} = 0$ if $i < v_p(n)$.

Proof. We use the description

$$\widetilde{\mathrm{TC}}(-)_p^\wedge \longrightarrow \widetilde{\mathrm{TF}}(-)_p^\wedge \xrightarrow{R-1} \widetilde{\mathrm{TF}}(-),$$

so shall first determine $\widetilde{\mathrm{TF}}(A[x]/(x^n), p)$, the homotopy inverse limit of $\tilde{T}(A[x]/(x^n))^{C_{p^n}}$ under the inclusion of fixed sets. For fixed m,

$$\bigvee_{s=1}^{\infty} \left(T(A)(V_s) \wedge S^1_+/C_s\right)^{C_{p^m}} \to \prod_{s=1}^{\infty} \left(T(A)(V_s) \wedge S^1_+/C_s\right)^{C_{p^m}} \tag{1}$$

is an equivalence of spectra. Indeed, since we are only interested in p-completions

$$T(A)(V_s) \wedge S^1_+/C_s \sim T(A)(V_s) \wedge S^1_+/C_{p^{v_p(s)}}$$

and

$$\left(T(A)(V_s) \wedge S^1_+/C_{p^{v_p(s)}}\right)^{C_{p^m}} \sim \left(\rho^\#_{C_{p^r}} T(A)(V_s)^{C_{p^r}} \wedge S^1_+\right)^{C_{p^{m-r}}}$$

with $r = \min(v_p(s), m)$. The action of $C_{p^{m-r}}$ on S^1_+ is free, so can be divided out, and when we use the S^1-action on $\rho^\#_{C_{p^r}} T(A)(V_s)^{C_{p^r}}$ to untwist the action, we get

$$\left(T(A)(V_s) \wedge S^1_+/C_s\right)^{C_{p^m}} \sim T(A)(V_s)^{C_{p^r}} \wedge S^1_+/C_{p^{m-r}}, \tag{2}$$

with $r = m$ when $v_p(s) \geq m$. The cofibration sequence of proposition 4.1.8 takes the form

$$T(A)(V_s)_{hC_{p^r}} \longrightarrow T(A)(V_s)^{C_{p^r}} \xrightarrow{R} T(A)(V_{s/p})^{C_{p^{r-1}}}$$

and inductive use show that the connectivity of $T(A)(V_s)^{C_{p^r}}$ tends to infinity with s. This proves (1). Next for fixed s,

$$\underset{\overleftarrow{F}}{\mathrm{holim}}\, (T(A)(V_s) \wedge S^1_+/C_s)^{C_{p^m}} \sim \Sigma T(A)(V_s)^{C_{p^{v_p(s)}}} \tag{3}$$

after p-completion. This follows from (2) with $r = v_p(s) \leq m$, since the F-map corresponds to the transfers

$$\Sigma^\infty_+(S^1/C_{p^{m+1-v_p(s)}}) \to \Sigma^\infty_+(S^1/C_{p^{m-v_p(s)}})$$

which fit into a cofibration diagram

$$\begin{tikzcd}
\Sigma^\infty(S^1/C_{p^{m+1-r}}) \arrow[r] \arrow[d, "\sim"] & \Sigma^\infty_+(S^1/C_{p^{m+1-r}}) \arrow[r] \arrow[d] & \Sigma^\infty(S^0) \arrow[d, "p"] \\
\Sigma^\infty(S^1/C_{p^{m-r}}) \arrow[r] & \Sigma^\infty_+(S^1/C_{p^{m-r}}) \arrow[r] & \Sigma^\infty(S^0)
\end{tikzcd}$$

Here $\Sigma^\infty_+(X)$ is the suspension spectrum of X_+. Now smash with $T(A)(V_s)^{C_{p^r}}$ to obtain (3), cf. proof of lemma 4.4.9. The above together with theorem 5.2.4 yields

$$\begin{aligned}\widetilde{\mathrm{TF}}\,(A[x]/(x^n),p) \sim & \prod_{n'\nmid s} \Sigma T(A)(V_s)^{C_{p^{v_p(s)}}} \\ & \times \prod_{n'\mid s} \mathrm{cof}\left(\Sigma T(A)(V_s)^{C_{p^{v_p(s)-v_p(n)}}} \xrightarrow{V_n^{(p)}} \Sigma T(A)^{C_{p^{v_p(s)}}}\right)\end{aligned}$$

after p-completion. The homotopy fiber of $R - \mathrm{id}$ corresponds to taking homotopy inverse limit over R. □

So far we have not specified the ground ring A, but to obtain explicit calculation we now restrict A to be a perfect field k of positive characteristic p, where we have the following result from [HM], sect. 8.1.

Proposition 5.2.7. *Let $V \subset \mathcal{U}$ be a complex S^1-module. The non-zero homotopy groups of $T(k)(V)^{C_{p^m}}$ is concentrated in even degrees greater that or equal to $\dim V^{C_p}$. They are explicitly given as*

$$\pi_{2i}T(k)(V)^{C_{p^m}} = W_s(k) \quad \text{if} \quad \dim V^{C_{p^{m-s+1}}} \leq 2i < \dim V^{C_{p^{m-s}}}$$

for $s = 1, \dots m$, and

$$\pi_{2i} T(k)(V)^{C_{p^m}} = W_{m+1}(k) \quad \textit{if} \quad 2i > \dim V.$$

Proof. The argument is similar to the proof of proposition 4.2.7 and (4.3.4). One first treats the case $k = \mathbb{F}_p$.

We remember that $T(k)(V) \sim T(k) \wedge S^V$. It follows that the inclusion $V^{C_p} \subset V$ induces an equivalence

$$\hat{\mathbb{H}}\,(C_{p^m}, T(k)(V^{C_p})) \xrightarrow{\sim} \hat{\mathbb{H}}\,(C_{p^m}, T(k)(V)).$$

Indeed, the cofiber is $\hat{\mathbb{H}}(C_{p^m}, T(k) \wedge S^{V - V^{C_p}})$ and $S^{V - V^{C_p}}$ is a free C_{p^m}-space, build up from free C_{p^m}-cells, and the obvious induction over cells reduces us to show that

$$\hat{\mathbb{H}}\,(C_{p^m}, T(k) \wedge (C_{p^m +} \wedge S^i)) = 0.$$

This follows e.g. from the spectral sequence of sect. 4.1, since Tate cohomology groups vanish on free modules.

We next use the following analogue of proposition 4.1.8:

$$\begin{tikzcd}
T(k)(V)_{hC_{p^m}} \arrow[r, "N"] \arrow[d, "\mathrm{id}"] & T(k)(V)^{C_{p^m}} \arrow[r, "R"] \arrow[d, "\Gamma_{m,V}"] & T(k)(V^{C_p})^{C_{p^{m-1}}} \arrow[d, "\hat{\Gamma}_{m,V}"] \\
T(k)(V)_{hC_{p^m}} \arrow[r, "N^h"] & T(k)(V)^{hC_{p^m}} \arrow[r, "R^h"] & \hat{\mathbb{H}}\,(C_{p^m}, T(k)(V))
\end{tikzcd} \tag{1}$$

For $m = 1$,

$$\hat{\Gamma}_{1,V} \colon T(k)(V^{C_p}) \to \hat{\mathbb{H}}\,(C_p, T(k)(V^{C_p})) \sim \hat{\mathbb{H}}\,(C_p, T(k)(V))$$

induces an equivalence on π_i for $i \geq \dim V^{C_p}$. This is simply lemma 4.2.6 suspended $\dim V^{C_p}$ times. Theorem 4.1.15 implies that $\Gamma_{m,V}$ and $\hat{\Gamma}_{m,V}$ induce isomorphisms on homotopy groups in the same range. The spectral sequence

$$\hat{H}^*\,(C_{p^m}, \pi_* T(\mathbb{F}_p)(V)) \Rightarrow \pi_* \hat{\mathbb{H}}\,(C_{p^m}, T(\mathbb{F}_p)(V))$$

is isomorphic to the spectral sequence for $V = 0$ reindexed by shifting bidegrees up by $(0, \dim V)$. Hence the argument of proposition 4.2.7 gives

$$\pi_* \hat{\mathbb{H}}\,(C_{p^m}, T(\mathbb{F}_p)(V)) = S_{\mathbb{Z}/p^m}\{\sigma, \sigma^{-1}\}[\dim V] \tag{2}$$

and it follows from (1) that $\pi_i T(\mathbb{F}_p)(V)^{C_{p^m}} = \mathbb{Z}/p^{m+1}$ if i is even and $i \geq \dim V^{C_p}$. Moreover, the upper horizontal sequence in (1) yields that

$$R \colon T(\mathbb{F}_p)(V)^{C_{p^m}} \to T(\mathbb{F}_p)(V^{C_p})^{C_{p^{m-1}}}$$

is $(\dim V - 1)$-connected, so induction on m gives the claimed homotopy groups for $k = \mathbb{F}_p$.

Finally, the argument going from $\mathbb{F}_p$ to the perfect field k is similar to the one presented in sect. 4.3. □

Let $\mathbf{W}(k)$ denote the big Witt vectors of k, i.e. $\mathbf{W}(k) = (1 + Xk[[X]])^\times$, the multiplicative group of power series which begins with 1. Write $\mathbf{W}_m(k)$ for the truncated Witt-vectors

$$\mathbf{W}_m(k) = (1 + Xk[[X]])^\times/(1 + X^{m+1}k[[X]])^\times.$$

Let $V_n\colon \mathbf{W}(k) \to \mathbf{W}(k)$ be the Verschiebung: it sends a polynomium $p(X)$ to $p(X^n)$, and induces an injection

$$V_n\colon \mathbf{W}_{m-1}(k) \to \mathbf{W}_{nm-1}(k).$$

Theorem 5.2.8. ([HM2]). *For a perfect field k of characteristic $p > 0$,*

$$K_{2m-1}\left(k[x]/(x^n); \mathbb{Z}_p\right) = \mathbf{W}_{nm-1}(k)/V_n\mathbf{W}_{m-1}(k)$$

and $K_{2m}(k[x]/(x^n); \mathbb{Z}_p) = 0$ for $m > 0$.

Proof. We are in a situation where $K_*(-;\mathbb{Z}_p)$ and $\mathrm{TC}_*(-;\mathbb{Z}_p)$ agree, and shall calculate the latter. I shall only treat the case $(p,n) \neq 1$; the other case is less complicated.

Suppose first $n' \mid l$ and choose m in the range

$$\dim_{\mathbb{C}} V_{p^{r-1}l} < m \leq \dim_{\mathbb{C}} V_{p^r l} \tag{1}$$

with notation as in theorem 5.2.6. By definition,

$$\dim_{\mathbb{C}} V_{p^{r-1}l} = \begin{cases} \left[\frac{p^r l}{n}\right] & \text{if } r < v_p(n) \\ \left[\frac{p^r l}{n}\right] - 1 & \text{if } r \geq v_p(n) \end{cases}$$

so the above condition is equivalent to

$$\begin{aligned} &p^{r-1}l - n < mn \leq p^r l - n \quad \text{if } r > v_p(n) \\ &p^{r-1}l < mn \leq p^r l - n \quad \text{if } r = v_p(n) \\ &p^{r-1}l < mn \leq p^r l \quad \text{if } r < v_p(n) \end{aligned} \tag{2}$$

Now

$$\pi_{2m-1}\left(\Sigma \operatorname{holim}_{\overleftarrow{R}} T(k)(V_{p^i l})^{C_{p^i}}\right) \cong \pi_{2m-2} \operatorname{holim}_{\overleftarrow{R}} T(k)(V_{p^i l})^{C_{p^i}}$$
$$\cong \pi_{2m-2} T(k)(V_{p^{r-1}l})^{C_{p^{r-1}}}$$
$$\cong W_r(k)$$

by theorem 5.2.7, and similarly

$$\pi_{2m-1}\left(\Sigma \operatorname{holim}_{\overleftarrow{R}} T(k)(V_{p^i l})^{C_{p^{i-v_p(n)}}}\right) \cong W_{r-v_p(n)}(k).$$

Thus the second factor in theorem 5.2.6 (where $n' \mid l$) contributes

$$\bigoplus_{(l,p)=1,\ n'|l} \operatorname{cok}\left(W_{r(m,l)-v_p(n)}(k) \to W_{r(m,l)}(k)\right)$$

to $\mathrm{TC}_{2m-1}(k[x]/(x^n);\mathbb{Z}_p)$, when $r = r(m,l)$ denotes the unique number which satisfies (2). In other words, the contribution is

$$\begin{aligned}&\bigoplus\{W_{v_p(n)}(k) \mid (l,p)=1,\ n' \mid l,\ l < pmn'\}\oplus \\ &\bigoplus\{W_{r(m,l)}(k) \mid (l,p)=1,\ n' \mid l,\ l > pmn'\}.\end{aligned} \tag{3}$$

Similar considerations show that the first factor in theorem 4.2.6 contributes

$$\bigoplus\{W_{r(m,l)}(k) \mid (p,l)=1,\ n' \nmid l\}, \tag{4}$$

where this time $r = r(m,l)$ is the unique number with $p^{r-1}l < mn \le p^r l$. Finally, it is easy to see that the direct sum of (3) and (4) is isomorphic to $\mathbf{W}_{mn-1}(k)/V_n(\mathbf{W}_{m-1}(k))$. □

Remark 5.2.9. The above argument shows that

$$\pi_{2m-1}\left(\prod_{(l,p)=1} \operatorname{holim}_{\overleftarrow{R}} T(k)(V_{p^i l})^{C_{p^i}}\right) = \mathbf{W}_{mn-1}(k)$$

and more generally that

$$\pi_{2m-1}\left(\prod_{(l,p)=1} \operatorname{holim}_{\overleftarrow{R}} T(k)(V_{p^i l})^{C_{p^{i-\nu}}}\right) = p^{\nu}\mathbf{W}_{mn-1}(k).$$

The difference between the two cases $(p,n)=1$ and $(p,n)\neq 1$ in theorem 5.2.8 is just that V_n in the first case gives an isomorphism on the subproduct with $n \mid l$; in the second case there is a cokernel whose size depends on the p-adic valuation of n.

I should point out that the low dimensional groups $K_i(k[x]/(x^n);\mathbb{Z}_p)$, $i \leq 3$, were determined previously, and that Thomas Geisser asked us to use the present techniques to work out the groups for general i; he even conjectured the correct answer.

5.3 Nil calculations.

McCarthy's relative theorem makes it possible to calculate the socalled Nil-groups of rings A which contain a nilpotent ideal I for which A/I is a regular ring. In this situation, we have the cofibration sequence

$$\mathrm{NK}(A)^\wedge \to \mathrm{TC}\,(A[t] \to A/I[t])^\wedge \to \mathrm{TC}\,(A \to A/I)^\wedge \tag{5.3.1}$$

with $\Omega\mathrm{NK}(A)^\wedge \sim \mathrm{Nil}(A)^\wedge$. I illustrate the situation with an explicit calculation for the rings $A_n = k[x]/(x^n)$ of the previous section. Further details and examples are to appear in [HM2].

Lemma 5.2.2 and (5.2.1) shows that

$$T(A[t]) \sim_{C_\infty} T(A) \wedge N^{\mathrm{cy}}(\Pi_\infty) \sim_{C_\infty} T(A) \wedge \bigvee_{s=0}^{\infty} S^1_+/C_s$$

and we can apply theorem 5.2.6 for the ring

$$A_n[t] = k[t,x]/(x^n).$$

This expresses $\mathrm{TC}(A_n[t])$ in terms of $\Sigma\underleftarrow{\mathrm{holim}}\, T(k[t])(V_{p^il})^{C_{p^i}}$ with $(p,l) = 1$. Write $\tilde{T}(k[t]) = T\,(k[t] \to k)$, and let $\sim_p$ denote equivalence after p-adic completions. Then

$$\begin{aligned}
\tilde{T}(k[t])(V_{p^il})^{C_{p^i}} &\sim \left(T(k)(V_{p^il})^{C_{p^i}} \wedge \bigvee_{s=1}^{\infty} S^1_+/C_s\right)^{C_{p^i}} \\
&\sim_p \bigvee_{(\nu,p)=1}\bigvee_{j=0}^{\infty}\left(T(k)(V_{p^il})^{C_{p^i}} \wedge S^1_+/C_{p^j}\right),
\end{aligned}$$

since $S^1/C_{p^j\nu} \sim_p S^1/C_{p^j}$ when $(\nu,p)=1$, and one has as in sect. 5.2:

$$\begin{aligned}
\left(T(k)(V_{p^il}) \wedge S^1_+/C_{p^j}\right)^{C_{p^i}} &\sim T(k)(V_{p^il})^{C_{p^{\min(i,j)}}} \wedge S^1_+/C_{p^{\max(i,j)}} \\
&\sim \Sigma T(k)(V_{p^il})^{C_{p^{\min(i,j)}}} \vee T(k)(V_{p^il})^{C_{p^{\min(i,j)}}}.
\end{aligned}$$

For fixed k,

$$\pi_k \operatorname*{holim}_{\overleftarrow{R}} T(k[t])(V_{p^i l})^{C_{p^i}} = \pi_k \Sigma T(k[t])(V_{p^i l})^{C_{p^i}}$$

if i is sufficiently large; the precise value of i is given in the proof of theorem 5.2.8. It follows from remark 5.2.9 and the above that

$$\begin{aligned}\pi_{2m-1} \prod_{(p,l)=1} \Sigma \operatorname*{holim}_{\overleftarrow{R}} T(k[t])(V_{p^i l})^{C_{p^i}} &\cong \pi_{2m} \left(\prod_{(p,l)=1} \operatorname*{holim}_{\overleftarrow{R}} T(k[t])(V_{p^i l}) \right)^{C_{p^i}} \\ &\cong \bigoplus_{(\nu,p)=1} \left(\bigoplus_{j=1}^{r} p^j \mathbf{W}_{mn-1}(k) \oplus \bigoplus^{\infty} \mathbf{W}_{mn-1}(k) \right),\end{aligned}$$

where p^r is the exponent of $\mathbf{W}_{mn-1}(k)$. This can also be written as

$$\bigoplus_{(\nu,p)=1} \mathbf{W}_{mn-1}[y]/\mathbf{W}_{mn-1}[py].$$

We divide out the image of the Verschiebung $V_n\colon \mathbf{W}_{m-1}(k) \to \mathbf{W}_{mn-1}(k)$ to get

Theorem 5.3.1. *The groups* $\mathrm{NK}_{2m}(k[x]/(x^n))$ *and* $\mathrm{NK}_{2m-1}(k[x]/(x^n))$ *are isomorphic and are given as an infinite sum of* $\Lambda_n[y]/\Lambda_n[py]$ *with* $\Lambda_n = \mathbf{W}_{mn-1}(k)/V_n\mathbf{W}_{m-1}(k)$. □

There are more canonical ways to present the result, e.g. by using the deRham-Witt complex of Deligne and Illusie. I refer the reader to [HM2].

5.4 On the K-theory of local class fields.

It is natural to attempt to generalize the calculations of the previous sections to rings of integers in local class fields, $A = \mathrm{int}(E)$ with $E/\mathbb{Q}_p$ abelian (or even to local number fields). Such fields appear as centers in group rings $\mathbb{Q}_p[G]$, and their integers are centers in the corresponding maximal orders $\mathcal{M}_p(G)$,

$$\mathbb{Z}_pG \subset \mathcal{M}_p(G) \subset \mathbb{Q}_pG.$$

If $E/\mathbb{Q}_p$ is unramified, then $A \cong W(\mathbb{F}_{p^s})$ is a factor in $\mathbb{Z}_p[C_f]$, $(p,f) = 1$, and one can use lemma 4.4.7,

$$T(\mathbb{Z}_p[C_f])^\wedge_p \sim (T(\mathbb{Z}_p) \wedge \Lambda BC_{f+})^\wedge_p \sim (T(\mathbb{Z}_p) \wedge C_{f+})^\wedge_p$$

to get the cofibration sequence

$$\mathrm{TC}(W(\mathbb{F}_{p^s}))_p^\wedge \longrightarrow \mathrm{TF}(\mathbb{Z}_p)_p^\wedge \xrightarrow{R^s-1} \mathrm{TF}(\mathbb{Z}_p)_p^\wedge,$$

cf. [BM2]. Thus the unramified case is of the same complexity as $A = \mathbb{Z}_p$, where one has the calculational methods of sect. 4.1. In outline the calculation of $\mathrm{TC}(\mathbb{Z}_p)$ is similar to the calculation of $\mathrm{TC}(\mathbb{F}_p)$, but the details are of a different magnitude of difficulties.

The first problem is to verify Conjecture 4.1.16 for the rings in question, i.e. to show that

$$\hat{\Gamma}\colon \mathrm{TH}(A)_p^\wedge \to \hat{\mathbb{H}}(C_p, \mathrm{TH}(A)) \tag{5.4.1}$$

induces isomorphisms on homotopy groups in non-negative degrees. This was done in [BM1], sect. 5 for $A = \mathbb{Z}_p$, p odd, and in [R] for $p = 2$. I will go through the p odd case below; it was not so well presented in [BM1]. First recall from [B2]:

Theorem 5.4.2. *The* mod p *homotopy groups of* $\mathrm{TH}(\mathbb{Z}_p)$ *are*

$$\pi_*(\mathrm{TH}(\mathbb{Z}_p); \mathbb{F}_p) \cong E\{e_{2p-1}\} \otimes S\{f_{2p}\}$$

where the subscripts indicate degrees. Moreover, the Bockstein operator on f_{2p} *is* e_{2p-1}, *and the reduction map from* $\mathrm{TH}(\mathbb{Z}_p)$ *to* $\mathrm{TH}(\mathbb{F}_p)$ *maps* f_{2p} *non-trivially.* □

The reader with no access to [B2] may consult [HM] for an outline. Recall from lemma 4.4.4 the S^1-map

$$\iota\colon \Sigma^\infty_{S^1}(S^0)_p^\wedge \to \mathrm{TH}(\mathrm{Id}_p)$$

which induces equivalence on all C_{p^n} fixed sets. The inclusion

$$\Sigma^\infty_{S^1}(S^0)^{S^1} \to \Sigma^\infty(S^0)$$

is split, and the splitting induces a map

$$f\colon \Sigma^\infty(S^0) \to \Sigma^\infty_{S^1}(S^0)^{S^1} \to \Sigma^\infty_{S^1}(S^0)^{hS^1} \to \mathrm{TH}(\mathrm{Id}_p)^{hS^1}.$$

The homotopy ring $\pi_*(\Sigma^\infty(S^0); \mathbb{Z}_p)$ is of course unknown, but it contains the direct summand

$$E\{a_{2p-3}\} \otimes S\{b_{2p-2}\} \subset \pi_*(\Sigma^\infty(S^0); \mathbb{F}_p), \quad p \text{ odd}. \tag{5.4.3}$$

The first element outside the direct summand lies in degree $2p^2 - 2p - 2$. There is a similar statement for $p = 2$. We compose f with the map

$$g\colon \mathrm{TH}(\mathrm{Id}_p)^{hS^1} \xrightarrow{L} \mathrm{TH}(\mathbb{Z}_p)^{hS^1} \longrightarrow F(S^3_+, \mathrm{TH}(\mathbb{Z}_p))^{S^1}$$

where L comes from linearization $\mathrm{Id}_p \to \tilde{\mathbb{Z}}_p$, and the second map is restriction to the skeleton $S^3_+ \subset ES^1_+$. The cofibration sequence $S^1_+ \to S^3_+ \to S^3/S^1$ and the S^1-equivalence $S^3/S^1 \sim S^1_+ \wedge S^2$ yield the fibration sequence

$$\Omega^2 \mathrm{TH}(\mathbb{Z}_p) \xrightarrow{i} F(S^3_+, \mathrm{TH}(\mathbb{Z}_p))^{S^1} \longrightarrow \mathrm{TH}(\mathbb{Z}_p).$$

The composition $g \circ f$ maps the homotopy fiber of $\Sigma^\infty(S^0) \to H\mathbb{Z}_p$ into the fiber $\Omega^2\mathrm{TH}(\mathbb{Z}_p)$,

$$l\colon hF(\Sigma^\infty(S^0) \to H\mathbb{Z}_p) \to \Omega^2\mathrm{TH}(\mathbb{Z}_p), \quad i \circ l = g \circ f.$$

On homotopy groups one has

$$l_*(a_{2p-3}) = \Omega^2 e_{2p-1}, \qquad l_*(b_{2p-2}) = \Omega^2 f_{2p}. \tag{5.4.4}$$

This is a consequence of the statement that the composition

$$S^1_+ \wedge H\mathbb{Z}_p \xrightarrow{\iota} \mathrm{TH}(\mathbb{Z}_p) \xrightarrow{\mathrm{proj}} \Sigma^{2p-1} H\mathbb{Z}/p,$$

with ι the inclusion of the cyclic 0-skeleton, represents the suspension of the first Steenrod operation P^1, cf. [BM1], lemma 5.3 for details. We next consider the diagram of spectral sequences

$$\begin{array}{ccc} E^r_{-p,q}\left(\mathrm{TH}(\mathrm{Id}_p)^{hS^1};\mathbb{F}_p\right) & \longrightarrow & E^r_{-p,q}\left(\mathrm{TH}(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p\right) \\ \Big\downarrow & & \Big\downarrow \\ E^r_{-p,q}\left(\mathrm{TH}(\mathrm{Id}_p)^{hC_p};\mathbb{F}_p\right) & \longrightarrow & E^r_{-p,q}\left(\mathrm{TH}(\mathbb{Z}_p)^{hC_p};\mathbb{F}_p\right) \end{array} \tag{5.4.5}$$

In fiber degree $q \le 2p^2 - 2p - 2$, the E^2-terms are:

$$\begin{aligned} E^2\left(\mathrm{TH}(\mathrm{Id}_p)^{hC_p};\mathbb{F}_p\right) &= E\{u_1\} \otimes S\{t\} \otimes E\{a_{2p-3}\} \otimes S\{b_{2p-2}\} \\ E^2\left(\mathrm{TH}(\mathbb{Z}_p)^{hC_p};\mathbb{F}_p\right) &= E\{u_1\} \otimes S\{t\} \otimes E\{e_{2p-1}\} \otimes S\{f_{2p}\} \\ E^2\left(\mathrm{TH}(\mathrm{Id}_p)^{hS^1};\mathbb{F}_p\right) &= S\{t\} \otimes E\{a_{2p-3}\} \otimes S\{b_{2p-2}\} \\ E^2\left(\mathrm{TH}(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p\right) &= S\{t\} \otimes E\{e_{2p-1}\} \otimes S\{f_{2p}\}. \end{aligned}$$

The vertical maps in (5.4.5) are the inclusions. It is well-known to homotopy theorists (see e.g. [BM1], sect. 3) that

$$E^2(\mathrm{TH}(\mathrm{Id}_p);\mathbb{F}_p) \cong E^{2(p-2)}(\mathrm{TH}(\mathrm{Id}_p);\mathbb{F}_p)$$

and that

$$d^{2p-2}(t) = t^p a_{2p-3}, \quad d^{2p-2}(u_1) = 0, \quad d^{2p-1}(u_1) = t^p b_{2p-2}. \tag{5.4.6}$$

The horizontal maps in (5.4.5) are zero (at least in fiber degrees $\leq 2p^2-2p-2$) but this is due to the filtration shift indicated by (5.4.4).

Proposition 5.4.7. *Let p be an odd prime. In the spectral sequence $E^r(\mathrm{TH}(\mathbb{Z}_p)^{hC_p};\mathbb{F}_p)$, the elements te_{2p-1} and tf_{2p} are infinite cycles. Moreover $E^2 = E^{2p}$ and*

$$d^{2p}(t) = t^{p+1}e_{2p-1}, \quad d^{2p}(u_1) = 0, \quad \text{and } d^{2p+1}(u_1) = t^{p+1}f.$$

Proof. Let $T = \mathrm{TH}(\mathbb{Z}_p)$ or $T = \mathrm{TH}(\mathrm{Id}_p)$. Consider the Postnikov tower

$$T[0,0] \leftarrow T[0,1] \leftarrow \cdots \leftarrow T[0,q] \leftarrow \cdots$$

with inverse limit T. Here $T[0,q]$ has homotopy groups precisely in degree t for $0 \leq t \leq q$, and in this range they are equal to the homotopy groups of T. The Postnikov tower can be taken to be functorial (e.g. by using J. Moore's simplicial construction of it), so each term has an S^1-action.

The homotopy groups of the Postnikov tower defines an exact couple, which gives the spectral sequence we are looking at. It has

$$E^2_{-p,q} = \pi_{q-p}F(EC_{p+},T[q,q])^{C_p} \cong \pi_{q-p}F(BC_{p+},T[q,q]) \cong H^p(BC_p,\pi_q T)$$

and the differentials d^{r+1} are induced from the additive relations

$$\pi_{q-p}F(EC_{p+},T[q,q])^{C_p} \longleftarrow \pi_{q-p}F(EC_{p+},T[q,q+r-1])^{C_p} \xrightarrow{\partial_*} \pi_{q-p-1}F(EC_{p+},T[q+r,q+r])^{C_p}.$$

Here ∂_* is the connecting homomorphism in the homotopy exact sequence of the fibration

$$T[q,q+r-1] \leftarrow T[q,q+r] \leftarrow T[q+r,q+r].$$

We shall now compare the situation for $\mathrm{TH}(\mathrm{Id}_p)$ and $\mathrm{TH}(\mathbb{Z}_p)$. To shorten notation, write

$$F[s,t] = F(EC_{p+},\mathrm{TH}(\mathrm{Id}_p)[s,t])^{C_p}$$
$$F_{\mathbb{Z}}[s,t] = F(EC_{p+},\mathrm{TH}(\mathbb{Z}_p)[s,t])^{C_p}$$

and let $\pi_*(-) = \pi_*(-;\mathbb{Z}_p)$. Then (5.4.4) translates as follows: the additive relations

$$\pi_{2p-3}F[2p-3,2p-3] \twoheadleftarrow \pi_{2p-3}F[2p-3,2p-1] \xrightarrow{l_*} \pi_{2p-3}F_{\mathbb{Z}}[2p-3,2p-1] \xleftarrow{\cong} \pi_{2p-3}F_{\mathbb{Z}}[2p-1,2p-1]$$
$$\pi_{2p-2}F[2p-2,2p-2] \xleftarrow{\cong} \pi_{2p-2}F[2p-2,2p] \xrightarrow{l_*} \pi_{2p-2}F_{\mathbb{Z}}[2p-2,2p] \hookleftarrow \pi_{2p-2}F_{\mathbb{Z}}[2p,2p] \qquad (1)$$

give well-defined maps (from left to right) which take a_{2p-3} and b_{2p-2} into the generators te_{2p-1} and tf_{2p} of

$$\pi_{2p-3}F_{\mathbb{Z}}[2p-1,2p-1] = H^2(BC_p;\pi_{2p-1}\mathrm{TH}(\mathbb{Z}_p))$$
$$\pi_{2p-2}F_{\mathbb{Z}}[2p,2p] = H^2(BC_p;\pi_{2p}\mathrm{TH}(\mathbb{Z}_p))$$

For example, the first additive relation is well-defined because l_* annihilates the generator $u_1b_{2p-2} \in H^1(BC_p;\pi_{2p-2}\mathrm{TH}(\mathrm{Id}_p))$: it maps to an element of filtration degree 3, according to (5.4.4).

The elements a_{2p-3} and b_{2p-2} are infinite cycles in the spectral sequence for $\mathrm{TH}(\mathrm{Id}_p)^{hC_p}$, being in the image of f_*. This means that they lift to elements of $\pi_{2p-3}F[2p-3,\infty]$ and $\pi_{2p-2}F[2p-2,\infty]$. It follows that te_{2p-1} and tf_{2p} lift to $\pi_{2p-3}F_{\mathbb{Z}}[2p-1,\infty]$ and $\pi_{2p-2}F_{\mathbb{Z}}[2p,\infty]$, so are infinite cycles.

Let us prove that $d^{2p}(u_1) = 0$ and $d^{2p+1}(u_1) = t^{p+1}f_{2p}$ and leave the easier differential $d^{2p}(t) = t^{p+1}e_{2p-1}$ to the reader. The additive relation defining $d^{2p}(u_1)$ is

$$\pi_{-1}F[0,0] \xleftarrow{\cong} \pi_{-1}F[0,2p-4] \leftarrowtail \pi_{-1}F[0,2p-3] \xrightarrow{\partial_*} \pi_{-2}F[2p-2,2p-2]. \quad (2)$$

Indeed u_1 lies in the subgroup $\pi_{-1}F[0,2p-3]$ because $d^{2p-2}(u_1) = 0$ (and not equal to $t^{p-1}u_1a_{2p-3}$). Because of the filtration shift represented by (5.4.4), it is better to consider the additive relations

$$\begin{array}{ccccc} A\colon \pi_{-1}F[0,0] & \longleftarrow & \pi_{-1}F[0,2p-4] & \xrightarrow{\partial'_*} & \pi_{-2}F[2p-3,2p] \\ \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle h_*} \\ A\mathbb{Z}\colon \pi_{-1}F_{\mathbb{Z}}[0,0] & \longleftarrow & \pi_{-1}F_{\mathbb{Z}}[0,2p-4] & \xrightarrow{\partial'_*} & \pi_{-2}F_{\mathbb{Z}}[2p-3,2p] \end{array} \quad (3)$$

where ∂'_* is the connecting homomorphism in the homotopy exact sequence of

$$F[0,2p-4] \leftarrow F[0,2p] \leftarrow F[2p-3,2p].$$

Theorem 5.4.2 and (5.4.3) gives

$$\pi_{-2}F[2p-3,2p] \cong \pi_{-2}F[2p-3,2p-2]$$
$$\pi_{-2}F[2p-3,2p] \cong \pi_{-2}F[2p-1,2p]$$

and hence exact sequences

$$\begin{array}{ccccccc} 0 \to \pi_{-2}F[2p-2,2p] & \xrightarrow{i} & \pi_{-2}F[2p-3,2p] & \xrightarrow{j} & \pi_{-2}F[2p-3,2p-3] \to 0 \\ \downarrow & & \downarrow{\scriptstyle l_*} & & \\ 0 \to \pi_{-2}F[2p,2p] & \xrightarrow{i_{\mathbb{Z}}} & \pi_{-2}F_{\mathbb{Z}}[2p-3,2p] & \xrightarrow{j_{\mathbb{Z}}} & \pi_{-2}F[2p-3,2p-1] \to 0 \end{array} \quad (4)$$

One has the following values of the groups involved:

$$\begin{aligned}
\pi_{-2}F[2p-2,2p] &\cong H^{2p}(BC_p;\pi_{2p-2}\mathrm{TH}(\mathrm{Id}_p)) &&= \mathbb{F}_p\langle t^p b_{2p-2}\rangle\\
\pi_{-2}F[2p-3,2p-3] &\cong H^{2p-1}(BC_p;\pi_{2p-3}\mathrm{TH}(\mathrm{Id}_p)) &&= \mathbb{F}_p\langle t^{p-1}u_1 a_{2p-3}\rangle\\
\pi_{-2}F[2p,2p] &\cong H^{2p+2}(BC_p;\pi_{2p}\mathrm{TH}(\mathbb{Z}_p)) &&= \mathbb{F}_p\langle t^{p+1} f_{2p}\rangle\\
\pi_{-2}F[2p-3,2p-1] &\cong H^{2p+1}(BC_p;\pi_{2p-1}\mathrm{TH}(\mathbb{Z}_p)) &&= \mathbb{F}_p\langle t^p u_1 e_{2p-1}\rangle
\end{aligned}$$

I claim that $j_{\mathbf{Z}}\circ l_*\circ i=0$, giving the left hand vertical arrow in (4). Indeed the generator of $\pi_{-2}F[2p-2,2p]$ is $t^p b_{2p-2}$, hence in the image of the product map

$$\pi_{-2p}F[0,0]\otimes\pi_{2p-2}F[2p-2,2p-1]\to\pi_{-2}F[2p-2,2p-1].$$

The homomorphism

$$\pi_{2p-2}F[2p-2,2p-1]\to\pi_{2p-2}F_{\mathbf{Z}}[2p-2,2p-1]$$

is zero by (5.4.4), and the claim follows by using the product

$$\pi_{-2p}F_{\mathbf{Z}}[0,0]\otimes\pi_{2p-2}F_{\mathbf{Z}}[2p-3,2p-1]\to\pi_{-2}F_{\mathbf{Z}}[2p-3,2p-1].$$

On the other hand

$$d^{2p-2}(u_1)=j\circ A(u_1),\qquad d^{2p}_{\mathbf{Z}}(u_1)=j_{\mathbf{Z}}\circ A_{\mathbf{Z}}(u_1)\tag{5}$$

and by (3), $j_{\mathbf{Z}}A_{\mathbf{Z}}(u_1)=j_{\mathbf{Z}}l_*A(u_1)$. Since $d^{2p-2}(u_1)=0$, one concludes that $d^{2p}_{\mathbf{Z}}(u_1)=0$. Finally the differential $d^{2p-1}(u_1)=t^p b$ shows that $l_*A(u_1)\neq 0$ and that it belongs to the image of i in diagram (4). But the left hand vertical arrow in (4) is an isomorphism; use of products as above and (5.4.4) completes the proof. □

Corollary 5.4.8. *For p odd,*

$$\pi_*\left(\hat{\mathbb{H}}(C_p,\mathrm{TH}(\mathbb{Z}_p));\mathbb{F}_p\right)=E\{e_{2p-1}\}\otimes S\{t^p,t^{-p}\}.$$

Moreover,

$$\hat{\Gamma}_*\colon\pi_*(\mathrm{TH}(\mathbb{Z}_p);\mathbb{F}_p)\to\pi_*\left(\hat{\mathbb{H}}(C_p,T(\mathbb{Z}_p));\mathbb{F}_p\right)$$

is an isomorphism in non-negative degrees.

Proof. The non-zero differentials in $E^r\left(\hat{\mathbb{H}}(C_p,\mathrm{TH}(\mathbb{Z}_p));\mathbb{F}_p\right)$ are by (5.4.7):

$$d^{2p}(t^i)=it^{i+p}e_{2p-1},\quad d^{2p+1}(u_1)=t^{p+1}f,\quad (i\in\mathbb{Z})$$

and a routine calculation gives $E^{2p-2}_{*,*} \cong E\{u_1\} \otimes S\{t^p, t^{-p}\}$. For degree reasons $E^{2p+2}_{*,*} = E^{\infty}_{*,*}$. This proves the first statement. The commutative diagram

$$\begin{tikzcd} T(\mathbb{Z}_p) \arrow[r] \arrow[d, "\hat{\Gamma}"] & T(\mathbb{F}_p) \arrow[d, "\hat{\Gamma}"] \\ \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p)) \arrow[r] & \hat{\mathbb{H}}(C_p, T(\mathbb{F}_p)) \end{tikzcd},$$

together with lemma 4.2.4 and theorem 5.4.2 tells us that $\hat{\Gamma}_*(f_{2p}) = t^{-2p}$ and $\hat{\Gamma}_*(e_{2p-1}) = e_{2p-1}$. $\square$

The corollary implies that

$$\mathrm{TH}(\mathbb{Z}_p) \xrightarrow{\hat{\Gamma}} \hat{\mathbb{H}}(C_p, \mathrm{TH}(\mathbb{Z}_p))[0,\infty)$$

is a p-adic equivalence, and theorem 4.1.15 then gives

$$\mathrm{TF}(\mathbb{Z}_p, p)^{\wedge}_p \sim \left(\operatorname*{holim}_{\overleftarrow{F}} T(\mathbb{Z}_p)^{hC_{p^n}}\right)^{\wedge}_p \sim (T(\mathbb{Z}_p)^{hS^1})^{\wedge}_p.$$

The homotopy groups $\pi_*(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p)$ and $\pi_*\left(\hat{\mathbb{H}}(S^1; T(\mathbb{Z}_p)); \mathbb{F}_p\right)$ were calculated in [BM2] by solving the involved spectral sequences, and

$$R^s_*\colon \pi_*\,(\mathrm{TF}(\mathbb{Z}_p, p); \mathbb{F}_p) \to \pi_*(\mathrm{TF}(\mathbb{Z}_p, p); \mathbb{F}_p)$$

was determined. This was enough to give the values of $\mathrm{TC}_*(W(\mathbb{F}_{p^s}); \mathbb{F}_p)$. The groups turn out to be v_1-periodic, i.e.

$$v_1\colon \mathrm{TC}_*\,(W(\mathbb{F}_{p^s}); \mathbb{F}_p) \xrightarrow{\cong} \mathrm{TC}_{*+2p-2}\,(W(\mathbb{F}_{p^s}); \mathbb{F}_p)$$

($v_1 = b_{2p-2}$), and this together with other tricks leads to the proof of theorem 1.5 of the introduction. I refer to [BM2] for the details.

For odd primes p, theorem 1.5 states that

$$\mathrm{TC}(\mathbb{Z}_p)^{\wedge}_p[0,\infty) \sim \operatorname{im} J_p \times B \operatorname{im} J_p \times SU^{\wedge}_p. \tag{5.4.9}$$

This is true as (-1)-connected spectra when one use the deloopings arising from Bott periodicity on the right hand side. For $p = 2$ there are added complications. For example, mod 2 homotopy groups of a ring spectrum does not in general form a ring. At the time of writing $\mathrm{TC}_*(\mathbb{Z}_2)$ has not been completely determined, but preliminary calculations of J. Rognes suggests that

$$\pi_*\,(\mathrm{TC}(\mathbb{Z}_2), \mathbb{F}_2) \cong \pi_*(\operatorname{im} J_2 \times B\mathrm{im}J_2 \times SU^{\wedge}_2; \mathbb{F}_2).$$

One expects that a twisted version of (5.4.9) is true for $p = 2$, cf. [BM2], sect. 6. I stress that $\operatorname{im} J_2$ is the *complex* J space at 2, i.e. the homotopy fiber of $\psi^k - 1\colon (BU \times \mathbb{Z})_2^\wedge \to BU_2^\wedge$.

For geometric reasons it is important to study the relative K-theory $K(\mathrm{Id} \to \mathbb{Z})$, by theorem 3.5.1 equal to $\mathrm{TC}(\mathrm{Id} \to \mathbb{Z})$. Indeed, a celebrated theorem due to F. Waldhausen states that

$$K(\mathrm{Id}) \sim \Omega^\infty S^\infty \times \mathrm{Wh}^{\mathrm{Diff}}(*)$$

where $\Omega^2 \mathrm{Wh}^{\mathrm{Diff}}(*) \sim \underrightarrow{\mathrm{holim}}\,\mathrm{Diff}(D^n, D^{n-1}_-)$ with $D^{n-1}_- \subset \partial D^n$ the lower hemisphere, and where $\Omega^\infty S^\infty$ is the zero'th term in the sphere spectrum.

Conjecture 5.4.10. For each odd p we have to split fibration

$$\mathrm{cok} J_p \to (\Omega^\infty S^\infty)_p^\wedge \underset{\longleftarrow}{\xrightarrow{e}} \operatorname{im} J_p. \tag{1}$$

There is a similar split fibration

$$X_p \to \Omega^\infty S^\infty (S^1 \wedge \mathbb{C}P^\infty)_p^\wedge \underset{\longleftarrow}{\xrightarrow{e'}} SU_p^\wedge. \tag{2}$$

Here the map from $S^1 \wedge \mathbb{C}P^\infty \to SU$ is adjoint to the map which classifies the reduced canonical line bundle, and e' is its 'universal' extension. The S^1-transfer

$$\tau\colon \Omega^\infty S^\infty(S^1 \wedge \mathbb{C}P^\infty) \to \Omega^\infty S^\infty$$

induces a map $\tau_p'\colon X_p \to \mathrm{cok} J_p$ and a map from $SU_p^\wedge$ to $\operatorname{im} J_p$ with fiber $SU_p^\wedge$. Let $\operatorname{im} \tilde{J}_p$ be the 0-connected cover of $\operatorname{im} J_p$. I conjecture that

$$\mathrm{TC}(\mathrm{Id} \to \mathbb{Z})_p^\wedge \sim \mathrm{cok} J_p \times B\mathrm{cok} J_p \times B \operatorname{im} \tilde{J}_p \times hF(\tau_p'). \tag{3}$$

The difficult part is to prove that the restriction of $\mathrm{TC}(\mathrm{Id}_p) \to \mathrm{TC}(\mathbb{Z}_p)$ to the $SU_p^\wedge$ factor of (2) is the deloop of $\psi^k - 1\colon BU_p^\wedge \to BU_p^\wedge$; this gives the factor $B \operatorname{im} \tilde{J}_p$ in (3).

The outstanding problem which remains is to determine $\mathrm{TC}(A)_p^\wedge$ in ramified situations. There are at least two approaches. One can attempt to use that A appears as the center in a maximal order $\mathcal{M}_p(G) \subset \mathbb{Q}_p[G]$, and use the ideas of sect. 5.1 to calculate $\mathrm{TC}(\mathbb{Z}_p[G])_p^\wedge$. But this leaves one with following problem, interesting in its own right:

Problem 5.4.11. Give a calculable trace description of $K(\mathbb{Z}_p[G] \to \mathcal{M}_p(G))$.

One knows by the localization theorem in K-theory a categorical description of $K(\mathbb{Z}_p[G] \to \mathbb{Q}_p[G])$, and hence of $K(\mathbb{Z}_p[G] \to \mathcal{M}_p(G))$, namely as the K-theory of cohomological trivial modules. But despite a lot of efforts by Bökstedt and the author, (5.4.11) remains unsolved (even for $G = C_p$).

A second approach is to follow sect. 4.1, starting with a calculation of $\mathrm{TH}(A)$. Recently, A. Lindenstrass has determined $\mathrm{TH}(A)$ for quadratic ramified extensions of $\mathbb{Z}_2$. In general one should have

Conjecture 5.4.12. *Let A be totally ramified and let $\pi \in A$ be the prime element ($A/\pi A = \mathbb{F}_p$). Then*

$$\pi_* \mathrm{TH}(A, A/\pi) \cong E_{\mathbb{F}_p}\{a_1\} \otimes S_{\mathbb{F}_p}\{a_2\}$$

with $\deg a_i = i$. □

Conjecture 5.1.12 yields $\pi_*(\mathrm{TH}(A); \mathbb{F}_p)$ as well, but I do not know if (5.4.1) is an equivalence in this case.

Finally of course there is the deep problem of determining the relative K-theory $K\left(\mathbb{Z}_{(p)} \to \mathbb{Z}_p\right)$ but this is a different story altogether.

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