THE METASTABLE HOMOTOPY OF S^n

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CHAPTER I

INTRODUCTION

1. The Adams spectral sequence [1] (see Chapter 2, section 1 for a summary) is the most powerful tool presently available for studying the stable homotopy of spheres. The Adams theory is essentially a stable one, in its present form, and so gives information about $\pi_j(S^n)$ only for $j < 2n-1$.

The next block of $n-1$ groups, i.e., for $2n-1 \leq j < 3n-2$, is called the metastable range and it too has many regular properties. But stable arguments do not in general apply. The main result now available for this range of groups is the following theorem of Toda.

THEOREM I [25;11.7]. The following sequence is exact for $j < 2n-2$ and is exact on the two component for $j < 3n-3$:

1.1

$$\rightarrow \pi_{j+n}(S^n) \xrightarrow{\Sigma^{k,n}} \pi_{j+k+n}(S^{n+k}) \xrightarrow{I_{k,n}} \pi_{j-1+n}(\Sigma^{n-1}P_n^{n+k-1}) \xrightarrow{P_{k,n}} \pi_{j-1}(S^n) \rightarrow$$

where $P_n^{n+k-1} = P^{n+k-1}/P^{n-1}$ and P^n is the real n-dimensional projective space.

Note that if $k > n+1$ and $j < n-2$ then $\pi_{j+k}(S^{n+k})$ and $\pi_{j-1}(\Sigma^{n-1}P_n^{n+k-1})$ are stable groups.

Our object is to bring to Toda's theorem the power of stable methods developed by Adams. One main result is

THEOREM A. Assume $k > n+1$. There is a map between Adams spectral sequences which on the E_2 level gives

$$\mathrm{Ext}_A^{s,t}(Z_2,Z_2) \xrightarrow{I^2_{k,n}} \mathrm{Ext}_A^{s-1,t}(\tilde{H}^*(P_n^{n+k-1}),Z_2)$$

for $t-s < 2n-2$ and projects in E_∞ to the same map to which $I_{k,n}$ of Toda's

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theorem projects for the same range. In addition if we restrict $t-s$ to be greater than $n-1$, $I^2_{k,n}$ is a mapping of $H^*(A)$ modules. $(H^*(A) = \mathrm{Ext}_A^{*,*}(Z_2,Z_2).)$

(Note that $\sum_{t-s > n-1} \mathrm{Ext}_A^{s,t}(Z_2,Z_2)$ is an $H^*(A)$ module.)

One can think of $I_{k,n}$ in theorem I as a generalized Hopf homomorphism and our primary interest will always center on the case where $k > n+1$, i.e., where we map the cokernel of the suspension from an unstable group to a stable group. The map $P_{k,n}$ in theorem I is a generalized Whitehead product and there have been several efforts to get general results about it [10], [11], and [19]. Theorem A gives a quick proof of all the results of [19] and substantial generalizations.

2. It is quite clear from theorem I that a detailed study of the homotopy of stunted projective spaces is central in the metastable homotopy of S^n. A second major object of this paper is to develop a technique which renders this a comparatively easy job if one knows Ext for a sphere. The details of the computation of $\pi_{k+p}(P_k)$ for $p \leq 29$ are given in Chapter III. The use of a large computer was important in this work; compare III section 8.* Table 4.1 tabulates these results. Detailed tables are given in Chapter III section 8.

Together with a proof of theorem A, Chapter II introduces a map between stable objects, $\lambda: P_1 \to S^0$. ($P_n^k = RP^k/RP^{n-1}$ where RP^k is a real k-dimensional projective space.) It is conjectured that this map is onto in homotopy (II.4.2) and this conjecture is verified as far as we have gone (Chapter IV).

In [4], Adams defines a collection of direct summands in certain stable stems. Table 1 gives a listing of them with names for the generators.

$i = 8j +$	-1	0	1	2	3
A summand of π_i^S	$Z_{\lambda(j)}$	Z_2	$Z_2 + Z_2$	Z_2	Z_8
Name of generator	ρ_j	$\eta\rho_j$**	$\eta^2\rho_j, \mu_j$	$\eta\mu_j$	ξ_j

Table 1.

*Dr. D. MacLaren did the programming using Cogent, a programming language developed by John Reynolds. Argonne National Laboratories supplied the machine time; compare [18].

**We actually will work with an element which is $\eta\rho_j$ modulo $2\pi_{8j}^S$.

Let $p(j)$ be defined by $8j \equiv 2^{p(j)-1}(2^{p(j)})$. In table 1, $\lambda(j) \equiv 2^{p(j)}$.

We will give particular representations of these elements in Chapter IV. Our representations are defined in such a way that $\eta^2\rho_j \in \operatorname{im} J$ for all $j > 1$ and some multiple of ρ_j is in $\operatorname{im} J$ if $j = 2^p$ for each p. The second statement is not proved here but will be discussed in another place and is not used here. It is believed that $\rho_j, \eta\rho_j, \eta^2\rho_j$ and ξ_j generate the real image of J. In particular we will prove

THEOREM B. It is possible to choose generators ρ_j $(j > 0)$, μ_j and ξ_j (for $j \geq 0$) in stems given in table 1 so that they have the following properties:

i) ρ_j has filtration $\geq 4j - p(j)$.

ii) $\eta^2\rho_j$ has filtration $\geq 4j$ for $j > 1$.

iii) μ_j has filtration $\geq 4j+1$.

iv) ξ_j has filtration $\geq 4j+1$.

v) $e(\rho_j) = 2^{-p(j)-1} \pmod{2^{-p(j)}}$.

vi) $d_R(\mu_j) \neq 0$.

vii) $e_R'(\xi_j) = \frac{1}{8} \pmod{\frac{1}{4}}$.

We will also investigate the Whitehead product structure for all these.

THEOREM C. Let α be an element in table 1. Suppose $[\iota_n,\alpha_n]$ is in $\pi_k(S^n)$ and $k < 4n-3$. Then the order of $[\iota_n,\alpha_n]$ is given by table 2 except if $i = 8p$, $\alpha = \rho_j$; $i = 8p-2$, $\alpha = \mu_j$; $i = 8p-3$, $\alpha = \eta\mu_j$ and $i = 8p-4$, $\alpha = \xi_j$. For these cases we require

i =	8p	8p − 2	8p − 3	8p − 4
8j <	8p − 6v + 2	8p − 6v − 2	8p − 6v − 5	8p − 6v − 7

where v is defined by $8(p+j) \equiv 2^v(2^{v+1})$.

Before we state theorem D we need some notation. Let n be an integer and let a and b be defined by $n = 4a+b$, $0 \leq b \leq 3$. Let $\varphi(n) = 8a + 2^b$. Let

$$\beta_n = \rho_{a+1} \qquad b = 3$$
$$= \xi_a \qquad b = 2$$
$$= \eta^2\rho_a \qquad b = 1$$
$$= \eta\rho_a \qquad b = 0.$$

$\alpha =$	ρ_j	$\eta\rho_j$	$\eta^2\rho_j$	μ_j	$\eta\mu_j$	ξ_j
$i = 0$	$\lambda_{(j)}$	2	2	2	2	8
1	2	2	2	2	2	0
2	$\lambda_{(j)}$	2	0	2	0	4
3	2	0	0	0	0	x
4	$\lambda_{(j)}$	2	2	2	2	8
5	2	2	x	2	2	0
6	$\lambda_{(j)}$	x	0	2	0	4
7	x	0	0	0	0	0

Table 2.

Notice that $\beta_n \varepsilon \pi^S_{\varphi(n)-1}$.

THEOREM D. If $n+\varphi(m)+1 \equiv 2^{m'}(2^{m+1})$, where $3 \leq m' \leq m$, then $[\iota_n,\beta_m] = 0$. If $n+\varphi(m)+1 \equiv 0 \pmod{2^{m+1}}$, then $[\iota_n,\beta_m]$ is either zero or of order 2.

<u>Conjecture</u>. $[\iota_n,\beta_m] \neq 0$ if $n+\varphi(m)+1 \equiv 0 \pmod{2^{m+1}}$ but $n+\varphi(m)+1 \neq 2^{m+1}$, and $[\iota_n,\beta_m] = 0$, $n+\varphi(m)+1 = 2^{m+1}$, iff $\{h_m^2\}$ is a permanent cycle in the Adams spectral sequence. In particular, we conjecture that if h_m^2 projects to a non-zero homotopy class a_m in the Adams spectral sequence then in the diagram

$$\begin{array}{ccc} \pi^S_{2^{m+1}-2} & \xrightarrow{I_n} & \pi_{2^{m+1}-3+n}(\Sigma^{n-1}P_n) \\ & & \uparrow i_* \\ & & \pi_{2^{m+1}-3+n}(S^{2n-1}) \end{array}$$

where I_n is as in theorem I, i is a generator and $n = 2^{m+1} - \varphi(m) - 1$, $I_n(a_m) = i_*\beta_m$.

Partial results supporting this conjecture are known but they will not be discussed here. In particular the conjecture is true for $m \leq 4$.

3. In addition to the above information we get detailed results on the first twenty or so unstable stems. In particular we give a table 4.2 which gives $\pi_j(S^n)$ for $23 \leq j \leq 40$ if $n > (j+3)/3$. These results follow easily from the collected calculations, and no detailed proof is given. We also can get rather strong statements about what the homomorphisms look like if $j > 40$, $28 \geq j-n \geq -1$. These are collected in tables 4.3 and 4.4. Propositions which make this explicit are given in Chapter V. The results there are sufficient to compute $[\iota_n, \alpha]$ for most $\alpha \in \pi_j(S^o)$, $j < 21$. The results would really be quite satisfying if a specific conjecture about $\mathrm{Ext}_A^{s,t}(Z_2,Z_2)$ could be verified, V.2.4. This conjecture is almost certainly true and it seems within range of present techniques. When verified the Whitehead product question for any element in $\pi_j(S^o) \leq 29$ with the exception of $\{c_1\}$ would be settled in the sense that $\mathrm{im}(I_{k,n})$ could be given.

4. This section contains the tables which collect the calculations made in the paper. The first table gives $\pi_{k+n}(P_n) \simeq \pi_{k+n}(V_{n+m,m})$ for $m > k+1$. By [8] we see that $\pi_{k+n}(BOS(n)) \simeq \pi_{k+n}(BSO)(n+m) \oplus \pi_{k+n}(V_{n+m,m})$ for $m > k+1$, $n > 13$, $k < n-1$. Thus table 1 also gives a table of the unstable homotopy groups of $BOS(n)$.

An element in table 1 consists of some powers of some integers. For example, for $n = 1$, $k = 19$ we have 8,2 as the entry. This means that $\pi_{n+19}(P_n) = Z_8 \oplus Z_2$ if $n \equiv 1 \pmod{16}$. In addition some entries contain the symbol A or B or C. If for a given k and n value the table lists $C,2^2$ this means that the group is $C(k,n) \oplus Z_2 \oplus Z_2$ where $C(k,n)$ (and A and B) are given by the following result.

PROPOSITION 4.1. a) Let $m(n,k)$ be defined by $n+k+1 \equiv 2^m \pmod{2^{m+1}}$. Let q be defined by $\varphi(q) \leq k < \varphi(q+1)$. Let $i(n,k) = \max(q-m(n,k),0)$. Then $A(k,n)$ is a cyclic group of order $2^{i(n,k)}$.

b) $B(k,n) = \overline{B}(k,n) \oplus Z_2$ if $m(n,k) = 4$ and $B(k,n) = \overline{B}(k,n)$ if $m = 4$. $\overline{B}(k,n)$ is a cyclic group of order 2^{m+1} if $q - m(n,k) \geq 0$ and the order of 1_k in tables III.8.i, $i = 2,\ldots,16$.

c) $C(k,n) = Z_2$ if $m(n,k) > 4$ and $C = 0$ if $m(n,k) = 4$.

Tables 2, 3 and 4 are quite clear. The kernel of the unstable J-homo-

n(8) =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k=0	2	∞	2	∞	2	∞	2	∞	2	∞	2	∞	2	∞	2	∞
1	2	4	0	2^2	2	4	0	2^2	2	4	0	2^2	2	4	0	2^2
2	8	0	2	2^2	8	0	2	2^2	8	0	2	2^2	8	0	2	2^2
3	2	4	2^2	4,16	2	4	2	8^2	2	4	2^2	16,4	2	4	2	8^2
4	0	2	16	2	0	0	8	2	0	2	16	2	0	0	8	2
5	2	16	2	0	0	8	2	0	2	16	2	0	0	8	2	0
6	16,2	2^2	2	0	8	4	0	2^2	16,2	2^2	2	0	8	4	0	2^2
7	2^3	2,16	2	16,4	2^2	16	2^2	32,8,2	2^3	2,16	2	16,4	2^2	16	2	16^2,2
8	2^4	4	8	2^4	2^3	4,A,2	2,32	5^2	2^4	4	8	2^4	2^3	4	16,2	2^5
9	8,2	8	2^2	2^5	4,2,A	$\bar{B}$,2	2^3	2^7	2,8	8	2^2	2^5	4,2	16,2	2^3	2^7
10	8	0	2^2	A,2^4	$\bar{B}$	2	2^4	8,2^2	8	0	2^2	2^3	16	2	2^4	8,2^2
11	0	8	A,2^2	$\bar{B}$,8	2	8,2^2	2,8	8^2	0	8	2^2	16,8	2	8,2^2	2,8	8^2
12	0	2^2,A	$\bar{B}$	2	2^2	8	8	0	0	2	16	2	2^2	8	8	0
13	2^2,A	$\bar{B}$	2	2^2	8	8	0	0	2	16	2	2^2	8	8	0	0
14	$\bar{B}$,2^2	2^2	4,2^2	8,2^2	8,2^2	2^2	2	2^3	16,2	2^3	4,2	2^2,8	8,2^2	2^2	0	2^3,A

Table 4.1. $\pi_{k+n}(P_n)$

n(8) =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k = 15	2^4	$4,2^2$ 32	$8,2^2$	$2^3,4$ 64	2^4	$32,2$	2^2	$16,2^2$ 32	2^3	$2,4$ 32	$2^2,4$	$2^3,4$ 64	2^4	32	$2^2,A$	$\bar{B},2^3$ 32
16	2^6	$4,8$	$8,2^2$	$2^4,4$	2^4	4	$16,2$	2^5	2^5	4^2	$2^2,8$	$2^4,4$	2^3	$A,4$	$\bar{B},2^2$	2^6
17	$8^2,2$	$4,8$ 2	$2^2,8$	2^8	$8,2$	$16,2$	2^3	2^9	$2,4$ 8	$8,4$ 2	$2^2,8$	2^7	$A,8$ 2	$\bar{B},2^2$	2^4	2^{10}
18	$2^3,8$	$4,8$	2^6	$4,2^2$ 16	$16,2$	4	2^5	$8^2,2^2$ 4	$2^3,8$	$8,4$	2^5	$A,2^2$ $16,4$	$B,2$	$2,4$	2^6	$8^3,2^2$
19	$8,2$	$2^4,8$	$16,2^2$	$8,2^2$ 16	0	$2^2,8$	$4,8$ 2	$8^2,2^4$	$8,2$	$8,2^3$	$A,2^2$ 16	$B,2^2$ 8	$C,2$	$2^2,4$ 8	$2,8^2$	$2^4,8^2$
20	2^5	$16,8$ 2	$16,2^3$	4	2^2	$8,4$ 2	$8,2^3$	$2,8^2$	2^4	$A,8$ $2,16$	$B,2^3$	$C,2$ 4	$4,2^3$	$8^2,4$	$8,2^3$	$8^2,2$
21	$16,2^4$	$2^3,4$ 16	2	2^2	$8,2^2$	$2^3,8$	$2,8$	2^5	$A,2^3$ 16	$B,2^3$ 4	$C,2$	$2^3,4$	$2,8^2$	$8,2^3$	$8,2$	2^7
22	$8,2^4$ 16	2^2	2	$2^3,4$	$8,2^2$	$4,2$	2^3	$A,2^5$ 8	$B,2^3$ 8	$C,2^2$	$4,2^2$	$8,2^2$ 4	$2^2,8$	$2,4$	2^5	$16,2^6$
23	2^5	$4,16$	$4,2^2$	$4^2,2^3$ 32	2^3	$16,8$	$A,2^3$ 8	$2^4,8^2$ $B,16$	$C,2^5$	$4^2,2$ 16	$2^2,4$ 8	$4^2,2^2$ 32	2^3	$4,8$ $16,2$	$16,2^4$	$16^2,8^2$ 2^5
24	2^2	2^2	$8,2$	2^4	2^2	$A,2^2$ 4	$B,2^3$ 8	$2^7,C$	$2^3,4$	$8,2^2$	$8,2$	2^4	$2^3,4$	$16,2^3$	$16,2^4$ 8	2^7
25	2^2	8	2^2	2^4	$A,2^4$	$B,2^2$ 8	$C,2^5$	$2^5,4$	$2^2,8$	8	2^2	$2^5,4$	$8,2^4$	$8,2^3$ 16	2^5	2^4
26	8	2^2	2^2	$2^5,A$	$B,2^2$ 8	$C,2^3$ 4	$2^3,4$	$8,2^4$	8	2^2	$2^3,4$	$2^5,8$	$8,2^3$ 16	$2^3,4$	2^2	2^4
27	2	8	$A,2^3$	$B,2^2$ $8,4$	$2^4,C$	$8,4$ 2	$8,2^2$	8^2	2	$8,4$ 2	$2^3,8$	$4,2^3$ $16,8$	2^4	8	2^2	$8,2$
28	0	$2^3,A$	$2^3,B$	$C,2^5$	$4,2$	$8,2^2$	8	2^2	$2^2,4$	$8,2^2$	$4,2^3$ 16	2^5	0	2^2	8	2
29	$2^2,A$	$2^3,B$	$2^4,C$	$4,2$	$8,2$	8	2	$2^2,4$	2^3	$16,2^4$	2^4	0	2	8	2	0

Table 4.1 (continued)

k \ n	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
23	$\pi_{24}(P_n) \oplus Z_2 \oplus Z_8 \oplus Z_{16}$																		
24	$\pi_{25}(P_n) \oplus Z_2^2$																		
25	$\pi_{26}(P_n) \oplus Z_2^2$																		
26	$\pi_{27}(P_n) \oplus Z_2^2$																		
27	$\pi_{28}(P_n) \oplus Z_8$																		
28	$\pi_{29}(P_n) \oplus Z_2$																		
29	$16,4,2^2$	8,2	4	2^2	2^3	2	2^2	2^3	2^4	$2^2,4$	4,2	$\pi_{30}(P_n)$ —							
30	2,8,16	2,8,32	$\pi_{31}(P_n)$ —									$2 \oplus \pi_{31}(P_n)$ —							
31	16 2 4	32 2 2	64 2^2 4	 2^3	 2^5	 2^3	 2	 0	 0	 0	 0	 2^2	 2^4	 2^2	 2	 0	 0	 0	 $\pi_{32}(P_n)$ —
32	2^2 2^3	2^2 2^3	2^3 — 2,4,8	 2^5	 2^9	 2^5	 4,2,32	 4,2	 2	 2	 2,8	 2^3	 2^6	 2^3	 16	 0	 — $\pi_{33}(P_n)$ —		2^4 —
33		2^4 2,8	$2^3,4$ — 8,4	 2,8	 $8^2,2$	 8,2	 4	 2^2		 0	 0	 2	 2^2	 2	 0	 0	$2^4,4$ — $\pi_{34}(P_n)$		
34		2^3 — $\pi_{35}(P_n)$ —													 $2^3,2$	 $2^3,4$	 $2^3,8$		
35			$\oplus\pi_{36}(P_n)\oplus 8,2^2$ —																
36			2 — 16 8^2 2	 16 8^2 4	 2^6	 2^4	 $2^3,8$	 2^5	 2^7	 2^3	 32	 0		 $\oplus\pi_{36}(P_n)$ —					
37			2 $4,2^2$	2^2 — $4,2^3$	 $4,2^5$	 $4,2^3$	 4,8,2	 $4,2^2$	 $2^3,16$	 4,2	 $\oplus\pi_{38}(P_n)$ —		2^3 —						

Table 4.2. $\pi_{n+k}(S^n)$

37		2	4								
		2	2^2	—						2^3 —	
		$4,2^2$	$4,2^3$	$4,2^5$	$4,2^3$	$4,8,2$	$4,2^2$	$2^3,16$	$4,2$	$\oplus \pi_{38}(P_n)$ —	

Table 4.2. $\pi_{n+k}(S^n)$

k \ n	12	14		16		18		20		22		24	26	28	30	32	34	36	38
38			2	2	2	2^2	2^2	2^2	2^3	2^2 —									
			16	16^2	16	16	16	16		4 —									
			8	8^2	8	4													
			2^2	2^3	2^2	2	2^2	8	$\oplus \pi_{39}(P_n)$ —										
								2											
39			Z_{16} —						16	16 —									
			Z_2^3						2^4	2^5 —									
			2^4	2^5	0	0	$\oplus \pi_{40}(P_n)$ —												
40			$8,4,2^2$	$8,4,2^2$	$8,4,2^2$	$8,4,2^3$	$8,4,2^4$ —												
			$\oplus \pi_{41}(P_n)$																

Table 4.2, continued. $\pi_{n+k}(S^n)$

Vertical lines indicate that the group on the left of the line is repeated until either another group is given or until the stable range is reached.

i \ k (−1, 0, 1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 25–28)		
−1	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
0	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
1	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	$\left(\pi_{n+k} \oplus \pi_{n+k+1}(P_n)/I_n(\pi_{n+k+1})\right)$
2	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	(π_{n+k}/Z_2)
3	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
4	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
5	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
6	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	$(\pi_{n+k}/(2^3))$
7	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	$\frac{\pi_{n+k}}{2}$
8	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
9	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
10	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
11	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
12	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
13	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	
14	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)$	

Table 4.3. $\pi_{2n+k}(S^n)$, $k+n \neq 2^j+1$, $n \geq 40$; $k+n \equiv 1 \pmod{16}$

Table 4.3. $\pi_{2n+k}(S^n)$, $k+n \neq 2^j+1$, $n \geq 40$; $k+n \equiv 1 \pmod{16}$

i	k = −1, 0, 1	k = 2–4	k = 5–7	k = 8–13	k = 14–17	k = 18–24
i = −1	SAA	$\pi_{n+k} \oplus \pi_{n+k+1}(P_n)/I_n\{h_1h_j\}$				
0	SAA	$\pi_{n+k}/\{h_1h_j\}$ $\oplus$ $\pi_{n+k+1}(P_n)$	$\pi_{n+k+1}(P_n)/I_n\{h_1^2h_j\}$			
1	SAA		$\pi_{n+k}/\{h_1^2h_j\}$ $\oplus$ $\pi_{n+k+1}(P_n)$	$\oplus$ $\pi_{n+k+1}(P_n)/I_n(h_1^3h_j)$ $j > 6$		
2	π_{n+k} $\oplus$ $\pi_{n+k+1}(P_n)$			$\pi_{n+k}/\{h_1^2h_j\}$		
3	SAA					
4	π_{n+k} $\oplus$ $\pi_{n+k+1}(P_n)$				$\pi_{n+k+1}(P_n)/I_n(h_2^2h_j)$	
5	π_{n+k} $\oplus$ $\pi_{n+k+1}(P_n)$				$\pi_{n+k}/\{h_2^2h_j\}$	
6 − 14	SAA					

Table 4.4. $\pi_{2n+k}(S^n)$, $k+n = 2^j + i$

morphism can be easily detected. In particular if the unstable group in one of these tables does not contain $\pi(P_n)$ then the unstable J-homomorphism has a kernel. Comparison with table 1 gives the kernel. The groups in parentheses in table 3 refer to undecided cases. Conjecture V.2.4 if true would decide in favor of the group not in parentheses. In addition the reader should be warned that not all the group extensions have been settled. This applies particularly to table 4.1.

5. The author would like to express his thanks to A. Luilevicius for many profitable conversations on the material of this paper.

CHAPTER II

THE ADAMS SPECTRAL SEQUENCE

1. INTRODUCTION. The purpose of this chapter is to summarize the Adams spectral sequence, section 2; to prove theorem A, section 3; and to introduce the map $\lambda: P_1 \to S^0$, section 4.

2. THE ADAMS SPECTRAL SEQUENCE. (See also [1].)

Suppose X is an $n-1$ connected space. By a resolution of X we will mean a system of fiber spaces

2.1
$$\begin{array}{ccccccccc} \cdots \longrightarrow & P_s & \xrightarrow{p_s} \cdots \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \longrightarrow X \\ & \uparrow & & \uparrow & & \uparrow & \\ & A_s & & A_2 & & A_1 & \end{array}$$

together with the system induced by 2.1 over a point

2.2
$$\begin{array}{ccccccc} \cdots \longrightarrow & B_s & \longrightarrow \cdots \longrightarrow & B_2 & \longrightarrow & A_1 & \longrightarrow * \\ & \uparrow & & \uparrow & & & \\ & A_s & & A_2 & & & \end{array}$$

Each space of 2.3 is the fiber of a composite map of 2.1, i.e.

$$B_s \longrightarrow P_s \longrightarrow X$$

is a fiber space. The Puppe sequence gives a map $f_s: \Omega X \to B_s$. It is clear that the system 2.2 together with the maps f_s define 2.1. Because of this we frequently will call 2.2 together with $\{f_s\}$ a resolution.

Associated with a resolution is a spectral sequence defined by the exact couple

2.3
$$\begin{array}{ccc} \sum_s \pi_*(P_s) & \xrightarrow{\Sigma p_s} & \sum_s \pi_*(P_s) \\ & \nwarrow_{\Sigma i_*} \quad \swarrow_{\Sigma\partial_*} & \\ & \sum_s \pi_*(A_s) & \end{array}$$

Of course in this generality nothing much can come from 2.3. There are several useful specializations. The first leads to

DEFINITION 2.4. A resolution (mod p) is called <u>admissible</u> through dimension $T < 2n-1$ if

1) Each A_s is a product of Eilenberg MacLane spaces ($K(Z,q)$ or $K(Z_p,q)$) of

dimensions less than T;

2) $\ker(f_s^*: \pi_*(\Omega X) \to \pi_*(B_s))$ is strictly monotonically decreasing.

The most important resolution has this

DEFINITION 2.5. A resolution is called an <u>Adams</u> <u>resolution</u> mod p if

1) it is admissible through dimension $2n-1$; and

2) each A_s is a product of $K(Z_p,q)$'s;

3) p_s^* is zero for each s with Z_p for coefficients through dimension T (in (2.1).

Because of 2.4.2 the spectral sequence associated with an admissible resolution of a $n-1$ connected space with finitely generated homotopy converges to a graded group associated with $\sum_{j<T} \pi_j(X)$, filtered by 2.1. Using both the s filtration and the q filtration of A_s we see that 2.3 is always begraded. In the case of an Adams resolution the $E_2^{s,t} = \mathrm{Ext}_A^{s-1,t}(\tilde{H}^*(X),Z_2)$ for $t-s < T-1$; for details see [1].

Related to the above is another notion which will be useful. Let $D \subset H^*(X;Z_p)$ such that D is a vector space over Z_p.

DEFINITION 2.6. We say X_D represents D if

1) X_D is a product of Eilenberg MacLane spaces;

2) there is a 1-1 correspondence with fundamental classes $\{\alpha\}$ of X_D and a homogeneous basis of D such that if $a \in D \cap H^j(X)$ then $\alpha_a \in H^{j-1}(X_D)$.

3) there is a map $f: X \to X_D$ such that $f^*(\alpha_a) = a$.

Given a subspace $D \subset H^*(X)$ there is always a fiber space

2.7 $$X_D \to Y \to X$$

with $\tau(\alpha_a) = a$ for each $a \in D$. For more details see [2; chapter 3].

3. THE CONSTRUCTION. Let $\Omega Y_n^{\,n}$ be the fiber of the $2n-2$ connected fiber space over S^n. That is, there is a map $f: S^n \to Y_n^{\,n}$ such that $f_*: \pi_j(S^n) \to \pi_j(Y_n^{\,n})$ is an isomorphism for $j < 2n-1$ and $\pi_j(Y_n^{\,n}) = 0$ for $j \geq 2n-1$. Since $Y_n^{\,n}$ has homotopy only through the stable range we can define an Ω-spectrum based on Y_n, i.e., $\Omega Y_{k+1}^{\,n} = Y_k^{\,n}$ for all $k \geq n$. Let $F_{n+k,k}$ be the fiber for the following map:

$$F_{n+k,k} \to \Sigma^k Y_n^{\,n} \to Y_{n+k}^{\,n}.$$

Note that n is a fixed integer and $Y_{n+k}^{\,n}$ depends on n. We will keep n fixed throughout the remainder of this section and thus suppress the superscript n. It will be understood throughout this section.

PROPOSITION 3.1. There is a homotopy equivalence through the $3n+k-2$ skeleton between $F_{n+k,k}$ and $\Sigma^{n+k}P_n^{n+k-1}$.

Proof. Consider

$$\Omega^k F_{n+k,k} \xrightarrow{i} \Omega^k\Sigma^k Y_n \to Y_n.$$

This fibration has a cross-section $\varphi: Y_n \to \Omega^k\Sigma^k Y_n$ given by $\varphi(y)(s_1,\dots,s_n) = (y,s_1,\dots,s_n)$ where $(y,s_1,\dots,s_n)$ is a point in $\Sigma^k Y_n$ in the standard representation. We can make φ into a fiber map giving

$$Q_{n+k,k} \to Y_n \to \Omega^k\Sigma^k Y_n$$

where Q is defined as the fiber. In any fibration $F \to E \to B$ the boundary homomorphism in homotopy can be realized by a map $f: \Omega B \to F$. Using this map we have

$$\Omega^{k+1}F_{n+k,k} \xrightarrow{i} \Omega^{k+1}\Sigma^k Y_n \xrightarrow{f} Q_{n+k,k}.$$

Since $\pi_j(Y_n) = 0$ for $j \geq 2n-1$, fi induces an isomorphism in homotopy for all dimension. Thus $\Omega^{k+1}F_{n+k,k}$ is homotopically equivalent to $Q_{n+k,k}$.

Now consider the following diagram of fibrations:

3.1.1
$$\begin{array}{ccccc} Q'_{n+k,k} & \to & S^n & \to & \Omega^k S^{n+k} \\ \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 \\ Q_{n+k,k} & \to & Y_n & \to & \Omega^k\Sigma^k Y_n. \end{array}$$

If $k = 1$ James [14] showed that $Q'_{n+1,1} = S^{2n-1}$ through homotopy dimension $3n-3$. While Barcus and Meyer [6] showed that $F_{n+1,1} = Y_n * Y_n = S^{2n+2}$ through dimension $3n$. Since $Q_{n+1,1} = \Omega^2 F_{n+1,1}$ and i_1 corresponds to $S^{2n-1} \to \Omega^2\Sigma^2 S^{2n-1}$ we see that i_1 is a homotopy equivalence through $3n-3$. We now proceed by induction. Consider

$$\begin{array}{ccccc} Q_{n+1,1} & \to & Y_n & \to & \Omega\Sigma Y_n \\ \downarrow & & \downarrow & & \downarrow \\ Q_{n+k,k} & \to & Y_n & \to & \Omega^k\Sigma^k Y_n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega Q_{n+k,k-1} & \to & Y_n & \to & \Omega^k\Sigma^{k-1} Y_{n+1} \end{array}$$

and

$$\begin{array}{ccccc} Q'_{n+1,1} & \longrightarrow & S_n & \longrightarrow & \Omega\Sigma S_n \\ \downarrow & & \downarrow & & \downarrow \\ Q'_{n+k,k} & \longrightarrow & S_n & \longrightarrow & \Omega^k\Sigma^k S_n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega Q'_{n+k,k-1} & \rightarrow & \Omega S_{n+1} & \rightarrow & \Omega^k\Sigma^{k-1}S^{n+1}. \end{array}$$

The natural maps between the two diagrams give

$$\begin{array}{ccccccc} \pi_j(Q_{n+1,1}) & \rightarrow & \pi_j(Q_{n+k,k}) & \rightarrow & \pi_{j-1}(Q_{n+k,k-1}) & \rightarrow & \pi_{j-1}(Q_{n+1,1}) \\ \uparrow j_1{}^* & & \uparrow j_2{}^* & & \uparrow j_3{}^* & & \uparrow j_1{}^* \\ \pi_j(Q'_{n+1,1}) & \rightarrow & \pi_j(Q'_{n+k,k}) & \rightarrow & \pi_{j-1}(Q'_{n+k,k-1}) & \rightarrow & \pi_{j-1}(Q'_{n+1,1}). \end{array}$$

By hypothesis $j_1{}^*$ and $j_3{}^*$ are isomorphisms for $j < 3n-3$ and $j < 3n-2$ respectively. Hence $j_2{}^*$ will be an isomorphism too for $j < 3n-3$. Theorem I completes the proof.

COROLLARY 3.2. $\pi_j(\Sigma^k Y_n) = \pi_j(Y_{n+k}) + \pi_j(\Sigma^{n+k}P_n^{n+k-1})$ for $j < 3n+k-3$.

Note that either one or the other group is zero in the range of interest. Let $\lambda\colon \pi_j(\Sigma^k Y_n) \to \pi_j(\Sigma^{n+k}P_n^{n+k-1})$ be the projection map. Of course it is defined only for $j < 3n+k-3$ and is not generated by any geometric map.

The following is an important corollary of the proof of 3.1.

PROPOSITION 3.3. The composite

$$\pi_j(S^{n+k}) \to \pi_j(\Sigma^k Y_n) \xrightarrow{\lambda} \pi_j(\Sigma^{n+k}P_n^{n+k-1})$$

is just

$$\pi_j(S^{n+k}) \xrightarrow{I_k} \pi_{j-1-k}(\Sigma^{n-1}P_n^{n+k-1}) \xrightarrow{\Sigma^{k+1}} \pi_j(\Sigma^{n+k}P_n^{n+k-1})$$

where I_k is the Toda map of theorem 1.

Proof. The proof is immediate from diagram 3.1.1.

Proposition 3.3 is the key to the proof of theorem A. The only thing left is to construct a suitable resolution of the cohomology of $\Sigma^k Y_n$ so as to be able to identify the copy of $\Sigma^{n+k}P_n^{n+k-1}$ which is present there.

Let

$$Y_k \rightarrow \cdots \rightarrow X_{s,k} \xrightarrow[\rho_{s,k}]{} \cdots \rightarrow X_{2,k} \xrightarrow[\rho_{2,k}]{} X_{1,k} \xrightarrow[\rho_{1,k}]{} K(Z,k)$$

be an Adams resolution of Y_k through dimension $2n+k-1$ (Def. 2.5). We require that $\Omega X_{s,k+1} = X_{s,k}$.

The rest of this section will be devoted to proving the existence of the following diagram with the properties we will require (and show) it to have.

$$\begin{array}{ccccccccccccc}
 & & \Sigma^k Y_n & \to \cdots \to & \Sigma^k X_{s,n} & \to & \Sigma^k X_{s-1,n} & \to \cdots \to & \Sigma^k X_{1,n} & \to & \Sigma^k K(Z,n) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{D_s} & \to & A_{s+1} & \to \cdots \to & A^s_{s+1} & & \downarrow & & \downarrow & & \downarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{D_{s-1}} & \to & A_s & \to \cdots \to & A^s_s & \to & A^{s-1}_s & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots & & \downarrow & & \downarrow \\
X_{D_1} & \to & A_2 & \to \cdots \to & A^s_2 & \to & A^{s-1}_2 & \to \cdots \to & A^1_2 & & \downarrow \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_{D_0} & \to & A_1 & \to \cdots \to & A_1^s & \to & A_1^{s-1} & \to \cdots \to & A_1^1 & \to & A_1^0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & Y_{n+k} & \to \cdots \to & X_{s,n+k} & \to & X_{s-1,n+k} & \to \cdots \to & X_{1,n+k} & \to & K(Z,n+k)
\end{array}$$

Diagram 3.4

The resolution of $\Sigma^k Y_n$, which will give a proof of theorem A, is the diagonal one in this diagram, i.e.,

$$\Sigma^k Y_n \to \cdots \to A_s{}^s \to \cdots \to A_1{}^1 \to K(Z_{n+k}).$$

Hence the tower induced by the left hand column over a point must be an Adams resolution of $F_{n+k,k}$. We will describe in detail the lower right corner and the general case involving the parameters s and s - 1. In everything that follows we will only consider cohomology through dimension $3n + k - 1$. $H^*(X)$ will mean $\sum_{0 < j < 3n+k} H^j(X)$.

First we need a lemma.

LEMMA 3.4.1. Let $F^s_{n+k,k}$ be the fiber of $\Sigma^k X_{s,n} \to X_{s,n+k}$ and let $f_s \colon F_{n+k,k} \to F^s_{n+k,k}$ be the natural map. For each s $f_s{}^*$ is surjective in dimension less than $3n + k$.

Proof. We proceed by induction. We need only show it for F^o. We have the following diagram:

$$\begin{array}{ccccc}
\Sigma^{k-1}F^{o}_{n+1,1} & \xrightarrow{i_1} & F^{o}_{n+k,k} & \xrightarrow{p_1} & F^{o}_{n+k,k-1} \\
\uparrow f_o{}'' & & \uparrow f_o & & \uparrow f_o{}' \\
\Sigma^{k-1}F_{n+1,1} & \longrightarrow & F_{n+k,k} & \xrightarrow{p_2} & F_{n+k,k-1}.
\end{array}$$

Since $F^{o}_{n+1,1} = K(Z,n) * K(Z,n)$ [6] $f_o{}''^{*}$ is surjective. The bottom cohomology sequence splits into a short sequence. By induction suppose $f_o{}'^{*}$ is surjective. Then $(p_2 f_o{}')^{*}$ is surjective in dimension for which $p_2{}^{*}$ is. Since $F_{n+k,k-1}$ is $2n+k$ connected $i_1{}^{*}$ is surjective in dimension $2n+k$, which completes the proof.

First the lower right corner. Consider the following diagram:

$$\begin{array}{ccccc}
F_{n+k,k} & \xrightarrow{i_2} & \Sigma^{k}Y_n & \longrightarrow & Y_{n+k} \\
\downarrow f & & \downarrow g & & \downarrow h \\
F^{o}_{n+k,k} & \xrightarrow{i_1} & \Sigma^{k}(K(Z,n)) & \longrightarrow & K(Z,n+k).
\end{array}$$

Let $H^{*}(F^{o}_{n+k,k}) = \ker f^{*} + D_0$ where D_0 is defined by this equation (although not uniquely). First observe that

3.5 $$\tau\colon D_0 \to H^{*}(K(Z,n+k))$$

is a monomorphism. Indeed if $a \in D_0$ satisfies $\delta^{*}a = 0$, then there is an a' such that $i_1{}^{*}a' = a$. But then $i_2{}^{*}g^{*}a' = f^{*}a \neq 0$ but g^{*} is clearly zero in dimension $\neq n+k$. Let X_{D_0} be a product of Eilenberg MacLane spaces which represents D_0 (2.6). We can form the fiber space

3.6 $$X_{D_0} \to A_1{}^{0} \to K(Z,n+k)$$

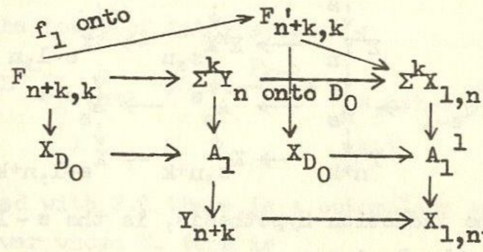
where the image of D_0 under transgression is given by 3.5.

The second row of 3.4 is induced by 3.6.

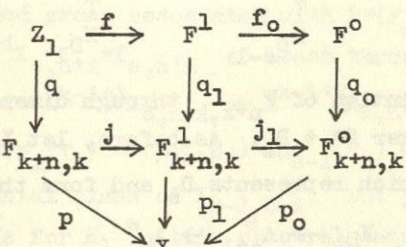
Now consider

$$\begin{array}{ccccccc}
 & Z_1 & \xrightarrow{f} & F^{1} & & & \\
 & & \searrow i & & \searrow & & \\
 & & \Sigma^{k}Y_n & \xrightarrow{\lambda_1} & \Sigma^{k}X_{1,n} & \xrightarrow{\lambda_1{}^{1}} & \Sigma^{k}K(Z,n) \\
 & & \downarrow & & \downarrow & & \downarrow \\
X_{D_0} & \to & A_1 & \xrightarrow{\lambda_1} & A_1{}^{1} & \xrightarrow{\lambda_1{}^{1}} & A_1{}^{0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Y_{n+k} & \xrightarrow{\lambda_0} & X_{1,n+k} & \xrightarrow{\lambda_0{}^{1}} & K(Z,n+k).
\end{array}$$

Let $H^*(F^1) = \ker f^* + D_1$ where D_1 is defined by this equation. Let X_{D_1} be a representation of D_1 and define $X_{D_1} \to A_2{}^1 \to A_1{}^1$ by requiring the transgression on D_1 to be the same as for the fibration $F^1 \to \Sigma^k X_{1,n} \to A_1{}^1$. The third row of 3.4 is induced by this fibration. In order to show that X_{D_1} is the correct fiber for the second stage of a resolution of $F_{n+k,k}$ we must show f^* is onto. First observe that i^* is zero. To see this consider the following diagram:



Since g^* is onto (because $f_o{}^*$ is onto according to 3.4.1), q^* is zero and thus i^* is zero. Now consider the diagram



where $F^1_{k+n,n}$ and $F^o_{k+n,k}$ are the fibers of $\Sigma^k X_{1,n} \to X_{1,n+k}$ and $\Sigma^k K(Z,n) \to K(Z,n+k)$ respectively. Now $\ker j_1{}^* = \ker (j,j)^*$. Indeed, $H^*(F^o_{k+n,k})$ is composed of the cohomology of $\Sigma^k K(Z,n)$ which is not in $\operatorname{im} H^*(K(Z,n+k))$, (i.e. suspension of cup product terms) together with the kernel of the map $H^*(K(Z,n+k)) \to H^*(\Sigma^k K(Z,n))$, i.e., those classes with an excess of greater than n in the Cartan basis representation [23]. Clearly $j_1{}^*$ maps to zero all cup product terms and all classes which transgress to classes with excess greater than n except those classes which transgress to Sq^i, $i > n$. But these classes which transgress to Sq^i are also mapped nontrivially by $j^* j_1{}^*$.

Hence if there is an $\alpha \in H^*(X_{D_0})$ such that $p^*\alpha = 0$, then $p_1{}^*\alpha$ is zero too. Since $q^* = 0$ this shows that f^* is onto.

The pattern of the above argument is repeated in each successive square. While it is clear that $\ker j_1{}^* = \ker(j_1 j)^*$ in this setting it is less clear later on because one does not have a hold on $H^*(X_{s,q})$, $q = n$ and $n+k$.

Now we will do the general case. Consider the following diagram:

3.7
$$\begin{array}{ccccccc} & & Z_s & \xrightarrow{f} & F^s & \to & F^{s-1} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Sigma^k Y_n & \to & \Sigma^k X_{s,n} & \to & \Sigma X_{s-1,n} \\ & & \downarrow & & \downarrow & & \downarrow \\ Q_{s-1} & \to & A_s & \to & A_s{}^s & \to & A_s{}^{s-1} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & Y_{n+k} & \to & X_{s,n+k} & \to & X_{s-1,n+k}. \end{array}$$

The fiber Q_{s-1}, in the induction hypothesis, is the $s-1$ space in a resolution of $F_{n+k,k}$, through $3n+k-3$, i.e.

3.8
$$\begin{array}{ccccccc} F_{n+k,k} \to \cdots \to & Q_{s-1} & \to \cdots \to & Q_1 & \to X_{D_0} \\ & \uparrow & & \uparrow & \\ & X_{D_{s-1}} & & X_{D_1} & \end{array}$$

where 3.8 is an Adams resolution of $F_{n+k,k}$ through dimension $3n+k-1$. Let D_s be defined by $H^*(F^s) = \ker f^* + D_s$. As before, let X_{D_s} be a product of Eilenberg MacLane spaces which represents D_s and form the fiber space

3.9
$$X_{D_s} \to A_{s+1}^s \to A_s{}^s$$

where the image of D_s under transgression is the same as in the fiber space

$$F^s \to \Sigma^k X_{s,n} \to A_s{}^s.$$

The $s+1$ row of 3.4 is induced by 3.9. All that remains is to show that X_{D_s} is the correct s^{th} fiber in 3.8 and this requires only that f^* be onto. For this we go to the previous stage obtaining the diagram below where the top row is defined in 3.7 while the middle row is the fiber from $\Sigma^k X_{s-1,n} \to X_{s-1,n+k}$, and so forth. The following lemma implies that f^* is onto as above.

LEMMA 3.10. In this diagram $\ker j_s{}^* = \ker(j_s j)^*$.

$$
\begin{array}{ccccc}
Z_s & \xrightarrow{f} & F^s & \xrightarrow{f_s} & F^{s-1} \\
\downarrow q & & \downarrow q_s & & \downarrow q_{s-1} \\
F_{k+n,k} & \xrightarrow{j} & F^s_{k+n,k} & \xrightarrow{j_s} & F^{s-1}_{k+n,k} \\
& {\scriptstyle p}\searrow & \downarrow p_s & \swarrow{\scriptstyle p_{s-1}} & \\
& & Q_{s-1} & &
\end{array}
$$

Proof. Consider the tower of fiber spaces

$$
\begin{array}{ccccccccc}
\Sigma^k X_{s,n} & \xrightarrow{p_o} & \Sigma^{k-1} X_{s,n+1} & \rightarrow & \cdots & \xrightarrow{p_{k-1}} & \Sigma X_{s,n+k-1} & \rightarrow & X_{s,n+k} \\
\uparrow & & \uparrow & & & & \uparrow & & \\
C^0 & & C^1 & & & & C^{k-1} & &
\end{array}
$$

Just as 2.3 is associated with 2.1 there is a cohomology spectral sequence associated with this tower whose E_1 term is

$$E_1 = \sum_i H^*(C^i)$$

and whose E_∞ is a graded group associated with $H^*(F^s_{n+k,k})$. Barcus and Meyer [6] prove that $C^i = \Sigma^{k-i} X_{s,n+i} * X_{s,n+i}$ at least through $3n+k-1$ dimensions, where C^i is the fiber of $\Sigma^{k-i+1} X_{s,n+i} \rightarrow \Sigma^{k-i+2} X_{s,n+i-1}$. If β_i, γ_i are $\varepsilon\ H^*(X_{s-1,n-i})$ then $j_s^*(\beta_i * \gamma_i) = (\rho_{s-1}^* \beta_i) * \rho_{s-1}^*(\gamma_i) = 0$ unless $\beta_i = \gamma_i = \alpha_i$ where α_i is the fundamental class of $X_{s-1,n-i}$. But $j^* j_s^*(\alpha_i * \alpha_i) \neq 0$ for each i, hence the lemma holds for E_1 in this spectral sequence. Now $(s^{k-i}\alpha_i * \alpha_i)$ projects to a non-zero class in E_∞ and $j^* j_s^*$ on these classes in E_∞ is an isomorphism. This implies the lemma.

The proof of this lemma completes the proof of the existence of 3.4 with 3.8 being a resolution of $F_{n+k,k}$.

Now consider the resolution

3.11 $$\Sigma^k Y_n \rightarrow \cdots \rightarrow A_s^{\ s} \rightarrow \cdots \rightarrow A_1^{\ 1} \rightarrow K(Z, n+k).$$

At each stage the fiber is a product of Eilenberg-MacLane spaces since in going from $A_s^{\ s}$ to $A_{s+1}^{\ s}$ the fiber consists of Eilenberg-MacLane spaces in dimensions above $2n+k$ while in going from $A_{s+1}^{\ s}$ to $A_{s+1}^{\ s+1}$ the dimensions of the homotopy in the fiber are all less than $2n+k$. Thus this is an admissible resolution. Let $E_r^{s,t}$ be the spectral sequence associated with it. Notice that

$$E_2^{s,t} = \mathrm{Ext}^{s,t}(Z_2,Z_2),\ t-s < n-1$$
$$= \mathrm{Ext}^{s,t}(\tilde{H}^*(P_n^{n+k-1}),Z_2),\ n \leq t-s < n-2.$$

Hence the spectral sequence based on 3.11 splits into the Adams spectral sequence for a sphere if $t-s < n-1$ and the one for P_n^{n+k-1} for $n \leq t-s < 2n-1$. There is a map of an Adams resolution of S^{n+k} into the resolution of 3.11 which induces a map between spectral sequences $\lambda\colon E_r^{s,t}(S^0) \to E_r^{s,t}$. The map of the theorem is obtained by just considering the portion of $E_r^{s,t}$ for $t-s > n-1$. This gives

$$I_k{}^r\colon E_r^{s,t}(S^0) \to E_r^{s,t}(P_n^{n+k-1}).$$

The module statement is clear by considering the entire spectral sequence as mapped by λ. Proposition 3.3 shows that I_k^{∞} is the map associated with I_k of theorem 1 and this completes the proof of theorem A.

4. THE MAP λ.

Adams [5] has shown that $\tilde{K}(P_1{}^n) = Z_{(2^{\varphi(n)})}$ where $\varphi(n)$ is a well defined function whose exact value is not important here. Let H_n be the generator of this group. It is well known that H_n can be chosen as the Hopf bundle over P_n. Let $T(jH_n)$ be the Thom complex of jH_n. It is easily seen that $T(jH_n) = P_j^{j+n}$. Hence $(2^{\varphi(n)}-1)H_n = P^{2^{\varphi(n)}-1+n}_{2^{\varphi(n)}-1}$. By James periodicity [13] $P^{2^{\varphi(n)}-1+n}_{2^{\varphi(n)}} = \Sigma^{2^{\varphi(n)}} P_0^{n-1}$ where $P_0 = P_1 \cup \{\mathrm{pt.}\}$ if n satisfies: $n' < n$ implies $\varphi(n') < \varphi(n)$. Consider the Puppe sequence

$$S^{2^{\varphi(n)}-1} \to P^{2^{\varphi(n)}-1+n}_{2^{\varphi(n)}-1} \to \Sigma^{2^{\varphi(n)}} P_0^{n-1} \xrightarrow{\lambda'} S^{2^{\varphi(n)}}.$$

The map λ^{n-1} clearly defines a map in the stable category of Adams giving $\lambda'\colon P_0 \to S^0$. Generally we will find the map $\lambda\colon P_1 \to S^0$ more useful where λ is the restriction.

There is another such map. James [15] has constructed a map $P_1{}^n \subset SO(n+1)$. The Whitehead J-homomorphism is induced by a map $SO(n+1) \subset \Omega^{n+1}S^{n+1}$. Let $\overline{\lambda}^{n+1}$ be the adjoint of the composition, i.e. $\overline{\lambda}^{n+1}\colon \Sigma^{n+1}P_1{}^n \to S^{n+1}$. This also defines a map in the stable category $\bar{\lambda}$.

Conjecture 4.1. a) λ and $\bar{\lambda}$ are the same.
b) λ_* (or $\bar{\lambda}_*$) is an epimorphism in both homotopy and in Ext.

In Chapter IV we will verify the conjecture as far as the computations go.

PROPOSITION 4.2. The following diagram is commutative:

$$\begin{array}{ccccccc}
\cdots \to \pi_j(S^n) & \xrightarrow{\Sigma^k} & \pi_{j+k}(S^{n+k}) & \xrightarrow{I_k} & \pi_{j-1}(\Sigma^{n-1}P_n^{n+k-1}) & \xrightarrow{P_k} & \cdots \\
\uparrow \lambda_*^n & & \uparrow \lambda_*^{n+k} & & \uparrow \simeq & & \\
\cdots \xrightarrow{a} \pi_j(\Sigma^n P_1^{n-1}) & \xrightarrow{b} & \pi_{j+k}(\Sigma^{n+k}P_1^{n+k-1}) & \xrightarrow{c} & \pi_{j-1}(\Sigma^{n-1}P_n^{n+k-1}) & \to & \cdots
\end{array}$$

for $j < 4n-3$ where a, b, and c are suspensions or desuspensions of corresponding maps in

$$\Sigma^n P_1^{n-1} \to \Sigma^n P_1^{n+k-1} \to \Sigma^n P_n^{n+k-1}$$

which is a fibration for our range of dimensions.

Proof. We will first prove the proposition for $k = 1$. Let $g\colon S^{n-1} \to P_1^{n-1}$ be the attaching map for the n-cell of P_1^n. Let $f\colon P_1^{n-1} \to BSO(n-1)$ be the classifying map for any $n-1$ plane bundle which is stably $(2^{\varphi(n)}-1)H_{n-1}$.

We have the diagram

$$\begin{array}{ccccc}
V_{n-1} & \xrightarrow{i} & BSO(n-1) & \xrightarrow{p} & BSO \\
\uparrow & & \uparrow f & \nwarrow h' & \uparrow h \\
S^{n-1} & \xrightarrow{g} & P_1^{n-1} & \xrightarrow{p'} & P_1^n
\end{array}$$

where h is the classifying map of $2^{\varphi(n)}-1$ (as a stable bundle). If $fg \simeq 0$ then h^1 would exist but since $h^*W_n \neq 0$, it cannot happen. But $pfg \simeq 0$ and so there is a map $\bar{f}\colon S^{n-1} \to V_{n-1}$. If n is even, then $[\bar{f}]$ generates $\pi_{n-1}(V_{n-1})$; if n is odd, then f can be chosen so that $[\bar{f}]$ generates $\pi_{n-1}(V_{n-1})$. Hence the bundle corresponding to fg is just the tangent bundle of S^{n-1}.

This gives

$$S^{n-1} \cup_{[\iota_{n-1},\iota_{n-1}]} e^{2n-2} \to T(f) = S^{n-1} \cup_{\lambda^{n-1}} \Sigma^n P_0^{n-2}$$

or

$$\begin{array}{ccc} S^{2n-3} & \xrightarrow{\tau} & \Sigma^{n-1}P_0^{n-2} \\ & \sigma \searrow & \swarrow \lambda^{n-1} \\ & S^{n-1} & \end{array}$$

where $\sigma \simeq [\iota_{n-1}, \iota_{n-1}]$. This gives

$$\begin{array}{ccccc} S^{2n-3} & \xrightarrow{[\iota_{n-1},\iota_{n-1}]} & S^{n-1} & \rightarrow & S^{n-1}/S^{2n-3} \\ & \searrow & \uparrow & & \uparrow \\ & & \Sigma^{n-1}P_0^{n-2} & \rightarrow & \Sigma^{n-1}P_0^{n-1}. \end{array}$$

James has shown that $S^{n-1}/S^{2n-3} \simeq \Omega S^n$ through the $3n-5$ skeleton and if we replace S^{n-1}/S^{2n-3} by ΩS^n the resulting sequence in homotopy is exact through dimension $4n-7$. This gives the proposition for $k = 1$. The induction argument is the same kind of argument as used to prove 3.1.

Note that this proposition is trivial for $\bar{\lambda}$.

REMARK. If conjecture 4.1.b were valid then this proposition would suffice for providing the kind of map described in theorem A. Since we do not have it we can use 4.2 for computation involving early stems (up to 44) and we use theorem A for any general result.

COROLLARY 4.3. Suppose $\alpha \in \pi_{j+k}(S^{n+k})$, k large, $j < 4n-3$, and $\alpha \in \operatorname{im} \lambda_*^{n+k}$. Then $I_k(\alpha) \neq 0$ iff $c(\bar{\alpha}) \neq 0$ for any $\bar{\alpha}$ such that $\lambda_*^{n+k}(\bar{\alpha}) = \alpha$.

This result suggests the following definition.

DEFINITION 4.4. Let α be an element of either Ext or π_* for a sphere. Let $i\colon S^n \to P_n$, the inclusion onto the bottom cell. Suppose there is a j such that for $P_{n-j} \xrightarrow{p} P_n$ $i_*(\alpha) \notin \operatorname{im} p_*$ stably. Let j be the smallest integer with this property. Then consider

$$\begin{array}{ccccc} S^{n-j} & \rightarrow & P_{n-j} & \xrightarrow{p_2} & P_{n-j+1} \\ & & & & \downarrow p_1 \\ & & & & P_n \end{array}$$

By the root of α, $\sqrt{\alpha}$ we mean $\alpha_*(\bar{\alpha})$ for any $\bar{\alpha}$ satisfying $p_1{}^*\bar{\alpha} = i^*\alpha$. Then $n-j$ is the dimension of the root.

PROPOSITION 4.5. Let α be as in 4.4. If $\alpha \in \operatorname{im} I_k$ and if there is an $\bar{\alpha}$ such that $I_k(\bar{\alpha}) = \alpha$ and $\bar{\alpha} \in \operatorname{im} \lambda_*^{n+k}$ then α has an imaginary root, i.e., $j \geq n$.

This is clear from 4.2 and the definition.

PROPOSITION 4.6. Suppose $\alpha \in \pi_q(S^n)$, and α has a root of dimension q' such that $3q' - 2 > q + n$ then $P_k(\alpha) \neq 0$.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\pi_j(S^n) & \longrightarrow & \pi_{j+k}(S^{n+k}) & \xrightarrow{I_k} & \pi_{j-1}(\Sigma^{n-1}P_n^{n+k-1}) \\
\uparrow & \nearrow & & \searrow I_{k'} & \uparrow p \\
\pi_{j-n+q'}(S^{q'}) & & & & \pi_{j-n+q'-1}(\Sigma^{q'-1}P_{q'}^{n+k-1})
\end{array}$$

Let $j = 2n + q$, then $i_*\alpha \in \pi_{j-1}(\Sigma^{n-1}P_n^{n+k-1})$. The restriction on q' which is important is $4q' - 3 > j - n + q' - 1$ or $3q' - 2 > q + n$. Then $i_*\alpha \not\in \operatorname{im} I_k$ since $i_*\alpha \not\in \operatorname{im} p_* I_{k'}$. Hence $P_k(\alpha) \neq 0$.

CHAPTER III

THE CALCULATION OF $\pi_{k+p}(P_k)$ FOR $p \leq 29$

1. In this chapter we introduce a spectral sequence which leads to an easy calculation of Ext for $H^*(P_k)$. We then compute all the differentials in this sequence and in Adams's spectral sequence which are needed to give the homotopy groups of P_k in a range of dimensions. The results are complete modulo group extensions through $p = 27$. Almost complete results are obtained for $p = 28$ and 29. Tables at the end of the chapter summarize these calculations. The explanation of the tables is in section 8. Frequent reference is made to the tables during the calculations and so some familiarity with section 8 is required to follow these arguments. These calculations extend the results announced in [12]. The method of calculation there is totally different.

2. Consider the collection of cofibrations

$$P_n^{n+k-1} \xrightarrow{i_k} P_n^{n+k} \xrightarrow{p_k} S^{n+k}$$

for a fixed n. The cohomology sequence of these cofibrations all break into short exact sequences of length 3.

Hence Ext_A applied to the cohomology gives a long exact sequence [2; 2.6.1],

$$\rightarrow \mathrm{Ext}_A^{s,t}(\tilde{H}^*(P_n^{n+k-1}),Z_2) \rightarrow \mathrm{Ext}_A^{s,t}(\tilde{H}^*(P_n^{n+k}),Z_2)$$
$$\rightarrow \mathrm{Ext}_A^{s,t}(\tilde{H}^*(S^{n+k}),Z_2) \rightarrow \mathrm{Ext}_A^{s+1,t}(\tilde{H}^*(P_n^{n+k-1}),Z_2) \rightarrow$$

The entire system gives rise to the following exact couple

2.1
$$\begin{array}{ccc} \sum_{k\geq 0} \mathrm{Ext}_A^{s,t}(\tilde{H}^*(P^{n+k}),Z_2) & \xrightarrow{\sum_k i_{k*}} & \sum_{k\geq 0} \mathrm{Ext}_A^{s,t}(\tilde{H}^*(P_n^{n+k}),Z_2) \\ \nwarrow {\scriptstyle \sum_k \delta_{k+1}} & & \swarrow {\scriptstyle \sum_k p_{k*}} \\ & \sum_{k\geq 0} \mathrm{Ext}_A^{s,t}(\tilde{H}^*(S^{n+k}),Z_2) & \end{array}$$

whose E_∞ term is a group associated with $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(P_n),Z_2)$. Since both are vector spaces over Z_2, the E_∞ term is isomorphic, as a vector space, to $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(P_n),Z_2)$. As a module over $H^*(A)$ the two are not isomorphic. We are able to recover much of the module structure by a more careful analysis of the couple together with some geometric considerations.

For the remainder of this chapter we take n to be fixed. It is convenient for notational purposes to decrease the t-filtration of each term of 2.1 by n. Then the E_1 term of 2.1 is

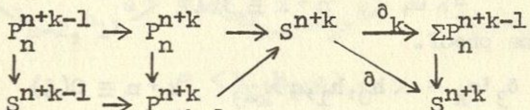
$$E_1^{s,t} = \sum_k \mathrm{Ext}_A^{s,t}(H^*(S^k), Z_2).$$

Clearly $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(S^k), Z_2) \simeq H^{s,t-k}(A)$ and so $E_1^{s,t,k} = H^{s,t-k}(A)$. Let $\alpha \in E_1^{s,t,k}$, then α is identified with $\tilde{\alpha}_{s,t-k} \in H^{s,t-k}(A)$. We will use the name of $\tilde{\alpha}$ as given in table 8.1 together with the additional subscript k. For example the non-zero element in $E_1^{1,k+2,k}$ is written $h_{1,k}$.* Hence generically α_k is in $E_1^{*,*+k,k}$ with α being the label of an element in $H^{*,*}(A)$.

The differentials δ_r of this spectral sequence are maps $\delta_r\colon E_r^{s,t,k} \to E_r^{s+1,t,k-r}$. Each class in $E_r^{s,t,k}$ has a representation α_k where $\alpha \in H^{s,t-k}(A)$ and we will describe $\delta_r\alpha_k$ by giving an operation $\delta_r'\colon H^{s,t-k}(A) \to H^{s+1,t-k+r}(A)$. It is clear that δ_r' describes completely δ_r.

PROPOSITION 2.1.
$$\begin{aligned} \delta_1'\alpha_k &= h_0\alpha_{k-1} & n+k &\equiv 0(2) \\ &= 0 & n+k &\equiv 1(2). \end{aligned}$$

Proof. The definition of a differential in an exact couple asserts that δ_1 is the composite $p_{k-1}{}^*\delta_k$ which we can think of as coming from the geometric maps $S^{n+k} \xrightarrow{\partial_k} \Sigma P_n^{n+k-1} \xrightarrow{p_{k-1}} S^{n+k}$. But consider



Hence δ_1 is just ∂_*. But $P_{n+k-1}^{n+k} = S^{n+k-1} \cup_{2\iota} e^{n+k}$ if $n+k \equiv 0(2)$ and it is a wedge otherwise. [2: 2,6,1] completes the proof.

PROPOSITION 2.3.
$$\begin{aligned} \delta_2'\alpha_k &= h_1\alpha_{k-2} & n+k &\equiv 0,1(4) \\ &= 0 & &\equiv 2,3(4). \end{aligned}$$

Proof. As in proof of 2.2 we get the diagram below. If $\alpha_k \in E_2$ then $\bar{p}_*\partial_*\alpha_k = 0$ and so $\partial_*\alpha_k = i_*\beta_{k-2}$ for some β_{k-2} and $\delta_2\alpha_k = \beta_{k-2}$. To determine

*In many places it is more convenient to index by a prefix and the symbol α_k and ${}_k\alpha$ are to be identified. The latter appears in the tables.

2.3.1

$$\begin{array}{ccccccc}
P_n^{n+k-1} & \to & P_n^{n+k} & \to & S^{n+k} & \to & \Sigma P_n^{n+k-1} & & \\
\downarrow & & \downarrow & \nearrow & & \searrow^{\partial} & \downarrow & & \\
P_{n+k-2}^{n+k-1} & > & P_{n+k-2}^{n+k} & & & & \Sigma P_{n+k-2}^{n+k-1} & \xrightarrow{\bar{p}} & S^{n+k} \\
& & & & & & \uparrow \Sigma \bar{i} & & \\
& & & & & & S^{n+k-1} = \Sigma P_{n+k-2}^{n+k-2} & &
\end{array}$$

β_{k-2} observe that if $n + k \equiv 0(2)$ we have

$$\begin{array}{ccccccc}
P_{n+k-2}^{n+k-1} & \to & P_{n+k-2}^{n+k} & \to & S^{n+k} & & \\
\bar{i}\uparrow\downarrow & & \downarrow & & f\downarrow & \searrow & \\
S^{n+k-2} & \to & CP_{(n+k-2)/2}^{(n+k)/2} & \xrightarrow{p} & S^{n+k} & \xrightarrow{\bar{\partial}} & S^{n+k-1},
\end{array}$$

and $\bar{i}\,\bar{\partial}\,f$ is just ∂ of 2.3.1 restricted to image of $\bar{i}$. But $\bar{\partial}_*\alpha = h_1\alpha$ if $n + k \equiv 0(4)$ and 0 if $n + k \equiv 2(4)$ by [2:2.6.1]. On the other hand, if $n+k \equiv 1(4)$ we have the James map [15] giving

$$\begin{array}{ccccccc}
P_{n+k-2}^{n+k-1} & \to & P_{n+k-2}^{n+k} & \to & S^{n+k} & \to & \Sigma P_{n+k-2}^{n+k-1} \\
\uparrow & & \uparrow & \nearrow & & \searrow^{\bar{\partial}} & \uparrow \bar{i} \\
S^{n+k-2} & \to & \Sigma CP_{(n+k-2)/2}^{(n+k-1)/2} & & & & S^{n+k-1}
\end{array}$$

Now $\bar{i}\,\bar{\partial} = \partial$ and so $\delta_2{}'\alpha_k = h_1\alpha_{k-2}$, $n + k \equiv 1(4)$

$= 0$, $\quad n + k \equiv 3(4)$

and this completes the proof.

PROPOSITION 2.4.
$$\begin{aligned}
\delta_3{}'\alpha_k &= \langle h_0, h_1, \alpha\rangle_{k-3} & k+n &\equiv 0(4) \\
&= \langle h_1, h_0, \alpha\rangle_{k-k} & k+n &\equiv 2(4) \\
&= 0 & k+n &\equiv 1,3(4).
\end{aligned}$$

Proof. Consider the sequence

$$\begin{array}{ccccccc}
P_{n+k-3}^{n+k} & \xrightarrow{p_3} & P_{n+k-2}^{n+k} & \xrightarrow{p_2} & P_{n+k-1}^{n+k} & \xrightarrow{p_1} & S^{n+k} \\
\uparrow & \swarrow_{\partial} & & & & & \\
S^{n+k-3} & & & & & &
\end{array}$$

Applying Ext to this diagram we get the following diagram.

If $\alpha_k \in \mathrm{Ext}^{s,t}(S^{n+k})$ is in E_3 of the spectral sequence then there is an $\bar{\alpha}_k \in \mathrm{Ext}^{s,t}(P_{n+k-2}^{n+k})$ such that $p_{1*}p_{2*}\bar{\alpha}_k = \alpha_k$. Also it is clear from the

$$\begin{array}{ccccccc}
 & & \mathrm{Ext}^{s+1,t}(S^{n+k-3}) & & & & \\
 & & \uparrow \delta & & & & \\
\mathrm{Ext}_A^{s,t}(\tilde{H}^*(P^{n+k}_{n+k-3}),Z_2) & \xrightarrow{p_{3*}} & \mathrm{Ext}^{s,t}(P^{n+k}_{n+k-2}) & \xrightarrow{p_{2*}} & \mathrm{Ext}^{s,t}(P^{n+k}_{n+k-1}) & \xrightarrow{p_{1*}} & \mathrm{Ext}^{s,t}(S^{n+k}) \\
\uparrow & & & & & & \\
\mathrm{Ext}^{s,t}(S^{n+k-3}). & & & & & &
\end{array}$$

definition that $\delta_3'\alpha_k = \delta\bar{\alpha}_k$. Now suppose $n+k \equiv 2(4)$. Then $p_1{*}\langle 1_{k-1},h_0,\alpha\rangle = \alpha_k$ and the Massey product is defined since $h_0\alpha_k = \delta_1'\alpha_k = 0$ because $\alpha_k \in E_2$. There is a class $\bar{1}_{k-1}$ such that $p_2{*}\bar{1}_{k-1} = 1_{k-1}$ and $h_0\bar{1}_{k-1} = 0$ and so we can form the product $\langle\bar{1}_{k-1},h_0,\alpha\rangle \in \mathrm{Ext}^{s,t}(P^{n+k}_{n+k-2})$. Now $\delta\langle\bar{1}_{k-1},h_0,\alpha\rangle = \langle\delta\bar{1}_{k-1},h_0,\alpha\rangle = \langle h_1,h_0,\alpha\rangle_{k-3}$ by the argument used in the proof of 2.3.

Now suppose $n+k \equiv 0(4)$. Then $p_1{*}\langle 1_{k-1},h_0,\alpha\rangle = \alpha_k$. As above there is a $\bar{1}_{k-1}$ such that $p_2{*}\bar{1}_{k-1} = 1_{k-1}$ but a simple direct calculation shows $h_0\bar{1}_{k-1} = h_1 1_{k-2} \neq 0$ since this formula is an immediate consequence of $Sq^2\alpha^{n+k-2} = Sq^1\alpha^{n+k-1} = \alpha^{n+k}$, $n+k \equiv 0(4)$.

$$p_2{*}\left\langle \begin{matrix} 1_{k-1},h_0, \\ 1_{k-2},h_1, \end{matrix}\ \alpha\right\rangle = \langle 1_{k-1},h_0,\alpha\rangle$$

and the left side exists since α_k is in E^3. Now

$$\delta\left\langle \begin{matrix} \bar{1}_{k-1},h_0, \\ 1_{k-2},h_1, \end{matrix}\ \alpha\right\rangle = \left\langle \begin{matrix} \delta\bar{1}_{k-1},h_0, \\ \delta 1_{k-2},h_1, \end{matrix}\ \alpha\right\rangle = \left\langle \begin{matrix} 0,h_0 \\ h_0,h_1, \end{matrix}\ \alpha\right\rangle$$
$$= \langle h_0,h_1,\alpha\rangle .$$

Since $P^{n+k}_{n+k-3} = S^{n+k-3} \vee P^{n+k-1}_{n+k-1} \vee S^{n+k}$ if $n+k \equiv 3(4)$, δ_3' is zero in this congruence. Finally suppose $n+k \equiv 1(4)$. We have, using the James map,

$$\begin{array}{ccccc}
P^{n+k}_{n+k-3} & & & \xrightarrow{\partial} & \Sigma P^{n+k-1}_{n+k-3} \\
\uparrow & \searrow & & \nearrow & \uparrow \bar{1} \\
 & & S^{n+k} & & \\
 & \nearrow & & \searrow & \\
\Sigma CP^{(n+k-1)/2}_{(n+k-2)/2} & & & \xrightarrow{\bar{\partial}} & S^{n+k-1}
\end{array}$$

Now δ_3 is defined by looking at $i\colon S^{n+k-2} \to \Sigma P^{n+k-1}_{n+k-3}$ and comparing im i_* with im ∂_*. Since im $i_* \cap$ im $\bar{1}_* = \{0\}$ and $\partial_* = (\bar{1}\bar{\partial})_*$, we see that im $\partial_* \cap$ im $i_* = \{0\}$ or $\delta_3' = 0$. This completes the proof.

PROPOSITION 2.5. $\delta_4{}'\alpha_k = h_2\alpha_k \quad k+n \equiv 0,1,2,3(8)$

$= 0 \quad k+n \equiv 4,5,6,7(8).$

The proof of this proposition follows closely the proof of 2.3 and we leave it to the reader.

This much of the computation is sufficient to get all of the calculations of Paechter [22].

3. SOME ALGEBRA EXTENSIONS.

Rather than continuing the step by step computations of the preceding section it is useful to recover some of the module structure and to use it to get further differentials.

PROPOSITION 3.1. Let $n+k \equiv 0(4)$. Then in $\mathrm{Ext}_A^{0,n+k-1}(\tilde{H}^*(P^{n+k}_{n+k-2}),Z_2)$ there is an element, 1_{k-1} and $h_0 1_{k-1} = h_1 1_{k-2}$.

Proof. It is easy to see that a basis of the Steenrod algebra for $\tilde{H}^*(P^{n+k}_{n+k-2})$ is given by α^{n+k-2} and α^{n+k-1}. The class represented by α^{n+k-1} in $\mathrm{Ext}_A(H^*(P^{n+k}_{n+k-2}),Z_2)$ is 1_{k-1}. Since $Sq^1\alpha^{n+k-1} = Sq^2\alpha^{n+k-2} = \alpha^{n+k}$ we see $h_0 1_{k-1} = h_1 1_{k-2}$.

PROPOSITION 3.2. Let $n+k \equiv 0(2)$. Then in

$$\mathrm{Ext}^{2,n+k+4}(\tilde{H}^*(P^{n+k}_{n+k-1}),Z_2),\ 1_{k-1}h_1{}^2 = \langle 1_{k-1},h_0,h_1\rangle h_0.$$

The proof is obvious in this context.

PROPOSITION 3.3. Let $n+k \equiv 1(4)$. Then in $\mathrm{Ext}^{2,n+k+6}(\tilde{H}(P^{n+k}_{n+k-2}),Z_2)$ there is a class β such that $h_0\beta \neq 0$ and

a) under $p: P^{n+k}_{n+k-2} \to S^{n+k}$, $p_*\beta = 1_k h_1{}^2$

b) under $p: P^{n+k}_{n+k-2} \to S^{n+k} \vee S^{n+k-1}$, $p_*\beta h_0 = 1_{k-1}h_1{}^3$.

Proof. Consider the sequence

$$S^{n+k-2} \to P^{n+k}_{n+k-2} \to S^{n+k} \vee S^{n+k-1}.$$

Applying the Ext functor we get a long exact sequence where $\delta 1_k = 1_{k-2}h_1$ and $\delta 1_{k-1} = 1_{k-2}h_0$ since $Sq^2\alpha^{n+k-2} = \alpha^{n+k}$ and $Sq^1\alpha^{n+k-2} = \alpha^{n+k-1}$. Hence $\delta(1_{k-1}h_0h_2 + 1_k h_1{}^2) = 0$ and this defines β which satisfies the proposition.

PROPOSITION 3.4. Let $n+k \equiv 1(4)$. Let $\alpha_k \in \mathrm{Ext}_A(\tilde{H}^*(S^{n+k}), Z_2)$ satisfy: $h_1\alpha_k = h_0\alpha_k = 0$. Then $\langle 1_{k-2}, h_1, \alpha\rangle$ projects to α_k under $P^{n+k}_{n+k-2} \to S^{n+k}$ and $\langle 1_{k-2}, h_1, \alpha\rangle h_0 = \langle h_1, \alpha, h_0\rangle_k$.

The proof is clear in view of the argument used for 3.3.

PROPOSITION 3.5. Let $n+k \equiv 0(8)$. In $\mathrm{Ext}(H^*(P^{n+k}_{n+k-4}), Z_2)$, $1_{k-1}h_0 = 1_{k-4}h_2$.

Proof. As before this follows directly from $Sq^1\alpha^{n+k-1} = Sq^4\alpha^{n+k-4}$ for the given congruence.

PROPOSITION 3.6. If $n+k \equiv 5(8)$ then $1_k h_2 = 1_{k+2}h_1$ in Ext for $\tilde{H}^*(P^{n+k+4}_{n+k})$.

Proof. It is sufficient to verify that $Sq^4\alpha^{n+k} = Sq^2\alpha^{n+k+2}$.

PROPOSITION 3.7. a) If $n \equiv 3(4)$ there is a class $j_2 \in \mathrm{Ext}^{7,n+k+33}(H^*(P^{n+2}_n), Z_2)$ such that in $S^n \xrightarrow[i]{} P^{n+2}_n \xrightarrow[p]{} S^{n+2}$, $p_*(j_2) = j$ and $h_0(j_2) = i_*P^1g$.

b) If $n \equiv 3(4)$ there is a class $i_2 \in \mathrm{Ext}^{7,n+k+30}(H^*(P^{n+2}_n), Z_2)$ such that $p_*i_2 = i$ and $h_0(i_2) = i_*P^1e_0$.

PROPOSITION 3.8. a) In Ext for P^{n+3}_n, $n \equiv 1(4)$ there is a class $(h_0h_2g)_3$ such that in $S^n \xrightarrow[i]{} P_n \xrightarrow[p]{} S^{n+3}$, $p_*(h_0h_2g)_3 = h_0h_2g$ and $h_0(h_0h_2g)_3 = i_*(j)$.

b) With the same data there is a class $(h_0{}^2g)_3$ such that $p_*(h_0{}^2g)_3 = h_0{}^2g$ and $h_0(h_0{}^2g)_3 = i_*(i)$.

Proof of 3.7 and 3.8. Consider the sequence

$$\begin{array}{ccccccccccc}
 & & & & S^5 & \xrightarrow{q_6} & \cdots & \xrightarrow{q_9} & P_5{}^9 & & \\
 & & & & \uparrow{\scriptstyle p_2} & & & & \uparrow{\scriptstyle p_6} & & \\
P_3{}^3 & \xrightarrow[i_1]{} & P_3{}^4 & \xrightarrow[i_2]{} & P_3{}^5 & \xrightarrow[i_3]{} & \cdots & \xrightarrow[i_7]{} & P_3{}^9 & \xrightarrow[i]{} & S^9
\end{array}$$

where the integers are intended to represent congruence classes mod 4 of $n+k$. By the computations made already and by the proof of the first part of proposition 4.2 (which does not use these propositions) we see that there is a class $(h_1{}^2)_9$ such that $i_*(h_1{}^2)_9 = h_1{}^2$ and $h_0{}^2(h_1{}^2)_9 = i_6 \dots i_{2*}\langle 1_3, h_0{}^4, h_3\rangle$. The class $\langle 1_3, h_0{}^4, h_3\rangle$ has the property that if $h_3\alpha = 0$ then $\langle 1_3, h_0{}^4, h_3\rangle\alpha = i_1{}^*P^1\alpha$. If we multiply $(h_1{}^2)_9$ by g then $(h_1{}^2)_9h_0{}^2g = i_6 \dots i_1{}^*(P^1g)$. Of course we cannot be sure that $(h_1^2)_9 h_0^2 g \neq 0$. In Ext for P_3^8 there is a class $(h_1^3)_8$ such that $i_{6*}(h_1^3)_8 = (h_1^2)_9 h_0$. Now $(h_1^3)_8 h_0 g = i_5 \ldots i_1{}^*(P^1 g)$ and by inspection this map is non-zero, since only $j \in E^{7,39}(S^6)$ could "kill" $(P^1 g)_3$ and it kills $h_0 j$. But this implies $i_5 \ldots i_1{}^*(P^1 g)$ can be divided by h_0. Again by inspection only j_5 satisfying $i_5{}^* j_5 = j$ could satisfy $i_5 \ldots i_3{}^*(j_5) h_0 = i_5 \ldots i_1{}^* P^1 g$. This proves 3.7 a) and in a similar fashion using e_0 instead of g proves 3.8 a).

We have shown that $q_9 \ldots q_6{}^* j = h_0 g P_6{}^*(h_1^2)_q$, i.e., $q_9 \ldots q_6{}^* j$ can be divided by h_0. A quick inspection of table 8.1 shows that the only possibility is $(h_0 h_2 g)_8$. This gives 3.7 b) and the same argument using e_0 and i shows 3.8 b).

PROPOSITION 3.9. If $n+k \equiv 15(16)$ then $h_0 l_k = h_3 l_{k-7}$ (together with what 3.5 implies) in Ext for P_{n+k-7}^{n+k+1}.

Proof. This is clear since $Sq^8 \alpha^{n+k-7} = \alpha^{n+k+1} = Sq^1 \alpha^{n+k}$ for this congruence.

PROPOSITION 3.10. $h_0(h_1 h_3)_k = (h_0 h_3^2)_{k-6}$ for $k+n \equiv 6(8)$.

Proof. Consider the sequence

$$S^{n+k-7} \to P_{n+k-7}^{n+k+\varepsilon} \xrightarrow{p} P_{n+k-6}^{n+k+\varepsilon},$$

with $\varepsilon = 0$ or 2, and for $k+n \equiv 6(15)$. Now $P_{n+k-6}^{n+k+2} = S^{n+k-6} \vee P_{n+k-5}^{n+k-2}$, $\delta l_{k-6} = h_0 l_{k-7}$ and $\delta l_{k+1} = h_3 l_{k-7}$. Hence $\delta(h_0 h_3 l_{k+1} + h_3^2 l_{k-6}) = 0$. Let β satisfy $p_*\beta = (h_0 h_3 l_{k+1} + h_3^2 l_{k-6})$. Then $p_* h_0 \beta = h_0 h_3^2 l_{k-6}$. Since $h_0 h_3 l_{k+1} = (h_1 h_3)_k$ the proposition is established for $n+k \equiv 6(15)$. Since it only involves six cells periodicity completes the proof.

PROPOSITION 3.11. If $n+k \equiv 3(8)$, then $h_2^2 l_k = h_1 (h_2)_{k+2}$.

Proof. Consider the sequence

$$P_{n+k}^{n+k+1} \xrightarrow{i} P_{n+k}^{n+k+2} \xrightarrow{p} S^{n+k+2}.$$

Then $p_* \langle l_k, h_1, h_2 \rangle = h_2$ so $\langle l_k, h_1, h_2 \rangle = (h_2)_k$. Now $\langle l_k, h_1, h_2 \rangle h_1 = i_* h_2^2$.

PROPOSITION 3.12. If $n+k \equiv 7(8)$ then $h_1 (h_2^2)_{k+4} = (h_2^3)_{k+2}$.

Proof. Consider the sequence

$$P^{n+k+1}_{n+k} \xrightarrow{i} P^{n+k+4}_{n+k} \xrightarrow{p} P^{n+k+4}_{n+k+2}.$$

By 2.5 we see $\delta 1_{k+4} = h_2 1_k$ while $\delta 1_{k+2} = h_1 1_k$. Hence $\delta(h_3{}^2)_{k+4} = (h_2{}^3)_k = \delta((h_1h_3)_{k+2})$. Hence $\delta[(h_2{}^2)_{k+4} + (h_1h_3)_{k+2}] = 0$. This class will represent $(h_2{}^2)_{k+4}$ in the spectral sequence and $h_1[(h_2{}^2)_{k+4} + (h_1h_3)_{k+2}] = (h_1{}^2h_3)_{k+2} = (h_2{}^3)_{k+2}$.

PROPOSITION 3.13. If $k+n \equiv 5(8)$ then $(h_3{}^2)_{k+4}h_1 = (c_1)_k$.

Proof. Consider the sequence

$$S^{k+n} \xrightarrow{i} P^{k+n+4}_{k+n} \xrightarrow{p} P^{k+n+4}_{k+n+1}.$$

Then $p_* \langle 1_k, h_2, h_3{}^2\rangle = (h_3{}^2)_{k+4}$ while $\langle 1_k, h_2, h_3{}^2\rangle h_1 = i_* \langle h_2, h_3{}^2, h_1\rangle = c_1$.

An argument similar to 3.11 gives

PROPOSITION 3.14. If $n+k \equiv 3(8)$, then $h_1(c_1)_{k+2} = (h_2c_1)_k$.

PROPOSITION 3.15. a) If $k+n \equiv 3(8)$ then $h_0(h_3{}^2)_k = (c_1)_{k-5}$.

b) If $k+n \equiv 6(8)$ then $h_0(c_1)_k = (h_2c_1)_{k-3}$.

Proof. Consider the sequence

$$S^{n+k-5} \xrightarrow{i} P^{n+k}_{n+k-5} \xrightarrow{p} P^{n+k}_{n+k-4} \xrightarrow{\bar{p}} S^{n+k}.$$

Then $\bar{p}_* p_* \langle 1_{n+k-4}, h_2, h_3{}^2\rangle = h_3{}^2$. Now $\langle 1_{n+k-4}, h_2, h_3{}^2\rangle = \langle 1_{n+k-4}, h_2, h_3\rangle h_3$ and $h_0 \langle 1_{n+k-4}, h_2, h_3\rangle = \langle 1_{n+k-5}, h_2, h_1h_3\rangle$. Multiplication by h_3 completes the proof. The proof of b is easy and similar.

4. The determination of $\delta_5{}'$ seems to be more complicated and some special attention is required. By inspection of table 8.1 we see that the only possibilities are:

(a) $(P^jh_0{}^ih_2)_k \to (P^jh_0{}^{i+1}h_3)_{k-5}$

(b) $(P^jh_0{}^3h_3)_k \to (P^jh_2)_{k-5}$

(c) $(h_0h_3{}^2)_k \to (h_0{}^2h_2h_4)_{k-5}$

(d) $(h_0h_3{}^2)_k \to (f_0)_{k-5}$

(e) $(P^id_0)_k \to (P^ih_0f_0)_{k-5}$

(f) $(P^i e_0)_k \longrightarrow (P^i h_1 g)_{k-5}$

Because of the parity either one side or the other of (a) and (c) is zero for each $k = E_5$; hence (a) and (c) contribute nothing. A check of the previous propositions shows that both sides of (b) are present in E_5 only if $n+k \equiv 4(8)$. Again a check of the previous propositions shows that both sides of (d) are defined when $k+n \equiv 0,2,4(8)$. Both sides of (e) are defined when $n+k \equiv 3(4)$ and finally both sides of (f) are present only if $n+k \equiv 1(2)$.

With these data we will prove

PROPOSITION 4.1. i) $\delta_5'(P^j h_0{}^3 h_3)_k = (P^j h_2)_{k-5}$, $k+n \equiv 4(8)$;

ii) $\delta_5'(h_0 h_3{}^2)_k = (f_0)_{k-5}$, $k+n \equiv 2,4(8)$;

iii) $\delta_5'(P^i d_0)_k = (P^i h_0 f_0)_{k-5}$, $k+n \equiv 3(8)$;

iv) $\delta_5'(P^i e_0)_k \equiv (P^i h_1 g)_{k-5}$, $k+n \equiv 1,3(8)$;

and δ_5' is zero on all other classes.

Proof. i) Consider the sequence

$$\begin{array}{ccccc} & & & & S^{n+k} \\ & & & & \uparrow \bar{p} \\ S^{n+k-5} & \xrightarrow{i} & P^{n+k}_{n+k-5} & \xrightarrow{p} & P^{n+k}_{n+k-4} \end{array}$$

In Ext for $\tilde{H}^*(P^{n+k}_{n+k-4})$ we can form $\langle 1_{k-1}, h_0{}^4, h_3\rangle = \beta$ and $\bar{p}_*\beta = h_0{}^3 h_3$. Since $\delta_* 1_{k-1} = 1_{k-5} h_2$ or zero if $n+k \equiv 4(8)$ or $\equiv 0(8)$ respectively i) is established for $j = 0$. The periodicity operator is defined on β giving $\bar{p}_* P^j \beta = P^j h_0{}^3 h_3$. Thus i) is established.

ii) Consider the diagram

$$\begin{array}{ccccccc} & \overset{i}{\nearrow} & \Sigma CP^{(n+k-4)/2}_{(n+k-6)/2} & \longrightarrow & \Sigma CP^{(n+k-2)/2}_{(n+k-6)/2} & \longrightarrow & S^{n+k-1} \\ & & \downarrow & & | & & \downarrow \\ S^{n+k-5} & \longrightarrow & P^{n+k-3}_{n+k-5} & \longrightarrow & P^{n+k}_{n+k-5} & \xrightarrow{p} & P^{n+k}_{n+k-2} \\ & & & & & & \downarrow \bar{p} \\ & & & & & & S^{n+k} \end{array}$$

In Ext for $\tilde{H}^*(P^{n+k}_{n+k-2})$ we can form $\langle 1_{k-1}, h_0{}^2 h_3, h_3\rangle = \beta$ and $\bar{p}_*\beta = h_0 h_3{}^2$. There are three cases. If $n+k \equiv 0(5)$ then $\delta_* 1_{k-1}$ in the top sequence is zero and

g) $h_1^2h_4 \to h_2C_1$

h) $f_0 \to h_2g$

i) $h_0f_0 \to h_0h_2g$

j) $h_1g \to h_2^2g$

k) $i \to P^1g$

Next, by checking both sides against earlier differentials we get the following table.

Formula	occurs when
a	$n+k \equiv 2 \bmod 8$
b, $i = 0$	never
b, $i > 0$	$n+k \equiv 3 \bmod 8$
c	never
d	$n+k \equiv 3,5 \pmod 8$
e and f	$n+k \equiv 2 \bmod 4$
g	$n+k \equiv 1 \bmod 8$
h	$n+k \equiv 3 \bmod 8$
i	$n+k \equiv 0,2,4 \pmod 8$
j	$n+k \equiv 0,1,4 \pmod 8$
k	$n+k \equiv 1 \bmod 2$

PROPOSITION 4.2. i) $\delta_6{}'h_1 = h_2^2$ $\quad n+k \equiv 2(8)$

$\delta_6{}'h_1d_0 = h_0^2g$

$\delta_6{}'h_1h_4 = h_3^3$

ii) $\delta_6{}'h_3^2 = C_1$ $\quad n+k \equiv 5(8)$

iii) $\delta_6{}'f_0 = h_2g$ $\quad n+k \equiv 3(8)$

iv) $\delta_6{}'h_0f_0 = h_0h_2g$ $\quad n+k \equiv 0,2(8)$

v) $\delta_6{}'P^ih_2 = P^{i-1}h_1d_0 = P^i(h_1h_3)^{i>0}$ $\quad n+k \equiv 3(8)$

vi) $\delta_6{}'h_1g = h_2^2g$ $\quad n+k \equiv 0,1(8)$

vii) $\delta_6{}'i = P^1g$ $\quad n+k \equiv 3,5(8).$

The rest are zero.

Proof. The three parts of i) are equivalent. Consider the sequence

$$P_{n+k-6}^{n+k-1} \to P_{n+k-6}^{n+k+2} \to P_{n+k}^{n+k+2}$$

where $n+k \equiv 2(8)$. In Ext for P_{n+k}^{n+k+2}, $h_0 l_{k+1} = h_1 l_k$ by 3.1. In the long exact sequence of Ext for this cofibration $\delta l_{k+1} = h_2 l_{k-3}$. Now in Ext for P_{n+k-6}^{n+k-1}, $h_0 l_{k-3} = h_2 l_{n+k-6}$ by 3.5. Hence $\delta h_0 l_{k+1} = \delta(h_1 l_k) = h_0 h_2 l_{k-3} = h_2^2 l_{n+k-6}$. This is i) but since $h_0^2 g = h_2^2 d_0$ and $h_3^3 = h_2^2 h_4$ this implies the other cases too. For cases ii) and iv) the congruence $n+k \equiv 6(8)$ is easily settled since l_{k+1} pulls back to Ext for P_{n+k-6}^{n+k-2} and so $h_0 d_0 l_{k+1}$ pulls back. This implies that $\delta_6'(h_1 d_0) = \delta_6'(h_1 h_4) = 0$.

This is also a good time to verify that $\delta_6'(h_1^2 h_4)_k = 0$ if $n+k \equiv 1(8)$. From the above discussion and 3.2 it is clear that $\delta_6'(h_1^2 h_4)_k = h_1(\delta_6' h_1 h_4)_{k-1}$ and $h_1 h_3^3 = 0$.

We now will prove ii).

$$\begin{array}{ccccccc} P^{n+k}_{n+k-6} & \xrightarrow{p_3} & P^{n+k}_{n+k-5} & \xrightarrow{p_2} & P^{n+k}_{n+k-2} & \xrightarrow{p_1} & S^{n+k} \\ \uparrow & \swarrow & & & & & \\ S^{n+k-6} & & & & & & \end{array}$$

If $n+k \equiv 5(8)$ then $p_1^* p_2^* \langle l_{n+k-2}, h_1 h_3^2\rangle = h_3^2$. But $\delta \langle l_{n+k-2}, h_1, h_3^2\rangle = \langle h_2, h_1, h_3^2\rangle = C_1$. If $n+k \equiv 3(8)$, then $p_1 p_2 p_3^* \langle l_{n+k-4}, h_2, h_3^2\rangle = h_3^2$ and so $\delta_6' h_3^2 = 0$ in this case.

To see iii) consider

$$\begin{array}{ccccc} & & P^{n+k-5}_{n+k-6} & & \\ & \swarrow & & \nwarrow & \\ P^{n+k}_{n+k-6} & \longrightarrow & P^{n+k}_{n+k-4} & \xrightarrow{p_1} & S^{n+k}. \end{array}$$

If $n+k \equiv 3(8)$ then $p_1^* \langle l_{k-4}, \begin{smallmatrix} h_2, f_0 \\ h_1, g \end{smallmatrix}\rangle = f_0$ since $h_1 g = h_2 f_0$. Now using 3.6 we see that $\delta \langle l_{k-4}, \begin{smallmatrix} h_2, f_0 \\ h_1, g \end{smallmatrix}\rangle = h_2 g$. If $n+k \equiv 7(8)$ then l_{n+k} pulls back to $\tilde{H}^*(P^{n+k}_{n+k-6})$ and so $\delta_6 f_0$ is zero. vi) is similar using 3.2 and $\langle h_2, h_1, P^1 h_2\rangle = h_1^2 d_0$.

To see iv) observe that if $n+k \equiv 2(8)$ iv) follows from i), indeed $h_0 f_0 = h_1 e_0$ and $h_0 h_2 g = h_2^2 e_0$. Consider the diagram below.

If $n+k \equiv 0(8)$ then $p_* \langle \begin{smallmatrix} l_{k-1}, h_0 \\ l_{k-4}, h_2 \end{smallmatrix}, h_1 e_0\rangle = h_1 e_0$ by 3.5. Finally,

$$\begin{array}{ccccc} P^{n+k}_{n+k-6} & \longrightarrow & P^{n+k}_{n+k-4} & \xrightarrow{p} & S^{n+k} \\ & \nwarrow & \swarrow \delta & & \\ & & P^{n+k-5}_{n+k-6} & & \end{array}$$

$\delta < \begin{matrix} 1_{k-1}, h_0, \\ 1_{k-4}, h_2 \end{matrix} h_1 e_0 > = < h_1, h_2, h_1 e_0 > = h_2{}^2 e_0$. On the other hand if $n+k \equiv 4(8)$ we get zero.

To see vi) we argue in a similar fashion using $< h_1, h_2, h_1 > = h_2{}^2$.

The last one, vii), requires a new argument. Consider the sequence

$$P^{n+k-3}_{n+k-7} \rightarrow P^{n+k}_{n+k-6} \rightarrow P^{n+k}_{n+k-2}$$

with $n+k \equiv 5(8)$. By 3.7(b) there is a class i_k in Ext for P^{n+k}_{n+k-3} and $h_0 i_k = (P^1 e_0)_{k-2}$. By 4.1.iv $\delta_5 (P^1 e_0)_{k-2} = (P^1 h_1 g)_{k-7} = h_0{}^2 k_{k-7}$. Hence $\delta_*(i_k) = a$ where $h_0 a = P^1 h_1 g_{k-7}$. Using 3.1 this gives $a = (P^1 g)_{k-6}$ which completes the proof. The same argument shows that $\delta_6{}'(i) = 0$ if $n+k \equiv 1(8)$. A similar discussion handles the cases $n+k \equiv 3$ and $7(8)$.

A check of table 8.1 together with a comparison of the earlier differentials gives the following table as the only possible for $\delta_7{}'$.

a)	$h_1{}^2 \rightarrow c_0$	$n+k \equiv 1(4)$
b)	$h_1{}^3 \rightarrow h_1 c_0$	$n+k \equiv 0(4)$
c)	$h_1 h_3 \rightarrow h_0 h_3{}^2$	never
d)	$c_0 \longrightarrow d_0$	6(8)
e)	$h_1 c_0 \rightarrow h_1 d_0$	5(8)
f)	$h_0 h_3{}^2 \rightarrow g$	6(8)
g)	$e_0 \longrightarrow h_2 g$	never
h)	$f_0 \longrightarrow h_1 h_4 c_0$	never
i)	$h_0{}^2 g \rightarrow j$	0,6(8)

PROPOSITION 4.3. i) $\delta_7{}' h_1{}^2 = c_0$ $\quad n+k \equiv 1(8)$

ii) $\delta_7{}' h_1{}^3 = h_1 c_0$ $\quad n+k \equiv 0(8)$

iii) $\delta_7{}' c_0 = d_0$ $\quad n+k \equiv 6(8)$

iv) $\delta_7'h_1c_0 = h_1d_0 \qquad n+k \equiv 5(8)$

v) $\delta_7'h_0h_3^2 = g \qquad n+k \equiv 6(8)$

vi) $\delta_7'h_0^2 g = j \qquad n+k \equiv 6,0(8)$

Proof. The proofs of all of these are similar and are based on "multiplication by h_0" considerations. We will prove only i). Consider the sequence

$$P^{n+k-1}_{n+k-7} \to P^{n+k+1}_{n+k-7} \to P^{n+k+1}_{n+k}$$

with $n+k \equiv 1(8)$. Then $h_0(h_1)_{k+1} = (h_1^2)_k$. Since $\delta_6'(h_1)_{k+1} = (h_2^2)_{k-5}$ and $h_0(h_2^2)_{k-5} = c_0$ by 3.4, $\delta_7'(h_1^2)_k = c_0$.

5. It is now convenient to group together all the differentials from δ_8' to δ_{15}'. Table 5.1 gives the listing of all possible differential homomorphisms as they would appear in E_8 of the spectral sequence.

8	9	10	11
$h_{3,3} \to h_3^2$	$h^3_{2,0} \to e_0$	$h_0h^2_{3,0} \to h_4c_0$	
$h^2_{3,3} \to h_3^3$	$h_1^3 d_{0,0} \to P^1e_0$		
$l_7 \to h_3$			
$h_{3,7} \to h_3^2$			
$h_{1,6} \to h_1h_3$			
$e_{0,5} \to h_1h_4c_0$			
$h_1^2 d_{0,1} \to i$			

12	13	14	15
$h_{2,5} \to h_3^2$	$h^2_{1,5} \to h_0h_3^2$	$h^2_{2,3} \to c_1$	$h^2_{2,1} \to h_4c_0$
$h_1h_{3,2} \to c_1$		$h^3_{2,1} \to h_2c_1$	$h^3_{1,4} \to e_0$
$h_0^3 h_{3,0} \to h_1e_0$		$h^2_{2,0} \to h_2c_1$	$P^1h^3_{1,4} \to P^1e_0$
		$h_1h_{3,6} \to h_3^3$	
		$P^1h^2_{1,5} \to i$	

Table 5.1

All possible differentials between 8 and 15.

The second subscript indicates the congruence class of $n+k$ mod 8.

PROPOSITION 5.2. i) $\delta_8{}'\alpha_k = \alpha_{k-8}h_3$ $\quad k+n \equiv 0,1,\ldots,7(16)$

ii) $\delta_8{}'e_{0,k} = (C_0h_1h_4)_{k-8}$ $\quad k+n \equiv 13(16)$

iii) $\delta_8{}'h_1{}^2d_{0,k} = i_{k-8}$ $\quad k+n \equiv 1(8)$

and all other $\delta_8{}'$'s are zero.

Proof. Part i) follows immediately since $Sq^8\alpha^{n+k} = \alpha^{n+k+8}$ if $n+k$ satisfies the required congruence and is zero otherwise.

In order to prove ii) we need a little more.

LEMMA 5.3. i) $\delta_{14}'(h_1h_3)_k = h_3{}^3$ if $n+k \equiv 6(15)$

ii) $\delta_{10}'(h_0h_3{}^2)_k = h_4c_0$ if $n+k \equiv 0(15)$.

Proof. Consider the sequence

$$P^{n+k-6}_{n+k-15} \rightarrow P^{n+k}_{n+k-14} \rightarrow P^{n+k}_{n+k-5},$$

where $n+k \equiv 7(8)$. Now $h_0h_3l_k = h_1h_3l_{k-1}$ and since $\delta_8{}'l_k = h_3l_{k-8}$, $\delta_8{}'h_3l_k = h_3{}^2l_{k-8}$ and $\delta_8{}'(h_0h_3l)_k = l_{k-8}h_0h_3{}^2$ by 3.9 $h_0l_{k-8} = h_3l_{k-15}$. Multiplying both sides by $h_3{}^2$ we have $h_0h_3{}^2l_{k-8} = h_3{}^3l_{k-15}$. This proves i).

Consider the sequence

$$P^{n+k-7}_{n+k-16} \rightarrow P^{n+k}_{n+k-16} \rightarrow P^{n+k}_{n+k-6}$$

for $n+k \equiv 6(15)$. Now $\delta_8{}'(h_1h_3)_k = (h_3{}^3)_{k-14}$ but $h_0(h_4c_0)_{k-1} = (h_1h_4c_0)_{k-8}$. Hence $h_0(h_0h_3{}^2)_{k+3} \neq 0$. The only possibilities are $(e_0)_k$ and $(h_1{}^3h_4)_{k-1}$. Since $h_0{}^3(h_4)_{k-4} = (h_1{}^3h_4)_{k-1}$ the latter choice is incompatible with the other requirements.

Consider the sequence

$$P^{n+k-1}_{n+k-8} \rightarrow P^{n+k+1}_{n+k-8} \rightarrow P^{n+k+1}_{n+k}$$

for $n+k \equiv 1(8)$. For this congruence we have $\delta_6(h_1d_0)_{k+1} = (h_0{}^2g)_{k-5}$ by 4.2.i. By 3.8.6 $h_0(h_0{}^2g)_{k-5} = i_{k-8}$ and by 3.2 $h_0(h_1d_0)_{k+1} = (h_1{}^2d_0)_k$. Hence $\delta h_0(h_1d_0)_{k+1} = \delta(h_1{}^2d_0)_k = h_0(h_0{}^2g)_{k-5} = i_{k-8}$.

PROPOSITION 5.4. $\delta_9{}'(h_2{}^3) = e_0$ and $\delta_9{}'(h_1{}^3d_0)_k = P^1e_0$ for $k+n \equiv 0(8)$.

Proof. Consider the sequence

$$P_{n+k-9}^{n+k-1} \to P_{n+k-9}^{n+k+1} \to P_{n+k}^{n+k+1}$$

for $n+k \equiv 0(8)$. By 5.2.iii, $\delta(h_1{}^2 d_0)_{k+1} = (i)_{k-7}$. Combining 3.7b and 3.3 one easily obtains the result for $h_1{}^3 d_0$. Noticing that $P^1 h_2{}^3 = h_1{}^3 d_0$ completes the proof.

Observe that Lemma 5.3 settles δ'_{10} and there is no δ'_{11}.

PROPOSITION 5.5. a) $\delta'_{12}(h_2)_k = h_3{}^2$ $\quad k+n \equiv 5(16)$

b) $\delta'_{12}(h_1 h_3)_k = c_1$ $\quad k+n \equiv 2(16)$

c) $\delta'_{14}(h_2{}^2)_k = c_1$ $\quad k+n \equiv 3(16)$

d) $\delta'_{14}(h_2{}^3)_k = h_2 c_1$ $\quad k+n \equiv 1(16)$.

Proof. By inspection we have seen that all entries in the equation of 5.5 are present in E_{12} (and for those that pertain to it, in E_{14}). Propositions 3.13, 14 and 15 relate the right hand sides by multiplication by h_0 and h_1 which corresponds exactly to the way 3.11, 12 and h_3 multiplied by the result of 3.2 relate the left hand side. Hence to prove all the formulas we must only start it someplace. But $\delta_8{}'(h_3)_k = h_3{}^2$ if $k+n \equiv 3(16)$ does start it, i.e., consider

$$P_{n+k-13}^{n+k-2} \to P_{n+k-13}^{n+k} \to P_{n+k-1}^{n+k}, \qquad n+k \equiv 3(8).$$

Now $\delta(h_{3_k}) = (h_3{}^2)_{k-8}$ hence $\delta h_0(h_3)_k = \delta(h_1 h_3)_{k-1} = h(h_3{}^2)_{k-8} = (c_1)_{k-13}$. Similar arguments work for the other cases too.

<u>Remark</u>. A computation such as this is needed to compute the entire 23-stem as Barratt or Toda do it. From this point of view the result was difficult and was settled using [19]. In particular (a) implies $[i_{21}, \nu] \neq 0$. (More general calculations of this sort are given in Chapter V.)

PROPOSITION 5.6. a) $\delta'_{13}(h_1{}^2)_k = h_0 h_3{}^2$ $\quad n+k \equiv 5(16)$

b) $\delta'_{14}(P^1 h_1{}^2)_k = i$ $\quad n+k \equiv 13(16)$

c) $\delta'_{15}(h_1{}^3)_k = P^i e_0$ $\quad n+k \equiv (4+i8)(16)$

d) $\delta'_{12} P^i(h_0{}^3 h_3)_k = P^i(h_1 e_0)_{k-12}$ $\quad n+k \equiv (0+i8)(16)$

e) $\delta'_{12} P^i(h_1)_k = P^{i-1}(h_0{}^2 g)_{k-12}$ $\quad n+k \equiv (6+i8)(16)$.

Proof. These follow immediately from multiplication by h_0. Indeed, $h_0(h_1)_{k+1} = (h_1^2)_k$ if $n+k \equiv 5(16)$. But $\delta_9'(h_1)_{k+1} = (h_1h_3)_{k-8}$ but $h_0(h_1h_3)_{k-8} = h_0h_3^2$ by 3.10 and this gives (a). To prove (b) one uses 3.8(b) and so forth. Tables 8.2 - 8.16 give copies of E_{16}. The tables are explained in section 8.

6. THE ADAMS DIFFERENTIALS, 1.

Recall again that the tables are not really copies of Ext for the stunted projective spaces but just E_{16} of a pre-spectral sequence whose E_∞ term is associated with Ext. We will call them Ext anyway. The composite last differential of this pre-spectral sequence is called δ_1. The task of evaluating the Adams differentials is not as extensive as it might seem at first. The pre-spectral sequence has the additional advantage of grouping elements together into families. We will evaluate the differentials by making much use of this interplay between the various stunted projective spaces.

First observe that if two classes, α and β, in Ext for a sphere are related by an Adams differential and their image in Ext for P_k under i_* induced by $S^k \to P_k$ is non-zero, then their images under i_* will be related by an Adams differential too. This occurs frequently when $k \equiv 0(2)$.

PROPOSITION 6.1. a) $\delta_2(h_0h_3^2)_k = (h_1d_0)_{k-2}$ $\quad n+k \equiv 4(8)$

b) $\delta_2(h_0^2g)_k = (P^1d_0)_{k-3}$ $\quad n+k \equiv 6(8)$

c) $\delta_2(i)_k = (P^1h_1d_0)_{k-1}$ $\quad n+k \equiv 3(8)$

Proof. These three are grouped together because results of section 3 imply that whenever both sides are present in Ext the following equations hold:

$$h_0(h_0h_3^2)_k = (e_0)_{k-3} \qquad n+k \equiv 4(8)$$

$$h_0(P^ie_0)_k = P^i(h_1e_0)_{k-1} \qquad n+k \equiv 1(8)$$

$$h_0(h_1e_0)_k = (h_0^2g)_{k-2} \qquad n+k \equiv 0(8)$$

$$h_0(h_0^2g)_k = i_{k-3} \qquad n+k \equiv 6(8)$$

$$h_0(i)_k = (P^1e_0)_{k-2} \qquad n+k \equiv 3(8)$$

Multiplying d_0 on both sides of the equations given in 3.1, 2 and 3 we get similar module extensions for the right side of the equations in 6.1. Hence we must only prove a similar result someplace in the sequence to get everything else by naturality. But $\delta_2 e_0 = h_1^2 d_0$ in a sphere and this completes the proof. (Compare Ext for P_1.)

PROPOSITION 6.2. a) $\delta_2(h_1 g)_k = i_{k-3}$ $\qquad n+k \equiv 0(8)$

b) $\delta_2(h_0 h_2 g)_k = (P^1 e_0)_{k-3}$ $\qquad n+k \equiv 6(8)$

Proof. The argument for these is similar to the above. The two families which are related by multiplication by h_0 in a fashion similar to the above are:

$$h_1 g,\ h_0 h_2 g,\ j,\ P^1 g \quad \text{and} \quad h_1 e_0,\ h_0^2 g,\ i,\ P^1 e_0,\ P^1 h_1 e_0$$

with the first beginning with $h_1 g_k$, $n+k \equiv 0(8)$ and the second beginning with $(h_1 e_0)_k$, $n+k \equiv 2(8)$. Again all we must do is to prove a result someplace in the sequence to get the proposition. To do this we need the following lemma.

PROPOSITION 6.3. In P_{n+k} for $n+k \equiv 6(8)$ $\delta_3(h_1 g)_{2+k} = (P_1 d_0)_k$.

Proof. In [19] it is shown that $\{P_1 d_0\} = \eta^2\{g\}$. Since $i_*\eta = 2\iota_1$ in $\pi_*(P_{n+k})$ where $i: S^0 \to \Sigma^{-(n+k)} P_{n+k}$ and ι_1 is a generator of $\pi_1(\Sigma^{-(n+k)} P_{n+k})$, $i_*\eta^2 g = 0$. This implies that either $i_* P_1 d_0 = 0$ or $i_* P_1 d_0$ is a boundary. There are two possibilities, $\delta_3(h_1 g)_2$ or $\delta_4(f_0)_5$. Consider the sequence

$$S^6 \vee S^7 \xrightarrow{i} P_6{}^8 \longrightarrow P_6{}^8 \xrightarrow{p} S^8.$$

In the homotopy exact sequence $\delta_* 1_8 = \eta\iota + 2\iota_7$. Hence there is a class in Ext for $P_6{}^8$ which maps to $h_1 g$ under p_*. Call this class $(h_1 g)_2$. (It clearly corresponds to $(h_1 g)_2$ in homotopy, hence $(h_1 g)_2$ cannot be a cycle for all r.) The only possibility is $\delta_3(h_1 g)_2 = i_*(P_1 d_0)$. By naturality this completes the proof of 6.3.

Now we return to 6.2. Consider the map

$$P_5 \xrightarrow{p} P_6.$$

Clearly $p_*(h_1 g)_3 = (h_1 g)_2$. If $\delta_2(h_1 g)_3 = 0$ then $p_*(\delta_3 h_1 g)_3 = p_*(0) \neq (P_1 d_0)_0$ which contradicts 6.3. Hence $\delta_3(h_1 g)_3 = h_0^2(h_0 f_0)_5 = i_0$ in Ext for P_5. This completes the proof of 6.2.

PROPOSITION 6.4. a) $\delta_3(h_0^2 g)_k = (P_1^2 h_1 d_0)_{k-4}$ $\qquad n+k \equiv 0(8)$

b) $\delta_3(h_0 f_0)_k = (P_1 d_0)_{k-5}$ $\qquad n+k \equiv 2(8)$.

Proof. These two are further consequences of the peculiar group extension in $\pi_{23}(S^0)$. By [20] $4\nu\{g\} = \{h_1P_1d_0\}$, since $4i_*\{g\} = 0$ where $i: S^{n+k} \to P_{n+k}$, $n+k \equiv 0(8)$ is the usual inclusion. $i_*\{P_1h_1d_0\} = 0$. Hence $(P_1h_1d_0)_0$ is a boundary and the only possibility is $\delta_3(h_0{}^2g)_4$. The argument for b) is similar.

PROPOSITION 6.5. a) $\delta_2(P_{i+1}h_1h_3)_k = (P_ih_1d_0)_{k-7}$ $\quad n+k \equiv 0(8)$

b) $\delta_2(h_1{}^2d_0)_k = (P_1d_0)_{k-7}$ $\quad n+k \equiv 1(8)$.

Proof. We will first prove $\delta_2(P_2h_1{}^2h_3)_k = (P_1h_1d_0)_{k-1}$. Clearly, $i_*(4\nu\{g\}) = 0$ where $i: S^1 \to P_1$. Hence $i_*P_1h_1d_0$ is a boundary for some Adams differential. The only possibility is the one claimed. Since $h_0{}^2(h_0{}^2g)_5 = P^1(e_0)_0$ and $\delta_2P^1e_0 = P^1h_1{}^3d_0$ we can conclude:

6.6. $\quad h_1(h_0{}^2h_2d_0)_k = P^1(e_0)_{k-7}$ and

$h_1(h_1{}^2h_3)_k = (e_0)_{k-7}$ $\quad$ if $k+n \equiv 0(8)$.

Indeed the first statement is now clear but since $P^1(h_1{}^2h_3) = h_1{}^3d_0$ the second is clear too. Using the second we complete the proof of a). The argument for b) follows 6.3 in concept.

Implicit in the above calculations are a few other module extensions such as 6.6. Most of them are indicated in the tables.

PROPOSITION 6.7. a) $\delta_2(h_2{}^2g)_k = P^1g_{k-3}$ $\quad n+k \equiv 6(8)$

b) $\delta_2(h_0h_2g)_k = P^1g_{k-5}$ $\quad n+k \equiv 0(8)$.

Proof. Both of these involve arguments "off the page" in the sense that we will need to look at Ext for $t-s = 29$. First observe that $h_2(h_1g)_k = h_2{}^2g_{k-2}$ where $k+n \equiv 0(8)$. Indeed consider the map

$$P_5{}^8 \xrightarrow{\lambda} CP_3{}^4.$$

By construction $\lambda_*(h_0(h_1g)_8) = \lambda_*(h_0h_2g)_6 = \overline{(h_0h_2g)}_6$ where the barred classes indicate elements in Ext for $CP_3{}^4$. Hence $\lambda_*(h_1g_8) = \overline{h_2g_6}$ and $\lambda_*h_2(h_1g_8) = \overline{h_2{}^2g_6}$ but this implies $h_2(h_1g)_8 = (h_2{}^2g)_i$ in Ext for $P_5{}^8$. Also by similar arguments one can show $h_2(h_0{}^2g_k) = h_0(h_0h_2g)_k$ for $n+k \equiv 0(8)$. Putting these together with 6.2 completes the proof of a). Using 3.1 we see that $h_0{}^3(h_0h_2g)_k = (h_0{}^2k)_{k-6}$ since $h_1P_1g = h_0{}^2k$. Now part b) follows by naturality.

7. ADAMS DIFFERENTIALS, 2.

Let m be an integer and let a and b be defined by $m = 4a + b$, $0 \leq b < 4$. Let $\varphi(m) = 8a + 2^b$. Notice that $\pi_{\varphi(m)-1}\, O(t) \neq 0$ for t large and each non-zero group appears for a suitable m.

We will prove several very general propositions in this section which will give the remaining Adams differentials. Indeed we essentially prove theorem C of the introduction but need some additional information first. This discussion is in Chapter IV.

Let $X = S^0 \cup_{2\iota} e^1$ and let $S^0 \xrightarrow{i} X \xrightarrow{p} S^1$ be the obvious cofibration. Elements in Ext for X are either in the image of i_* or else map under p_* to a non-zero class. Let $\overline{\alpha}$ be any class in Ext for X such that $p_*\overline{\alpha} = \alpha$.

PROPOSITION 7.1. The following table identifies the element in Ext for X which projects in E_∞ to the element to which β_m projects.

	$m = 1$	2	3	4	5
$i_*\beta_m = \{\alpha\}$ for $\alpha =$	i_*h_1	i_*h_2	i_*h_3	$i_*h_1h_3$	$i_*h_1^2h_3$
	$\equiv 2(4)$	$\equiv 3(4)$	$\equiv 0(4)$	$\equiv 1(4)$	
	$i_*P^a h_2$	$\overline{P^a h_2^2}$	$i_*P^{a-1}c_0$	$i_*P^{a-1}h_1c_0$	

where the last four entries require that $m \geq 6$.

Proof. The first five entries are obvious. Next notice that in $\mathrm{Ext}^{4,12}(H^*(X), Z_2)$ there is a class $\overline{h_0^3h_3}$. This class behaves like a periodicity operator in the sense that if we multiply $\overline{h_0^3h_3}$ by α where $h_0^3h_3\alpha = 0$ we get $i_*(P^1\alpha)$. As a homotopy class $\overline{h_0^3h_3}$ projects to $\langle i_*\iota, 2\iota, 8\sigma\rangle$ where ι generates the 1-stem, and σ generates the seven stem. By the Bott periodicity we see $\langle i_*\iota, 2\iota, 8\sigma\rangle\, \beta_m = i_*\beta_{m+4}(i_*(2\pi_{\varphi(m+4)-1}(S^0))) = i_*\beta_{m+4}$. Hence if we identify the elements corresponding to $i_*\beta_m$ for $6 \leq m \leq 9$ we will be finished. The argument fails to apply to the first five cases since the periodic copies for $m = 4$ are not permanent cycles, the image of the periodic copy of $m = 3$ and 5 is zero in Ext for X (since it is h_0 of something) and the image of the periodic copy of h_1 is exceptional for several reasons. Also there is some difficulty if $m \equiv 3(4)$ since the element in Ext is not in im i_*. We will discuss this in a moment.

Since the 11-stem contains only P^1h_2, $i_*P^1h_2$ must represent the image of

J. This settles all $m \equiv 2(4)$, $m \geq 6$.

To see the next case we look at Ext for $t-s = 14$, 15 and 16 for X.

14			h_3^2		d_0				
15	h_5			$\overline{h_0h_3^2}$			$\overline{h_0^2d_0}$		
16						$\overline{h_1d_0}$		P^1c_0	$\overline{h_0^7h_4}$
	0	1	2	3	4	5	6	7	8

Since $i_*\beta_7 = \{\overline{h_0^3h_3}\}\{h_3\}$, the s filtration of $i_*\beta_7$ must be at least 5 (and equal to 5 only if $P^1h_3 \neq 0$) and hence greater than 5. Thus the only possibility is $\overline{h_0^2d_0} = \overline{P^1h_2^2}$. Now $\langle i_*1, h_0, h_0^2d_0\rangle h_1 = i_*\langle h_0, h_0^2d_0, h_1\rangle = P^1c_0$ and since $\eta\beta_7 = \beta_8$ and $\eta\beta_8 = \beta_9$ we have completed the proof of the proposition for $m \equiv 0, 1(4)$.

We must be a little more careful with the case $m \equiv 3(4)$. By induction suppose $i_*\beta_m = \{\overline{P_{\frac{m-3}{4}}h_2^2}\}$. Then there is a map $S^{2m+1} \to S^0 \cup_{2\iota} e_1$ such that $(P_{\frac{m-3}{4}}h_2^2) = p_*(1)$, i.e., $\{p\}$ is in the homotopy class of $i_*\beta_m$. Clearly $h_3(\overline{P_{\frac{m-3}{4}}h_2^2}) = 0$ and $\sigma(i_*\beta_m) = 0$. (Note that neither statement follows from the other but both follow from the fact that $\overline{P_{\frac{m-3}{4}}h_2^2}$ is on the Adams edge and so the composition can be checked.) Hence we can form

$$\begin{array}{ccccc} & & & S^0 & \\ & & \nearrow^{\tilde{p}_1} & & \searrow^{i} \\ S^{2m+9} & \xrightarrow[\overline{16\iota}]{} & e^{2m+9} \cup_\sigma S^{2m+1} & \xrightarrow[\tilde{p}]{} & S^0 \cup_{2\iota} e^1 . \end{array}$$

By the Bott periodicity we know that $\beta_{m+4} = \tilde{p}_1\,\overline{16\iota}(16\pi_{2m+9}(S^0))$ and hence $\tilde{p}\,\overline{16\iota} = i_*\beta_{m+4}$. But in Ext the map $\overline{16\iota}$ raises the s filtration by 4 and so leaves as the only possibility $i_*\beta'_{m+4} = \{\overline{P_{\frac{m+1}{4}}h_2^2}\}$. This completes the proof of 7.1.

In order to get the remaining differentials we will use this result together with the following theorem which uses this diagram:

$$\begin{array}{ccc} \pi_{n+\varphi(m)}(S^{n+\varphi(m)}) & \xrightarrow{\partial_*} & \pi_{n+\varphi(m)-1}(V_{n+\varphi(m),\varphi(m)}) \\ \uparrow p & & \uparrow i \\ \pi_{n+\varphi(m)}(V_{n+\varphi(m)+1,\varphi(m)}) & \xrightarrow{\partial_1{}_*} & \pi_{n+\varphi(m)-1}(S^n) \end{array}$$

THEOREM 7.2. Suppose $n + \varphi(m) + 1 \equiv 2^m(2^{m+1})$. Then there is a class $\iota_{n+\varphi(m)}$ such that $p_*(\iota_{n+\varphi(m)}) = \iota$ and $\partial_1 * \iota_{n+\varphi(m)} = \beta_m$.

This is just a recast of 4.3.2 of [19] which was a recast of a theorem of Toda [26] and Adams [5]. The following proof is included for completeness.

Proof. Let $k = n + \varphi(m)$ and consider the following diagram:

7.2.1

$$\begin{array}{ccccc} P_n^{k-1} & \xrightarrow{i''} & P_n^k & \xrightarrow{p} & S^k \\ i \uparrow & & g \uparrow & & \uparrow i' \\ S^n & \rightarrow & S^n \cup_{\beta_m} e^k & \xrightarrow{p'} & S^k \end{array}$$

where p' is the obvious projection. If we can prove that 7.2.1 exists with i', the identity map, then in the stable range at least $i_* \partial_2 \iota = \partial_1(\iota)$ and clearly $\partial_2 \iota = \beta_m$, where ∂_1 refers to the boundary homomorphism in the top sequence. This implies the theorem except if $k + 1 = 16$. A detailed hand calculation is needed and can be found in Toda [25]. Using Spanier Whitehead duality* we see that 7.2.1 exists with i' the identity map if and only if the dual diagram exists. Now $\mathcal{D}(P_n^{k-1}) = P^{\kappa}_{2^m+1}$, $\mathcal{D}(P_n^{k-1}) = P^{\kappa}_{2^m-1}$ and $\mathcal{D}(S^n \cup_{\beta_m} e^k) = S^{2m} \cup_{\beta_m} e^{\kappa}$, where $\kappa = 2^m + \varphi(m)$.* Thus the dual diagram is

7.2.2

$$\begin{array}{ccccc} S^{2^m-1} & \xrightarrow{\mathcal{D}(p)} & P^{\kappa}_{2^m-1} & \xrightarrow{\mathcal{D}(i'')} & P^{\kappa}_{2^m-1} \\ \downarrow \mathcal{D}(i') & & \downarrow \mathcal{D}(g) & & \downarrow \mathcal{D}(i) \\ S^{2^m-1} & \xrightarrow[\mathcal{D}(P')]{} & S^{2^m-1} \cup_{\beta_m} e^{\kappa} & \longrightarrow & S^{\kappa} \end{array}$$

with $\mathcal{D}(i')$ and $\mathcal{D}(i)$ being maps of degree 1. Clearly $P^{\kappa}_{2^m-1}$ is the Thom Complex of 2^{m-1} times the canonical line bundle over $P_1^{\varphi(m)}$. Since this bundle is trivial over the $\varphi(m) - 1$ skeleton the classifying map factors through $P_1^{\varphi(m)} \rightarrow S^{\varphi(m)} \rightarrow BSO_{2^m-1}$ where the first map is the usual projection and the second map generates $\pi_{\varphi(m)}(BSO)$. Passing to Thom complexes we have 7.2.2 and this completes the proof of 7.2.

In our language 7.2 becomes

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*See, for example, E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966, p. 462 and in particular Ex. F and Ex. F-6.

PROPOSITION 7.3. Suppose $n+\varphi(m)+1 \equiv 2^m(2^{m+1})$. Then in the Adams spectral sequence for P_n we have

a) $m > 5$, $\delta_r 1_{n+\varphi(m)} = 0$ for $r < m-1$ and $\delta_{m-1}(1_{n+\varphi(m)}) = j_*\alpha_m$ where $j\colon P_n^{n+1} \to P_n$ and α_m is the entry in the table of 7.1 corresponding to m.

b) $m = 5$, $\delta_r 1_{n+\varphi(5)} = 0$, $r < 4$, and $\delta_4 1_{n+\varphi(5)} = j_*(i_* h_1 c_0)$.

c) $m = 4$, $\delta_2 1_{n+\varphi(4)} = i_* h_1 h_3$ where $i\colon S^n \to P_n$.

Proof. Theorem 7.2 implies that $i_*\beta_m' \neq 0$ where $i\colon S^n \to P_n^{n+\varphi(m)-1}$ but $i_1\colon S^n \to P_n$ satisfies $i_1{}^*\beta_m' = 0$. Hence look at $S_n \cup_{2\iota} e^{n+1} \to P_n^{n+\varphi(m)-1} \to P_n$. Suppose $m > 5$. In Ext for $S^n \cup_{2\iota} e^{n+1}$ we see that α_m is on the edge or one below it and so its image in $P_n^{n+\varphi(m)-1}$ is well defined and must represent β_m'. Therefore α_m is a surviving permanent cycle. The only change possible in Ext for $P_n^{n+\varphi(m)-1}$ and Ext for P_n in the $n+\varphi(m)$ stem is the addition of 1 representing the $n+\varphi(m)$ cell. Hence the differential must behave as described in the proposition, part a.

Now suppose $m = 5$ and consider

$$i\colon S^n \cup e^{n+1} \to P_n \quad \text{for } n \equiv 21(64).$$

A glance at table 8.6 shows $i_*(h_1{}^2h_3) = 0$. Yet theorem 7.2 implies $i_*\beta_5' \neq 0$. Hence the class which represents β_5' has filtration higher than $h_1{}^2h_3$, i.e., higher than 3. There are two possibilities, $\delta_4(1)_{10} = (h_1c_0)_0$ or $\delta_5(1_0) = (P^1h_1{}^2)_1$. The second would imply the corresponding differential in the spectral sequence for P_{n+1}, contradicting 7.2. Hence $\delta_4(1_{10}) = (h_1c_0)_0$ in the sequence for P_n.

The case for $m = 4$ proceeds just like the case for $m > 5$, using the appropriate part of 7.1.

The most important corollary of 7.3 is the following result.

Let n be fixed and let 1_k be a class in Ext for P_n (Ext here means E_{16} of the pre-spectral sequence). Suppose $n+k+1 = 2^m(2^{m+1})$. This defines $m(n,k)$. Let q be defined by $\varphi(q) \leq k < \varphi(q+1)$ and let $i(n,k) = \max(q-m(n,k),0)$.

THEOREM 7.4. Suppose $m \geq 3$ and if $m = 3$, $k \geq 9$ or $m = 4$, $k \geq 10$. Suppose also that $k-\varphi(q) = 0$ if $q \not\equiv 3(4)$ and $k-\varphi(q) = 1$ if $q \equiv 3(4)$. Let j: $P_n^{n+1} \rightarrow P_n$ be the usual inclusion. Then $h_0{}^{i+1}1_k$ is a surviving cycle and $\delta_{m-1}(h_0{}^{i}1_k) = j_*\alpha_q$ where α_q is given by table 7.1 and $i = i(nk)$.

Proof. We will prove the theorem by induction. Fix $n+k$ and the induction will be done on q. First observe that for $q < m$ the theorem follows directly from 7.3. Now suppose $m \geq 5$. Then for $m = q$, $i = 0$ and again 7.4 is just 7.3. Now suppose 7.4 is true for $q > m$. Consider

$$\begin{array}{ccc} P_{k-\varphi(q+1)} & \xrightarrow{p} & P_{k-\varphi(q)} \\ \uparrow i_2 & & \uparrow i_1 \\ S^{k-\varphi(q+1)} & & S^{k-\varphi(q)} \end{array}$$

$i_1{}^*\alpha_q \neq 0$ and there is a class γ such that $p_*\gamma = \alpha_q$. Using one of 3.1, 3.2, 3.4 or 3.5 we see that $i_2{}^*\alpha_{q+1} = h_0\gamma$. Naturality of Adams differentials with respect to multiplication now completes the proof.

Now suppose $m = 4$. If we require $k \geq 10$ the induction argument is identical with the above. There is a difficulty starting because 7.3 is not quite the right statement. On the other hand consider

$$P_{n-1} \rightarrow P_n \rightarrow P_{n+10} \quad \text{with } n+10 \equiv 15(32).$$

Let $\partial_1{}^*$ be the boundary homomorphism into P_n^{n+9} and $\partial_2{}^*$ into P_{n-1}^{n+9}. Now 7.2 says $\partial_1{}^*\iota_{n+10} = \eta\sigma\iota_n$. Now consider

$$\begin{array}{ccc} P_{n-1}^{n+9} & \xrightarrow{p} & P_n^{n+9} \\ i_3\uparrow & \overset{i_2}{\nwarrow} & \uparrow i_1 \\ S^{n-1} & & S^n \end{array}$$

From table 8.6 we see that $i_1{}^*\eta \in \operatorname{im} p_*$ and if $p_*\gamma = i_1{}^*\eta$ then $i_3{}^*\eta^2 = 2\gamma$. Hence $p^*\sigma\gamma = i_1{}^*\eta\sigma$ and $2\sigma\gamma = i_3{}^*\eta^2\sigma$. By inspection of table 8.4 we see that $i_3{}^*h_1{}^2h_3 = 0$ and so the class representing $i_3{}^*\eta^2\sigma$ must have filtration greater than 3. It is not hard to see that $i_3h_1c_0$ must represent $i_3{}^*\eta^2\sigma$. This begins the induction and the argument is completed as above. The argument for $m = 3$ is similar and we leave it to the reader. This completes the proof of 7.4.

Theorem 7.4, of course, is a very general proposition holding for stems of all orders. In this section we will use it to complete the discussion of the Adams differentials for our calculations. Quite directly 7.4 gives the order of direct summand of homotopy generated by $h_0{}^i 1_k$. Obviously it gives the order of what is subtracted from the $k-1$ stem for each P_n. The only remaining statement to verify is simply what happens when $k+n \equiv 15(32)$. In this case if $k \geq 16$, $\delta_1 1_k = h_4$ and whatever this implies. Putting all of this together we have the following proposition which defines A, B and C. We will always have $A_k \subset \pi_{k-1}(\Sigma^{-n}P_n)$, $B_k \subset \pi_k(\Sigma^{-n}P_n)$ and $C_k \subset \pi_{k+1}(\Sigma^{-n}P_n)$. We will always use $m(k,n)$ defined by $n+k+1 \equiv 2^m(2^{m+1})$ and $i(n,k) = \max(q-m(n,k),0)$.

PROPOSITION 7.5.

A_k is cyclic group of order 2^i where $i = i(n,k)$.

$B_k = \bar{B}_k + Z_2$ if $m > 4$ and $B_k = \bar{B}_k$ if $m = 4$ with $\bar{B}_k$ being a cyclic group of order 2^{m+1} if $q-m \geq 0$ and of the order of 1_k as given in the table if $q-m < 0$.

$C_k = Z_2$ for $m > 4$ and $= 0$ for $m = 4$.

This completes the calculation of $\pi_*(P_n)$ except for Proposition 7.6.

PROPOSITION 7.6.
$$\delta_3(h_0 f_0)_k = P^1 g \quad n+k \equiv 12(64),$$
$$\delta_4(h_4)_k = h_3{}^3 \quad n+k \equiv 7(16).$$

We do not have a natural proof of this nor do we know what happens in the other congruences. We will deduce this from a general proposition in the next chapter. We have tried to avoid using this general proposition for the calculations. It seems clear that this differential could easily be settled if Ext were computed further. With this one exception we are finished!

8. In the pages which follow are 16 tables. The first table gives a copy of $\mathrm{Ext}_A^{s,t}(Z_2,Z_2)$ for $t-s \leq 44$. Slanting lines to the right indicate multiplication by h_1 and vertical lines indicate multiplication by h_0. Slanting lines to the left indicate Adams differentials. The first table is included for reference and the details are to be found in [20], [21] and [24].

The next fifteen tables are print outs of E_{16} of the pre-spectral sequence. The only missing differential is δ_{16} which is handled as a δ_1 in the Adams spectral sequence. The tables are given for P_k, $k \equiv 1,\ldots,15(16)$. Since
$$E_{16}{}^{s,t}(P_k) \simeq \mathrm{Ext}_A^{s,t}(Z_2,Z_2) + E_{16}{}^{s,t+1}(P_{k+1})$$
for $k \equiv 0(16)$ no table is given for this case. Also $\pi_*(P_k)$ is a group exten-

sion of $\pi_*(P_{k+1})$ and $\pi_*(S^k)$ if $k \equiv 0(32)$ and deviates from this by the δ_1 in the case $k \equiv 16(32)$. Table 16 has the groups for both P_k, $k \equiv 15$ and $16(16)$.

Above each table is a sequence of groups. This sequence is just the homotopy sequence of

$$S^k \to P_k \to P_{k+1}$$

with the image of ∂_* written as a fourth line. Using [8] this sequence is just the homotopy sequence of $SO(n) \to SO(n+1) \to S^n$ in the metastable range. From the EHP sequence it is also clear that these homomorphisms represent just E, H, and P too, i.e.,

$$\begin{array}{ccccccc}
\pi_j(S^n) & \xrightarrow{E} & \pi_{j+1}(S^{n+1}) & \xrightarrow{H} & \pi_{j+1}(S^{2n+1}) & \xrightarrow{P} & \\
\uparrow P_n & & \uparrow P_{n+1} & & \uparrow \Sigma^2 & & \\
\pi_j(\Sigma^{n-1}P_n) & \xrightarrow{\Sigma p_*} & \pi_{j+1}(\Sigma^n P_{n+1}) & \xrightarrow{\partial_*} & \pi_{j-1}(S^{2n-1}) & \xrightarrow{i_*} &
\end{array}$$

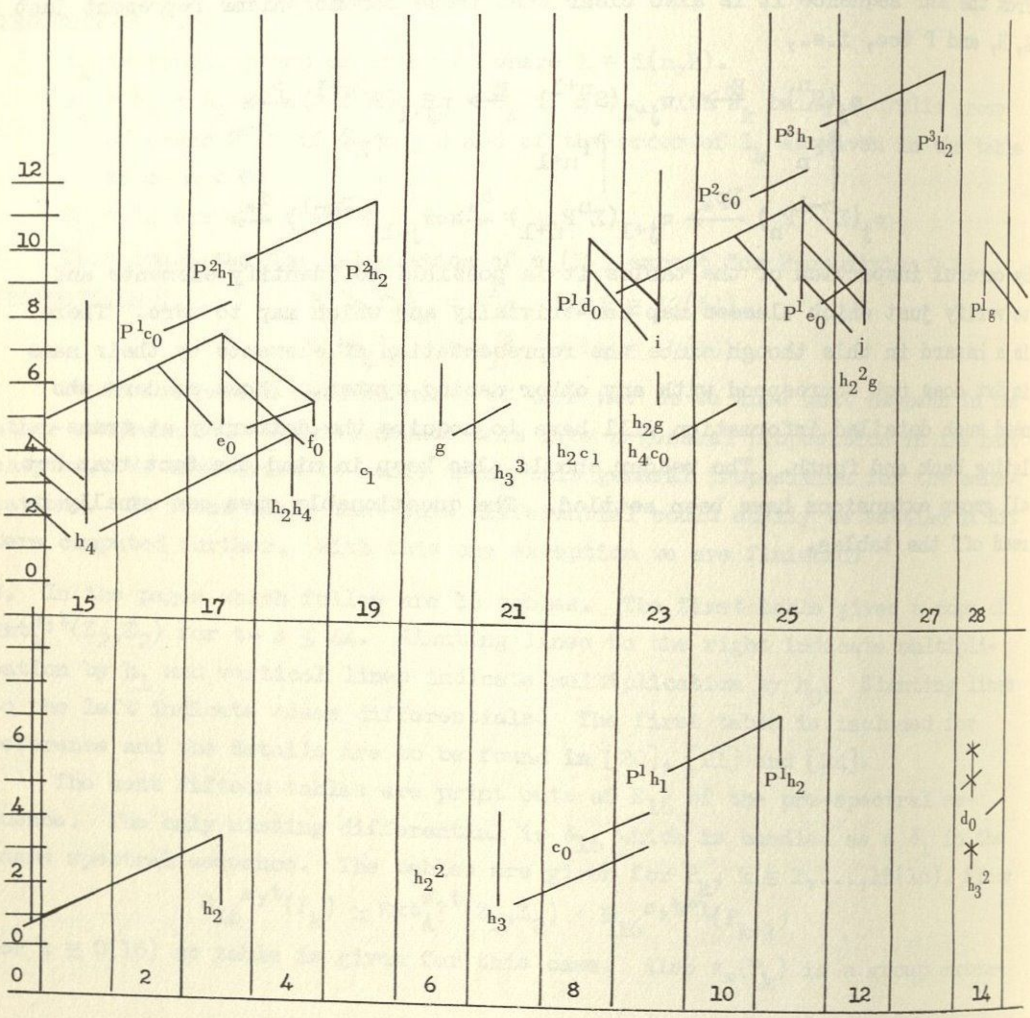
By careful inspection of the tables it is possible to identify elements and to verify just which classes map non-trivially and which map to zero. There is a hazard in this though since the representation of elements by their name in Ext does not correspond with any other naming system. Those readers who need such detailed information will have to acquire the dexterity at translating back and forth. The reader should also keep in mind the fact that not all group extensions have been settled. The questionable ones can usually be read off the tables.

TABLE 8.1

$\mathrm{Ext}(Z_2, Z_2)$

$$E_{16}^{s,t}(\Sigma^{-16}P_{16}) = \mathrm{Ext}^{s,t}(Z_2,Z_2) + E_{16}^{s,t-1}(\Sigma^{-1}P_1)$$

Homotopy groups of P_k, $k \equiv 0(16)$ are easily obtained from this splitting.



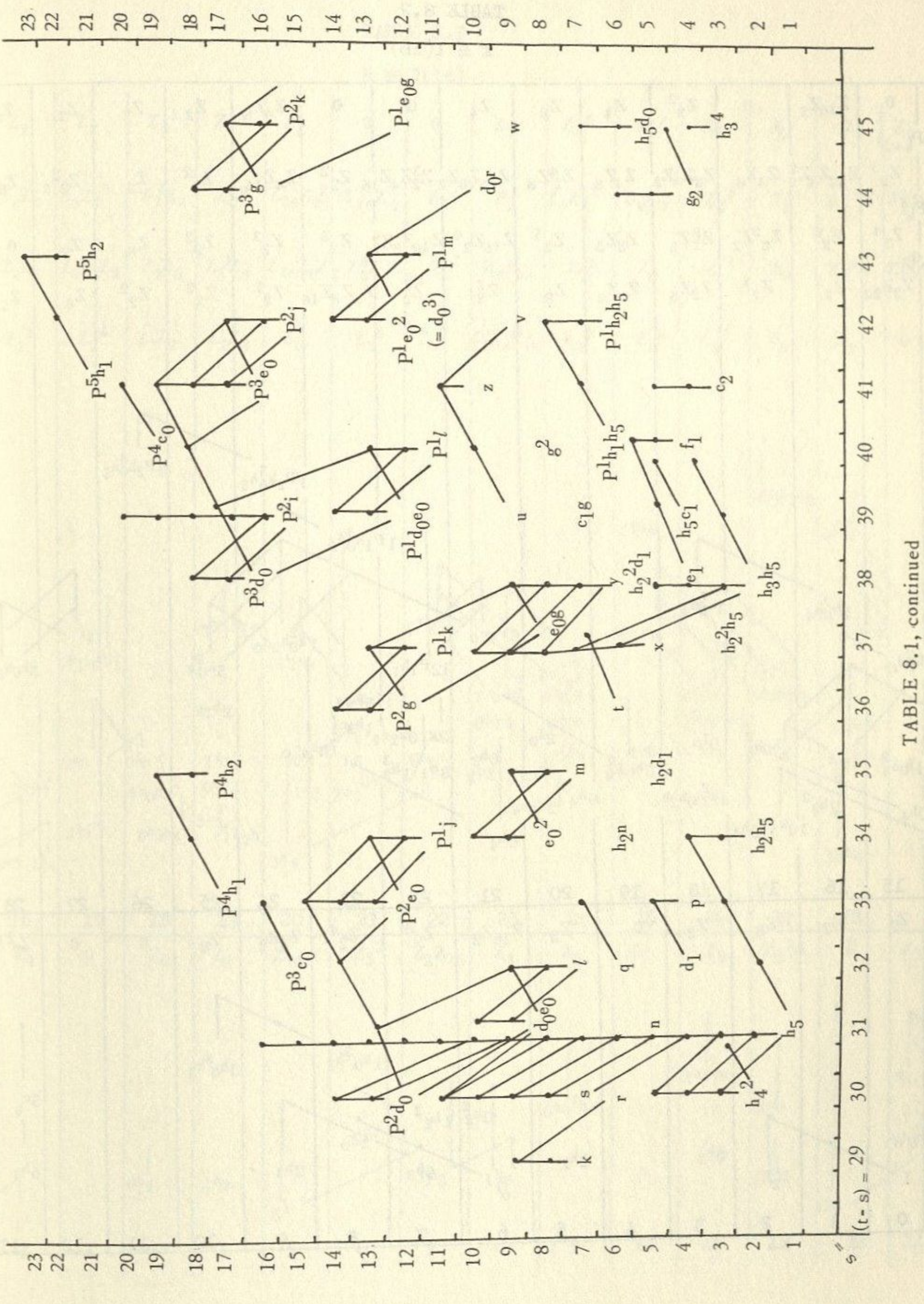
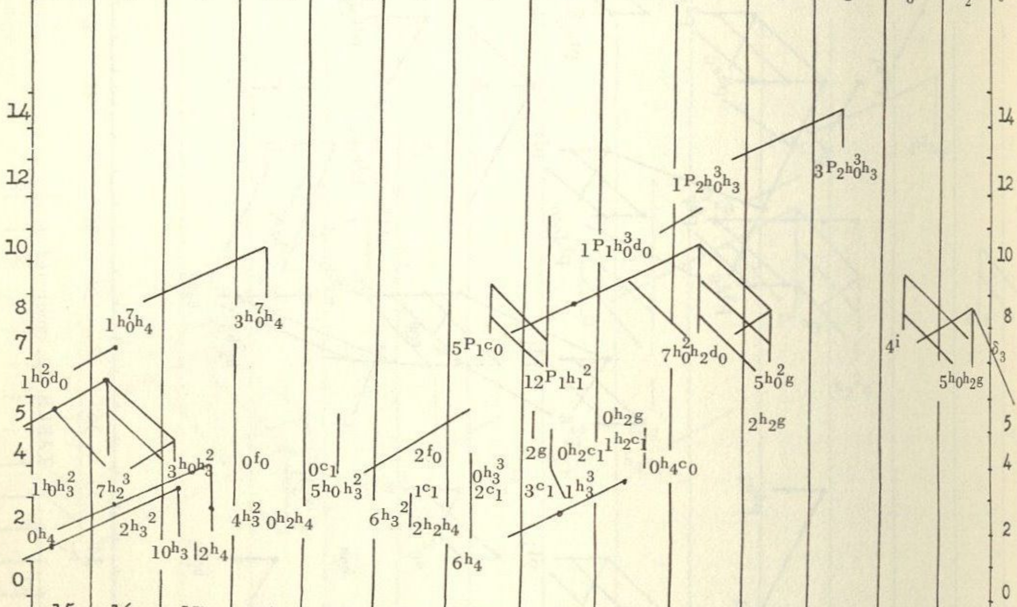


TABLE 8.1, continued

TABLE 8.2

$k \equiv 1(16)$

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
	0	$Z_{16}Z_2$	0	$Z_2{}^2$	Z_4	Z_8	Z_4	0	0	Z_4Z_8	Z_2	Z_2	Z_2	Z_8	Z_2
$\pi_*(P_{k+1})$															
	$Z_2{}^3$	$Z_{32}Z_4Z_2^2$	Z_4Z_8	$Z_8Z_4Z_2$	Z_4Z_8	$Z_2^4Z_8$	$Z_{16}Z_8Z_2$	$Z_2^3Z_4Z_{16}$	$Z_2{}^2$	Z_4Z_{16}	$Z_2{}^2$	Z_8	$Z_2{}^2$	Z_8	Z_2^3A
$\pi_*(P_k)$															
	$Z_2{}^4$	$Z_2{}^6$	$Z_8{}^2Z_2$	$Z_2^3Z_8$	Z_8Z_2	$Z_2{}^5$	$Z_{16}Z_2{}^4$	$Z_{16}Z_8Z_2^4$	$Z_2{}^5$	$Z_2{}^2$	$Z_2{}^2$	Z_8	Z_2	0	Z_2^2A
$\pi^*(S^0)$															
	Z_2Z_{32}	$Z_2{}^2$	$Z_2{}^4$	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	$Z_2{}^2$	$Z_2{}^2$	$Z_2{}^2$	Z_8	Z_2	0



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Z_2	Z_2	Z_8	Z_2	0	Z_2	$Z_{16}Z_2$	$Z_2{}^3$	$Z_2{}^4$	Z_8Z_2	Z_8	0	0	$Z_2{}^2A$	$\bar{B}Z_2^2$

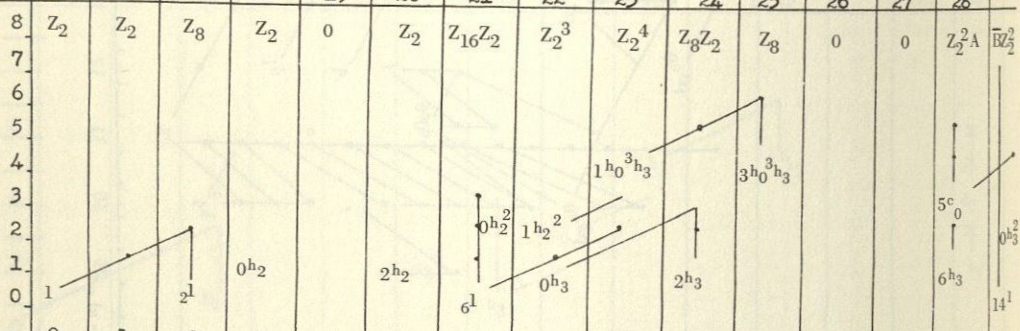
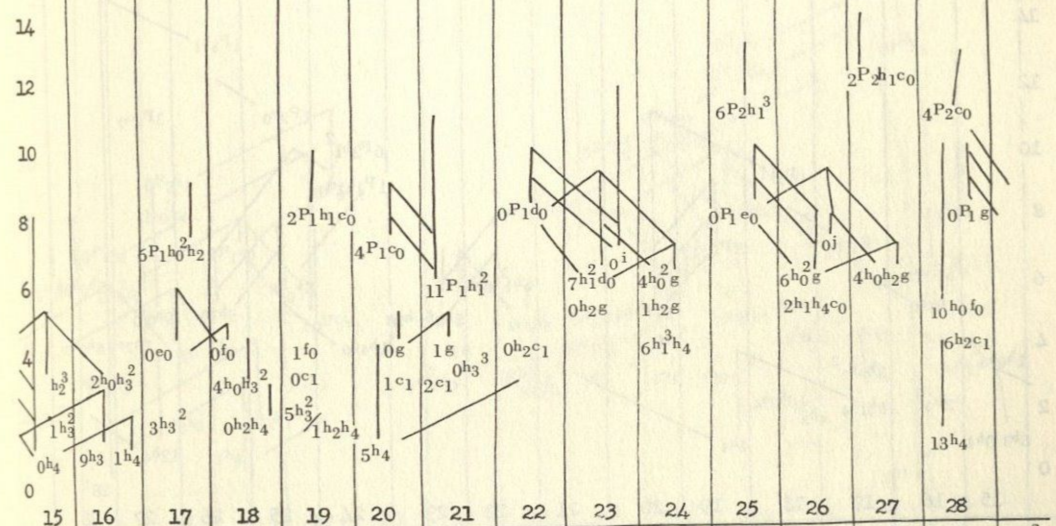


TABLE 8.3

$k \equiv 2(16)$

0	Z_2	Z_2	$Z_2{}^2$	$Z_2{}^2$	Z_2	0	0	Z_2	$Z_2{}^2$	$Z_2{}^2$	Z_2	Z_2	Z_2	Z_2
$\pi_*(P_{k+1})$														
$Z_4Z_2{}^2$	$Z_8Z_2{}^2$	$Z_8Z_2{}^2$	$Z_2^2Z_8$	$Z_2{}^6$	$Z_{16}Z_2{}^2$	$Z_{16}Z_2^3$	Z_2	Z_2	$Z_4Z_2{}^2$	Z_8Z_2	$Z_2{}^2$	$Z_2{}^2$	$Z_2{}^3A$	Z_2^3B
$\pi_*(P_k)$														
$Z_4Z_2^2Z_{32}$	Z_4Z_8	$Z_8Z_4Z_2$	Z_4Z_8	$Z_2^4Z_8$	$Z_{16}Z_8Z_2$	$Z_2^3Z_4Z_{16}$	$Z_2{}^2$	Z_4Z_{16}	$Z_2{}^2$	Z_8	$Z_2{}^2$	Z_8	$Z_2{}^3A$	Z_2^3B
$\pi^*(S_0)$														
Z_2Z_{32}	$Z_2{}^2$	$Z_2{}^4$	Z_2Z_8	Z_2Z_8	Z_8	$Z_2{}^2$	$Z_2{}^2$	$Z_2Z_8Z_{16}$	$Z_2{}^2$	$Z_2{}^2$	$Z_2{}^2$	Z_8	Z_2	0



Z	Z_4	0	Z_4	Z_2	Z_{16}	$Z_2{}^2$	Z_2Z_{16}	Z_4	Z_8	0	Z_8	$Z_2{}^2A$	$\overline{B}$	$Z_2{}^2$

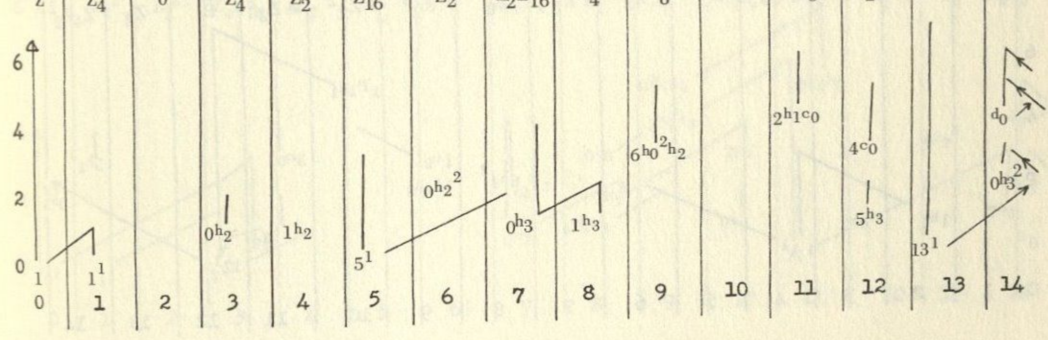
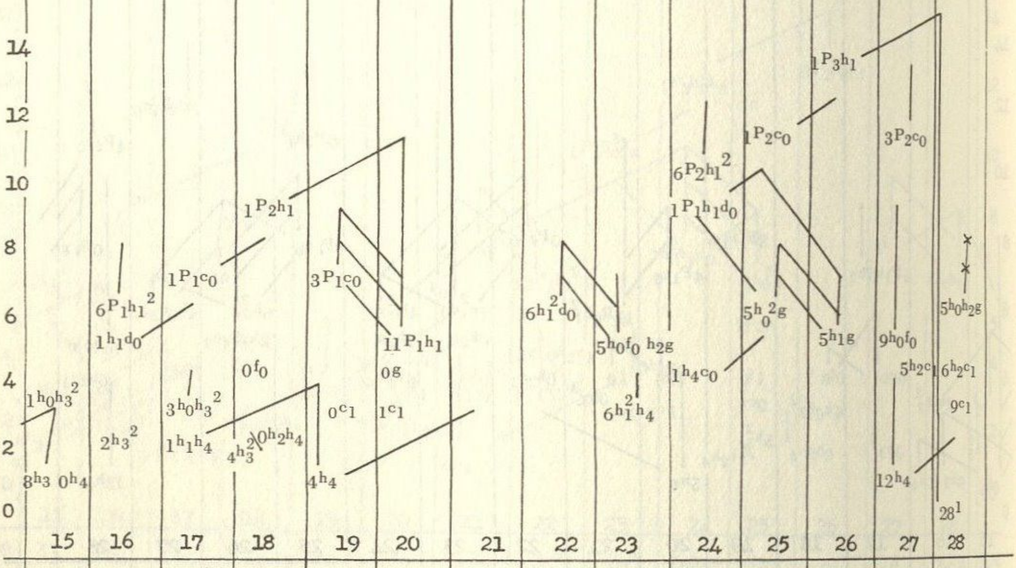


TABLE 8.4

$k \equiv 3(16)$

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
	0	$Z_{32}Z_2$	Z_2^2	Z_2^3	Z_4Z_2	Z_8	Z_4	Z_2	Z_2^2	$Z_{16}Z_4Z_2$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2
$\pi_*(P_{k+1})$	$Z_8Z_2^2$	$Z_{64}Z_4Z_2^3$	$Z_2^4Z_4$	Z_2^8	$Z_4Z_{16}Z_2^2$	$Z_{16}Z_8Z_2^2$	Z_4	Z_2^2	$Z_2^3Z_4$	$Z_{32}Z_4^2Z_2^2$	Z_2^4	Z_2^4	$Z_2^5 A$	$BZ_8Z_4Z_2^2$	CZ_2^5
$\pi_*(P_k)$	$Z_8Z_2^2$	$Z_8Z_2^2$	$Z_2^2Z_8$	Z_2^6	$Z_{16}Z_2^2$	$Z_{16}Z_2^3$	Z_2	Z_2	$Z_4Z_2^2$	Z_8Z_2	Z_2^2	Z_2^2	AZ_2^3	Z_3^3B	CZ_2^4
$\pi_*(S^0)$	Z_2Z_{32}	Z_2^2	Z_2^4	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2	0



1	2	3	4	5	6	7	8	9	10	11	12	13	14
Z_2 0	Z_2	Z_2^2	Z_{16}	Z_2	Z_2	Z_2	Z_8	Z_2^2	Z_2^2	AZ_2^2	$\bar{B}$	Z_2	$Z_4Z_2^2$

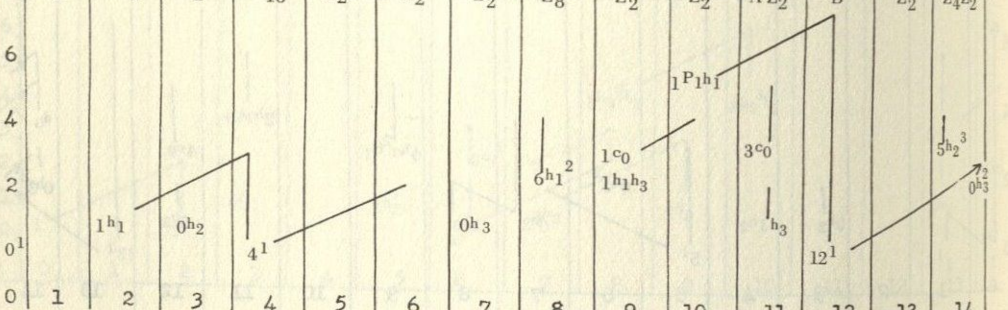
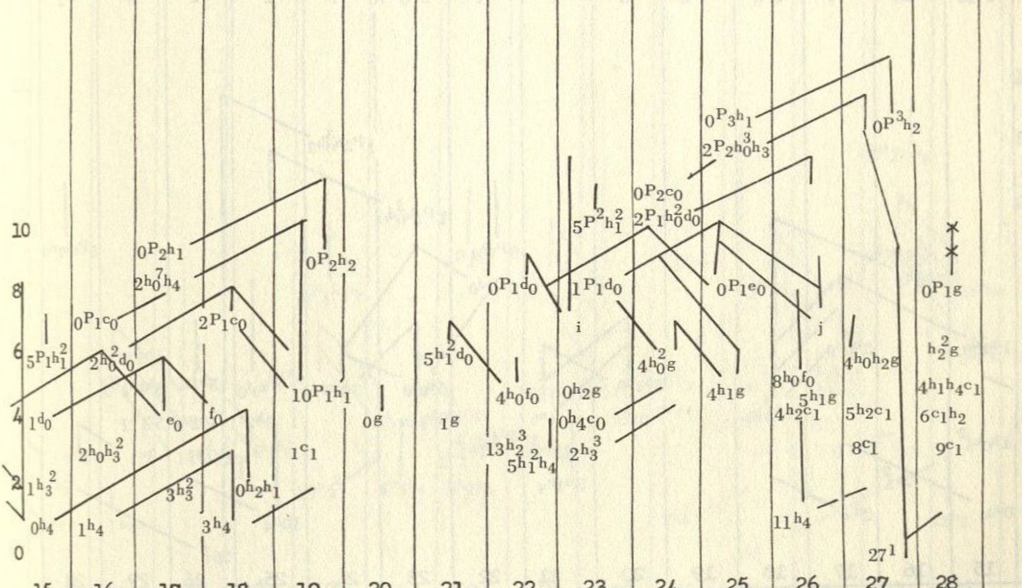


TABLE 8.5

$k \equiv 4(16)$

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
	0	0	0	0	0	0	Z_2	Z_2	Z_2	Z_2	0	0	Z_2	0	0
$\pi_*(P_{k+1})$	$Z_2^2Z_8$	Z_2^4	Z_2^4	Z_8Z_2	$Z_{16}Z_2$	0	Z_2^2	$Z_8Z_2^2$	$Z_8Z_2^2$	Z_2^3	Z_2^2	AZ_2^4	$BZ_2^2Z_8$	Z_2^4C	Z_4Z_2
$\pi_*(P_k)$	$Z_{64}Z_2^3Z_4$	$Z_2^4Z_4$	Z_2^8	$Z_4Z_{16}Z_2^2$	$Z_{16}Z_8Z_2^2$	Z_4	Z_2^2	$Z_2^3Z_4$	$Z_{32}Z_4^2Z_2^2$	Z_2^4	Z_2^4	Z_2^5A	$BZ_8Z_4Z_2^2$	CZ_2^5	Z_4Z_2
$\pi_*(S^0)$	Z_2Z_{32}	Z_2^2	Z_2^4	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2	0



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Z	Z_2^2	Z_2^2	Z_4Z_{16}	Z_2	0	0	$Z_{16}Z_4$	Z_2^4	Z_2^5	AZ_5^4	$\bar{B}Z_8$	Z_2	Z_2^2	$Z_8Z_2^2$

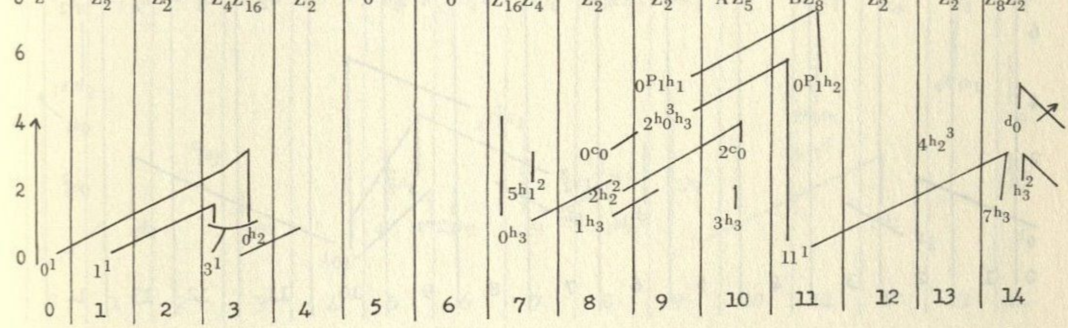


TABLE 8.6

$k \equiv 5(16)$

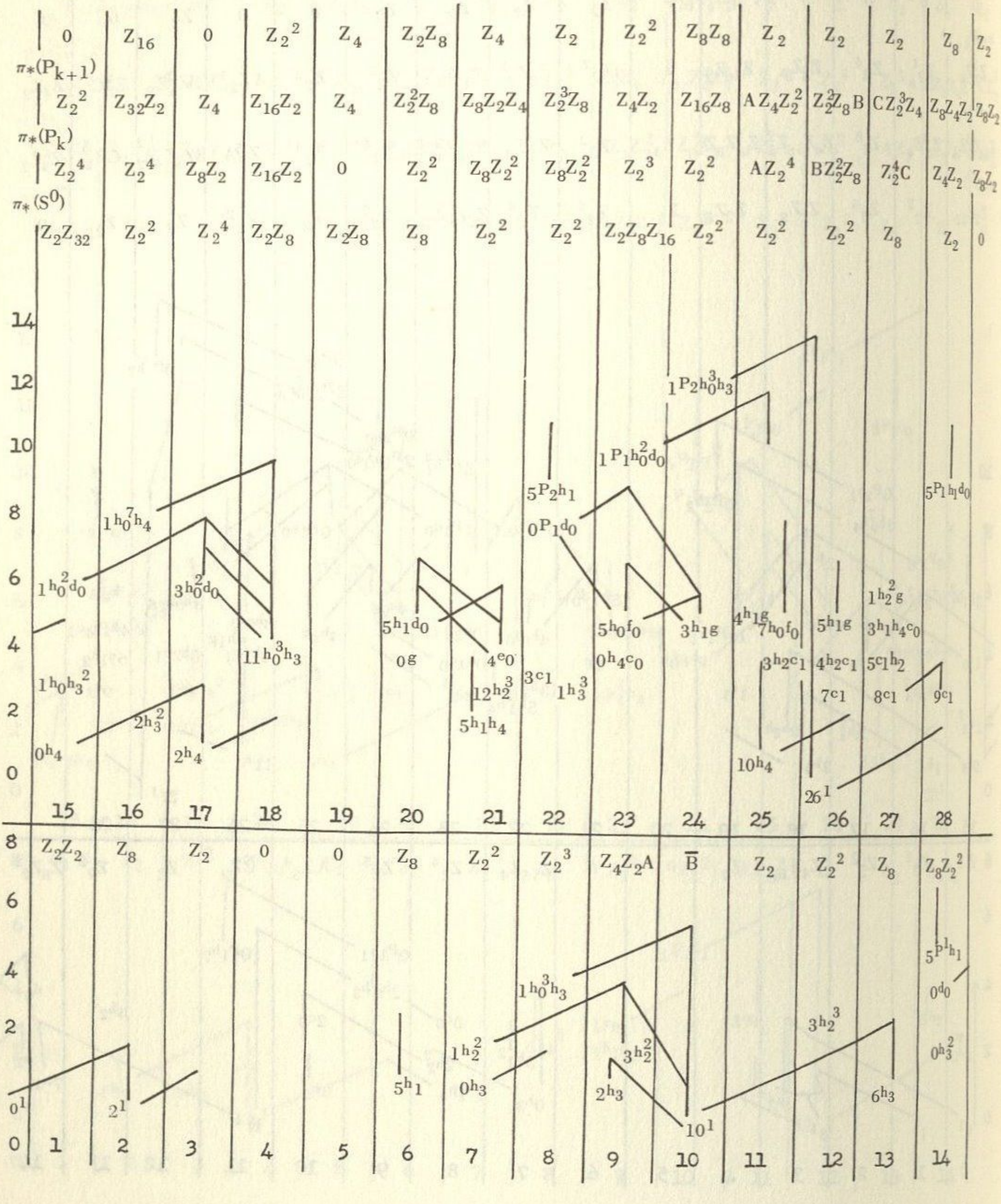


TABLE 8.7

$k \equiv 6(16)$

	0	Z_2	Z_2	$Z_2{}^3$	$Z_2{}^2$	$Z_2{}^2$	Z_2	Z_2	$Z_2{}^2$	$Z_2{}^2$	$Z_2{}^2$	Z_2	Z_2	Z_2	Z_2
$\pi_*(P_{k+1})$															
	Z_2	$Z_2{}^2$	$Z_{16}Z_2$	$Z_2{}^3$	$Z_2{}^5$	$Z_4Z_8Z_2$	$Z_8Z_2{}^3$	Z_2Z_8	$Z_2{}^3$	$AZ_8Z_2^3$	$BZ_2^3Z_8$	Z_2^5C	$Z_2^3Z_4$	$Z_8Z_2^2$	Z_8
$\pi_*(P_k)$															
	$Z_{32}Z_2$	Z_4	$Z_{16}Z_2$	Z_4	$Z_2^2Z_8$	$Z_8Z_4Z_2$	$Z_2^3Z_8$	Z_4Z_2	$Z_{16}Z_8$	$AZ_4Z_2^2$	$Z_2^2Z_4B$	$CZ_2^3Z_4$	$Z_8Z_4Z_2$	$Z_8Z_2{}^2$	Z_4
$\pi_*(S^0)$															
	Z_2Z_{32}	$Z_2{}^2$	$Z_2{}^4$	Z_2Z_8	Z_2Z_8	Z_8	$Z_2{}^2$	$Z_2{}^2$	$Z_2Z_8Z_{16}$	$Z_2{}^2$	$Z_2{}^2$	$Z_2{}^2$	Z_8	Z_2	0

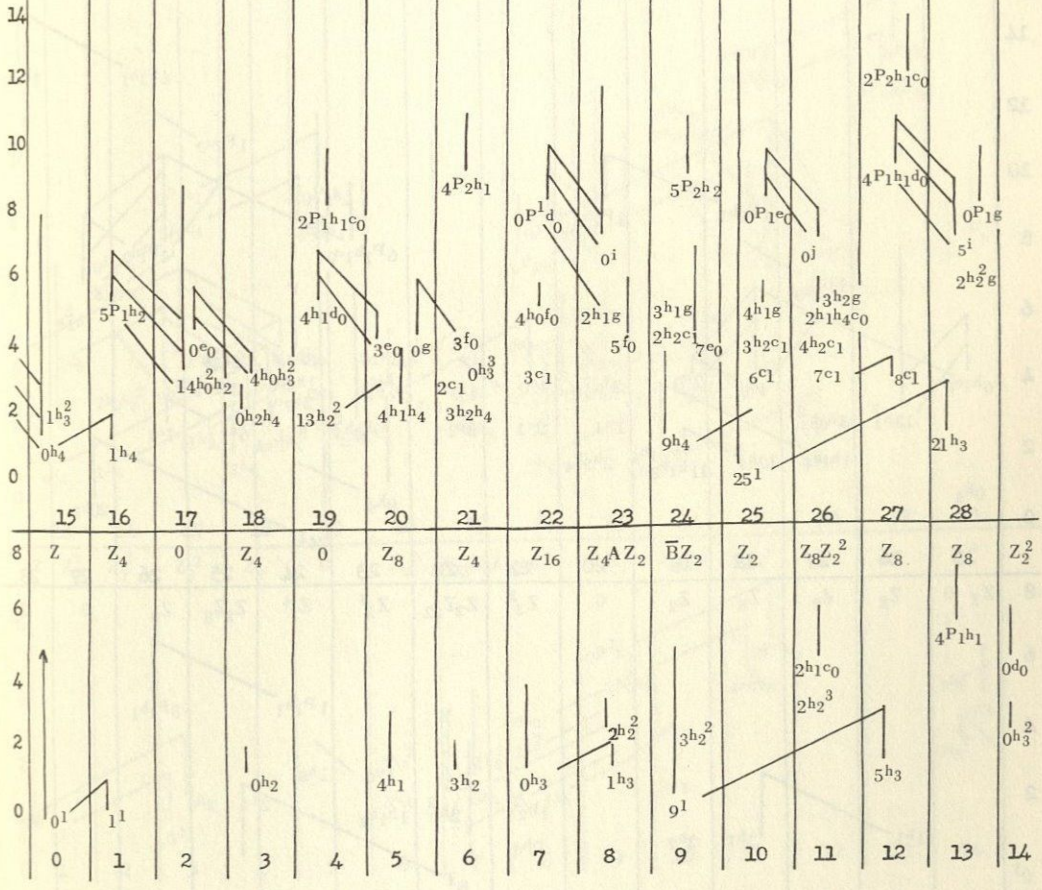


TABLE 8.8

$k \equiv 7(16)$

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
	Z_2	$Z_{32}Z_2$	$Z_2^{\,3}$	$Z_2^{\,4}$	Z_8Z_2	Z_8Z_2	Z_8	$Z_2^{\,2}$	$Z_2^{\,2}$	$Z_2Z_{16}Z_8$	$Z_2^{\,2}$	$Z_2^{\,2}$	Z_2^2	Z_8	Z_2
$\pi_*(P_{k+1})$	$Z_2^{\,3}$	$Z_{32}Z_{16}Z_2^2$	$Z_2^{\,5}$	$Z_2^{\,9}$	$Z_8^2Z_4Z_2^2$	$Z_8^2Z_2^{\,4}$	$Z_2Z_8^{\,2}$	$Z_2^{\,5}$	$AZ_2^5Z_8$	$BZ_{16}Z_2^4Z_8^2$	$CZ_2^{\,7}$	$Z_2^5Z_4$	$Z_8Z_2^4$	Z_8^2	Z_2^2
$\pi_*(P_k)$	$Z_2^{\,2}$	$Z_{16}Z_2$	$Z_2^{\,3}$	$Z_2^{\,5}$	$Z_4Z_8Z_2$	$Z_8Z_2^{\,3}$	Z_2Z_8	$Z_2^{\,3}$	$AZ_8Z_2^3$	$BZ_2^3Z_8$	$CZ_2^{\,5}$	$Z_2^3Z_4$	$Z_8Z_2^2$	Z_8	Z_2
$\pi_*(S^0)$	Z_2Z_{32}	$Z_2^{\,2}$	$Z_2^{\,4}$	Z_2Z_8	Z_2Z_8	Z_8	$Z_2^{\,2}$	$Z_2^{\,2}$	$Z_2Z_8Z_{16}$	$Z_2^{\,2}$	$Z_2^{\,2}$	$Z_2^{\,2}$	Z_8	Z_2	0

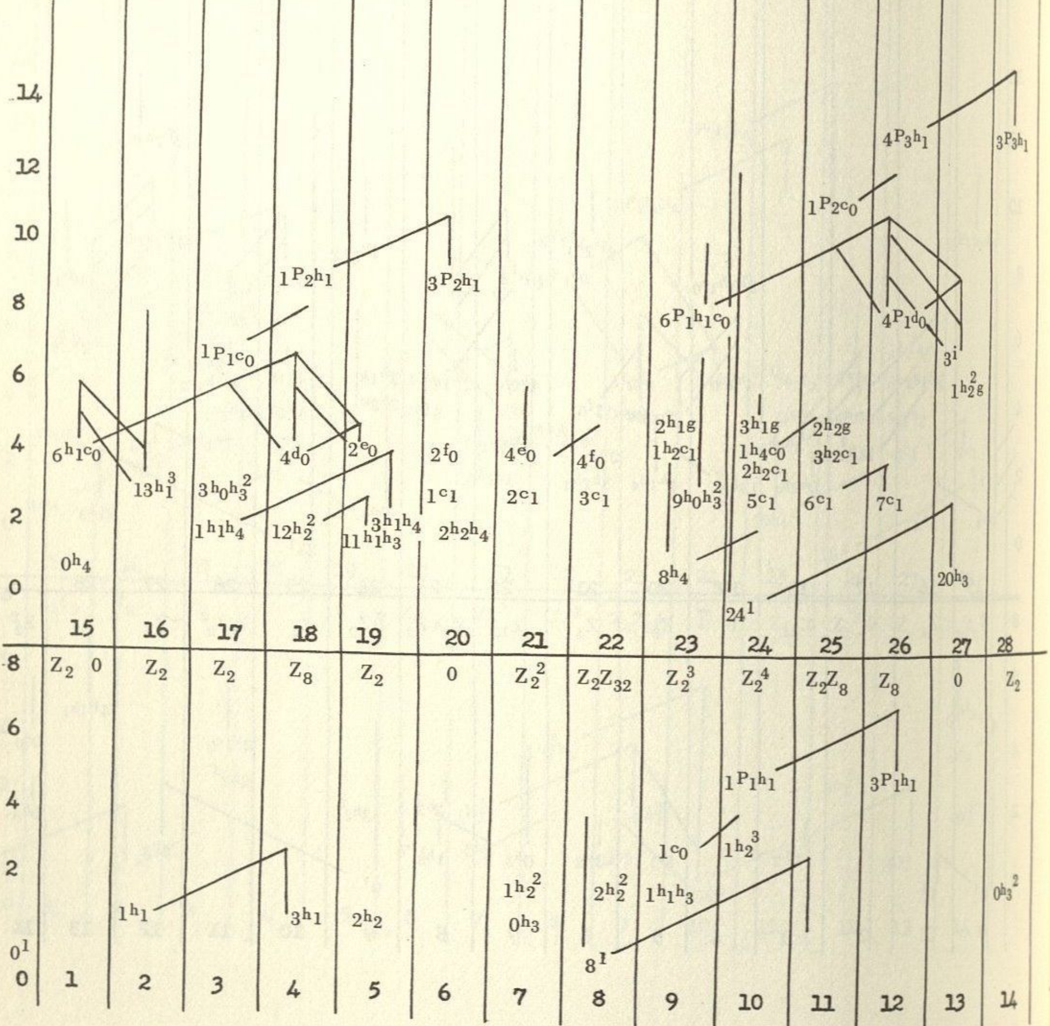


TABLE 8.9

$k \equiv 8(16)$

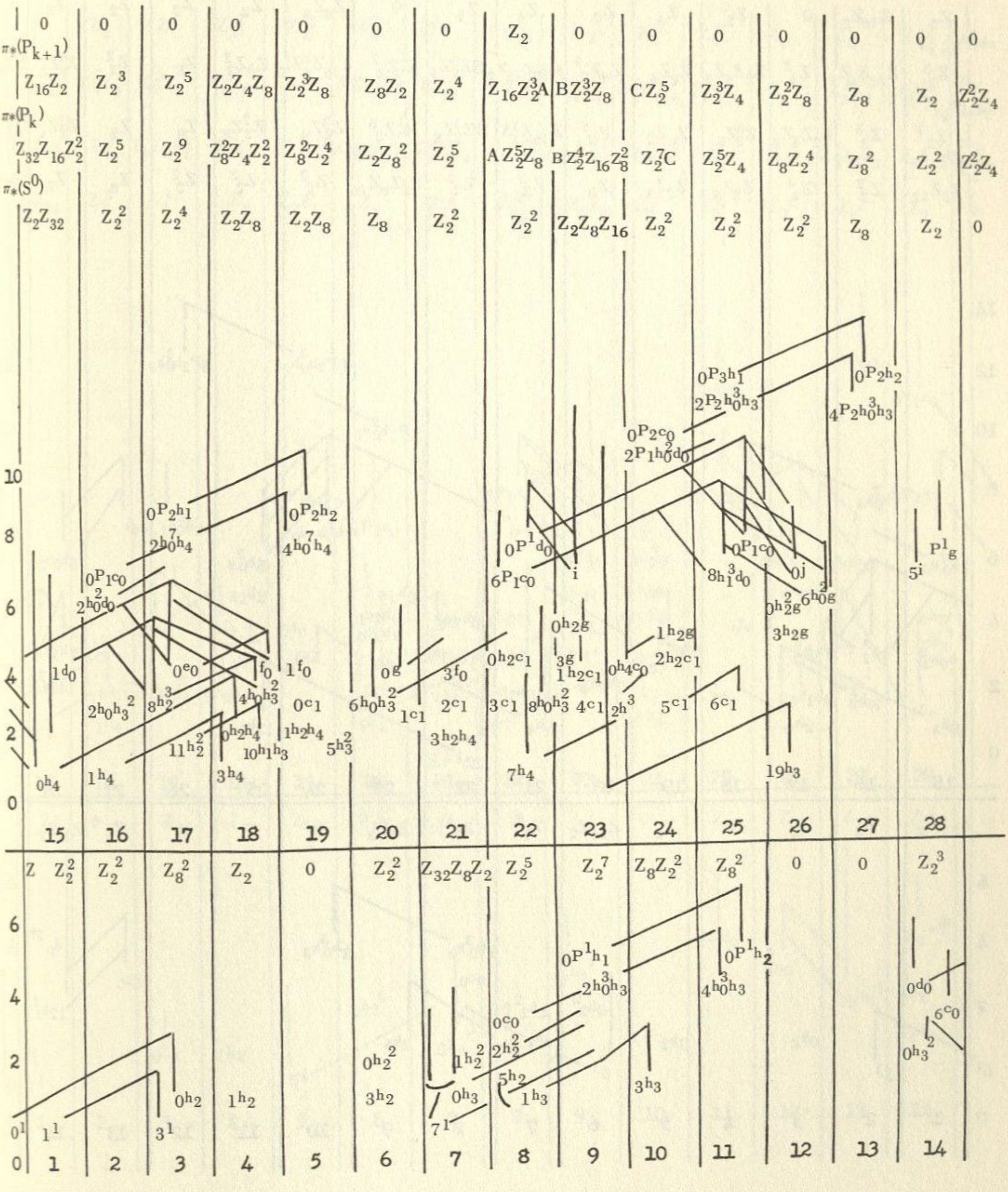
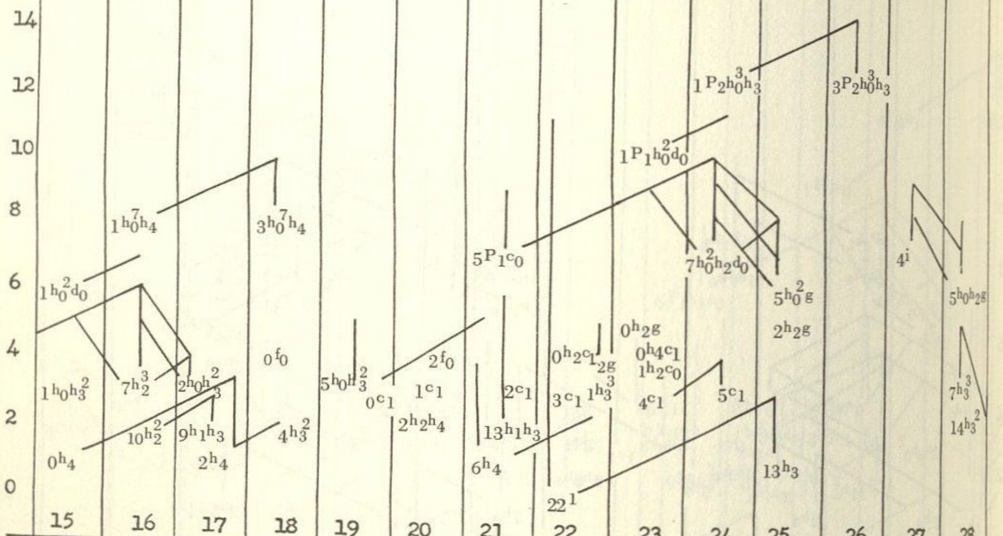


TABLE 8.10

$k \equiv 9(16)$

	Z_2	$Z_{16}Z_2$	0	Z_2^2	Z_4	Z_8	Z_4	Z_2	0	Z_8Z_4	Z_2	Z_2	Z_2	Z_8	0
$\pi_*(P_{k+1})$	Z_2^3	$Z_{32}Z_4Z_2$	Z_4^2	$Z_8Z_4Z_2$	Z_8Z_4	$Z_8Z_2^3$	$A\,Z_8Z_{16}Z_2$	$BZ_2^3Z_4$	CZ_2^2	$Z_{16}Z_4^2Z_2$	$Z_8Z_2^2$	Z_8	Z_2^2	$Z_8^4Z_2$	Z^2Z_8
$\pi_*(P_k)$	Z_2^3	Z_2^5	$Z_2Z_4Z_8$	$Z_2^3Z_8$	Z_8Z_2	Z_2^4	$Z_{16}Z_2^3A$	$BZ_2^3Z_8$	CZ_2^5	$Z_2^3Z_4$	$Z_2^2Z_8$	Z_8	Z_2	$Z_2^2Z_4$	Z_2^3
$\pi_*(S^0)$	Z_2Z_{32}	Z_2^2	Z_2^4	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2	0



Z_2	Z_2	Z_8	Z_2	0	Z_2	$Z_{16}Z_2$	Z_2^3	Z_2^4	Z_2Z_8	Z_8	0	0	Z_2	$Z_{16}Z_2$

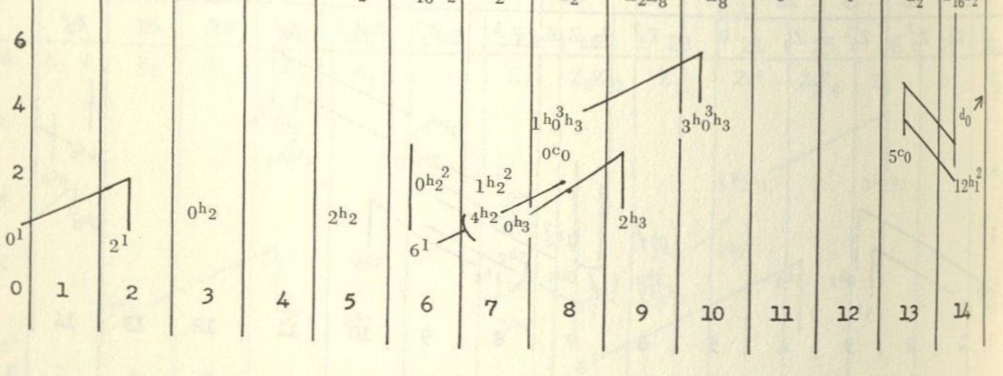
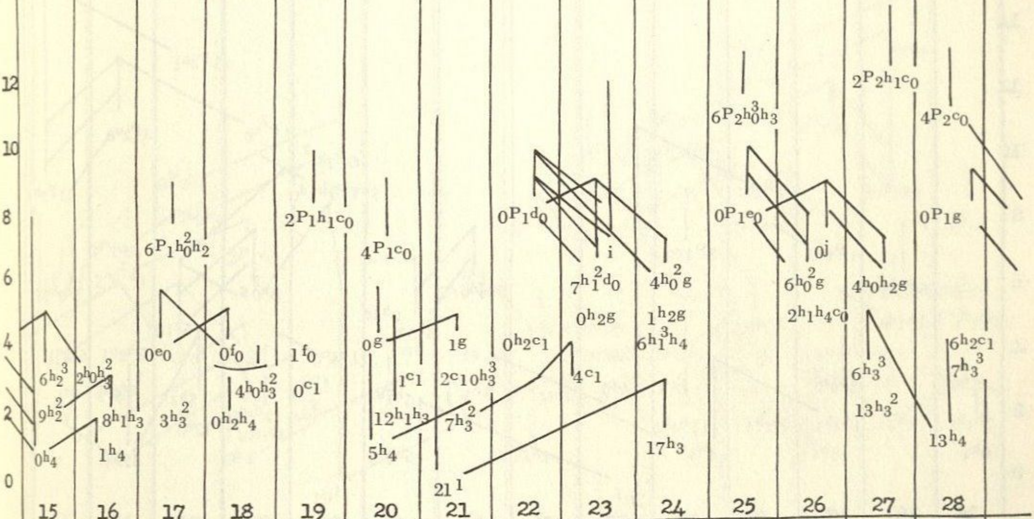


TABLE 8.11

$k \equiv 10(16)$

	0	Z_2	Z_2	Z_2^2	Z_2^2	Z_2	0	0	Z_2	Z_2^2	Z_2^2	Z_2	Z_2	Z_2	Z_2
$\pi_*(P_{k+1})$	Z_4Z_2	$Z_2^2Z_4$	$Z_2^2Z_8$	$Z_2^2Z_8$	Z_2^5	$AZ_{16}Z_2^2$	BZ_2^3	CZ_2	$Z_4Z_2^2$	$Z_2^2Z_4Z_8$	Z_8Z_2	Z_2^2	$Z_2^3Z_4$	$Z_2^3Z_8$	$Z_{16}Z_4Z_2^3$
$\pi_*(P_k)$	$Z_{32}Z_2Z_4$	Z_4^2	$Z_8Z_4Z_2$	Z_8Z_4	$Z_8Z_2^3$	$AZ_8Z_2Z_{16}$	$BZ_2^3Z_4$	CZ_2^2	$Z_{16}Z_4^2Z_2$	$Z_8Z_2^2$	Z_8	Z_2^2	$Z_8Z_4Z_2$	$Z_8Z_2^2$	$Z_{16}Z_2^4$
$\pi_*(S^0)$	Z_2Z_{32}	Z_2^2	Z_2^4	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2	0



Z Z_4 0 Z_4 Z_2 Z_{16} Z_2^2 Z_2Z_{16} Z_4 Z_8 0 Z_8 Z_2 Z_{16} Z_2^3

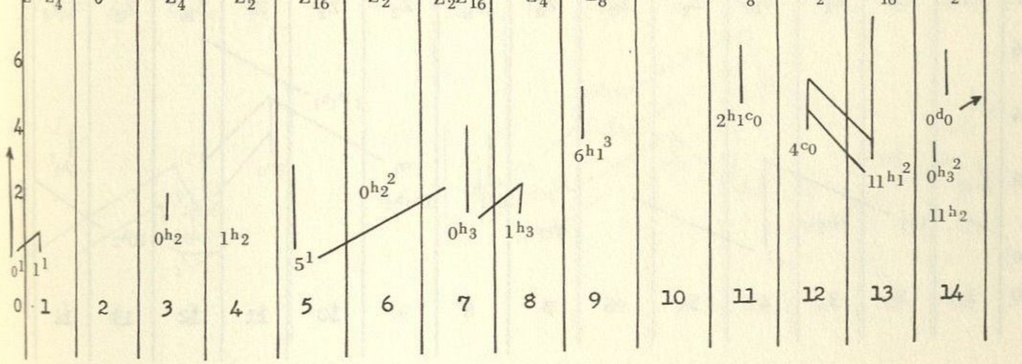


TABLE 8.12

$k \equiv 11(16)$

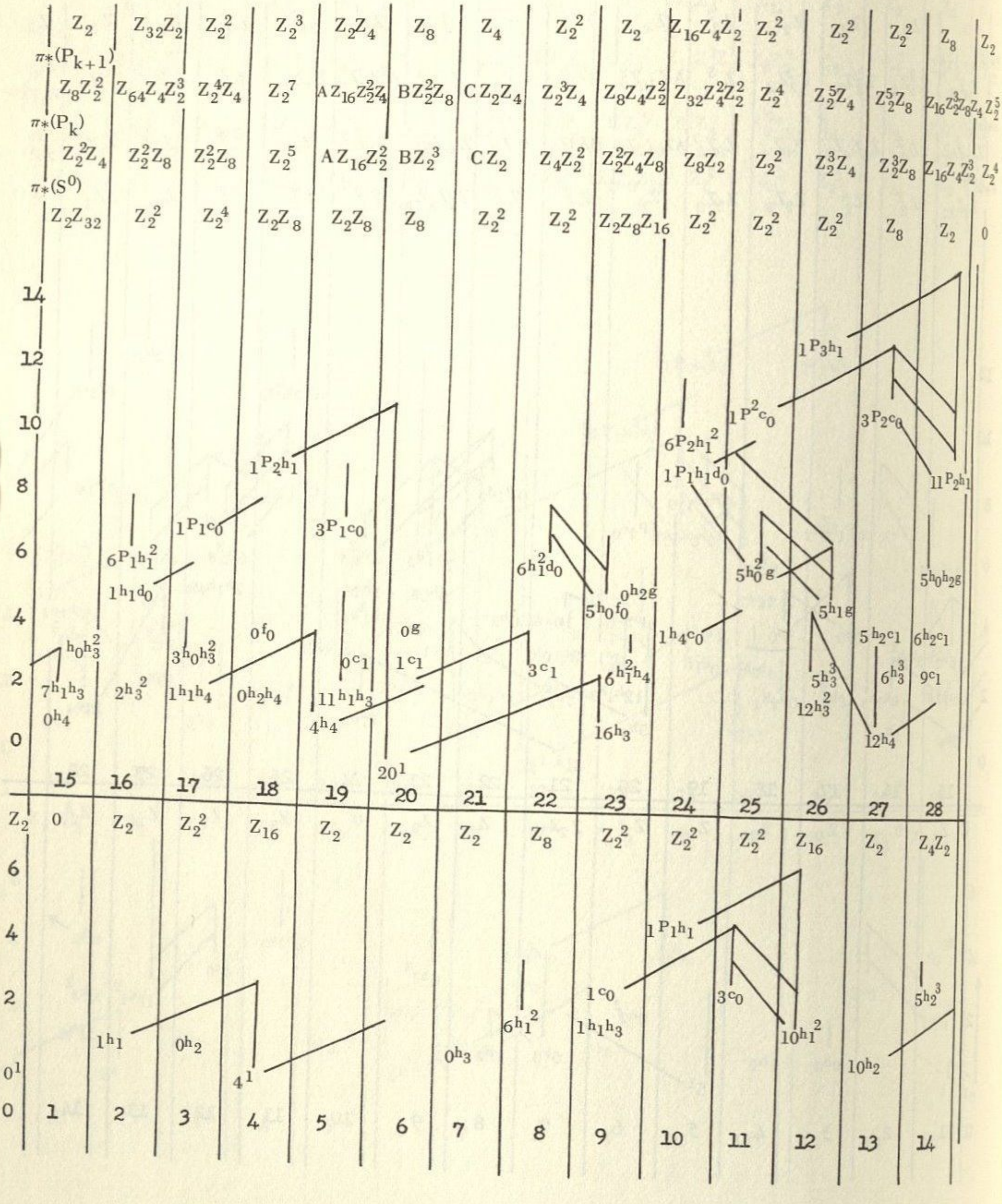


TABLE 8.13

$k \equiv 12(16)$

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
	0	0	0	0	0	0	Z_2	Z_2	Z_2	Z_2	0	0	Z_2	0	
$\pi_*(P_{k+1})$	$Z_2^2Z_8$	Z_2^4	Z_2^3	AZ_8Z_2	BZ_2	CZ_2	$Z_4Z_2^3$	$Z_2Z_8^2$	$Z_2^2Z_8$	Z_2^3	$Z_2^3Z_4$	Z_8Z_2	$Z_{16}Z_8Z_2^3$	Z_2^4	0
$\pi_*(P_k)$	$Z_{64}Z_2^3Z_4$	$Z_2^4Z_4$	Z_2^7	$AZ_{16}Z_4Z_2^2$	$BZ_2^2Z_8$	CZ_2Z_4	$Z_2^3Z_4$	$Z_8Z_4Z_2^2$	$Z_{32}Z_4^2Z_2^2$	Z_2^4	$Z_2^5Z_4$	$Z_2^5Z_8$	$Z_{16}Z_4Z_8Z_2^3$	Z_2^5	0
$\pi_*(S^0)$	Z_2Z_{32}	Z_2^2	Z_2^4	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2	0

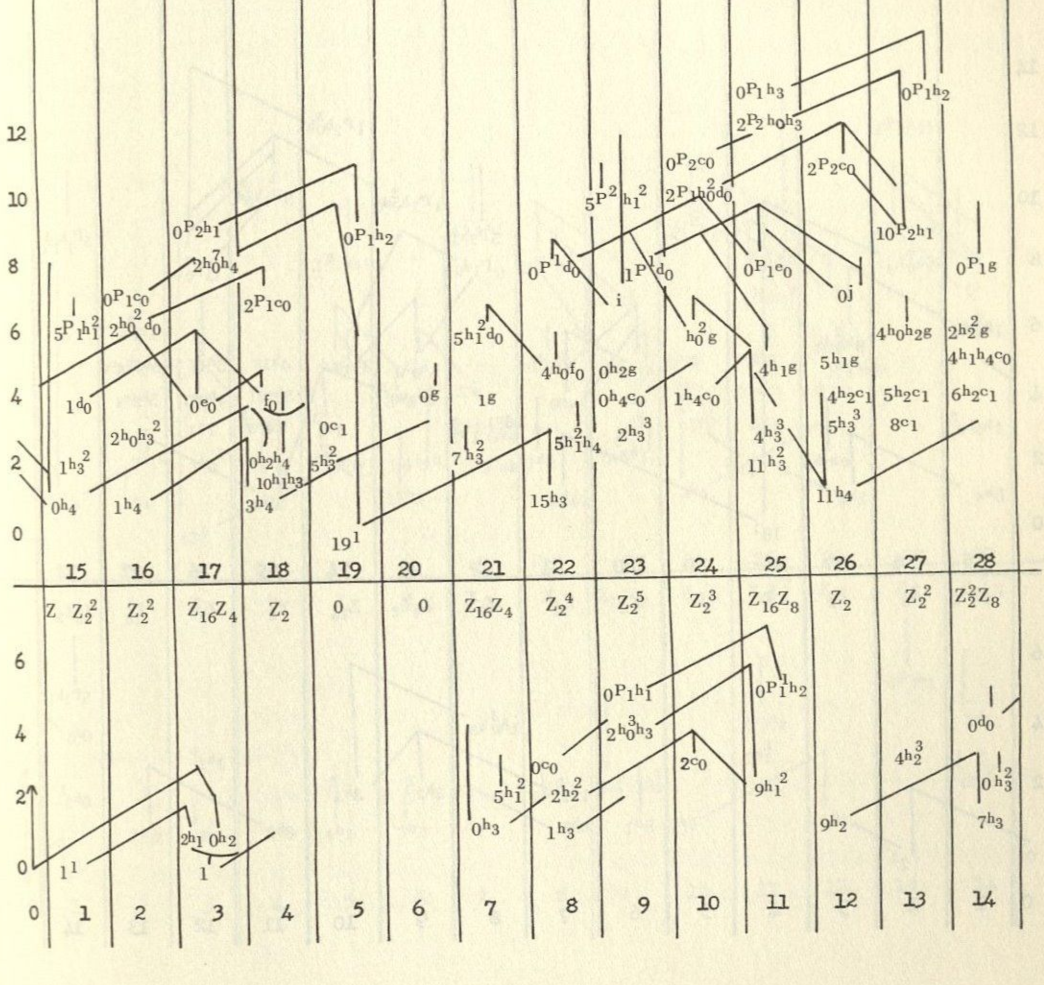


TABLE 8.14

$k \equiv 13(16)$

	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
	0	Z_{16}	0	Z_2^2	CZ_4	Z_8	Z_4	Z_2	Z_2^2	Z_4Z_8	Z_2	Z_2	Z_2	Z_8	Z_2
$\pi_*(P_{k+1})$	Z_2^2	Z_{32}	AZ_4	$\bar{B}Z_2^2$	Z_2Z_4	$Z_2^2Z_4Z_8$	$Z_8^2Z_4$	$Z_8Z_2^3$	Z_2Z_4	$Z_{16}Z_2Z_4Z_8$	$Z_4Z_2^3$	$Z_{16}Z_8Z_2^3$	$Z_2^3Z_4$	Z_8	Z_2^2
$\pi_*(P_k)$	Z_2^4	Z_2^3	AZ_8Z_2	BZ_2	CZ_2	$Z_4Z_2^3$	$Z_2Z_8^2$	$Z_2^2Z_8$	Z_2^3	$Z_2^3Z_4$	$Z_2Z_2^4$	$Z_{16}Z_8Z_2^3$	Z_2^4	0	Z_2
$\pi_*(S^0)$	Z_2Z_{32}	Z_2^2	Z_2^4	Z_2Z_8	Z_2Z_8	Z_8	Z_2^2	Z_2^2	$Z_2Z_8Z_{16}$	Z_2^2	Z_2^2	Z_2^2	Z_8	Z_2	0

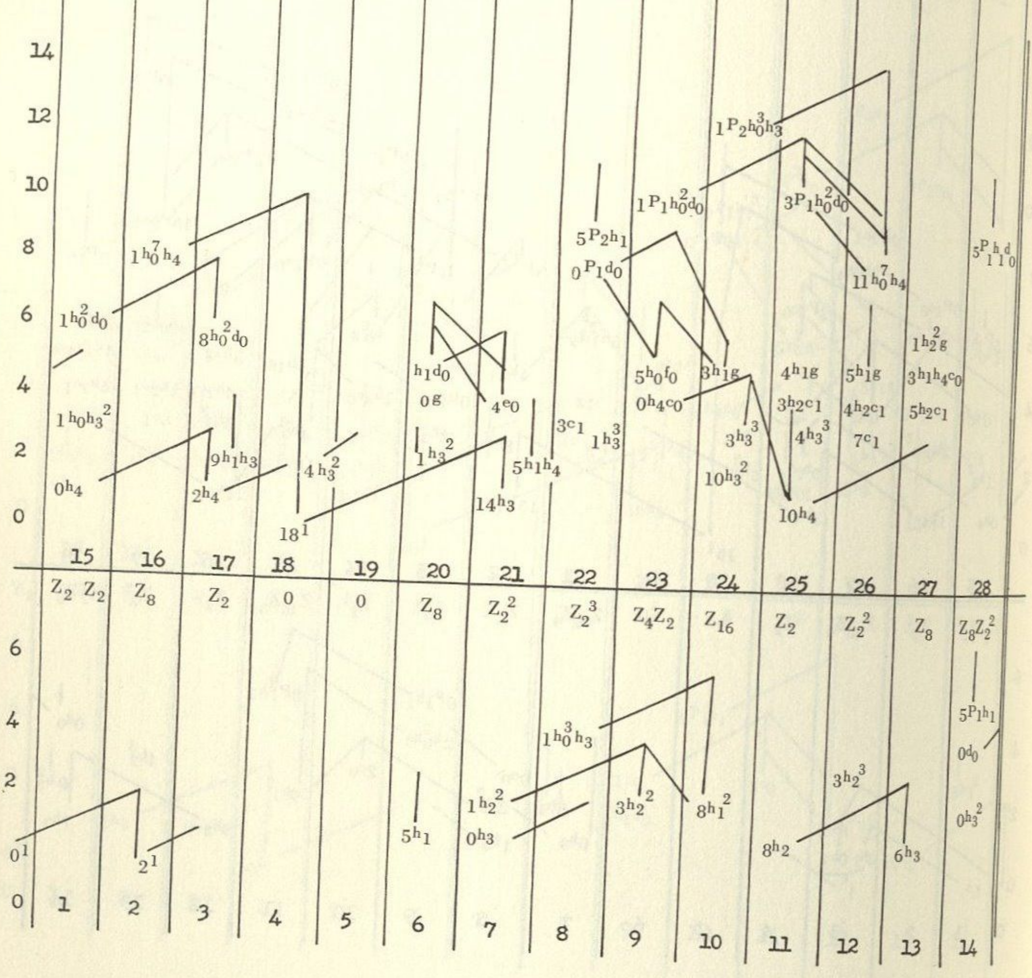


TABLE 8.15

$k \equiv 14(16)$

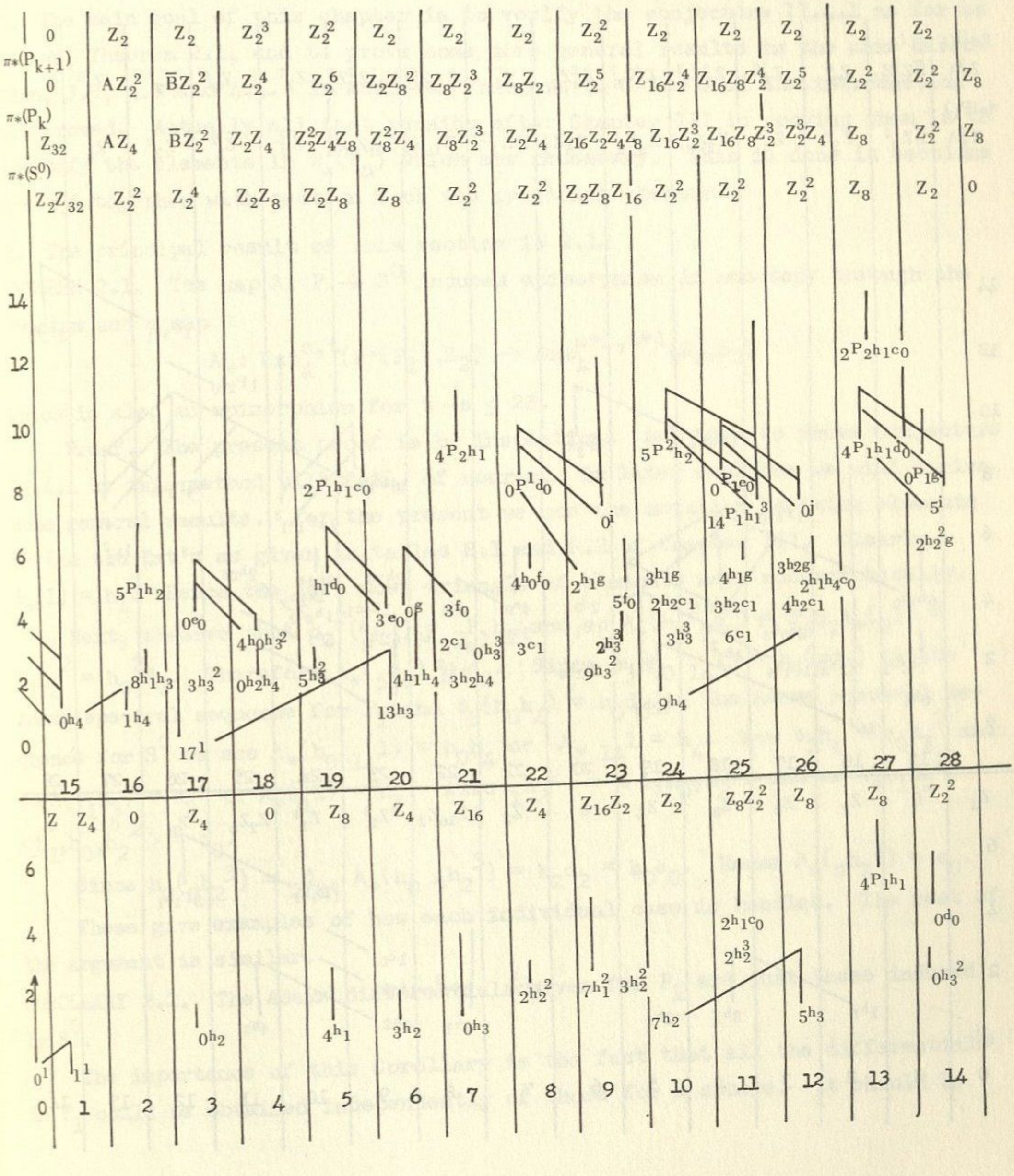
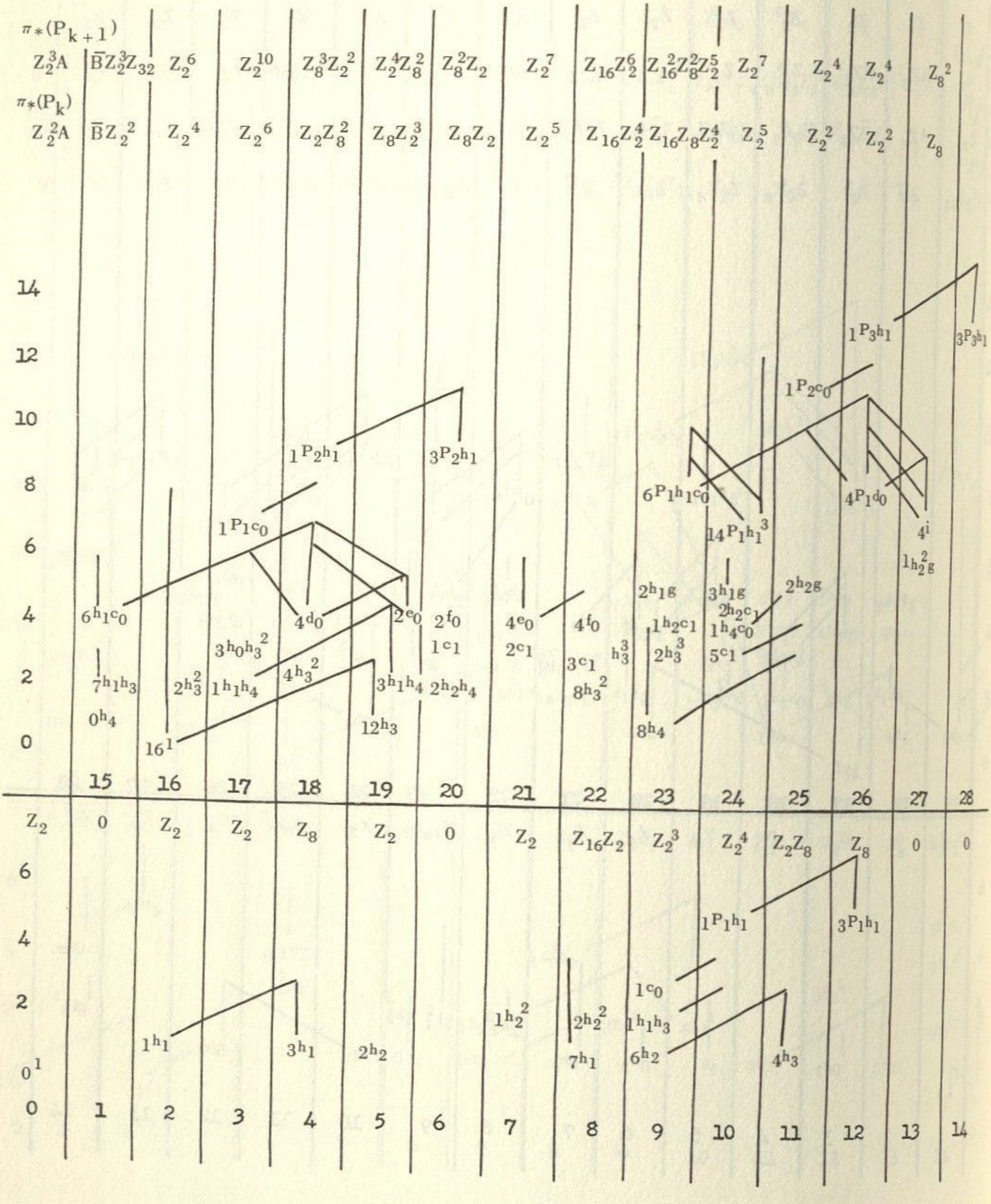


TABLE 8.16

$k \equiv 15(16)$



CHAPTER IV

SOME PROPERTIES OF λ

1. The main goal of this chapter is to verify the conjecture II.4.1 as far as we can, Theorem 2.1, and to prove some more general results in the same direction, 3.3, 4.3 and 4.4. In addition Theorems B, C and D of the introduction are proved. Actually all that remains after Chapter III in proving them is to identify the elements in $\pi_*(P_n)$ which are necessary. This is done in sections 4 and 5 together with section 7 of the preceding chapter.

2. The principal result of this section is 2.1.

THEOREM 2.1. The map $\lambda: P \to S^0$ induces epimorphism in homotopy through the 29-stem and a map

$$\lambda_*: \mathrm{Ext}_A^{s,t}(H^*(P_1),Z_2) \to \mathrm{Ext}_A^{s+1,t+1}(Z_2,Z_2)$$

which is also an epimorphism for $t-s \leq 28$.

Proof. The present proof is by inspection. Any hope to prove Conjecture II.4.1 by this method will fail, of course. In later sections we will derive some general results. For the present we use the notation defining elements in the two Ext's as given in tables 8.1 and 8.2 of Chapter III. Clearly $\lambda_*(1) = h_1$. Hence the left most triangle of elements maps monomorphically.

Next, observe that $h_0^2({}_5c_0) = {}_2P_1h_2$ and so $\lambda_*(h_0^2{}_5c_0) = P_1h_2\lambda_*({}_21) = P_1h_2^2 = h_0^2d_0$. Therefore $\lambda_*({}_5c_0) = d_0$. Since $\delta_3(h_0\,{}_{14}1) = h_0({}_5c_0)$ in the Adams spectral sequence for P_1 and $\delta_3(h_0h_4) = h_0d_0$ in the Adams spectral sequence for S^0 we see $\lambda_*(h_0\,{}_{14}1) = h_0h_4$ or $\lambda_*\,{}_{14}1 = h_4$. Now $\delta_2h_4 = h_0h_3^2$ and so $\lambda_*({}_ih_3) = h_3^2$ or $\lambda_*({}_i1) = h_3$. Also ${}_1h_2^2 = \langle 1,h_0,h_2^2\rangle$ and so $\lambda_*({}_1h_2^2) = \langle h_1,h_0,h_2^2\rangle = c_0$.

Since $h_0({}_7h_2^3) = {}_2d_0$, $\lambda_*(h_0\,{}_7h_2^3) = h_2d_2 = h_0e_0$. Hence $\lambda_*({}_7h_2^3) = e_0$.

These give examples of how each individual case is handled. The rest of the argument is similar.

COROLLARY 2.2. The Adams differentials given for P_1 are just those induced by S^0.

The importance of this Corollary is the fact that all the differentials in P_1 could be obtained independently of those for a sphere. It should be noted that the argument of Chapter III does not do this. There the result for a sphere of $\delta_2(e_0)$ is used to start the induction. We will prove here

LEMMA 2.3. In P_1, $\delta_2({}_3h_0h_3{}^2) = {}_1h_1d_0$.

REMARK 2.4. $\lambda_*({}_3h_0h_3{}^2) = f_0$ and all the differentials for a sphere throughout $t-s \leq 29$ not pertaining to P_ih_j can easily be obtained from this.

Proof. First consider

$$S^3 \xrightarrow{i} P_3{}^4 \to S^4.$$

Using [2:2.6.1] it is easy to construct Ext for $P_3{}^4$ in homotopy dimension 18. The following classes appear: ${}_2h_4$, ${}_3h_0h_3{}^2$, ${}_2h_1d_0$, ${}_3h_0{}^2d_0$. The homotopy exact sequence shows that $\langle i, 2\iota, \sigma\sigma\rangle$, $\langle i, 2\iota, \kappa\rangle$, $i_*\eta\kappa$ and $i_*\rho$ are all non-zero classes in $\pi_{18}(P_3{}^4)$ with $2\langle i, 2\iota, \kappa\rangle = i_*\eta\kappa$. It is not hard to verify that $\{{}_3h_0h_3{}^2\}$ projects to $\langle i, 2\iota, \kappa\rangle$ (particularly in view of 7.1). Now consider

$$S^2 \vee S^3 \to P_2{}^4 \to S^4.$$

In homotopy $\partial_*\kappa = {}_2\eta\kappa$ since $\partial_*\iota = {}_2\eta + {}_12\iota$. The only way in the Adams spectral sequence to accomplish this is for $\delta_2({}_3h_0h_3{}^2) = ({}_1h_1d_0)$. This completes the proof of the lemma.

This lemma and the discussion before it suggest strongly that all differentials in the Adams spectral sequence are direct consequences of the Hopf invariant, one problem which, from our point of view, is just the vector field problem.

3. Using the Adams periodicity we see that the edge of Ext for P_1 is continued periodically. In particular, in each $8j-2$ stem there is a collection of at least four elements connected by h_0 and ending with filtration $4j-1$. For example, if $t-s = 22$, $({}_{12}P_1h_1{}^2)$ generates such a family. A portion of this family can be described using the periodicity theorem (theorem 5 of [3]).

Let $Ⓟ_j$ be the periodicity operator raising $t-s$ by $8j$ and s by $4j$.

PROPOSITION 3.1. In $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(P_1), Z_2)$,

$$Ⓟ_{k\cdot 2_{j+1}} h_0{}^i({}_{2^{j+3}}1)$$

is non-zero for $k \geq 0$, $n \geq 0$ and $0 \leq i < 2^{j+2}$.

Proof. The definition of λ implies $\lambda_*({}_{2^{j+3}}1) = h_{j+3}$. Naturality and

II.6.15 of [21] complete the proof.

DEFINITION 3.2. In Ext for P_1 with $t-2 = 8k-2$, $8k \equiv 2^j(2^{j+1})$, there is a class b_k and an integer i such that $h_0{}^i b_k = ⓟ_q(_{2}j{-}2^1)$ for an appropriate q and $b_k \neq h_0\alpha$ for any α.

THEOREM 3.3. $\lambda_* b_k \neq 0$.

The proof is clear.

4. In $\mathrm{Ext}_A^{4,12}(H^*(P_1^2),Z_2)$ there is a class $_1h_0{}^3h_3$ which is a permanent surviving cycle and represents a coextension of 8σ by 2ι. Let $i\colon P_1^2 \subset P_1$ and let $\mu_1' = i_*\{_1h_0{}^3h_3\}$. (In table III 8.2 the symbol $_1h_0{}^3h_3$ represents μ_1'.) Define $\mu_k' = \langle \mu_{k-1}', 2\iota, 8\sigma\rangle$ where the coextension of 8σ is always taken to be $\{_1h_0{}^3h_3\}$. Let $\mu_k = \lambda_*\mu_k'$.

PROPOSITION 4.1. i) $\mu_k \neq 0$.

ii) P^kh_1 is a surviving permanent cycle and $\mu_k = \{P^kh_1\}$.

iii) $d_R(\mu_k) = \frac{1}{2}(1)$ where R is the Adams invariant.

iv) $\eta\mu_k$ and $\eta^2\mu_k \neq 0$.

Proof. Clearly iii) will imply i). But $\mu_k \in \langle \mu_{k-1}, 2\iota, 8\sigma\rangle$ and so if we show $e_c(\mu_1) = 1/2$ we are done. But $\lambda_*\{_1h_0{}^3h_3\} \in \langle \eta, 2\iota, 8\sigma\rangle$ and thus satisfies $e_c(\mu_1) = \frac{1}{2} \bmod 1$. By [4] $d_R = e_c$ in this case.

Notice that our requirement that the coextension of 8σ used always has filtration (4,12) implies that the filtration of μ_k' is (4k,12k) and hence μ_k must have filtration (4k+1,12k+2), which means that it must project to P^kh_1, proving ii).

Since our μ_k is essentially the same as Adams μ_{8k+1} (they are defined by the same Toda bracket), iv) follows from [4], 12.14 and 12.17. This completes the proof.

Proposition 4.1.iv implies that $\{P^kh_1{}^3\} \neq 0$ and hence $\{P^kh_2\} \neq 0$. Let $\xi_k = \{P^kh_2\}$.

PROPOSITION 4.2. $\eta^2\mu_k = 4\xi_k$ and $\xi_k \in \operatorname{im}\lambda_*$.

The proof is clear and this proves vii) of theorem B since $e(\eta^2\mu) = 1/2$ and e is a homomorphism.

In Theorem III.7.4 let $n = -1$ and consider the sequence $P^0_{-1} \to P_{-1} \xrightarrow{\tilde{i}} P_1$.* Suppose $k \equiv 2^m(2^{m+1})$, $m \geq 3$. Then $i(-1,k) = \frac{k}{2} - m - 1$ and $\tilde{i}_*(h_0{}^i 1_k)$ is a permanent cycle. Let $\lambda_*\{\tilde{i}_* h_0{}^i 1_k\} = \rho_{k/8}$.

PROPOSITION 4.3. i) $\rho_j \neq 0$, indeed $e(\rho_j) = 2^{-m+2}(\text{mod } 1)$ where $8j = k \equiv 2^m(2^{m+1})$.

ii) $\eta\rho_j = \{P^{j-1}c_0\}$ modulo elements divisible by 2.

iii) order $\rho_j = 2^{m-2}$.

Proof. Let $a_1 \in \text{Ext}_A^{3,9}(H^*(P_1), Z_2)$ be the non-zero class. $\lambda_*\{a_1\} = 8\sigma \neq 0$. Also $8\rho_1 = 8\sigma$. Let $\{a_j\} = \langle\{a_{j-1}\}, 2\iota, 8\sigma\rangle$ where we use $\{{}_1h_0^3h_3\}$ as the particular coextension of 8σ. Clearly $e(\lambda_* a_j\} = 1/2$ and therefore $\{a_j\} \neq 0$. By a filtration consideration then $\{a_j\} = 2^{m-3}\{h_0{}^i 1_k\}$ where k, j and m are related as above. Hence $e(\lambda_*\{h_0{}^i 1_k\}) = 2^{-m+2}(\text{mod } 1)$.

Consider the map $\bar{\lambda}: P_1 \xrightarrow{\lambda} S^0 \xrightarrow{i} S^0 \cup e^1$. An argument essentially paralleling the proof of III.7.4 but in homotopy shows that $\bar{\lambda}_*\{h_0{}^i 1_k\} = \{\overline{P^{j-1}h_2{}^2}\}$. Since $h_1\overline{P^{j-1}h_2{}^2} = P^{j-1}c_0$, part ii) is established. Clearly $2\{a_j\} = 0$ and so $2^{m-2}\{h_0{}^i 1_k\} = 0$ but $2^{m-3}\{h_0{}^i 1_k\} = a_j \neq 0$. This completes the proof of the proposition.

This proposition completes the definition of the Adams collection of elements, table I.1 and the proof of theorem B.

5. We will now prove theorem C. The main tool is II.4.6 and the results of Chapter III. Notice that if $n \equiv 0(2)$ $[\iota_n, 2^p\rho_j] \neq 0$ if $p < m-3$ and $[\iota_n, 2^p\xi_j] \neq 0$ if $p \leq 2$ while if $n \equiv 1(2)$ $[\iota_n, 2\rho_j] = [\iota_n, 2\xi_j] = 0$. Now the results of Chapter III prove theorem C except for the following cases: $i \equiv 0(8)$, $2^{m-3}\rho_j$; $i \equiv 6(8)$, μ_j; $i \equiv 5(8)$, $\eta\mu_j$; and $i \equiv 4(8)$; $\eta^2\mu_j$.

First consider $2^{m-3}\rho_j$ for $i = 8p$. This produces a class in $\pi_{8j-1+8p}(P_{8p})$ with s-filtration $4j$. Suppose $8(j+p) \equiv 2^v(2^{v+1})$. Let $q = v - 1 + 4j$ and let β_q be as defined in Chapter I. Finally let $n = 8p - \varphi(q) + 8j - 1$. Then the

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*The complex P_{-1} as a stable object is the Thom complex of the bundle over ΣP_1 induced by the adjoint of λ_1, $\lambda: \Sigma P_1 \to BSO$.

homotopy version of III 7.4 asserts that in the sequence

$$\begin{array}{ccccc} & & \pi_{8(j+p)-1}(S^{8p}) & & \\ & & \downarrow d_* & & \\ \pi_{8j+8p-1}(P_n) & \rightarrow & \pi_{8(j+p)-1}(P_{8p}) & \xrightarrow{\partial_*} & \pi_{8(j+p)-2}(P_n^{8p-1}) \\ & & & & \uparrow c_* \\ & & & & \pi_{8(j+p)-2}(S^n) \end{array}$$

$\partial_* d_*(2^{m-3}\rho_j) = c_*(\beta_q)$. Thus by II.4.6 if $3n-2 > 8(j+2p)-1$ the theorem holds. By an easy calculation this is $8p > 8j+6v-2$. Now consider $i = 8p-2$ and μ_j. This produces a class in $\pi_{8(j+p)-1}(P_{8p-2})$ with s-filtration $4j+1$. Let v be as above and let $q = v+4j$ with β_q defined as above. Let $n = 8p-\varphi(q)+8j-1$. Then we have the same diagram as above with the same conclusion. The estimate again comes out $8p > 8j+6v+2$.

The other two cases are done in a similar fashion, with the estimates being $8p > 8j+6v+5$ and $(i = 8p-3)$ $8p > 8j+6v+7$ $(i = 8p-4)$ respectively.

The above argument also completes the proof of theorem D.

CHAPTER V

UNSTABLE GROUPS

1. The purpose of this chapter is to give the general results on $\pi_{k+2n}(S^n)$ for $-1 \leq k \leq 27$ that can easily be obtained from the calculations of Chapter III. The results are not as sharp as one would like because of the lack of a particular calculation in Ext for spheres, conjecture 2.4. It seems to us that the argument proving 2.6 is the most valuable contribution in this Chapter.

Throughout this Chapter the maps $P_{k,n}$ and $I_{k,n}$ are the ones in the Toda sequence

$$\pi_{j+k}(S^{n+k}) \xrightarrow{I_{k,n}} \pi_{j-1}(\Sigma^{n-1}P_n^{n+k-1}) \xrightarrow{P_{k,n}} \pi_{j-1}(S^n).$$

Using the propositions of section 3 and the tables of Chapter III almost all Whitehead products among elements in the first 20 stems can be determined. There seems to be little point in tabulating them. Recall also that $P_{k,n}$ is essentially the unstable J-homomorphism.

In connection with [19] one should compare 3.5 and 3.17 together with the observation that for all other congruence classes the tables of Chapter III settle the Whitehead products discussed there. Also among unstable groups the homomorphism described in the tables of Chapter III is just the one for the EHP sequence with exceptions as noted (compare section 3).

As a useful exercise we have the following table which is given without proof. The details consist just of gathering together all that we have done in Chapter III and the latter section of Chapter II. The determination of the stable groups is given in [20]. Let $I_n : \pi_p(S^0) \to \pi_{p-1}(\Sigma^{n-1}P_n)$. The Toda sequence requires that $n > (p+3)/3$.

Table 1.1

The Hopf invariant of some stable homotopy classes through the 40 stem. The element β which is the image of α under I_n is defined by what it looks like for the largest n for which $I_n(\alpha) \neq 0$.

$p = 23, n \geq 9 \quad I_n = 0$

$p = 24, n \geq 10 \quad I_n = 0$

$p = 25, n \geq 10 \quad I_n = 0$

$p = 26, n \geq 10$	$I_n = 0$	
$p = 27, n \geq 11$	$I_n = 0$	
$p = 28, n \geq 11$	$I_n = 0$	
$p = 29, n \geq 11$	$I_n = 0$	
$p = 30, n \geq 12$	$I_n(\{h_4^2\}) = \eta\sigma$	
$p = 31, n \geq 12$	$I_n(\{h_1h_4^2\}) = \{h_2h_4\}$	$22 \geq n \geq 12$
	$I_n(\{h_0^{10}h_5\}) = \{P^2h_1^2\}$	$n = 12, 13$
	$I_n(\{h_0^{11}h_5\}) = \{P^2h_1\}$	$n = 12, 13$
$p = 32, n \geq 12$	$I_n(\{h_1h_5\}) = \{h_2\}$	$n = 12$
$p = 33, n \geq 13$	$I_n(\{d_1\}) = \{c_1\}$	$29 \geq n \geq 12$
	$I_n(\{p\}) = \{c_1\}$	$n = 13$
	$I_n(\{h_1^2h_5\}) = \{h_2^2\}$	$n = 13, 14$
$p = 34, n \geq 13$	$I_n(\{h_2h_5\}) = \{h_3\}$	$27 \geq n \geq 13$
	$I_n(\{h_0h_2h_5\}) = \{h_1h_3\}$	$27 \geq n \geq 13$
	$I_n(\{h_0^2h_2h_5\}) = \{h_1^2h_3\}$	$26 \geq n \geq 13$
		$25 \geq n \geq 13$
$p = 35, n \geq 13$	$I_n = 0$	
$p = 36, n \geq 14$	$I_n = 0$	
$p = 37, n \geq 14$	$I_n(\{h_2^2h_5\}) = \{h_3^2\}$	$23 \geq n \geq 14$
	$I_n(\{x\}) = \{h_4c_0\}$	$n = 14$
$p = 38, n \geq 14$	$I_n(\{h_0^2h_3h_5\}) = \{h_1^2h_4\}$	$21 \geq n \geq 14$
	$I_n(\{h_0^3h_3h_5\}) = \{h_1^3h_4\}$	$20 \geq n \geq 14$
	$I_n(\{e_1\}) = \{h_3^3\}$	$17 \geq n \geq 14$
$p = 39, n \geq 15$	$I_n(\{h_1h_3h_5\}) = \{h_2h_5\}$	$21 \geq n \geq 15$
	$I_n(\{h_5c_0\}) = \{c_1\}$	$20 \geq n \geq 15$
$p = 40, n \geq 15$	$I_n(\{h_1^2h_3h_5\}) = \{h_2^2h_4\}$	$19 \geq n \geq 15$
	$I_n(\{h_5c_0h_1\}) = \{h_2c_1\}$	

2. Consider the table of Ext for P_k, $k \equiv 3(16)$. We will study the following subset of that table:

s \ t − s	12	13	14	15	16	17	18	19	20	21
4								${}_1h_1{}^3h_4$		
3				${}_6h_2{}^3$				${}_2h_1{}^2h_4$		${}_0h_2{}^2h_4$
2			${}_8h_2{}^2$	${}_7h_1h_3$			${}_4h_3{}^2$	${}_3h_1h_4$	${}_2h_2h_4$	
1		${}_{10}h_2$		${}_8h_3$				${}_4h_4$		
0	${}_{12}1$									

Table 2.1

First we recall the results on multiplication by h_0 and h_1.

a) $h_1{}^3({}_{12}1) = h_1{}^2({}_{10}h_2) = h_1({}_8h_2{}^2) = {}_6h_2{}^3 = h_0{}^2({}_8h_3) = h_0({}_7h_1h_3)$ and

b) $h_0{}^3({}_4h_4) = h_0{}^2({}_3h_1h_4) = h_0{}^1({}_2h_1{}^2h_4) = {}_1h_1{}^3h_4$.

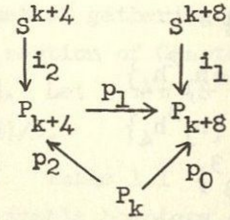
c) $h_1{}^2({}_4h_4) = h_1({}_2h_2h_4) = {}_0(h_2{}^2h_4)$.

Then we will prove

PROPOSITION 2.2. $h_2{}^2({}_{12}1) = h_2{}^1({}_8h_3) = ({}_4h_3{}^2)$.

Proof. That $h_2({}_{12}1) = {}_8h_3$ follows immediately from $Sq^4\alpha^{15+16k} = Sq^8\alpha^{11+16k} = \alpha^{19+16k}$.

Consider the diagram



By definition $p_{0*}({}_8h_3) = i_{1*}h_3$. But clearly $p_{1*}\langle 1_{k+4}, h_2h_3\rangle = i_{1*}h_3$ too. Now $\langle 1_{k+4}, h_2, h_3\rangle h_2 = p_{2*}(h_2({}_8h_3)) = i_{2*}\langle h_2h_3h_2\rangle = i_{2*}h_3{}^2$.

PROPOSITION 2.3. Suppose $k \equiv 19(32)$. Then $h_3({}_{12}1) = {}_4h_4$ and hence $h_2({}_4h_3{}^2) = {}_0h_2{}^2h_4$.

The proposition follows immediately from $Sq^8\alpha^{31+32k} = Sq^{16}\alpha^{23+32k} \neq 0$.

From May's calculations it seems likely that the following is true.

Conjecture 2.4. For $k \geq 6$ $\beta h_k^{\,2} \neq 0$ for β any non-zero monomial in h_i with $t-s \leq 10$.

If $k = 6$ Tangora has verified the conjecture (unpublished) while if $k = 5$ we have the following:

LEMMA 2.5 (Tangora [24]). If $k = 5, \beta h_k^{\,2} \neq 0$ for α any non-zero monomial in h_i with $t-s \leq 3$.

The main result of this section is the next theorem.

THEOREM 2.6. Suppose $k+13 = (2^i + j)(2^{i+1})$, $j \equiv 1(2)$, and $j > 0$. If $\beta h_i^{\,2} \neq 0$ in $H^*(A)$, for $\beta \in H^*(A)$ $(t-s < 2^i$ for $\beta)$, then $\beta({}_{12}1)$ is not in the image of p_* where

$$p\colon P_{(j-1)2^{i+1}+2^i+1} \to P_k.$$

Proof. The map $\lambda\colon P \to S^0$ satisfies $\lambda_*({}_{2}i - {}_{2}1) = h_i$. Hence $\lambda_*({}_{2}i - {}_{2}h_i) = h_i^{\,2}$. Thus if $\beta h_i^{\,2} \neq 0$, then $\beta({}_{2}i - {}_{2}h_i) \neq 0$. Hence in E_{2^i} of the pre-spectral sequence the same conclusion must hold. Now $E_{2^i}(P_1) \simeq E_{2^i}(P_{(j-1)2^{i+1}+2^i+1})$ since as modules over A involving cohomology operation which raise dimension by less than 2^i, $H^*(P_1) \simeq H^*(P_q)$ where $q = (j-1)2^{i+1} + 2^i + 1$. But in $E_{2^i}(P_q)$, $\delta_{2^i}({}_{(2^{i+1}-2)}1) = {}_{(2^i-2)}h_{2^i}$ since

$$Sq^{2^i}\alpha^{q+2^i-2} = \alpha^{k+12}.$$

Hence the theorem follows.

Using [2] we know that $h_i h_j^{\,2} \neq 0$ if $i \leq j-3$.

3. Using theorem 2.6 we will now investigate the first few unstable groups. First we show

LEMMA 3.1. a) Applying Ext to the sequence $P_{k-6}^{k-1} \to P_{k-6} \to P_k$ where $k \equiv 5(16)$ and letting δ be the coboundary in the resulting sequence we have $\delta({}_{8}c_1) = d_1$ and $\delta({}_{9}c_1) = p$.

b) Applying Ext to the sequence $P_{k-8}^{k-1} \to P_{k-8} \to P_k$ with the other notation as above we have $\delta(h_0({}_{9}c_1)) = h_2 d_1$.

Proof. Recall $d_1 = \langle h_3, h_2, h_1, c_1 \rangle$. Consider the diagram

$$P_{k+8} \xleftarrow{p_1} P_{k+6} \xleftarrow{p_2} P_{k+2} \xleftarrow{p_3} P_k \xleftarrow{p_4} P_{k-6}$$

$$(p_2p_3)_*({}_8c_1) = \langle 1_{k+6}, h_1, c_1\rangle = \langle 1_{k+6}, h_1, \langle h_2, h_3, h_1h_3\rangle\rangle$$
$$= \langle\langle 1_{k+6}, h_1, h_2\rangle, h_3, h_1h_3\rangle .$$

Now $p_{2*}\langle 1_{k+2}, \langle h_2, h_1, h_2\rangle, h_3, h_1h_3\rangle = \langle\langle 1_{k+6}, h_1, h_2\rangle, h_3, h_1h_3\rangle$. Finally in the sequence for $P^{k+1} \to P_{k-6} \to P_{k+2}$ we have $\delta_* 1_{k+2} = h_3$ or $\delta_*\langle 1_{k+2}, h_1h_3, h_3, h_1h_3\rangle = d_1$. Since $h_1 1_{k-6} = h_2 1_{k-8}$ we get part b) of the proposition while the second part of a) then follows from the module extension property given by 3.15.

LEMMA 3.2. a) Applying Ext to the sequence $P_{k-10}^{k-1} \to P_{k-10} \to P_k$ where $k \equiv 9(16)$ we have $\delta(h_0({}_7h_3{}^3)) = {}_0x$.

b) If $k \equiv 10(16)$, $\delta({}_7h_3{}^3) = {}_0e_1$.

LEMMA 3.3. If $k \equiv 12(16)$ then ${}_8c_1 = {}_{19}1\ c_0$.

These propositions are proved just as 3.1.

The calculations tabulated in tables I.4.2 and I.4.3 now follow by looking at

$$p_*: \pi_{j+29}(P_{j+k}) \to \pi_{j+29}(P_j)$$

and finding

a) the smallest k such that p_* is zero, or

b) a k such that there is a k' for which $p_*': \pi_{j+29}(P_{j+k}) \to \pi_{j+29}(P_{j-k'})$ is zero.

Inspection of the tables gives the first statement and Lemmas 3.1, 2 and 3 together with 2.6 supply the answers to the second part. The tables give the easy results possible by this method.

The details of this calculation are omitted but we give one case to illustrate the procedure.

PROPOSITION 3.4. If $k + n \neq 2^j + 2$, but $k + n \equiv 2(\text{mod } 16)$ then $\pi_{2n+k}(S^n) = \pi^S_{n+k} \oplus \pi_{n+k+1}(P_n)$ for $k \leq 6$ and for $k \leq 28$ if 2.4 holds.

Proof. Consider the sequence

$$\pi^S_{29+n} \xrightarrow{I_{29}} \pi_{29+n}(P_n) \to \pi_{28+2n}(S^n) \to \pi^S_{28+n} \xrightarrow{I_{28}} \pi_{28+n}(P_n) \to$$

for $n \equiv 6(16)$. Since $\pi_{29+n}(P_n) = Z_8$ and by a simple check of the differentials we see that

$$p^*: \pi_{29+n}(P_{n-5}) \rightarrow \pi_{29+n}(P_n)$$

is zero. Thus $I_{29} = 0$.

Also by inspection we see that $\mathrm{im}(p^*: \pi_{28+n}(P_n) \rightarrow \pi_{28+n}(P_{n+3}))$ is generated by $\{_{18}h_3\} = \{_{22}1\ h_2\}$. Conjecture 2.4 and theorem 2.6 show I_{28} is zero. Without 2.4 we know that $\mathrm{im}\{I_{28}\}$ could only be $\{_{22}1_{20}h_0{}^2h_2\}$, a Z_2 group, since 2.4 is verified through s filtration 3.

BIBLIOGRAPHY

1. J. F. Adams, On the structure and applications of the Steenrod algebra, Commit. Math. Helv. 32(1958), 180-214.
2. ——, On the non-existence of elements of Hopf invariant one, Ann. of Math. 72(1960), 20-103.
3. ——, Stable Homotopy Theory, Springer Verlay, Berlin, 1964.
4. ——, On the groups J(X), IV, Topology 5(1966), 21-71.
5. ——, Vector fields on spheres, Ann. of Math. 75(1962), 603-632.
6. W. D. Barcus and J. P. Meyer, The suspension of a loop space, Amer. J. Math. 80(1958), 895, 920.
7. M. G. Barratt, Homotopy operations and homotopy groups, mimeographed notes, Seattle Topology Conference, 1963.
8. —— and M. E. Mahowald, The metastable homotopy of O(n), Bull. Am. Math. Soc. 70(1964), 758-760.
9. R. Bott, The stable homotopy of classical groups, Ann. of Math. 70(1959), 313-337.
10. P. J. Hilton, A note on the P homomorphism in homotopy groups of spheres, Proc. Camb. Phil. Soc. 59(1955), 230-233.
11. —— and J. H. C. Whitehead, Note on the Whitehead product, Ann. of Math. 58(1953), 429-442.
12. C. S. Hoo and M. E. Mahowald, Some homotopy groups of Stiefel Manifolds, Bull. Am. Math. Soc., 71(1965), 661-667.
13. I. M. James, Cross section of Stiefel manifolds, Proc. Lon. Math. Soc., 8(1958), 536-547.
14. ——, On the iterated suspension, Quart. J. of Math. Oxford (2) 5(1954), 1-10.
15. ——, Spaces associated with Stiefel manifolds, Proc. Lon. Math. Soc. (3) 9(1959), 115-140.
16. ——, On the suspension triad, Ann. of Math. 63(1956), 191-247.
17. M. Kervaire, An interpretation of Whitehead's generalization of the Hopf invariant, Ann. of Math. 69(1959), 345-365.
18. M. P. MacLaren and M. E. Mahowald, A Machine Calculation of a Spectral Sequence, Sym. on Applied Math., Am. Math. Soc. (to appear).

19. M. E. Mahowald, Some Whitehead Products in S^n, Topology 4(1965), 17-26.

20. ——— and M. C. Tangora, Some differentials in the Adams spectral sequence, Topology (to appear).

21. J. P. May, The cohomology of Restricted Lie Algebras and of Hopf Algebras, Ph.D. Thesis, Princeton University (1964).

22. G. Paechter, The groups $\pi_r(V_{n,m})$, Quart. J. Math. Oxford 7(1956), 249-268.

23. N. E. Steenrod and D. B. Epstein, Cohomology operation, Ann. of Math. Studies 50, Princeton, New Jersey.

24. M. C. Tangora, The cohomology of the Steenrod Algebra, Ph.D. Thesis, Northwestern University (1966).

25. H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies, 49, Princeton, New Jersey.

26. ———, Vector fields on spheres, Bull. Amer. Math. Soc. 67(1961), 408-412.