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# Homology Fibrations and the "Group-Completion" Theorem

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A topological monoid $M$ has a classifying-space $BM$, which is a space with a base-point. There is a canonical map of $H$-spaces $M \to \Omega BM$ from $M$ to the space of loops on $BM$, and it is a homotopy-equivalence if the monoid of connected components $\pi_0 M$ is a group. The "group-completion" theorem ([2–4, 6, 9]) describes the relationship between $M$ and $\Omega BM$ in general. Let us regard $\pi = \pi_0 M$ as a multiplicative subset of the Pontrjagin ring $H_*(M)$, using singular integral homology. The map $M \to \Omega BM$ induces a homomorphism of Pontrjagin rings, and (because $\pi_0(\Omega BM)$ is a group) the image of $\pi$ in $H_*(\Omega BM)$ consists of units.

**Proposition 1.** *If $\pi$ is in the centre of $H_*(M)$ then*

$$H_*(M)[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega BM).$$

Although several proofs of this theorem have appeared its importance for the process of "Quillenization"[1] perhaps justifies our publishing the present one, which is simple and conceptual. We shall prove, moreover, a stronger statement than Proposition 1 in the two respects described in Remarks 1 and 2 below. Our method was suggested by Quillen's second unpublished proof, and by conversations with him for which we are very grateful. The use of homology fibrations arose from [5]. We have listed some examples and applications of the theorem at the end.

*Remark 1.* In Proposition 1 one need not assume that $\pi$ is in the centre of $H_*(M)$, but only that $H_*(M)[\pi^{-1}]$ can be constructed by right fractions. Recall that if $\pi$ is a multiplicative subset of a ring $A$ one says that $A[\pi^{-1}]$ can be constructed by right fractions if every element of it can be written $a p^{-1}$ with $a \in A$, $p \in \pi$, and if $a_1 p_1^{-1} = a_2 p_2^{-1}$ if and only if $a_1 p_1' = a_2 p_2'$ and $p_1 p_1' = p_2 p_2'$ for some $p_1', p_2' \in \pi$. A typical example is when $\pi$ consists of the powers of an element $x \in A$ such that $a x = x \alpha(a)$ for all $a \in A$, where $\alpha$ is an endomorphism of $A$. This arises as the Pontrjagin ring of the monoid of all maps $S^n \to S^n$ whose degrees are powers of a prime $p$, as we shall see below.

[1] This word is due to I.M. Gel'fand.

We shall prove Proposition 1 by constructing a space $M_\infty$ whose homology is obviously $H_*(M)[\pi^{-1}]$, and a homology equivalence $M_\infty \to \Omega BM$. The basic example is the case when $M = \coprod_{n \geq 0} B\Sigma_n$, where $\Sigma_n$ is the $n^{\text{th}}$ symmetric group, and the monoid structure of $M$ comes from juxtaposition $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$. Then $M_\infty$ will be $\mathbb{Z} \times B\Sigma_\infty$.

*Remark 2.* To say that a map $f: X \to Y$ is a homology equivalence may have at least two meanings. The weaker one is that $f$ induces an isomorphism of integral homology. The stronger is that $f_*: H_*(X; f^*A) \xrightarrow{\cong} H_*(Y; A)$ for every coefficient system $A$ of abelian groups on $Y$. The map $M_\infty \to \Omega BM$ we shall construct will be a homology equivalence in the stronger sense. Thus $\Omega BM$, whose components have of course abelian fundamental groups, is a "Quillenization" of $M_\infty$. The advantage of allowing twisted coefficient systems is that one can conclude that $\tilde{M}_\infty \to \widetilde{\Omega BM}$ is a homology equivalence as well as $M_\infty \to \Omega BM$, where $\widetilde{\Omega BM}$ is the universal covering space of $\Omega BM$, and $\tilde{M}_\infty$ is its pull-back to $M_\infty$. This means that the fundamental group of $\tilde{M}_\infty$ must be perfect, and so our method incorporates a general proof that the commutator subgroup of $\pi_1(M_\infty)$ is perfect. If isolated this would reduce to Wagoner's argument in [11].

Everything we say below is true if homology equivalence is given either of the above meanings. Nevertheless it will be convenient to adopt a middle definition, allowing only *abelian* coefficient systems $A$ on $Y$, i.e. those such that for each $y \in Y$ the group of automorphisms of the coefficient group $A_y$ at $y$ induced by the action of $\pi_1(Y, y)$ is abelian. Of course any system coming from $\Omega BM$ is abelian.

Our main idea is that of a *homology fibration*. In [5] a homology fibration was defined as a map $p: E \to B$ such that for each $b \in B$ the natural map $p^{-1}(b) \to F(p, b)$ from the fibre at $b$ to the homotopical fibre at $b$ is a homology equivalence. ($F(p, b)$ is defined as the fibre-product $P_b \times_B E$, where $P_b$ is the space of paths in $B$ beginning at $b$.) In this language to obtain a homology equivalence $M_\infty \to \Omega BM$ it is enough to produce a homology fibration $E \to BM$ with $E$ contractible and with fibre $M_\infty$ at the base-point.

If $M$ is a topological group which acts on a space $X$ one often considers the space $X_M$ fibred over $BM$ with fibre $X$, associated to the universal bundle $EM \to BM$. But the construction of $X_M$ makes sense even if $M$ is only a topological monoid, for $X_M$ can be described as the realization of the topological category whose space of objects is $X$ and whose space of morphisms is $M \times X$, a pair $(m, x)$ being thought of as a morphism from $x$ to $mx$. (Here, and in constructing $BM$ also, we use the "thick" realization of simplicial spaces, denoted by $\| \;\; \|$ in the appendix to [9].)

Our main result is

**Proposition 2.** *If $M$ is a topological monoid which acts on a space $X$, and for each $m \in M$ the map $x \mapsto mx$ from $X$ to itself is a homology equivalence, then $X_M \to BM$ is a homology fibration with fibre $X$.*

This should be compared with the fact that if $x \mapsto xm$ is a homotopy equivalence for each $m$ then $X_M \to BM$ is a quasifibration. (When $M$ is discrete this is a particular case of [7] (Lemma p. 98); in general it is a particular case of [9] (1.5).)

Notice that in the basic example the left action of $M=\coprod_{n\geq 0} B\Sigma_n$ on $M_\infty=\mathbb{Z}\times B\Sigma_\infty$ is essentially the "shift" maps $B\Sigma_\infty\to B\Sigma_\infty$ induced by embedding $\Sigma_\infty$ in $\Sigma_\infty$ as the permutations of $\{n, n+1, \ldots\}$. These are homology equivalences but not homotopy equivalences, even though they induce the identity on $[K; B\Sigma_\infty]$ for any compact space $K$. They would not be homology equivalences if we had allowed non-abelian coefficient systems.

To see how the group completion theorem follows from Proposition 2 let us begin with the case when $\pi_0 M$ is the natural numbers $\mathbb{N}$. Choose $m\in M$ in the component $1\in\mathbb{N}$, and let $X$ be the telescope $M_\infty$ formed from the sequence $M\to M\to M\to\cdots$, where each map is right multiplication by $m$. The homology of $M_\infty$ is the direct limit of

$$H_*(M)\to H_*(M)\to H_*(M)\to\cdots,$$

which is precisely $H_*(M)[\pi^{-1}]$ because we have assumed the latter can be formed by right fractions. For the same reason the action of $M$ on $M_\infty$ on the left is by homology equivalences. The space $(M_\infty)_M$ is the telescope of a sequence of copies of $M_M$, which is canonically contractible. (It is the standard $EM$ of [8].) So $(M_\infty)_M$ is contractible, and the homotopical fibre of $(M_\infty)_M\to BM$ is $\Omega BM$, and Proposition 2 yields Proposition 1.

The general case of Proposition 1 reduces at once to that where $\pi_0 M$ is finitely generated, for both $H_*(M)[\pi^{-1}]$ and $H_*(\Omega BM)$ are continuous under direct limits. But if $\{s_1, \ldots, s_k\}$ generate $\pi$ then $H_*(M)[\pi^{-1}]=H_*(M)[s^{-1}]$, where $s=s_1 s_2\ldots s_k$, and the preceding argument applies, defining $M_\infty$ as the telescope generated by multiplication by any element $m$ in the component $s$.

We come to the proof of Proposition 2. For technical convenience we shall adopt a stronger definition of homology-fibration than that of [5]. It is appropriate only for base-spaces $B$ which are locally contractible in the sense that each point has arbitrarily small contractible neighbourhoods. But if $M$ has this property then $BM$ has; and restricting to such $M$ is immaterial for our purposes, as both $H_*(M)$ and $H_*(\Omega BM)$ are unchanged if $M$ is replaced by the realization of its singular complex.

*Definition.* A map $p\colon E\to B$ is a *homology-fibration* if each $b\in B$ has arbitrarily small contractible neighbourhoods $U$ such that the inclusion $p^{-1}(b')\to p^{-1}(U)$ is a homology-equivalence for each $b'$ in $U$.

To justify this definition we must show that such a map is a homology-fibration in the earlier sense. This will be done in Proposition 5 below.

The advantage of the new definition is that it makes the following proposition obvious. (Cf. [5] (5.2).)

**Proposition 3.** *If*

$$\begin{array}{ccccc} E_1 & \longleftarrow & E_0 & \longrightarrow & E_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_2} \\ B_1 & \underset{f_1}{\longleftarrow} & B_0 & \underset{f_2}{\longrightarrow} & B_2 \end{array}$$

*is a commutative diagram in which* $p_0, p_1, p_2$ *are homology-fibrations, and* $p_0^{-1}(b) \to p_i^{-1}(f_i(b))$ *is a homology-equivalence for each* $b \in B_0$, *then the induced map of double-mapping-cylinders*

$$p: \operatorname{cyl}(E_1 \leftarrow E_0 \to E_2) \to \operatorname{cyl}(B_1 \leftarrow B_0 \to B_2)$$

*is a homology-fibration.*

*Proof.* Each point of the lower cylinder has arbitrarily small neighbourhoods $U$ in the form of mapping-cylinders of maps $V_0 \to V_i$ ($i=0, 1$ or $2$), and $p^{-1}(U)$ is the mapping-cylinder of $p_0^{-1}(V_0) \to p_i^{-1}(V_i)$.

Exactly as in [9] (1.6) one deduces

**Proposition 4.** *If* $p$: $E \to B$ *is a map of simplicial spaces such that* $E_k \to B_k$ *is a homology-fibration for each* $k \geqq 0$, *and for each simplicial operation* $\theta$: $[k] \to [l]$ *and each* $b \in B_l$ *the map* $p^{-1}(b) \to p^{-1}(\theta^* b)$ *is a homology-equivalence, then the map of realizations* $\|E\| \to \|B\|$ *is a homology-fibration.*

*Proof.* This follows from Proposition 3 because the realizations $\|E\|$ and $\|B\|$ can be made up skeleton by skeleton, and $\|B\|_{(k)}$ is the double-mapping-cylinder of $(\|B\|_{(k-1)} \leftarrow \dot{\Delta}^k \times B_k \to \Delta^k \times B_k)$, and so on.

Proposition 2 is a particular case of Proposition 4, for $X_M$ and $BM$ are the realizations of simplicial spaces $E$ and $B$ such that $E_k = X \times B_k$ and $B_k = M^k$.

To conclude we need the following justifying proposition.

**Proposition 5.** *If* $B$ *is a paracompact locally contractible space, and* $p$: $E \to B$ *is a homology-fibration, then* $p^{-1}(b) \to F(p, b)$ *is a homology-equivalence for each* $b \in B$.

*Proof.* Let $P$ be the space of paths in $B$ beginning at $b$, and let $f$: $P \to B$ be the end-point map, a Hurewicz fibration. Then $f^* E$ is $F(p, b)$. Choose a basis $\mathscr{B}$ for the topology of $B$ consisting of contractible sets. Then there is a basis $\mathscr{B}^*$ for the topology of $P$ consisting of contractible sets $U$ such that $f(U) \in \mathscr{B}$ and $f$: $U \to f(U)$ is a Hurewicz fibration. $\mathscr{B}^*$ consists of sets $P(t_1, \dots, t_k;\ U_1, \dots, U_k;\ V_1, \dots, V_k)$, where $0 = t_0 < t_1 < \dots < t_k = 1$, and $U_1 \supset V_1 \subset U_2 \supset V_2 \subset \dots \subset U_k \supset V_k$ belong to $\mathscr{B}$; a path $\alpha$ belongs to this set if $\alpha(t_i) \in V_i$ and $\alpha([t_{i-1}, t_i]) \subset U_i$ for $i = 1, \dots, k$. Because $f$: $U \to f(U)$ is both a homotopy-equivalence and a Hurewicz fibration when $U \in \mathscr{B}^*$, the pull-back $f^* E | U$ is homotopy-equivalent to $E | f(U)$. Thus $f^* E \to P$ is a homology-fibration in our sense, and Proposition 5 follows from the particular case:

**Proposition 6.** *If* $p$: $E \to B$ *is a homology-fibration (with* $B$ *paracompact and locally contractible), and* $B$ *is contractible, then* $p^{-1}(b) \to E$ *is a homology-equivalence for each* $b \in B$.

*Proof.* Let $\mathscr{B}$ be a basis for $B$ consisting of contractible sets $U$ such that $p^{-1}(b) \to p^{-1}(U)$ is a homology equivalence for each $b \in U$. There is a Leray spectral sequence for the covering of $E$ by the $p^{-1}(U)$. One obtains it as in [8] by forming a space $E_{\mathscr{B}}$ homotopy-equivalent to $E$ which maps to the nerve $|\mathscr{B}|$ so that above a point of the open simplex $[U_0 \subset U_1 \subset \dots \subset U_p]$ of the nerve one has $p^{-1}(U_0)$.

The spectral sequence comes from the filtration of $E_{\mathscr{B}}$ by the inverse-images of the skeletons of $|\mathscr{B}|$. It is $H_p(|\mathscr{B}|; \mathscr{H}_q) \Rightarrow H_*(E)$, where $\mathscr{H}_q$ is the local coefficient system $U \mapsto H_q(p^{-1}(U))$ on $\mathscr{B}$. But $|\mathscr{B}|$ is homotopy-equivalent to $B$, which is contractible, so $H_0(|\mathscr{B}|; \mathscr{H}_q) \cong H_q(E)$, as we want.

*Examples.* (i) If $M$ is a discrete monoid whose enveloping group is $G$, and $G$ can be constructed from $M$ as the set of formal fractions $m_1 m_2^{-1}$ with $m_1$ and $m_2$ in $M$, then Proposition 2 implies that $BM \simeq BG$.

(ii) The case $M = \coprod_{n \geq 0} B\Sigma_n$, where $\Sigma_n$ is the $n^{\text{th}}$ symmetric group, has already been mentioned. It is closely related to the basic example of algebraic $K$-theory, where $M = \coprod_P B\,\mathrm{Aut}(P)$, and $P$ runs through the finitely generated projective modules over a fixed discrete ring $A$, and the composition law in $M$ comes from the direct sum of modules. Then $M_\infty$ can be taken to be $K_0(A) \times BGL_\infty(A)$, as one can form the telescope $M \to M \to M \to \cdots$ by successively adding the free $A$-module on one generator. As with $\Sigma_\infty$ the shifts $GL_\infty(A) \to GL_\infty(A)$ induce homology isomorphisms because they are conjugate to the identity on each $GL_n(A)$.

(iii) If $M = \coprod_{k \geq 0} G_n(p^k)$, where $G_n(p^k)$ is the space of maps $S^{n-1} \to S^{n-1}$ of degree $p^k$ (for some prime $p$), and the composition is composition of maps, then one has an example where $\pi$ is not in the centre of $H_*(M)$. Each component of $M$ is the telescope of

$$G_n(1) \to G_n(p) \to G_n(p^2) \to \cdots,$$

where the maps are composition on the left with a standard map of degree $p$. This telescope is the same up to homotopy as one component of the space of maps from $S^{n-1}$ to the telescope $S^{n-1} \to S^{n-1} \to S^{n-1} \to \cdots$ whose maps have degree $p$, i.e. as one component of $\mathrm{Map}(S^{n-1}; S^{n-1}[p^{-1}])$, where $S^{n-1}[p^{-1}]$ is $S^{n-1}$ localized away from $p$. Comparing homotopy groups one finds that $M_\infty$ can be identified with $\mathbb{Z} \times G_n(1)[p^{-1}]$. The right-hand action of $M$ on $M_\infty$ is by homotopy equivalences, so the homology fibration of Proposition 2 is actually a quasifibration, and $M_\infty \simeq \Omega BM$. Thus enlarging the monoid of homotopy equivalences of $S^{n-1}$ to the monoid of maps of degree $p^k$ has the effect of localizing the classifying space, a result essentially equivalent to the "mod $p$ Dold theorem" of Adams [1].

In this example because the right-hand action of $M$ on $M_\infty$ is by homotopy equivalences $H_*(M)[\pi^{-1}]$ can be formed by left fractions. But it cannot be formed by right fractions. For example $G_2(p^k)$ is homotopically a circle, and composition on the right with a map of degree $p$ is a homotopy equivalence $G_2(p^k) \to G_2(p^{k+1})$, and the telescope formed from it is not local for the left action.

(iv) A closely related example is $M = \coprod_{k \geq 0} B\Sigma_{p^k}$, where composition comes from the cartesian product of permutations. Then $M_\infty \simeq \mathbb{Z} \times B\Pi$, where $\Pi = \varinjlim \Sigma_{p^k}$ is the group of periodic permutations of $\mathbb{Z}$ whose period is a power of $p$. But $\Omega BM$ is $\mathbb{Z} \times Q[p^{-1}]$, where $Q$ is one component of $\Omega^\infty S^\infty$. This follows from the Barratt-Priddy-Quillen homology isomorphism $B\Sigma_\infty \to Q$; for $B\Sigma_{p^k}$ has the homology of $Q$ up to a dimension tending to infinity with $k$, and in the telescope defining $M_\infty$ the map $B\Sigma_{p^k} \to B\Sigma_{p^{k+1}}$ corresponds to multiplying by $p$ in the $H$-space structure of $Q$.

Examples (iii) and (iv) have been studied by Tornehave and Snaith in works to appear.

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