NORMS OF EILENBERG–MAC LANE SPECTRA AND REAL BORDISM

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ABSTRACT. We provide a new method to compute the (homotopy) fixed-points of the permutation action on $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ by relating it to Real bordism. More precisely, we identify the C_4-pullback of the C_2-spectrum $N_e^{C_2} H\mathbb{F}_2$ with a localization of $N_{C_2}^{C_4} MU_{\mathbb{R}}$. This allows us to use the localized slice spectral sequence for the computation of $\pi_{\bigstar}^{C_2} N_e^{C_2} H\mathbb{F}_2$. From this we compute the first eight homotopy groups and deduce an infinite family of differentials in the homotopy fixed point spectral sequence.

CONTENTS

1. INTRODUCTION

The Segal conjecture is a deep result in equivariant homotopy theory. In its original formulation, it was proven by Lin [Lin80] for the group C_2 and by Carlsson [Car84] for all finite groups, building in particular on the work of [MM82] and [AGM85]. Focusing on the group C_2, the most general formulation can be found in [LNR11] and [NS18]: for every bounded below spectrum X, the map $X \to (N_1^2 X)^{tC_2}$ into the Tate construction is a 2-adic equivalence. Here, we denote by $N_1^2 X$ the norm of X, i.e. the spectrum $X \wedge X$ with the permutation C_2-action. Moreover, Nikolaus–Scholze observed that the complete result follows from the case when X is $H\mathbb{F}_2$, which also has been reproved recently in [HW19].

The equivalence $N_1^2(H\mathbb{F}_2)^{tC_2} \simeq H\mathbb{F}_2$ is mysterious from the point of view of the Tate spectral sequence. The E_2-page is the C_2-Tate cohomology of the conjugation action on the dual Steenrod algebra $\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$ and thus highly non-trivial. However, on the E_∞-page, everything is concentrated at a single spot, namely an $\mathbb{F}_2$ at $(0, 0)$. The pattern of differentials that achieves this is unknown. In the present work, as a consequence of our main theorems, we compute the Tate spectral sequence in a range and exhibit the first infinite family of differentials (see Theorem 6.8).

One may ask what use there is for a partial spectral sequence computation when the target is already known. One reason is that the Tate spectral sequence essentially contains the homotopy fixed points spectral sequence, which computes $\pi_* N_1^2(H\mathbb{F}_2)^{hC_2}$. Indeed, the computation of $\pi_* N_1^2(H\mathbb{F}_2)^{hC_2}$ is one of our main goals and we obtain the following result.

Theorem 1.1. *(Theorem 4.4) The first 8 stems of $\pi_* N_1^2(H\mathbb{F}_2)^{hC_2}$ are*

i	0	1	2	3	4	5	6	7	8
π_i	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

Given the current knowledge of differentials in the slice spectral sequence of $BP^{(\!(C_4)\!)}$ [HHR17, HSWX18], we can actually use the method described in this paper to compute the first 30 stems of $\pi_* N_1^2(H\mathbb{F}_2)^{hC_2}$. The goal of the current paper is to describe our method and give low dimensional computations. In a future paper, we will focus on the higher stem computations.

Before discussing our method, we would like to remark that there are two interesting natural C_2-actions on the spectrum $H\mathbb{F}_2 \wedge H\mathbb{F}_2$. The first action is the trivial C_2-action, where C_2 acts trivially on the two factors. For this action, one can completely compute $\pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{hC_2}$ by using the homotopy fixed point spectral sequence. In fact, Hu and Kriz [HK01] completely computed $\pi_\bigstar^{C_2}(H\underline{\mathbb{F}_2} \wedge H\underline{\mathbb{F}_2})$ and determined the Hopf algebroid structure of $(H\underline{\mathbb{F}_2}_\bigstar, (H\underline{\mathbb{F}_2} \wedge H\underline{\mathbb{F}_2})_\bigstar)$. Their computation is a crucial input for the C_2-equivariant Adams spectral sequence.

We consider instead the permutation C_2-action on $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, where the two factors are permuted. Unlike the first action, this C_2-action induces a nontrivial action on the dual Steenrod algebra $\mathcal{A}_*$. As a consequence, the homotopy fixed point spectral sequence is much harder to compute. For instance, even a complete algebraic presentation of the E_2-page is not known (though it can be computed by a computer program in any finite range). The Segal conjecture implies that the pattern of differentials must be complicated. This stands in constract to the trivial C_2-action, where there are no differentials and the spectral sequence degenerates after the E_2-page.

While Theorem 1.1 is a significant advance on our knowledge of the spectrum $N_1^2(H\mathbb{F}_2)^{hC_2}$, we deem our methods as more interesting than our result. Note first that the Segal conjecture implies an equivalence $N_1^2(H\mathbb{F}_2)^{hC_2} \simeq N_1^2(H\mathbb{F}_2)^{C_2}$ as $N_1^2(H\mathbb{F}_2)^{\Phi C_2} \simeq H\mathbb{F}_2$ agrees with the Tate construction. This allows for an attack using methods from genuine equivariant homotopy theory.

Our starting point is the equivalence $\Phi^{C_2} BP_\mathbb{R} \simeq H\mathbb{F}_2$ for the Real Brown–Peterson spectrum $BP_\mathbb{R}$ from [HK01]. This suggests the relevance of norms of $BP_\mathbb{R}$ for the study of $N_1^2(H\mathbb{F}_2)$. Let $P^*_{C_4/C_2}(-)$ denote the pullback functor $\mathrm{Sp}_{C_4/C_2} \to \mathrm{Sp}_{C_2}$ (see [Hil12, Definition 4.1]). The following theorem is a special case of Theorem 2.2:

Theorem 1.2. *Let λ denote the irreducible 2-dimensional real representation of C_4 and let a_λ denote the Euler class $S^0 \to S^\lambda$. There is an equivalence*

$$a_\lambda^{-1} BP^{(\!(C_4)\!)} \simeq P^*_{C_4/C_2} N_1^2(H\mathbb{F}_2),$$

where $BP^{(\!(C_4)\!)} := N_{C_2}^{C_4} BP_\mathbb{R}$.

This theorem implies in particular an isomorphism between $\pi_*^{C_2} N_1^2(H\mathbb{F}_2)$ and $\pi_*^{C_4} a_\lambda^{-1} BP^{(\!(C_4)\!)}$. Moreover, it can be easily generalized to obtain a similar equivalence between a pullback of $N_1^{2^k}(H\mathbb{F}_2)$ and a localization of $BP^{(\!(C_{2^{k+1}})\!)}$.

The slice spectral sequence, invented by Hill–Hopkins–Ravenel in their solution of the Kervaire invariant one problem, is an excellent tool to compute norms of $BP_\mathbb{R}$ and $MU_\mathbb{R}$ (see [HHR16] and [HHR17]). In order to compute $a_\lambda^{-1} BP^{(\!(C_4)\!)}$, we introduce a new spectral sequence which is a variant of the original slice spectral sequence. We call it the *localized slice spectral sequence.*

The E_2-term of this spectral sequence is computable due to [Zen]. The convergence is provided by the following theorem:

Theorem 1.3 (Theorem 3.1)**.** *Let X be a C_{2^n}-spectrum, and let $\{P^\bullet\}$ denote the slice tower for X. Let V be an orthogonal C_{2^n}-representation such that $V^{C_{2^n}} = 0$ and $a_V : S^0 \to S^V$ be the Euler class. Consider the tower*

$$\{Q^\bullet\} := \{a_V^{-1} P^\bullet\}$$

obtained by localizing $\{P^\bullet\}$ at a_V. The spectral sequence associated to $\{Q^\bullet\}$ converges strongly to the homotopy groups of $a_V^{-1}X$.

Importing and extending differentials from [HHR16] and [HHR17], we compute the localized slice spectral sequence for $a_\lambda^{-1} BP^{(\!(C_4)\!)}$ in a range. It is actually from this that we deduce Theorem 1.1 and the differentials in the homotopy fixed points spectral sequence and the Tate spectral sequence (see Figure 7 and Theorem 6.8).

In computing the localized slice spectral sequence, the norm map plays an essential role. However, localizing at an Euler class such as a_λ will never preserve the commutative ring structure on which the norm map is based, because the underlying spectrum of such localization is always contractible. To overcome this problem, we apply the theory of N_∞-operads from [BH15]. More precisely, in Section 2.3, by establishing a criterion generalizing the result of [HH14], we show that a_V-localization preserves the algebra structure over a certain N_∞-operad, which depends on the class a_V. Therefore, the homotopy of the a_V-localization of an equivariant commutative ring such as $MU^{(\!(G)\!)}$ forms an incomplete Tambara functor [BH18], and the norm map essential to our computation is still available. Furthermore, in Section 3.4, we draw consequences of the behavior of norms in the localized slice spectral sequence.

As an outlook, we will comment about how our results fit into the grander scheme of things. The Real bordism spectrum and its norms are central to understanding chromatic homotopy theory. By the Goerss–Hopkins–Miller theorem, the Lubin–Tate spectra E_n are acted upon by the Morava stabilizer group, and in particular they can be viewed as genuine G-spectra for any finite subgroup G of the Morava stabilizer group. The higher real K-theories $E_n^{hC_{2^k}}$ play a crucial role in approaches to understand the chromatic tower and the $K(n)$-local sphere. Moreover, a computation of $E_4^{hC_8}$ could possibly resolve the last remaining open case of the Kervaire invariant one problem.

To study these higher real K-theories, Hahn and the second author [HS20] proved that at the prime 2, the classical complex orientation for E_n can be refined to a Real orientation $MU_{\mathbb{R}} \longrightarrow E_n$. If we further localize at the prime 2, then the Real orientation becomes

$$BP_{\mathbb{R}} \longrightarrow E_n.$$

Furthermore, for any finite subgroup G containing C_2, the Real orientation extends to a G-equivariant orientation

$$BP^{(\!(G)\!)} \longrightarrow E_n,$$

where $BP^{(\!(G)\!)} := N_{C_2}^G BP_{\mathbb{R}}$. This makes the computation of $\pi_*^G BP^{(\!(G)\!)}$ a major open problem in stable homotopy theory as it contains crucial information about computational problems of Lubin–Tate spectra. In particular, by the recent work of Beaudry–Hill–Shi–Zeng [BHSZ20], computational problems about the Lubin–Tate spectra can be turned into computations with norms of $BP_{\mathbb{R}}$ and its quotients. Essentially, this implies that computation of $E_n^{hC_{2^k}}$ would follow from that of $\pi_*^{C_{2^k}} BP^{(\!(C_{2^k})\!)}$ at all heights n.

When $G = C_2$, the homotopy groups of $E_n^{hC_2}$ have been completely computed at all heights [HS20]. When $G = C_4$, the homotopy groups of $E_2^{hC_4}$ have been computed by Behrens–Ormsby [BO16], Hill–Hopkins–Ravenel [HHR17], and Beaudry–Bobkova–Hill–Stojanoska [BBHS19]; the homotopy groups of $E_4^{hC_4}$ has been computed by Hill–Shi–Wang–Xu [HSWX18].

The computations above present the slice spectral sequence as a powerful tool in computing $BP^{(\!(C_{2^n})\!)}$ and the higher real K-theories. The slice spectral sequence is stratified into different regions, each containing classes corresponding to the different representations of C_{2^n}. The localized spectral sequence is a more refined spectral sequence than the slice spectral sequence because it can analyze each of these regions separately. By choosing to localize at different Euler classes (which corresponds to smashing with $\widetilde{E}\mathcal{F}[C_{2^i}]$ for different subgroups $C_{2^i} \subseteq C_{2^n}$), the localized slice spectral sequence will only contain specific regions of the original slice spectral sequence (which corresponds to computing the geometric fixed points at each level). As a consequence, one can localize at different Euler classes to study each of the regions in the original slice spectral sequence separately. In the extreme case when we are smashing with $\tilde{E}G \cong \widetilde{E}\mathcal{F}[C_{2^0}]$ (when $i = 0$), the differentials in the localized spectral sequence recovers all of the differentials in the original slice spectral sequence.

In a future paper, we will focus on exploiting the localized slice spectral sequence to demonstrate the interplay of differentials in the slice spectral sequences of $BP^{(\!(C_{2^n})\!)}$ as n varies. This will give an inductive approach to computing $BP^{(\!(C_{2^n})\!)}$ and higher real K-theories. In particular, differentials in the smaller height and smaller group spectral sequences can be directly imported to the bigger height, bigger group spectral sequences.

While the flow of information in this paper is mainly from $BP^{(\!(C_4)\!)}$ to $N_1^2(H\mathbb{F}_2)$, there is also significant potential for a flow of information in the other direction. If one can compute the homotopy of $N_1^2(H\mathbb{F}_2)$ by other methods, it will greatly help in understanding the localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$. Besides the localized slice spectral sequence and the homotopy fixed point spectral sequence, there are several other approaches to the computation of $N_1^2(H\mathbb{F}_2)$. In particular, one can use the Adams spectral sequence, the C_2-equivariant Adams spectral sequence, and a new THH-based spectral sequence by [HW19]. An Adams spectral sequence computation of the first six stems was actually already obtained a few years ago in unpublished work of Quigley, who observed that in the Adams spectral sequence for $N_1^2(H\mathbb{F}_2)_{hC_2}$ there cannot be any differentials in this range. It is a current project of Bruner, Quigley and the third author to study the interplay between the Adams spectral sequence, the localized slice spectral sequence and the Tate spectral sequence. So far, the localized slice spectral sequence is the most effective spectral sequence, and its interplay with the other spectral sequences may allow for further advances in the computation of $N_1^2(H\mathbb{F}_2)$ and hence of $BP^{(\!(C_4)\!)}$.

Outline of paper. We now turn to a summary of the contents of this paper. In Section 2, we recall a few basics of equivariant homotopy theory. In particular, we discuss the interplay between the norm functor, the geometric fixed point functor, and the pull back functor. We prove Theorem 2.2, from which Theorem 1.2 directly follows as a special case. We also investigate the multiplicative structure of localizations and give a criterion for a localization at an element to preserve multiplicative structures.

In Section 3, we recall the spectrum $MU^{(\!(G)\!)}$, $BP^{(\!(G)\!)}$, and their slice spectral sequences. We then introduce the main computational tool for this paper, the localized slice spectral sequence. Theorem 3.1 proves the strong convergence of the localized slice spectral sequence. We also discuss extensions and norms in the localized slice spectral sequence.

Sections 4 and 5 are dedicated to the computation of the localized slice spectral sequence of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$. In Section 4, we give an outline of the computation and list our main results (Theorem 4.1 and Theorem 4.4). The detailed computations are in Section 5. While computing differentials, we make full use of the Mackey functor structure of the spectral sequence. Certain differentials are proven using exotic extensions and norms by methods established in Section 3.3 and 3.4.

In Section 6, we turn our attention to the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$. We use the computation of the localized slice spectral sequence of $BP^{(\!(C_4)\!)}$ to prove families of differentials and compute the Tate spectral sequence in a range. In particular, Theorem 6.8 describes the first infinite family of differentials in the Tate spectral sequence.

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Conventions.

(1) Given a finite group G, all representations will be finite-dimensional and orthogonal. Per default actions will be from the left.
(2) We denote by ρ_G the real regular representation of a finite group G and we abbreviate ρ_{C_2} to ρ_2.
(3) All spectral sequences use the Adams grading.

2. Equivariant stable homotopy theory

2.1. **A few basics.** We work in the category of genuine G-spectra for a finite group G, and our particular model will be the category of orthogonal G-spectra Sp_G. For us these will be simply G-objects in orthogonal spectra as in [Sch14], which will often be just called *G-spectra*. This category is equivalent to the categories of orthogonal G-spectra considered in [MM02] and [HHR16]. In particular, we are able to evaluate a G-spectrum at an arbitrary G-representation to obtain a G-space. We refer to the three cited sources for general background on G-equivariant stable homotopy theory, of which we will recall some for the convenience of the reader.

For each G-representation V, we denote by S^V its one-point compactification. Denoting further by ρ_G the regular representation, we obtain for each subgroup $H \subset G$ and each G-spectrum its homotopy groups

$$\pi_n^H(X) = \mathrm{colim}_k [S^{k\rho_G+n}, X(k\rho_G)]^H.$$

These assemble into a Mackey functor $\underline{\pi}_n(X)$. A map of G-spectra is an *equivalence* if it induces an isomorphism on all $\underline{\pi}_n$. Inverting the equivalences of G-spectra in the 1-categorical sense yields the genuine equivariant stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ and inverting them in the ∞-categorical sense the ∞-category of G-spectra Sp_G^∞. These constructions are well-behaved as there is a stable model structure on Sp_G with the weak equivalences we just described [MM02, Theorem III.4.2]. The fibrant objects are precisely the Ω-G-spectra.

By [MM02, Proposition V.3.4], the categorical fixed point construction $\mathrm{Sp}_G \to \mathrm{Sp}$ is a right Quillen functor. We call the right derived functor $(-)^G\colon \mathrm{Sp}_G^\infty \to \mathrm{Sp}^\infty$ the *(genuine) fixed points.* We can define fixed point functors for subgroups $H \subset G$ by applying first the restriction functor

$\mathrm{Sp}_G \to \mathrm{Sp}_H$ and then the H-fixed point functor. One easily shows that $\pi_n X^H \cong \pi_n^H X$. Thus, a map is an equivalence if it is an equivalence on all fixed points.

Note that if $H \subset G$ is normal, the categorical fixed points carry a residual G/H-action. The resulting functor $\mathrm{Sp}_G \to \mathrm{Sp}_{G/H}$ is a right Quillen functor as well [MM02, p. 81] and thus H-fixed points actually define a functor $\mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty$. The left adjoint of this is the inflation functor p^* associated to the projection $p\colon G \to G/H$.

As π_n^H translates filtered homotopy colimits into colimits, we see that fixed points $\mathrm{Sp}_G^\infty \to \mathrm{Sp}^\infty$ preserve filtered homotopy colimits. As they preserve homotopy limits as well (as they are induced by a Quillen right adjoint) and are a functor between stable ∞-categories, they preserve all finite homotopy colimits [Lur17, Proposition 1.1.4.1] and hence all homotopy colimits [Lur09, Proposition 4.4.2.7]. By the associativity of fixed points, the same is true for $(-)^H\colon \mathrm{Sp}_G^\infty \to \mathrm{Sp}_{G/H}^\infty$ for a normal subgroup $H \subset G$.

2.2. **Norms and pullbacks.** In this section, we will identify certain localizations of norm functors with pullbacks of norms from quotient groups. In the case of $BP^{(\!(G)\!)}$ this is a central ingredient of this paper.

First, we will recall the norm construction. For a group G, let $\mathcal{B}G$ denote the category with one object and having G as morphisms. Given an arbitrary symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$, there is for a subgroup $H \subset G$ a norm functor

$$\mathcal{C}^{\mathcal{B}H} \to \mathcal{C}^{\mathcal{B}G}, \qquad X \mapsto X^{\otimes_H G}$$

from H-objects to G-objects, where the G-action is induced by the right G-action on G. In the case of spaces or sets, one can identify $X^{\times_H G}$ with $\mathrm{Map}_H(G, X)$ and for based spaces or sets, one can likewise identify $X^{\wedge_H G}$ with $\mathrm{Map}_H^*(G, X)$. In the case of orthogonal spectra, one can by [HHR16, Proposition B.105] left derive the functor $(-)^{\wedge_H G}$ to obtain a functor N_H^G. (Often, N_H^G is also used for the corresponding underived functor, but the derived functor will be more important for us.) The functor N_H^G commutes with filtered (homotopy) colimits by [HHR16, Propositions A.53, B.89]. Note moreover that $N_H^G \Sigma^\infty X \simeq \Sigma^\infty \mathrm{Map}_H^*(G, X)$ (if X is cofibrant or at least well-pointed) as Σ^∞ is symmetric monoidal.

Lemma 2.1. *Let G be a finite group, $K, H \subset G$ be two subgroups and X be a (based) topological H-space. Let $H\backslash G/K = \{Hg_1K, \ldots, Hg_lK\}$. Then there are (based) homeomorphisms*

$$\mathrm{Map}_H(G, X)^K \cong X^{g_1 K g_1^{-1} \cap H} \times \cdots \times X^{g_l K g_l^{-1} \cap H}$$

and

$$\mathrm{Map}_H^*(G, X)^K \cong X^{g_1 K g_1^{-1} \cap H} \wedge \cdots \wedge X^{g_l^{-1} K g_l^{-1} \cap H}, \tag{1}$$

where the K-action on the mapping spaces is induced by the right K-action on G. In particular, if $H = K$ is normal, we obtain a G/H-equivariant homeomorphism

$$\mathrm{Map}_H^*(G, X)^H \cong \mathrm{Map}^*(G/H, X^H).$$

Proof. The first two statements follow from the H-K-equivariant decomposition of G into $\coprod_{i=1}^l Hg_iK$. For the last one observe that if $H = K$ is normal, $H\backslash G/K = G/H$ and G/H permutes the factors of the decomposition in (1). $\square$

Theorem 2.2. *Let $H \subset G$ be a normal subgroup and X be an H-spectrum. Then we have an equivalence of G-spectra*

$$\widetilde{E}\mathcal{F}[H] \wedge N_H^G X \simeq P_{G/H}^*(N_e^{G/H} \Phi^H(X)).$$

Proof. As a first step to construct a natural map from the left hand side to the right hand side, we observe that

$$\Phi^H N_H^G X \simeq N_e^{G/H} \Phi^H X \tag{2}$$

for all H-spectra X. Indeed: If X is a suspension spectrum, this reduces to the space-level statement $\mathrm{Map}^*_H(G, X)^H \simeq \mathrm{Map}^*(G/H, X^H)$, which is part of Lemma 2.1. Both sides of (2) are symmetric monoidal and commute with filtered homotopy colimits. As $\mathrm{Ho}(\mathrm{Sp}^G)$ is generated by the $S^{-V} \wedge \Sigma^\infty Z$, the claim follows.

As noted above, the functor $P^*_{G/H}$ is right adjoint to $\Phi^H\colon \mathrm{Ho}(\mathrm{Sp}^G) \to \mathrm{Ho}(\mathrm{Sp}^{G/H})$. Thus, the equivalence $\Phi^H N_H^G X \simeq N_e^{G/H}\Phi^H X$ induces a natural map

$$N_H^G X \to P^*_{G/H}\Phi^H N_H^G X \simeq P^*_{G/H}(N_e^{G/H}\Phi^H(X)).$$

As smashing with $\widetilde{E}\mathcal{F}[H]$ is idempotent, this in turn induces a map

$$\widetilde{E}\mathcal{F}[H] \wedge N_H^G X \to P^*_{G/H}(N_e^{G/H}\Phi^H(X)), \tag{3}$$

which we will show to be an equivalence, first for suspension spectra and then in general. For $X = \Sigma^\infty Z$, the map (3) is obtained by applying Σ^∞ to the map

$$\widetilde{E}\mathcal{F}[H] \wedge Z^{\wedge_H G} \to \widetilde{E}\mathcal{F}[H] \wedge (Z^H)^{\wedge G/H}. \tag{4}$$

If we apply K-fixed points for $H \subset K \subset G$, the map becomes equivalent to the K-fixed points of $Z^{\wedge_H G} \to (Z^H)^{\wedge G/H}$ and this is an equivalence by Lemma 2.1; moreover, $\widetilde{E}\mathcal{F}[H]^K \simeq *$ if H is not contained in $K \subset G$. Thus, the map (4) is an equivalence after taking K-fixed points for every subgroup $K \subset G$ and hence a G-equivalence. This shows that (3) is an equivalene if X is a suspension spectrum.

By [Hil12, Corollary 4.6], we have for an arbitrary H-representation V an equivalence $S^{-V^{\otimes_H G}} \wedge P^*_{G/H}(-) \simeq P^*_{G/H}(S^{-(V^H)^{\otimes G/H}} \wedge (-))$, using the isomorphism $(V^H)^{\otimes G/H} \cong (V^{\otimes_H G})^H$. We thus obtain for $X = S^{-V} \wedge \Sigma^\infty Z$ a chain of equivalences

$$\begin{aligned}
\widetilde{E}\mathcal{F}[H] \wedge N_H^G X &\simeq \widetilde{E}\mathcal{F}[H] \wedge S^{-V^{\otimes_H G}} \wedge N_H^G \Sigma^\infty Z \\
&\simeq S^{-V^{\otimes_H G}} \wedge P^*_{G/H}(N_e^{G/H}\Phi^H \Sigma^\infty Z) \\
&\simeq P^*_{G/H}(S^{-(V^H)^{\otimes G/H}} \wedge N_e^{G/H}\Phi^H\Sigma^\infty Z) \\
&\simeq P^*_{G/H}(N_e^{G/H}\Phi^H(S^{-V}\Sigma^\infty Z)),
\end{aligned}$$

showing that the map (3) is also an equivalence if $X = S^{-V} \wedge \Sigma^\infty Z$. It remains to observe that both sides of (3) commute with filtered homotopy colimits as Sp^∞_H is generated under filtered colimits by H-spectra of the form $S^{-V} \wedge \Sigma^\infty Z$. □

Corollary 2.3. *Let $K \subset H \subset G$ be subgroups and assume that $H \subset G$ is normal. Let moreover X be a K-spectrum. Then there is an equivalence of G-spectra*

$$\widetilde{E}\mathcal{F}[H] \wedge N_K^G X \simeq P^*_{G/H}(N_e^{G/H}\Phi^K(X)).$$

Proof. This follows from Theorem 2.2 by applying it to $N_K^H X$. Here, we use $N_K^G X \simeq N_H^G N_K^H X$ and $\Phi^H N_K^H X \simeq \Phi^K X$. □

As we will recall below, there is a C_2-spectrum $BP_{\mathbb{R}}$ with geometric fixed points $H\mathbb{F}_2$. For $G = C_4$ and $H = C_2$, we can express $\widetilde{E}\mathcal{F}[H]$ as $S^{\infty\lambda}$, where λ is the 2-dimension representation of C_4 rotating by an angle of $\frac{\pi}{2}$. Denoting the norm $N_{C_2}^{C_4} BP_{\mathbb{R}}$ by $BP^{(\!(C_4)\!)}$, we obtain the following result already stated in slightly different form in the introduction.

Corollary 2.4. *There is an equivalence*

$$BP^{(\!(C_4)\!)} \wedge S^{\infty\lambda} \simeq P^*_{C_4/C_2} N_1^2(H\mathbb{F}_2).$$

2.3. **Multiplicative structures of localizations.** In many cases, smashing with $\widetilde{E}\mathcal{F}[H]$ is equivalent to localization at a certain element in $\pi_*^G\mathbb{S}$ (for example if G is cyclic). The goal of this section is to investigate which kind of multiplicative structure localization at such an element preserves. More specifically let us fix an N_∞-operad $\mathcal{O}$, i.e. an operad $\mathcal{O}$ in (unbased) G-spaces such that each $\mathcal{O}(n)$ is a universal space for a family $\mathcal{F}_n$ of graph subgroups of $G \times \Sigma_n$, containing all $H \times \{e\}$. This notion was introduced in [BH15]. In the maximal case, we speak of a *G-E_∞-operad* and by [BH15, Theorem A.6] every algebra over such an operad can be strictified to a commutative G-spectrum. In the minimal case, we speak of a *(naive) E_∞-operad*.

Essentially, the different versions of N_∞-operads encode which norms we see in the homotopy groups of an $\mathcal{O}$-algebra. To be more precise, call an H-set T *admissible* if the graph of the H-action on T lies in $\mathcal{F}_{|T|}$. By [AB18, Remark 5.15] an $\mathcal{O}$-algebra R admits norms $N_K^H\colon \pi_V^K R \to \pi^H_{\mathrm{Ind}_K^H V} R$ if H/K is admissible, and the groups $\pi_\bigstar^H R$ assemble into an $RO(G)$-graded incomplete Tambara functor.

As already observed in [McC96], localizations only need to preserve naive E_∞-structures, but not G-E_∞-structures. Later, [HH14] gave a criterion when localizations indeed preserve G-E_∞-structures and this was extended in [Böh19] to N_∞-algebras, albeit only for localizations of elements in degree 0. In this section, we will extend this work to elements in non-trivial degree and follow the proof strategy of [Böh19, Proposition 2.30].

Let us first recall what localizing at some $x \in \pi_V^G\mathbb{S}$ means. We say that a G-spectrum X is *x-local* if x acts invertibly on E or, equivalently, on $\pi_*^G E$. Given a G-spectrum E, we construct its *x-localization* as

$$x^{-1}E = \mathrm{hocolim}\left(E \xrightarrow{x} \Sigma^{-V}E \xrightarrow{x} \Sigma^{-2V}E \xrightarrow{x} \cdots\right).$$

Note that $x^{-1}E \simeq E \wedge x^{-1}\mathbb{S}$.

Example 2.5. *Given a G-representation V, let $a_V\colon S^0 \to S^V$ be the Euler class. Then $a_V^{-1}\mathbb{S} \simeq S^{\infty V}$ and hence in general $a_V^{-1}E \simeq S^{\infty V} \wedge E$. In particular, we can reformulate Corollary 2.4 as*

$$a_\lambda^{-1} BP^{(\!(C_4)\!)} \simeq P^*_{C_4/C_2} N_1^2(H\mathbb{F}_2).$$

A map $f\colon E \to F$ is an *x-local equivalence* if $f \wedge x^{-1}\mathbb{S}$ is an equivalence; by abuse of notation, we call for $H \subset G$ a map of H-spectra an x-equivalence if it is a $\mathrm{Res}_H^G(x)$-equivalence.

Definition 2.6. *Localization at x* preserves $\mathcal{O}$-algebras *if for every $\mathcal{O}$-algebra R, we can lift the morphism $R \to x^{-1}R$ in $\mathrm{Ho}(\mathrm{Sp}^G)$ (up to isomorphism) to a morphism in $\mathrm{Ho}(\mathcal{O} - \mathrm{Alg})$.*

We will use the following specialization of a criterion of [GW18, Corollary 7.10]:

Proposition 2.7. *Localization at x preserves $\mathcal{O}$-algebras if and only if*

$$N_K^H \mathrm{Res}_K^G\colon \mathrm{Sp}_\infty^G \to \mathrm{Sp}_\infty^H$$

preserves x-equivalences for every $K \subset H \subset G$ such that H/K is admissible as an H-set.

To reformulate this criterion, we need the following lemma.

Lemma 2.8. *There is an equivalence* $N_K^H \operatorname{Res}^G_K(x^{-1}\mathbb{S}) \simeq (N_K^H \operatorname{Res}^G_K(x))^{-1}(\mathbb{S}_H)$ *for* $\mathbb{S}_H$ *the* H*-equivariant sphere spectrum.*

Proof. Applying $N_K^H \operatorname{Res}^G_K$ to

$$\mathbb{S} \xrightarrow{x} \Sigma^{-V}\mathbb{S} \xrightarrow{x} \Sigma^{-2V}\mathbb{S} \xrightarrow{x} \cdots,$$

we obtain precisely

$$\mathbb{S}_H \xrightarrow{N_K^H \operatorname{Res}^G_K(x)} \Sigma^{-\operatorname{Ind}_K^H \operatorname{Res}^G_K V}\mathbb{S}_H \xrightarrow{N_K^H \operatorname{Res}^G_K(x)} \Sigma^{-2\operatorname{Ind}_K^H \operatorname{Res}^G_K V}\mathbb{S}_H \xrightarrow{N_K^H \operatorname{Res}^G_K(x)} \cdots$$

Here we have used that the norm of a representation sphere is computed by induction. As both N_K^H and Res^G_K preserve filtered homotopy colimits, the result follows. □

Proposition 2.9. *Localization at* x *preserves* $\mathcal{O}$*-algebras if and only if* $N_K^H \operatorname{Res}^G_K(x)$ *divides a power of* $\operatorname{Res}^G_H(x)$ *for every* $K \subset H \subset G$ *such that* H/K *is admissible as an* H*-set.*

Proof. Let $K \subset H \subset G$ be subgroups such that H/K is admissible as an H-set. By Proposition 2.7, we have to show that $N_K^H \operatorname{Res}^G_K(x)$ divides a power of $\operatorname{Res}^G_H(x)$ if and only if

$$N_K^H \operatorname{Res}^G_K \colon \operatorname{Sp}^G_\infty \to \operatorname{Sp}^H_\infty$$

preserves x-equivalences.

Assume first that $N_K^H \operatorname{Res}^G_K$ preserves x-equivalences. By the preceding lemma, we see in particular that $\mathbb{S}_H \to (N_K^H \operatorname{Res}^G_K(x))^{-1}\mathbb{S}_H$ is an x-equivalence, i.e. $N_K^H \operatorname{Res}^G_K(x)$ becomes a unit after inverting $\operatorname{Res}^G_H(x)$ and just must divide a power of it.

Assume now that $N_K^H \operatorname{Res}^G_K(x)$ divides a power of $\operatorname{Res}^G_H(x)$. Then the map $\mathbb{S}_H \to \operatorname{Res}^G_H(x)^{-1}\mathbb{S}_H$ factors over the standard map $\mathbb{S}_H \to (N_K^H \operatorname{Res}^G_K(x))^{-1}\mathbb{S}_H$.

Let now $f\colon E \to F$ be an x-equivalence of G-spectra, i.e. we assume that $f \wedge x^{-1}\mathbb{S}$ is an equivalence. As N_K^H and Res^G_H are symmetric monoidal, we see that $N_K^H \operatorname{Res}^G_H(f \wedge x^{-1}\mathbb{S})$ is equivalent to $N_K^H \operatorname{Res}^G_H(f) \wedge (N_K^H \operatorname{Res}^G_K(x))^{-1}\mathbb{S}_H$, which is thus an equivalence. Tensoring with $\operatorname{Res}^G_H(x)^{-1}\mathbb{S}_H$ over $(N_K^H \operatorname{Res}^G_K(x))^{-1}\mathbb{S}_H$ yields the result. □

We specialize now to the case that x is the Euler class $a_V\colon S^0 \to S^V$. In this case we have $N_K^G \operatorname{Res}^G_H a_V = a_{\operatorname{Ind}_K^G \operatorname{Res}^G_H V}$. Thus to see which multiplicative structure localization at a_V preserves, we only have to understand divisibility relations between Euler classes. In particular, we obtain the following corollary:

Corollary 2.10. *Let* V *be a* G*-representation. Assume that* $\operatorname{Ind}_K^H \operatorname{Res}^G_K V$ *is a summand of a multiple of* $\operatorname{Res}^G_H V$ *for every* $K \subset H \subset G$ *such that* H/K *is an admissible* H*-set. Then localizing at* a_V *preserves* $\mathcal{O}$*-algebras.*

Example 2.11. Let $G = C_{2^n}$ and $\lambda = \lambda^n$ be the two-dimensional representation of C_{2^n} given by rotation by an angle of $\frac{2\pi}{2^n}$. We observe that $\operatorname{Res}^{C_2^n}_{C_{2^k}} \lambda^n = \lambda^k$ and $\operatorname{Ind}^{C_2^m}_{C_{2^k}} \lambda^k = 2^{m-k}\lambda_m$ unless $k = 1$. Thus localizing at a_λ preserves $\mathcal{O}$-algebras if the following holds: H/K is H-admissible if and only if $K \neq e$. In particular, we see that for any commutative C_{2^n}-spectrum R, the localization $a_\lambda^{-1}R$ admits norms from $\pi_*^{C_{2^k}}$ to $\pi_*^{C_{2^n}}$ for $0 < k < n$, but will not admit norms from π_*^e unless the target is zero. The example we care most about is $a_\lambda^{-1}MU^{((C_{2^n}))}$.

3. The slice spectral sequence and the localized slice spectral sequence

3.1. The slice spectral sequence of $MU^{((C_{2^n}))}$ and $BP^{((C_{2^n}))}$. Our main computational tool in this paper is a modification of the equivariant slice spectral sequence of Hill–Hopkins–Ravenel. In this subsection, we list some important facts about the slice filtration for norms of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$, which we will need for the rest of the paper. For a detailed construction of the slice spectral sequence and its properties, see [HHR16, Section 4] and [HHR17].

Let $G = C_{2^n}$ be the cyclic group of order 2^n, with generator γ. The spectrum $MU^{((G))}$ is defined as

$$MU^{((G))} := N_{C_2}^G MU_{\mathbb{R}}.$$

The underlying spectrum of $MU^{((G))}$ is the smash product of 2^{n-1}-copies of MU.

Hill, Hopkins, and Ravenel [HHR16, Section 5] constructed elements

$$\overline{r}_i \in \pi_{i\rho_2}^{C_2} MU^{((G))}$$

such that

$$\pi_{*\rho_2}^{C_2} MU^{((G))} \cong \mathbb{Z}[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots],$$

Here, $G \cdot x$ denotes the set $\{x, \gamma x, \gamma^2 x, \ldots, \gamma^{2^{n-1}} x\}$, and the Weyl action is given by

$$\gamma \cdot \gamma^j \overline{r}_i = \begin{cases} \gamma^{j+1}\overline{r}_i & 0 \leq j \leq 2^{n-1} - 2 \\ (-1)^i \overline{r}_i & j = 2^{n-1} - 1. \end{cases}$$

Adjoint to each map

$$\overline{r}_i : S^{i\rho_2} \longrightarrow i_{C_2}^* MU^{((G))}$$

is an associative algebra map from the free associative algebra

$$S^0[\overline{r}_i] = \bigvee_{j \geq 0} (S^{i\rho_2})^{\wedge j} \longrightarrow i_{C_2}^* MU^{((G))}.$$

Applying the norm and using the norm-restriction adjunction, this gives a G-equivariant associative algebra map

$$S^0[G \cdot \overline{r}_i] = N_{C_2}^G S^0[\overline{r}_i] \longrightarrow MU^{((G))}.$$

Smashing these maps together produces an associative algebra map

$$A := S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots] = \bigwedge_{i=1}^{\infty} S^0[G \cdot \overline{r}_i] \longrightarrow MU^{((G))}.$$

Note that by construction, A is a wedge of representation spheres, indexed by monomials in the $\overline{r}_i$s. By the Slice Theorem [HHR16, Theorem 6.1], the slice filtration of $MU^{((G))}$ is the filtration associated with the powers of the augmentation ideal of A. The slice associated graded for $MU^{((G))}$ is the graded spectrum

$$S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots] \wedge H\underline{\mathbb{Z}},$$

where the degree of a summand corresponding to a monomial in the $\overline{r}_i$ generators and their conjugates is the underlying degree.

As a consequence of the slice theorem, the slice spectral sequence for the $RO(G)$-graded homotopy groups of $MU^{((G))}$ has E_2-term the $RO(G)$-graded homology of $S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots]$ with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. To compute this, note that $S^0[G \cdot \overline{r}_1, G \cdot \overline{r}_2, \ldots]$ can be decomposed into a wedge sum of slice cells of the form

$$G_+ \wedge_{H_p} S^{\frac{|p|}{|H_p|}\rho_{H_p}},$$

where p ranges over a set of representatives for the orbits of monomials in the $\gamma^j \bar{r}_i$ generators, and $H_p \subset G$ is the stabilizer of p (mod 2). Therefore, the E_2-page of the integer graded slice spectral sequence can be computed completely by writing down explicit equivariant chain complexes for the representation spheres $S^{\frac{|p|}{|H_p|}\rho_{H_p}}$.

The exact same story holds for norms of $BP_{\mathbb{R}}$ as well. By [HK01, Theorems 2.25, 2.33], the classical Quilllen idempotent $MU \longrightarrow MU$ lifts to a multiplicative idempotent $MU_{\mathbb{R}} \to MU_{\mathbb{R}}$ with image $BP_{\mathbb{R}}$, resulting in particular in a multiplicative C_2-equivariant map

$$MU_{\mathbb{R}} \longrightarrow BP_{\mathbb{R}}.$$

Taking the norm $N_{C_2}^{G}(-)$ of this map produces a multiplicative G-equivariant map

$$MU^{(\!(G)\!)} \longrightarrow BP^{(\!(G)\!)} =: N_{C_2}^{G} BP_{\mathbb{R}}.$$

The exact same technique in [HHR16, Section 5] show that there are generators

$$\bar{t}_i \in \pi^{C_2}_{(2^i-1)\rho_2} BP^{(\!(G)\!)}$$

such that

$$\pi^{C_2}_{*\rho_2} BP^{(\!(G)\!)} \cong \mathbb{Z}_{(2)}[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots].$$

For a precise definition of these generators, see formula (1.2) in [BHSZ20].

Just like $MU^{(\!(G)\!)}$, we can build an equivariant refinement

$$S^0[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots] \longrightarrow BP^{(\!(G)\!)}$$

from which the Slice Theorem implies that the slice associated graded for $BP^{(\!(G)\!)}$ is the graded spectrum $S^0[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots] \wedge H\underline{\mathbb{Z}}_{(2)}$.

Since the slice filtration is an equivariant filtration, the slice spectral sequence is a spectral sequence of $RO(G)$-graded Mackey functors. Moreover, the slice spectral sequences for $MU^{(\!(G)\!)}$ and $BP^{(\!(G)\!)}$ are multiplicative spectral sequences and the natural maps between them are multiplicative as well (see [HHR16, Section 4.7]), and the slice spectral sequence for $BP^{(\!(G)\!)}$ is a spectral sequence of modules over the spectral sequence of $MU^{(\!(G)\!)}$ in Mackey functors.

3.2. **The localized spectral sequence.** In this subsection, we introduce a variant of the slice spectral sequence which we call the localized slice spectral sequence. This will be our main computational tool to compute $a_{\lambda}^{-1} BP^{(\!(C_4)\!)}$ in the later sections.

Let $\lambda_{2^{n-i}}$ denote the 2-dimensional real C_{2^n}-representation corresponding to rotation by $\left(\frac{\pi}{2^{n-i}}\right)$ and σ denote the real sign representation of C_{2^n}. Given a C_{2^n}-spectrum X, we have an equivalence

$$\widetilde{E}\mathcal{F}[C_{2^i}] \wedge X \simeq S^{\infty \lambda_{2^{n-i}}} \wedge X \simeq a^{-1}_{\lambda_{2^{n-i}}} X$$

for all $1 \leq i \leq n$. For example, there are equivalences

$$\begin{aligned}
\widetilde{E}\mathcal{F}[C_{2^n}] \wedge X &\simeq a^{-1}_{\lambda_1} X = a^{-1}_{2\sigma} X = a^{-1}_{\sigma} X, \\
\widetilde{E}\mathcal{F}[C_{2^{n-1}}] \wedge X &\simeq a^{-1}_{\lambda_2} X, \\
\widetilde{E}\mathcal{F}[C_{2^{n-2}}] \wedge X &\simeq a^{-1}_{\lambda_4} X.
\end{aligned}$$

The following theorem shows that one can compute the homotopy groups of $\widetilde{E}\mathcal{F}[C_{2^i}] \wedge X = a^{-1}_{\lambda_{2^{n-i}}} X$ by smashing the slice tower of X with $\widetilde{E}\mathcal{F}[C_{2^i}]$. The resulting localized slice spectral sequence will converge to the homotopy groups of $a^{-1}_{\lambda_{2^{n-i}}} X$.

Theorem 3.1. *Let X be a C_{2^n}-spectrum, and let $\{P^\bullet\}$ denote the slice tower for X. Consider the tower*

$$\{Q^\bullet\} := \{\widetilde{E}\mathcal{F}[C_{2^i}] \wedge P^\bullet\}$$

obtained by smashing $\{P^\bullet\}$ with $\widetilde{E}\mathcal{F}[C_{2^i}]$. The spectral sequence associated to $\{Q^\bullet\}$ converges strongly to the homotopy groups of $\widetilde{E}\mathcal{F}[C_{2^i}] \wedge X$.

Proof. Let $\lambda := \lambda_{2^{n-i}}$. Consider the tower

$$\begin{array}{ccccc}
S^{\infty\lambda} \wedge X & \longrightarrow & \varprojlim(S^{\infty\lambda} \wedge P^\bullet X) & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & S^{\infty\lambda} \wedge P^n X & \longleftarrow & S^{\infty\lambda} \wedge P^n_n X \\
 & & \downarrow & & \\
 & & S^{\infty\lambda} \wedge P^{n-1} X & \longleftarrow & S^{\infty\lambda} \wedge P^{n-1}_{n-1} X \\
 & & \downarrow & & \\
 & & \vdots & &
\end{array}$$

We will first show that the spectral sequence converges to the limit, $\varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$. Since smash products commute with colimits, we have the equivalence

$$\varinjlim(S^{\infty\lambda} \wedge P^\bullet X) \simeq *.$$

The slices $P^n_n X$ satisfy $P^n_n X \geq n$ for all n. Furthermore, since $S^{\infty\lambda} \geq 0$, we also have

$$S^{\infty\lambda} \wedge P^n_n X \geq n$$

by [HHR16, Proposition 4.26]. Applying Proposition 4.40 in [HHR16] to $S^{\infty\lambda} \wedge P^n_n X$ shows that the homotopy groups

$$\underline{\pi}_k(S^{\infty\lambda} \wedge P^n_n X) = 0 \text{ if } \begin{cases} n \geq 0 \text{ and } k < \lfloor \frac{n}{|G|} \rfloor, \\ n < 0 \text{ and } k < n. \end{cases}$$

This gives a vanishing line on the E_2-page of the spectral sequence. Since the colimit of the tower is contractible, the spectral sequence converges strongly to the homotopy groups of the limit, $\underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$ [Boa99, Section 5-6].

To finish our proof, it suffices to show that the map

$$S^{\infty\lambda} \wedge X \longrightarrow \varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$$

is an equivalence.

Consider the cofiber sequence

$$P_{n+1}X \longrightarrow X \longrightarrow P^n X$$

used in the definition of the slice tower. In the cofiber sequence, $P_{n+1}X \geq n+1$ and $P^n X \leq n$. Smashing this cofiber sequence with $S^{\infty\lambda}$ produces a new cofiber sequence

$$S^{\infty\lambda} \wedge P_{n+1}X \longrightarrow S^{\infty\lambda} \wedge X \longrightarrow S^{\infty\lambda} \wedge P^n X.$$

Since $S^{\infty\lambda} \geq 0$, [HHR16, Proposition 4.26] implies that

$$S^{\infty\lambda} \wedge P_{n+1}X \geq n+1.$$

Applying [HHR16, Proposition 4.40] to $S^{\infty\lambda} \wedge P_{n+1}X$ shows that

$$\underline{\pi}_k(S^{\infty\lambda} \wedge P_{n+1}X) = 0 \text{ if } \begin{cases} n+1 \geq 0 \text{ and } k < \lfloor \frac{n+1}{|G|} \rfloor, \\ n+1 < 0 \text{ and } k < n+1. \end{cases}$$

The cofiber sequence above induces the following long exact sequence in homotopy groups:

$$\underline{\pi}_k(S^{\infty\lambda} \wedge P_{n+1}X) \longrightarrow \underline{\pi}_k(S^{\infty\lambda} \wedge X) \longrightarrow \underline{\pi}_k(S^{\infty\lambda} \wedge P^nX) \longrightarrow \underline{\pi}_{k-1}(S^{\infty\lambda} \wedge P_{n+1}X) \longrightarrow \cdots$$

It follows from this long exact sequence and the discussion above that

$$\underline{\pi}_k(S^{\infty\lambda} \wedge X) \cong \underline{\pi}_k(S^{\infty\lambda} \wedge P^nX) \text{ if } \begin{cases} n+1 \geq 0 \text{ and } k < \lfloor \frac{n+1}{|G|} \rfloor, \\ n+1 < 0 \text{ and } k < n+1. \end{cases}$$

This means that for any k, the kth homotopy groups of $S^{\infty\lambda} \wedge X$ and $S^{\infty\lambda} \wedge P^nX$ will be isomorphic when n is large enough. In particular, the map $S^{\infty\lambda} \wedge P^{n+1}X \to S^{\infty\lambda} \wedge P^nX$ will induce an isomorphism on $\underline{\pi}_k$. It is then immediate that the system $\underline{\pi}_k(S^{\infty\lambda} \wedge P^\bullet X)$ satisfies the Mittag–Leffler condition and therefore

$$\underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X) \cong \varprojlim \underline{\pi}_k(S^{\infty\lambda} \wedge P^\bullet X) \cong \underline{\pi}_k(S^{\infty\lambda} \wedge P^nX)$$

for n large.

Another way to observe this is by using the localized slice spectral sequence. As we have shown, the spectral sequence associated to the tower $\{Q^\bullet\} := \{S^{\infty\lambda} \wedge P^\bullet\}$ converges to the homotopy groups of $\varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$. It takes the form

$$E_2^{s,n} = \underline{\pi}_{n-s}(S^{\infty\lambda} \wedge P_n^nX) \Longrightarrow \underline{\pi}_{n-s} \varprojlim(S^{\infty\lambda} \wedge P^\bullet X).$$

By [HHR16, Proposition 4.40], the homotopy groups

$$\underline{\pi}_{n-s}(S^{\infty\lambda} \wedge P_n^nX)$$

do not contribute to $\underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$ when $n \geq 0$ and $k < \lfloor \frac{n}{|G|} \rfloor$, or when $n < 0$ and $k < n$ (see Figure 1). Therefore,

$$\underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X) \cong \underline{\pi}_k(S^{\infty\lambda} \wedge P^nX) \text{ if } \begin{cases} n \geq 0 \text{ and } k < \lfloor \frac{n}{|G|} \rfloor, \\ n < 0 \text{ and } k < n. \end{cases}$$

For any k, consider the diagram

$$\begin{array}{ccc} \underline{\pi}_k(S^{\infty\lambda} \wedge X) & \longrightarrow & \underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X) \\ & \searrow^{\cong} & \downarrow \cong \\ & & \underline{\pi}_k(S^{\infty\lambda} \wedge P^nX) \end{array}$$

We have proven that when n is large enough ($n > k$), the vertical arrow and the diagonal arrow are isomorphisms. Therefore, the horizontal arrow induces an isomorphism

$$\underline{\pi}_k(S^{\infty\lambda} \wedge X) \cong \underline{\pi}_k \varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$$

for all k. It follows that $S^{\infty\lambda} \wedge X \simeq \varprojlim(S^{\infty\lambda} \wedge P^\bullet X)$, as desired. □

From the discussion in [HHR16, Section 4.7] and our discussion in Section 3.1, it follows that the localized slice spectral sequences of $MU^{(\!(G)\!)}$ and $BP^{(\!(G)\!)}$ are multiplicative spectral sequences.

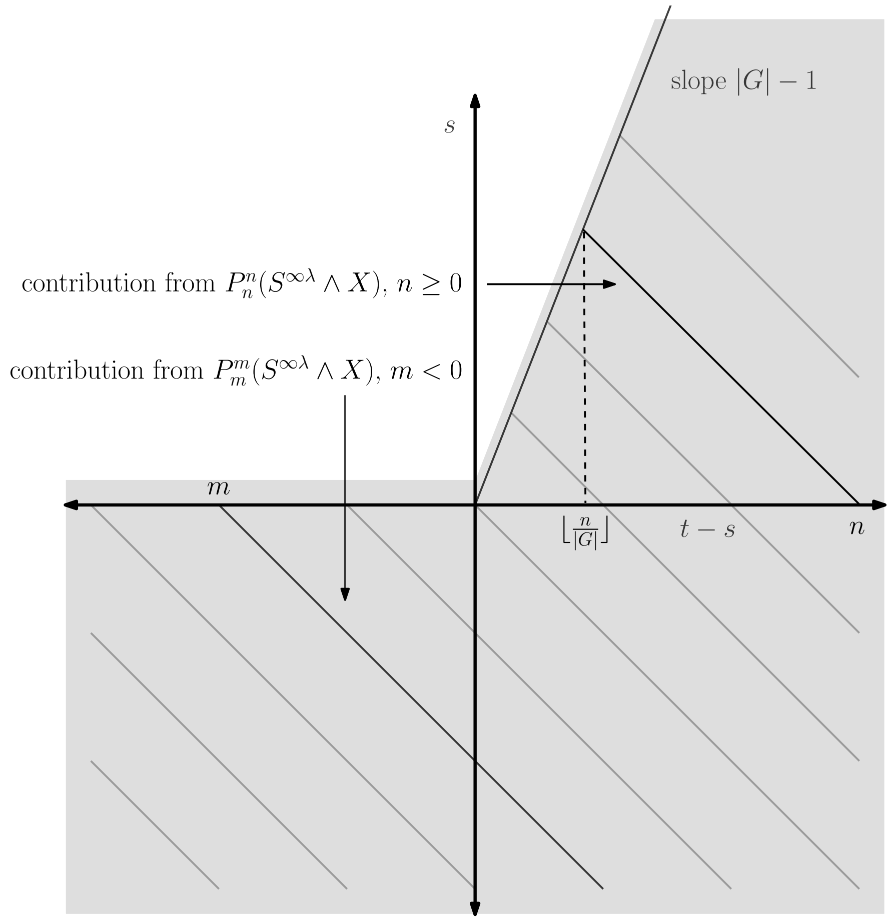


FIGURE 1. Spectral sequence associated to the tower $\{\widetilde{E}\mathcal{F}[C_{2^i}] \wedge P^\bullet\}$.

3.3. **Exotic transfers.** If the transfer of a given class in the slice spectral sequence is zero, it might still support a non-trivial *exotic transfer* in a a higher filtration. Understanding these is both crucial for understanding the Mackey functor structure of the spectral sequence and also quite helpful to deduce differentials and extensions inside the spectral sequence. While the concept of exotic transfers is pretty transparent for permanent cycles, it is slightly more subtle for exotic transfers just happening on finite pages. Following the lead of [BBHS19] (in the case of the Picard spectral sequence), we will give a precise definition of this phenomenon and show how it behaves with respect to differentials. It turns out that it is no more difficult to treat a more general setting, which specializes to several different known spectral sequences and allows also for more general operations than just transfers.

We consider a tower

$$\cdots \to X^{i+1} \to X^i \to X^{i-1} \to \cdots$$

of G-spectra. Recall that to this we can associate a spectral sequence as follows: Let $X^m_n = \mathrm{fib}(X^m \to X^{n-1})$. For V a virtual G-representation of dimension t, we set $E_2^{s,V} = \underline{\pi}_{V-s}(X^t_t)$ and more generally

$$E_r^{s,V} = \mathrm{im}(\underline{\pi}_{V-s}X_t^{t+(r-2)} \to \underline{\pi}_{V-s}X^t_{t-r+2}).$$

The differentials $d_r \colon E_r^{s,V} \to E_r^{s+r,V+r-1}$ are defined as the restrictions of the boundary maps $\delta\colon \underline{\pi}_{V-s}X^t_{t-r+2} \to \underline{\pi}_{V-s-1}X^{t+r-1}_{t+1}$. See e.g. [Lur17, Section 1.2.2] for some details in the setting of an ascending filtration. Our setting specializes in particular to the following spectral sequences:

(1) Given a spectrum Z with a G-action, set $X^i = (\tau_{\leq i}Z)^{EG_+}$. We recover the homotopy fixed point spectral sequence.
(2) Given a spectrum Z with a G-action, set $X^i = (\tau_{\leq i}Z \wedge \tilde{E}G)^{EG_+}$. We recover the Tate spectral sequence.
(3) Given a G-spectrum Z, set $X^i = P^iZ$, the slice tower. We obtain the slice spectral sequence.
(4) Given a C_{2^n}-spectrum Z and $1 \leq j \leq n$, set $X^j = \widetilde{E}\mathcal{F}[C_{2^j}] \wedge P^jZ$. We obtain the localized slice spectral sequence. This will be the main example of relevance for us.

We fix an arbitrary map $\Sigma^\infty G/K \to \Sigma^\infty G/H$ and denote the resulting operation $\pi_n^H \to \pi_n^K$ by w. The most important case for us will be $H \subset K$ and $w = \mathrm{Tr}_H^K$. But equally well w might be a restriction map, multiplication by a fixed element like 2, or any combination of these.

Definition 3.2. *Let $x \in E_r^{s,t}(G/H)$. By definition, we may lift the corresponding element in $\pi^H_{t-s}X^t_{t-r+2}$ to an element $\widetilde{x} \in \pi^H_{t-s}X_t^{t+r-2}$. Given $0 \leq p \leq r-2$, consider the image of $w(\widetilde{x})$ in $\pi^K_{t-s}X^{t+p}_{t+p-r+2}$. If this image lies in $E_r^{s+p,t+p}(G/K)$, we call it a* w-operation of x of filtration jump p. *If $p > 0$, we speak of an* exotic w-operation, *which, depending on w, might be an* exotic transfer, exotic restriction *etc.*

Note that with x and $\widetilde{x}$ fixed, a w-operation of filtration jump p can only exist if all w-operations of lower filtration jump vanish. Indeed, if the image of $w(\widetilde{x})$ in $\pi^K_{t-s}X^{t+p}_{t+p-r+2}$ lies in $E_r^{s+p,t+p}(G/K)$, it is in the image of $\pi^K_{t-s}X^{t+p+r-2}_{t+p}$. The map from this group to $\pi^K_n X^{t+p-1}_{t+p-r+1}$ factors through $\pi^K_{t-s}X^{t+p-1}_{t+p} = 0$.

Remark 3.3. *A different viewpoint on this definition may make it more transparent: With x as above, we consider the modified tower $\widetilde{X}^\bullet$ with $\widetilde{X}^i = X^i_t$ for $i \leq t+r-1$ and $\widetilde{X}^i = X^{t+r-2}_t$ if $i \geq s+r-2$ and denote the associated spectral sequence by $\widetilde{E}^{*,*}_*$. Note that $E_r^{s,t}(G/K)$ is a quotient of $\widetilde{E}_r^{s,t}(G/H)$ and any lift of x becomes a permanent cycle in the modified spectral sequence, represented by some $\widetilde{x} \in \pi^H X^{t+r-2}_t$. If $w(\widetilde{x})$ is nonzero, it must be detected in some $\widetilde{E}_r^{s+p,t+p}$, the result being a w-operation in $\widetilde{E}$ of filtration jump p. If $\widetilde{x}$ is fixed, the resulting class is well-defined.*

The fixing of $\widetilde{x}$ is essential though: There might be, for example, a class z in $E_2^{s+q,t+q}$ with $0 < q < p$ whose class $[z]$ supports a non-trivial non-exotic transfer, then the exotic transfer associated to $\widetilde{x} + z$ has filtration jump q. Thus, exotic w-operations are not well-defined in general. In the extreme case a vanishing class might support a non-trivial exotic transfer of some filtration jump p, caused by another transfer of smaller filtration jump – thus it might be better to think of transfers of filtration jump p as having filtration jump at most *p.*

Proposition 3.4. *Let $x \in E_r^{s,t}(G/H)$ and z a class with $d_r(z) = x$. Suppose $d_{r+q}(w(z))$ is zero for $q < p$. Then $d_{r+p}(w(z))$ is a w-operation of x of filtration jump p.*

Proof. We choose a lift of $z \in \pi^H_{t-s+1} X^{t-r+1}_{t-2r+3}$ to $\widetilde{z} \in \pi^H_{t-s+1} X^{t-1}_{t-r+1}$. As $\delta(\widetilde{z})$ in the diagram below is a lift of x, contemplating the fate of $w(\widetilde{z})$ passing along the two different travel paths from the upper left corner to the lower right corner proves the proposition.

$$\begin{array}{ccc}
\pi^K_{t-s+1} X^{t-1}_{t-r+1} & \xrightarrow{\quad\delta\quad} & \pi^K_{t-s} X^{t+r-2}_{t} \\
\downarrow & & \downarrow \\
E_r^{s-r,t-r+1} \subseteq \pi^K_{t-s+1} X^{t-r+1}_{t-2r+3} & & \pi^K_{t-s} X^{t+p}_{t+p-r+2} \supseteq E_r^{s+p,t+p} \\
\downarrow & & \downarrow \\
E_{r+p}^{s-r,t-r+1} \subseteq \pi^K_{t-s+1} X^{t-r+1}_{t-2r+3-p} & \xrightarrow{\ \delta\ } & \pi^K_{t-s} X^{t+p}_{t-r+2} \supseteq E_{r+p}^{s+p,t+p}
\end{array}$$

□

While the definition and results so far are very general (and our proofs would also apply to other settings than equivariant homotopy theory), we want also to formulate a result specific to cylic 2-groups. Both statement and proof are a variant of those of [HHR17, Proposition 4.4], but also work for exotic transfers and restrictions on finite pages.

Proposition 3.5. *Let $H \subset G$ be an index 2 subgroup of a cyclic 2-group and $V \in RO(G)$.*

(i) Let $y \in E^{s,V}_{r+1}(G/G)$ with $a_\sigma y = 0 \in E^{s+1,V+1-\sigma}_{r+1}(G/G)$. Then y is an exotic transfer of filtration jump (at most) $r-1$.

(ii) Let $z \in E^{s,V}_{r+1}(G/H)$ with $\mathrm{Tr}(z) = 0 \in E^{s,V}_{r+1}(G/G)$. Then z is an exotic restriction from $E^{s-r+1,V-r+2-\sigma}_{r+1}$ of filtration jump (at most) $r-1$.

Proof. For the first part, fix the bidegree of y to be $(0,0)$. The term $E^{0,0}_{r+1}(G/G)$ injects into $\pi^G_0 X^{r-1}_0$. Using the long exact sequence induced by $G/H_+ \to S^0 \xrightarrow{a_\sigma} S^\sigma$, we see that $a_\sigma y = 0$ implies $y = \mathrm{Tr}(\widetilde{w})$ with $\widetilde{w} \in \pi^H_0 X^{r-1}_0$. By definition, this defines an element $w \in E^{-r+1,-r+1}_{r+1}(G/H)$ such that y is an exotic transfer of w.

For the second part, fix the bidegree of z to be $(0,0)$. We see that z is the restriction of some $\widetilde{v} \in \pi^G_{1-\sigma} X^{r-1}_0$. By definition, this defines an element $v \in E^{-r+1,-r+2-\sigma}_{r+1}(G/G)$ such that z is an exotic restriction of v. □

3.4. **The behaviour of norms.** This section is about the behaviour of norms in the (regular) slice spectral sequence and its localized variant. We will formulate a generalization of [Ull13] and then discuss how it applies both to Ullman's original setting, the regular slice spectral sequence, and to the localized slice spectral sequence.

We will first work in an abstract setting: Let (X^i) be a tower of G-spectra and $E^{*,*}_*$ be the associated spectral sequence as in the preceding subsection. Set $X^\infty = \lim_i X^i$ and $X_n = X^\infty_n$.

Let $H \subset G$ be a subgroup of index h. We assume that we have maps $N^G_H X_n \to X_{hn}$ and $N^G_H X^n_n \to X^{hn}_{hn}$ that are (up to homotopy) compatible with the maps $X_n \to X_{n-1}$ and $X_n \to X^n_n$. We call it a *norm structure*. It induces norm maps $N^G_H \colon E^{s,V+s}_2 \to E^{hs,\mathrm{Ind}^G_H V+hs}_2$.

Proposition 3.6. *Let $x \in E_2(G/H)$ be an element representing zero in $E_{r+2}(G/H)$. Then $N^G_H(x)$ represents zero in $E_{rh+2}(G/G)$.*

Proof. The proof is the same as that of [Ull13, Proposition I.5.17]. □

Example 3.7. *Our first example of this setting is the regular slice tower of [Ull13], which coincides with the slice tower of [HHR16] for norms of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$ – thus there should be no danger of confusion if we use the same notation P^iX for the regular slice tower.*

Ullman constructs in [Ull13, Corollaries I.5.10 and I.5.11] for every H-spectrum X natural compatible maps $N_H^G P_n X \to P_{nh} N_H^G X$ and $N_H^G P_n^n X \to P_{nh}^{nh} N_H^G X$. Moreover the square

$$\begin{array}{ccc} N_H^G P_n X & \longrightarrow & P_{hn} N_H^G X \\ \downarrow & & \downarrow \\ N_H^G P_{n-1} X & \longrightarrow & P_{hn-h} N_H^G X \end{array}$$

commutes, as $N_H^G P_n X$ is $\geq hn$ by [Ull13, Corollary I.5.8] and both maps into $N_H^G P_n X \to P_{hn-h} N_H^G X$ are compatible with the respective maps to $N_H^G X$.

Given now a G-commutative ring spectrum R, we obtain a map $N_H^G \operatorname{Res}_H^G R$. Setting $X = \operatorname{Res}_H^G R$, the composite $N_H^G P_n \operatorname{Res}_H^G X \to P_{nh} N_H^G \operatorname{Res}_H^G R \to P_{nh} R$ and its analogue for P_n^n define a norm structure on the regular slice tower of R.

Example 3.8. *Let R be a G-commutative ring spectrum with $G = C_{2^n}$. We will define a norm structure on the tower $X^i = a_\lambda^{-1} P^i X$ defining the localized regular slice spectral sequence. Using the observations above for the regular slice spectral sequence, it suffices to produce natural maps $N_H^G \operatorname{Res}_H^G a_\lambda^{-1} P_n R \to a_\lambda^{-1} N_H^G \operatorname{Res}_H^G P_{hn} R$ and similarly for P_n^n. As N_H^G and Res_H^G are monoidal and by Lemma 2.8 it thus suffices to provide a natural map*

$$a_\lambda^{-1} \mathbb{S}_G \simeq a_\lambda^{-1} N_H^G \mathbb{S}_H \to N_H^G \operatorname{Res}_H a_{\lambda^{-1}} \mathbb{S}_G \simeq a^{-1}_{\operatorname{Ind}_H^G \operatorname{Res}_H^G \lambda} \mathbb{S}_G$$

As observed before, $\operatorname{Ind}_H^G \operatorname{Res}_H^G \lambda$ is a multiple of λ if $H \neq e$ and contains a trivial summand if $H = e$. This produces the norm structure if $H \neq e$. In contrast for $H = e$, all norms would have to be zero.

We remark that we have not used the full strength of our considerations in Section 2.3 here, but we expect that these will be necessary for deeper considerations about norms.

We will use the following proposition without further comment.

Proposition 3.9. *Both in the regular slice spectral sequence and in the localized regular slice spectral sequence of a G-commutative ring spectrum, the norms are multiplicative: $N_H^G(xy) = N_H^G(x) N_H^G(y)$.*

Proof. This follows from the commutativity of

$$\begin{array}{ccc} N_H^G(P_m X \wedge P_n Y) & \longrightarrow & N_H^G(P_{m+n} X \wedge Y) \\ \downarrow & & \downarrow \\ P_{hm} N_H^G X \wedge P_{hn} N_H^G Y & \longrightarrow & P_{hm+hn} N_H^G (X \wedge Y) \end{array}$$

for G-spectra X and Y. This in turn follows as there is up to homotopy just one map

$$N_H^G(P_m X \wedge P_n Y) \to P_{hm+hn} N_H^G (X \wedge Y)$$

compatible with the maps to $N_H^G(X \wedge Y)$ as $N_H^G(P_m X \wedge P_n Y) \geq h(m+n)$ by [Ull13, Corollaries I.4.2 and I.5.8]. ∎

4. The localized slice spectral sequences of $BP^{(\!(G)\!)}$: summary of results

We now turn to analyze the localized slice spectral sequence of $BP^{(\!(G)\!)}$ for $G = C_{2^n}$. From now on, everything will be implicitly 2-localized. In this section, we list our main results and give an outline of the computation. Detailed computations of the results stated in this section are in Section 5.

As we discussed in Section 3, the Slice Theorem [HHR16, Theorem 6.1] implies that the slice associated graded of $BP^{(\!(C_{2^n})\!)}$ is

$$H\underline{\mathbb{Z}}[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots],$$

where $\bar{t}_i \in \pi^{C_2}_{(2^i-1)\rho_2} BP^{(\!(C_{2^n})\!)}$ (see also [HHR16, Section 2.4] for details).

For the rest of the paper, we use λ for the 2-dimensional real representation of C_{2^n} which is rotation by $\left(\frac{\pi}{2^{n-1}}\right)$, and σ for the 1-dimensional sign representation of G. We use σ_2 for the sign representation of the unique *subgroup* C_2 in G. Let $i < j \leq n$, we will use $\mathrm{Res}^{2^j}_{2^i}$, $\mathrm{Tr}^{2^j}_{2^i}$ and $N^{2^j}_{2^i}$ for restrictions, transfers and norms between C_{2^i} and C_{2^j} as subgroups of G. If their subscript and superscript are omitted, they mean the restriction, transfer and norm between C_2 and C_4.

Theorem 4.1.

(1) *Let $G = C_{2^n}$ and $H = C_2$ be the subgroup of order 2 inside G. There is a $RO(G/H)$-graded spectral sequence of Mackey functors a_λ^{-1}SliceSS$(BP^{(\!(G)\!)})$ that converges to the $RO(G/H)$-graded homotopy Mackey functor of $N_e^{G/H} H\mathbb{F}_2$. The E_2-page of this spectral sequence is*

$$a_\lambda^{-1} H\underline{\mathbb{Z}}_\bigstar[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \cdots].$$

(2) *The integral E_2-page of a_λ^{-1}SliceSS$(BP^{(\!(G)\!)})$ is bounded by the vanishing lines $s = (2^n - 1)(t-s)$ and $s = -(t-s)$ in Adams grading. In other words, at stem $t-s$, the classes with filtrations greater than $(2^n-1)(t-s)$ or less than $-(t-s)$ are all zero.*

(3) *On the integral E_2-page, the a_λ-localizing map*

$$\textit{SliceSS}(BP^{(\!(G)\!)}) \to a_\lambda^{-1}\textit{SliceSS}(BP^{(\!(G)\!)})$$

induces an isomorphism of classes in positive filtrations. The kernel of this map consists of transfer classes in SliceSS$(BP^{(\!(G)\!)})$ from the trivial subgroup in filtration 0. These classes are all permanent cycles.

Proof. By Theorem 3.1, a_λ^{-1}SliceSS$(BP^{(\!(G)\!)})$ computes the homotopy of $\tilde{E}G \wedge BP^{(\!(G)\!)}$. By Proposition 2.2 and the fact that $\Phi^{C_2}(BP_\mathbb{R}) \simeq H\mathbb{F}_2$,

$$\tilde{E}G \wedge BP^{(\!(G)\!)} \simeq P^*_{G/C_2}(N_1^{G/C_2} H\mathbb{F}_2).$$

Since the E_2-page of the slice spectral sequence of $BP^{(\!(G)\!)}$ has the form

$$H\underline{\mathbb{Z}}_\bigstar[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots],$$

the E_2-page of a_λ^{-1}SliceSS$(BP^{(\!(G)\!)})$ is

$$a_\lambda^{-1} H\underline{\mathbb{Z}}_\bigstar[G \cdot \bar{t}_1, G \cdot \bar{t}_2, \ldots]$$

Together with Theorem 2.2 and Theorem 3.1 this proves (1).

The top vanishing line $s = (2^n-1)(t-s)$ follows from the fact that $\underline{\pi}_i(S^{k\rho_G + l\lambda} \wedge H\underline{\mathbb{Z}}) = 0$ for $k, l \geq 0$ and $i < k$ (See [HHR16, Theorem 4.42]). For the second vanishing line $y = -x$, note that in stem $t-s$, classes in filtration less than $-(t-s)$ are contributed by slices of negative dimension, but $BP^{(\!(G)\!)}$ has no negative slices. This proves (2).

To prove (3), by unpacking the description of the E_2-page, we need to show that for $k, l \geq 0$, the a_λ-multiplication map

$$a_\lambda : \pi_i^G(S^{k\rho_G + l\lambda} \wedge H\underline{\mathbb{Z}}) \longmapsto \pi_i^G(S^{k\rho_G+(l+1)\lambda} \wedge H\underline{\mathbb{Z}})$$

is an isomorphism for $k \leq i < k|G| + 2l$ and is surjective with kernel consisting of transfer classes from trivial subgroup for $i = k|G| + 2l$. This is a direct consequence of the cellular chain computation of the representation spheres. Since the underlying tower of the slice tower is the Postnikov tower, all the class in the trivial subgroup and their transfers are permanent cycles. □

Remark 4.2. *In fact,* (2) *and* (3) *of Theorem 4.1 hold in a greater generality. For instance, they are true for any* (-1)*-connected* G*-spectrum. We will investigate properties of the localized slice spectral sequences in a future paper.*

By [LNR11] and [BBLNR14], all C_{2^n} norms of $H\mathbb{F}_2$ are cofree, therefore we will not distinguish between their fixed points and homotopy fixed points.

Corollary 4.3. *The* 0*-th homotopy group of* $(N_1^{2^{n-1}} H\mathbb{F}_2)^{hC_{2^{n-1}}}$ *is isomorphic to* $\mathbb{Z}/2^n$*.*

Proof. In $a_\lambda^{-1}\mathrm{SliceSS}(BP^{((G))})$, the only Mackey functor contributing to the 0-stem is $\underline{\pi}_0(a_\lambda^{-1} H\underline{\mathbb{Z}})$, and we claim that

$$\pi_0^G(a_\lambda^{-1} H\underline{\mathbb{Z}})(G/G) \cong \mathbb{Z}/2^n.$$

Indeed, the maps $\pi_0^G(S^{n\lambda} \wedge H\underline{\mathbb{Z}}) \to \pi_0^G(S^{(n+1)\lambda} \wedge H\underline{\mathbb{Z}})$ are isomorphisms for $n \geq 1$ and $\pi_0^G(S^\lambda \wedge H\underline{\mathbb{Z}})$ is the cokernel of the transfer $\mathrm{Tr}_1^{2^n} : \pi_0^e H\underline{\mathbb{Z}} \to \pi_0^{C_{2^n}} H\underline{\mathbb{Z}}$, i.e. of multiplication by 2^n on $\mathbb{Z}$. □

For the rest of the paper, we focus on the case $G = C_4$. We compute the first 8 stems of $a_\lambda^{-1}\mathrm{SliceSS}(BP^{((C_4))})$.

Theorem 4.4. *The first* 8 *stems of* $\pi_*^{C_4}(a_\lambda^{-1} BP^{((C_4))}) \cong \pi_*^{C_2} N_1^2 H\mathbb{F}_2$ *are shown in the following chart:*

i	0	1	2	3	4	5	6	7	8
π_i	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$

On the E_∞*-page of the localized spectral sequence, the black subgroups are those generated by non-exotic transfers from* $\mathcal{A}_* = \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$*, and the red subgroups consist of everything else. For the Mackey functor structure, see Figure 6.*

Modulo transfers from $\mathcal{A}_*$*, the homotopy groups has the following generators:*

(1) π_1 *is generated by* $\eta = N(\bar{t}_1)a_\lambda a_\sigma$*, the image of the first Hopf invariant one element;*
(2) π_2 *is generated by* $\frac{\eta^2}{2} = \frac{2u_\lambda}{a_\lambda}$*;*
(3) π_3 *is generated by* $\nu = N(\bar{t}_2)a_\lambda^3 a_\sigma^3$*, the image of the second Hopf invariant one element;*
(4) π_6 *is generated by* $\frac{\nu^2}{2} = \frac{2u_\lambda^3}{a_\lambda^3}$*;*
(5) π_7 *is generated by* $N(\bar{t}_3)a_\lambda^7 a_\sigma^7$ *and* $N(\bar{t}_2)u_\lambda u_{2\sigma} a_\lambda^2 a_\sigma$*, and one of them detects the third Hopf invariant one element* σ*.*
(6) π_8 *is generated by* $\mathrm{Tr}_2^4(\bar{t}_2^2 \bar{t}_1^2 a_{\sigma_2}^8) + \mathrm{Tr}_2^4(\bar{t}_3 \bar{t}_1 a_{\sigma_2}^8) + N(\bar{t}_2)N(\bar{t}_1)u_{2\sigma}^2 a_\lambda^4$*.*

In [Rog], Rognes shows that the unit map $S^0 \to (N_1^2 H\mathbb{F}_2)^{hC_2}$ induces a splitting injection on mod 2 homology as an $\mathcal{A}_*$-comodule thus a splitting injection on the E_2-page of the Adams spectral sequence. Therefore, the ring spectrum $(N_1^2 H\mathbb{F}_2)^{hC_2} \simeq (a_\lambda^{-1} BP^{((C_4))})^{C_4}$ detects all Hopf invariant one elements. They all restrict to 0, since the underlying Adams spectral sequence of

$H\mathbb{F}_2 \wedge H\mathbb{F}_2$ is concentrated in filtration 0. Therefore, they are detected by red subgroups in the corresponding degree.

The proof of Theorem 4.4 is by computing $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$ and is given in the next section. The most relevant differentials in the spectral sequence are listed in the following table:

Differential	Formula	Proof
d_3	$d_3(u_{2\sigma_2}) = a_{\sigma_2}^3(\bar{t}_1 + \gamma\bar{t}_1)$ $d_3(u_\lambda) = \mathrm{Tr}_2^4(a_{\sigma_2}^3\bar{t}_1)$	Proposition 5.8
d_5	$d_5(u_{2\sigma}) = N(\bar{t}_1)a_\lambda a_\sigma^3$	Theorem 5.7
d_5	$d_5(u_\lambda^2) = N(\bar{t}_1)u_\lambda a_\lambda^2 a_\sigma$	Proposition 5.11
d_7	$d_7(u_{2\sigma_2}^2) = a_{\sigma_2}^7(\bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2)$ $d_7(2u_\lambda^2) = \mathrm{Tr}_2^4(a_{\sigma_2}^7\bar{t}_1^3)$	Theorem 5.4 Proposition 5.16
d_7	$d_7(u_\lambda^4) = \mathrm{Tr}_2^4(\bar{t}_1^3 u_{2\sigma_2}^2 a_{\sigma_2}^7)$	Proposition 5.17
d_{13}	$d_{13}(u_\lambda^4 a_\sigma) = N(\bar{t}_2 + \bar{t}_1^3 + \gamma(\bar{t}_2))u_{2\sigma}^2 a_\lambda^7$	Proposition 5.21
d_{15}	$d_{15}(2u_\lambda^4) = \mathrm{Tr}_2^4(\bar{t}_3^{C_2} a_{\sigma_2}^{15})$	Proposition 5.22

5. Computing the localized slice spectral sequences of $BP^{(\!(G)\!)}$

5.1. **Computing the E_2-page.** In this section, we compute $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$ and prove Theorem 4.4. Our approach is similar to that of [HHR17] and [HSWX18]. Before we start our computation, we give a complete algebraic description of the E_2-page of $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$ in terms of generators and relations. To do so, by Theorem 4.1, we need to describe the C_2-homotopy groups $\underline{\pi}_\bigstar(a_{\sigma_2}^{-1}H\underline{\mathbb{Z}})$ and the C_4-homotopy groups $\underline{\pi}_\bigstar(a_\lambda^{-1}H\underline{\mathbb{Z}})$.

Proposition 5.1. *We have*

$$\pi_\bigstar^{C_2}(a_{\sigma_2}^{-1}H\underline{\mathbb{Z}}) = \mathbb{F}_2[u_{2\sigma_2}, a_{\sigma_2}^{\pm 1}].$$

The Mackey functor structure is determined by the contractibility of the underlying spectrum.

This proposition is proved by a standard Tate cohomology computation, see [Gre18, Section 2.C] for details.

In order to compute $\underline{\pi}_\bigstar(a_\lambda^{-1}H\underline{\mathbb{Z}})$ for $G = C_4$, note that we only need to consider representations of the form $a+b\sigma$, as multiplications by powers of a_λ induce isomorphisms between representations with nontrivial λ components.

Let S be the subring of

$$R = \mathbb{Z}/4[a_\sigma, u_{2\sigma}^{\pm 1}, u_\lambda a_\lambda^{-1}]/(2a_\sigma, u_\lambda a_\lambda^{-1} a_\sigma^2 = 2u_{2\sigma})$$

generated by the elements $\{a_\sigma, u_{2\sigma}, u_\lambda a_\lambda^{-1}, 2u_{2\sigma}^k, u_{2\sigma}^k u_\lambda a_\lambda \mid k < 0\}$, and let $M = \mathbb{Z}/2[u_{2\sigma}^{\pm 1}, u_\lambda a_\lambda^{-1}, a_\sigma^{\pm 1}]/(u_{2\sigma}^\infty, a_\sigma^\infty)$ be considered as a module over S. Here, $R[x^{\pm 1}]/(x^\infty)$ is the cokernel of the map $R[x] \to R[x^{\pm 1}]$.

Proposition 5.2. *We have*

$$\pi_\bigstar^{C_4}(a_\lambda^{-1}H\underline{\mathbb{Z}}) = (S \oplus \Sigma^{-1}M)[a_\lambda^{\pm 1}],$$

where $S \oplus \Sigma^{-1}M$ is the square-zero extension of M over S of degree -1.

The Green functor structure is determined by the following facts:

1. *The C_2-restriction of $a_\lambda^{-1}H\underline{\mathbb{Z}}$ is the spectrum $a_{\sigma_2}^{-1}H\underline{\mathbb{Z}}$ in Proposition 5.1.*
2. *The C_2-restrictions of the classes u_λ and $u_{2\sigma}$ are $u_{2\sigma_2}$ and 1, respectively.*

(3) Given $V \in RO(C_4)$, there is an exact sequence (see [HHR17, Lemma 4.2])

$$\pi^{C_2}_{i^*_{C_2}V}X \xrightarrow{\mathrm{Tr}^4_2} \pi^{C_4}_V X \xrightarrow{a_\sigma} \pi^{C_4}_{V-\sigma}X \xrightarrow{\mathrm{Res}^4_2} \pi^{C_2}_{i^*_{C_2}V-1}X.$$

In other words, the kernel of a_σ-multiplication is the image of the transfer from C_2 to C_4, and the image of a_σ-multiplication is the kernel of the restriction from C_4 to C_2.

The proof of Proposition 5.2 and a more explicit presentation of the Mackey functor are given in [Zen, Proposition 6.7]. Fortunately, in most of the paper we only need the "positive cone" of the coefficient Green functor, that is, the part $\bigstar = a + b\sigma + c\lambda$ for $b \leq 0$. The Green functor structure of this part is computed in [HHR17, Section 3]. However, the other part also plays an important role on the computation, see for example proofs of Proposition 5.14, 5.20 and 6.7.

The relation $\frac{u_\lambda}{a_\lambda}a_\sigma^2 = 2u_{2\sigma}$ and its integral version $u_\lambda a_\sigma^2 = 2u_{2\sigma}a_\lambda$ are commonly called the *gold relation* (see [HHR17, Lemma 3.6]).

Figure 2 gives the Lewis diagrams (first introduced in [Lew88]) we use for C_4-Mackey functors, where restrictions Res^G_H map downwards and transfers Tr^G_H map upwards. These notations are consistent with [HHR17, Section 5].

Symbol	$\circ$	$\blacktriangle$	$\blacktriangledown$
Lewis Diagram	$\mathbb{Z}/4$ $1\downarrow\ \uparrow 2$ $\mathbb{Z}/2$ $\downarrow\ \uparrow$ 0	$\mathbb{Z}/2$ $1\downarrow\ \uparrow 0$ $\mathbb{Z}/2$ $\downarrow\ \uparrow$ 0	$\mathbb{Z}/2$ $0\downarrow\ \uparrow 1$ $\mathbb{Z}/2$ $\downarrow\ \uparrow$ 0
Symbol	$\bullet$	$\hat{\bullet}$	$\bar{\bullet}$
Lewis Diagram	$\mathbb{Z}/2$ $\downarrow\ \uparrow$ 0 $\downarrow\ \uparrow$ 0	$\mathbb{Z}/2$ $\triangle\downarrow\ \uparrow\triangledown$ $\mathbb{Z}/2[C_4/C_2]$ $\downarrow\ \uparrow$ 0	0 $\downarrow\ \uparrow$ $\mathbb{Z}/2$ $\downarrow\ \uparrow$ 0

FIGURE 2. Table of C_4-Mackey functors

Figure 3 shows $\underline{\pi}_{a+b\sigma}(a_\lambda^{-1}H\underline{\mathbb{Z}})$ in the range $-6 \leq a, b \leq 6$. In the figure, the horizontal coordinate is a and the vertical coordinate is b. Vertical lines are a_σ-multiplications, where solid lines are surjections and the dash lines represent maps of the form $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$.

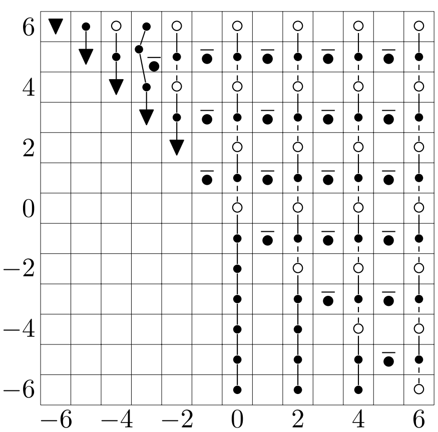


FIGURE 3. $\underline{\pi}_{a+b\sigma}(a_\lambda^{-1}H\underline{\mathbb{Z}})$ for $-6 \le a, b \le 6$.

Although we only care the most about the C_4-equivariant homotopy groups of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$, there are two advantages for computing $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$ as a spectral sequence of Mackey functors:

(1) The Mackey functor structure can transport certain differentials on the C_2-level to differentials on the C_4-level.
(2) The Mackey functor structure and d_r-differentials can result in exotic extensions of filtration $r-1$ (see Section 3.3).

We will see (1) in the computations of d_3, d_7, and d_{15}-differentials below. (2) will be used to prove certain extensions forming the $(\mathbb{Z}/4)$s in Theorem 4.4, see Proposition 5.14 and 5.20.

Notation 5.3. *Let $V \in RO(H)$ be a virtual representation that is in the image of the restriction $i_H^* : RO(G) \to RO(H)$. Then for any preimage W of V, there is a transfer map*

$$\mathrm{Tr}_H^{G,W} : \pi_V^H X \to \pi_W^G X,$$

as a part of the homotopy Mackey functor structure. In our computation we will omit writing W when it is clear from the context what W is.

5.2. **The C_2-spectral sequence.** We start our computation with the C_2-underlying spectral sequence of $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$.

Theorem 5.4.

(1) *The underlying C_2-spectral sequence of a_λ^{-1} SliceSS$(BP^{(\!(C_4)\!)})$ is $a_{\sigma_2}^{-1}$ SliceSS$(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$. Its E_2-page is*

$$a_{\sigma_2}^{-1}H\underline{\mathbb{Z}}_\bigstar[\bar{t}_1, \gamma\bar{t}_1, \bar{t}_2, \gamma\bar{t}_2, \cdots].$$

More precisely, the E_2-page of the underlying non-equivariant spectral sequence is trivial, and the E_2-page of the C_2-spectral sequence is

$$\mathbb{F}_2[u_{2\sigma_2}, a_{\sigma_2}^{\pm 1}][\bar{t}_1, \gamma\bar{t}_1, \bar{t}_2, \gamma\bar{t}_2, \cdots].$$

The elements $u_{2\sigma_2}, \bar{t}_i$ and $\gamma\bar{t}_i$ have filtration 0, while a_{σ_2} has filtration 1.[1]

(2) *All the differentials in $a_{\sigma_2}^{-1}$SliceSS$(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$ are determined by a_{σ_2}, $\bar{t}_i$ and $\gamma\bar{t}_i$ being permanent cycles, the differentials*

$$d_{2^{k+1}-1}(u_{2\sigma_2}^{2^{k-1}}) = a_{\sigma_2}^{2^{k+1}-1} \sum_{i=0}^{k} \bar{t}_{k-i}^{2^i} \gamma\bar{t}_i, \quad k \geq 1$$

and the Leibniz formula (for notational convenience, we let $\bar{t}_0 = \gamma\bar{t}_0 = 1$). The $E_{2^{k+1}}$-page has the form

$$\mathbb{F}_2[u_{2\sigma_2}^{2^k}, a_{\sigma_2}^{\pm 1}][\bar{t}_1, \gamma\bar{t}_1, \cdots]/(\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_k)$$

where $\bar{v}_k = \sum_{i=0}^{k} \bar{t}_{k-i}^{2^i} \gamma\bar{t}_i$.

(3) *The E_∞-page of $a_{\sigma_2}^{-1}$SliceSS$(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$ is*

$$\mathbb{F}_2[a_{\sigma_2}^{\pm 1}][\bar{t}_1, \gamma\bar{t}_1, \cdots]/(\bar{v}_1, \bar{v}_2, \cdots)$$

In particular, in the integral grading, all the stem-n non-trivial permanent cycles are located in filtration n.

Proof. For (1), note that since $i_{C_2}^* BP^{(\!(C_4)\!)} = BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$, the C_2-underlying slice spectral sequence of SliceSS$(BP^{(\!(C_4)\!)})$ is SliceSS$(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$. Moreover, $i_{C_2}^* a_\lambda = a_{\sigma_2}^2$. Therefore inverting a_λ in the C_4-spectral sequence inverts a_{σ_2} in the underlying C_2-spectral sequence.

For (2), we use the Hill–Hopkins–Ravenel slice differential theorem [HHR16, Theorem 9.9] and the formula in [BHSZ20, Theorem 3.1] that expresses the $\bar{v}_i$-generators in terms of the $\bar{t}_i$-generators. The Hill–Hopkins–Ravenel slice differential theorem states that in the slice spectral sequence of $BP_{\mathbb{R}}$, there are differentials

$$d_{2^{k+1}-1}(u_{2\sigma_2}^{2^{k-1}}) = \bar{v}_i a_{\sigma_2}^{2^{k+1}-1}, \quad k \geq 1.$$

The formula in [BHSZ20, Theorem 3.1] shows that under the left unit map $BP_{\mathbb{R}} \to BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$,

$$\bar{v}_k = \sum_{i=0}^{k} \bar{t}_{k-i}^{2^i} \gamma\bar{t}_i \bmod (2, \bar{v}_1, \cdots, \bar{v}_{k-1}).$$

The left unit map induces a map

$$a_{\sigma_2}^{-1}\text{SliceSS}(BP_{\mathbb{R}}) \longrightarrow a_{\sigma_2}^{-1}\text{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$$

of spectral sequences. We will use naturality and induction to obtain the differentials and the description of the $E_{2^{k+1}}$-page.

To start the induction process, note that the description of the E_2-page is already given in (1). Now assume that we have obtained a description of the E_{2^k}-page. For degree reasons, the next potential differential is of length exactly $2^{k+1} - 1$. The differential formula for $a_{\sigma_2}^{-1}$SliceSS$(BP_{\mathbb{R}})$ above shows that for any polynomial $P \in \mathbb{F}_2[\bar{t}_1, \gamma\bar{t}_1, \cdots]/(\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_{k-1})$ and l an odd number, we have the differential

$$d_{2^{k+1}-1}(P u_{2\sigma_2}^{2^k l}) = P \bar{v}_k u_{2\sigma_2}^{2^k (l-1)} a_{\sigma_2}^{2^{k+1}-1}$$

in $a_{\sigma_2}^{-1}$SliceSS$(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$. The source and the target of this differential are always non-zero on the E_{2^k}-page because the sequence $(\bar{v}_1, \bar{v}_2, \cdots)$ is a regular sequence in the polynomial ring

[1] We recall the convention here that the filtration of an element in $\pi_V^H P_n^n X$ in the slice spectral sequence for some X is in filtration $n - \dim_{\mathbb{R}} V$. In particular the classes a_V will be always in filtration $\dim_{\mathbb{R}} V$.

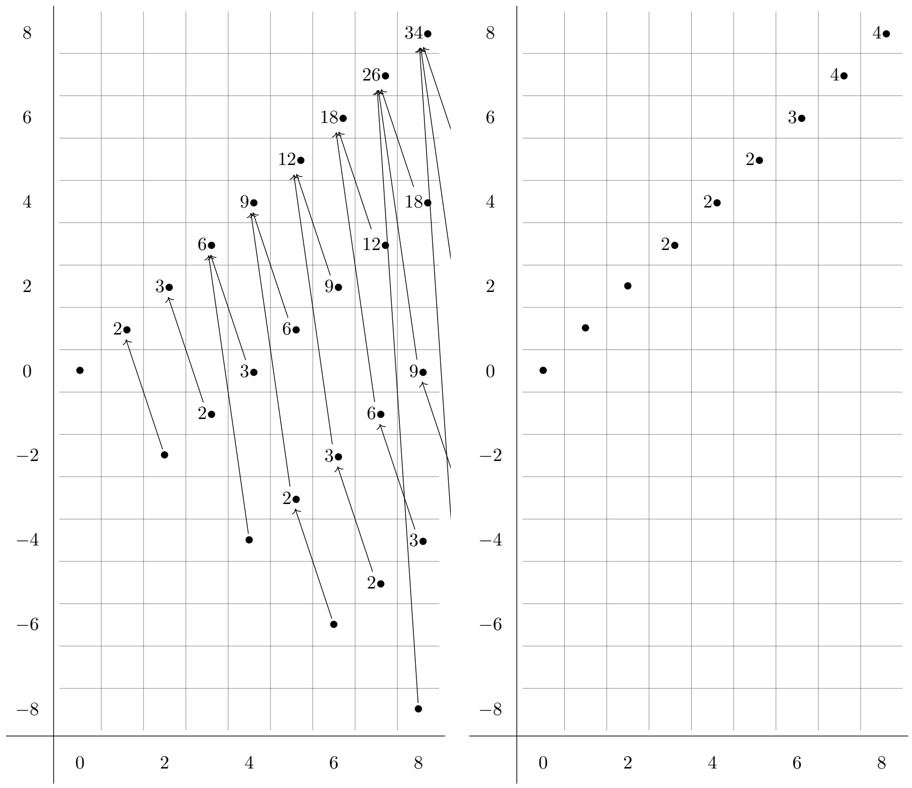


FIGURE 4. The integral E_2- and E_∞-pages of $a_{\sigma_2}^{-1}\mathrm{SliceSS}(BP_{\mathbb{R}} \wedge BP_{\mathbb{R}})$

$\mathbb{F}_2[\bar{t}_1, \gamma\bar{t}_1, \cdots]$. Taking the quotient of the kernel and cokernel of this differential, we see that the $E_{2^{k+1}}$-page has the above description.

(3) is a direct consequence of (2) by letting $k \to \infty$. See Figure 4 for the integral E_2 and E_∞-pages of this spectral sequence. □

Remark 5.5. *One can show that the C_2-geometric fixed points of the $\bar{t}_i$ and $\gamma\bar{t}_i$ generators are the ξ_i and ζ_i generators in the mod 2 dual Steenrod algebra $\mathcal{A}_*$. More precisely, $\Phi^{C_2}(\bar{t}_i) = \xi_i$ and $\Phi^{C_2}(\gamma\bar{t}_i) = \zeta_i$, and the formula*

$$\overline{v}_k = \sum_{i=0}^{k} \bar{t}_{k-i}^{2^i}\gamma\bar{t}_i \ \mathit{mod}\ (2, \overline{v}_1, \cdots, \overline{v}_{k-1})$$

reduces to Milnor's conjugation formula relating ξ_i and ζ_i in $\mathcal{A}_*$. Although we don't need this fact in this paper, it is one of the observations that originally motivated this project.

5.3. **The C_4-spectral sequence: d_3, d_5 and d_7-differentials.** The rest of this section is dedicated to computing the first 8 stems of the C_4-Mackey functor homotopy groups of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$. The result is stated in Theorem 4.4. By Section 3.4, we are free to use the norm structure from C_2 to C_4 in the localized slice spectral sequence.

As a consequence of the slice theorem [HHR16, Theorem 6.1], the 0-th slice of $MU^{((G))}$ is $H\underline{\mathbb{Z}}$ and $\underline{\pi}_0 MU^{((G))} \cong \underline{\mathbb{Z}}$. Therefore, every Mackey functor in the (localized) slice spectral sequence and the homotopy of any $MU^{((G))}$-module is a module over $\underline{\mathbb{Z}}$. By [TW95, Theorem 16.5], we have the following proposition.

Proposition 5.6. *Let $K \subset H \subset G$, and x be an element in the G/H-level of a Mackey functor either in the (localized) slice spectral sequence or the homotopy of a $MU^{((G))}$-module, then*

$$\mathrm{Tr}_K^H(\mathrm{Res}_K^H(x)) = [H : K]x.$$

Before getting to the page-by-page computation, we note that all the differentials on the classes $u_{2\sigma}^{2^k}$, $k \geq 0$ are already known by the work of Hill–Hopkins–Ravenel. Their theorem is originally formulated for the slice spectral sequence for $MU^{((C_4))}$ and the exact same statement and proof carries over to $\mathrm{SliceSS}(BP^{((C_4))})$ and $a_\lambda^{-1}\mathrm{SliceSS}(BP^{((C_4))})$.

Theorem 5.7 ([HHR16, Theorem 9.9])**.** *For $k \geq 0$ and $i < 2^{k+3} - 3$, $d_i(u_{2\sigma}^{2^k}) = 0$ and*

$$d_{2^{k+3}-3}(u_{2\sigma}^{2^k}) = N(\bar{t}_{k+1})a_\lambda^{2^{k+1}-1}a_\sigma^{2^{k+2}-1}.$$

Now we will start the page-by-page computation. First, note for degree reasons all the differential lengths will be odd.

Proposition 5.8.

$$d_3(u_\lambda) = \mathrm{Tr}_2^4(\bar{t}_1 a_{\sigma_2}^3)$$

Proof. By Theorem 5.4, the restriction $\mathrm{Res}_2^4(u_\lambda) = u_{2\sigma_2}$ supports the differential

$$d_3(u_{2\sigma_2}) = (\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3$$

in the C_2-spectral sequence. By naturality and degree reasons, the class u_λ must also support a d_3-differential in the C_4-spectral sequence whose target restricts to the class $(\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3$. The only class that restricts to $(\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3$ with $RO(C_4)$-degree $1 - \lambda$ is $\mathrm{Tr}_2^4(\bar{t}_1 a_{\sigma_2}^3)$. □

In Figure 5, this proposition gives all d_3 coming out of ◦, namely $u_\lambda a_\lambda^{-1}$ at $(2, -2)$, $N(\bar{t}_1)^2 u_\lambda u_{2\sigma} a_\lambda$ at $(6, 2)$ and $u_\lambda^3 a_\lambda^{-3}$ at $(6, -6)$.

Corollary 5.9. *Let P be a polynomial of $\bar{t}_i$, $\gamma\bar{t}_i$, a_{σ_2}, then*

$$d_3(u_\lambda^{2k+1} Tr_2^{4,V}(P)) = Tr_2^{4,V-2\lambda+2}(P(\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^3)u_\lambda^{2k}$$

for all $k > 0$ and any $V \in RO(C_4)$ restricting to the $RO(C_2)$-degree of P.

Proof. This is a direct consequence of Proposition 5.8, the Frobenius relation [HHR17, Definition 2.3] and the Leibniz rule. □

In Figure 5, this corollary gives all other d_3-differentials. We now explain them in detail.

In terms of Mackey functors, the d_3-differentials give the following exact sequences:

$$0 \to \bullet \to \circ \xrightarrow{d_3} \hat{\bullet} \to \bar{\bullet} \to 0$$

$$0 \to \hat{\bullet} \xrightarrow{d_3} \hat{\bullet} \to 0$$

$$0 \to \bar{\bullet} \xrightarrow{d_3} \hat{\bullet} \to \blacktriangledown \to 0.$$

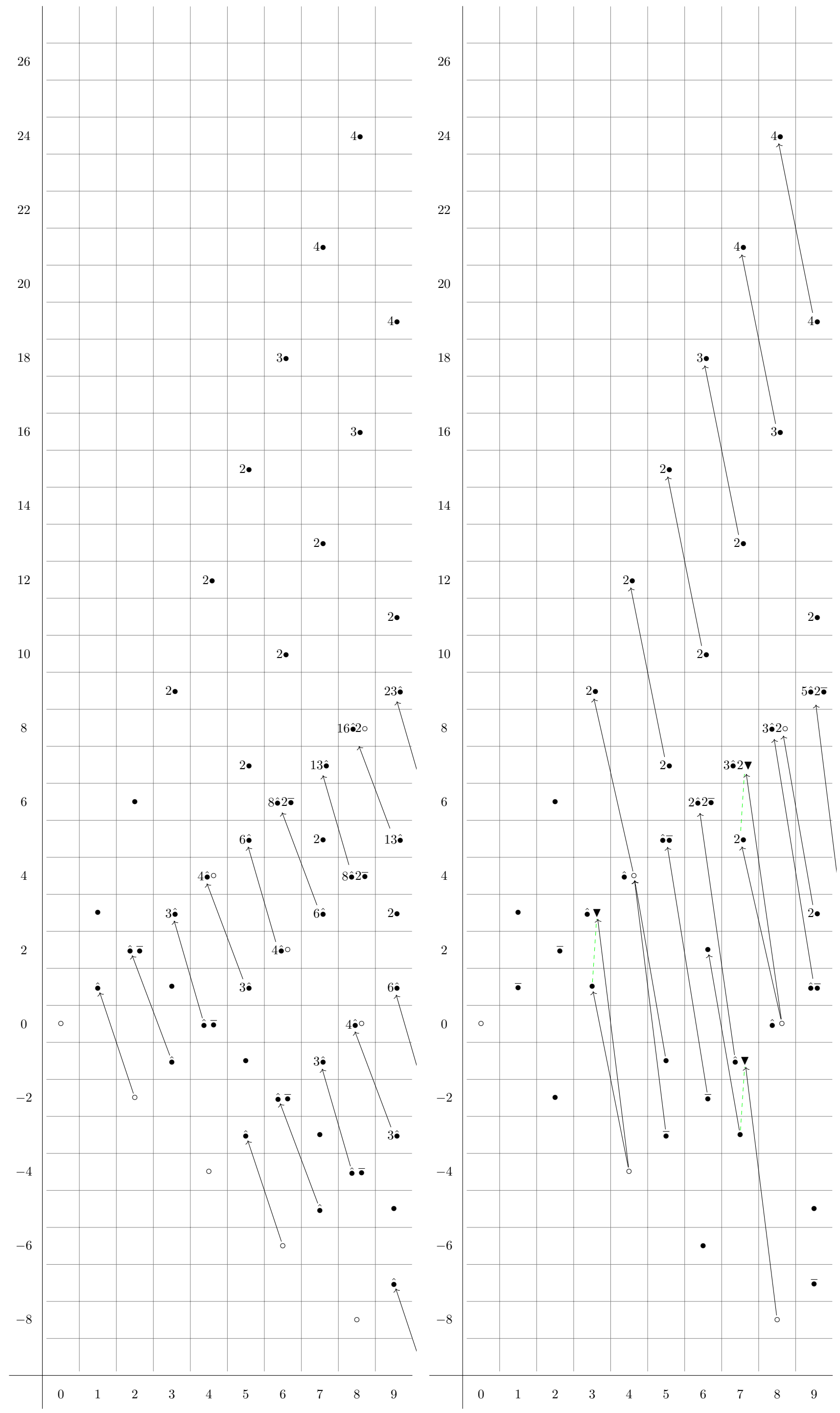


FIGURE 5. Left: d_3-differentials in $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$.
Right: d_5- and d_7-differentials in $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$.

Here are examples of d_3-differentials corresponding to each exact sequence above:

$$d_3(u_\lambda) = \mathrm{Tr}_2^4(\bar{t}_1 a_{\sigma_2}^3)$$
$$d_3(\mathrm{Tr}_2^4(\bar{t}_1 a_{\sigma_2})u_\lambda) = \mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^4)$$
$$d_3(u_{2\sigma_2} a_{\sigma_2}) = (\bar{t}_1 + \gamma\bar{t}_1)a_{\sigma_2}^4.$$

Note that the last differential is a C_2-differential, but it has an effect on C_4-level Mackey functor structure. By results in Section 3.3, the d_3-differentials also give certain exotic restrictions of filtration jump at most 2 (that is, the image of the restriction is of filtration at most 2 higher than the source). For example, consider the element $N(\bar{t}_1)u_\lambda a_\sigma$ at $(3,1)$. This class is a d_3-cycle. By Proposition 5.8, the class $N(\bar{t}_1)u_\lambda$ supports the d_3-differential

$$d_3(N(\bar{t}_1)u_\lambda) = \mathrm{Tr}_2^4(\bar{t}_1^2 \gamma\bar{t}_1 a_{\sigma_2}^3).$$

By Proposition 3.5, the class $\bar{t}_1^2\gamma\bar{t}_1 a_{\sigma_2}^3$ receives an exotic restriction of filtration jump at most 2 in integral degree, and the only possible source is $N(\bar{t}_1)u_\lambda a_\sigma$. The same argument applies to all 2-torsions classes with $(t-s,s)$-bidegrees $(3+4i+4j, 1+4i-4j)$ for $i,j \geq 0$. The exotic restrictions are represented by the vertical green dashed lines in Figure 5.

Remark 5.10. These exotic restrictions are the first family of examples of an interesting phenomenon in the $RO(G)$-graded spectral sequence of Mackey functors. Exotic restrictions and transfers can imply nontrivial abelian group extensions. As Mackey functors, these extensions are of the form

$$0 \to \bullet \to \circ \to \blacktriangledown \to 0,$$

which represents a nontrivial extension

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

if one evaluates the exact sequence of Mackey functors at C_4/C_4.

For readers who are familiar with Lubin-Tate E-theories and topological modular forms, the family of 2-extensions above is a generalization of the type of 2-extension between the class ν at $(3,1)$ and the class 2ν at $(3,3)$ in the homotopy fixed points spectral sequences of $E_2^{hC_4}$ and $TMF_0(5)$ (see [BBHS19] and [BO16]).

In summary, the d_3-differentials can be described as follows:

(1) On C_2-level, it is the first differential in Theorem 5.4.
(2) The Green functor structure of the spectral sequence gives d_3-differentials on the C_4-level, by Proposition 5.8 and Corollary 5.9. After these d_3-differentials, there is no room for further d_3-differentials.
(3) Every d_3-differential of the form $\bar{\bullet} \to \hat{\bullet}$ gives an extension of filtration 2 by the above remark.

Now we will prove the d_5-differentials. There are two different types of d_5-differentials. The first type is given by Theorem 5.7:

$$d_5(u_{2\sigma}) = N(\bar{t}_1)a_\lambda a_\sigma^3.$$

On the integral page for our range, it gives the following d_5-differential at $(4,4)$:

$$d_5(N(\bar{t}_1)^2 u_{2\sigma} a_\lambda^2) = N(\bar{t}_1)^2 a_\lambda^3 a_\sigma^3,$$

and it repeats by multiplying by $N(\bar{t}_1)a_\lambda a_\sigma$. In Figure 5, these are the d_5-differentials with sources on or above the line of slope 1.

The second type of d_5-differentials is given by the following proposition.

Proposition 5.11.

$$d_5(u_\lambda^2) = N(\bar{t}_1)u_\lambda a_\lambda^2 a_\sigma,$$
$$d_5(u_\lambda^2 a_\sigma) = 2N(\bar{t}_1)u_{2\sigma}a_\lambda^3.$$

Proof. The restriction $\mathrm{Res}_2^4(u_\lambda^2) = u_{2\sigma_2}^2$ supports the d_7-differential

$$d_7(u_{2\sigma_2}) = (\bar{t}_2 + \gamma\bar{t}_2 + \bar{t}_1^3)a_{\sigma_2}^7$$

by Theorem 5.4. By naturality, u_λ^2 must support a differential of length at most 7. For degree reasons, the length of this differential can only be 5 or 7. If the length of this differential is 7, the target must restrict to the class $(\bar{t}_2 + \gamma\bar{t}_2 + \bar{t}_1^3)a_{\sigma_2}^7$. However, this class is not in the image of the restriction map Res_2^4. Therefore, u_λ^2 must support a d_5-differential. The only possible target of this d_5-differential is $N(\bar{t}_1)u_\lambda a_\lambda^2 a_\sigma$. This proves the first d_5-differential

Multiplying with a_σ on both sides of the first d_5-differential gives

$$d_5(u_\lambda^2 a_\sigma) = N(\bar{t}_1)u_\lambda a_\lambda^2 a_\sigma^2.$$

Applying the gold relation $u_\lambda a_\sigma^2 = 2u_{2\sigma}a_\lambda$ gives the second d_5-differential. □

In Figure 5, the d_5-differentials in Proposition 5.11 can be seen on the following classes:

(1) $\frac{u_\lambda^2}{a_\lambda^2}$ at $(4, -4)$,
(2) $N(\bar{t}_1)u_\lambda^2 a_\lambda^{-1} a_\sigma$ at $(5, -1)$,
(3) $N(\bar{t}_1)^2 u_\lambda^2 u_{2\sigma}$ at $(8, 0)$,
(4) $N(\bar{t}_1)^3 u_\lambda^2 u_{2\sigma} a_\lambda a_\sigma$ and $N(\bar{t}_2)u_\lambda^2 u_{2\sigma} a_\lambda a_\sigma$ at $(9, 3)$.

Remark 5.12. *Although u_λ^2 and $u_\lambda^2 a_\sigma$ support differentials of the same length, this is not true in general. For example, we will see soon that u_λ^4 supports a d_7-differential, while $u_\lambda^4 a_\sigma$ supports a d_{13}-differential.*

Corollary 5.13.

$$d_5(u_\lambda^3 a_\sigma) = 2N(\bar{t}_1)u_\lambda u_{2\sigma}a_\lambda^3.$$

Proof. First, we will show that $u_\lambda a_\sigma$ is a nontrivial permanent cycle. Since the target of the d_3-differential on u_λ is a transfer class, it is killed by a_σ, and therefore $u_\lambda a_\sigma$ is a d_3-cycle. The only potential non-trivial differential that $u_\lambda a_\sigma$ can support is the d_5-differential

$$d_5(u_\lambda a_\sigma) = N(\bar{t}_1)a_\lambda^2 a_\sigma^2.$$

If this differential happens, then multiplying a_σ on both sides and using the gold relation will produce the differential

$$d_5(2u_{2\sigma}a_\lambda) = N(\bar{t}_1)a_\lambda^2 a_\sigma^3.$$

This is a contradiction to Theorem 5.7.

Applying the Leibniz rule on the first d_5-differential in Proposition 5.11 with the class $u_\lambda a_\sigma$ produces the d_5-differential

$$d_5(u_\lambda^3 a_\sigma) = u_\lambda a_\sigma d_5(u_\lambda^2) = N(\bar{t}_1)u_\lambda^2 a_\lambda^2 a_\sigma^2 = 2N(\bar{t}_1)u_\lambda u_{2\sigma}a_\lambda^3.$$ □

In Figure 5, this d_5-differential implies the d_5-differential on the class $N(\bar{t}_1)u_\lambda^3 a_\lambda^{-2} a_\sigma$ at $(7,-3)$. Notice that the class $N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^2$ supports a d_3-differential and the class $2N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^2$ is killed by a d_5-differential. In the integral grading, this happens to the $\mathbb{Z}/4$ in $(6,2)$.

There are extensions of filtration jump 4 induced by the d_5-differentials.

Proposition 5.14. *There is an exotic transfer of filtration jump* 4 *from* $(2,2)$ *to* $(2,6)$*:*

$$\mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^2) = N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2.$$

There is an exotic restriction of filtration jump 4*, from* $(2,-2)$ *to* $(2,2)$*:*

$$\mathrm{Res}_2^4\left(2u_\lambda a_\lambda^{-1}\right) = \bar{t}_1^2 a_{\sigma_2}^2.$$

Proof. We use Proposition 3.5 to prove both extensions.

For the first claim, note that $d_5(N(\bar{t}_1)u_{2\sigma}a_\lambda) = N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^3$, and $N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2$ is a nontrivial d_5-cycle. Therefore, $N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2$ is the target of an exotic transfer of filtration jump 4 in E_6, and the only possible source is $\bar{t}_1^2 a_{\sigma_2}^2$.

For the second claim, first note that by Proposition 5.2 (also see Figure 3) and the gold relation,

$$2u_\lambda a_\lambda^{-1} = \left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2} a_\sigma\right) a_\sigma.$$

We have the d_5-differential

$$d_5\left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2} a_\sigma\right) = \mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^2).$$

To prove this differential, consider the class $\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2}$. This class supports a d_5-differential because after multiplying it by $u_{2\sigma}^2 a_\lambda^2$ (which is a d_5-cycle), the class $u_\lambda^2 u_{2\sigma}$ supports the d_5-differential

$$d_5(u_\lambda^2 u_{2\sigma}) = N(\bar{t}_1)u_\lambda u_{2\sigma} a_\lambda^2 a_\sigma$$

by Proposition 5.11. Therefore

$$d_5\left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2}\right) = N(\bar{t}_1)\frac{u_\lambda}{u_{2\sigma}} a_\sigma.$$

Multiplying both sides by a_σ, we have

$$d_5\left(\frac{u_\lambda^2}{u_{2\sigma}} a_\lambda^{-2} a_\sigma\right) = N(\bar{t}_1)\frac{u_\lambda}{u_{2\sigma}} a_\sigma^2 = 2N(\bar{t}_1)a_\lambda = \mathrm{Tr}_2^4(\mathrm{Res}_2^4(N(\bar{t}_1)a_\lambda)) = \mathrm{Tr}_2^4(\bar{t}_1\gamma\bar{t}_1 a_{\sigma_2}^2) = \mathrm{Tr}_2^4(\bar{t}_1^2 a_{\sigma_2}^2)$$

The last equation holds because by Theorem 5.4, $\bar{t}_1 = \gamma\bar{t}_1$ after the d_3-differentials in the C_2-spectral sequence.

Therefore, $\bar{t}_1^2 a_{\sigma_2}^2$ must receive an exotic restriction of filtration jump 4 in the integral degree, and the only source of the restriction is $2u_\lambda a_\lambda^{-1}$. □

In Figure 6, the exotic restrictions and transfers are the green and blue dashed lines, respectively.

Remark 5.15. Similar to Remark 5.10, the exotic restrictions and transfers also give extensions of abelian groups on the C_4-level. The situation is more subtle here because each individual exotic extension doesn't involve non-trivial extensions of abelian groups at any level. When we combine the two extensions together, however, we obtain an abelian group extension of filtration 8 from $(2,-2)$ to $(2,6)$:

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0,$$

and $2(\frac{2u_\lambda}{a_\lambda}) = N(\bar{t}_1)^2 a_\lambda^2 a_\sigma^2$ in homotopy. This extension is similar to the extension in the 22-stem of $E_2^{hC_4}$ and $TMF_0(5)$. (See [BBHS19, Figure 10] and [BO16, Section 2]).

We will now prove the d_7-differentials. While we state them first in some $RO(C_4)$-graded page first, we recommend that the reader multiplies with appropriate powers of a_λ whenever possible to visualize the arguments in Figure Fig. 5.

Proposition 5.16. *We have the following d_7-differentials*

$$\begin{aligned} d_7(2u_\lambda^2) &= \mathrm{Tr}_2^{4,3-2\lambda}(a_{\sigma_2}^7 \bar{t}_1^3), \\ d_7(2u_\lambda^2 u_{2\sigma}) &= \mathrm{Tr}_2^{4,5-2\lambda-2\sigma}(a_{\sigma_2}^7 \bar{t}_1^3), \end{aligned}$$

(see Notation 5.3 for the transfer notations).

Proof. We will prove the first differential. The second differential is proven by the exact same method. On the C_2-level, we have the d_7-differential

$$d_7(u_{2\sigma_2}^2) = (\bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2)a_{\sigma_2}^7$$

by Theorem 5.4. Taking transfer on the target and using naturality, the class

$$\mathrm{Tr}_2^{4,3-2\lambda}(a_{\sigma_2}^7(\bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2)) = \mathrm{Tr}_2^{4,3-2\lambda}(a_{\sigma_2}^7 \bar{t}_1^3)$$

must be killed by a differential of length at most 7. For degree reasons, it must be the d_7-differential with source $2u_\lambda^2$. □

In Figure 5, The d_7-differentials in Proposition 5.16 and the underlying C_2-level d_7-differentials in Theorem 5.4 are supported by the classes at $(4+i, -4+i)$ for $i \geq 0$.

Proposition 5.17.

$$d_7(u_\lambda^4) = u_\lambda^2 \,\mathrm{Tr}_2^4(\bar{t}_1^3 a_{\sigma_2}^7).$$

Proof. We will prove in Proposition 5.21 that there is a nontrivial d_{13}-differential on the class $u_\lambda^4 a_\sigma$ (we can already prove it at this point, but for organization reasons we prove it later). This implies that the class u_λ^4 must support a differential of length at most 13. For degree reasons, the claimed d_7-differential is the only possibility. □

In Figure 5, the d_7-differential in Proposition 5.17 gives the d_7-differential supported by the class $\frac{u_\lambda^4}{a_\lambda^4}$ at $(8, -8)$.

5.4. **The C_4-spectral sequence: higher differentials and extensions.** We will now prove the higher differentials in our range (see Figure 6). The next possible differential is a d_{13}-differential from Theorem 5.7:

$$d_{13}(u_{2\sigma}^2) = N(\bar{t}_2)a_\lambda^3 a_\sigma^7.$$

However, we won't see this differential in Figure 6. This is because its first appearance in the integer graded spectral sequence is on the class $(10, 14)$, which is outside of our range. Note also that even though some classes at $(8, 8)$ contain $u_{2\sigma}^2$, they don't support d_{13}-differentials. We will give a detailed discussion of the classes at $(8, 8)$ in Section 5.5.

Proposition 5.18.

$$d_{13}(u_\lambda^4 u_{2\sigma}) = N(\bar{t}_2)u_\lambda u_{2\sigma}^2 a_\lambda^6 a_\sigma$$

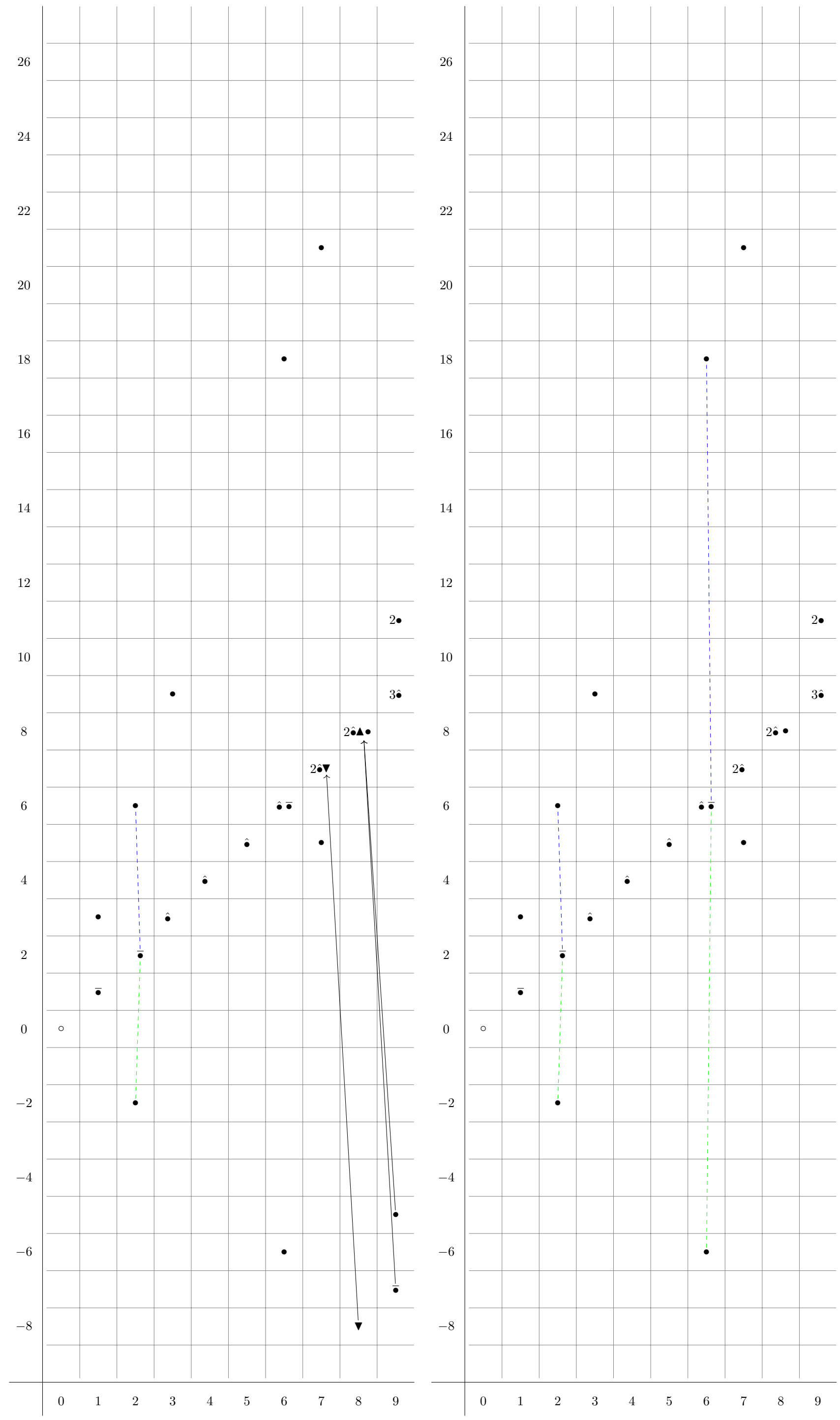


FIGURE 6. Left: d_{13}- and d_{15}-differentials in $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$.
Right: E_∞-page of $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$ with all extensions.

Proof. On the C_2-level, the restriction $\mathrm{Res}_2^4(u_\lambda^4 u_{2\sigma}) = u_{2\sigma_2}^4$ supports a d_{15}-differential hitting the class $\overline{v}_3 a_{\sigma_2}^{15} = (\bar{t}_3 + \bar{t}_2^2 \bar{t}_1 + \bar{t}_1^4 \gamma \bar{t}_2 + \gamma \bar{t}_3) a_{\sigma_2}^{15}$. Since this class is not in the image of the restriction after the d_3-differentials, by naturality the class $u_\lambda^4 u_{2\sigma}$ must support a differential of length shorter than 15. After computing the first few pages, we see that for degree reasons the potential targets are the following classes:

(1) $\mathrm{Tr}_2^4((\bar{t}_2 + \bar{t}_1^3 + \gamma \bar{t}_2) u_{2\sigma_2}^2 a_{\sigma_2}^7)$ in filtration 7;
(2) $N(\bar{t}_1)^3 u_\lambda u_{2\sigma}^2 a_\lambda^6 a_\sigma$ in filtration 13;
(3) $N(\bar{t}_2) u_\lambda u_{2\sigma}^2 a_\lambda^6 a_\sigma$ in filtration 13.

We will first prove that the class $\mathrm{Tr}_2^4((\bar{t}_2 + \bar{t}_1^3 + \gamma \bar{t}_2) u_{2\sigma_2}^2 a_{\sigma_2}^7)$ supports the d_{11}-differential

$$d_{11}(\mathrm{Tr}_2^4((\bar{t}_2 + \bar{t}_1^3 + \gamma \bar{t}_2) u_{2\sigma_2}^2 a_{\sigma_2}^7)) = N(\bar{t}_1)^4 u_{2\sigma}^2 a_\lambda^8 a_\sigma^2.$$

To prove this, first note that

$$\mathrm{Tr}_2^4((\bar{t}_2 + \bar{t}_1^3 + \gamma \bar{t}_2) u_{2\sigma_2}^2 a_{\sigma_2}^7) = \mathrm{Tr}_2^4(\bar{t}_1^3 u_{2\sigma_2}^2 a_{\sigma_2}^7)$$

since the class $(\bar{t}_2 + \gamma \bar{t}_2) a_\sigma$ transfers to 0 in the homotopy. On the C_2-level, we have the d_7-differential

$$d_7(\bar{t}_1^3 u_{2\sigma_2}^2 a_{\sigma_2}^7) = \bar{t}_1^3 (\bar{t}_2 + \bar{t}_1^3 + \gamma \bar{t}_2) a_{\sigma_2}^{14}.$$

The transfer of the target, $\mathrm{Tr}_2^4(\bar{t}_1^3(\bar{t}_2 + \bar{t}_1^3 + \gamma \bar{t}_2) a_{\sigma_2}^{14}) = \mathrm{Tr}_2^4(\bar{t}_1^6 a_{\sigma_2}^{14})$, is zero. This is because after the C_2-level d_3-differentials, the class $\bar{t}_1^6 a_{\sigma_2}^{14}$ is identified with the class $\bar{t}_1^3 \gamma \bar{t}_1^3 a_{\sigma_2}^{14}$, which transfers to 0. We will show that the class $\bar{t}_1^6 a_{\sigma_2}^{14}$ actually supports an exotic transfer of filtration jump 4. Let $x = N(\bar{t}_1)^3 a_\lambda^7 u_{2\sigma}^3$. We have the d_5-differential from Theorem 5.7

$$d_5(x) = N(\bar{t}_1)^4 u_{2\sigma}^2 a_\lambda^8 a_\sigma^3.$$

By Proposition 3.5, $N(\bar{t}_1)^4 u_{2\sigma}^2 a_\lambda^8 a_\sigma^2$ receives an exotic transfer of jump 4, and the only possible source is $\bar{t}_1^6 a_\sigma^{14}$. Combining the C_2-level d_7 and this exotic transfer, we prove the claimed d_{11}.

The class $N(\bar{t}_1)^3 u_\lambda u_{2\sigma}^2 a_\lambda^6 a_\sigma$ in filtration 13 is killed by a d_5-differential from Proposition 5.11:

$$N(\bar{t}_1)^3 u_\lambda u_{2\sigma}^2 a_\lambda^6 a_\sigma = d_5(N(\bar{t}_1)^2 u_\lambda^2 u_{2\sigma}^2 a_\lambda^4).$$

It follows that the class $N(\bar{t}_2) u_\lambda u_{2\sigma}^2 a_\lambda^6 a_\sigma$ is the only possible target. □

Remark 5.19. The class $u_\lambda^4 u_{2\sigma}$ is a permanent cycle in the homotopy fixed points spectral sequence of $E_2^{hC_4}$ (see [BBHS19, Proposition 5.23]) because $N(\bar{t}_2)$ is zero there.

Although this d_{13} doesn't imply any differentials in our range, it is used in proving extensions.

Proposition 5.20. *(1) There is an exotic transfer in stem* 6 *of filtration* 12*,*

$$\mathrm{Tr}_2^4(\bar{t}_2 \gamma \bar{t}_2 a_{\sigma_2}^6) = N(\bar{t}_2)^2 a_\lambda^6 a_\sigma^6.$$

(2) There is an exotic restriction in stem 6 *of filtration* 12*,*

$$\mathrm{Res}_2^4(2 u_\lambda^3 a_\lambda^{-3}) = \bar{t}_2 \gamma \bar{t}_2 a_{\sigma_2}^6.$$

Proof. The proof is similar to that of Proposition 5.14. The exotic transfer comes from applying Proposition 3.5 to the d_{13}-differential

$$d_{13}\left(N(\bar{t}_2) u_{2\sigma}^2 a_\lambda^3\right) = N(\bar{t}_2)^2 a_\lambda^6 a_\sigma^7$$

in Theorem 5.7.

For the exotic restriction, first note that $2u_\lambda^3 a_\lambda^{-3} = \left(\frac{u_\lambda^4}{u_{2\sigma}} a_\lambda^{-4} a_\sigma\right) a_\sigma$ by the gold relation. We will prove that the class $\frac{u_\lambda^4}{u_{2\sigma}} a_\lambda^{-4} a_\sigma$ supports a d_{13}-differential. To do so, we multiply this class by $u_{2\sigma}^2 a_\lambda^4$. After multiplying the differential in Proposition 5.18 by a_σ, we have

$$d_{13}(u_\lambda^4 u_{2\sigma} a_\sigma) = 2N(\bar{t}_2) u_{2\sigma}^3 a_\lambda^7.$$

As by the gold relation u_λ^2 kills $d_{13}(u_{2\sigma}^2 a_\lambda^4)$, we can use the Leibniz rule to obtain the d_{13}-differential

$$d_{13}\left(\frac{u_\lambda^4}{u_{2\sigma}} a_\lambda^{-4} a_\sigma\right) = 2N(\bar{t}_2) u_{2\sigma} a_\lambda^3.$$

On the E_2-page, $2N(\bar{t}_2) u_{2\sigma} a_\lambda^3 = Tr_2^4(\bar{t}_2 \gamma \bar{t}_2 a_{\sigma_2}^6)$. By Proposition 3.5, $\bar{t}_2 \gamma \bar{t}_2 a_{\sigma_2}^6$ must receive an exotic restrion of filtration jump 12, and the only possible source is $2u_\lambda^3 a_\lambda^{-3}$ (see Figure 6). □

In Figure 6, they are the exotic restriction from the class $(6,-6)$ to $(6,6)$ and the exotic transfer from $(6,6)$ to $(6,18)$. Since these extensions involve elements containing $\bar{t}_2$, we expect similar extensions in the homotopy fixed points spectral sequence of $E_4^{hC_4}$ by [BHSZ20, Theorem 1.1].

Proposition 5.21.

$$d_{13}(u_\lambda^4 a_\sigma) = N(\bar{t}_2 + \bar{t}_1^2 \gamma \bar{t}_1 + \gamma \bar{t}_2) u_{2\sigma}^2 a_\lambda^7.$$

Proof. Consider the C_2-differential

$$d_7(u_{2\sigma_2}^2) = \bar{t}_2^{C_2} a_{\sigma_2}^7.$$

Applying Proposition 3.6 to its target, we see that its norm $N(\bar{t}_2 + \bar{t}_1^2 \gamma \bar{t}_1 + \gamma(\bar{t}_2)) a_\lambda^7$ must be killed by a differential of length 13 or shorter. Since the restriction of this element is killed by d_7, it must be killed by a differential of length between 7 and 13. Since $u_{2\sigma}^2$ supports a d_{13}, if $d_r(x) = N(\bar{t}_2 + \bar{t}_1^2 \gamma \bar{t}_1 + \gamma(\bar{t}_2)) a_\lambda^7$ happens for $r < 13$, one can multiply both sides by $u_{2\sigma}^2$. However, for degree reasons $N(\bar{t}_2 + \bar{t}_1^2 \gamma \bar{t}_1 + \gamma \bar{t}_2) u_{2\sigma}^2 a_\lambda^7$ cannot be hit by a differential shorter than a d_{13}. Thus this element and hence also $N(\bar{t}_2 + \bar{t}_1^2 \gamma \bar{t}_1 + \gamma(\bar{t}_2)) a_\lambda^7$ must be hit by a d_{13} and the only possible source is $u_\lambda^4 a_\sigma$. □

On the integer graded page, this contributes to the d_{13}-differential supported by the class $N(\bar{t}_1) u_\lambda^4 a_\lambda^{-3} a_\sigma$ at $(9,-5)$.

The last differential in our range is a d_{15}-differential.

Proposition 5.22. *We have the d_{15}-differential*

$$d_{15}(2u_\lambda^4) = \mathrm{Tr}_2^4(\bar{t}_3^{C_2} a_{\sigma_2}^{15}).$$

Proof. In the the C_2-spectral sequence, we have the d_{15}-differential

$$d_{15}(u_{2\sigma_2}^4) = \bar{t}_3^{C_2} a_{\sigma_2}^{15}.$$

Applying the transfer shows that the class $\mathrm{Tr}_2^4(\bar{t}_3^{C_2} a_{\sigma_2}^{15})$ must be killed by a differential of length at most 15. By naturality and degree reasons, the only possible source is the class $2u_\lambda^4 = \mathrm{Tr}_2^4(u_{2\sigma_2}^4)$. □

In Figure 6, this contributes to the d_{15}-differential supported by the class $\frac{2u_\lambda^4}{a_\lambda^4}$ at $(8,-8)$ (the d_{15}-differential supported by the class at $(9,-7)$ is a C_2-level differential).

These are all the differentials and extensions in the first 8 stems. Now we will discuss in detail the generators and relations in degree $(8,8)$ after each differential in order to illustrate the technical aspect of tracking differentials in the localized slice spectral sequences.

5.5. **The classes at** $(8,8)$**.** Since our discussion here focuses on a single degree, we will omit the powers of a_V and u_V classes on each monomial, except in formulas of differentials. That is, we omit $u_{2\sigma}^2 a_\lambda^4$ on C_4-classes and $a_{\sigma_2}^8$ on C_2-classes.

On the E_3-page, there are 2 $\circ$ and 16 $\hat{\bullet}$. The 2 $\circ$ are $N(\bar{t}_1)^4$ and $N(\bar{t}_2)N(\bar{t}_1)$. The 16 $\hat{\bullet}$ are

(1) $\mathrm{Tr}_2^4(\bar{t}_1^8)$, $\mathrm{Tr}_2^4(\bar{t}_1^7\gamma\bar{t}_1)$, $\mathrm{Tr}_2^4(\bar{t}_1^6\gamma\bar{t}_1^2)$, $\mathrm{Tr}_2^4(\bar{t}_1^5\gamma\bar{t}_1^3)$;
(2) $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^5)$, $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^4\gamma\bar{t}_1)$, $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^3\gamma\bar{t}_1^2)$, $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^2\gamma\bar{t}_1^3)$, $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1\gamma\bar{t}_1^4)$, $\mathrm{Tr}_2^4(\bar{t}_2\gamma\bar{t}_1^5)$;
(3) $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$, $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1\gamma\bar{t}_1)$, $\mathrm{Tr}_2^4(\bar{t}_2^2\gamma\bar{t}_1^2)$;
(4) $\mathrm{Tr}_2^4(\bar{t}_2\gamma\bar{t}_2\bar{t}_1^2)$;
(5) $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$, $\mathrm{Tr}_2^4(\bar{t}_3\gamma\bar{t}_1)$.

At the C_2-level, the d_3-differentials identifies $\bar{t}_1$ with $\gamma\bar{t}_1$. At the C_4-level, the effect of the d_3-differentials are as follows:

(1) All the classes in (1) are identified with $2N(\bar{t}_1)^4$;
(2) all the classes in (2) are identified to be the same;
(3) all the classes in (3) are identified to be the same;
(4) the class $\mathrm{Tr}_2^4(\bar{t}_2\gamma\bar{t}_2\bar{t}_1^2)$ is identified with $2N(\bar{t}_2)N(\bar{t}_1)$;
(5) all the classes in (5) are identified to be the same.

Therefore after the d_3-differentials, there are 2 $\circ$, generated by $N(\bar{t}_1)^4$ and $N(\bar{t}_2)N(\bar{t}_1)$, and 3 $\hat{\bullet}$, generated by $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^5)$, $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$, and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$.

On the E_5-page, by Proposition 5.11, we have the following two d_5-differentials:

$$\begin{aligned} d_5(N(\bar{t}_1)^3 u_\lambda^2 u_{2\sigma} a_\lambda a_\sigma) &= 2N(\bar{t}_1)^4 u_{2\sigma}^2 a_\lambda^4, \\ d_5(N(\bar{t}_2) u_\lambda^2 u_{2\sigma} a_\lambda a_\sigma) &= 2N(\bar{t}_2)N(\bar{t}_1) u_{2\sigma}^2 a_\lambda^4. \end{aligned}$$

It follows that after the d_5-differentials, the 2 $\circ$ become 2 $\blacktriangle$, with the same generators. In total, there are 2 $\blacktriangle$ and 3 $\hat{\bullet}$ at $(8,8)$ after the d_5-differentials (with the same generator as before).

Now we will discuss the d_7-differentials. At $(9,1)$, there are two classes on the E_7-page: a $\hat{\bullet}$ generated by $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^2)$ and a $\overline{\bullet}$ generated by $\bar{t}_1^5$ (it only exists on the C_2-level). Since $\bar{v}_2 = \bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2$, the d_7-differential on the class $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^2)$ hits the class

$$\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^2(\bar{t}_2 + \bar{t}_1^3 + \gamma\bar{t}_2)) = \mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2) + \mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^5) + \mathrm{Tr}_2^4(\bar{t}_2\gamma\bar{t}_2\bar{t}_1^2) = \mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2) + \mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^5).$$

In other words, it identifies the classes $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^5)$.

The d_7-differential on the class $\bar{t}_1^5$ hits the class

$$\begin{aligned} \bar{t}_1^5(\bar{t}_2 + \gamma\bar{t}_2 + \bar{t}_1^3) &= \bar{t}_2\bar{t}_1^5 + \gamma\bar{t}_2\bar{t}_1^5 + \bar{t}_1^8 \\ &= \mathrm{Res}_2^4(\mathrm{Tr}_2^4(\bar{t}_2\bar{t}_1^5)) + \mathrm{Res}_2^4(N(\bar{t}_1)^4) \\ &= \mathrm{Res}_2^4(\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)) + \mathrm{Res}_2^4(N(\bar{t}_1)^4). \end{aligned}$$

As Mackey functors, we have

$$\hat{\bullet}\overline{\bullet} \xrightarrow{d_7} 2\blacktriangle 3\hat{\bullet} \twoheadrightarrow \bullet\blacktriangle 2\hat{\bullet}.$$

In the quotient we need to choose our generators carefully: The $\bullet$ is generated by $N(\bar{t}_1)^4 + \mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$, because the image of $\overline{\bullet}$ identifies the restriction of $N(\bar{t}_1)^4$ with the restriction of

$\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$. Therefore their sum is the unique element in C_4-level that has trivial restriction. The $\blacktriangle$ is generated by $N(\bar{t}_2)N(\bar{t}_1)$, as it still has nontrivial restriction. The two $\hat{\bullet}$ are generated by $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$.

The next differential is a d_{13}-differential supported by the class $N(\bar{t}_1)u_\lambda^4 a_\lambda^{-3}a_\sigma$ at $(9,-5)$. By Proposition 5.21, the target of this differential is the class $N(\bar{t}_1)N(\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)u_{2\sigma}^2 a_\lambda^4$. The restriction of this class is

$$\bar{t}_1\gamma\bar{t}_1(\bar{t}_2+\bar{t}_1^3+\gamma\bar{t}_2)(\gamma\bar{t}_2+\gamma\bar{t}_1^3-\bar{t}_2),$$

which, after the d_3-differentials, is

$$\bar{t}_2^2\bar{t}_1^2+\gamma\bar{t}_2^2\bar{t}_1^2+\bar{t}_1^8=\mathrm{Res}_2^4(\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2))+\mathrm{Res}_2^4(N(\bar{t}_1)^4).$$

As we have discussed above, this class is killed by the d_7-differentials supported by the class $\bar{t}_1^5$. It follows that the target of the d_{13}-differential is the generator of $\bullet$, the unique nontrivial element that restricts to 0.

There is another possible d_{13}-differential supported by some classes at $(8,8)$ that is induced by the differential

$$d_{13}(u_{2\sigma}^2)=N(\bar{t}_2)a_\lambda^3 a_\sigma^7.$$

However, in $(8,8)$ every monomial containing $u_{2\sigma}^2$ also contains $N(\bar{t}_1)$. By [HHR16, Corollary 9.13],

$$d_{13}(N(\bar{t}_1)u_{2\sigma}^2)=N(\bar{t}_1)N(\bar{t}_2)a_\lambda^3 a_\sigma^7=d_5(N(\bar{t}_2)u_{2\sigma}a_\lambda^2 a_\sigma^4).$$

This makes all elements containing $u_{2\sigma}^2$ in $(8,8)$ d_{13}-cycles.

In summary, after the d_{13}-differentials, we have two $\hat{\bullet}$, generated by $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$, and $\blacktriangle$, generated by $N(\bar{t}_2)N(\bar{t}_1)$.

Our final differential is a d_{15}-differential on the C_2-level supported by the class at $(9,-7)$:

$$\begin{aligned}
d_{15}(\bar{t}_1 u_{2\sigma_2}^4 a_{\sigma_2}^{-7}) &= \bar{t}_1(\bar{t}_3+\bar{t}_2^2\bar{t}_1+\gamma\bar{t}_2\bar{t}_1^4+\gamma\bar{t}_3)a_{\sigma_2}^8\\
&= (\bar{t}_3\bar{t}_1+\gamma\bar{t}_3\bar{t}_1)a_{\sigma_2}^8+(\bar{t}_2^2\bar{t}_1^2+\gamma\bar{t}_2\bar{t}_1^5)a_{\sigma_2}^8\\
&= \mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)+(\bar{t}_2^2\bar{t}_1^2\gamma\bar{t}_2^2\bar{t}_1^2+\bar{t}_2\gamma\bar{t}_2\bar{t}_1^2)a_{\sigma_2}^8\\
&= \mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)+\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)+\mathrm{Tr}_2^4\mathrm{Res}_2^4(N(\bar{t}_2)N(\bar{t}_1)).
\end{aligned}$$

The map in Mackey functors is

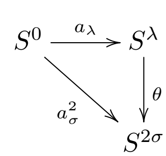
$$\bar{\bullet}\xrightarrow{d_{15}}\blacktriangle 2\hat{\bullet}\twoheadrightarrow\bullet 2\hat{\bullet}.$$

On the E_∞-page, $(8,8)$ is given by $\bullet 2\hat{\bullet}$. The generators for the two $\hat{\bullet}$ are $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)$ and $\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)$. The generator for $\bullet$ is $\mathrm{Tr}_2^4(\bar{t}_2^2\bar{t}_1^2)+\mathrm{Tr}_2^4(\bar{t}_3\bar{t}_1)+N(\bar{t}_2)N(\bar{t}_1)$.

5.6. **A family of permanent cycles.** We will now present families of nontrivial permanent cycles in $a_\lambda^{-1}\mathrm{SliceSS}(BP^{(\!(C_4)\!)})$. These families will be used in the proof of Theorem 6.8.

Lemma 5.23. *In $\pi_\bigstar^{C_4}a_\sigma^{-1}\mathbb{S}$, the element a_λ is invertible.*

Proof. Unstably, we have the following commutative diagram



where θ is the C_4-equivariant 2-folded branched cover. Since $\theta a_\lambda = a_\sigma^2$ is invertible, a_λ is invertible. □

Proposition 5.24. *In $\pi_\bigstar^{C_4} a_\lambda^{-1} BP^{(\!(C_4)\!)}$, the classes $N(\bar{t}_k)a_\sigma^i$ for $k > 0$ and $0 \leq i < 2^{k+1}-1$ are non-zero.*

Proof. By Lemma 5.23 we have a map of spectral sequences

$$a_\lambda^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)}) \longrightarrow a_\sigma^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)}).$$

Notice that in $a_\sigma^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)})$, the differentials in Theorem 5.7 completely determine the spectral sequence (See [HHR16, Remark 9.11]). In particular, we have the following differentials in $a_\sigma^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)})$:

$$d_{2^{k+2}-3}(u_{2\sigma}^{2^{k-1}} a_\lambda^{-(2^k-1)} a_\sigma^{-(2^{k+1}-1)+i}) = N(\bar{t}_k)a_\sigma^i.$$

On $E_{2^{k+2}-3}$-page, this is the only differential happens in this degree.

By Proposition 5.2 and the gold relation, the class $u_{2\sigma}^{2^{k-1}} a_\sigma^{-(2^{k+1}-1)+i}$ is in the image of

$$\pi_\bigstar^{C_4} a_\lambda^{-1} H\underline{\mathbb{Z}} \to \pi_\bigstar^{C_4} a_\sigma^{-1} H\underline{\mathbb{Z}}$$

only when a_σ has a non-negative power, i.e. $i \geq 2^{k+1}-1$. Therefore by naturality, if the class $N(\bar{t}_k)a_\sigma^i$, $0 \leq i < 2^{k+1}-1$ is killed in $a_\lambda^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)})$, the differential killing it must be of length longer than $2^{k+2}-3$. By Proposition 5.2 and Theorem 4.1, the source of such a differential is trivial in the E_2-page. It follows that the classes $N(\bar{t}_k)a_\sigma^i$ for $k > 0$ and $0 \leq i \leq 2^{k+1}-1$ are nontrivial permanent cycles. □

Remark 5.25. Note that after inverting a_λ, the element $N(\bar{t}_k)a_\sigma^{2^{k+1}-1}$ is zero by Theorem 5.7.

6. The Tate spectral sequence of $N_1^2 H\mathbb{F}_2$

In this section, we use the computation of $a_\lambda^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)})$ to understand the Tate diagram of $N_1^2 H\mathbb{F}_2$. We use the computation in the previous section to prove families of differentials in the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$. By Corollary 2.4, the group C_2 here is the *quotient group* of C_4 rather than the subgroup, and the $RO(C_4)$-graded $a_\lambda^{-1}\text{SliceSS}(BP^{(\!(C_4)\!)})$ computes the $RO(C_4/C_2)$-graded homotopy groups of $N_1^2 H\mathbb{F}_2$. Therefore, we use σ, instead of σ_2, for the sign representation of the quotient group C_2.

By the generalized Segal conjecture for C_2 [LNR11], in the bottom row of the Tate diagram of $N_1^2 H\mathbb{F}_2$

$$\begin{array}{ccccc}
(N_1^2 H\mathbb{F}_2)_{hC_2} & \longrightarrow & (N_1^2 H\mathbb{F}_2)^{C_2} & \longrightarrow & \Phi^{C_2} N_1^2 H\mathbb{F}_2 \\
\downarrow & & \downarrow & & \downarrow \\
(N_1^2 H\mathbb{F}_2)_{hC_2} & \xrightarrow{T} & (N_1^2 H\mathbb{F}_2)^{hC_2} & \longrightarrow & (N_1^2 H\mathbb{F}_2)^{tC_2}
\end{array}$$

the map T induces an isomorphism of homotopy groups except for π_0. In π_0, the long exact sequence of homotopy groups becomes the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

We can identify the negative filtration part of the Tate spectral sequence with the homotopy orbit spectral sequence (HOSS) and the non-negative part of the Tate spectral sequence with the homotopy fixed points spectral sequence (HFPSS). All differentials in the Tate spectral sequence originating from negative filtration to non-negative filtration represent nontrivial elements mapping to each other under the Tate norm map T.

Note that the Tate spectral sequence of $N_1^2H\mathbb{F}_2$ and of $N_1^2H\mathbb{F}_2 \wedge S^{-b\sigma}$ agree up to a filtration shift. Thus, modulo transfers the homotopy fixed point spectral sequence computing $\pi^{C_2}_{*+b\sigma}N_1^2H\mathbb{F}_2$ also embeds into the Tate spectral sequence, only with a filtration shift of b.

The folllowing proposition makes these observations more precise. For an element $x \in \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \cong \mathcal{A}_*$, we denote by $\overline{x}$ its conjugate under the permutation C_2-action.

Proposition 6.1. *Consider $\pi^{C_2}_\bigstar(N_1^2H\mathbb{F}_2)$ as a commutative algebra over $\mathbb{Z}[a_\sigma]$, where a_σ acts via multiplication by $a_\sigma \in \pi^{C_2}_{-\sigma}(S)$.*

1. *Let $I = \{0\} \cup \{\mathrm{Tr}_1^{2,V}(x) \mid x \in \pi^e_*(N_1^2H\mathbb{F}_2), \overline{x} \neq x\}$, then I is an ideal of $\pi^{C_2}_\bigstar(N_1^2H\mathbb{F}_2)$.*
2. *In $\pi^{C_2}_\bigstar(N_1^2H\mathbb{F}_2)/I$, every a_σ-tower except for the one on ± 1 is finite.*
3. *The a_σ-towers in $\pi^{C_2}_\bigstar(N_1^2H\mathbb{F}_2)/I$ are in one-to-one correspondence with differentials in the C_2-Tate spectral sequence of $N_1^2H\mathbb{F}_2$. More precisely, given an a_σ-tower starting at degree $a+b\sigma$ with length r, it corresponds to a differential d_r in the Tate spectral sequence, hitting a cycle in bidegree (a, b).*

Proof. For (1), we first need to check that I is closed under addition. If x and x' are not equal to $\overline{x}$ and $\overline{x}'$, respectively, then they span together with their conjugates $\mathbb{F}_2[C_2]^2$. Its C_2-fixed points are 2-dimensional and spanned by $x + \overline{x}$ and $x' + \overline{x}'$, whose transfers are zero (as they are in the image of Tr Res Tr). Thus $\overline{x + x'} = x + x'$ is only possible if $\mathrm{Tr}_1^{2,V}(x + x') = 0$.

Let now $y \in \pi^{C_2}_W(N_1^2H\mathbb{F}_2)$. By the Frobenius relation [HHR17, Definition 2.3], we have

$$yTr_1^{2,V}(x) = Tr_1^{2,V+W}(Res_1^2(y)x).$$

Notice that $\mathcal{A}_*$ is a polynomial ring, thus an integral domain. Thus either $\mathrm{Res}_1^2(y) = 0$ (and then $y\,\mathrm{Tr}_1^{2,V}(x) = 0 \in I$) or $\mathrm{Res}_1^2(y) \neq 0$ and hence $\mathrm{Res}_1^2(y)x$ is not fixed by conjugation; this implies $yTr_1^{2,V}(x) \in I$ as well.

For (2), since

$$(a_\sigma^{-1}N_1^2H\mathbb{F}_2)^{C_2} \simeq \Phi^{C_2}(N_1^2H\mathbb{F}_2) \simeq H\mathbb{F}_2,$$

all elements other than a_σ^k in $\pi^{C_2}_\bigstar(N_1^2H\mathbb{F}_2)/I$ are killed by some power of a_σ. (Note that the standard sequence with transfer, a_σ-multiplication and restriction implies that $a_\sigma x = a_\sigma^k$ is only possible if $x = a_\sigma^{k-1}$ with $k \geq 1$ or $x = -1$.) On the other hand, $N_1^2H\mathbb{F}_2$ being cofree implies that there is no element that is infinitely divisible by a_σ: such an element x would be the image of 1 under a map $\Sigma^{|x|}a_\sigma^{-1}N_1^2H\mathbb{F}_2 \to N_1^2H\mathbb{F}_2$, which is necessarily zero as cofreeness is equivalent to being a_σ-complete.

For (3), first note that the homotopy fixed points spectral sequence computing the $* + b\sigma$-homotopy groups maps to the homotopy fixed points spectral sequence computing the $* + (b-1)\sigma$-homotopy groups, and this map is multiplication by a_σ.

Let x be an element of degree $a + b\sigma$ in $\pi^{C_2}_\bigstar(N_1^2H\mathbb{F}_2)/I$ that is not divisible by a_σ and r be the minimal number such that $a_\sigma^r x = 0$. Then x corresponds to a permanent cycle of filtration 0 in the homotopy fixed points spectral sequence computing the $* + b\sigma$-homotopy groups. Now $a_\sigma^r x = 0$ means that when mapping to the spectral sequence computing the $*+(b-r)\sigma$-homotopy groups it is killed by an differential. Since $a_\sigma^{r-1}x \neq 0$, the differential must be an d_r.

Conversely, assume that there is a Tate differential $d_r(\tilde{y}) = \tilde{x}$ where $\tilde{x}$ is in bidegree (a, b). Since modulo I, the map from the homotopy fixed points spectral sequence to the Tate spectral sequence is injective, $\tilde{x}$ represents a permanent cycle $x \in \pi^{C_2}_{a+b\sigma} N_1^2 H\mathbb{F}_2/I$. Now if $a_\sigma^{r'} x = 0$ for $r' < r$ and we choose r' to be the smallest positive integer that kills x, then $\tilde{x}$ must be killed in the homotopy fixed point spectral sequence computing the $a + (b - r')\sigma$-homotopy groups. By the injectivity mentioned above, it implies that $\tilde{x}$ is killed by a $d_{r'}$-differential, which is a contradiction. □

In the HFPSS, the ideal I corresponds to the image of the Tate norm map T in the E_2-page.

For the rest of this section we compute certain families of differentials in the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$. We give multiple arguments for the same differential if possible, not only to show that the differentials are indeed correct, but also to illustrate how the localized slice spectral sequence and the Tate spectral sequence interact. Since the HOSS is locally finite on the E_2-page but the HFPSS is not, we apply Theorem 1.1 to HOSS to deduce differentials. For notation convenience, in the rest of this section, we use HO to refer to $(N_1^2 H\mathbb{F}_2)_{hC_2}$, whose C_2-equivariant $RO(C_2)$-graded homotopy groups are isomorphic to the C_4-equivariant $RO(C_4)$-graded homotopy groups of $a_\lambda^{-1} BP^{((C_4))}$ except in degrees $n - n\sigma$.

The E_2-page of the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$ is of the form

$$\widehat{H}^*(C_2; \mathcal{A}_*) \cong \widehat{H}^0(C_2; \mathcal{A}_*)[x^{\pm 1}],$$

where $x \in \widehat{H}^{-1}(C_2, \mathcal{A}_0) \cong \widehat{H}^{-1}(C_2, \mathbb{F}_2)$ and C_2 acts on $\mathcal{A}_*$ via conjugation. Moreover, $\widehat{H}^0(C_2; \mathcal{A}_*)$ are precisely the fixed points of $\mathcal{A}_*$ under the conjugation action modulo transfers. These can be computed by a computer program in any finite range. Here we use the computation of Bruner in [Bru]. In our range of interest, there are algebra generators

$$\begin{aligned} b_1 &= \xi_1 \\ b_6 &= \xi_2 \xi_1^3 + \xi_2^3 = \xi_2 \bar{\xi}_2 \\ b_9 &= \xi_3 \xi_1^2 + \xi_2^3. \end{aligned}$$

with relations

$$\begin{aligned} b_1^3 &= 0 \\ b_1 b_6 &= 0 \\ b_1 b_9 &= 0. \end{aligned}$$

In general, a closed formula of the E_2-page of the Tate spectral sequence is not known, see [CW00]. Figure 7 shows the E_2-page of the Tate spectral sequence with known differentials in a range. The red class at $(0, 0)$ is the only nontrivial permanent cycle surviving to the E_∞-page.

One can read both HFPSS and HOSS of $N_1^2(H\mathbb{F}_2)$ from Figure 7. If we ignore all elements with filtration ≥ 0 and add elements in I into filtration -1, then we obtain (up to a shift of degree 1) the E_2-page of the HOSS. Every element that supports a differential hitting filtration ≥ 0 becomes a nontrivial permanent cycle in the HOSS. Similarly, we can ignore all elements with filtration < 0 and add elements in I into filtration 0 to obtain the homotopy fixed points spectral sequence. This can also be done in an $RO(C_2)$-graded manner: If we ignore all elements with filtration $\geq b$ for any $b \in \mathbb{Z}$ and add I in filtration $b - 1$, then we obtain the HOSS computing the $(* + b\sigma)$ stems of the homotopy orbit, and similarly for the homotopy fixed points.

On the other hand, one can compute the $(* + b\sigma)$ stems of the homotopy orbit or fixed points by computing $a_\lambda^{-1}\mathrm{SliceSS}(BP^{((C_4))})$ in degree $* + b\sigma + c\lambda$, making full use of Proposition 5.2. This

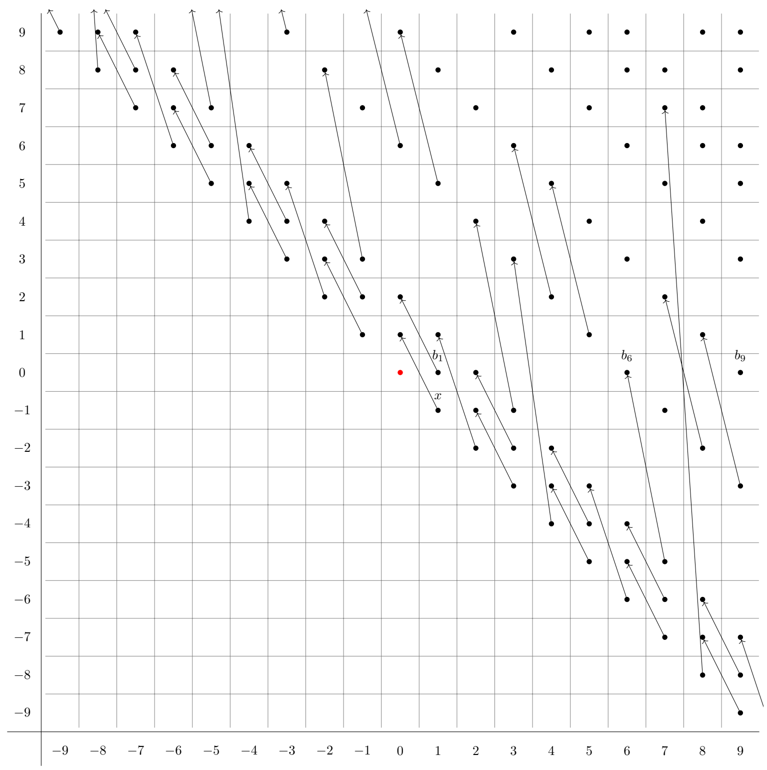


FIGURE 7. E_2-page of the Tate spectral sequence of $N_1^2 H\mathbb{F}_2$ with known differentials

$RO(C_4/C_2)$-graded comparison gives more information than the integral graded comparison, but at the cost of running computations similar (but not identical) to Section 5 for each $b \in \mathbb{Z}$. We will only use this approach in the proof of Proposition 6.7, where the integral graded comparison is not sufficient.

Proposition 6.2. *There is a differential*

$$d_2(x) = b_1 x^{-1}$$

and therefore differentials $d_2(x^{2k+1}) = b_1 x^{2k-1}$ *for all* $k \in \mathbb{Z}$.

Proof. We provide two arguments.

(1) By Proposition 5.2, the element 2 is (uniquely) divisible by a_σ^2 in $\pi_\bigstar^{C_4}(a_\lambda^{-1} H\underline{\mathbb{Z}})$, but the class $\frac{2}{a_\sigma^2}$ supports a d_3 as $\mathrm{Res}_2^4(\frac{2}{a_\sigma^2}) = \frac{u_{2\sigma_2}}{a_{\sigma_2}^2}$ does. Since $2a_\sigma = 0$ not only in $a_\lambda^{-1} H\underline{\mathbb{Z}}$ but also in the homotopy of $BP^{(\!(C_4)\!)}$ and 2 must be divisible by a_σ as its restriction is trivial, the a_σ-tower involving 2 has length 2. By Proposition 6.1, $b_1 x^{-1}$ in bidegree (0,1) is killed by a d_2, and the only possible source is x.

(2) By Theorem 4.4 $\pi_1(HO) = \mathbb{Z}/2$. That means the class $b_1 x$ in $(2, -1)$ must be a cycle, otherwise $b_1 x$ and x^2 (which cannot support a d_2 since the differential on it must be longer than the one on x) both support differentials into filtration ≥ 0, forcing $\pi_1(HO)$

to have order at least 4. The only possible differential that can kill b_1x is $d_2(x^3) = b_1x$. By the Leibniz rule, we deduce that $d_2(x) = b_1x^{-1}$.

□

Proposition 6.3. *There is a differential*

$$d_2(b_1) = b_1^2x^{-2}$$

and therefore differentials $d_2(b_1x^{2k}) = b_1^2x^{2k-2}$ *for all* $k \in \mathbb{Z}$.

Proof. We provide two arguments.

(1) b_1 is not a target of any differential, since the only possible differential that can kill it is $d_2(x^2)$, but x^2 is a d_2-cycle. Now if $d_2(b_1) = 0$, then by the Leibniz rule $d_2(b_1x) = b_1^2x^{-1} \neq 0$, which contradicts to the previous proposition stating that b_1x is a cycle. Therefore $d_2(b_1) \neq 0$ and $b_1^2x^{-2}$ is the only potential target.
(2) By Theorem 4.4 $\pi_3(HO) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, but one of $\mathbb{Z}/2$s comes from transfer. Since $d_2(x^5) = b_1x^3$, this forces a nontrivial differential $d_2(b_1x^4) = b_1^2x^2$. Since x^4 is a d_2-cycle, we must have $d_2(b_1) = b_1^2x^{-2}$.

□

Proposition 6.4. *There is a differential*

$$d_3(x^2) = b_1^2x^{-1}$$

and therefore differentials $d_3(x^{4k+2}) = b_1^2x^{4k-1}$.

Proof. We provide two arguments.

(1) The element $N(\bar{t}_1)$ is not divisible by a_σ modulo transfers and the a_σ-tower involving it has length 3: $N(\bar{t}_1)a_\sigma^2 \neq 0$ since $N(\bar{t}_1)^2a_\lambda^2a_\sigma^2 \neq 0$ in the integral page of a_λ^{-1}SliceSS. $N(\bar{t}_1)a_\sigma^3 = 0$ is the direct consequence of Theorem 5.7. By Proposition 6.1 the element in $(1,1)$ of the Tate spectral sequence receives a d_3, and the only possible differential is $d_3(x^2) = b_1^2x^{-1}$.
(2) By Theorem 4.4, $\pi_4(HO) = \mathbb{Z}/2$ and it is from transfer. Combining with earlier differentials, we see $d_3(x^6) = b_1^2x^3$ must happen. Therefore $d_3(x^2) = b_1^2x^{-1}$.

□

Proposition 6.5. *There is a differential*

$$d_5(b_1^2x) = b_6x^{-4}$$

and therefore differentials $d_5(b_1^2x^{4k+1}) = b_6x^{4k-4}$.

Proof. In $\pi_6(HO)$, the $\mathbb{Z}/4$ indicates the starting point of an a_σ-tower, therefore the element in $(6,0)$ of the Tate spectral sequence is a cycle. The only differential that kills it is $d_5(b_1^2x^5) = b_6$. Since x^4 is a d_5-cycle, we have $d_5(b_1^2x) = b_6x^{-4}$. □

Proposition 6.6. *There is a differential*

$$d_7(x^4) = b_6x^{-3}$$

and therefore differentials $d_7(x^{8k+4}) = b_6x^{8k-3}$.

Proof. We provide two arguments.

(1) Consider the element $N(\bar{t}_2)$. Its restriction in $\mathcal{A}_*$ is $\xi_2\bar{\xi}_2 = \xi_2^2 + \xi_2\xi_1^3$, which is non-zero modulo transfers. Therefore it is a starting point of an a_σ-tower. On the other hand, $N(\bar{t}_2)a_\sigma^7 = 0$ by Theorem 5.7 and $N(\bar{t}_2)a_\sigma^6 \neq 0$ since $N(\bar{t}_2)^2a_\sigma^6 \neq 0$ in the integral degrees. It follows that the a_σ-tower on $N(\bar{t}_2)$ has length 7, which implies the differential $d_7(x_4) = b_6x^{-3}$.
(2) One can compute the $(*+4\sigma)$-graded slice spectral sequence and see that in degree $3+4\sigma$ there is nothing other than transfers, which implies that b_6x^{-3} is a cycle and x_4 is the only element that can kill it.

□

Proposition 6.7. *There is a differential*

$$d_4(b_6x^2) = b_9x^{-2}$$

and therefore differentials $d_4(b_6x^{4k+2}) = b_9x^{4k-2}$.

Proof. One can compute stem $7+2\sigma$ of $a_\lambda^{-1}BP^{(\!(C_4)\!)}$ by a_λ^{-1}SliceSS, and see that modulo elements in I of Proposition 6.1, the only non-trivial element is $N(\bar{t}_3)a_\lambda^7a_\sigma^5$, which by the proof of Theorem 6.8 below, corresponds to the element x^8 in the Tate spectral sequence. Therefore, b_6x^2 must support a differential of at most length 4, and from the E_2-page there is only one possibility. □

Finally, we describe an infinite family of differentials in the Tate spectral sequence.

Theorem 6.8. *In the Tate spectral sequence of $N_1^2(H\mathbb{F}_2)$, the element x^{2^k} supports a nontrivial differential of length exactly $2^{k+1}-1$.*

To prove Theorem 6.8, we will first prove the following lemma.

Lemma 6.9. *In $\pi^{C_2}_{2(2^k-1)\rho_2}(a_\lambda^{-1}BP^{(\!(C_4)\!)})$, the element $\bar{t}_k\gamma\bar{t}_k$ cannot be written as $x+\gamma x$ for any element $x \in \pi^{C_2}_{2(2^k-1)\rho_2}(a_\lambda^{-1}BP^{(\!(C_4)\!)})$*

Proof. We follow the computation in [CW00]. By Theorem 5.4 $\gamma\bar{t}_k = \bar{t}_k + P$ where P doesn't involve $\bar{t}_k$. Therefore, writing $\bar{t}_k\gamma\bar{t}_k$ as a polynomial of $\bar{t}_1,\ldots,\bar{t}_k$, $\bar{t}_k^2$ appears as one of the monomials. Consider a monomial x of $\bar{t}_1,\bar{t}_2,\ldots,\bar{t}_k$ in this degree ($\bar{t}_{k+1}$ cannot appear since its degree is one larger than $\bar{t}_k^2$), the only possible monomial that $\bar{t}_k^2$ might appear as a part of $x+\gamma x$ is when $x = \bar{t}_k^2$. However, $\bar{t}_k^2$ is not a monomial of $\bar{t}_k^2 + \gamma\bar{t}_k^2$, by Theorem 5.4. □

Proof of Theorem 6.8. Consider the class $N(\bar{t}_k)$ in a_λ^{-1}SliceSS$(BP^{(\!(C_4)\!)})$. The restriction of this class is $\bar{t}_k\gamma\bar{t}_k$, which is nontrivial modulo transfer by the lemma. Therefore the corresponding element of $N(\bar{t}_k)$ in $(N_1^2H\mathbb{F}_2)^{hC_2}$ is not divisible by a_σ modulo I, so it is the starting point of its a_σ-tower. By Proposition 5.24, $N(\bar{t}_k)a_\sigma^i = 0$ only when $i \geq 2^{k+2}-1$. Hence this a_σ-tower starts in bidegree $(2^k-1, 2^k-1)$ of the Tate spectral sequence and is killed by $d_{2^{k+2}-1}$. The only possible source of this differential is $x^{2^{k+1}}$. □

Currently we cannot describe the target of these differentials, because we don't have a complete description of the E_2-page, and we don't understand many shorter differentials. However, it is very likely that the target of these differentials are the elements in the Tate cohomology of $\mathbb{F}_2$ generated by the elements $\xi_k\bar{\xi}_k$ in the dual Steenrod algebra. This can be verified for $k \leq 4$ by direct computation.

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