SOME ALGEBRAIC ASPECTS OF THE ADAMS-NOVIKOV SPECTRAL SEQUENCE

by

Haynes Robert Miller

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CONTENTS

INTRODUCTION

The computation of the stable homotopy groups of a space has been one of the touchstones of algebraic topology since Hopf discovered in 1935 that $\pi_3(S^2) \neq 0$. A major conceptual advance in the study of these groups occurred with Adams' ([1], 1958) construction of a spectral sequence abutting to (roughly) the p-component of $\pi_*^s(X)$, with

$$E_2^{**} = \mathrm{Ext}_{A^*}^{**}(H^*(X; \mathbb{F}_p), \mathbb{F}_p),$$

where A^* is the mod p Steenrod algebra.

In time, Adams (1964) and Novikov ([50], 1967) showed how to use various associative ring-spectra E to construct spectral sequences with E_2-terms depending only on the behavior of primary E-theory operations on $E^*(X)$. Under the assumption that E_*E is flat over E_*, Adams [3] gave a description of E_2 as

$$\mathrm{Ext}_{E_*E}^{**}(E_*, E_*(X))$$

defined in terms of a certain "coaction" of the "coalgebra" E_*E on $E_*(X)$.

In this thesis we study various algebraic tools for computing this Ext group. Chapter I describes the construction of a general Adams spectral sequence. In Chapter II we study homological algebra over gadgets like E_*E, which we call "Hopf algebroids." By exhibiting a cosimplicial structure on a "cobar" construction for E_*E, we show that $\mathrm{Ext}_{E_*E}^{**}(E_*, E_*)$ supports $\cup_i$-products and, if $pE_* = 0$, Steenrod operations. We construct an analogue of the Cartan-Eilenberg change-of-rings spectral sequence, and study some special classes of Hopf algebroids.

Chapter III introduces a spectral sequence obtained by filtering the cobar construction by powers of an ideal I in E_* invariant under the operations in E-theory. This generalizes a construction of Novikov, and we call it an algebraic Novikov spectral sequence. In case E is the Brown-Peterson summand BP of the unitary Thom spectrum MU localized at p, and I is the kernel of the Thom homomorphism $BP_*(\text{point}) \longrightarrow H_*(\text{point}; \mathbb{F}_p)$, the spectral sequence is shown to have

$$E_2^{***} = \mathrm{Ext}_{P_*}^{**}(\mathbb{F}_p, E_0^{**}BP_*(X))$$

relative to a certain canonical coaction of the dual Steenrod reduced powers P_* on $E_0^{**}BP_*(X)$. For p odd, and under restrictive torsion assumptions on X, this E_2^{**}-term is merely a regrading of the classical Adams E_2-term $\mathrm{Ext}_{A_*}^{**}(\mathbb{F}_p, H_*(X; \mathbb{F}_p))$. Thus differentials in the algebraic Novikov spectral sequence may be regarded as BP-theoretic differentials in the mod p Adams spectral sequence.

In the last chapter we use these techniques to prove a vanishing theorem for $\mathrm{Ext}_{BP_*BP}^{**}(BP_*, BP_*(X))$ and to compute $\mathrm{Ext}_{BP_*BP}^{**}(BP_*, BP_*)$ in a band of width $p^2(2p-2)$ above the vanishing edge. These results are combined in an appendix with known facts about π_*^s to calculate $\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*)$ through degree $t = 96$ for $p = 3$ and $t = 472$ for $p = 5$. This computation reveals several nontrivial differentials in the BP-Adams spectral sequence besides the well-known Cohen-Toda differential [62].

CHAPTER ONE

ADAMS SPECTRAL SEQUENCES

§1.0. Introduction

Adams spectral sequences arise from resolutions in a category with exact triangles. In this chapter we construct an Adams spectral sequence in this generality. In the first section we study the behavior of injective classes (after Eilenberg-Moore [18]) in a triangulated category (after Verdier [59]). Then we show how to lift a resolution to a "filtration" or dual Postnikov system whose associated fibers are the objects of the resolution. A "homology theory" carries this structure, dubbed an Adams complex, to an exact couple. Associated to this exact couple is the Adams spectral sequence.

Our interest is of course principally in the stable homotopy category. We shall use Adams' formulation of this category [5]. In §1.3 we study the Adams spectral sequence briefly in this context, and recall a known convergence theorem.

Despite this homotopy-theoretic orientation, we hope that the Adams spectral sequence, in the generality presented here, will find other applications, perhaps for instance to algebraic geometry.

§1.1. Injective classes and triples in triangulated categories

In this section we recall Verdier's definition [59] of a context for "first-order" stable homotopy theory, and study the behavior of injective classes and triples in such a situation. Much of this material can be found in [60] or [49].

(1.1.1) Let $\underline{X}$ be a pre-additive category (i.e., a category enriched [35] over the closed category $\underline{Ab}$ of Abelian groups). Let $\Sigma\colon \underline{X} \to \underline{X}$ be an additive automorphism, called <u>suspension</u>. There arises a new category $\underline{X}_*$, enriched over the category $\mathbb{Z}\underline{Ab}$ of graded Abelian groups, with the same objects and with $\underline{X}_*(X,Y)_n = \underline{X}(\Sigma^n X, Y)$ and the obvious composition. Given any functor $F\colon \underline{X} \to \underline{A}$ let $F_*\colon \underline{X}_* \to \mathbb{Z}\underline{A}$ by $F_n(X) = F_*(X)_n = F(\Sigma^{-n}X)$ for X in $\underline{X}$ and analogously for morphisms.

A <u>triangle</u> is a diagram $X \to Y \to Z \to \Sigma X$, which we frequently denote by

(1.1.2)
$$\begin{tikzcd} X \arrow[rr, "u"] & & Y \arrow[dl, "v"] \\ & Z \arrow[ul, "w"] \arrow[u, phantom, "(-1)"] & \end{tikzcd}$$

where (-1) indicates that $w\colon Z \to X$ has degree -1. A morphism of triangles is a triple (f,g,h) of maps in $\underline{X}$ such that

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

commutes.

<u>Definition 1.1.3</u> [59], [22], [20]. A <u>triangulated category</u> $\underline{X} = (\underline{X}, \Sigma, \underline{\Delta})$ is an additive category $\underline{X}$, an automorphism Σ, and a class $\underline{\Delta}$ of "exact"

triangles, satisfying (1.1.4) - (1.1.6).

(1.1.4) Every triangle isomorphic to an exact triangle is exact. Every morphism $u: X \to Y$ lies in an exact triangle (1.1.2). The triangle

$$X \xrightarrow{X} X \longrightarrow * \longrightarrow \Sigma X$$

is exact, if $*$ is the point in $\underline{X}$.

(1.1.5) The triangle (1.1.2) is exact iff

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is exact.

(1.1.6) Let (1.1.2) and (1.1.2)$'$ be exact triangles, and let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

commute. Then there exists $h: Z \to Z'$ making (f, g, h) a morphism of triangles.

Verdier ([59] [22]) states another axiom, the octahedral axiom (which is stronger than the "octahedral axiom" of [20] in an essential way). This axiom is useful in providing the Adams spectral sequence with a "Rees system"; but, having nothing new to say about convergence, we make no use of this axiom.

Definition 1.1.7 [18],[19],[23]. Let $\underline{X}$ be a pointed category. A (two-morphism) sequence is a diagram

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

such that gf is the trivial map. An <u>injective class</u> $\underline{I}$ is a class $\underline{I}$-obj of objects and a class $\underline{I}$-seq of sequences such that:

(1.1.8) $I \in \underline{I}$-obj iff for all $X' \to X \to X'' \in \underline{I}$-seq, (1.1.9) is exact:

(1.1.9) $$\underline{X}(X', I) \leftarrow \underline{X}(X, I) \leftarrow \underline{X}(X'', I)$$

(1.1.10) $X' \to X \to X'' \in \underline{I}$-seq iff for all $I \in \underline{I}$-obj, (1.1.9) is exact;

(1.1.11) for all $X' \to X$ there exists $X' \to X \to I \in \underline{I}$-seq with $I \in \underline{I}$-obj.

A longer sequence is <u>$\underline{I}$-exact</u> iff each two-morphism subsequence is in $\underline{I}$-seq.

<u>Definition 1.1.12</u> [49]. An injective class $\underline{I}$ in the triangulated category $\underline{X}$ is <u>stable</u> iff $\underline{I}$-obj is stable under Σ, or equivalently, if $\underline{I}$-seq is stable under Σ.

The following lemmas are proved in [49].

<u>Lemma 1.1.13</u>. Let $\underline{I}$-obj be a class of objects in a triangulated category $\underline{X}$. Suppose $\underline{I}$-obj is closed under retraction and suspension. Let $\underline{I}$-seq be the class of sequences $X' \to X \to X''$ in $\underline{X}$ such that for all $I \in \underline{I}$-obj, (1.1.9) is exact. Then ($\underline{I}$-obj, $\underline{I}$-seq) is a stable injective class iff for all $X \in \operatorname{ob} \underline{X}$ there exists $* \to X \to I \in \underline{I}$-seq with $I \in \underline{I}$-obj.

□

<u>Lemma 1.1.14</u>. Let $\underline{I}$ be a stable injective class in a triangulated category $\underline{X}$. Let $* \to X' \to X \in \underline{I}$-seq and let $X' \to X \to X'' \to \Sigma X'$ be an exact triangle. Then $* \to X' \to X \to X'' \to *$ is $\underline{I}$-exact.

□

(1.1.15) Let $\mathbb{T} = (T, \eta, \mu)$ be a triple [19] on the category $\underline{X}$. Recall that Eilenberg and Moore [19] have associated to $\mathbb{T}$ the category $\underline{X}^{\mathbb{T}}$ of $\mathbb{T}$-algebras. An object of $\underline{X}^{\mathbb{T}}$ is a pair (X, φ), $X \in \mathrm{ob}\, \underline{X}$, $\varphi: TX \longrightarrow X$, such that φ is unitary and associative with respect to η and μ. Let $T': \underline{X} \longrightarrow \underline{X}^{\mathbb{T}}$ be defined by $T'X = (TX, \mu_X)$ and $S': \underline{X}^{\mathbb{T}} \longrightarrow \underline{X}$ by $S'(X, \varphi) = X$. Then $T = S'T'$.

For further important properties of S', T', see [19]. They form an adjoint pair "generating" $\mathbb{T}$, and are universal as such.

Now let $\underline{X}$ be pointed. Let $\mathbb{T}$-seq be the class of sequences $X' \longrightarrow X \longrightarrow X''$ such that for all $Y \in \underline{X}$,

$$\underline{X}(X', TY) \longleftarrow \underline{X}(X, TY) \longleftarrow \underline{X}(X'', TY)$$

is exact.

<u>Lemma 1.1.16.</u> Let $f: X \longrightarrow X''$. The following are equivalent.

(i) $* \longrightarrow X \xrightarrow{f} X'' \in \mathbb{T}$-seq.

(ii) $\eta_X = \theta f$ for some $\theta: X'' \longrightarrow TX$.

(iii) $T'f: T'X \longrightarrow T'X''$ is split mono in $\underline{X}^{\mathbb{T}}$.

(iv) $Tf: TX \longrightarrow TX''$ is split mono in $\underline{X}$.

<u>Proof:</u> (i) $\Longrightarrow$ (ii) by definition of $\mathbb{T}$-seq, with $Y = X$.

(ii) $\Longrightarrow$ (iii) follows from the commutative diagram

$$\begin{tikzcd}
 & T'X \arrow[dl, "T'f"'] \arrow[d, "T'\eta_X"] \arrow[dr, "T'X"] & \\
T'X'' \arrow[r, "T'\theta"'] & T'TX \arrow[r, "\mu'_X"'] & T'X
\end{tikzcd}$$

where μ'_X is μ_X regarded (using associativity of $\mathbb{T}$) as a map in $\underline{X}^{\mathbb{T}}$.

(iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (i). Let $g\colon X \longrightarrow TY$ and let $s\colon TX'' \longrightarrow TX$ split Tf. Commutativity of the following diagram implies that $g = (\mu_Y \cdot Tg \cdot s \cdot \eta_{X''})f$.

$$\begin{tikzcd}
X'' \arrow[d, "\eta_{X''}"'] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\eta_X"] & TY \arrow[dd, "\eta_{TY}"'] \arrow[ddr, "TY"] & \\
TX'' \arrow[dr, "s"'] & TX \arrow[l, "Tf"'] \arrow[d, "TX"] & & \\
 & TX \arrow[r, "Tg"] & T^2Y \arrow[r, "\mu_Y"] & TY
\end{tikzcd}$$

$\square$

<u>Definition 1.1.17.</u> Let $\underline{X}, \underline{Y}$ be triangulated categories. A functor $F\colon \underline{X} \longrightarrow \underline{Y}$ is <u>stable</u> iff it is additive and commutes with suspension. A natural transformation $\theta\colon F \longrightarrow G$ of stable functors is <u>stable</u> iff θ is additive and commutes with suspension, i.e., $\theta_{\Sigma X} = \Sigma\theta_X$. A triple $\mathbb{T} = (T, \eta, \mu)$ on $\underline{X}$ is <u>stable</u> iff T, η, and μ are all stable.

<u>Lemma 1.1.18.</u> Let $\mathbb{T}$ be a stable triple on a triangulated category $\underline{X}$. Then $\mathbb{T}$-seq is the class of sequences of a stable injective class $\underline{I}(\mathbb{T})$ in $\underline{X}$. $\underline{I}(\mathbb{T})$-obj is the class of objects X of $\underline{X}$ such that η_X is split mono.

Proof. We first show that this class $\underline{I}(\mathbb{T})$-obj is the class of retracts of objects of the form TX. $I \in \underline{I}(\mathbb{T})$-obj is a retract of TI by definition. For the other inclusion suppose that I, X are in $\underline{X}$ and that

$$\begin{tikzcd} I \arrow[r, "i"] \arrow[dr, "I"'] & TX \arrow[d, "s"] \\ & I \end{tikzcd}$$

commutes. Then commutativity of

$$\begin{tikzcd} & TI \arrow[drr, "Ti"] & & \\ I \arrow[ur, "\eta_I"] \arrow[r, "i"] \arrow[dr, "I"'] & TX \arrow[rr, "\eta_{TX}"] \arrow[d, "s"] \arrow[drr, "TX"'] & & T^2X \arrow[d, "\mu_X"] \\ & I & & TX \arrow[ll, "s"] \end{tikzcd}$$

implies that $s \cdot \mu_X \cdot Ti$ splits η_I.

Thus $\underline{I}(\mathbb{T})$-obj is closed under retraction and determines $\mathbb{T}$-seq via (1.1.9). But

$$* \longrightarrow X \xrightarrow{\eta_X} TX \in \mathbb{T}\text{-seq}$$

since it satisfies (1.1.16.ii), so (1.1.13) completes the proof. □

(1.1.19) Recall [19] that a triple $\mathbb{T}$ on a category $\underline{X}$ determines a functor $S_{\mathbb{T}}: \underline{X} \to s^{\circ}\underline{X}$ into cosimplicial objects over $\underline{X}$, with T^{n+1} in degree n and, for $0 \leq i \leq n$,

$$s_i = T^i \mu T^{n-i}: S_{\mathbb{T}}^{n+1} \longrightarrow S_{\mathbb{T}}^n$$

$$d_i = T^i \eta\, T^{n-i} : S_{\mathbb{T}}^{n-1} \longrightarrow S_{\mathbb{T}}^n .$$

$S_{\mathbb{T}}$ is the standard construction of $\mathbb{T}$.

The associated cochain complex $A_{\mathbb{T}}$, with $A_{\mathbb{T}}^n = S_{\mathbb{T}}^n$ for $n \geq 0$ and

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : A_{\mathbb{T}}^n \longrightarrow A_{\mathbb{T}}^{n+1} ,$$

is called the standard complex of $\mathbb{T}$. A slight modification of the second proof of [19], Prop. 4.1, using exact triangles and Lemma 1.1.14 in place of cokernels, gives

Proposition 1.1.20. Let $\underline{X}$ be a triangulated category and $\mathbb{T}$ a stable triple on $\underline{X}$. The standard complex $A_{\mathbb{T}}X$ at $X \in \underline{X}$ is an $\underline{I}(\mathbb{T})$-injective resolution of X. □

For another useful resolution see Example 1.2.4.

§1.2. Adams resolutions

Throughout this section $\underline{X}$ will be a triangulated category.

Definition 1.2.1. An Adams complex is a sequence

$$(1.2.2)\qquad \begin{tikzcd} \cdots & X_1 \arrow[dr] & & X_0 \arrow[ll, "i_0"'] \arrow[dr] & & X_{-1} \arrow[ll, "i_{-1}"'] & \cdots \\ & & I_0 \arrow[ur, "-1"'] & & I_{-1} \arrow[ur, "-1"'] & & \end{tikzcd}$$

of exact triangles in $\underline{X}$; the labels "-1" denote maps of degree -1. The

Adams complex (1.2.2) is negative iff i_n is iso for $n \geq 0$, or equivalently, if $I_n \cong *$ for $n > 0$. All Adams complexes we consider will be negative, so we suppress the modifier.

The associated (augmented) complex of (1.2.2) is

(1.2.3) $$* \longrightarrow I_0 \longrightarrow \Sigma^{-1} I_{-1} \longrightarrow \Sigma^{-2} I_{-2} \longrightarrow \cdots$$

(with augmentation $X_0 \longrightarrow I_0$).

Let $\underline{I}$ be a stable injective class in $\underline{X}$. An Adams complex is $\underline{I}$-injective iff its associated complex is $\underline{I}$-injective, and $\underline{I}$-exact iff its associated augmented complex is $\underline{I}$-exact. It is an Adams resolution relative to $\underline{I}$ iff it is $\underline{I}$-injective and $\underline{I}$-exact, i.e., iff its associated complex is an $\underline{I}$-resolution.

Example 1.2.4. Let $\mathbb{T} = (T, \eta, \mu)$ be a stable triple on $\underline{X}$. The Adams complex (1.2.2) is canonical iff for all $n \leq 0$ there exists an isomorphism $I_n \xrightarrow{\cong} TX_n$ such that

$$\begin{tikzcd} & I_n \arrow[d, "\cong"] \\ X_n \arrow[ur] \arrow[r, "\eta_{X_n}"'] & TX_n \end{tikzcd}$$

commutes. From Lemma 1.1.18, $* \longrightarrow X_n \longrightarrow TX_n \in \underline{I}\text{-}(\mathbb{T})\text{-seq}$ and $TX_n \in \underline{I}(\mathbb{T})\text{-obj}$. Thus if (1.2.2) is canonical then it is an Adams resolution relative to $\underline{I}(\mathbb{T})$.

Note that any two canonical Adams resolutions are isomorphic, but not by a unique isomorphism.

Lemma 1.2.5. The Adams complex (1.2.2) is $\underline{I}$-exact iff $* \to X_n \to I_n$ is $\underline{I}$-exact for all $n \leq 0$.

Proof. "If" is clear, since $\underline{I}$-exact sequences paste together, and $* \to X_n \to I_n \to \Sigma X_{n-1} \to *$ is $\underline{I}$-exact by Lemma 1.1.14. The converse is proved by the following induction on n.

$n = 0$: $* \longrightarrow X_0 \longrightarrow I_0 \in \underline{I}$-seq by hypothesis, so by Lemma 1.1.14, $* \longrightarrow X_0 \longrightarrow I_0 \longrightarrow \Sigma X_{-1} \longrightarrow *$ is $\underline{I}$-exact.

$n < 0$: Assume that $* \to X_{n+1} \to I_{n+1} \to \Sigma X_n \to *$ is $\underline{I}$-exact. Given $J \in \underline{I}$-obj and $f\colon X_n \to J$,

$$\begin{tikzcd}
* \arrow[dr] & & & & * & \\
& X_{n+1} \arrow[dr] & & X_n \arrow[ll] \arrow[ur] \arrow[rr, "f"] \arrow[dr, "j"] & & J \\
I_{n+2} \arrow[rr, "d"'] & & I_{n+1} \arrow[ur, "k"] \arrow[rr, "d"'] & & I_n \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

we seek g such that $f = gj$. Now $fkd = *$ so, since I_* is $\underline{I}$-exact, g such that $fk = gd = gjk$ exists. By hypothesis $I_{n+1} \xrightarrow{k} X_n \to * \in \underline{I}$-seq, so $\underline{X}(I_{n+1}, J) \leftarrow \underline{X}(X_n, J)$ injects, and $f = gj$ follows. $\underline{I}$-exactness everywhere now results from Lemma 1.1.14.

□

Proposition 1.2.6. Any $\underline{I}$-resolution is the associated augmented complex of some Adams resolution, which is unique up to (non unique) isomorphism of Adams complexes.

Proof. Let $X \to I_*$ be an $\underline{I}$-resolution. Build the Adams complex inductively; start by forming the exact triangle

$$\begin{tikzcd}
X = X_0 \arrow[dr] & & X_{-1} \arrow[ll] \\
& I_{-1} \arrow[ur, "-1"'] &
\end{tikzcd}.$$

Now suppose the complex is defined up to

$$\begin{tikzcd}
X_{n+1} \arrow[dr] & & X_n \arrow[ll] \arrow[dr, dashed, "j"] & \\
& I_{n+1} \arrow[ur] \arrow[rr, "d"'] & & I_n
\end{tikzcd}$$

(omitting degrees). $I_{n+1} \to \Sigma X_n \to * \in \underline{I}$-seq by induction, and $I_n \in \underline{I}$-obj, so d lifts to j. Arguing as in Lemma 1.2.5, we see that $* \to X_n \to I_n \in \underline{I}$-seq. Complete the triangle to get X_{n-1}. Uniqueness follows from [59], p. 4, and Lemma 1.1.14 now completes the proof. □

Note that injectivity and exactness both play crucial roles in this construction.

Proposition 1.2.7. Let $\underline{I}$ be a stable injective class in $\underline{X}$. Let (1.2.2), (1.2.2)$'$ be Adams resolutions for X, X', and let $f: X \to X'$ lift to $h_*: I_* \to I'_*$. Then h_* lifts to a morphism of Adams complexes.

Proof. Suppose the lift has been constructed up to $f_n: X_n \to X'_n$. Consider

$$\begin{tikzcd}
X_n \arrow[dr] \arrow[dd, "f_n"'] & & X_{n-1} \arrow[ll] \arrow[dr, "j"] \arrow[dd, dashed, "f_{n-1}"] & \\
& I_n \arrow[ur, "k"'] \arrow[dd, "h_n"] & & I_{n-1} \arrow[dd, "h_{n-1}"] \\
X'_n \arrow[dr] & & X'_{n-1} \arrow[ll] \arrow[dr, "j'"] & \\
& I'_n \arrow[ur, "k'"'] & & I'_{n-1}
\end{tikzcd}$$

($|k| = -1 = |k'|$). Complete the map of triangles to f_{n-1}; we claim $j'g_{n-1} = f_{n-1}j$. By construction $f_{n-1}jk = j'k'f_n = j'g_{n-1}k$. But $I_n \to X_{n-1} \to *$ $\in \underline{I}$-seq and $I'_{n-1} \in \underline{I}$-obj together allow us to cancel the k. □

§1.3. Adams spectral sequences

We now transfer an Adams complex in a triangulated category $\underline{X}$ to an exact couple in an Abelian category $\underline{A}$ by means of an exactness-preserving functor.

Definition 1.3.1. [20] A functor $V_0: \underline{X} \to \underline{A}$ is homological iff it is additive and such that for every exact triangle $X' \to X \to X'' \to \Sigma X'$ in $\underline{X}$, $V_0X' \to V_0X \to V_0X''$ is exact in $\underline{A}$.

Let $\mathbb{Z}\underline{A}$ be the category of graded objects over $\underline{A}$. $V_0: \underline{X} \to \underline{A}$ defines a functor $V_*: \underline{X}_* \to \mathbb{Z}\underline{A}$ by $V_n(X) = V_*(X)_n = V(\Sigma^{-n}X)$. Then $V_*(\Sigma X) = \sigma V_*(X)$ where σ is the suspension automorphism on $\mathbb{Z}\underline{A}$ given by $(\sigma A)_n = A_{n-1}$.

Let

$$(1.3.2)\qquad \begin{tikzcd}[column sep=small] X = X_0 & & X_{-1} \arrow[ll] \arrow[dr] & & X_{-2} \arrow[ll] & \cdots \\ & I_0 \arrow[ur, "-1"] \arrow[from=ul] & & I_{-1} \arrow[ur, "-1"] & & \end{tikzcd}$$

be an Adams complex in $\underline{X}$. Applying V_* we arrive at an exact couple

$$(1.3.3)\qquad \begin{tikzcd} A^1 \arrow[dr, "{(0,0)}"'] & & A^1 \arrow[ll, "{(1,-1)}"'] \\ & E^1 \arrow[ur, "{(-1,0)}"'] & \end{tikzcd}$$

over $\underline{A}$, with maps of the indicated (filtration, complimentary) bidegrees, and with

$$(1.3.4)\qquad \begin{aligned} A^1_{s,t} &= V_{s+t}(X_s) \\ E^1_{s,t} &= V_{s+t}(I_s)\ . \end{aligned}$$

$d^1 : E^1_{s,t} \longrightarrow E^1_{s-1,t}$ is induced from $I_s \longrightarrow \Sigma X_{s-1} \longrightarrow \Sigma I_{s-1}$.

Let $\underline{I}$ be a stable injective class in $\underline{X}$ and suppose (1.3.2) is an Adams resolution relative to $\underline{I}$. Then in the spectral sequence associated to (1.3.3),

$$(1.3.5)\qquad E^2_{s,t} = R^{\underline{I}}_s\ V_{s+t}(X)$$

where $R^{\underline{I}}_* V_n$ denotes the right derived functor of V_n relative to $\underline{I}$; see [18],[23].

If (1.3.2)′ is an Adams resolution of X' and $f\colon X' \to X$, let $f_*\colon X'_* \to X_*$, $h_*\colon I'_* \to I_*$ lift f as in (1.2.7). We obtain a map of exact couples, which depends only on h at E^2. Thus the associated <u>Adams spectral sequence</u> depends functorially on X. Without implying convergence, we shall denote it by

$$R_*^{\underline{I}}\, V_*(X) \Rightarrow V_*(X).$$

Of course, our principal interest is in the stable homotopy category $\underline{S}$. We refer the reader to [5] for a description of this triangulated category. Let $[X,Y]_* = \underline{S}(X,Y)_*$. $\underline{S}$ possesses a coherently associative and commutative smash product $\wedge$ for which the sphere spectrum S is the unit. Let $\pi_*(\ \ ) = [S,\ \ ]_*\colon \underline{S} \to \mathbb{Z}\underline{Ab}$ denote the stable homotopy functor. Recall [61],[5], that a spectrum E defines a homological functor $E_*(\ \ ) = \pi_*(E \wedge\ \ )$ on $\underline{S}$.

<u>Definition 1.3.6</u>. A <u>ring-spectrum</u> is a spectrum E together with morphisms $\eta\colon S \to E$ and $\mu\colon E \wedge E \to E$ which are associative and unitary, and such that $\pi_*(E)$ is commutative under the product

$$\pi_*(E) \otimes \pi_*(E) \longrightarrow \pi_*(E \wedge E) \xrightarrow{\pi_*(\mu)} \pi_*(E).$$

A ring-spectrum $E = (E, \eta, \mu)$ defines a triple $(E \wedge\ \ , \eta \wedge\ \ , \mu \wedge\ \ )$ on $\underline{S}$ which we denote simply by E. Write $R_*^E V_0$ for the derived functor of V_0 relative to the E-injective class $\underline{I}(E)$. Note that any spectrum V defines a homological functor $V_0\colon \underline{S} \to \underline{Ab}$, $\underline{Ab}$ the category of Abelian groups.

We lose little by restricting attention to the derived functors of the stable homotopy functor π_*. In fact we have:

<u>Proposition 1.3.7.</u> Let E be a ring-spectrum and V any spectrum. Then the Adams spectral sequences

$$R_*^E V_*(X) \Rightarrow V_*(X) \tag{1.3.8}$$

$$R_*^E \pi_*(V \wedge X) \Rightarrow \pi_*(V \wedge X) \tag{1.3.9}$$

are naturally isomorphic in a way compatible with the identity $V_*(X) = \pi_*(V \wedge X)$. If we are given a ring-spectrum structure on $E \wedge V$ and a map $\theta: E \longrightarrow E \wedge V$ of ring-spectra, then the natural morphism

$$\begin{tikzcd} R_*^E \pi_*(V \wedge X) \arrow[r, Rightarrow] \arrow[d, "R_*^\theta \pi_*(V \wedge X)"'] & \pi_*(V \wedge X) \arrow[d, "="] \\ R_*^{E \wedge V} \pi_*(V \wedge X) \arrow[r, Rightarrow] & \pi_*(V \wedge X) \end{tikzcd} \tag{1.3.10}$$

is an isomorphism.

<u>Proof.</u> Form a canonical Adams resolution, (1.2.4), relative to E (in which $\Sigma\overline{E}$ completes the exact triangle $S \longrightarrow E \longrightarrow \Sigma\overline{E} \longrightarrow \Sigma S$):

$$\begin{tikzcd} X \simeq X_0 \arrow[dr] & & X_{-1} \arrow[ll] \arrow[dr] & & X_{-2} \arrow[ll] \\ & E \wedge X \arrow[ur, "-1"] & & E \wedge \overline{E} \wedge X \arrow[ur, "-1"] & \end{tikzcd} \tag{1.3.11}$$

Then $V \wedge (1.3.11)$ is a canonical Adams resolution for $V \wedge X$, and the exact couple V_* (1.3.11) is clearly naturally isomorphic to the exact couple $\pi_*(V \wedge (1.3.11))$. The first statement follows.

For the second, we want $V \wedge (1.3.11)$ to be an Adams resolution relative to $E \wedge V$. From the ring-spectrum map $\theta: E \longrightarrow E \wedge V$ we deduce that it is $E \wedge V$-exact. On the other hand each term in the associated complex has an obvious $E \wedge V$-module structure, and hence is injective. $\square$

Let E be a spectrum and V a Moore-spectrum with integral homology $H_*(V)$ equal to the graded Abelian group A. Following [3], write EA for the spectrum $E \wedge V$.

Example 1.3.11. Let H be the Eilenberg-MacLane spectrum with homotopy $\pi_*(H) = \mathbb{Z}_{(p)}$, the integers localized at p. Thus we are suppressing the localization from our notation. Let $\mathbb{F}_p$ be the Galois field with p elements, and let V be the sphere spectrum. Then

$$R_*^{H\mathbb{F}_p} \pi_*(X) \Rightarrow \pi_*(X)$$

is the classical mod p Adams spectral sequence.

Next let $E = H$ and $V = S\mathbb{F}_p$. Then (1.3.8), for p odd, was considered by J. Neisendorfer [52]. The proposition shows that his spectral sequence at X agrees with the usual Adams at $V \wedge X$. Neisendorfer notes that the mod p stable homotopy functor V_* carries a natural Bockstein differential induced by the cofibration

$$S \xrightarrow{p} S \longrightarrow V.$$

This induces a differential ∂ in the spectral sequences (1.3.8) - (1.3.10). That is, E^r has two differentials, ∂ and d^r, such that $\partial d^r = -d^r \partial$, and ∂ on E^r induces ∂ on E^{r+1}. The isomorphisms in Proposition 1.3.7 respect this structure by virtue of naturality in V.

Another case of special interest occurs when V is the Spanier-Whitehead dual [5] Λ^* of a finite spectrum Λ. The functors

$$[\Lambda, \quad]_* \cong V_*(\quad)$$

are naturally isomorphic, so the first part of (1.3.7) asserts that

$$R^E_*[\Lambda, \quad]_*(X) \Rightarrow [\Lambda, X]_*$$

and

$$R^E_* \pi_*(\Lambda^* \wedge X) \Rightarrow \pi_*(\Lambda^* \wedge X)$$

are isomorphic compatibly with the duality isomorphism.

Since the homological degree is negative, we will henceforth systematically raise it in our notation. Because of the special importance of π_*, we fix the following notation:

(1.3.12) $$E^{s,t}(X;E) = R^E_{-s}\pi_{t-s}(X) = R^s_E \pi_{t-s}(X).$$

(1.3.13) We now summarize the work of Adams, Bousfield, and others on the convergence of Adams spectral sequences. Let E be a (-1)-connected ring-spectrum, and let P be the set of rational primes invertible in $\pi_0(E)$. The unit map $\mathbb{Z} \to \pi_0(E)$ factors through the subring $\mathbb{Z}[P^{-1}]$ of $\mathbb{Q}$ consisting of fractions whose denominators are products of primes in P. For any connective spectrum X, define a spectrum X^E as follows. If $\mathbb{Z} \to \pi_0(E)$

injects, then

$$X^E = X\,\mathbb{Z}[P^{-1}].$$

Otherwise,

$$X^E = \prod_{p \notin P} X_p^{\wedge}$$

where $X_p^{\wedge}$ is the obvious (cf. [11] VI, 6.5(ii)) stabilization of the Bousfield-Kan $\mathbb{F}_p$-completion [11]. X^E is the "E-completion" in the sense of Adams [5]; see [10].

Theorem 1.3.14 (Adams [5]). Let E be a (-1)-connected ring-spectrum and let P be the set of rational primes invertible in $\pi_0(E)$. Assume that $\pi_0(E)$ is solid (i.e., multiplication: $\pi_0(E) \otimes_{\mathbb{Z}} \pi_0(E) \longrightarrow \pi_0(E)$ is an isomorphism) and that $H_*(E; \mathbb{Z})$ is of finite type over $\mathbb{Z}[P^{-1}]$. Then for any connective spectrum X, the spectral sequence

$$E_2(X; E) \cong E_2(X^E; E) \Rightarrow \pi_*(X^E)$$

converges, in the sense that the filtration on $\pi_*(X^E)$ is bicomplete. $\square$

CHAPTER TWO

HOPF ALGEBROIDS

§2.0. Introduction

Let E be a ring-spectrum. Then E^*E is the algebra of stable operations on E_*. In fact given $x: S \to E \wedge X$ and $\theta: E \to E$ we may form $(\theta \wedge X)x: S \to E \wedge X$. This gives an action

$$E^*E \otimes E_*(X) \longrightarrow E_*(X)$$

natural in X.

Proper treatment of the algebra E^*E requires the use of topological modules over a topological algebra [62].

To avoid this unpleasantness J. F. Adams proposed in [3] to study the dual object E_*E and its coaction on $E_*(X)$, in analogy with Milnor's work [44] on the dual Steenrod algebra.

The description of this coaction ψ is slightly tricky. A natural candidate for ψ is the composite

$$E_*X = \pi_*(E \wedge X) \xrightarrow{\eta_*} E_*(E \wedge X) \xleftarrow{\nu} E_*E \otimes_{E_*} E_*X.$$

Here η_* is the Hurewicz homomorphism and ν is the usual Künneth map carrying $\xi: S \to E \wedge E$, $x: S \to E \wedge X$, to the composite

$$\begin{tikzcd}
S = S \wedge S \arrow[r, "\xi \wedge x"] \arrow[drr] & (E \wedge E) \wedge (E \wedge X) \arrow[r, "E \wedge T \wedge X"] & E \wedge E \wedge E \wedge X \arrow[d, "\mu \wedge E \wedge X"] \\
& & E \wedge E \wedge X
\end{tikzcd}$$

.

To invert ν we assume that E_*E is right-flat over E_*; then ν is a map of homology theories which is an isomorphism on the sphere and hence everywhere.

The problem with this attempt is that in order to study homological properties, ψ must be associative. To consider associativity, ψ must be left E_*-linear. Now ν is left E_*-linear, but η_* is in general not; so ψ generally fails to have this property.

Adams overcame this obstacle by introducing a new algebraic object, which we recall in §2.1 and christen a "Hopf algebroid" for reasons stated there. We define comodules over a Hopf algebroid, indicate some of their properties, and recall the connection with geometry.

In §2.2 we study homological algebra in the category of comodules and construct a cobar resolution. §2.3 shows how the situation studied by Landweber in [29] fits into this context.

With §2.5 we come to the main application, which is to complex bordism and Brown-Peterson homology. We recall the structure and certain elementary properties of the Hopf algebroid BP_*BP, and give yet another proof of the Stong-Hattori theorem.

§2.1. Hopf algebroids

Let E be a ring-spectrum such that E_*E is flat over E_*. We saw above that E_*E cannot in general be made into a Hopf algebra in a meaningful way.

Recall [45] that a commutative Hopf algebra (with involution) is precisely a cogroup in the category of commutative algebras. Recall also that a groupoid is a small category in which every morphism is an isomorphism; thus a group is a groupoid with one object.

What Adams realized [3] was that E_*E is generally not a cogroup, but rather a cogroupoid in the category of commutative algebras.

Let R be a commutative ring. Throughout this chapter "R-algebra" will mean "commutative graded R-algebra."

Definition 2.1.1. A Hopf algebroid over R is a cogroupoid in the category of R-algebras. Explicitly, it consists in two R-algebras, A and Γ (the algebras of "objects" and of "morphisms"), called the coefficient algebra and the cooperation algebra, and R-algebra morphisms

$\eta_L: A \to \Gamma$	left unit, "target"
$\eta_R: A \to \Gamma$	right unit, "source"
$\epsilon: \Gamma \to A$	counit, "identity"
$\Delta: \Gamma \to \Gamma \otimes_A \Gamma$	diagonal, "composition"
$c: \Gamma \to \Gamma$	conjugation, "inverse"

Here Γ is a left A-module via η_L and a right A-module via η_R, and $\Gamma \otimes_A \Gamma$ is the usual tensor product of bimodules. We require of these maps that Δ and ϵ be A-bimodule maps and that the following diagrams commute.

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\simeq"'] \arrow[d, "\Delta"] \arrow[dr, "\simeq"] & \\
A \otimes_A \Gamma & \Gamma \otimes_A \Gamma \arrow[l, "\epsilon \otimes_A \Gamma"] \arrow[r, "\Gamma \otimes_A \epsilon"'] & \Gamma \otimes_A A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\eta_L"] \arrow[dr, "\eta_R"'] & \Gamma \arrow[d, "c"] & A \arrow[l, "\eta_R"'] \arrow[dl, "\eta_L"] \\
 & \Gamma &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "\eta_R"'] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & A \arrow[d, "\eta_L"] \\
\Gamma & \Gamma \otimes_A \Gamma \arrow[l, "\mu(c \otimes_A \Gamma)"] \arrow[r, "\mu(\Gamma \otimes_A c)"'] & \Gamma
\end{tikzcd}$$

We leave to the reader the amusement of interpreting these as cogroupoid axioms. We will frequently let the coefficient algebra A be understood, and write simply Γ for (A, Γ).

Suppose $\eta_L = \eta_R : A \longrightarrow \Gamma$. Then all tensor products are simply tensor products of left modules over the commutative algebra A. The product $\Gamma \otimes \Gamma \longrightarrow \Gamma$ factors through $\Gamma \otimes \Gamma \longrightarrow \Gamma \otimes_A \Gamma$ and Γ becomes simply a Hopf algebra over A. This extends the classical notion only in that A is graded.

Much of this chapter could be carried through in greater generality. In fact, let A be an R-algebra. The category (A-mod-A) of A-bimodules has an internal tensor product $\otimes_A$ which is coherently associative and unitary, but is not commutative. We may study $\otimes_A$-coalgebras in (A-mod-A).

We refrain from this generalization because we see no use for it. We neglect also to point out which results are valid for any <u>cocategory</u> in the category of R-algebras, and which depend on the inverse.

<u>Definition 2 1.2</u>. Let E and V be ring-spectra; see (1.3.6). $V_*(E \wedge E)$ is a V_*E-bimodule by virtue of the ring-maps

(2.1.3) $$\eta_L: V_*E \cong V_*(E \wedge S) \xrightarrow{V_*(E \wedge \eta)} V_*(E \wedge E),$$

(2.1.4) $$\eta_R: V_*E \cong V_*(S \wedge E) \xrightarrow{V_*(\eta \wedge E)} V_*(E \wedge E).$$

E is <u>flat relative to</u> V iff $V_*(E \wedge E)$ is left-flat over V_*E. In case V is the sphere-spectrum S, E will be called <u>flat</u>.

Generalizing a construction of Adams [3], we provide $(V_*E, V_*(E \wedge E))$ with a Hopf algebroid structure with left and right units as in (2.1.3) and (2.1.4), and

(2.1.5) $$\varepsilon: V_*(E \wedge E) \xrightarrow{V_*\mu} V_*E,$$

(2.1.6) $$c: V_*(E \wedge E) \xrightarrow{V_*T} V_*(E \wedge E),$$

where T switches factors. To define Δ first let X be any spectrum and construct ψ_X in

$$\text{(2.1.7)}\qquad \begin{tikzcd} V_*(E\wedge X) \cong V_*(E\wedge S\wedge X) \arrow[r, "V_*(E\wedge\eta\wedge X)"] \arrow[dr, "\psi_X"'] & V_*(E\wedge E\wedge X) \\ & V_*(E\wedge E)\otimes_{V_*E} V_*(E\wedge X) \arrow[u, "m_X", "\cong"'] \end{tikzcd}$$

where m_X carries $f\colon S\to V\wedge E\wedge E$, $g\colon S\to V\wedge E\wedge X$, to the composite

$$\begin{tikzcd} S\cong S\wedge S \arrow[r, "f\wedge g"] \arrow[drr] & V\wedge E\wedge E\wedge V\wedge E\wedge X \arrow[r, "T'"] & V\wedge V\wedge E\wedge E\wedge E\wedge X \arrow[d, "\mu\wedge E\wedge\mu\wedge X"] \\ & & V\wedge E\wedge E\wedge X \end{tikzcd} .$$

Here T' exchanges the neighboring copies of $E\wedge E$ and V. Now let

(2.1.8) $$\Delta = \psi_E\colon V_*(E\wedge E)\longrightarrow V_*(E\wedge E)\otimes_{V_*E} V_*(E\wedge E).$$

The map ψ_X of (2.1.7) prompts the following:

<u>Definition 2.1.9.</u> Let (A,Γ) be a Hopf algebroid. A left (A,Γ)-<u>comodule</u> is a left A-module M together with an A-module map $\psi\colon M\longrightarrow \Gamma\otimes_A M$ making commutative:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[dr, "\cong"'] & \Gamma\otimes_A M \arrow[d, "\epsilon\otimes_A M"] \\ & A\otimes_A M \end{tikzcd} ,$$

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & \Gamma\otimes_A M \arrow[d, "\Gamma\otimes_A\psi"] \\ \Gamma\otimes_A M \arrow[r, "\Delta\otimes_A M"'] & \Gamma\otimes_A\Gamma\otimes_A M \end{tikzcd} .$$

Γ-comodules form a category (Γ-comod) in an obvious way. A is a Γ-comodule with coaction

$$(2.1.10) \qquad A \xrightarrow{\eta_L} \Gamma \cong \Gamma \otimes_A A .$$

Γ is a Γ-comodule with coaction $\Delta: \Gamma \longrightarrow \Gamma \otimes_A \Gamma$.

If $M, N \in (\Gamma\text{-comod})$, then the set of comodule morphisms from M to N forms a sub-R-module of the R-module of A-linear maps.

There is an obvious suspension automorphism σ lifting the usual suspension on (A-mod). Let $\mathrm{Hom}_\Gamma(M,N)$ denote the graded R-module with the R-module of Γ-comodule maps $\sigma^n M \longrightarrow N$ in dimension n.

As an important special case let $M = A$ as in (2.1.10). Then

$$(2.1.11) \qquad \mathrm{Hom}_\Gamma(A,N) = \{n \in N: \psi(n) = 1 \otimes_A n\}.$$

If $N = A$ as well, then $\psi(n) = \eta_L(a) \otimes_A 1$ while $1 \otimes_A n = \eta_R(a) \otimes_A 1$; so

$$(2.1.12) \qquad \mathrm{Hom}_\Gamma(A,A) = \{a \in A: \eta_L(a) = \eta_R(a)\}.$$

<u>Definition 2.1.13</u>. The Hopf algebroid (A, Γ) is <u>flat</u> iff Γ is left or equivalently right flat over A. (A,Γ) is <u>connected</u> iff A and Γ are positively graded and $\epsilon_0: \Gamma_0 \longrightarrow A_0$ is an isomorphism. More generally, (A,Γ) is $(n-1)$-<u>connected</u> iff A and Γ are positively graded and $\epsilon: \Gamma \longrightarrow A$ is $(n-1)$-connected.

If (A,Γ) is flat then the category (Γ-comod) is Abelian.

(2.1.14) Let E and V be ring-spectra such that E is flat relative to V, so that $V_*(E \wedge E)$ is a Hopf algebroid. Then [3] the coaction ψ_X gives $V_*(E \wedge X)$ a $V_*(E \wedge E)$-comodule structure natural in X. Furthermore, $V_*(E \wedge \Sigma X) \cong \sigma V_*(E \wedge X)$.

Example 2.1.15. The modp Eilenberg-MacLane spectrum $H\mathbb{F}_p$ is flat since $\pi_*(H\mathbb{F}_p) = \mathbb{F}_p$ is a field. The coefficient algebra is just the ground-ring $\mathbb{F}_p$, so $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ is actually a Hopf algebra: in fact, it is just the dual Steenrod algebra A_*.

If $p \geq 5$ then the Moore-spectrum $M = S\mathbb{F}_p$ is an associative ring-spectrum, and the $\mathbb{Z}_{(p)}$-Eilenberg-MacLane spectrum $H = H\mathbb{Z}_{(p)}$ is M-flat. The associated Hopf algebra is the quotient

$$A(1)_* = A_*/(\tau_0)$$

of the Steenrod algebra considered by J. Neisendorfer [49]. It may also be constructed in this manner for $p = 3$; what is really needed in (2.1.3) - (2.1.8) is that V is a not necessarily associative ring-spectrum such that $E \wedge V$ is homotopy-associative and $\pi_*(E \wedge V)$ is commutative. The projection $A_* \longrightarrow A(1)_*$ is π_* applied to the composite

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \cong M \wedge H \wedge M \wedge H \xrightarrow{M \wedge T \wedge H} M \wedge M \wedge H \wedge H \xrightarrow{\mu \wedge H \wedge H} M \wedge H \wedge H.$$

Since $M_*(X)$ supports a natural Bockstein differential, $A(1)_*$ is a differential Hopf algebra.

To better understand the definition (2.1.13) of connectivity, we have

Lemma 2.1.16. Let E be a flat ring-spectrum. Then the Hopf algebroid

E_*E is connected iff E is (-1)-connected and $\pi_0(E) = E_0$ is solid.

(Recall [11] that a commutative ring R is solid iff the unit $\mathbb{Z} \to R$ is a categorical epimorphism, or equivalently, iff multiplication: $R \otimes_{\mathbb{Z}} R \longrightarrow R$ is an isomorphism.)

Proof. We must show that if E_* is positively graded then $\varepsilon_0 = \mu_0 : (E \wedge E)_0 \longrightarrow E_0$ is an isomorphism iff E_0 is solid. This will follow if the Künneth map $E_0 \otimes E_0 \longrightarrow (E \wedge E)_0$ is an isomorphism. Consider the Atiyah-Hirzebruch spectral sequence for E_* at E. We see that $(E \wedge E)_0 \cong H_0(E; E_0)$, and the universal coefficient theorem completes the proof. $\square$

Proposition 2.1.17. ("Nakayama" [45], Prop. 2.5). Let (A, Γ) be a flat connected Hopf algebroid. Let $f: M \to M'$ be a Γ-comodule map and suppose M is connective. Then f is monomorphic iff $\mathrm{Hom}_\Gamma(A, f)$ is monomorphic.

Proof. For a Γ-comodule N let $F_0 N = \mathrm{Hom}_\Gamma(A, N)$. Suppose N is $(n-1)$-connected and $N_n \neq 0$. Unitarity (2.1.9) reads, in degree n,

$$\begin{tikzcd}
N_n \arrow[r, "\psi_n"] \arrow[dr, "\cong"'] & (\Gamma \otimes_A N)_n \arrow[d, "(\varepsilon \otimes_A N)_n"] & = \Gamma_0 \otimes_{A_0} N_n \arrow[d, "\varepsilon_0 \otimes_{A_0} N_n"] \\
 & (A \otimes_A N)_n & = A_0 \otimes_{A_0} N_n
\end{tikzcd}$$

Since ε_0 is an isomorphism, $(F_0 N)_n = N_n$. In particular, if $F_0 N = 0$ then $N = 0$.

Since $(\Gamma\text{-comod})$ is Abelian, we can construct the kernel

$$0 \longrightarrow N \longrightarrow M \longrightarrow M'.$$

Assume $F_0 M \longrightarrow F_0 M'$ is monomorphic. Since F_0 is left-exact, $F_0 N = 0$. Since M is connective, N is too; so $N = 0$ and $M \longrightarrow M'$ is monomorphic. $\square$

(2.1.18) Let $(A, \Gamma) \longrightarrow (B, \Sigma)$ be a map of Hopf algebroids. For a left Γ-comodule M, give the B-module $B \otimes_A M$ a Σ-coaction by extending the A-linear map

$$M \xrightarrow{\psi} \Gamma \otimes_A M \longrightarrow \Sigma \otimes_A M \cong (\Sigma \otimes_B B) \otimes_A M$$

to a B-linear map

$$B \otimes_A M \longrightarrow \Sigma \otimes_B (B \otimes_A M).$$

We have defined a functor

(2.1.19) $$B \otimes_A : (\Gamma\text{-comod}) \longrightarrow (\Sigma\text{-comod}).$$

Now $(A \otimes A, \Gamma \otimes \Gamma)$ is a Hopf algebroid in an obvious way, and $(\mu, \mu): (A \otimes A, \Gamma \otimes \Gamma) \longrightarrow (A, \Gamma)$ is a morphism of Hopf algebroids. If M and N are Γ-comodules, then $M \otimes N$ is a $\Gamma \otimes \Gamma$-comodule. Define

(2.1.20) $$M \otimes_A^L N = A \otimes_{A \otimes A} (M \otimes N).$$

Note that the underlying A-module is the tensor-product of left modules over the commutative algebra A. The functor $\otimes_A^L$ is a coherently associative and commutative tensor product on $(\Gamma\text{-comod})$ with unit A. We have a similar functor

$$\otimes_A^R : (\text{comod-}\Gamma) \times (\text{comod-}\Gamma) \longrightarrow (\text{comod-}\Gamma).$$

The product $\mu: \Gamma\otimes\Gamma \to \Gamma$ factors through maps $\mu_L: \Gamma\otimes_A^L\Gamma \to \Gamma$ and $\mu_R: \Gamma\otimes_A^R\Gamma \to \Gamma$. Now in analogy with the theory of groups [15] or of Hopf-algebras [8], we have:

Lemma 2.1.21. There is an isomorphism

$$\Gamma\otimes_A N \cong \Gamma\otimes_A^L N$$

of Γ-comodules, natural in the Γ-comodule N.

Proof. The following composites are inverse isomorphisms.

$$h: \Gamma\otimes_A^L N \xrightarrow{\Gamma\otimes_A^L\psi} \Gamma\otimes_A^L(\Gamma\otimes_A N) \xrightarrow{\cong} (\Gamma\otimes_A^L\Gamma)\otimes_A N \xrightarrow{\mu_L\otimes_A N} \Gamma\otimes_A N$$

$$k: \Gamma\otimes_A N \xrightarrow{\Gamma\otimes_A\psi} \Gamma\otimes_A(\Gamma\otimes_A N) \xrightarrow{\Gamma\otimes c\otimes N} (\Gamma\otimes_A^R\Gamma)\otimes_A^L N \xrightarrow{\mu_R\otimes_A N} \Gamma\otimes_A^L N. \qquad \square$$

In the geometric setting we have the following "Cartan formula."

Lemma 2.1.22. Let E, V be ring-spectra with E V-flat. Then the Künneth map

$$\nu: V_*(E\wedge X)\otimes_{V_*E}^L V_*(E\wedge Y) \longrightarrow V_*(E\wedge X\wedge Y)$$

sending $f: S \to V\wedge E\wedge X$, $g: S\to V\wedge E\wedge Y$, to the composite

$$\begin{tikzcd}
S = S\wedge S \arrow[r, "f\wedge g"] \arrow[drr, "\nu(f\otimes g)"'] & V\wedge E\wedge X\wedge V\wedge E\wedge Y \arrow[r, "T"] & V\wedge V\wedge E\wedge E\wedge X\wedge Y \arrow[d, "\mu\wedge\mu\wedge X\wedge Y"] \\
 & & V\wedge E\wedge X\wedge Y
\end{tikzcd}$$

is a map of $V_*(E \wedge E)$-comodules. □

§2.2 Homological algebra

We turn now to homological algebra in the category of (A, Γ)-comodules.

The forgetful functor $(\Gamma\text{-comod}) \longrightarrow (A\text{-mod})$ has a right adjoint taking M to $\Gamma \otimes_A M$ with coaction $\Delta \otimes_A M$. This adjoint pair pulls the split injective class [23] in (A-mod) back to an injective class in $(\Gamma\text{-comod})$. An object is injective iff it is a direct summand of $\Gamma \otimes_A M$ for some A-module M. A sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

of comodules is exact iff it is split-exact in (A-mod); that is, the unique map $\operatorname{coker}(f) \longrightarrow M''$ factoring g is split mono in (A-mod).

Thus for Γ-comodules M and N we may define the bigraded R-module with

$$\operatorname{Ext}_\Gamma^{s,t}(M,N) = H_{-s,t}\operatorname{Hom}_\Gamma(M,I)$$

where $N \longrightarrow I$ is any injective resolution.

Since $\operatorname{Hom}_\Gamma(M, \quad)$ is left-exact, $\operatorname{Ext}_\Gamma^0(M,N) = \operatorname{Hom}_\Gamma(M,N)$.

It will often be convenient to have an explicit and natural resolution. Observe that the injective class in $(\Gamma\text{-comod})$ arises from the triple $\mathbb{T} = (\Gamma \otimes_A, \eta, \mu)$, where

$$\eta_M = \psi_M : M \longrightarrow \Gamma \otimes_A M$$

(2.2.1)

$$\mu_M = \Gamma \otimes_A \epsilon \otimes_A M : \Gamma \otimes_A \Gamma \otimes_A M \longrightarrow \Gamma \otimes_A M.$$

Thus we have the standard (cosimplicial) construction $S_{\mathbb{\Gamma}}(M)$ and its associated cochain complex $A_{\mathbb{\Gamma}}(M)$, the standard resolution recalled above (1.1.19). It is usual to work in the slightly smaller canonical resolution $W(\Gamma;M)$, which is the (co)normalized complex [39] of the cosimplicial Γ-comodule $S_{\mathbb{\Gamma}}(M)$.

Explicitly,

$$W(\Gamma;M) = \Gamma \otimes_A \overline{\Gamma} \otimes_A \overset{n}{\cdots} \otimes_A \overline{\Gamma} \otimes_A M$$

where

$$\overline{\Gamma} = \mathrm{Ker}\ (\varepsilon: \Gamma \longrightarrow A)$$

as an A-bimodule. Write $\gamma[\gamma_1|\cdots|\gamma_n]m$ for an element of $W(\Gamma;M)^n$. The differential is:

$$d\gamma[\gamma_1|\cdots|\gamma_n]m = \Sigma\gamma'[\gamma''|\gamma_1|\cdots|\gamma_n]m$$

$$+ \sum_{i=1}^{n} (-1)^{\varepsilon(i)} \gamma[\gamma_1|\cdots|\gamma_i'|\gamma_i''|\cdots|\gamma_n]m$$

$$+ (-1)^{\varepsilon(n+1)} \Sigma\, \gamma[\gamma_1|\cdots|\gamma_n|m']m''$$

where

$$\Delta\gamma = \Sigma\, \gamma' \otimes \gamma'' \quad ,$$

$$\psi m = \Sigma\, m' \otimes m'' \quad ,$$

$$\varepsilon(i) = |\gamma| + |\gamma_1| + \cdots + |\gamma_{i-1}| + i.$$

Let

(2.2.2) $$\Omega(\Gamma;M) = \mathrm{Hom}\ (A, W(\Gamma;M))$$

so that

$$\mathrm{Ext}_{\Gamma}(A,M) = H(\Omega(\Gamma;M)).$$

Write $\Omega(\Gamma;A) = \Omega\Gamma$. The adjointness mentioned above implies

$$\Omega(\Gamma;M)^n = \bar{\Gamma} \otimes_A \overset{n}{\cdots} \otimes_A \bar{\Gamma} \otimes_A M$$

and the differential is now

$$d[\gamma_1|\cdots|\gamma_n]m = [1|\gamma_1|\cdots|\gamma_n]m$$

(2.2.3) $$+ \sum_{i=1}^{n} (-1)^{\varepsilon(i)} [\gamma_1|\cdots|\gamma_i'|\gamma_i''|\cdots|\gamma_n]m$$

$$+ \quad (-1)^{\varepsilon(n+1)} \Sigma[\gamma_1|\cdots|\gamma_n|m']m'' \ .$$

In particular for $n = 0$,

$$d[\ \ ]m = [1]m - \Sigma[m']m'' .$$

If $M = A$ with the comodule structure (2.1.5),

(2.2.4) $$\begin{aligned} d[\ \ ]a &= [1]a - [\eta_L a]1 \\ &= [\eta_R a - \eta_L a] \ . \end{aligned}$$

The existence of this cobar resolution has several useful consequences.

Proposition 2.2.5. Let (A, Γ) be a flat Hopf algebroid. Then any sequence of comodules

(2.2.6) $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

which is short-exact as R-modules induces a long exact sequence in $\mathrm{Ext}_\Gamma(A,\ \ )$.

Proof. Since Γ is right-flat over A, so is $\Omega(\Gamma;A)$. But non-differentially, $\Omega(\Gamma;M) \cong \Omega(\Gamma;A) \otimes_A M$; so (2.2.6) induces the short exact sequence

$$0 \to \Omega(\Gamma;M') \to \Omega(\Gamma;M) \to \Omega(\Gamma;M'') \to 0$$

of DG R-modules. Hence the result.

□

<u>Proposition 2.2.7.</u> Let (A, Γ) be flat and let

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be a sequence of Γ-comodules which is R-exact and such that each I^j is Γ-injective. Then

$$\mathrm{Ext}_\Gamma(A, M) \cong H(\mathrm{Hom}_\Gamma(A, I)).$$

<u>Proof.</u> Form the double complex $\Omega(\Gamma; I)$. Filter on degree in I; in the associated spectral sequence

$$E_\infty = E_2 = H(\mathrm{Hom}_\Gamma(A, I)).$$

Then filter on degree in Ω; in the associated spectral sequence

$$E_\infty = E_2 = \mathrm{Ext}_\Gamma(A, M).$$

The result follows.

□

<u>Proposition 2.2.8.</u> If (A, Γ) is $(n-1)$-connected (2.1.5) and M is an $(m-1)$-connected Γ-comodule, then for $s \geq 0$, $\mathrm{Ext}^s_\Gamma(A, M)$ is $(ns+m-1)$-connected.

<u>Proof.</u> The cobar construction $\Omega(\Gamma; M)$ has this connectivity.

□

(2.2.9) Let (A, Γ), (B, Σ) be two Hopf algebroids over R, and let M be a Γ-comodule and M' a Σ-comodule. Let $M \longrightarrow I$, $M' \longrightarrow I'$ be injective resolutions. Since $\eta \colon M \longrightarrow I$ is A-split there are A-maps $\epsilon \colon I^0 \longrightarrow M$, $s \colon I^n \longrightarrow I^{n-1}$, such that

$$\varepsilon\eta = M, \quad sd + ds = 1 - \eta\varepsilon, \quad s^2 = 0.$$

So also for $M' \to I'$, over B. Then $M \otimes M' \to I \otimes I'$ is $A \otimes B$-split by the retraction $\varepsilon \otimes \varepsilon' : I^0 \otimes (I')^0 \to M \otimes N$ and the homotopy $(s \otimes 1 + \eta\varepsilon \otimes s')$. Clearly $(I \otimes I')^n$ is $\Gamma \otimes \Sigma$-injective; so we have an injective resolution.

Now for any A-module X,

$$\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A X) \cong \mathrm{Hom}_A(A, X) \cong X.$$

Therefore for any B-module Y,

$$\begin{aligned} &\mathrm{Hom}_{\Gamma \otimes \Sigma}(A \otimes B, (\Gamma \otimes_A X) \otimes (\Sigma \otimes_B Y)) \\ \cong\ &\mathrm{Hom}_{\Gamma \otimes \Sigma}(A \otimes B, (\Gamma \otimes \Sigma) \otimes_{A \otimes B} (X \otimes Y)) \\ \cong\ &X \otimes Y \\ \cong\ &\mathrm{Hom}_\Gamma(A, \Gamma \otimes_A X) \otimes \mathrm{Hom}_\Sigma(B, \Sigma \otimes_B Y). \end{aligned}$$

Hence in the situation above,

$$\mathrm{Hom}_{\Gamma \otimes \Sigma}(A \otimes B, I \otimes I') \cong \mathrm{Hom}_\Gamma(A, I) \otimes \mathrm{Hom}_\Sigma(B, I').$$

Note that $\Omega(\Gamma; M)$ is R-flat if M is R-flat and (A, Γ) is flat. Therefore we have ([15], XVII) :

<u>Proposition 2.2.10</u> Let (A, Γ) and (B, Σ) be two Hopf algebroids over R, and let M be a Γ-comodule and N a Σ-comodule. Suppose (A, Γ) is flat and M is R-flat. Then there is a convergent spectral sequence

$$\mathrm{Tor}^R(\mathrm{Ext}_\Gamma(A, M), \mathrm{Ext}_\Sigma(B, N)) \Longrightarrow \mathrm{Ext}_{\Gamma \otimes \Sigma}(A \otimes B, M \otimes N).$$

□

Now suppose $(A,\Gamma) = (B,\Sigma)$. Then the edge-homomorphism of (2.2.11) composes with the map induced by $(\mu,\mu)\colon (A\otimes A, \Gamma\otimes\Gamma) \longrightarrow (A,\Gamma)$ (see 2.1.20)) to give

(2.2.11) $$\nu\colon \mathrm{Ext}_\Gamma(A,M)\otimes \mathrm{Ext}_\Gamma(A,N) \longrightarrow \mathrm{Ext}_\Gamma(A, M\otimes_A^L N).$$

ν is associative and commutative in the obvious sense.

(2.2.12) In order to relate this product to the cobar construction, we need a tensor algebra.

If X is an A-bimodule, let $T_A(X)$ be the R-algebra with underlying R-module

$$\coprod_{n\geq 0} X\otimes_A \overset{n}{\cdots} \otimes_A X$$

and multiplication

$$(x_1\otimes_A\cdots\otimes_A x_i)\otimes(y_1\otimes_A\cdots\otimes_A y_j)$$
$$= x_1\otimes_A\cdots\otimes_A x_i\otimes_A y_1\otimes_A\cdots\otimes_A y_j\,.$$

Note that $\Omega\Gamma \cong T_A(\sigma^{-1}\overline{\Gamma})$ as bigraded R-modules; here σ^{-1} gives a graded R-module a second degree of -1. Inspection shows that this isomorphism makes $\Omega\Gamma$ into a differential R-algebra and $\Omega(\Gamma;M)$ into a differential $\Omega\Gamma$-module. Thus $\mathrm{Ext}_\Gamma(A,A)$ is naturally an R-algebra and $\mathrm{Ext}_\Gamma(A,M)$ an $\mathrm{Ext}_\Gamma(A,A)$-module; and these maps agree with those of (2.2.11).

(2.2.13) There is another expression for the cobar resolution which reveals more structure. Notice that Γ is a comodule-algebra over (A,Γ) with unit η_L and multiplication μ_L (see (2.1.20)). Hence

$$\mathbb{\Gamma}_L = (\Gamma \otimes_A^L \;, \eta_L \otimes_A^L \;, \mu_L \otimes_A^L \;)$$

is a triple on (Γ-comod). Recall from (2.1.20) that the functors $\Gamma \otimes_A$ and $\Gamma \otimes_A^L$ are naturally equivalent. It is easy to check that this equivalence extends to an isomorphism of triples $\mathbb{\Gamma} \cong \mathbb{\Gamma}_L$. Hence the associated canonical resolutions are naturally isomorphic $\mathbb{D}\mathbb{G}$ (A, Γ)-comodules.

If M is a Γ-comodule-algebra then $\mathrm{Hom}_\Gamma(A, S_{\mathbb{\Gamma}_L}(M))$ is a cosimplicial commutative R-algebra. Since $\mathrm{Ext}_\Gamma(A, M)$ is the homology of the normalized complex, it has $\cup_i$-products; and if $R = \mathbb{F}_p$, then it has Steenrod operations. See [41].

(2.2.14) Suppose now that E and V are ring-spectra such that E is V-flat; see (2.1.2). Then $(V_*E, V_*(E \wedge E))$ is a Hopf algebroid. We deduce from (2.1.21) and (2.1.22) that for any spectrum X, the Künneth isomorphism

$$m_X\colon V_*(E \wedge E) \otimes_{V_*E} V_*(E \wedge X) \xrightarrow{\cong} V_*(E \wedge E \wedge X)$$

of (2.1.7) is a map of $V_*(E \wedge E)$-comodules. Thus

$$\mathrm{Hom}_{V_*(E \wedge E)}(V_*E, V_*(E \wedge E \wedge X)) \cong V_*(E \wedge X).$$

By additivity, then, $V_*(E \wedge I)$ is $V_*(E \wedge E)$-injective for any E-injective I, and

$$\mathrm{Hom}_{V_*(E \wedge E)}(V_*E, V_*(E \wedge I)) \cong V_*(I).$$

Furthermore, let $* \longrightarrow X \longrightarrow X''$ be E-exact: i.e., $E \wedge X \longrightarrow E \wedge X''$ is a split monomorphism of E-module-spectra, by (1.1.16). Then

$V_*(E \wedge X) \rightarrow V_*(E \wedge X'')$ is a split monomorphism of V_*E-modules.

It follows that $V_*(E \wedge \ )$ carries a resolution of X relative to E to a resolution of $V_*(E \wedge X)$ as a $V_*(E \wedge E)$-comodule, and that

(2.2.15) $$R_E^* V_*(X) \cong \mathrm{Ext}^{**}_{V_*(E \wedge E)}(V_*E, V_*(E \wedge X)).$$

Arguments analogous to those of [16] prove that for spectra X and Y there is a pairing

$$\nu: E_r(X; E) \otimes E_r(Y; E) \longrightarrow E_r(X \wedge Y; E)$$

of spectral sequences agreeing at E_2 with the composite of (2.2.11) and (2.1.22) and associated at E_∞ with the pairing

$$\nu: V_*(X) \otimes V_*(Y) \longrightarrow V_*(X \wedge Y).$$

(2.2 16) If the spectral sequence

(2.2.17) $$\mathrm{Ext}_{V_*(E \wedge E)}(V_*E, V_*(E \wedge X)) \Longrightarrow V_*(X)$$

is known to converge at the connective spectrum X and if $V_*(E \wedge E)$ is 1-connected (2.1.13), then (2.2.17) converges classically by (2.2.8). For example let V be the Moore-spectrum with homology $\mathbb{F}_p$, p odd prime, and $E = H\mathbb{Z}_{(p)}$. The associated Hopf algebroid $A(1)_*$, (2.1.15), is (2p-3)-connected, so here convergence is classical. So also if $V = S$ and $E = MU$ or $E = BP$ for any prime p. Indeed, (2.2.8) gives the Novikov-Zahler vanishing line [50], [62] for MU and for BP at $p = 2$.

§2.3 Split Hopf algebroids

We describe here an important special class of Hopf algebroids. For motivation let G be a group (in the category of sets) and let G act from the right on the set X. Define a groupoid $X \tilde{\times} G$ with object set X and, for $x, y \in X$,

$$(X \tilde{\times} G)(x, y) = \{g \in G: xg = y\}.$$

The structure maps are the obvious ones. A groupoid is <u>split</u> if it is isomorphic to one of this form.

Now recall the situation envisaged by Landweber [29]. Let S be a commutative Hopf algebra (with involution) over a commutative ring R and let A be a right S-comodule-algebra with coaction ψ. (Remember that all our R-algebras are commutative.) Thus S is a cogroup in the category of R-algebras, coacting from the right on A. We construct a Hopf algebroid $(A, A \tilde{\otimes} S)$ with cooperation algebra $A \otimes S$ and structure maps given by

(2.3.1):
$$\eta_L = A \otimes \eta: A \cong A \otimes R \longrightarrow A \otimes S$$
$$\eta_R = \psi: A \longrightarrow A \otimes S$$
$$\epsilon = A \otimes \epsilon: A \otimes S \longrightarrow A \otimes R \cong A$$
$$\Delta = A \otimes \Delta: A \otimes S \longrightarrow A \otimes S \otimes S \cong (A \otimes S) \otimes_A (A \otimes S)$$
$$c = (A \otimes \mu)(\psi \otimes c): A \otimes S \longrightarrow A \otimes S \otimes S \longrightarrow A \otimes S\ .$$

A Hopf algebroid is <u>split</u> iff it is isomorphic to one of this form.

The Hopf structure of S equips (comod-S) with an internal "diagonal" tensor product $\otimes^\Delta$. A is just a commutative $\otimes^\Delta$-algebra. Still following Landweber, consider the category (A-mod/S) of left A-modules over S:

that is, A-modules M together with a right S-coaction ψ such that $A \otimes^{\Delta} M \longrightarrow M$ is an S-comodule map.

We wish to compare (A-mod/S) with $(A \tilde{\otimes} S\text{-comod})$. To this end, define for any R-module M two R-module maps, f and g, as follows.

$$g: M \otimes S \longrightarrow (A \tilde{\otimes} S) \otimes_A M$$

by $$g(m \otimes s) = (-1)^{|m|\,|s|} \otimes c(s) \otimes m.$$

$$f: (A \tilde{\otimes} S) \otimes_A M \longrightarrow M \otimes S$$

by $$f(a \otimes s \otimes m) = \Sigma(-1)^{|m|(|a''|+|s|)} a'm \otimes a''c(s),$$

where $\psi: A \longrightarrow A \otimes S$ by $\psi(a) = \Sigma\, a' \otimes a''$. The reader may check that f actually factors through the tensor product over A. Now an easy verification yields:

Lemma 2.3.2. The correspondences

$$G: (M, \psi) \longmapsto (M, g\psi)$$

$$(M, f\Psi) \longleftarrow\!\shortmid\; (M, \Psi): F$$

define inverse functors

$$(A\text{-mod}/S) \underset{F}{\overset{G}{\rightleftarrows}} (A \tilde{\otimes} S\text{-comod}).$$

$\square$

This equivalence frequently reduces $\mathrm{Ext}_{A \tilde{\otimes} S}$ to a "classical" object. To see this we first construct adjoint pairs

$$(2.3.3)\qquad \begin{tikzcd} (A\text{-mod}) \arrow[r, shift left, "\otimes S"] & (A\text{-mod}/S) \arrow[l, shift left, "u_A"] \arrow[r, shift left, "u_S"] & (\text{comod-}S) \arrow[l, shift left, "A\otimes"] \end{tikzcd}$$

$$u_A \dashv \otimes S \qquad\qquad A\otimes \dashv u_S$$

where u_A, u_S are the forgetful functors. The A-actions φ and S-coactions ψ are given, for A-module Y and S-comodule X, by

$$\varphi_{A\otimes X} = \mu\otimes X : A\otimes A\otimes X \longrightarrow A\otimes X$$

$$\begin{tikzcd} A\otimes X \arrow[r, "\psi_A\otimes\psi_X"] \arrow[drr, "\psi_{A\otimes X}"'] & A\otimes S\otimes X\otimes S \arrow[r, "A\otimes T\otimes S"] & A\otimes X\otimes S\otimes S \arrow[d, "A\otimes X\otimes\mu"] \\ & & A\otimes X\otimes S \end{tikzcd}$$

$$\begin{tikzcd} A\otimes Y\otimes S \arrow[r, "\psi_A\otimes Y\otimes S"] \arrow[drr, "\varphi_{Y\otimes S}"'] & A\otimes S\otimes Y\otimes S \arrow[r, "A\otimes T\otimes S"] & A\otimes Y\otimes S\otimes S \arrow[d, "\varphi_Y\otimes\mu"] \\ & & Y\otimes S \end{tikzcd}$$

$$\psi_{Y\otimes S} = Y\otimes\Delta : Y\otimes S \longrightarrow Y\otimes S\otimes S .$$

The adjunction morphisms may easily be written down.

Recall the adjoint pair

$$\begin{tikzcd} (A\text{-mod}) \arrow[r, shift left, "(A\tilde{\otimes} S)\otimes_A"] & (A\tilde{\otimes} S\text{-comod}) \arrow[l, shift left, "U"] \end{tikzcd}$$

$$U \dashv (A\tilde{\otimes} S)\otimes_A \quad .$$

Since the inverse functors F and G leave the A-action alone, $UF = u_A$, and so for an A-module Y,

$$F((A \tilde{\otimes} S) \otimes_A Y) \cong Y \otimes S$$

in (A-mod/S). But the S-comodule $u_S(Y \otimes S)$ is injective. Thus $u_S F$ carries injectives to injectives; and it is clearly exact.

So let X be a right S-comodule and N an $A \tilde{\otimes} S$-comodule, and let $N \rightarrowtail I$ be an injective resolution. Then

$$\begin{aligned} &\mathrm{Ext}_{A \tilde{\otimes} S}(G(A \otimes X), N) \\ &\quad = H\,\mathrm{Hom}_{A \tilde{\otimes} S}(G(A \otimes X), I) \\ &\quad = H\,\mathrm{Hom}_{A/S}(A \otimes X,\ FI) \\ &\quad = H\,\mathrm{Hom}_S(X, u_S FI) \\ &\quad = \mathrm{Ext}_S(X, u_S FN). \end{aligned}$$

Here $\mathrm{Hom}_{A/S}$ denotes the obvious graded R-module of A-linear, S-colinear maps.

Since F, G, and u_S leave the R-module structure untouched, we omit them in stating

<u>Proposition 2.3.4.</u> Let X be a right S-comodule and N an $A \tilde{\otimes} S$-comodule. Then

$$\mathrm{Ext}_{A \tilde{\otimes} S}(A \otimes X, N) \cong \mathrm{Ext}_S(X, N).$$

□

We remark that throughout this section R could have been graded; the results would be the same. The proof actually yields a natural, multiplicative isomorphism

$$\Omega(A\tilde{\otimes}S;N) \cong \Omega(S;N)$$

between cobar constructions.

(2.3.5) Finally we indicate a different type of Hopf algebroid, which will be useful in §2.5. It is analogous to a groupoid with exactly one morphism between any two objects. So let B be a commutative R-algebra. The unicursal Hopf algebroid on B is $(B, B\otimes_R B)$ with $\eta_L(b) = b\otimes 1$, $\eta_R(b) = 1\otimes b$, $\varepsilon(b_1\otimes b_2) = b_1 b_2$, $\Delta(b_1\otimes b_2) = (b_1\otimes 1)\otimes(1\otimes b_2)$, $c(b_1\otimes b_2) = (-1)^{|b_1|\,|b_2|}\,b_2\otimes b_1$.

Let $C(B)$ be the core [11] subalgebra

$$C(B) = \{b\in B\colon b\otimes 1 = 1\otimes b \in B\otimes_R B\}.$$

Lemma 2.3.6. We have

$$\mathrm{Ext}^q_{B\otimes B}(B,B) = \begin{cases} C(B) & \text{if } q = 0 \\ 0 & \text{if } q > 0\,. \end{cases}$$

Proof. Since $0 \longrightarrow B \longrightarrow B \longrightarrow 0$ is an injective resolution, the result follows from (2.1.12). □

§2.4 Cotensor and change of Hopf algebroids

It will be convenient to have a cotensor product over the Hopf algebroid (A,Γ). So let M be a right Γ-comodule and N a left Γ-comodule. The R-module $M\square_\Gamma N$ is the difference-kernel of

$$M \otimes_A N \underset{M \otimes_A \psi}{\overset{\psi \otimes_A N}{\rightrightarrows}} M \otimes_A \Gamma \otimes_A N .$$

For example if $M = A$ then (cf. (2.1.11))

(2.4.1) $\quad A \,\square_\Gamma\, N = \{n \in N \colon \psi(n) = 1 \otimes_A n\} = \mathrm{Hom}_\Gamma(A, N).$

The functors $M \,\square_\Gamma$ and $\square_\Gamma\, N$ are left-exact; and the functor $\square_\Gamma$ is balanced [23] (relative to the injective class of §2.2). Let $\mathrm{Cotor}_\Gamma(M,N)$ denote the derived functor; $\mathrm{Cotor}^0_\Gamma(M,N) = M \,\square_\Gamma\, N$. From (2.4.1), there is a canonical isomorphism

(2.4.2) $$\mathrm{Cotor}_\Gamma(A, N) \cong \mathrm{Ext}_\Gamma(A, N).$$

Let (A, Γ), (B, Σ) be two Hopf algebroids. In the "situation" (cf. [15], II §3)

$$L_\Gamma, \quad {}_\Gamma M_\Sigma, \quad {}_\Sigma N,$$

we have a natural associativity isomorphism

(2.4.3) $$L \,\square_\Gamma\, (M \,\square_\Sigma\, N) \cong (L \,\square_\Gamma\, M) \,\square_\Sigma\, N.$$

Let $\varphi \colon (A, \Gamma) \longrightarrow (B, \Sigma)$ be a map of Hopf algebroids. Recall from (2.1.19) the functor

$$\otimes_A B \colon (\text{comod-}\Gamma) \longrightarrow (\text{comod-}\Sigma).$$

Note that $\Gamma \otimes_A B$ has a natural Γ-Σ-bicomodule structure. For a Γ-comodule M we have a natural isomorphism

(2.4.4) $$M \square_{\Gamma} (\Gamma \otimes_A B) \cong M \otimes_A B$$

of Σ-comodules.

Proposition 2.4.5. Let $\varphi\colon (A, \Gamma) \longrightarrow (B, \Sigma)$ be a map of Hopf algebroids, M a right Γ-comodule, and N a left Σ-comodule. There is a natural spectral sequence converging to

$$\mathrm{Cotor}_{\Sigma}(M \otimes_A B, N)$$

with

$$E_2 = \mathrm{Cotor}_{\Gamma}(M, \mathrm{Cotor}_{\Sigma}(\Gamma \otimes_A B, N)).$$

Proof. Let $M \longrightarrow I$, $N \longrightarrow J$ be injective resolutions over Γ, Σ. Form

$$X = (I \otimes_A B) \square_{\Sigma} J \cong I \square_{\Gamma} ((\Gamma \otimes_A B) \square_{\Sigma} J)$$

(using (2.4.3) and (2.4.4)).

Filter X on homological degree in J. Since $0 \longrightarrow M \longrightarrow J$ is A-split exact, $0 \longrightarrow M \otimes_A B \longrightarrow J \otimes_A B$ is exact and

$$E_1 = (M \otimes_A B) \square_{\Sigma} J;$$

$$H(X) = E_2 = \mathrm{Cotor}_{\Sigma}(M \otimes_A B, N).$$

Filter X on homological degree in I. Since I is Γ-injective,

$$E_1 = I \square_{\Gamma} \mathrm{Cotor}_{\Sigma}(\Gamma \otimes_A B, N),$$

$$E_2 = \mathrm{Cotor}_{\Gamma}(M, \mathrm{Cotor}_{\Sigma}(\Gamma \otimes_A B, N)).$$

$\square$

(2.4.6) To motivate an important application of (2.4.5) let G be a groupoid (in the category of sets). Call $X \subset \mathrm{ob}\, G$ invariant iff all morphisms starting in X end in X. Then the restriction of G to X is the groupoid $G|X$ with object set X and $(G|X)(x,y) = G(x,y)$.

A special case of the corresponding construction for Hopf algebroids is the following. Let (A,Γ) be a Hopf algebroid. An ideal $I \subset A$ is invariant iff $I\Gamma = \Gamma I$ as submodules of Γ. Then the structure maps of (A,Γ) factor to give a Hopf algebroid structure to (B,Σ), where

(2.4.7) $$B = A/I, \quad \Gamma/I\Gamma = \Sigma = \Gamma/\Gamma I.$$

Let M be an (A,Γ)-comodule such that $IM = 0$. Then the Γ-coaction can be written

$$\psi\colon M \longrightarrow \Gamma \otimes_A M = \Gamma \otimes_A (B \otimes_B M) = \Sigma \otimes_B M.$$

The first part of the next proposition is then clear.

Proposition 2.4.8. With the above notations, the category of (A,Γ)-comodules annihilated by I is equivalent to the category of (B,Σ)-comodules. Furthermore we have an isomorphism

$$\mathrm{Cotor}_\Gamma(M,N) \cong \mathrm{Cotor}_\Sigma(M/MI, N),$$

natural in M_Γ, ${}_\Sigma N$.

Proof. In the spectral sequence (2.4.5), $E_2 = \mathrm{Cotor}_\Gamma(M,N)$ is concentrated on an axis; hence the edge-homomorphism is an isomorphism. □

§2.5 The Hopf algebroids of MU and BP

In this section we recall some standard notation and basic data about operations in complex bordism $MU_*(\ \ )$ and in Brown-Peterson homology $BP_*(\ \ )$. For background see [43], [13], [53], [4].

There exist elements $x_{2i} \in MU_{2i}$, $b_{2i} \in MU_{2i}(MU)$, such that

$$MU_* = \mathbb{Z}[x_2, x_4, \dots],$$
$$MU_*MU = MU_*[b_2, b_4, \dots].$$

Thus MU_*MU is flat over MU_*, and (MU_*, MU_*MU) is a Hopf algebroid. It is split in a well-known way. In fact MU_* is a comodule-algebra over the dual Landweber-Novikov algebra $S_* = \mathbb{Z}[b_2, b_4, \dots]$ ([27], [50], [46]) in such a way that

$$MU_*MU = MU_* \tilde{\otimes} S_*.$$

From Proposition 2.3.3 we have therefore:

<u>Corollary 2.5.1</u>. For any MU_*MU-comodule M,

$$\mathrm{Ext}_{MU_*MU}(MU_*, M) \cong \mathrm{Ext}_{S_*}(\mathbb{Z}, M).$$

□

In particular, for any spectrum X,

$$E_2(X; MU) \cong \mathrm{Ext}_{S_*}(\mathbb{Z}, MU_*(X)).$$

We turn now to the Brown-Peterson spectrum [13]. Let p be any rational prime. Quillen [53] constructs a ring-spectrum BP and ring-spectrum maps

$$BP \xrightarrow{\iota} MU_{(p)} \xrightarrow{\pi} BP$$

such that (i) $\pi\iota = BP$ and (ii) the idempotent $\varepsilon = \iota\pi$ acts in $\pi_*(MU)$ by

$$\epsilon_*(P_n) = \begin{cases} P_n & \text{if } n = p^j - 1 \text{ for some } j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where P_n is the bordism class of complex projective n-space. It follows that

$$\pi_*(BP) = \mathbb{Z}_{(p)}[\pi_* x_{2p-2}, \pi_* x_{2p^2-2}, \ldots]$$

and $\pi_*(MU_{(p)}) \cong B \otimes \pi_*(BP)$ where

$$B = \mathbb{Z}_{(p)}[x_{2i} : i \neq p^j - 1 \text{ for any } j].$$

Thus the natural maps

$$MU_{(p)*} \otimes_{BP_*} BP_*(X) \longrightarrow MU_{(p)*}(X)$$

$$BP_* \otimes_{MU_*} MU_*(X) \longrightarrow BP_*(X)$$

are isomorphisms which respect external products. In particular

$$\begin{aligned} \mathbb{Z}_{(p)} \otimes MU_*MU &\cong MU_{(p)*} MU_{(p)} \\ &\cong MU_{(p)*} \otimes_{BP_*} BP_*BP \otimes_{BP_*} MU_{(p)*}. \end{aligned}$$

Thus this Hopf algebroid splits as a tensor-product

$$\mathbb{Z}_{(p)} \otimes MU_*MU \cong (B \otimes B) \otimes BP_*BP$$

where $(B, B \otimes B)$ is the unicursal Hopf algebroid (2.3.5) on B. Now the core (2.3.5) of the $\mathbb{Z}_{(p)}$-algebra B is $\mathbb{Z}_{(p)}$, so (2.3.6) and (2.2.10) imply that for any BP_*BP-comodule M

$$\begin{aligned}
&\mathrm{Ext}_{MU_*MU}(MU_*, MU_{(p)*} \otimes_{BP_*} M) \\
&\cong \mathrm{Ext}_{\mathbb{Z}_{(p)} \otimes MU_*MU}(MU_{(p)*}, MU_{(p)*} \otimes_{BP_*} M) \\
&\cong \mathrm{Ext}_{(B \otimes B) \otimes BP_*BP}(B \otimes BP_*, B \otimes M) \\
&\cong \mathrm{Ext}_{BP_*BP}(BP_*, M).
\end{aligned}$$

In particular, if X is a spectrum then

$$\mathbb{Z}_{(p)} \otimes \mathrm{Ext}_{MU_*MU}(MU_*, MU_*(X)) \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X)).$$

To describe (BP_*, BP_*BP), recall ([43], [53], [4]) that it embeds in the Hopf algebroid $(H_*(BP), H_*(BP \wedge BP))$. We have, with

$$m_i = \frac{\pi_* P_{2p^i-2}}{2p^i-1},$$

$$H_*(BP) = \mathbb{Z}_{(p)}[m_1, m_2, \ldots],$$

$$H_*(BP \wedge BP) = H_*(BP)[t_1, t_2, \ldots],$$

where $|m_i| = 2p^i-2 = |t_i|$ and where the t_i's are described inductively by (2.5.3) below. Then:

(2.5.2) $$\eta_L(m_k) = m_k$$

(2.5.3) $$\eta_R(m_k) = \sum_{i+j=k} m_i t_j^{p^i}$$

(2.5.4) $$\varepsilon(t_k) = 0$$

(2.5.5) $$\sum_{i+j=k} m_i (\Delta t_j)^{p^i} = \sum_{h+i+j=k} m_h t_i^{p^h} \otimes t_j^{p^{h+i}}$$

(2.5.6) $$m_k = \sum_{h+i+j=k} m_h t_i^{p^h} (ct_j)^{p^{h+i}} .$$

These formulae may be effectively translated to BP_* by means of the generators v_i, $i \geq 1$, of Hazewinkel [21], described inductively by

(2.5.7) $$pm_k = \sum_{i=0}^{k-1} m_i v_{k-i}^{p^i} .$$

It is frequently convenient to set $v_0 = p$.

We insert here a short remark on why (MU_*, MU_*MU) and (BP_*, BP_*BP) are cogroupoids in the category of R-algebras. Quillen's theorem [53] asserts that MU_* represents the functor Λ carrying a $\mathbb{Z}$-algebra R to the set of commutative one-dimensional formal groups of degree -2 over R ("formal groups over R," for short). Morava [46] shows how to interpret the dual Landweber-Novikov algebra S_* as the $\mathbb{Z}$-algebra representing the functor Γ_0 carrying the $\mathbb{Z}$-algebra R to the group of strict isomorphisms of formal groups over R — i.e., the group of formal power series of the form $1 + a_1 T + \ldots$, $a_i \in R_{2i}$, under composition. The diagonal in S_* represents composition. Now the groupoid of formal groups and strict isomorphisms over R is, in the notation of (2.3.1), $\Lambda(R) \tilde{\times} \Gamma_0(R)$. This splitting is functorial, so the representing Hopf algebroid is split: $MU_* \tilde{\otimes} S_*$. This is MU_*MU.

Passing to BP, P. S. Landweber [31] has shown that the Hopf algebroid (BP_*, BP_*BP) represents the functor carrying a $\mathbb{Z}_{(p)}$-algebra R to the groupoid of p-typical formal groups and strict isomorphisms over R.

If M is a BP_*BP-comodule with coaction

$$\psi: M \longrightarrow BP_*BP \otimes_{BP_*} M$$

let

(2.5 8) $$\psi(m) = \sum_E c(t^E) \otimes r_E(m)$$

define for any multiindex $E = (e_1, e_2, \ldots)$, e_i non-negative integers almost all of which are zero, an operation $r_E: M \longrightarrow M$. Recall that BP_* is a left comodule by means of

$$\eta_L: BP_* \longrightarrow BP_*BP \cong BP_*BP \otimes_{BP_*} BP_*.$$

Now a short computation reveals that for any $x \in BP_*$,

(2.5.9) $$\eta_R(x) = \sum_E r_E(x) t^E.$$

Let I_n denote the ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$; $I_0 = (0)$; $I = (p, v_1, \ldots)$. It is easy to see that these are invariant ideals. Let M be a BP_*BP-comodule annihilated by I_n. By (2.4.8) there is an isomorphism, natural in M:

(2.5.10) $$\mathrm{Ext}_{BP_*BP}(BP_*, M) \cong \mathrm{Ext}_{BP_*BP/I_n}(BP_*/I_n, M).$$

The following computation is of fundamental importance.

Theorem 2.5.11. (Landweber [28]; see also [26].)

$$\mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*) \cong \mathbb{Z}_{(p)}$$

$$\mathrm{Ext}^0_{BP_*BP}(BP_*, BP_*/I_n) \cong \mathbb{F}_p[v_n], \quad n \geq 1.$$

$\square$

For an application, let G be the Adams summand of connective K-theory localized at the prime p ([3],[7],[25]). There is a ring-spectrum map $BP \longrightarrow G$, and

$$G_* = BP_*/(v_2, v_3, \ldots) \cong \mathbb{Z}_{(p)}[v_1].$$

<u>Corollary 2.5.12</u>. (Stong-Hattori [7]). The Hurewicz map

$$\pi_*(BP) \longrightarrow G_*(BP)$$

is a split monomorphism.

<u>Proof</u>. It suffices to show that for each prime p,

$$\eta\colon BP_*(S\mathbb{F}_p) \longrightarrow BP_*(G\mathbb{F}_p)$$

is a monomorphism. Since $G\mathbb{F}_p$ is a BP-module-spectrum it is BP-injective, so

$$\begin{aligned}\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(G\mathbb{F}_p)) &= E_2(G\mathbb{F}_p; BP)\\ &= \pi_*(G\mathbb{F}_p) = \mathbb{F}_p[v_1]\end{aligned}$$

concentrated in homological degree 0. From (2.5.11) (n=1), we see that $\mathrm{Ext}^0_{BP_*BP}(BP_*, \eta)$ is an isomorphism, so by the Nakayama lemma (2.1.17), η is a monomorphism.

<u>Remarks</u>. Larry Smith's original proof [55] of (2.5.11) (n=1) relied on the Stong-Hattori theorem. From our point of view this implication is immediate. On the other hand, [26] gives a proof of another version of the Stong-Hattori theorem using the classical Nakayama lemma.

CHAPTER THREE

THE ALGEBRAIC NOVIKOV SPECTRAL SEQUENCE

§3.0 Introduction

In this chapter we construct a spectral sequence by filtering the cobar construction of a Hopf algebroid by powers of an invariant ideal in the coefficient algebra. It may be regarded as an analogue of the May spectral sequence [37],[38]. It was first constructed, in the dual context and in a special case, by Novikov [50], and we refer to it as the algebraic Novikov spectral sequence. The special case of BP_*BP is studied in Section 3.2.

§3.1 The Novikov spectral sequence

We begin with some remarks on filtrations. To keep the indices under control, we relapse briefly to lower indices.

Let R be a commutative ring, A a graded R-algebra, and $I \subset A$ an ideal. Every A-module M is naturally filtered by the <u>I-adic</u> filtration

$$\begin{aligned} F_{-n}M &= I^n M \qquad 0 < n \\ F_{-n}M &= M \qquad -\infty \le n \le 0. \end{aligned}$$

In this filtration A is a filtered R-algebra and M is a filtered module over this filtered algebra. Thus E^0A is naturally a bigraded R-algebra and E^0M is naturally an E^0A-module.

Let M be a right A-module and N a left A-module. The tensor-product $M \otimes_A N$ is filtered by

$$F_{-n}(M \otimes_A N) = \operatorname{im} \sum_{i+j=n} MI^i \otimes_A I^j N$$

$$= \operatorname{im}(M \otimes_A F_{-n} N).$$

Thus if M is flat then

(3.1.1)
$$\begin{aligned} E^0(M \otimes_A N) &\cong M \otimes_A E^0 N \\ &\cong (M \otimes_A E_0^0 A) \otimes_{E_0^0 A} E^0 N \end{aligned}$$

where A acts on $E^0 N$ through $E_0^0 A$. But for any M

$$E^0 M \otimes_{E^0 A} E_0^0 A \cong E_0^0 M \cong M \otimes_A E_0^0 A.$$

Thus:

Lemma 3.1.2. Let $I \subset A$ be an ideal, M a flat right A-module, and N a left A-module. Then

$$E^0(M \otimes_A N) \cong E^0 M \otimes_{E^0 A} E^0 N.$$

□

Now let (A, Γ) be a Hopf algebroid over R. Suppose that $I \subset A$ is an invariant ideal: that is, $I\Gamma = \Gamma I$ as submodules of Γ. Then the right and left I-adic filtrations on Γ agree and filter Γ by sub-A-bimodules. By naturality of the filtration, the structure maps η_L, η_R, ε, c, and Δ are all filtration-preserving.

Suppose (A, Γ) is flat: that is, Γ is A-flat. Then by Lemma 3.1.2,

$$E^0(\Gamma \otimes_A \Gamma) \cong E^0 \Gamma \otimes_{E^0 A} E^0 \Gamma.$$

We may therefore form a bigraded Hopf algebroid $(E^0A,\ E^0\Gamma)$. Furthermore any (A,Γ)-comodule M, when filtered by the I-adic filtration, gives an $(E^0A,\ E^0\Gamma)$-comodule E^0M.

We now filter the cobar construction. Recall first that if X is a filtered graded R-module, then the t-fold suspension $\sigma^t X$ is filtered by

$$F_n(\sigma^t X) = \sigma^t F_{n-t} X, \quad -\infty < n \leq \infty,$$

so that, using (filtration, total) degrees,

$$E^0(\sigma^t X) = \sigma^{t,t} E^0 X.$$

Filter $\Omega(\Gamma; M) = T_A(\sigma^{-1}\overline{\Gamma}) \otimes_A M$ accordingly. It is easy to check that the differential strictly decreases filtration, so there arises a spectral sequence with

$$E^1 = E^0 = E^0\,\Omega(\Gamma; M).$$

By the considerations above,

$$E^1 \cong \Omega(E^0\Gamma;\ E^0M)$$

and one may check that the differential is correct. Thus

$$\text{(3.1.3)} \qquad E^2 = \operatorname{Ext}^{E^0\Gamma}(E^0A,\ E^0M) \Longrightarrow \operatorname{Ext}^{\Gamma}(A, M).$$

This is a fourth-octant homology spectral sequence. At this point we return to upper indices, so it becomes an eighth-octant cohomology spectral sequence.

Since the filtration is negative, it is clear that

$$(E_0\Gamma \otimes_{E_0A} E_0\Gamma)^0 = E_0^0\Gamma \otimes_{E_0^0A} E_0^0\Gamma .$$

Let $E_0^0\Gamma = S$, $E_0^0A = K$. Then clearly (K, S) forms a Hopf algebroid. Furthermore, since Γ is flat over A, (3.1.1) gives

$$\begin{aligned} E_0\Gamma &= E_0A \otimes_K (K \otimes_A \Gamma) \\ &= E_0A \otimes_K S . \end{aligned}$$

Suppose that

$$E_0^0\eta_R = E_0^0\eta_L : K \longrightarrow S.$$

Then S is a Hopf algebra over K (§2.1). $(E_0A, E_0\Gamma)$ may also be regarded as a Hopf algebroid over K. Under

$$E_0\eta_R : E_0A \longrightarrow E_0\Gamma = E_0A \otimes_K S,$$

E_0A becomes an S-comodule-algebra and (§2.3)

$$(E_0A, E_0\Gamma) \cong (E_0A, E_0A \tilde{\otimes}_K S).$$

Thus in this situation, Proposition 2.3.3 gives

(3.1.4) $$E_2 = \mathrm{Ext}_S(K, E_0M)$$

in the algebraic Novikov spectral sequence.

(3.1.5) Suppose $M = A$. Then the I-adic filtration on $\Omega\Gamma = \Omega(\Gamma; A)$ respects the algebra structure (2.2.12), and (3.1.3) is a spectral sequence of algebras.

We mention two convergence conditions.

(3.1.6) Suppose that (A, Γ) is connected (2.1.13), that $I_0 = 0$, and that M is connective. Then the convergence is "classical" : the filtration is finite in each degree, and for any tridegree (s,t,u), there exists r such that that for $r' \geq r$, $E_r^{s,t,u} = E_{r'}^{s,t,u}$.

(3.1.7) Suppose that R is Noetherian, that $J \subset R$ is an ideal such that $I = JA$, and that A, Γ, and M are of finite type over R. Let $\hat{R}$ be the completion of R at J. Then the completion of $\Omega(\Gamma; M)$ at I is $\hat{R} \otimes \Omega(\Gamma; M)$; so the spectral sequence converges in the sense of [17] to $\hat{R} \otimes \mathrm{Ext}_\Gamma(A, M)$.

§3.2 The Novikov spectral sequence for BP_*BP

We refer to §2.5 for notations and definitions surrounding the Hopf algebroid BP_*BP. In particular, $I = (p, v_1, \ldots) \subset BP_*$ is an invariant ideal, so an algebraic Novikov spectral sequence is defined for any BP_*BP-comodule M. Furthermore $E_0^0 BP_* = k = \mathbb{F}_p$, so $E_0^0 \eta_L = E_0^0 \eta_R$, and (3.1.3) gives

(3.2.1) $$E_2 = \mathrm{Ext}_{P_*}(k, E_0 M)$$

where

(3.2.2) $$P_* = E_0^0 BP_*BP = k[t_1, t_2, \ldots].$$

We compute the diagonal Δ in P_* and the coaction of P_* on

(3.2.3) $$Q_*^* = E_0 BP_* = k[v_0, v_1, \ldots].$$

Here $v_n \in Q^1_{2p^n-2}$ (as in (3.1.3)) and v_0 is the class of $p \in BP_0$.

Proposition 3.2.4. In BP_*BP,

(i) $$\eta_R v_n \equiv pt_n + \sum_{i=1}^{n} v_i t_{n-i}^{p^i} \qquad \text{mod } I^2$$

(ii) $$\Delta t_n \equiv \sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i} \qquad \text{mod } I .$$

Proof. (i) Consider

(3.2.5)
$$\begin{tikzcd} \pi_*(BP) \arrow[r, "h"] \arrow[d] & H_*(BP; \mathbb{Z}_{(p)}) \arrow[d] \\ \pi_*(BP)/I^2 \arrow[r, dashed, "\bar{h}"] & H_*(BP; \mathbb{Z}/p^2\mathbb{Z}) \end{tikzcd}$$

The definition (2.5.7) of Hazewinkle's generators implies that for all $n \geq 1$,

(3.2.6) $$p \mid h(v_n),$$

(3.2.7) $$p^2 \nmid h(v_n).$$

By (3.2.6) $\bar{h}$ exists and by (3.2.7) it injects. The right unit η_R maps (3.2.5) to the similar diagram featuring $BP \wedge BP$. Let a bar denote any of the vertical reduction homomorphisms. Then (2.5.7) implies

$$p\,\overline{m}_n = \bar{h}(\bar{v}_n) .$$

Thus using (2.5.3),

$$\bar{h}\bar{\eta}_R(\bar{v}_n) = \bar{\eta}_R\,\bar{h}(\bar{v}_n)$$

$$= \bar{\eta}_R(p\,\overline{m}_n)$$

$$= \sum_{i=0}^{n} p\,\overline{m}_i\,\bar{t}_{n-i}^{p^i} .$$

This implies (i).

The idea of working in $H_*(; \mathbb{Z}/p^2\mathbb{Z})$ is due to Dave Johnson.

(ii) Using (3.2.6) construct $\bar{h}$, $\bar{h}'$ in (3.2.8), and using (3.2.7) show they inject.

(3.2.8)
$$\begin{tikzcd} \pi_*(BP \wedge BP)/I \arrow[r, "\bar{h}"] \arrow[d, "\Delta/I"] & H_*(BP \wedge BP; \mathbb{Z}/p\mathbb{Z}) \arrow[d, "\Delta/(p)"] \\ \pi_*(BP \wedge BP \wedge BP)/I \arrow[r, "\bar{h}'"] & H_*(BP \wedge BP \wedge BP; \mathbb{Z}/p\mathbb{Z}). \end{tikzcd}$$

Thus we compute using (2.5.5) mod p. Clearly $\Delta t_1 = t_1 \otimes 1 + 1 \otimes t_1$; and the result follows by induction. □

Corollary 3.2.9. With the above notations,

(i) Q_*^* is a right P_*-comodule-algebra with coaction determined by

$$\psi v_n = \sum_{i=0}^{n} v_i \otimes t_{n-i}^{p^i} ;$$

(ii) the diagonal in P_* is given by

$$\Delta t_n = \sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i} .$$

□

Notice that P_* is precisely the Hopf algebra of dual Steenrod reduced powers, with $t_n = \overline{\xi}_n$ conjugate to the Milnor generator ξ_n. The Thom map $BP \longrightarrow H = H\mathbb{F}_p$ is compatible with this identification: consider

$$\begin{array}{ccc}
E_0 BP_*(X) & \longrightarrow & H_*(X) \\
\downarrow{\scriptstyle E_0\psi} & & \downarrow{\scriptstyle \psi} \\
E_0 BP_*BP \otimes_{Q_*} E_0 BP_*(X) & \longrightarrow & A_* \otimes H_*(X) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\
E_0 BP_*(X) \otimes P_* & \longrightarrow & H_*(X) \otimes A_*
\end{array}$$

in which each horizontal arrow is induced from the Thom map and f is as in §2.3. Thus $E_0 BP_*(X) \longrightarrow H_*(X)$ is a map of right A_*-comodules, using the "corestricted" coaction on the left and the right coaction conjugate to the usual one on $H_*(X)$.

Now suppose that $H_*(X; \mathbb{Z}_{(p)})$ is torsion-free of finite type. Then (1) the Atiyah-Hirzebruch spectral sequence converging to $BP_*(X)$ collapses, and (2) the A_*-coaction on $H_*(X)$ factors through P_*. (2) implies that the H-Adams spectral sequence at $BP \wedge X$ has

$$\begin{aligned} E_2 &= \mathrm{Ext}_{A_*}(k, P_* \otimes^{\Delta} H_*(X)) \\ &\cong \mathrm{Ext}_{E_*}(k, k) \otimes H_*(X), \end{aligned}$$

$E_* = E[\overline{\tau}_0, \overline{\tau}_1, \ldots]$; and (1) implies that $E_2 = E_\infty$ by a dimension count. Thus the H-Adams filtration on $BP_*(X)$ coincides with the I-adic filtration.

We may identify

(3.2.10) $$\mathrm{Ext}_{E_*}(k,k) = Q_*^* .$$

Then the multiplicative structure of the H-Adams spectral sequence implies that

(3.2.11) $$E_0 BP_*(X) \cong Q_*^* \otimes H_*(X)$$

as Q_*^*-modules.

Combining these two remarks, we see that (3.2.11) is an isomorphism of Q_*^*-modules over P_*, and the algebraic Novikov spectral sequence has

(3.2.12) $$E_2 = \mathrm{Ext}_{P_*}(k, Q_*^* \otimes^{\Delta} H_*(X)).$$

We now describe a reinterpretation of this group, due to Novikov [50]. Consider the multiplicative extension sequence

(3.2.13) $$P_* \longrightarrow A_* \longrightarrow E_* .$$

It is noncocentral; in fact, the P_*-coaction on $\mathrm{Ext}_{E_*}(k,k)$ agrees with the P_*-coaction (3.2.9) on Q_*^* under the identification (3.2.10). Thus (3.2.12) is also E_2 of the Cartan-Eilenberg spectral sequence converging to

$$\mathrm{Ext}_{A_*}(k, H_*(X)) .$$

Give A_* a second gradation, the "Cartan degree," by setting

$$|\xi_n| = (0,\ 2p^n-2)$$

$$|\tau_n| = (1,\ 2p^n-2).$$

The resulting bigraded object A_{**} remains a Hopf algebra, and $H_*(X)$ may be regarded as concentrated in Cartan degree 0 (since $H_*(X; \mathbb{Z}_{(p)})$ is torsion-free!). Then the Cartan-Eilenberg spectral sequence collapses for degree reasons. In fact,

$$\mathrm{Ext}_{P_*}(k, Q_*^* \otimes^{\Delta} H_*(X)) \cong \mathrm{Ext}_{A_{**}}(k, H_*(X))$$

as trigraded algebras with Steenrod operations and Massey products.

Thus the H-Adams E_2-term may be regraded to give E_2 of the algebraic Novikov spectral sequence. This observation may be exploited to compute "BP-theoretic" differentials in the H-Adams spectral sequence.

(3.2.14) We investigate next the behavior of the Bockstein in the algebraic Novikov spectral sequence. So let M be a BP_*BP-comodule. Since $(p) \subset BP_*$ is invariant, $\overline{M} = M/pM$ is a BP_*BP-comodule, and we have a spectral sequence

(3.2.15) $$\mathrm{Ext}_{P_*}(k, E_0\overline{M}) \Longrightarrow \mathrm{Ext}_{BP_*BP}(BP_*, \overline{M}).$$

Suppose M is flat over $\mathbb{Z}_{(p)}$. Then the short exact sequence of comodules

(3.2.16) $$0 \longrightarrow M \xrightarrow{p} M \longrightarrow \overline{M} \longrightarrow 0$$

generates a Bockstein spectral sequence with $E_1 = \mathrm{Ext}_{BP_*BP}(BP_*, \overline{M})$ and differentials ∂_r of degree $(1,0)$.

If $M = BP_*(X)$ for a finite-type spectrum X then ([25], 3.10) M is $\mathbb{Z}_{(p)}$-flat iff it is BP_*-flat. So assume that M is BP_*-flat. Then tensoring the exact sequence

$$0 \longrightarrow BP_* \xrightarrow{p} I \longrightarrow \bar{I} \longrightarrow 0\ ,$$

where $I = (p, v_1, \ldots)$ and $\bar{I} = I/pI$, with $I^{n-1}M$, we obtain an exact sequence

$$0 \longrightarrow I^{n-1}M \longrightarrow I^n M \longrightarrow I^n \bar{M} \longrightarrow 0.$$

Apply the 3×3-lemma to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I^n M \arrow[r, "p"] \arrow[d] & I^{n+1}M \arrow[r] \arrow[d] & I^{n+1}\bar{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n-1}M \arrow[r, "p"] \arrow[d] & I^n M \arrow[r] \arrow[d] & I^n \bar{M} \arrow[r] \arrow[d] & 0 \\
 & E_0^{n-1}M \arrow[r, "v_0"] \arrow[d] & E_0^n M \arrow[r] \arrow[d] & E_0^n \bar{M} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.2.17}$$

The short exact bottom row generates a Bockstein spectral sequence with $E_1' = \mathrm{Ext}_{P_*}(k, E_0\bar{M})$ and differentials ∂_r' of degree $(s,t,u) = (1, 1-r, 0)$.

Lemma 3.2.18. Let M be a BP_*-flat BP_*BP-comodule. The spectral sequence (3.2.15) carries a natural differential ∂ of degree $(s,t,u) = (1,0,0)$ which agrees in E_2 with ∂_1' and which at E_∞ is associated to the differential ∂_1 of (3.2.16).

Proof. In the exact couple

$$\begin{tikzcd}
A_1 \arrow[rr, "u"] & & A_1 \arrow[dl, "v"] \\
 & E_1 \arrow[ul, "w"] &
\end{tikzcd}$$

defining (3.2.15),

$$A_1 = H\,\Omega(BP_*BP;\, I^*\,\overline{M})$$

$$E_1 = H\Omega(BP_*BP;\, E_0\,\overline{M})$$

$$\cong H\Omega(P_*;\, E_0\,\overline{M}).$$

The horizontal sequences of (3.2.17) induce differentials in both terms, and u and v respect these differentials. The equation $w\partial = -\partial w$ follows from [15], III, 4.1. The result now follows easily. $\square$

CHAPTER FOUR

EXT NEAR THE VANISHING LINE

§4.0 Introduction

In this chapter we apply the tools of Chapters II and III to compute part of $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n)$. If the spectrum $V(n-1)$ exists, this is its Novikov E_2-term.

In §4.1 we display a vanishing line for $\mathrm{Ext}_{BP_*BP}(BP_*, M)$, for any connective comodule M. §4.2 sets up certain Bockstein spectral sequences by specializing the construction of Chapter III, and shows that if P_* is the Hopf algebra of dual reduced powers then $\mathrm{Ext}_{P_*}(\mathbb{F}_p, \mathbb{F}_p)$ is an approximation to $\mathrm{Ext}_{BP_*}(BP_*, BP_*/I_n)$ in a certain range. A portion of this algebra is computed in §4.3, and in §4.4 is fed into the Bockstein spectral sequences. Certain results of §4.4 are tabulated in an Appendix. Comparison with results of Tangora [56] and Oka [51], [52] on the stable homotopy algebra yields the E_2 term in a larger range and implies the existence of several "new" nontrivial differentials.

§4.1 The vanishing line

Here we show that the vanishing line stated in [62] as (7.1) for the BP Adams spectral sequence at the sphere holds for any connective spectrum X. We also recall various facts from homological algebra which will be useful later.

The following vanishing theorem for coalgebras is proved in straightforward analogy with [2], Lemma 2.3 and Proposition 2.5.

Proposition 4.1.1. Let B be a connected coalgebra over the field k. Let $S \geq 1$, and let $T(s)$, $1 \leq s \leq S$, be integers such that $\mathrm{Ext}_B^{s,*}(k,k)$ is $(T(s)-1)$-connected for $0 \leq s \leq S$. Extend T to $\mathbb{N} = \{0,1,\ldots\}$ by

$$T(aS+b) = aT(S) + T(b), \qquad 0 \leq b \leq S.$$

Let M be an $(m-1)$-connected B-comodule. Then $\mathrm{Ext}_B^{s,*}(k,M)$ is $(T(s)+m-1)$-connected for all $s \geq 0$. □

The proof of [2], Theorem 3.1, extends to give:

Proposition 4.1.2. Let $f\colon B \longrightarrow C$ be a map of connected coalgebras over the field k. Suppose that B is injective as a right C-comodule. Let f be $(u-1)$-connected, and let $T\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a nondecreasing function such that $\mathrm{Ext}_B^{s,*}(k,k)$ is $(T(s)-1)$-connected for $s \geq 0$. Let M be an $(m-1)$-connected B-comodule. Then

$$\mathrm{Ext}_f^{s,*}(k,M)\colon \mathrm{Ext}_B^{s,*}(k,M) \longrightarrow \mathrm{Ext}_B^{s,*}(k,M)$$

is $(T(s-1)+u+m-1)$-connected for $s \geq 0$. □

(4.1.3) Next we need to recall (from [33] for example) the structure of the loop-homology of the primitive Hopf algebra $k[t]$, where $k = \mathbb{F}_p$, p is odd, and $|t| = n$ is even. We have

$$\mathrm{Ext}_{k[t]}(k,k) = S[h_i,\ \lambda_i : i \geq 0]$$

where S is the free commutative bigraded k-algebra functor and where $|h_i| = (1, p^i n)$, $|\lambda_i| = (2, p^{i+1} n)$. The classes h_i, λ_i are represented in the cobar construction by

$$\left[t_1^{p^i}\right], \quad \sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} \left[t_1^{jp^i} \middle| t_1^{(p-j)p^i}\right],$$

respectively. The action of the Steenrod operations (in Liulevicius' indexing [33]) is determined by the Cartan formula, the Adam relations, the expression $P^s x = x^p$ for $|x| = (2s, t)$, t even, and the formulae

$$P^0 h_i = h_{i+1},$$

$$\beta P^0 h_i = \lambda_i .$$

(4.1.4) Notice also that the quotient Hopf algebra $k[t]/(t^p)$ has loop-homology $S[h_0, \lambda_0]$.

(4.1.5) Let P_* denote the Hopf algebra of Steenrod reduced powers. As in Chapter III, let t_n be the Hopf conjugate of the Milnor generator ξ_n, so that $P_* = k[t_1, t_2, \ldots]$ as algebras. Then $D = k[t_1]/(t_1^p)$ is a quotient Hopf algebra by a $(pq-1)$-connected projection $f: P_* \longrightarrow D$, $q = 2p-2$. From (4.1.2) with $T(s) = 0$ for all s, we see that

(4.1.6) $$f_*\colon \mathrm{Ext}^{s,*}_{P_*}(k,k) \longrightarrow \mathrm{Ext}^{s,*}_{D}(k,k)$$

is (pq-1)-connected for $s \geq 0$. But (4.1.4) implies that $\mathrm{Ext}^{s,*}_{D}(k,k)$ is $(T(s)-1)$-connected for $s = 0, 1, 2$, where $T(0) = 0$, $T(1) = q$, $T(2) = pq$. Thus (4.1.2) implies:

Proposition 4.1.7. Let M be an (m-1)-connected P_*-comodule. Then $\mathrm{Ext}^{s,*}_{P_*}(k,M)$ is $(T(s)+m-1)$-connected, where, for $n \geq 0$,

$$T(2n) = npq$$
$$T(2n+1) = (np+1)q.$$

□

It will be convenient later to set $T(s) = 0$ for $s < 0$.

(4.1.8) Let M be an (m-1)-connected BP_*BP-comodule. Then in the augmentation filtration of §3.2, $E^t_0 M$ is (m-1)-connected for all $t \geq 0$; so in the E_2-term of the algebraic Novikov spectral sequence,

$$\mathrm{Ext}^{s,*}_{P_*}(k, E^t_0 M)$$

is $(T(s)+m-1)$-connected for all $s, t \geq 0$. Now if M is of finite type, then the spectral sequence converges by (3.1.7), and

$$\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, M)$$

is $(T(s)+m-1)$-connected for $s \geq 0$. In the general case, note that M is a union of comodules of finite type. Thus:

Theorem 4.1.9. Let M be an (m-1)-connected BP_*BP-comodule. Then

$$\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, M)$$

is $(T(s)+m-1)$-connected for $s \geq 0$.

$\square$

§4.2 Bockstein spectral sequences

Here we set up several spectral sequences which, in various contexts, add one Bockstein at a time. The prime p remains odd.

(4.2.1) Let A_{**} be the dual Steenrod algebra bigraded as in §3.2, and consider the quotient Hopf algebra

$$A(n) = A_{**}/(\tau_0, \ldots, \tau_{n-1}), \qquad 0 \leq n .$$

There is a noncocentral multiplicative extension sequence

$$E[\tau_n] \longrightarrow A(n) \longrightarrow A(n+1)$$

which gives rise to a Cartan-Eilenberg spectral sequence with

$$E_1 = k[a_n] \otimes \mathrm{Ext}_{A(n+1)}(k, k),$$

$|a_n| = (1,\ 1,\ 2p^n - 2)$.

(4.2.2) Let $Q^*_* = k[v_0, v_1, \ldots]$ be the right P_*-comodule-algebra studied in §3.2, and consider the quotient comodule-algebra

$$Q(n) = Q^*_*/(v_0, \ldots, v_{n-1}), \qquad 0 \leq n .$$

Then

$$\mathrm{Ext}^{s+t,t,u}_{A(n)}(k, k) = \mathrm{Ext}^{s,u}_{P_*}(k,\ Q(n)^t).$$

Filter the cobar construction $\Omega(Q(n), P_*, k)$ by powers of the ideal

$(v_n) \subset Q(n)$. There arises a spectral sequence identical with the Cartan-Eilenberg spectral sequence above. The grading is such that

$$E_1^{0,j,t,u} = \mathrm{Ext}_{P_*}^{j,u}(k, Q(n+1)^t),$$

$$a_n = \{v_n + F^2\} \in E_1^{1,-1,1,2p^n-2}.$$

(4.2.3) Note that a_n survives to $v_n \in \mathrm{Ext}_{P_*}^{0,*}(k, Q(n)^1)$. Let $k(a_n)$ be the graded field of fractions of the graded integral domain $k[a_n]$. Then the differential modules

(4.2.4)
$$k(a_n) \otimes_{k[a_n]} E_r$$

form a spectral sequence, since $k(a_n)$ is flat over $k[a_n]$.

The exact sequence of P_*-comodules

$$0 \longrightarrow Q(n) \xrightarrow{v_n} Q(n) \longrightarrow Q(n+1) \longrightarrow 0$$

gives rise to an exact couple in Ext_{P_*}. The associated spectral sequence may be identified, after tensoring over k with $k(a_n)$, with (4.2.4).

(4.2.5) Let (BP_*, BP_*BP) be the Hopf algebroid of BP cooperations described in §2.5, and consider the quotient Hopf algebroid

$$(B(n), \Gamma(n)) = (BP_*/I_n, BP_*BP/I_n), \qquad 0 \leq n,$$

where I_n is the invariant ideal $(p, v_1, \ldots, v_{n-1}) \subset BP_*$. In particular, $(B(\infty), \Gamma(\infty)) = (k, P_*)$. From (2.4.8) we have

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n) = \mathrm{Ext}_{\Gamma(n)}(B(n), B(n)).$$

Now $I_k(n) = (v_n, \ldots, v_{k-1}) \subset B(n)$, $n \leq k$, is an invariant ideal, and we wish to study the algebraic Novikov spectral sequence associated to $I_{n+1}(n) = (v_n)$. As algebras,

$$E_0 B(n) = k[a_n] \otimes B(n+1)$$

$$E_0 \Gamma(n) = k[a_n] \otimes \Gamma(n+1)$$

For degree reasons, $E_0 \eta_R(a_n) = a_n$; and $E_0^0 \eta_R$ and $E_0^0 \Delta$ agree with η_R and Δ on $(B(n+1), \Gamma(n+1))$. That is, $E_0 \Gamma(n)$ is obtained from $\Gamma(n+1)$ by extending the ground-ring from k to $k[a_n]$. Therefore, in the Novikov spectral sequence,

$$E_2 = k[a_n] \otimes \mathrm{Ext}_{\Gamma(n+1)}(B(n+1), B(n+1)).$$

It is convenient to regrade this slightly so that E_r becomes E_{r-1}. This is accomplished by neglecting the shift in filtration of a suspension in the construction (p. 56) of the spectral sequence. The algebra structure is unharmed, and we obtain a cohomological spectral sequence lying in the 6th, 7th, and 8th octants, with

$$E_1^{0,j,u} = \mathrm{Ext}^{j,u}_{\Gamma(n+1)}(B(n+1), B(n+1)),$$

$$a_n \in E_1^{1,-1,2p^n-2} .$$

(4.2.6) Note that a_n survives to $v_n \in \mathrm{Ext}^{0,*}_{\Gamma(n)}(B(n), B(n))$. The differential modules

(4.2.7) $$k(a_n) \otimes_{k[a_n]} E_r$$

form a spectral sequence since $k(a_n)$ is flat over $k[a_n]$.

The exact sequence of BP_*BP-comodules

$$0 \longrightarrow BP_*/I_n \xrightarrow{v_n} BP_*/I_n \longrightarrow BP_*/I_{n+1} \longrightarrow 0$$

gives rise to an exact couple in Ext_{BP_*BP}. The associated spectral sequence may be identified, after tensoring over k with $k(a_n)$, with (4.2.7).

We shall call all these spectral sequences "Bockstein spectral sequences."

(4.2.8) There are also Novikov spectral sequences

$$\mathrm{Ext}_{P_*}(k, Q(n)) \Longrightarrow \mathrm{Ext}_{\Gamma(n)}(B(n), B(n)).$$

A proof identical to that of (3.2.18) shows that the differentials ∂_1 in the respective Bockstein spectral sequences are respected by this spectral sequence.

To get started on this ladder of spectral sequences let us note the exact sequences

$$0 \longrightarrow J(n) \longrightarrow Q(n) \longrightarrow k \longrightarrow 0$$

$$0 \longrightarrow I(n) \longrightarrow B(n) \longrightarrow k \longrightarrow 0$$

of P_*, resp. $\Gamma(n)$, -comodules. The appropriate long exact sequence in Ext then proves that

(4.2.9) $$\mathrm{Ext}^{s,*}_{P_*}(k, Q(n)^t) \longrightarrow \mathrm{Ext}^{s,*}_{P_*}(k,k)$$

is $(T(s) + (2p^n - 2)t - 1)$-connected and (using (4.1.9)) that

(4.2.10) $$\mathrm{Ext}^{s,*}_{\Gamma(n)}(B(n), B(n)) \longrightarrow \mathrm{Ext}^{s,*}_{P_*}(k,k)$$

is $(T(s) + (2p^n - 2) - 1)$-connected. Notice also that for $t = 0$,

$$\mathrm{Ext}^{**}_{P_*}(k, Q(n)^0) \xrightarrow{\cong} \mathrm{Ext}^{**}_{P_*}(k, k).$$

So our first goal is to compute a portion of the loop-homology of the reduced powers.

§4.3 A portion of $\mathrm{Ext}_{P_*}(k,k)$

Consider the sub Hopf algebra

$$\iota : k[t_1, t_2] \subset P_*,$$

with reduced diagonal determined by $\bar{\Delta} t_2 = t_1 \otimes t_1^p$. We shall compute $\mathrm{Ext}_{P_*}(k,k)$ in the modest range where ι_* maps isomorphically. To define that range, note that in the Cartan-Eilenberg spectral sequence defined by ι, $E_2^{i,j,*}$ is $(T(i) + (p^2+p+1)T(j) - 1)$-connected. Then it is obvious that the edge-homomorphism ι_* is $(T(s-2) + (2p^3-2) - 1)$-connected in degree s.

To compute part of $\mathrm{Ext}_{k[t_1,t_2]}(k,k)$ we use the cocentral multiplicative extension sequence

$$k[t_1] \longrightarrow k[t_1, t_2] \longrightarrow k[t_2].$$

From (4.1.3),

$$E_2 = S[h_i, \lambda_i : i \geq 0] \otimes S[h_{2,i}, \lambda_{2,i} : i \geq 0].$$

In our area of interest we may restrict our attention to

$S[h_0, h_1, h_2, \lambda_0, \lambda_1] \otimes [h_{20}, h_{21}, \lambda_{20}]$ where $[X]$ denotes the span of X.

Following [33], from the cobar construction of $k[t_1, t_2]$ we have

$$d_2 h_{2,i} = -h_i h_{i+1}.$$

Hence the Steenrod action implies that $d_2\lambda_{20} = 0$ and

$$d_3\lambda_{20} = h_2\lambda_0 - h_1\lambda_1 .$$

In our range of E_3 there are new generators

$$\mu_0 = \{-h_0 h_{20}\} = \langle h_1, h_0, h_0\rangle$$

$$\nu_0 = \{h_1 h_{20}\} = \langle h_0, h_1, h_1\rangle.$$

Passage to E_4 merely kills λ_{20} and adds the relation $h_1\lambda_0 = h_1\lambda_1$. Thus in our range E_∞ has generators

$$h_0,\quad h_1,\quad h_2,\quad \lambda_0,\quad \lambda_1,\quad \mu_0,\quad \nu_0,$$

and relations

$$h_0h_1,\quad h_1h_2;$$

$$h_0\mu_0,\quad h_1\nu_0,\quad h_1\mu_0 - h_0\nu_0,\quad \mu_0^2,\quad \mu_0\nu_0,\quad \nu_0^2;$$

$$h_2\lambda_0 - h_1\lambda_1 .$$

For degree reasons all algebra extensions are trivial if $p > 3$. If $p = 3$, recall the Massey triple product [40]

$$\lambda_i = \langle h_i,\ h_i,\ h_i\rangle.$$

Now compute, using [40] Cor. 3.2 (iii),

$$\begin{aligned} h_1\lambda_0 &= h_1 \langle h_0, h_0, h_0\rangle \\ &= \langle h_1, h_0, h_0\rangle h_0 = \mu_0 h_0, \end{aligned}$$

and similarly $h_0\lambda_1 = \nu_0 h_1$ Using [40] Cor. 3.2 (i),

$$\begin{aligned} \mu_0^2 &= \mu_0 \langle h_1, h_0, h_0\rangle = \langle \mu_0 h_1, h_0, h_0\rangle \\ &= \langle \nu_0 h_0, h_0, h_0\rangle = \nu_0 \langle h_0, h_0, h_0\rangle = \nu_0\lambda_0, \end{aligned}$$

and similarly $\nu_0^2 = \mu_0\lambda_1$. Finally,

$$(h_0\mu_0)\nu_0 = h_1\lambda_1\nu_0 = (h_0\nu_0)\lambda_0 = h_0\lambda_0\lambda_1$$

so $\mu_0\nu_0 = \lambda_0\lambda_1$. We have proved:

Proposition 4.3.1. In the range $u < T(s-2) + (2p^3-2)$, $\mathrm{Ext}_{P_*}^{s,u}(k,k)$ has generators

$$h_0, \quad h_1, \quad h_2, \quad \lambda_0, \quad \lambda_1, \quad \mu_0, \quad \nu_0,$$

and relations:

$$h_0h_1, \quad h_1h_2, \quad h_1\mu_0 - h_0\nu_0, \quad h_2\lambda_0 - h_1\lambda_1;$$

if $p > 3$,

$$\mu_0^2, \quad h_0\mu_0, \quad \mu_0\nu_0, \quad h_1\nu_0, \quad \nu_0^2;$$

and if $p = 3$,

$$\mu_0^2 - \nu_0\lambda_0, \quad h_0\mu_0 - h_1\lambda_0, \quad \mu_0\nu_0 - \lambda_0\lambda_1,$$

$$h_1\nu_0 - h_0\lambda_1, \quad \nu_0^2 - \mu_0\lambda_1.$$

□

§4.4 A computation

Let $k = \mathbb{F}_p$, $Q(n) = Q_*^*/(v_0, \ldots, v_{n-1})$, $\Gamma = BP_*BP$, and $B(n) = BP_*/I_n$, as in §4.2. Let

$$H(P_*; Q(n)) = \mathrm{Ext}_{P_*}(k, Q(n)),$$

$$H(\Gamma; B(n)) = \mathrm{Ext}_\Gamma(BP_*, B(n)).$$

We summarize our intentions in this section by the following diagram of spectral sequences.

$$
(4.4.1)\qquad
\begin{array}{ccccccc}
H(P_*;Q(3)) & \Rrightarrow & H(P_*;Q(2)) & \Rrightarrow & H(P_*;Q(1)) & \Rrightarrow & H(P_*;Q(0)) \\
 & & \Downarrow & & \Downarrow & & \\
H(\Gamma;B(3)) & \Rrightarrow & H(\Gamma;B(2)) & \Rrightarrow & H(\Gamma;B(1)) & \Rrightarrow & H(\Gamma;B(0))
\end{array}
$$

Here the horizontal triple arrows denote Bockstein spectral sequences. For lack of time we shall be brief.

By (4.2.9) and (4.2.10), Proposition 4.3.1 provides us with the left-most algebras in (4.4.1) in the range $u < T(s-2) + (2p^3-2)$.

(4.4.2) The first Bockstein spectral sequences must collapse in this region. The Cartan degree implies that in the homology spectral sequence, all algebra extensions are trivial. In the BP case, note that the given relations in $\mathrm{Ext}_{P_*}(k,k)$ all arise from boundaries of chains in ΩP_* involving only t_1 and t_2. Modulo (p, v_1), t_1 and t_2 have the same diagonal in Γ as in P_*, so the same relations must hold. Hence there are no extensions in the BP spectral sequence either.

(4.4.3) Pass now to the homology Bockstein spectral sequence

$$k[a_1] \otimes H(P_*;Q(2) \Longrightarrow H(P_*;Q(1)).$$

For reasons of degree, $v_2 \in H^{0,1}(P_*;Q(1)^1)$ is the only generator which may support a nonzero ∂_1. From the cobar construction $\Omega(P_*;Q(1))$ (with the right comodule $Q(1)$ as coefficients) we have

$$\partial_1 v_2 = a_1 h_1 .$$

Thus E_2 has generators

$$h_2, \qquad \lambda_0, \qquad \lambda_1, \qquad \mu_0,$$

$$v_2^p = \{v_2^p\}$$

$$m_i = \{v_2^{i-1} h_0\} \qquad 1 \leq i \leq p$$

$$b_i = \{iv_2^{i-1} h_1\} \qquad 1 \leq i \leq p-1$$

$$\overline{b}_p = \{v_2^{p-1} h_1\}$$

$$b_i' = \{i(i-1) v_2^{i-2} \nu_0\} \qquad 2 \leq i \leq p-1$$

$$\overline{b}_p' = \{v_2^{p-2} \nu_0\}$$

so that

$$m_1 = \{h_0\}, \qquad m_i = \langle m_{i-1}, h_1, a_1 \rangle$$

$$b_1 = \{h_1\}, \qquad b_i = \frac{i-1}{i} \langle b_{i-1}, h_1, a_1 \rangle$$

$$b_2' = \{\nu_0\}, \qquad b_i' = \frac{i-2}{i} \langle b_{i-1}', h_1, a_1 \rangle$$

(with no indeterminancy).

Now the chain

$$v_2^2[t_1] + 2v_1 v_2[t_1^{p+1} - t_2] + 2v_1^2 [t_1^{2p+1} - t_1^p t_2] \in \Omega^1(P_*; Q(1)^2)$$

is a representative mod filtration 3 for m_3. This chain has boundary $v_1^2(2[t_2 | t_1^p] + [t_1 | t_1^{2p}])$, which is a representative for $-v_1^2 \nu_0$. Consequently,

$$\partial_2 m_3 = -\frac{1}{2} a_1^2 b_2' .$$

Hence ([40], Cor. 4.4) for $4 \leq i \leq p$,

$$\begin{aligned}\partial_2 m_i &\simeq -\langle \partial_2 m_{i-1}, h_1, a_1\rangle \\ &\simeq a_1^2 \langle b'_{i-2}, h_1, a_1\rangle \\ &\simeq a_1^2 b'_{i-1}\end{aligned}$$

by induction; here $\simeq$ indicates equality up to a nonzero scalar factor.

We now skip ahead in the spectral sequence. This is the Cartan-Eilenberg spectral sequence of a multiplicative extension, so Steenrod operations behave in the expected way; cf. [33], [41]. In particular, there is a Kudo theorem, which asserts that if $r - 1 = 2s$ and $x \in E_r^{0,2s}$ transgresses to $d_r x = y \in E_r^{2s+1,0}$, then $x^{p-1}y$ transgresses to

$$d_{qs+1}\{x^{p-1}y\} = -\{\beta P^s y\}$$

for $q = 2p-2$. Consequently, from $\partial_1 v_2 = a_1 h_1$ it follows that

$$\partial_{p-1}\{a_1 v_2^{p-1} h_1\} = -a_1^p \lambda_1$$

and so

$$\partial_{p-1}\{v_2^{p-1} h_1\} = -a_1^{p-1}\lambda_1 .$$

Also,

$$\partial_p v_2^p = \partial_p P^1 v_2 = P^1 \partial_1 v_2 = a_1^p h_2 .$$

Now look back. Suppose $p > 3$. Then

$$0 = \partial_{p-1}(v_2^{p-1} h_0 h_1) = -a_1^{p-1} h_0 \lambda_1 .$$

To avoid a contradiction, we must have

$$\partial_{p-2}(\overline{b}'_p) \simeq a_1^{p-2} h_0 \lambda_1 .$$

If $p = 3$, we have

$$\partial_1(v_2\nu_0) = a_1 h_1 \nu_0 = a_1 h_0 \lambda_1 .$$

In either case, this differential introduces a new indecomposable in E_{p-1}:

$$\phi = \{h_0 \bar{b}'_p\} = \{v_2^{p-2} h_0 \nu_0\} = \{v_2^{p-2} h_1 \mu_0\}.$$

Taking cognizance of the Cartan degree one can see that no differentials enter the region of interest from an uncomputed group. Thus for example we know that

$$\partial_{p-1}(v_2^{p-1} h_1 \mu_0) = a_1^{p-1} \mu_0 \lambda_1 .$$

The spectral sequence collapses at E_{p+1}, and the abutment is displayed in Table I of the Appendix.

(4.4.4) Consider next the algebraic Novikov spectral sequence

$$H(P_*; Q(1)) \Rightarrow H(\Gamma; B(1)) .$$

For reasons of degree, the only generators on which d_2 may be nontrivial are m and μ_0. The class m is represented in $F^2\,\Omega(\Gamma; B(1))$ by

$$\bar{m} = [v_1 t_2] - [v_1 t_1^{p+1}] - [v_2 t_1] .$$

Now by §2.5, we have modulo p:

$$\bar{\Delta}\, t_1 = 0$$

$$\bar{\Delta}\, t_2 = t_1 \otimes t_1^p - v_1 \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} [t_1^i \,|\, t_1^{p-i}]$$

$$\eta_R v_1 = v_1$$

$$\eta_R v_2 = v_2 + v_1 t_1^p - v_1^p t_1 .$$

Therefore

$$d\overline{m} = [v_1^p t_1 | t_1] + \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} [v_1^2 t_1^i | t_2^{p-i}]$$

$$\equiv v_1^2 \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} [t_1^i | t_1^{p-i}] \mod F^4.$$

Consequently,

$$d_2 \overline{m} = v_1^2 \lambda_0 \ .$$

A similar argument shows that

$$d_2 \mu_0 = -2 v_1 h_0 \lambda_0 \ .$$

For $p > 3$, $\{v_1^{p-3} \mu_0 \lambda_1\}$ becomes an indecomposable class in E_3. There are no further differentials, for reasons of degree. We indicate some extensions in the abutment.

$$h_0 b_1 = -v_1 \lambda_0$$

$$h_0 b_i = -v_1 b_i', \qquad 2 \leq i \leq p-1$$

$$h_0 h_2 \lambda_0 = -v_1 \lambda_1 \lambda_0$$

$$p > 3: \quad b_i' b_j' \simeq \{v_1^{p-3} \mu_0 \lambda_1\}, \qquad i+j = p+1$$

$$p > 3: \quad h_0 b_2' b_{p-1}' \simeq \phi \lambda_0$$

Most of these can be seen from the behavior of the Bockstein differential in $H(\Gamma; B(1))$, which we take up in (4.4.6).

(4.4.5) It is now an easy matter to decide the structure of the BP Bockstein spectral sequence

$$k[a_1] \otimes H(\Gamma; B(2)) \implies H(\Gamma; B(1)).$$

We indicate the differentials:

$$\partial_1 v_2 = a_1 h_1$$

$$\partial_1 \mu_0 = a_1 h_0 \lambda_0$$

$$\partial_1 \nu_0 = -a_1 h_1 \lambda_0$$

$$\partial_2 (v_2 h_0) = a_1^2 \lambda_0$$

$$\partial_2 (v_2^i h_0) = -i a_1^2 v_2^{i-2}((i-1)\nu_0 + v_2 \lambda_0), \quad 2 \leq i \leq p-1$$

$$\partial_{p-2}(v_2^{p-2} \nu_0) \simeq a_1^{p-2} h_0 \lambda_1$$

$$\partial_{p-1}(v_2^{p-1} h_1) = -a_1^{p-1} \lambda_1$$

$$\partial_p (v_2^p) = a_1^p h_2$$

Consider the extension $h_0 h_1 = -v_1 \lambda_0$. Apply P^0: $h_1 h_2 = -v_1^p \lambda_1 = 0$ by the behavior of ∂_{p-1}. Thus $h_i h_{i+1} = 0$ in $H(\Gamma; B(1))$ for $i \geq 1$.

(4.4.6) In the homology Bockstein spectral sequence

$$k[a_0] \otimes H(P_*; Q(1)) \implies H(P_*; Q(0))$$

we have easily:

$$\partial_1 m = \mu_0$$

$$\partial_1 b_i = b_i', \qquad 2 \leq i \leq p-1.$$

We now use (3.2.18) to feed these differentials into the BP Bockstein spectral sequence. Thus $\partial_1 b_i = b_i'$, $2 \leq i \leq p-1$. In addition, $\Omega(\Gamma; B(1))$ is the normalization of the mod p reduction of a cosimplicial commutative

$\mathbb{Z}_{(p)}$-algebra which is $\mathbb{Z}_{(p)}$-flat in each degree. Hence [41], Prop. 2.3(v), for all $s \geq 0$,

$$\beta P^s = \partial_1 P^s .$$

In particular, for $i \geq 0$,

$$\partial_1 h_{i+1} = \partial_1 P^0 h_i = \beta P^0 h_i = \lambda_i .$$

We find that E_2 modulo a_0-torsion has basis (in this range)

$$\begin{array}{ll} v_1^{pi} & , \quad 0 \leq i \leq p+1 , \\ v_1^{pi-1} h_0 , & 1 \leq i \leq p+1 ; \\ v_1^{p-1} h_0 h_2 , & \\ \phi & . \end{array}$$

It is known [62] that $\partial_2 (v_1^{p-1} h_0 h_2) \simeq \phi$, but we shall not prove this here. Turn instead to the first collection of classes.

Let $A(n)$ be as in §4.2. Consider

$$\pi : A(1) \longrightarrow S[t_1, \bar{\tau}_1]/(t_1^p) = L.$$

Then $\mathrm{Ext}_L(k, k) = S[h_0, \lambda_0, v_1]$ and π_* hits the generators. Thus π_* surjects. By placement, $S[h_0, v_1]$ survives nontrivially in the mod p algebraic Novikov spectral sequence

$$H(A(1); k) \Longrightarrow H(\Gamma; B(1)).$$

Recall the formula of W. Browder, [12] Theorem 5.11 and [41] Prop. 6.8: if $x \in E_r^{2n}$ in the Bockstein spectral sequence of a sufficiently homotopy-commutative DG $\mathbb{Z}_{(p)}$-algebra has $d_r x = y$, then $d_{r+1}\{x^p\} = \{x^{p-1} y\}$. The cosimplicial structure of $\Omega(\Gamma)$ guarantees infinite homotopy commutativity.

Now $\partial_1 v_1 = h_0$; so by induction,

$$\partial_r v_1^{p^r} = v_1^{p^r - 1} h_0 .$$

Thus we have

<u>Proposition 4.4.7.</u> (Novikov, [50]). If $p > 2$, $(a,p) = 1$, and $q = 2p-2$, then

$$H^{1, ap^s q}(\Gamma) = \mathbb{Z}/p^s \mathbb{Z} .$$

(4.4.8) The end-result of this computation is included in Tables II and III of the Appendix for the primes 3 and 5. In this range, $u < T(s-2) + (2p^3-2)$, primes ≥ 5 behave uniformly.

Table I. We give an $\mathbb{F}_p$-basis for

$$\mathrm{Ext}_{P_*}^{s,u}(\mathbb{F}_p, Q_*^t/(v_0))$$

in the range

$$u < T(s-2) + (2p-2)t + (2p^3-2).$$

Degree s is plotted horizontally and t vertically. Vertical lines indicate multiplication by

$$v_1 \in \mathrm{Ext}_{P_*}^{0,2p-2}(\mathbb{F}_p, Q_*^1/(v_0)).$$

Arrows of bidegree $(1,1)$ indicate differentials in the algebraic Novikov spectral sequence converging to

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*/(p)).$$

We display degrees $s = 0, \ldots, 4$; thereafter multiplication by

$$\lambda_0 \in \mathrm{Ext}_{P_*}^{2,2p(p-1)}(\mathbb{F}_p, Q_*^0/(v_0))$$

is an isomorphism.

Tables II and III. We display

$$\mathrm{Ext}_{BP_*BP}^{s,2u(p-1)}(BP_*, BP_*)$$

for $p = 3$, $u \leq 23$, and $p = 5$, $u \leq 59$, combining results of Chapter IV with those of Tangora [56] and Oka [51], [52]. Homological degree is plotted vertically and (internal degree)/(2p-2) horizontally. In addition to the listed relations, one has

$$\alpha_{i+1}\epsilon_j = \alpha_1\epsilon_{i+j}$$

$$\alpha_{i+1}\epsilon_{2,j} = \alpha_1\epsilon_{2,i+j}$$

$$k\ell\beta_i\beta_j = ij\beta_k\beta_\ell$$

and possibly others; for example, it is likely that $\zeta \simeq \alpha_1\beta_4$ for $p = 3$. The symbol $\simeq$ indicates equality up to a nonzero factor. Differentials in the Adams-Novikov spectral sequence are determined by:

$p = 3$:

$$d_5\epsilon_0 \simeq \alpha_1\beta_1^3$$

$$d_5\beta_4 \simeq \alpha_1\beta_1^2\epsilon_0$$

$$d_5\eta = \beta_1^3\beta_2$$

$$d_9\zeta = \beta_1^6$$

$p = 5$:

$$d_9\epsilon_0 \simeq \alpha_1\beta_1^5$$

$$d_9\eta_i = \beta_1^5\beta_{i+1} \qquad 1 \leq i \leq 4 \quad (5?).$$

TABLE I.

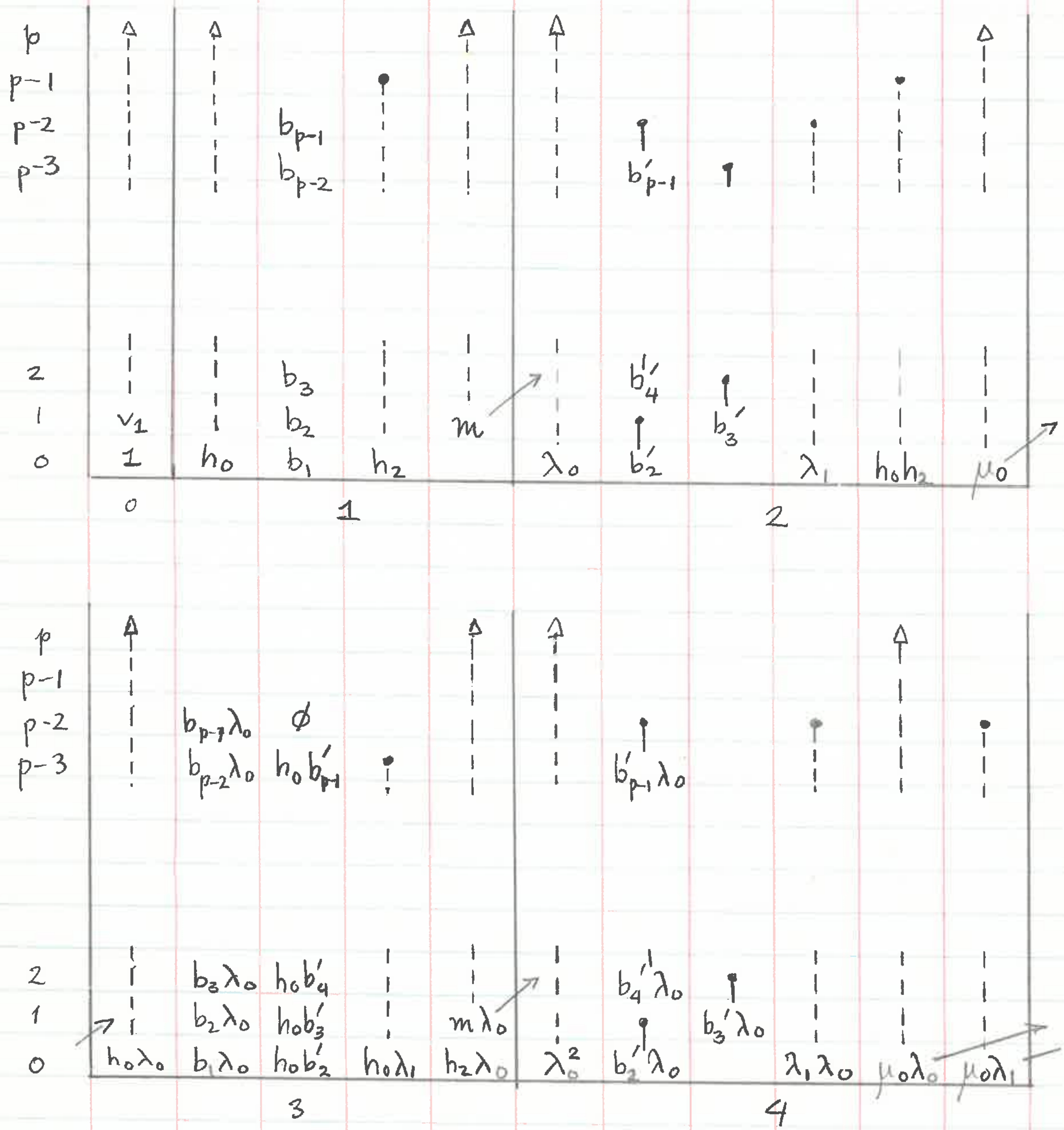
p
p-1
p-2
p-3
2
1
0
v_1
1
h_0
b_{p-1}
b_{p-2}
b_3
b_2
b_1
h_2
m
λ_0
b'_{p-1}
b'_4
b'_2
b'_3
λ_1
h_0h_2
μ_0
0
1
2
$h_0\lambda_0$
$b_{p-1}\lambda_0$
$b_{p-2}\lambda_0$
$b_3\lambda_0$
$b_2\lambda_0$
$b_1\lambda_0$
ϕ
$h_0b'_{p-1}$
$h_0b'_4$
$h_0b'_3$
$h_0b'_2$
$h_0\lambda_1$
$m\lambda_0$
$h_2\lambda_0$
λ_0^2
$b'_{p-1}\lambda_0$
$b'_4\lambda_0$
$b'_2\lambda_0$
$b'_3\lambda_0$
$\lambda_1\lambda_0$
$\mu_0\lambda_0$
$\mu_0\lambda_1$
3
4

TABLE II: p = 3

7											$\alpha_1\beta_1^3$	
6										β_1^3		
5								$\alpha_1\beta_1^2$				$\alpha_1\beta_1\beta_2$
4							β_1^2				$\beta_1\beta_2$	
3					$\alpha_1\beta_1$				$\alpha_1\beta_2$		$\alpha_1\varepsilon_0$	
2				β_1				β_2		ε_0	ε_1	β_3
1		α_1	α_2	α_3'	α_4	α_5	α_6'	α_7	α_8	α_9''	α_{10}	α_{11}
0	1											

TABLE II: p = 3

15											$\alpha_1\beta_1^7$	
14										β_1^7		
13								$\alpha_1\beta_1^6$				$\alpha_1\beta_1^5\beta_2$
12							β_1^6				$\beta_1^5\beta_2$	
11					$\alpha_1\beta_1^5$				$\alpha_1\beta_1^4\beta_2$		$\alpha_1\beta_1^4\varepsilon_0$	
10				β_1^5				$\beta_1^4\beta_2$		$\beta_1^4\varepsilon_0$		$\beta_1^3\beta_2^2$
9		$\alpha_1\beta_1^4$				$\alpha_1\beta_1^3\beta_2$		$\alpha_1\beta_1^3\varepsilon_0$		$\alpha_1\beta_1^2\beta_2^2$		
8	β_1^4				$\beta_1^3\beta_2$		$\beta_1^3\varepsilon_0$		$\beta_1^2\beta_2^2$		$\alpha_1\beta_1^2\eta$	
7			$\alpha_1\beta_1^2\beta_2$		$\alpha_1\beta_1^2\varepsilon_0$		$\alpha_1\beta_1\beta_2^2$			$\beta_1^2\eta$	$\beta_1^2\xi$	$\beta_1\mu$
6		$\beta_1^2\beta_2$		$\beta_1^2\varepsilon_0$		$\beta_1\beta_2^2$		$\alpha_1\beta_1\eta$		$\beta_1^2\beta_4$ $\alpha_1\mu$		$\alpha_1\beta_2\eta$
5		$\alpha_1\beta_1\varepsilon_0$		$\alpha_1\beta_2^2$			$\beta_1\eta$	$\beta_1\xi$	μ		$\beta_2\eta$	
4	$\beta_1\varepsilon_0$		β_2^2		$\alpha_1\eta$		$\beta_1\beta_4$ λ				$\alpha_1\nu$ $\alpha_1\gamma_2$	
3	ϕ			η	ξ					ν γ_2	$\alpha_1\varepsilon_{20}$	?
2				β_4				β_5		ε_{20}	ε_{21}	β_6
1	α_{12}'	α_{13}	α_{14}	α_{15}'	α_{16}	α_{17}	α_{18}''	α_{19}	α_{20}	α_{21}'	α_{22}	α_{23}

$p\phi = \alpha_1\beta_3$

$\alpha_1\beta_2^2 = \beta_1\phi$

$\beta_1\beta_4 = \varepsilon_0^2$

$\alpha_1\mu = \beta_2^3 = \beta_1\lambda$

$\alpha_1\eta = \beta_2\varepsilon_0$

$\alpha_1\nu = \beta_1\beta_5$

TABLE III: p = 5

9										$\alpha_1\beta_1^4$		
8									β_1^4			
7					$\alpha_1\beta_1^3$						$\alpha_1\beta_1^2\beta_2$	
6				β_1^3						$\beta_1^2\beta_2$		
5						$\alpha_1\beta_1\beta_2$						$\alpha_1\beta_1\beta_3$
4					$\beta_1\beta_2$						$\beta_1\beta_3$	
3							$\alpha_1\beta_3$					
2	$\alpha_1\beta_2$					β_3						β_4
1	α_{12}	α_{13}	α_{14}	α_{15}'	α_{16}	α_{17}	α_{18}	α_{19}	α_{20}'	α_{21}	α_{22}	α_{23}

5												$\alpha_1\beta_1^2$
4											β_1^2	
3							$\alpha_1\beta_1$					
2						β_1						β_2
1		α_1	α_2	α_3	α_4	α_5'	α_6	α_7	α_8	α_9	α_{10}'	α_{11}
0	1											

TABLE III: p = 5

14												β_1^7
13								$\alpha_1\beta_1^6$				
12							β_1^6					
11			$\alpha_1\beta_1^5$						$\alpha_1\beta_1^4\beta_2$			
10		β_1^5						$\beta_1^4\beta_2$				
9				$\alpha_1\beta_1^3\beta_2$						$\alpha_1\beta_1^3\beta_3$		
8			$\beta_1^3\beta_2$						$\beta_1^3\beta_3$			
7					$\alpha_1\beta_1^2\beta_3$						$\alpha_1\beta_1^2\beta_4$	
6				$\beta_1^2\beta_3$						$\beta_1^2\beta_4$		$\beta_1^2\varepsilon_0$
5						$\alpha_1\beta_1\beta_4$		$\alpha_1\beta_1\varepsilon_0$				$\alpha_1\beta_2\beta_4$
4					$\beta_1\beta_4$		$\beta_1\varepsilon_0$				$\beta_2\beta_4$	
3	$\alpha_1\beta_4$		$\alpha_1\varepsilon_0$	$\alpha_1\varepsilon_1$	$\alpha_1\varepsilon_2$		ϕ					η_1
2		ε_0	ε_1	ε_2	ε_3	β_5						β_6
1	α_{24}	α''_{25}	α_{26}	α_{27}	α_{28}	α_{29}	α'_{30}	α_{31}	α_{32}	α_{33}	α_{34}	α'_{35}

$p\phi = \alpha_1\beta_5$ $\qquad$ $\beta_1\phi \doteq \alpha_1\beta_2\beta_4$

TABLE III: p = 5

19											$\alpha_1\beta_1^9$	
18										β_1^9		
17						$\alpha_1\beta_1^8$						$\alpha_1\beta_1^7\beta_2$
16					β_1^8						$\beta_1^7\beta_2$	
15	$\alpha_1\beta_1^7$						$\alpha_1\beta_1^6\beta_2$					
14						$\beta_1^6\beta_2$						$\beta_1^6\beta_3$
13		$\alpha_1\beta_1^5\beta_2$						$\alpha_1\beta_1^5\beta_3$				
12	$\beta_1^5\beta_2$						$\beta_1^5\beta_3$					
11			$\alpha_1\beta_1^4\beta_3$						$\alpha_1\beta_1^4\beta_4$		$\alpha_1\beta_1^4\varepsilon_0$	
10		$\beta_1^4\beta_3$						$\beta_1^4\beta_4$		$\beta_1^4\varepsilon_0$		
9				$\alpha_1\beta_1^3\beta_4$		$\alpha_1\beta_1^3\varepsilon_0$				$\alpha_1\beta_1^2\beta_2\beta_4$		
8			$\beta_1^3\beta_4$		$\beta_1^3\varepsilon_0$				$\beta_1^2\beta_2\beta_4$		$\alpha_1\beta_1^2\eta_1$	
7	$\alpha_1\beta_1^2\varepsilon_0$				$\alpha_1\beta_1\beta_2\beta_4$					$\beta_1^2\eta_1$		$\beta_1\kappa_1$
6				$\beta_1\beta_2\beta_4$		$\alpha_1\beta_1\eta_1$				$\beta_1^2\beta_6$		$\alpha_1\beta_1\eta_2$
5					$\beta_1\eta_1$		κ_1				$\beta_1\eta_2$	
4	$\alpha_1\eta_1$				$\beta_1\beta_6$		$\alpha_1\eta_2$				$\beta_1\beta_7$	
3	$\alpha_1\beta_6$					η_2	$\alpha_1\beta_7$					η_3
2						β_7						β_8
1	α_{36}	α_{37}	α_{38}	α_{39}	α'_{40}	α_{41}	α_{42}	α_{43}	α_{44}	α'_{45}	α_{46}	α_{47}

$\alpha_1\eta_1 = \beta_2\varepsilon_0$

$\alpha_1\eta_2 = \beta_3\varepsilon_0$

TABLE III: p = 5

20			β_1^{10}						$\beta_1^9\beta_2$			
19					$\alpha_1\beta_1^8\beta_2$						$\alpha_1\beta_1^8\beta_3$	
18				$\beta_1^8\beta_2$						$\beta_1^8\beta_3$		
17						$\alpha_1\beta_1^7\beta_3$						$\alpha_1\beta_1^7\beta_4$
16					$\beta_1^7\beta_3$						$\beta_1^7\beta_4$	
15	$\alpha_1\beta_1^6\beta_3$						$\alpha_1\beta_1^6\beta_4$		$\alpha_1\beta_1^6\varepsilon_0$			
14						$\beta_1^6\beta_4$		$\beta_1^6\varepsilon_0$				$\beta_1^5\beta_2\beta_4$
13		$\alpha_1\beta_1^5\beta_4$		$\alpha_1\beta_1^5\varepsilon_0$				$\alpha_1\beta_1^4\beta_2\beta_4$				
12	$\beta_1^5\beta_4$		$\beta_1^5\varepsilon_0$				$\beta_1^4\beta_2\beta_4$		$\alpha_1\beta_1^4\eta_1$			
11			$\alpha_1\beta_1^3\beta_2\beta_4$					$\beta_1^4\eta_1$		$\beta_1^3\kappa_1$		
10		$\beta_1^3\beta_2\beta_4$		$\alpha_1\beta_1^3\eta_1$				$\beta_1^4\beta_6$		$\alpha_1\beta_1^3\eta_2$		
9			$\beta_1^3\eta_1$		$\beta_1^2\kappa_1$				$\beta_1^3\eta_2$		$\beta_1^2\kappa_2$	
8			$\beta_1^3\beta_6$		$\alpha_1\beta_1^2\eta_2$				$\beta_1^3\beta_7$		$\alpha_1\beta_1^2\eta_3$	
7				$\beta_1^2\eta_2$		$\beta_1\kappa_2$			$\alpha_1\beta_1\varepsilon_0^2$	$\beta_1^2\eta_3$		$\beta_1\kappa_3$ $\beta_2\beta_4\varepsilon_0$
6				$\beta_1^2\beta_7$		$\alpha_1\beta_1\eta_3$		$\beta_1\varepsilon_0^2$		$\beta_1^2\beta_8$		
5	κ_2			$\alpha_1\varepsilon_0^2$	$\alpha_1\varepsilon_0\varepsilon_1$ $\beta_1\eta_3$	$\alpha_1\beta_1\beta_8$	κ_3				$\beta_2\eta_3$	
4	$\alpha_1\eta_3$		ε_0^2	$\varepsilon_0\varepsilon_1$	$\varepsilon_0\varepsilon_2$ $\beta_1\beta_8$	$\varepsilon_0\varepsilon_3$			$\alpha_1\gamma_2$		$\beta_1\beta_9$	
3	$\alpha_1\beta_8$						$\alpha_1\beta_9$?	ν γ_2	$\alpha_1\varepsilon_{20}$	$\alpha_1\varepsilon_{21}$	$\alpha_1\varepsilon_{22}$	η_5?
2						β_9		ε_{20}	ε_{21}	ε_{22}	ε_{23}	β_{10}
1	α_{48}	α_{49}	α''_{50}	α_{51}	α_{52}	α_{53}	α_{54}	α'_{55}	α_{56}	α_{57}	α_{58}	α_{59}

$\alpha_1\eta_3 = \beta_4\varepsilon_0$

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ABSTRACT

Let E be a ring-spectrum such that E_*E is flat over $E_* = \pi_*(E)$. In this thesis we study homological algebra over the "Hopf algebra" of cooperations E_*E, after Adams. We construct a cobar construction and use it to produce $\cup_i$-products, and if $pE_* = 0$ Steenrod operations, in Ext_{E_*E}. An ideal $I \subset E_*$ invariant under the E_*E-coaction determines a spectral sequence by filtering the resolution by powers of I. Cases of this construction are various Bockstein spectral sequences and the "algebraic Adams spectral sequence" of Novikov. With these tools we obtain a vanishing line for $\mathrm{Ext}_{BP_*BP}(BP_*, M)$ for a comodule M, and we compute $\mathrm{Ext}_{BP_*BP}(BP_*BP_*)$ in a band of width p^2q, $q = 2p-2$, above the vanishing line. Combined with known facts about $\pi_*^s(S^0)$, this yields $\mathrm{Ext}^{s,tq}_{BP_*BP}(BP_*, BP_*)$ in a larger range: through $t = 23$ for $p = 3$ and through $t = 49$ for $p = 5$. This comparison reveals several nontrivial differentials besides the Cohen-Toda-Zahler differential.