THE MODULI VARIETY FOR FORMAL GROUPS

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Fix an algebraically closed field k. A (one-parameter, commutative) formal group law over k is an element $F \in k[[X, Y]]$ such that

1. $F(X, Y) = F(Y, X) = X + Y +$ higher order terms.
2. $F(F(X, Y), Z) = F(X, F(Y, Z))$.

Let Λ denote the set of such formal group laws. By a theorem of Lazard [4], there exists a ring L carrying a *universal* formal group law; consequently Λ can be identified with the set of ring homomorphisms $L \to k$. This may be used to enrich the structure of the set Λ; it gains a Zariski topology, and a sheaf of rings making it into a ringed space; in fact with this data, Λ would be an algebraic space [7] in the sense of Serre, except that it is not noetherian.

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Now let Γ denote the group of formal power series $f \in k[[T]]$ of the form $f(T) = uT +$ higher order terms, $u \neq 0$, the group operation being composition. This is a (nonnoetherian) algebraic group, which acts on Λ by changing coordinates: if $f \in \Gamma$, $F \in \Lambda$, then we define $F^f(X, Y) = f^{-1}F(fX, fY)$. It is not hard to see that $(f, F) \to F^f$ defines a morphism $\Gamma \times \Lambda \to \Lambda$ of (pro)-algebraic spaces of Serre. In this note we discuss this group action when the characteristic of k is positive (the char 0 case being trivial). In a succeeding note we will apply these results to the study of unitary cobordism, via Quillen's theorem, which identifies the ring L with the unitary cobordism ring of a point [6]. We remark here that Γ is the algebraic group underlying the Landweber-Novikov algebra of operations for cobordism.

THEOREM 1. *Λ is stratified into orbits, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_\infty$, such that*

(a) $\bar{\Lambda}_n = \bigcup_{m \geq n} \Lambda_m$; *$\Lambda_1$ is open, and Λ_∞ is closed.*

(b) *$\bar{\Lambda}_n$ is a complete intersection of hyperplanes.*

(c) *Λ_n is homogeneous under Γ; for finite n, there exists a p-adic Lie group G_n such that $\Gamma/G_n \xrightarrow{\sim} \Lambda_n$.*

(d) *The normal bundle of Λ_n in Λ is given by an $(n-1)$-dimensional representation of G_n over k.*

REMARK. The Λ_n can be described explicitly in terms of Milnor's generators of the unitary cobordism ring: Λ_n is the locus where $p = p_1 = \cdots = p_{n-1} = 0$ and p_n is a unit, where p_k is a Milnor generator of dimension $2(p^k - 1)$. A theorem of Landweber [3] is a corollary: the ideals $(p, p_1, \cdots, p_n)$ in the unitary cobordism ring are the only prime ideals invariant under the Landweber-Novikov algebra.

To complete the description of the orbit structure we must identify the groups G_n and the representations in (d). For this we recall that central simple division algebras over the p-adic numbers Q_p are completely classified by the rank (as Q_p-vector spaces) and Brauer invariant, which lies in Q/Z.

Let D_n be such a division algebra of rank n^2 and Brauer invariant $1/n$. There is a natural valuation on D_n, $v: D_n^* \to Q_p^* \to Z$, the former arrow being the norm, and the latter being p-adic valuation. Let $E_n = \{x \in D_n \mid v(X) \geq 0\}$ denote the ring of integers of D_n.

Now consider the twisted polynomial algebra $k\langle \tilde{\delta} \rangle$ defined by the relation $\lambda^p \tilde{\delta} = \tilde{\delta}\lambda$, $\lambda \in k$; we abbreviate by M_n the quotient ring modulo the (two-sided) ideal generated by $\tilde{\delta}^n$. It can be shown [1, p. 80] that E_n/pE_n is isomorphic to $F_q\langle \tilde{\delta} \rangle/(\tilde{\delta}^n)$, where $q = p^n$ and $F_q\langle \tilde{\delta} \rangle$ is defined like $k\langle \tilde{\delta} \rangle$. Consequently M_m is a *right* E_n-module, whenever $n \geq m$.

THEOREM 2. *The stabilizer G_n is isomorphic to the group of units of E_n. The representation of* (1d) *has M_{n-1} as underlying k-vector space, with G_n-action given by*

$$g(v) = vg^{-1}, \qquad v \in M_{n-1}, g \in G_n \subset E_n.$$

REMARKS. G_n is a profinite group over F_p, but it can be given a p-adic analytic structure. As such it is a form, in the sense of Galois cohomology, of $GL(n, Z_p)$.

From the description of the normal bundle of Λ_n in Λ just given, it is possible to read off the normal representation of Λ_n in Λ_m, $n \geqq m$. It is also possible to identify the stabilizer of the infinite stratum: $\Lambda_\infty = \Gamma/G_\infty$, where G_∞ is the group of units of $k\langle\langle \mathfrak{F}_\tau \rangle\rangle$, considered as a proalgebraic group over F_p. This is just the group underlying the reduced Steenrod algebra (at the prime p).

NOTES ON THE PROOFS. Theorem 1(a) is just a restatement of a theorem of Lazard [4]: two formal group laws over an algebraically closed field are isomorphic iff they are of the same height. Thus Λ_n is the moduli variety of formal groups of height n, and part (b) is proved by applying Lazard's techniques to the moduli functor of formal groups of height n. Part (d) is trivial.

To prove (c), we use a theorem of Grothendieck [2, III, §3]. Let G, X be respectively a groupscheme and a scheme upon which G acts, both noetherian over k, with X smooth. Then X is a homogeneous space of G iff $G(k)$ acts transitively on $X(k)$. (Here $G(k)$ and $X(k)$ are the k-valued points of G, X.)

Now the Λ_n are represented by localizations of polynomial rings and are smooth, but neither they nor Γ are noetherian. However the above result can be extended to *proalgebraic* group actions of a certain kind. Thus we let $\Lambda_n(\deg r)(A)$ denote the set of r-buds of a formal group over A, of height n ($r \geqq p^n$). Then $\Lambda_n = \text{proj lim}_r \Lambda_n(\deg r)$, the maps being surjections for any A; it is not hard to see that $\Lambda_n(\deg r)$ is a smooth, noetherian scheme. Similarly, $\Gamma_r(A)$ is the set of invertible series in $A[T]/(T^{r+1})$, and acts on $\Lambda_n(\deg r)$, compatibly with truncations. Using these approximations systematically, we prove (c).

The identification of G_n is due to Dieudonné and Lubin; see also [1, Theorem 3, p. 72]. To identify the normal representation, we show by direct computation, following [5], that the tangent space to Λ at F is the group $Z_s^2(F; k)$ of 2-cocycles of F, while the tangent space to Λ_n at F, n being the height of F, is the group $B_s^2(F; k)$ of 2-coboundaries. Thus the normal bundle is given by the 2-cohomology representation $H_s^2(F; k)$. A basis for this group is approximately known, and one checks directly that the representation is as indicated.

BIBLIOGRAPHY

1. A. Fröhlich, *Formal groups*, Lecture Notes in Math. vol. 74, Springer, New York, 1968. MR 39 #4164.

2. P. Gabriel and M. Demazure, *Groupes algébriques*, Tom. I, North-Holland, Amsterdam, 1970.

3. P. S. Landweber, *Annihilator ideals and primitive elements in complex cobordism*, Rutgers University, (preprint).

4. M. Lazard, *Sur les groupes de Lie formels à un paramètere*, Bull. Soc. Math. France 83 (1955), 251–274. MR 17, 508.

5. J. Lubin and J. Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France 94 (1966) 49–59. MR 39 #214.

6. D. Quillen, *Elementary proofs of some results in cobordism theory* III, Advances in Math. 7 (1971).

7. J. P. Serre, *Faisceaux algébrique cohérentes*, Ann. of Math. (2) 61 (1955), 197–278. MR 16, 953.

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