0. Introduction. This paper examines the homotopy type of the Thom spectrum $MSU$ associated with special unitary cobordism. For odd primes $p$, standard methods show that the $p$-localization $MSU(p)$ is equivalent to a wedge of suspensions of the Brown-Peterson spectrum $BP$. For $p = 2$, however, this is not the case, and our work is devoted to determining the 2-primary homotopy type of $MSU$. This involves a new indecomposable spectrum, and our main results are the following.

There is an indecomposable 2-local spectrum, which we call $BoP$, such that $MSU(2)$ is equivalent to a wedge of suspensions of $BoP$ and $BP$. Under the equivalence, the Thom class lies in a $BoP$ summand. As a comodule over the dual Steenrod algebra $A$ [11], $H_*(BoP; \mathbb{Z}/2)$ is a sum of suspensions of $B = \mathbb{Z}/2[\xi_1^2, \xi_2^2, \ldots, \xi_j^2, \ldots] \subset A$, where $\xi_j$ is the conjugate of Milnor's generator $\xi_j$. There is one suspension of $B$ beginning in each nonnegative dimension divisible by 8.

$BoP$ bears strong similarities to $BP$ and the $(-1)$-connected $K$-theory spectra $bo$ and $bu$. In particular, in Section 6 we show there is a map $BoP \to bo(2)$ inducing an epimorphism $v_*$ of homotopy groups. In fact, $v_*$ induces an isomorphism of torsion subgroups, and its torsion free kernel is concentrated in even dimensions.

A brief summary of our methods is as follows. In Sections 1 and 2 we describe the Adams spectral sequence for $\pi_*MSU(2)$, including a computation of the differentials, with particular attention paid to the product structure. Anderson, Brown, and Peterson [4] gave a computation for these differentials, but their proof requires some correction, and in any case we will need the more extensive knowledge of the product structure.

In Sections 3 to 5, we construct $BoP$ and show it is indecomposable. To produce $BoP$, first the Sullivan-Baas construction is applied to $MSU$, yielding a spectrum representing a bordism theory of $SU$-manifolds with
certain singularities. Then we produce a map from the 2-localization of this spectrum to a wedge of \(BP\) suspensions, and the fibre is the desired spectrum \(BoP\).

Sections 6 and 7 are devoted to producing a homotopy equivalence between \(MSU(2)\) and a wedge of \(BoP\) and \(BP\) suspensions. Maps from \(MSU(2)\) to a suspension of \(BoP\) are somewhat difficult to construct, so maps to a suspension of \(bo(2)\) are constructed first, using the Adams spectral sequence, and then lifted to \(BoP\) by obstruction theory.

Several of the indecomposable spectra which, like \(BoP\), appear as summands in cobordism Thom spectra, have proven extremely useful in homotopy theory. The most notable are the Eilenberg-MacLane and Brown-Peterson spectra, upon which the Adams and Novikov spectral sequences are based. Hopefully, \(BoP\) too will have a useful role to play in homotopy theory. In particular, a generalized Adams-Novikov spectral sequence based on \(BoP\) has the advantage that the Hopf map \(\eta \in \pi_1 \Sigma^0\) appears on the zero line. To apply \(BoP\) effectively, it would be useful to find a canonical description for it similar to Quillen’s construction [1] of \(BP\), and to know that \(BoP\) is a commutative ring spectrum. It must also be shown that \(BoP\) has better flatness properties than \(bo\).

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1. The Mod Two Homology of MSU. The Thom spectrum \(MSU\) is a commutative ring spectrum. Thus \(H_*(MSU; \mathbb{Z}/2)\) is a graded left \(A\) comodule algebra [16], whose structure we will describe below.

Henceforth, ‘\(A\) algebra’ means ‘graded left \(A\) comodule algebra’, unstated coefficient groups are \(\mathbb{Z}/2\), and \(\otimes\) means \(\otimes_{\mathbb{Z}/2}\). We give the polynomial algebra \(C = \mathbb{Z}/2[x_8, x_{10}, \ldots, x_{2i} (i \neq 2^j - 1), \ldots]\) an \(A\) algebra structure by letting \(x_{2i}\) be in grade \(2i\), and defining the coaction map by \(\psi x_{4i} = \sum_1 \otimes x_{4i-2} + 1 \otimes x_{4i}\) for \(i \neq 2^j\), \(\psi x_{4i-2} = 1 \otimes x_{4i-2}\), and \(\psi x_{2j} = 1 \otimes x_{2j}\).

Since the subalgebra \(B \subset A\) defined in the introduction is in fact a sub-\(A\) algebra of \(A\), we can give \(B \otimes C\) the natural \(A\) algebra structure of a tensor product. In previous work [13], we showed that

**Theorem 1.1.** There is an isomorphism \(H_* MSU \cong B \otimes C\) of \(A\) algebras.
We identify $H_\ast MSU$ and $B \otimes C$ via such an isomorphism, and now proceed to analyze $B \otimes C$. Let $E(x)$ denote the primitive exterior Hopf algebra on $x$, and let $E$ be the quotient Hopf subalgebra $E(\xi^2)$ of $A$. Let $P$ be the sub-$A$ algebra $\mathbb{Z}/2[\xi^2, \xi^2_2, \ldots, \xi^2_{j_1}, \ldots] \subset A$, which is isomorphic to $H_\ast BP$ and dual to the quotient $A^\ast/A^\ast Sq^1 A^\ast$ [7].

**Lemma 1.2.** There is an $A$ algebra isomorphism $B \cong P \otimes E \mathbb{Z}/2$. 

**Proof.** The isomorphism $\mathbb{Z}/2[\xi^2, \xi^2_2, \ldots] \cong A \otimes_{E(\xi_1)} \mathbb{Z}/2$ is well known [16, p. 511], and squaring provides a Hopf algebra isomorphism $A \cong P$. □

We will also need the following presumably well-known fact.

**Proposition 1.3.** If $H$ is a connected graded commutative $\mathbb{Z}/2$ Hopf algebra, $D$ is an $H$ algebra, and $E$ is a quotient Hopf algebra of $H$, then $(m \otimes 1) \circ (1 \otimes x \otimes 1) \circ (1 \otimes \psi_D) : H \otimes E \otimes D \to (H \otimes E \mathbb{Z}/2) \otimes D$ is defined and is an $H$ algebra isomorphism.

**Proof.** After taking $\mathbb{Z}/2$ duals, the formula for the map and the fact that it is an isomorphism follow from the special case $N = \mathbb{Z}/2$ of Proposition 1.7 in [10] along with the commutativity of $H$. It is straightforward to check that it is an $H$ algebra map. □

Together Lemma 1.2 and Proposition 1.3 imply

**Corollary 1.4.** $(m \otimes 1) \circ (1 \otimes x \otimes 1) \circ (1 \otimes \psi) : P \otimes E \mathbb{C} \to B \otimes C$ is a $P$ algebra isomorphism and hence an $A$ algebra isomorphism.

Next we examine $P \otimes E \mathbb{C}$. By analyzing the $E$ comodule structure of $C$, we will be able to express $P \otimes E \mathbb{C}$ as a sum of cocyclic $A$ comodules. Let $\bar{\psi}$ be the quotient $E$-coaction map, and define $Sq^2 : C \to C$ by $\bar{\psi}c = \xi^2_1 \otimes Sq^2 c + 1 \otimes c$. $Sq^2$ is a differential and a derivation. Define

$$y_{8i} = \begin{cases} x_{8i} & \text{if } i = 2^l \\ x_{4i}^2 & \text{if } i \neq 2^l \end{cases} \text{ for } i \geq 1,$$

and let $Y = \mathbb{Z}/2[y_{8}, \ldots, y_{8i}, \ldots] \subset C$. Note $Sq^2$ acts trivially on $Y$. For $i \neq 2^l$ let $R_i$ be the subspace of $C$ spanned by $\{x_{4i-2}^n, x_{4i}x_{4i-2}^n | n \geq 0\}$. $R_i$ is closed under $Sq^2$. Let $R = \bigotimes_{i \neq 2^l} R_i$ with diagonal $Sq^2$ action. The natural map $R \otimes Y \to C$ is clearly an $E(Sq^2)$ module isomorphism.

Since $H_\ast(R_1, Sq^2) = \mathbb{Z}/2$, $R$ is a sum of a single trivial summand in
grade zero and a free $E$ comodule with primitives $Sq^2 R$. We will write $R'$ for $Sq^2 R$. The desired description of $P \square_E C$ now follows:

**Proposition 1.5.** There is a sequence of $A$ comodule isomorphisms

$$P \square_E C \cong (P \square_E R) \otimes Y \cong (P \square_E (\mathbb{Z}/2 \oplus (E \otimes R'))) \otimes Y$$

$$\cong (B \oplus (P \otimes R')) \otimes Y.$$  

We will need to know something about the algebra structure in this description. Notice $P \triangleleft_E C$ contains the $A$ subalgebra $B \otimes Y$. The $A$ algebra inclusion $B \subseteq P$ provides an obvious $B$ module structure on $B \oplus (P \otimes R')$, and hence a $B \otimes Y$ module structure on $(B \oplus (P \otimes R')) \otimes Y$. The composite isomorphism of (1.5) is clearly a $B \otimes Y$ module map.

2. The Adams Spectral Sequence for $\pi_* MSU_{(2)}$. The $E_2$ term of the 2-primary Adams spectral sequence [1] converging to $\pi_* MSU_{(2)}$ is given by $\text{Ext}^*_{\mathcal{A}}(\mathbb{Z}/2, H_* MSU)$. We abbreviate $\text{Ext}^*_{\mathcal{A}}(\mathbb{Z}/2, -)$ as $\text{Ext}(-)$. By (1.5) we need only know $\text{Ext}(B)$ and $\text{Ext}(P)$ to describe $E_2$. They are given by

**Theorem 2.1.** [9]. There are isomorphisms

$$\text{Ext}(B) \cong \mathbb{Z}/2[q_0, h, q_1^2q_0, q_1^4, q_2, \ldots, q_k, \ldots]/(q_0h, h^3)$$

and

$$\text{Ext}(P) \cong \mathbb{Z}/2[q_0, q_1, q_2, \ldots, q_k, \ldots],$$

where $h \in \text{Ext}^{1,2}$ and $q_i \in \text{Ext}^{1,2i+1-1}$. Under the isomorphisms the map $B \to P$ induces the obvious algebra map.

Applying Theorem 2.1 to the description of $H_* MSU$ provided by (1.5) immediately yields

**Theorem 2.2.**

$$\text{Ext}(H_* MSU) \cong (\text{Ext}(B) \oplus (\text{Ext}(P) \otimes R')) \otimes Y$$

$$\cong (\mathbb{Z}/2[q_0, h, q_1^2q_0, q_1^4, q_2, \ldots, q_k, \ldots]/(q_0h, h^3)$$

$$\oplus (\mathbb{Z}/2[q_0, q_1, q_2, \ldots, q_k, \ldots] \otimes R')) \otimes Y,$$
with the $\text{Ext}(B) \otimes Y$ module structure induced on the tensor product in the obvious way using the algebra map $\text{Ext}(B) \to \text{Ext}(P)$ described in (2.1).

We will now use this description of $\text{Ext}(H_*MSU)$, along with a result of Conner and Floyd, to determine the differentials in the spectral sequence. The reader is urged to construct a picture of the $E_2$ term as described in (2.2). We will make frequent use of the following three lemmas, all proven in [4].

**Lemma 2.3.** $h$ is a permanent cycle, and for each $r$, $E_r^{s,t} \xrightarrow{h} E_r^{s+1,t+2}$ is an epimorphism if $t - s$ is even, a monomorphism if $t - s$ is odd.

**Lemma 2.4.** $d_2$ is zero on the summand $\text{Ext}(P) \otimes R' \otimes Y$ of $E_2$.

**Lemma 2.5.** $q_1 \cdot q_0$ and $q_4 \cdot q_1$ are permanent cycles.

Since $d_2$ is a derivation, it is completely determined by (2.4), (2.5), and the following theorem.

**Theorem 2.6.** There are elements

$$q'_j \in E_2^{1,2j+1-1}, \text{ for } j \geq 2, \text{ and}$$

$$y'_i \in E_2^{0,8i}, \text{ for } i \geq 1,$$

of the form

$$q'_j = q_j + \text{decomposables in } \mathbb{Z}/2[q_2, \ldots, q_k, \ldots] \otimes Y,$$

and

$$y'_i = y_i + \text{decomposables in } Y,$$

such that

$$d_2q'_j = 0,$$

and

$$d_2y'_i = hq'_{i-1},$$

and

$$d_2y'_i = 0 \text{ if } 8i \text{ is not a power of two.}$$
Before proving the theorem, we will see how it immediately leads to the description of $E_2$ and $d_2$ that we desire. Let $\text{Ext}(B)'$ denote the subalgebra $\mathbb{Z}/2[q_0, h, q_1q_0, q_4, q_2', \ldots, q_k, \ldots]$ of $E_2$ in which the element $q_k$ in $\text{Ext}(B)$ has been replaced by $q_k'$ for $k \geq 2$. From (2.2) and (2.6) we have

**Corollary 2.7.** There is an isomorphism $E_2 \cong (\text{Ext}(B)'+(\text{Ext}(P) \otimes R')) \otimes Y$ with $Y = \mathbb{Z}/2[y_1', \ldots, y_i', \ldots]$, and the differential $d_2$ is described explicitly by (2.4), (2.5), and (2.6).

**Proof of Theorem 2.6.** Suppose inductively that appropriate $q_j'$ and $y_k'$ have been found for all $j$ such that $2^{j+1} < 8k$ and all $i$ such that $8i < 8k$.

Let $l$ be the largest integer such that $2^l < 8k$. Let $G_k' = \mathbb{Z}/2[q_4, q_2, q_2', q_1'] \otimes \mathbb{Z}/2[y_1', \ldots, y_{8(k-1)}'] \subset E_2'^{*,*}$. Define a derivation $d: G_k' \to G_k' + 1$ by letting $dy_j' = q_j' - 1$ for $3 < j < l$ and letting $d = 0$ on all the other polynomial generators of $G_k'$. Notice that on $G_k'$, $d_2 = h \cdot d$ by the inductive hypothesis, and $d$ is a differential. $H^{*,*}(G_k; d)$ is easily computed using the Künneth theorem, and we find that $H^{s,t}$ is nonzero only if $t - s \equiv 0 \pmod{8}$.

**Case I.** $8k$ is not a power of two. Consider the ‘column’ $E_2'^{s,t}$ with $t - s = 8k - 1$, and the map

$$h \cdot (\ker d)^{s-1,t-2}/h \cdot (\im d)^{s-1,t-2} \to (\ker d_2)^{s,t}/(\im d_2)^{s,t}.$$  

Using (2.3) we see the numerators are equal. The left group is zero since $H^{s-1,t-2}(G_k; d) = 0$, and thus the two denominators are equal. So $d_2y_{8k} = d_2y$ for some $y \in G_k^{0,8k}$. Defining $y_{8k}' = y_{8k} + y$ completes the inductive step.

**Case II.** $8k = 2^m$ for some $m$. First we will examine the ‘column’ $E_2'^{s,t}$ with $t - s = 2^m - 3$. The same argument as in Case I shows that $d_2q_{m-1}' = d_2x$ for some $x \in G_k^{1,2m-1}$. Define $q_{m-1}' = q_{m-1} + x$, so $d_2q_{m-1}' = 0$. The same argument also shows that $E_3'^{s,t} = 0$ for $t - s = 2^m - 3$, so $q_{m-1}'$ is a permanent cycle, and thus so is $h \cdot q_{m-1}' \in E_2^{2m+1,2m+1}$. From Conner and Floyd’s work [8] we know $\pi_{2m-1} \text{MSU}(2) = 0$, so $h \cdot q_{m-1}'$ must be in the image of a differential. The only possibility is $d_2: E_2'^{0,2m} \to E_2^{2,2m+1}$. Now $E_2^{0,2m} \cong G_k^{0,2m} \oplus \{y_{2m}\} \oplus (R' \otimes Y)^{0,2m}$. From (2.4) we know $d_2 = 0$ on the rightmost summand. Inductively $d_2$ has been determined on $G_k^{0,2m}$, and $h \cdot q_{m-1}'$ is clearly not in the image. Thus there must
be an element $y \in G_k^{0,2m}$ such that $d_2(y_{2m} + y) = h \cdot q_{m-1}$. Let $y_{2m} = y_{2m} + y$.

Finally, we will show that all higher differentials are zero. Let $G_{**}^* = \mathbb{Z}/2[q^4, q^6, \ldots, q^{2k}, \ldots] \otimes \mathbb{Z}/2[y^4, y^6, \ldots, y^8, \ldots, y^i, \ldots] \subset E^2_{**}^*$, with $d$ as in the proof of (2.6). $H_{**}^*(G; d)$ is easily computed, and $H^s(G; d) = 0$ unless $t - s \equiv 0 \pmod{8}$. Now if $t - s$ is odd, $E_3^{s,t} \equiv H^{s-1,t-2}(G; d)$, since the map

$$h \cdot (\ker d)^{s-1,t-2}/h \cdot (\im d)^{s-1,t-2} \to (\ker d_2)^{s,t}/(\im d_2)^{s,t}$$

is an isomorphism, and multiplication by $h$ maps $G$ monomorphically in $E_2$. Thus $E_3$ is rather sparse in the sense that $E_3^{s,t} = 0$ for $t - s \equiv 3, 5, \text{ or } 7 \pmod{8}$. This will enable us to prove

**Proposition 2.8.** All the higher differentials $d_r: E^{s,t}_r \to E^{s+r,t+r-1}_r$, for $r \geq 3$, are zero.

**Proof.** By the sparseness of $E_3$ this is obvious except when $t - s \equiv 1$ or $2 \pmod{8}$.

**Case I.** $t - s \equiv 1 \pmod{8}$. Given $x \in E^{s,t}_r$, by (2.3) $x = h \cdot y$ for some $y$. Now $d_r y = 0$ by sparseness, so $d_r x = h \cdot d_r y = 0$.

**Case II.** $t - s \equiv 2 \pmod{8}$. Given $x \in E^{s,t}_r$, by (2.3) we have $h \cdot d_r x = d_r (h \cdot x) = d_r (0) = 0$, so $d_r x = 0$, again by (2.3). $\square$

3. The Sullivan-Baas Construction. In this section we will apply the Sullivan-Baas construction [6] to MSU to produce a spectrum whose 2-localization is closely related to the indecomposable spectrum $BoP$ we seek.

First we describe a sequence of elements in $\pi_* MSU$ for use with the Sullivan-Baas construction. Define $z_{8i} \in Y$ for $i \geq 2$ by

$$z_{8i} = \begin{cases} y_{8i}^2 & \text{if } i \neq 2j \\ y_{4i} & \text{if } i = 2j \end{cases}$$

In our description of the Adams spectral sequence, $z_{8i} \in E_2^{0,8i}$ survives to $E_\infty^{0,8i}$ by (2.6) and (2.8), and is thus represented by an element $\hat{z}_{8i} \in \pi_{8i} MSU$. If $h: \pi_* MSU \to H_* MSU$ is the Hurewicz homomorphism, $h(\hat{z}_{8i}) = z_{8i}$. 

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Lemma 3.1. The sequence \( \{z_{8i}\}_{i \geq 2} \) is a regular sequence in \( H_*MSU \).

Proof. We will follow \( z_{8i} \in Y \) under the composite \( S: Y \cong \mathbb{Z}/2 \otimes Y \subset P \boxtimes E C \to B \otimes C \) involving the map \((m \otimes 1) \circ (1 \otimes X \otimes 1) \circ (1 \otimes \psi)\) of (1.4). Notice that

\[
S'(y_{8i}) = \begin{cases}
S(x_{4i}^2) = x_{4i}^2 & \text{if } i \neq 2j \\
S(x_{8i}) = 1 \otimes x_{8i} & \text{if } i = 2j.
\end{cases}
\]

So if we write \( B \otimes C = \mathbb{Z}/2[\xi_2, \xi_4, \ldots, \xi_{2i}, \ldots] \otimes \mathbb{Z}/2[x_8, x_{10}, \ldots, x_{2i} \ (i \neq 2j - 1), \ldots] \) as \( \mathbb{Z}/2[w_4, w_6, w_8, \ldots, w_{2i}, \ldots] \) in the obvious way, with \( w_{2i} \) in grade \( 2i \), we see that \( S(Y) = \mathbb{Z}/2[w_{12}^2 + w_4w_{10}, \ldots, w_{4i}^2 + w_4w_{4i-2} \ (i \neq 2j), \ldots, w_8, w_{16}, \ldots, w_{2j}, \ldots] \), and

\[
S'(z_{8i}) = \begin{cases}
w_{4i}^2 + w_4w_{4i-2} + \text{decomposables in } S(Y) & \text{if } i \neq 2j \\
(w_{4i} + \text{decomposables in } S(Y))^2 & \text{if } i = 2j.
\end{cases}
\]

\( \{S'(z_{8i})\}_{i \geq 2} \) is clearly a regular sequence in \( \mathbb{Z}/2[w_4, w_6, w_8, \ldots, w_{2i}, \ldots] \).

Those aspects of the Sullivan-Baas construction relevant to our needs are summarized by

Theorem 3.2. Let \( \{*, [M_1], \ldots, [M_n], \ldots\} \) be a sequence of elements in \( \Omega_*^{SU} \) (point) \( \cong \pi_*MSU \), and let \( m_n = h([M_n]) \in H_*MSU \). Suppose \( \{m_n\}_{n \geq 1} \) is a regular sequence in the algebra \( H_*MSU \). Then there are CW-spectra \( M(n) \) for \( n \geq 0 \) and maps \( M(n) \overset{p_n}{\longrightarrow} M(n + 1) \) with \( M(0) = MSU \), such that the composite \( M(0) \overset{p_n}{\longrightarrow} M(1) \overset{p_{n-1}}{\longrightarrow} \cdots \overset{p_2}{\longrightarrow} M(n) \) is an epimorphism in mod 2 homology with kernel the ideal generated by \( \{m_1, \ldots, m_n\} \).

Proof. Let \( S_n = \{*, [M_1], \ldots, [M_n]\} \) and let \( MSU(S_n)_*(-) \) be the bordism theory of \( SU \)-manifolds with singularity set \( S_n \) (see Baas [6]). There are long exact sequences

\[
\cdots \to MSU(S_n)_*(-) \overset{\beta_n = \{M_{n+1}\}}{\longrightarrow} MSU(S_n)_* + \dim M_{n+1}(-) \overset{\gamma_n}{\longrightarrow} MSU(S_{n+1})_* + \dim M_{n+1}(-) \overset{\delta_n}{\longrightarrow} MSU(S_n)_*-1(-) \to \cdots
\]
relating the corresponding (reduced) generalized homology theories $MSU(S_n)_*(-)$ and $MSU(S_{n+1})_*(-)$ on the category of CW-spectra. It is clear from the bordism definition of $MSU(S_n)_*(-)$ that it satisfies the direct limit axiom. Thus [3] it is represented by a CW-spectrum $M(n)$, and there is a map $M(n) \xrightarrow{p_n} M(n + 1)$ inducing the natural transformation $\gamma_n$. Of course $M(0) = MSU$.

If we apply the sequences (3.3) to the Eilenberg-MacLane spectrum $K(\mathbb{Z}/2)$, then $\beta_n$ is just multiplication by $m_{n+1} = h([M_{n+1}])$, so we see inductively that it is a monomorphism, and conclude that the composite $M(0) \xrightarrow{p_0} M(1) \rightarrow \cdots \xrightarrow{p_{n-1}} M(n)$ has the desired property. \square

Letting $M = \lim_{\rightarrow \mathcal{P}_n} M(n)$, we have

**Corollary 3.4.** With the hypotheses of Theorem 3.2, there is a CW-spectrum $M$ and a map $MSU \xrightarrow{p} M$, such that $p$ is an epimorphism in mod 2 homology with kernel the ideal generated by $\{m_1, \ldots, m_n, \ldots\}$.

Applying (3.4) to the sequence $\{z_{8i}\}_{i \geq 2} \subset \pi_*MSU$ already described, using (3.1) and localizing at the prime 2, we obtain

**Proposition 3.5.** There is a 2-local CW-spectrum $X$, and a map $MSU(2) \xrightarrow{p} X$, such that $p$ is an epimorphism in mod 2 homology with kernel the ideal generated by $\{z_{8i}\}_{i \geq 2}$.

**4. The Adams Spectral Sequence for $\pi_*X$.** We now examine the 2-primary Adams spectral sequence converging to $\pi_*X$ by studying the map of spectral sequences induced by $MSU(2) \xrightarrow{p} X$. We begin by examining the map of $E_2$ terms.

Recall that Proposition 1.5 identified the $A$ comodule $H_*MSU$ as $(B \oplus (P \otimes R')) \otimes Y$, with the obvious $B \otimes Y$ module structure from the subalgebra $B \otimes Y$. Let $Z = \mathbb{Z}/2[z_{16}, \ldots, z_{8i}, \ldots]$ be the polynomial subalgebra of $Y$ generated by the regular sequence introduced in Section 3. It follows from (3.5) that in homology $p$ induces the natural map

$$(B \oplus (P \otimes R')) \otimes Y \xrightarrow{p_2} (B \oplus (P \otimes R')) \otimes Y \mathbin{\|} Z,$$

where $Y \mathbin{\|} Z$ denotes the algebra quotient by the ideal generated by $\bar{Z}$. The induced map of $E_2$ terms is the natural projection

$$(\text{Ext}(B) \oplus (\text{Ext}(P) \otimes R')) \otimes Y \rightarrow (\text{Ext}(B) \oplus (\text{Ext}(P) \otimes R')) \otimes Y \mathbin{\|} Z.$$

To grasp the behavior of the differentials $d_2$ we must first interpret
the behavior of this map when Ext(H*MSU) is identified as in (2.7). Clearly the $Y$ module action on $(Ext(B)' \oplus (Ext(P) \otimes R')) \otimes Y$ induced from that on Ext($H*MSU$) is still just multiplication in the right factor, so in fact

**Proposition 4.1.** $p: MSU(2) \rightarrow X$ induces a natural projection

$$(Ext(B)' \oplus (Ext(P) \otimes R')) \otimes Y \rightarrow (Ext(B)' \oplus (Ext(P) \otimes R')) \otimes Y \rightarrow Z$$

of Ext$_A(Z/2, \ldots)$ groups.

Since $p$ induces a map of spectral sequences, the differentials $d_2$ on Ext($H*X$) are completely determined by those on Ext($H*MSU$) already described in Section 2. The description of Ext($H*X$) provided by (4.1) has a natural algebra structure respected by Ext($p*$), and $d_2$ is a derivation on Ext($H*X$).

**Proposition 4.2.** The map Ext($p*$) has a splitting which is a map of $d_2$ chain complexes.

**Proof.** From the definition of the sequence $\{z_{8i}\}_{i \geq 2} \subset Y$ it is clear that $Y \rightarrow Z$ is an exterior algebra with generators represented by $\{y_{ij}\}_{j \geq 3}$. So there is an obvious identification of $Y \rightarrow Z$ with the subspace of $Y$ spanned by the monomials in the $y_{ij}$'s in which no $y_{ij}$ appears to a power greater than one. This splitting of $Y \rightarrow Y \rightarrow Z$ provides a splitting of Ext($p*$), and from the form of the differentials, as described in (2.7), it is clear the splitting commutes with $d_2$. \[\square\]

**Proposition 4.3.** The map of $E_3$ terms induced by $p$ is an epimorphism. All the differentials $d_r$ for $r \geq 3$ in the spectral sequence for $\pi*X$ vanish.

**Proof.** The splitting of (4.2) shows the map of $E_3$ terms is onto. The proposition now follows from (2.8). \[\square\]

**5. The Indecomposable Spectrum BoP.** In this section we will produce BoP from $X$, examine the Adams spectral sequence converging to $\pi*BoP$, and show BoP is indecomposable.

Recall from Section 4 that as an $A$ comodule

$$H*_X \cong (B \oplus (P \otimes R')) \otimes Y \cong (B \otimes Y \otimes Z) \oplus (P \otimes R' \otimes Y \otimes Z).$$
From now on we identify $H \ast X$ with this direct sum. Since the $A$ algebra $P$ is isomorphic to $H \ast BP$, it appears that $X$ may decompose into two wedge summands, one a spectrum with homology $B \otimes Y \| Z$, the other a wedge of $BP$ summands. We will show that it lies in a fibration between two such spectra.

**Proposition 5.1.** Let $W$ denote the graded vector space $R' \otimes Y \| Z$, and let $m_\ast$ denote the canonical projection $H \ast X \cong (B \otimes Y \| Z) \oplus (P \otimes W) \to P \otimes W$. Then there is a map $X \overset{\tilde{m}}{\to} BP \wedge W$ inducing $m_\ast$ in mod 2 homology.

**Proof.** The projection $H \ast X \overset{\tilde{m}}{\to} P \otimes W \overset{\epsilon \otimes id}{\to} \mathbb{Z}/2 \otimes W \cong W$ corresponds naturally to a map $X \overset{m}{\to} K(\mathbb{Z}/2) \wedge W$, since $W$ is of finite type and $H^\ast(X; \mathbb{Z})$ is concentrated in even dimensions and is torsion free. In mod 2 homology, $m$ induces the natural map $H \ast X \overset{m}{\to} P \otimes W - A \otimes W$, since a map of graded bounded below comodules is uniquely determined by its composition with any projection of the target onto its $A$ primitives. The obstructions to lifting $m$ into $BP \wedge W$ lie in zero groups, so we obtain the lift $\tilde{m}$ we seek.

Now define $BoP$ to be the fibre of $\tilde{m}$, and let $i: BoP \to X$ be the inclusion of the fibre. We immediately have

**Proposition 5.2.** In mod 2 homology, $i_\ast$ is a monomorphism onto the left summand $B \otimes Y \| Z$ of $(B \otimes Y \| Z) \oplus (P \otimes W) \cong H \ast X$.

We identify $H \ast BoP$ with this summand, and remark that since $Y \| Z$ is an exterior algebra on generators $y_{ij}$ with $j \geq 3$, $H \ast BoP$ is a sum of copies of the cocyclic $A$ comodule $B$, with one copy beginning in each dimension divisible by 8.

Next we compute the Adams spectral sequence for $i_\ast BoP$. Not only does Ext$(H \ast X)$ split into the two summands Ext$(B)' \otimes Y \| Z$ and Ext$(P) \otimes W$, but the form of the differentials, described by (4.1), (4.3), and (2.7), shows this is a splitting of $d_2$ chain complexes, and hence the entire spectral sequence for $i_\ast X$ splits. Thus we have

**Proposition 5.3.** In the Adams spectral sequence converging to $\pi \ast BoP$, with $E_2 \cong Ext(B)' \otimes Y \| Z$, the differentials are as follows. $d_2$ is zero on $\{q_0, h, q_1q_0, q_4, q_2, \ldots, q_k, \ldots\}$, $d_2 y_{ij} = h_{q_{j-1}}$, and $d_2$ is a derivation with respect to the natural algebra structure of Ext$(B)' \otimes Y \| Z$. The higher differentials $d_r$ for $r \geq 3$ are all zero.
The indecomposability of BoP will follow from

**Theorem 5.4.** Let $p$ be a prime, and suppose $F$ is a CW-spectrum satisfying

1. $F$ is $p$-local;
2. $F$ is bounded below;
3. $H_*(F; \mathbb{Z})$ is of finite type over $\mathbb{Z}(p)$;
4. The image of the $\mathbb{Z}/p$ Hurewicz homomorphism $\pi_* F \to H_*(F; \mathbb{Z}/p)$ has rank one.

Then $F$ has no nontrivial wedge decomposition.

**Proof.** Suppose $F = F' \vee F''$. Since the Hurewicz homomorphism is additive on wedges, the Hurewicz isomorphism theorem along with (2) and (4) shows that one of the wedge summands, say $F'$, has zero mod $p$ homology. $H_*(F'; \mathbb{Z}) \cong H_*(F'; \mathbb{Z}(p))$ is of finite type over $\mathbb{Z}(p)$, so $H_*(F'; \mathbb{Z})$ is a sum of copies of $\mathbb{Z}(p)$ and $\mathbb{Z}/p^n$ ($n \geq 1$). If it were nonzero, then $H_*(F'; \mathbb{Z}/p)$ would be nonzero. Thus $H_*(F'; \mathbb{Z}) = 0$, so by the Whitehead Theorem, $F'$ is homotopy equivalent to a point.

**Corollary 5.5.** BoP has no nontrivial wedge decomposition.

**Proof.** Clearly BoP satisfies (1), (2), (3) of (5.4). The image of the mod 2 Hurewicz homomorphism is given by the zero line $E_\infty^{0,*}$ in the Adams spectral sequence for $\pi_* \text{BoP}$. But the differentials, as described in (5.3), show this is of rank one.

6. **Maps from MSU$_{(2)}$ to Suspensions of BoP.** We ultimately intend to show MSU$_{(2)}$ is a wedge of BoP and BP suspensions by producing a map to such a wedge and showing it is a homotopy equivalence. In this section, we produce the necessary maps from MSU$_{(2)}$ to various suspensions of BoP. Such maps are difficult to produce directly. As an intermediate step, we first produce certain maps to suspensions of bo$_{(2)}$. It seems that the KO-theory techniques of [4, 17] would provide sufficient maps to bo$_{(2)}$, but we will use the Adams spectral sequence since this method will also produce a fundamental map we require from BoP to bo$_{(2)}$.

If $M$ is a graded comodule or vector space, let $M_n$ denote the $n^{th}$ grade, $M^n$ the $n$-skeleton. If a graded vector space is concentrated in even dimensions, we call it an evenly graded vector space (abbreviated egvs).

* A priori the Adams spectral sequence with $E_2$ term isomorphic to $\text{Ext}_A^{*,*}(H_\text{MSU}, H_\text{bo})$ doesn’t necessarily converge to $[\text{MSU}_{(2)}, \text{bo}_{(2)}]_*$, since MSU is not a finite complex. We intend to obtain elements of
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by examining \( \lim_n [MSU^{2n}, bo(2)]_* \), where the spectra \( MSU^{2n} \) are an increasing sequence of finite subspectra of \( MSU \) (a choice of \( 2n \)-skeleta for \( MSU \)) such that \( MSU^{2n} \to MSU \) induces an inclusion onto \( (H_*MSU)^{2n} \) in homology (such subspectra exist since \( H_*MSU; \mathbb{Z} \) is torsion free and even dimensional).

The \( E_2 \) term for the Adams spectral sequence converging to \( [MSU^{2n}, bo(2)]_* \) is given by \( \text{Ext}^{*,*}_{A_*}((H_*MSU)^{2n}, H_*bo) \), which we now examine.

Let \( A_1^* \) denote the subalgebra of \( A_* \) generated by \( Sq^1 \) and \( Sq^2 \). Then \( H_*bo \cong A_*^* \otimes A_1^* \mathbb{Z}/2 \) [14; 15, Chapter XI], so \( H_*bo \cong A_1^* \mathbb{Z}/2 \cong \mathbb{Z}/2(\xi_1^4, \xi_2^2, \xi_3, \ldots, \xi_j, \ldots) \) [15, p. 324], with \( A_1 = \mathbb{Z}/2(\xi_1, \xi_2)/(\xi_1^4, \xi_2^2) \).

Using the change of rings theorem [16, p. 498], we have

**Lemma 6.1.**

\[
\text{Ext}^{*,*}_{A_*}((H_*MSU)^{2n}, H_*bo) \cong \text{Ext}^{*,*}_{A_1^*}((H_*MSU)^{2n}, \mathbb{Z}/2). 
\]

To compute these groups we must analyze the structure of \( H_*MSU \equiv (B \otimes Y) \oplus (P \otimes R' \otimes Y) \) as a comodule over \( A_1^* \). The coaction in fact makes \( H_*MSU \) a comodule over the Hopf subalgebra \( E = E(\xi_1^2) \) of \( A_1^* \). So, as in Section 1, we need only understand the induced \( Sq^2 \) action on \( H_*MSU \).

**Proposition 6.2.** As an \( A_1^* \) comodule, \( B \equiv \mathbb{Z}/2 \oplus (E \otimes V') \), with \( V' \) an egvs.

**Proof.** \( Sq^2 \) is a derivation and differential on \( B \) with \( Sq^2(\xi_j^2) = \xi_j^4 \xi_{j-1} \). The Künneth theorem yields \( H_*(B; Sq^2) = \mathbb{Z}/2 \), so \( B \) is as described. \( \square \)

**Proposition 6.3.** As an \( A_1^* \) comodule, \( P \equiv E \otimes V'' \), with \( V'' \) an egvs.

**Proof.** \( H_*(P; Sq^2) = 0. \) \( \square \)

**Corollary 6.4.** As an \( A_1^* \) comodule, \( H_*MSU \equiv Y \oplus (E \otimes V) \), where \( V \) is an egvs.

We immediately deduce that

**Corollary 6.5.** As an \( A_1^* \) comodule, \( (H_*MSU)^{2n} \equiv Y^{2n} \oplus V_{2n} \oplus (E \otimes V^{2n-2}) \).

To analyze \( \lim_n [MSU^{2n}, bo(2)]_* \), we now consider, for each \( k \geq 0 \), the inclusion \( r: MSU^{8k+6} \to MSU^{8(k+1)+6} \), and the induced map of Adams spectral sequences converging to

\[
[MSU^{8(k+1)+6}, bo(2)]_* \to [MSU^{8k+6}, bo(2)]_* .
\]
From (6.1) and (6.5) we see we can describe the $E_2$ terms of the two spectral sequences provided we know $\text{Ext}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ and $\text{Ext}^{*,*}(E, \mathbb{Z}/2)$. It is well known [9] that

$$\text{Ext}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[q_0, h, q_1^2, q_1^4]/(q_0h, h^3),$$

with the generators in the same bidegrees as in Section 2. Regarding $\text{Ext}^{*,*}(E, \mathbb{Z}/2)$, it will suffice to know that

**Lemma 6.7.** $\text{Ext}^{*,*}(E, \mathbb{Z}/2) = 0$ if $t - s$ is odd.

*Proof.* Consider the unreduced, normalized cobar resolution [2] $A_1 \otimes F(A_1)$ for $\mathbb{Z}/2$ over $A_1$. The differential on the cobar resolution induces a differential on $\text{Hom}^{*,*}(E, A_1 \otimes F(A_1)) \cong \text{Hom}^{*,*}(E, F(A_1)) \cong E(Sq^2) \otimes F(A_1)$, with homology isomorphic to $\text{Ext}^{*,*}(E, \mathbb{Z}/2)$. If we filter the chain complex $E(Sq^2) \otimes F(A_1)$ by the skeletal filtration on $E(Sq^2)$, we obtain a spectral sequence converging to the desired Ext group, with $E_1$ term isomorphic to $E(Sq^2) \otimes \text{Ext}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$. Moreover, it is easy to check that the only nonzero differential, $d_2$, is given by $d_2(1 \otimes x) = Sq^2 \otimes hx$ and $d_2(Sq^2 \otimes x) = 0$, for $x \in \text{Ext}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$. The lemma now follows, since $\text{Ext}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ is onto if $(t - s)$ is even, one-to-one if $(t - s)$ is odd. \hfill \Box

Now we can examine the induced map $\text{Ext}(r_*)$ of $E_2$ terms.

**Proposition 6.8.** $\text{Ext}^{*,*}(r_*)$ is onto if $t - s$ is congruent to 7 or 0 mod 8.

*Proof.* Using the descriptions of $(H_\bullet MSU)^{8(k+1)+6}$ and $(H_\bullet MSU)^{8k+6}$ provided by (6.5), we first notice that the composite

$$Y^{8k+6} \oplus (E \otimes V^{8k+4}) \rightarrow (H_\bullet MSU)^{8k+6} \rightarrow (H_\bullet MSU)^{8(k+1)+6}$$

is a split $A_1$ comodule monomorphism, so $\text{Ext}(r_* \circ \text{Ext}(i))$ is onto. But $\text{Ext}(i)$ is an isomorphism for $t - s$ congruent to 7 or 0 mod 8 since $V^{8k+6}$ is concentrated in grade $8k + 6$ and $\text{Ext}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) = 0$ for $t - s$ congruent to 5 or 6 mod 8. \hfill \Box

When (6.5), (6.6), and (6.7) are combined with the fact that $Y$ is concentrated in dimensions divisible by 8, we find that

**Proposition 6.9.** For any $k \geq 0$, $\text{Ext}^{*,*}(H_\bullet MSU)^{8k+6}, \mathbb{Z}/2) = 0$ if $t - s$ is congruent to 7 mod 8.
So the ‘columns’ with \( t - s \equiv 0 \) (mod 8) in the two spectral sequences survive in their entirety to \( E_{\infty} \), and if we let \( E_{\infty}^{s,t}(r) \) denote the induced map of \( E_{\infty} \) terms, it follows from (6.8) and (6.9) that

**Corollary 6.10.** \( E_{\infty}^{s,t}(r) \) is onto for \( t - s \equiv 0 \) (mod 8).

Now we need a technical lemma which ensures that the epimorphism at the \( E_{\infty} \) level really means the induced homomorphism of mapping groups is an epimorphism. We prove something slightly more general for use later in this section.

**Lemma 6.11.** Suppose \( U, V_1, V_2 \) (respectively \( U_1, U_2, V \)) are 2-local spectra with \( U \) (respectively \( U_1, U_2 \)) finite and \( V_1, V_2 \) (respectively \( V \)) of finite type and bounded below. Consider the Adams spectral sequences for \([U, V_1]_*\) and \([U, V_2]_*\) (respectively \([U_1, V]_*\) and \([U_2, V]_*\)). If \( V_1 \xrightarrow{f} V_2 \) (respectively \( U_2 \xrightarrow{f} U_1 \)) induces an epimorphism of \( E_{\infty}^{s,t} \) terms, for all groups with \( t - s = i \) for some fixed \( i \), then \([U, V_1]_i \xrightarrow{f} [U, V_2]_i\) (respectively \([U_1, V]_i \xrightarrow{f} [U_2, V]_i\)) is onto.

**Proof.** Let \( F_j^s \), \( s \geq 0 \), denote the groups in the decreasing Adams filtration of \([U, V_j]_i\) (respectively \([U_j, V]_i\)) for \( j = 1, 2 \). Since the spectral sequences converge, \( E_{\infty}^{s,t} \equiv F_j^s / F_j^{s+1} \) and \( \cap F_j^s = 0 \) for all \( i \) and \( j \). Since the source spectra are finite dimensional and the targets are bounded below, the Adams and 2-adic filtrations induce equivalent topologies on the mapping groups [12, p. 189]. So in particular, for \( i \) fixed, we can choose \( s \) such that \( F_j^s \subset 2^{-j} F_i^0 \) for \( j = 1, 2 \).

Consider the induced commutative square

\[
\begin{array}{ccc}
1F_i^0 / F_i^s & \longrightarrow & 1F_i^0 / 2 \cdot 1F_i^0 \\
\downarrow & & \downarrow \\
2F_i^0 / 2F_i^s & \longrightarrow & 2F_i^0 / 2 \cdot 2F_i^0.
\end{array}
\]

The lower horizontal is clearly onto. The left vertical is onto because the Adams filtrations induce finite filtrations on the two groups, and by assumption the map is onto when one passes to filtered quotients. Thus the right vertical is onto. It now follows from Nakayama’s Lemma [5, Proposition 2.6] that \( 1F_i^0 \xrightarrow{f} 2F_i^0 \) is onto, as desired. \( \square \)

Now we are equipped to produce the maps we desire from \( MSU(2) \) to suspensions of \( bo(2) \).
THEOREM 6.12. If $\tilde{\lambda}: Y \to \Sigma^8iZ/2$ is a graded homomorphism from $Y$ to the $8i^{th}$ suspension of $Z/2$, then there is a map $MSU(2) \to \Sigma^8i bo(2)$ inducing the obvious composition

$$\lambda: H_\ast MSU$$

$$\equiv (B \otimes Y) \oplus (P \otimes R' \otimes Y) \to B \otimes Y \to \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \ldots, \xi_j, \ldots] \otimes Y$$

$$\equiv H_\ast bo \otimes Y^{id \otimes \tilde{\lambda}} \Rightarrow H_\ast bo \otimes \Sigma^8iZ/2 \equiv \Sigma^8i H_\ast bo.$$

**Proof.** $\lambda$ is clearly an element of $\text{Hom}_A^{-8}(H_\ast MSU, H_\ast bo) = \text{Ext}_A^{0,-8}(H_\ast MSU, H_\ast bo)$. The restrictions of $\lambda$ to the skeleta $(H_\ast MSU)^{8k+6}$ for $k \geq 0$ are elements of $\text{Ext}_A^{0,-8}((H_\ast MSU)^{8k+6}, H_\ast bo)$, compatible with one another under restriction. Using (6.10), (6.11), and the fact that

$$\text{Hom}_A^{0,-8}((H_\ast MSU)^{8(k+1)+6}, \mathbb{Z}/2) \cong \text{Hom}_A^{0,-8}((H_\ast MSU)^{8k+6}, \mathbb{Z}/2)$$

is clearly an isomorphism if $k \geq i$, we see there is a sequence of elements $\hat{\lambda}_k \in [MSU^{8k+6}, bo(2)]_{-8i}$, compatible with each other under restriction, and each inducing the restriction of $\lambda$ in mod 2 homology. Since $[MSU(2), bo(2)]_{-8i} \to \lim_p [MSU(2n), bo(2)]_{-8i}$ is onto, the theorem follows. □

Recall that $H_\ast Bop \equiv B \otimes Y || Z$. Thus the Bop wedge summands in the desired decomposition of $MSU(2)$ ought to be indexed by $Z$. As a first step towards a projection $MSU(2) \to Bop \wedge Z$, we construct an appropriate map $MSU(2) \to bo(2) \wedge Z$ by fitting together maps made available by (6.12). Since $Z$ is concentrated in dimensions divisible by 8, it follows from (6.12) that

**THEOREM 6.13.** If $Y \to Z$ is a (graded) projection of $Y$ onto the subspace $Z$, then there is a map $\mu: MSU(2) \to bo(2) \wedge Z$ inducing the obvious composition

$$H_\ast MSU$$

$$\equiv (B \otimes Y) \oplus (P \otimes R' \otimes Y) \to B \otimes Y \to \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \ldots, \xi_j, \ldots] \otimes Y$$

$$\equiv H_\ast bo \otimes Y^{id \otimes \mu} \Rightarrow H_\ast bo \otimes Z$$

in mod 2 homology.
Using a map of the type produced in (6.13) we wish to produce a map $MSU(2) \to BoP \wedge Z$ which will serve as a projection to the $BoP$ summands we claim exist in $MSU(2)$. First we need a suitable map $BoP \to bo(2)$.

**Theorem 6.14.** There is a map $BoP \to bo(2)$ inducing the obvious map

$$B \otimes Y||Z \stackrel{\text{id} \otimes \epsilon}{\to} B \otimes \mathbb{Z}/2 \cong B \leftrightarrow H_*bo$$

in mod 2 homology.

**Proof.** All the properties of $MSU(2)$ used in the proof of (6.12) are also satisfied by $BoP$. So the same argument produces the desired map. □

**Theorem 6.15.** $\nu_* : \pi_*boP \to \pi_*bo(2)$ is an epimorphism for all $i$ and an isomorphism for $i$ odd.

**Proof.** In (5.3) we computed the $E_2$ term of the Adams spectral sequence converging to $\pi_*boP$, and all the differentials in the spectral sequence. The $E_2$ term of the Adams spectral sequence converging to $\pi_*bo(2)$ is $\text{Ext}^*_{\mathbb{A}}(\mathbb{Z}/2, H_*bo) \cong \text{Ext}^*_{\mathbb{A}_1}(\mathbb{Z}/2, \mathbb{Z}/2)$, described in (6.6). There is no room for nonzero differentials in the spectral sequence. Combining (5.3), (6.6), and (6.14), we see that $\nu$ induces an epimorphism of $E_\infty$ terms, and hence, by (6.11), of homotopy groups. From the description of the two spectral sequences it is clear that $\pi_i bo(2)$ and $\pi_i boP$ are zero for $i$ odd unless $i \equiv 1 \pmod{8}$, in which case both groups are $\mathbb{Z}/2$. The theorem follows. □

Let $F$ be the fibre of $\nu : BoP \to bo(2)$. From (6.15) and the long exact homotopy sequence we immediately have

**Corollary 6.16.** $\pi_*F$ is concentrated in even dimensions.

In fact, Massey product and Toda bracket arguments can be used to show that $\nu_*$ induces an isomorphism of torsion subgroups, so $\pi_*F$ is torsion free. We will not prove this here.

Now we are prepared to produce the map $MSU(2) \to BoP \wedge Z$ we seek. Note that for any such map, the induced map $f_*$ in homology carries the primitives in $H_*MSU$ into the primitives $Y||Z \otimes Z$ in $H_*(BoP \wedge Z)$.

**Theorem 6.17.** There is a map $f : MSU(2) \to BoP \wedge Z$ such that

$$Z \stackrel{f_*}{\to} (Y||Z) \otimes Z \cong Z \otimes Z \cong Z$$

is the identity.
Proof. Consider any map \( \mu: MSU(2) \to bo(2) \wedge Z \) satisfying (6.13). We would like to lift \( \mu \) to \( f \) making

\[
\begin{array}{ccc}
MSU(2) & \xrightarrow{\mu} & bo(2) \wedge Z \\
\downarrow{f} & & \downarrow{\nu \wedge id} \\
BoP \wedge Z & \xrightarrow{} & MSU(2) \wedge Z
\end{array}
\]

commute. The obstructions to such a lift lie in the groups \( H^p(MSU(2); \pi_{p-1}F) \). Since \( H^*(MSU; Z) \) is torsion free and even dimensional, it follows from (6.16) that the obstruction groups are all zero. From (6.13) and (6.14) it is clear \( f_* \) has the desired property.

7. The Decomposition of \( MSU(2) \). In this section we will show \( MSU(2) \) is homotopy equivalent to a wedge of suspensions of \( BoP \) and \( BP \).

The map \( f: MSU(2) \to BoP \wedge Z \) produced in (6.17) will serve as the projection to the \( BoP \) summands. But first we need to know more about the homology behavior of this map. So far we only know that the restriction of \( f_* \) to \( Z \subset Y \) is such that \( Z \to (Y/Z) \otimes Z/2 \otimes Z \cong Z \) is the identity. We will in fact need to know that \( Y \to (Y/Z) \otimes Z \) is an isomorphism.

We will show this is forced by the differentials in the Adams spectral sequences for \( \pi_*MSU \) and \( \pi_*(BoP \wedge Z) \). The idea is roughly as follows. Since we know \( f_1(1) = 1 \otimes 1 \), it 'ought' to follow that \( Ext(f_*)(q^j) = q^j \otimes 1 \). Since \( d_2 \) commutes with \( Ext(f_*)(q^j) \) and \( d_2y^j = q^j \) in both the spectral sequence for \( \pi_*MSU(2) \) and for \( \pi_*BoP \), it follows that \( Ext(f_*)(y^j) = y^j \otimes 1 \), so \( f_*(y^j) = y^j \otimes 1 \), as desired, etc. The details are rather technical, and we will relegate them to a lemma, from which our main result will follow easily.

To state the lemma, we need a few preliminaries. Recall that \( Y/Z \) is isomorphic to the exterior algebra \( E(y^j, \ldots, y^j, \ldots) \). Let \( L: Y/Z \to Y \) be the obvious splitting of the projection \( \rho: Y \to Y/Z \). In other words,

\[
L(\prod_k y^j_{jk}) = \prod_k y^j_{jk} \quad \text{for} \quad j_1 < j_2 < \cdots
\]

Now consider \( J: (Y/Z) \otimes Z \xrightarrow{\otimes 1} Y \otimes Y \xrightarrow{m} Y \). Clearly \( J \) is an isomorphism and a map of right \( Z \) modules. Let \( I \) be the inverse of \( J \). Specifically, any monomial in the \( y^j_{jk} \)'s can be written in the form
\[
y = \left( \prod_k y_{2j_k} \right) \otimes z,
\]

with \( z \in Z \) and \( j_k < j_{k+1} \) for all \( k \). Then

\[
I(y) = \left( \prod_k y_{2j_k} \right) \otimes z.
\]

Let \( F_{8r} Z \) denote \( \bigoplus Z_i \) (and define \( F_{8r} Y \) and \( F_{8r} Y \| Z \) similarly). For \( r \) fixed, let \( \pi : Z \to Z_{8r} \) be the natural projection to grade \( 8r \). So \( id \land \pi \) maps \( BoP \land Z \) onto \( BoP \land Z_{8r} \). The technical lemma is as follows.

**Lemma 7.1.** The diagram

\[
\begin{array}{ccc}
Y \to (Y \| Z) \otimes Z \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
Y \cdot F_{8r} Z \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
Y \to (Y \| Z) \otimes Z_{8r} \\
\end{array}
\]

commutes.

Before proving the lemma, we will show how the decomposition of \( MSU(2) \) follows from it.

**Corollary 7.2.** \( f^* : Y \to (Y \| Z) \otimes Z \) is a monomorphism.

**Proof.** Let \( 0 \neq y \in Y \). Since \( Y = Y \cdot F^0 Z \), and \( \cap Y \cdot F_{8r} Z = 0 \), there is some \( r \) with \( y \in Y \cdot F_{8r} Z \) but \( y \notin Y \cdot F_{8(r+1)} Z \). If \( f^*(y) = 0 \), then by (7.1), \( (id \land \pi)^* \circ I(y) = 0 \). But \( I(Y \cdot F_{8r} Z) \subset (Y \| Z) \otimes F_{8r} Z \) since \( I \) is a \( Z \) module map, so

\[
I(y) \in ((Y \| Z) \otimes F_{8r} Z) \cap \ker((id \land \pi)^*) = (Y \| Z) \otimes F_{8(r+1)} Z.
\]

Thus \( y = J(I(y)) \in Y \cdot F_{8(r+1)} Z \), a contradiction. \( \square \)

**Lemma 7.3.** There is a map \( g : MSU(2) \to BP \land (R' \otimes Y) \) such that

\[
(B \otimes Y) \oplus (P \otimes R' \otimes Y) \cong H_\ast MSU(2) g_\ast H_\ast (BP \land (R' \otimes Y))
\]

\[
\cong P \otimes R' \otimes Y
\]

is projection onto the second factor.
Proof. The proof of (5.1) carries over word for word to produce the desired map. □

We can combine the maps $f$ from (6.17) and $g$ from (7.3) to produce a map $h : MSU(2) \to (BP \land Z) \lor (BP \land (R' \otimes Y))$ that induces $f$ and $g$ when projected onto the left and right summands of the target.

**Theorem 7.4.** $h$ is a homotopy equivalence.

Proof. In homology, consider the restriction $h_* : Y \oplus (R' \otimes Y) \to ((Y \| Z) \otimes Z) \oplus (R' \otimes Y)$ to the $A$ primitives. Let $x \in Y \oplus (R' \otimes Y)$ with $x \neq 0$. If $x \notin Y$, then by (7.3), $g_*(x) \neq 0$, so $h_*(x) \neq 0$. If $x \in Y$, then $f_*(x) \neq 0$ by (7.2), so $h_*(x) \neq 0$. Thus $h_*$ is a monomorphism on primitives, and hence a monomorphism. But the graded ranks of the source and target of $h_*$ are the same, so $h_*$ is an isomorphism. Since both source and target are even dimensional, $H_*(MSU(2); Z)$ and $H*((BP \land Z) \lor (BP \land (R' \otimes Y)); Z)$ are torsion free and $h$ must induce an isomorphism in integral homology. Thus by Whitehead’s Theorem $h$ is a homotopy equivalence. □

**Proof of Lemma 7.1.** We will prove the diagram commutes when restricted to $Y \cdot Z_{8r+s}$ for any $s \geq 0$. For each $s$ we will do this inductively by showing it commutes on $Y \cdot Z_{8r+s}$ provided it commutes on $Y \cdot Z_{8r+s}$ for $n < i$. To begin the induction we must show it commutes on $Z_{8r+s}$. In $Ext^*(H_*(MSU), Z_{8r+s} \subset ker(d_2)$. So

$$f_*(Z_{8r+s}) \subset ker(d_2) \cap Ext^{0,8r+s}(H_*(BP \land Z)) = Z/2 \otimes Z_{8r+s}.$$ 

Thus $(id \land \pi)_* \circ f_*(Z_{8r+s}) = 0$ if $s > 0$, and if $s = 0$ $(id \land \pi)_* \circ f_* : Z_{8r} \to (Y \| Z) \otimes Z_{8r}$ is, by hypothesis, the natural inclusion. In either case, the diagram commutes, beginning the induction.

Define $j, k$ by $2^j \leq 8i < 2^{j+1}$ and $2^k \leq 8i + s < 2^{k+1}$. Let $\tilde{f} = (id \land \pi) \circ f : MSU(2) \to BP \land Z \to BP \land Z_{6r}$. Consider the diagram

$$B \otimes [((Y \| Z) / F^{8i-2^j}(Y \| Z)] \otimes Z_{8r}$$

involving the induced map $\tilde{f}_*$ in homology, with $\tau$ the natural projection.
The composite $\alpha \circ \bar{f}_* \circ \beta$ is trivial, since the target has nonzero primitives only in grades less than $8i - 2j + 8r + s$, the source is concentrated only in grades at least as large as this number, and an $A$ comodule map (between bounded below graded $A$ comodules) is determined by its composition with a projection of the target onto the $A$ primitives. Thus $\bar{f}_*$ factors as shown because $j \leq k$.

Now consider the restriction of $\tau \circ \bar{f}_*$ to the primitives. This is obviously zero except (possibly) in grade $8i - 2j + 8r + s$, and in this grade it equals $\tau \circ (id \wedge \pi)_* \circ (id \otimes I)$ by the inductive assumption. But $\tau \circ \bar{f}_*$ is uniquely determined by its restriction to the primitives, since the source is primitive in all grades less than or equal to $8i - 2j + 8r + s$, and these are the only grades in which the target has nonzero primitives. Thus $\tau \circ \bar{f}_* = \tau \circ (id \wedge \pi)_* \circ (id \otimes I)$.

With this in hand let us consider (see Figure 1) the map induced by $\bar{f}$ between relevant portions of the $E_2$-terms of the Adams spectral sequences converging to $\pi_*MSU(2)$ and $\pi_*(BoP \wedge Z_{8r})$. The commutative diagram in the figure requires some justification.

The vertical equalities are valid since $(F^{8i-2j} Y) \cdot Z_{8r+s}$ is an ideal in $Y$, and $F^{8i+s-2k} (Y \| Z)$ and $F^{8i+s-2k+8} (Y \| Z)$ are ideals in $Y \| Z$. To see that $d_2$ and $d_2 \otimes id$ really land in the subgroups shown, recall that on $Y$, $d_2 = h \cdot d$, with the derivation $d$ as described in Section 2 (this is true also on $Y \| Z$ in the spectral sequence for $\pi_*BoP$; note $dZ = 0$ so $d$ passes naturally to a derivation on $Y \| Z$). We leave it to the reader to check that the definition of $d$ on $\{y_{8i}\}$, the fact that $d$ is a derivation, and the fact that multiplication in $Y$ adds filtrations $F*Y$, ensure that $d(Y_{8i}) \subset Z/[q_i, \ldots, q_i, \ldots] \otimes F^{8i-2j} Y$. Thus since $dZ = 0$, $d_2(Y_{8i} \cdot Z_{8r+s}) \subset \text{Ext}(B)' \otimes (F^{8i-2j} Y) \cdot Z_{8r+s}$, as claimed. By the same reasoning $d_2 \otimes id$ behaves as shown.

The lowest horizontal $\text{Ext}^{0, *}(\bar{f}_*)$ is the map we wish to determine. We detect its behavior as follows.
Let \((j_1, \ldots, j_l)\) be the unique increasing integer sequence such that

\[ 8i + s = \sum_{m=1}^{l} 2^j_m. \]

Note \(j_i = k\). Let

\[ y = \prod_{m=1}^{l} y_{2j_m} \in Y_{8i+s}. \]

Of course \(\rho(y)\) is the generator of \((Y \| Z)_{8i+s} = \mathbb{Z}/2\).

Now

\[ \text{Ext}(\tau) \circ (d_2 \otimes id)(\rho(y) \otimes z) = (\text{Ext}(\tau)) \left( h \otimes \sum_{n=1}^{l} q_{n-1} \otimes \rho \left( \prod_{m \neq n} y_{2j_m} \right) \otimes z \right) \]

\[ = h \otimes q_{k-1} \otimes \rho \left( \prod_{m=1}^{l-1} y_{2j_m} \right) \otimes z, \]

since \(\text{Ext}(\tau)\) kills all but one of the terms in the sum. Thus \(\text{Ext}(\tau) \circ (d_2 \otimes id)\) is a monomorphism. So any map \(\Phi\) such that \(\text{Ext}(\tau) \circ (d_2 \otimes id) \circ \Phi = \text{Ext}(\tau) \circ \text{Ext}(\bar{f}_m) \circ d_2\) must equal the map \(\text{Ext}(0, \ast f_m)\) of interest. We now show \((id \wedge \pi)_\ast \circ I\) is such a \(\Phi\), which will complete the proof.

Earlier in the proof we showed that \(\tau \circ \bar{f}_\ast = \tau \circ (id \wedge \pi)_\ast \circ (id \otimes I)\), so applying \(\text{Ext}(\tau)\) yields \(\text{Ext}(\tau) \circ \text{Ext}(\bar{f}_\ast) = \text{Ext}(\tau) \circ \text{Ext}((id \wedge \pi)_\ast) \circ (id \otimes I)\). Now \((id \otimes I) \circ d = (d \otimes id) \circ I\) since \(dZ = 0\), so \((id \otimes I) \circ d_2 = (d_2 \otimes id) \circ I\). Thus \(\text{Ext}(\tau) \circ \text{Ext}(\bar{f}_\ast) \circ d_2 = \text{Ext}(\tau) \circ \text{Ext}((id \wedge \pi)_\ast) \circ (d_2 \otimes id) \circ I = \text{Ext}(\tau) \circ (d_2 \otimes id) \circ (id \wedge \pi)_\ast \circ I\), as claimed. \(\square\)

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