Cocycle schemes and $MU[2k,\infty)$–orientations

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Abstract. We recall the study of $MU[2k,\infty)$–orientations as elucidated by Ando, Hopkins, and Strickland. Their work prompts us to investigate a particular algebraic moduli which, after 2–localization, we (together with Adam Hughes and JohnMark Lau) fully describe for all values of k. It gives a strikingly good (but imperfect) approximation of our topological motivator.

1. Motivation: Integration in extraordinary cohomology theories

There is a whole industry in algebraic topology dedicated to understanding "orientations" of ring spectra, defined as follows:

Definition 1.1. Suppose that $G(n) \to O(n)$ is a multiplicative system of structure groups:

$$\begin{tikzcd} \cdots \arrow[r] & G(n) \arrow[r] \arrow[d] & G(n+1) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & O(n) \arrow[r] & O(n+1) \arrow[r] & \cdots . \end{tikzcd}$$

A *G–orientation* of a ring spectrum E is a map φ of ring spectra

$$\varphi\colon\ MG \to E,$$

where MG is the Thom spectrum associated to the system of structure groups.

Example 1.2. There are orientations $MO \to H\mathbb{F}_2$ and $MSO \to H\mathbb{Z}$ given by coconnective truncation.

This definition is stanadard, but it may not make plain to an interloper why this is of any special interest. The first observation (and the usual motivation) is that evaluating such a map on a point gives rise to an E_*–valued genus

$$\varphi_*\colon\ MG_* \to E_*,$$

i.e., a ring homomorphism which converts G–structured bordism classes to elements of E_*.

Example 1.3. The induced maps $\pi_*MO \to \pi_*H\mathbb{F}_2 = \mathbb{F}_2$ and $\pi_*MSO \to \pi_*H\mathbb{Z} = \mathbb{Z}$ count the number of (signed) points in a dimension 0 (oriented) manifold.

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However, this is really just scratching the surface, and there are techniques for getting a lot more out of an orientation than this simple invariant. For example, Kochman [**Koc78**, Section 4] models the G–structured bordism groups $MG_m(X)$ via a chain complex of monoids

$$\begin{aligned}\cdots \xrightarrow{\partial} &\left\{ Z^n \to X \,\middle|\, \begin{array}{c} Z^n \text{ a compact } n\text{–manifold} \\ \text{with } G\text{–structure} \end{array} \right\} \\ \xrightarrow{\partial} &\left\{ Z^{n-1} \to X \,\middle|\, \begin{array}{c} Z^{n-1} \text{ a compact } (n-1)\text{–manifold} \\ \text{with } G\text{–structure} \end{array} \right\} \\ \xrightarrow{\partial} &\cdots,\end{aligned}$$

where each map "∂" is given by restricting to the boundary of the source manifold. Under this definition, we then consider the bordism homology group $MG_{n-m}(\underline{E}_n)$, where $\underline{E}_n$ is the n^{th} space in the Ω–spectrum for the G–oriented ring spectrum E. Using this model, one computes

$$MG_n \underline{E}_{n-m} = \left\{ M^n \xrightarrow{\omega} \underline{E}_{n-m} \,\middle|\, \begin{array}{c} M \text{ a closed } n\text{–dimensional } G\text{–manifold,} \\ \omega \in E^{n-m}(M) \text{ a class of codimension } m \end{array} \right\} \Big/ \sim .$$

In terms of stable homotopy theory, the spectrum

$$MG \wedge \Sigma^{-(n-m)} \Sigma^\infty_+ \underline{E}_{n-m}$$

contains in π_{-m} information about G–structured bordism classes of dimension n which are equipped with m–codimensional E–cohomology classes. Incorporating the orientation φ, one can then build the composite

$$\mathbb{S}^{-m} \xrightarrow{(M,\omega)} MG \wedge \Sigma^{-(n-m)} \Sigma^\infty_+ \underline{E}_{n-m} \xrightarrow{\operatorname{colim}_{n\to\infty}} MG \wedge E \xrightarrow{\varphi \wedge 1} E \wedge E \xrightarrow{\mu} E,$$

which gives a recipe for an element of $\pi_{-m}E$. Classically, this element is identifiable.

Lemma 1.4. *For $\varphi_{MO}\colon MO \to H\mathbb{F}_2$ and $\varphi_{MSO}\colon MSO \to H\mathbb{Z}$, this assignment is only interesting in the case $m = 0$, where*

$$\varphi_{MO}\colon (M^n, \omega \in H^n(M;\mathbb{F}_2)) \mapsto \int_M \omega \in \mathbb{F}_2 = \pi_0 H\mathbb{F}_2,$$
$$\varphi_{MSO}\colon (M^n \textit{ oriented}, \omega \in H^n(M;\mathbb{Z})) \mapsto \int_M \omega \in \mathbb{Z} = \pi_0 H\mathbb{Z}. \quad \square$$

In general, we think of this as a way to extract an integral for E–cohomology classes on G–structured manifolds.

Remembering the Pontryagin–Thom equivalence $\mathbb{S} \simeq M\mathrm{Framed}$, this construction suddenly gains widespread application: to *any* ring spectrum E, the unit map $\eta_E : \mathbb{S} \to E$ associates a theory of integration for E–cohomology classes on framed bordism classes by the orientation

$$M\mathrm{Framed} \simeq \mathbb{S} \xrightarrow{\eta_E} E.$$

This also gives a further interpretation of orientations in general: the sequence of ring maps

$$\mathbb{S} \to MSO \xrightarrow{\varphi} H\mathbb{Z}$$

can then be thought of as a *factorization* which witnesses some overdeterminacy of the naturally occurring framed integral. After all, given a framed manifold one can integrate a top-dimensional integral cohomology class, but actually a mere orientation of the tangent bundle is all that is required to produce an integral.[1]

Recognizing that O and SO form the beginning of the Postnikov tower for O, we are inspired to ask the following:

Question 1.5. Given a ring spectrum $\eta_E \colon \mathbb{S} \to E$, through what stage of the Postnikov filtration does η_E factor?

Here's a somewhat longer list of known orientations of cohomology theories:

$$\begin{tikzcd}
\mathbb{S} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & MO[8,\infty) \arrow[r] \arrow[d, equal] & MO[4,\infty) \arrow[r] \arrow[d, equal] & MO[2,\infty) \arrow[r] \arrow[d, equal] & MO[1,\infty) \arrow[d, equal] \\
M\mathrm{Framed} \arrow[r] \arrow[drr] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr] & \cdots \arrow[r] & M\mathrm{String} \arrow[r] \arrow[d] & M\mathrm{Spin} \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\
 & & tmf & kO & H\mathbb{Z} & H\mathbb{F}_2.
\end{tikzcd}$$

The first two vertical maps are classical; the orientation $M\mathrm{Spin} \to kO$ is due to Atiyah, Bott, and Shapiro [**ABS64, Joa97**]; and the orientation

$$\sigma\colon\ M\mathrm{String} \to tmf$$

is due to Ando, Hopkins, Rezk, and Strickland [**AHR, AHS01, AS01, HAS99**]. One can see an interesting correspondence beginning to take form: the level of Postnikov filtration on the top is matched (imperfectly) by the chromatic height of the spectrum on the bottom. The precise nature of this correspondence is pretty widely open, and our goal is to shed computational light on one aspect of it, motivated by the construction of the String–orientation of *tmf*. Their first step is to instead consider maps

$$\begin{tikzcd}
M\mathrm{String} \arrow[d, dotted, "\sigma"'] \arrow[dr, dotted] & MU[6,\infty) \arrow[l] \arrow[d] \arrow[dl, dotted] \\
tmf \arrow[r] & E.
\end{tikzcd}$$

approximating σ on both sides. The replacement of $M\mathrm{String}$ by $MU[6,\infty)$ is meant to be simpler because $MU[6,\infty)$ is closer to MU, whose theory of orientations is exceedingly well–understood: this the theory of *complex orientations*. Accordingly, E is chosen to be a complex-orientable spectrum approximating *tmf*. Their game, then, is to scale the Postnikov tower from orientations $MU \to E$ to gain an understanding of orientations $MU[6,\infty) \to E$, then use this knowledge to show that wisely chosen approximations suffice to construct the desired map σ.

2. The algebraic geometry of the Thom construction

Chromatic homotopy theory, the study of complex-oriented cohomology theories, is naturally cast as a marriage of arithmetic geometry and algebraic topology. According to the taste of the author, the exposition can vary widely in which of these two fields is favored, and the version set forward by Ando, Hopkins, and Strickland is one of the more arithmetically dense. Their basic object is:

[1] Take this lightly, as this factorization is also a witness that oriented manifolds which *cannot* be framed support such integrals as well.

Definition 2.1 ([**Str99**, Definition 8.11], [**AHS01**, Section 2.1.2])**.** For a space X and ring spectrum E, we write X_E for the formal affine scheme

$$X_E = \operatorname{Spf} E^0 X$$

topologized by the lattice of compact subspaces of X.

This visibly requires some niceness — for instance, E^0X has to be a commutative ring, E has to be sufficiently self-entwined so that "E^0" carries all of the interesting information, and the formal topology on E^0X has to be handled delicately. Details are available for the interested reader [**Str99**, Section 8.2], but for our purposes it suffices to say that they arrange their situation so that this is the case. Here is the basic sort of theorem you can expect to see in their language:

Theorem 2.2 ([**Str99**, Section 8.6]). *There is a bijection between ring spectrum maps $MUP \to E$ and isomorphisms between $\mathbb{CP}^\infty_E$ and the affine line $\widehat{\mathbb{A}}^1 = \operatorname{Spf} E^0[\![x]\!]$.* □

Before discussing bordism orientations further, we will take the time to translate some other theorems about complex-orientable cohomology theories into this language. Of course, the main point of choosing a complex orientation is to provide Thom isomorphisms for complex vector bundles, so this gives us a wide body of results to work on translating. For instance, the complex-oriented E–cohomology of the projectivization $\mathbb{P}(V)$ of a rank n vector bundle V on a space X has a presentation as

$$E^*\mathbb{P}(V) \cong E^*X[\![t]\!]/c_V(t),$$

where $c_V(t)$ is the total Chern class of V, i.e.,

$$c_V(t) = t^n - c_1(V)t^{n-1} + c_2(V)t^{n-2} - \cdots + (-1)^n c_n(V).$$

In their language, one writes:

Lemma 2.3 ([**Str99**, Proposition 8.28])**.** *The formal scheme $\mathbb{P}(V)_E$ is finite (of degree n) and free over X_E, and it forms a closed subscheme — i.e., it is an* effective Weil divisor *— of the curve $\mathbb{CP}^\infty_E \times X_E$ (thought of as an X_E–scheme).* □

Corollary 2.4 ([**Str99**, Proposition 8.31])**.** *The induced map $BU(n)_E \to \operatorname{Div}_n^+ \mathbb{CP}^\infty_E$ is an isomorphism, where $\operatorname{Div}_n^+ \mathbb{CP}^\infty_E$ is the formal scheme classifying effective Weil divisors of degree n.* □

Projectivization leads to another essential theorem in vector bundle geometry: the splitting principle. Fixing V over X, there is a map $f\colon Y \to X$ such that f^*V splits naturally as a sum of line bundles and the induced map E^*f, for any complex-orientable E, is injective.

Lemma 2.5. *Fixing a vector bundle V over a space X, there is a map $f\colon Y \to X$ such that $Y_E \to X_E$ is finite and faithfully flat, and $f_E^*\mathbb{P}(V)_E = \mathbb{P}(f^*V)_E$ splits as a sum of points.* □

We can also directly interpret the cohomology of the Thom spectrum X^V, although this takes a little more vocabulary. In classical language, a Thom isomorphism is an E^0X–module isomorphism $\widetilde{E}^0 X^V \cong E^0X$, and the entire theory of modules is recast by algebraic geometers as the theory of quasicoherent sheaves over affine schemes.

Lemma 2.6 ([**Str99**, Definition 8.33])**.** *The quasicoherent sheaf* $\mathbb{L}(V)$ *over* X_E *determined by* E^0X^V *is a trivializable line bundle.* □

A remarkable feature of the algebro-geometric language employed in this section is that many things we are first taught to think of as structures in topology — here is a collection of Thom isomorphisms, for instance, or a consequence of a calculation that begins with invoking the existence of some basis — can be easily recast as properties of the associated geometric objects — here are *trivializable invertible sheaves*, without a chosen trivializing section. This allows us to make concise distinctions between facts about a complex-*oriented* cohomology theory and facts about a complex-*orientable* cohomology theory, *without a particular complex orientation taken as reference.* In fact, the theorem at the beginning of this section can be stated without such a reference object:

Theorem 2.7 ([**Str99**, Section 8.6], [**AHS01**, Example 2.53]). *Let* $\mathcal{L}$ *denote the tautological line bundle on* $\mathbb{C}\mathrm{P}^\infty$*. The sheaf* $\mathbb{L}(\mathcal{L})$ *on* $\mathbb{C}\mathrm{P}^\infty_E$ *is invertible exactly when* E *is complex-orientable, in which case it is trivializable. Trivializations then biject with ring maps* $MUP \to E$*.*

Sketch of proof. A trivialization of $\mathbb{L}(\mathcal{L})$ gives rise to a trivialization of

$$\mathbb{L}(\mathcal{L}^{\times n} \downarrow (\mathbb{C}\mathrm{P}^\infty)^{\times n}) \cong \mathbb{L}(\mathcal{L})^{\boxtimes n} \downarrow (\mathbb{C}\mathrm{P}^\infty_E)^{\times n}.$$

Applying the splitting principle to a vector bundle V over a space X to produce a map $f\colon Y \to X$, the resulting sheaf $\mathbb{L}(f^*V)$ pulls back from $\mathbb{L}(\mathcal{L}^{\times n})$, hence receives a trivialization. Finally, trivializations descend along finite flat maps. To see that these trivializations are suitably compatible, write $\zeta\colon X_E \to \mathbb{C}\mathrm{P}^\infty_E \times X_E$ for the zero-section and $\mathcal{I}(\mathbb{P}(V)_E)$ for the ideal sheaf of functions vanishing on $\mathbb{P}(V)_E$. There is then is an equivalence $\zeta^*\mathcal{I}(\mathbb{P}(V)_E) \cong \mathbb{L}(V)$. The projectivization construction converts sums of bundles to sums of divisors, and hence the trivialization of $\mathbb{L}(V \oplus W)$ is the product of the trivializations for $\mathbb{L}(V)$ and $\mathbb{L}(W)$. Applying these constructions to the universal bundles over the spaces $BU(n)$, this data constructs a ring map $MUP \to E$. □

3. Orientations for MSU and $MU[6,\infty)$

We now return to the study of ring maps $MU[2k,\infty) \to E$ for positive k. Because we went through the analysis above, we can see where the crucial pieces of input lie: first we should develop a version of the splitting principle for these structured bundles, and then we should tease apart the resulting mess of formal schemes.

Suppose that V is a vector bundle over X and that $Y \to X$ is the space over X guaranteed to us by the splitting principle for V. Now consider first the case of $k = 1$, where V has been lifted as in

$$\begin{tikzcd} BU \arrow[r, "+n\cdot 1"] & BU \times \mathbb{Z} \\ & X. \arrow[ul] \arrow[u, "V"'] \end{tikzcd}$$

The analog of the splitting principle for BU–structured bundles then says that over Y as in Lemma 2.5, there is an isomorphism

$$f^*(V - n \cdot 1) \cong \left(\bigoplus_{j=1}^{n} \mathcal{L}_j \right) - n \cdot 1 \cong \bigoplus_{j=1}^{n} (\mathcal{L}_j - 1),$$

where now both the left– and right–hand sides have a natural interpretation as vector bundles of virtual rank 0.

For the case $k = 2$, select a further lift

$$\begin{tikzcd} BSU \arrow[r] & BU \arrow[r, "+n\cdot 1"] & BU \times \mathbb{Z} \\ & & X. \arrow[u, "V"'] \arrow[ul] \arrow[ull, dotted, "\widetilde{V}"] \end{tikzcd}$$

We record the following analogous Lemma and proof:

Lemma 3.1. *The bundle $f^*\widetilde{V}$ is equivalent (as BSU–classes) to a sum of bundles of the form $(\mathcal{H} - 1)(\mathcal{H}' - 1)$ for $\mathcal{H}$, $\mathcal{H}'$ complex line bundles.*

PROOF. Over Y, we may split two lines off of $V = \mathcal{L}_1 + \mathcal{L}_2 + V'$. Because $(\mathcal{L}_1 - \varepsilon)$ and $(\mathcal{L}_2 - \varepsilon)$ both have natural BU–structures, they can both be interpreted as classes in $kU^2(Y) = [Y, BU]$. It follows that their product $(\mathcal{L}_1 - \varepsilon)(\mathcal{L}_2 - \varepsilon)$ has a natural SU–structure, where we cup the two classes in $kU^2(Y)$ to form a class in $kU^4(Y)$, and hence the sum

$$\widetilde{V} + (\mathcal{L}_1 - \varepsilon)(\mathcal{L}_2 - \varepsilon)$$

has a natural SU–structure as well.[2] The underlying bundle has the form

$$\begin{aligned} (V - n\varepsilon) + (\mathcal{L}_1 - \varepsilon)(\mathcal{L}_2 - \varepsilon) &= (V' + \mathcal{L}_1 + \mathcal{L}_2 - n\varepsilon) + (\mathcal{L}_1 - \varepsilon)(\mathcal{L}_2 - \varepsilon) \\ &= (V' + \mathcal{L}_1 + \mathcal{L}_2 - n\varepsilon) \\ &\quad - (\mathcal{L}_1 - \varepsilon) - (\mathcal{L}_2 - \varepsilon) + (\mathcal{L}_1\mathcal{L}_2 - \varepsilon) \\ &= (V' + \mathcal{L}_1\mathcal{L}_2) - (n-1)\varepsilon. \end{aligned}$$

This is the virtualization of a vector bundle of one rank fewer, so we can induct.

To ground the induction, we need only know that all rank 1 bundles with SU–structure are trivial. This follows because admitting an SU–structure is identical to having vanishing first Chern class, but complex line bundles are determined by their first Chern classes. □

Lemma 3.2 (Hopkins[3])**.** *In the case $k = 3$, $BU[6, \infty)$–bundles decompose into sums of $BU[6, \infty)$–bundles of the form $(\mathcal{H} - 1)(\mathcal{H}' - 1)(\mathcal{H}'' - 1)$.* □

In light of these enhanced splitting principles for the structure groups SU and $U[5, \infty)$, we are moved to study the universal maps

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[2] In general, the ring structure gives a lift $(\mathbb{C}\mathrm{P}^\infty)^{\times k} \to BU[2k, \infty)$ of the k–fold external product of the reduced tautological bundle. This lift does not actually depend on the ring structure: it also exists because the bottom several Chern classes of the external product vanish, and it is unique because there are no odd-degree classes in $kU^*(\mathbb{C}\mathrm{P}^\infty)^{\times k}$.

[3] This has been claimed without proof [**Hop95**, pg. 558]. However, I do not know an elementary proof of this, along the lines of Lemma 3.1.

$$(\mathbb{C}\mathrm{P}^\infty_E)^{\times 2} \xrightarrow{(\mathcal{L}_1-1)(\mathcal{L}_2-1)} BSU_E,$$

$$(\mathbb{C}\mathrm{P}^\infty_E)^{\times 3} \xrightarrow{(\mathcal{L}_1-1)(\mathcal{L}_2-1)(\mathcal{L}_3-1)} BU[6,\infty)_E.$$

These maps have some evident properties:

(1) They are *symmetric*: trading $\mathcal{L}_j$ and $\mathcal{L}_k$ gives an isomorphic bundle.
(2) They are *rigid*: replacing $\mathcal{L}_j$ by the trivial line gives the zero bundle.
(3) They are $\operatorname{Div}\mathbb{C}\mathrm{P}^\infty_E$*–linear*: one can act by a line bundle $\mathcal{H} \in kU^0\mathbb{C}\mathrm{P}^\infty$ using the product map

$$\underline{kU}_0 \times \underline{kU}_6 \xrightarrow{\mu} \underline{kU}_6$$

which gives simultaneous decompositions

$$\begin{aligned}(\mathcal{L}_1-1)\mathcal{H}(\mathcal{L}_2-1) &= (\mathcal{L}_1-1)(\mathcal{H}\mathcal{L}_2-\mathcal{H}) = (\mathcal{L}_1-1)(\mathcal{H}\mathcal{L}_2-1)-(\mathcal{H}-1)(\mathcal{L}_1-1) \\ &= (\mathcal{H}\mathcal{L}_1-\mathcal{H})(\mathcal{L}_2-1) = (\mathcal{H}\mathcal{L}_1-1)(\mathcal{L}_2-1)-(\mathcal{H}-1)(\mathcal{L}_2-1).\end{aligned}$$

Equating these last two expressions gives a kind of 2–cocycle condition.

With these observations in hand, one is led to investigate the following functions:

Definition 3.3 ([**AHS01**, Definition 2.42])**.** A *k–variate symmetric 2–cocycle for* E is a trivialization of the line bundle $\mathbb{L}(\bigotimes_{j=1}^k(\mathcal{L}_j-1))$ on $(\mathbb{C}\mathrm{P}^\infty)^{\times k}_E$ satisfying a symmetry condition, a rigidity condition, and the linearity condition[4]

$$\frac{f(t, x +_{\mathbb{C}\mathrm{P}^\infty_E} y, \ldots)}{f(t,x,\ldots)} = \frac{f(t +_{\mathbb{C}\mathrm{P}^\infty_E} x, y, \ldots)}{f(x,y,\ldots)}.$$

We write $C^k(\mathbb{C}\mathrm{P}^\infty_E; \mathcal{I}(0))$ for the set of such.

These are the sections which one "expects" to come from $BU[2k,\infty)$–structures. The main theorem of Ando, Hopkins, and Strickland is that these considerations are sufficient:

Theorem 3.4 ([**AHS01**, Corollary 2.52]). *For E a complex-orientable ring spectrum and $k \leq 3$ an integer, such a trivializing section determines a ring map $MU[2k,\infty) \to E$.*

Notes on proof. They reduce from general E to $E = MU$, then from there to $E = Hk$ for k a prime field. They then perform the brutal calculation of the available such sections in the algebraic model and compare with $\operatorname{Spec} Hk_*MU[2k,\infty)$, made accessible by Singer's calculation below. They find that the topological and algebraic objects have the same graded ranks and that one surjects onto the other. □

In the case of $k = 0$ and $k = 1$, this degenerates to exactly the classical analysis. In the case where E is an *elliptic spectrum*, they further show that the associated elliptic curve begets a canonical section in the $k = 3$ case.[5] This unicity is the main punchline in the construction of the (complex, nonparametrized) σ–orientation. It's also possible, by placing more hypotheses on E, to extend this same mode of analysis to ring maps $M\mathrm{String} \to E$ [**HAS99**].

[4] To explain the notation in this last condition, $\mathbb{C}\mathrm{P}^\infty_E$ is a smooth formal group, owing to the tensor product of line bundles, and we denote its group operation by "$+_{\mathbb{C}\mathrm{P}^\infty_E}$".

[5] Remarkably, attaching elliptic data to E does *not* give a canonical section for $k < 3$!

4. Orientations for $MU[2k,\infty)$?

Several things go wrong in trying to extend this analysis to $MU[2k,\infty)$ for $k \ge 4$, which forms the real subject of this note. In fact, something immediately obstructs any attempt to continue the algebro-geometric analysis described. The following theorem is key to their result:

THEOREM 4.1 ([**Sin68, Sto63**]). *There are classes $\theta_{2j} \equiv c_j$ (mod decomposables) such that*

$$H\mathbb{F}_2{}^*(BU[2k,\infty)) \cong \frac{H\mathbb{F}_2{}^*(BU)}{\langle \theta_{2j} \mid \sigma_2(j-1) < k-1 \rangle} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-3}],$$

where $\mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-3}]$ is the Steenrod–Hopf sub-algebra of $H\mathbb{F}_2{}^*K(\mathbb{Z},2k-3)$ generated by $\mathrm{Sq}^3 \iota_{2k-3}$. □

Corollary 4.2 ([**HLP13**, Remark 8.1])**.** *There are odd classes in $H\mathbb{F}_{2*}BU[2k,\infty)$ for $k \ge 4$.*

PROOF. The class $\mathrm{Sq}^7 \mathrm{Sq}^3 \iota_{2k-3} \in \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-3}]$ is non-zero once $k \ge 4$. □

By consequence, there are also odd classes in $H\mathbb{F}_{2*}MU[2k,\infty)$, and there are similar statements at primes $p \ge 3$. This means that we're not allowed to form the affine scheme "$\operatorname{Spec} H\mathbb{F}_{p*}MU[2k,\infty)$" for $k \ge 4$, and so the Ando–Hopkins–Strickland program is immediately stymied.

Nonetheless, we can still perform a purely algebraic analysis. In order to find more concrete footing, consider the following result of Ando, Hopkins, and Strickland:

Lemma 4.3 ([**AHS01**, Theorem 2.50])**.** *For $k \le 3$, the Thom diagonal map*

$$E_*MU[2k,\infty) \otimes_{E_*} E_*BU[2k,\infty) \leftarrow E_*MU[2k,\infty)$$

models the tensor product map

$$\begin{aligned} C^k(\mathbb{CP}^\infty_E;\mathcal{I}(0)) \times C^k(\mathbb{CP}^\infty_E;\mathcal{O}) &\to C^k(\mathbb{CP}^\infty_E;\mathcal{I}(0)\otimes\mathcal{O}) \\ &\xrightarrow{\cong} C^k(\mathbb{CP}^\infty_E;\mathcal{I}(0)), \end{aligned}$$

where $C^k(\mathbb{CP}^\infty_E;\mathcal{O})$ denotes the set of k–variate $\mathbb{G}_m$–valued functions satisfying the linearity, symmetry, and 2–cocycle conditions. □

This presents $C^k(\mathbb{CP}^\infty_E;\mathcal{I}(0))$ as a torsor for $C^k(\mathbb{CP}^\infty_E;\mathcal{O})$. In particular, there is a noncanonical isomorphism between them, and we may as well study the untwisted version, which is a considerable gain in concreteness. Moreover, our case of interest is $E = H\mathbb{F}_2$, so that $\mathbb{CP}^\infty_{H\mathbb{F}_2} = \widehat{\mathbb{G}}_a$ over $\operatorname{Spec}\mathbb{F}_2$, where we too can perform the brutal calculation analogous to the proof of Theorem 3.4:

THEOREM 4.4 ([**HLP13**, Theorem 7.7]). *There is a presentation of $\mathbb{F}_2$–algebras*

$$\begin{aligned} \mathcal{O}(\underline{C}^k(\widehat{\mathbb{G}}_a;\mathcal{O})) = \mathbb{F}_2[z_n \mid \nu_2\varphi(n,k) \le \nu_2 n] &\otimes \\ \otimes\, \Gamma[b_{n,\gamma_2(n,k)} \mid \nu_2\varphi(n,k) > \nu_2 n] &\otimes \\ \otimes\, \mathbb{F}_2[b_{n,i} \mid \gamma_2(n,k) < i < D_{n,k}]/\langle b_{n,i}^2 \rangle&, \end{aligned}$$

where $n \ge k$ ranges over integers, $D_{n,k}$ is the coefficient of the generating function

$$\prod_{i=0}^{\infty} \frac{1}{1-tx^{2^i}} = \sum_{n,k} D_{n,k} x^n t^k,$$

and $\gamma_p(n,k) = \max\{0, \min\{k - \sigma_p(n), \nu_p(n)\}\}$ *counts the number of divided power classes introduced already.*

The same construction as the universal map in Section 3 begets a ring map

$$\mathcal{O}(\underline{C}^k(\widehat{\mathbb{G}}_a;\mathcal{O})) \to H\mathbb{F}_{2*}BU[2k,\infty).$$

More than this, this map is equivariant for the coaction of $\mathcal{O}(\operatorname{Aut}\widehat{\mathbb{G}}_a)$, i.e., it is a map of comodules for the dual Steenrod algebra. Accordingly, it factors

$$\begin{tikzcd}
\mathcal{O}(\underline{C}^k(\widehat{\mathbb{G}}_a;\mathcal{O})) \arrow[rr] \arrow[drr] \arrow[ddrr] & & H\mathbb{F}_{2*}BU[2k,\infty) \\
& & \text{even part} \arrow[u] \\
& & \text{even sub-Steenrod-Hopf algebra.} \arrow[u]
\end{tikzcd}$$

Conjecture 4.5 ([**HLP13**, Remark 8.2])**.** The longest map is an isomorphism.

Evidence. The Poincaré series of the source of the isomorphism is accessible from Theorem 4.4. Singer's method gives partial information about the coaction of the dual Steenrod algebra on the topological side, which is enough to make an educated guess about the Poincaré series of the restricted algebraic target. As far out as we can check (through some thousands of bidegrees) these series agree.

This means that the algebraic approximation is actually pretty good — much, much closer than you might initially think! This begets a natural question, if we want to use this calculation in homotopy theory:

Question 4.6 ([**HLP13**, Remark 8.5])**.** Is there a space $X(k)$ with $\operatorname{Spec} E_*X(k) \cong \underline{C}^k(\widehat{\mathbb{G}}_E;\mathcal{O})$ for E a complex–orientable cohomology theory? In particular, is there a space whose $H\mathbb{F}_2$–homology realizes this particular subset of the $H\mathbb{F}_2$–homology of $BU[2k,\infty)$?

These spaces should probably have some other nice properties if they're to be truly useful. However, because these hypothetical spaces are so close to such already-famous spaces, there are many dangerous theorems to steer around.

Lemma 4.7. *The spaces $X(k)$ cannot collectively form the spaces of an Ω–spectrum.*

Proof. The results of Adams and Priddy [**AP76**] show that any connective spectrum delooping BSU is p–locally equivalent to $kU[4,\infty)$. Since we have $X(2) = BSU$, this would force $X(k) = BU[2k,\infty)$ for all k. However, we know that $BU[8,\infty)$ does not model $X(4)$. □

Nonetheless, the putative spaces $X(k)$ may individually be infinite loopspaces, with associated spectra $x(k)$. These, too, are prohibited from being too interesting. Recall that part of the Ando–Hopkins–Strickland program was to study a particular lifting problem, which $K(2)$–locally looks like

$$\begin{tikzcd}
MO[8,\infty) \arrow[d, dotted] \arrow[dr, dotted] & MU[6,\infty) \arrow[l] \arrow[d] \arrow[dl, dotted] \\
EO_2 \arrow[r] & E_2,
\end{tikzcd}$$

where E_2 is a Morava E–theory associated to a supersingular elliptic curve and EO_2 is its homotopy fixed points under the action of a maximal finite subgroup of the automorphisms of the curve. These sorts of factorizations are largely prohibited at higher chromatic heights.

Lemma 4.8. *There cannot exist a diagram of ring spectra*

$$\begin{array}{ccccccc} \mathbb{S} & \longrightarrow & MO[j,\infty) & \longrightarrow & xo(k) & \longrightarrow & EO_{p-1} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & MU[j,\infty) & \longrightarrow & x(k) & \longrightarrow & E_{p-1}. \end{array}$$

for any prime $p \geq 5$ and for any *choice of k and j.*

Proof. Already the "outer" rectangle is prohibited from existing, with the $xo(k) \to x(k)$ column removed. Namely, Hovey has shown [**Hov97**, Proposition 2.3.2] that there cannot exist a map of ring spectra $MO[j,\infty) \to EO_{p-1}$ because the homotopy element $\alpha_{1/1}$ is visible in $\pi_* EO_{p-1}$ but not in $\pi_* MO[j,\infty)$. $\square$

This means that, even if $x(k) \to E_{p-1}$ does exist in any sense, there cannot exist a map $xo(k) \to EO_{p-1}$ factoring it which participates at all naturally with orientations by connective real bordism theories.

In spite of these results, there is an interesting sequence of infinite loopspaces that overlaps with the well–understood part of this analysis.

Definition 4.9. A *Wilson space* is a connected p–local H–space which is indecomposable and which has $\mathbb{Z}_{(p)}$–free homotopy and homology.

Theorem 4.10 ([**Wil75**, Corollary 6.8]). *Wilson spaces* all *appear as $\underline{BP\langle n\rangle}_k$ for varying values of n and k.* $\square$

At $p = 2$, the first few Wilson spaces are indicated in the following table:

k	1	2	3	4	$\cdots$
Y_k	BU	BSU	$BU[6,\infty)$	$\underline{BP\langle 2\rangle}_8$	$\cdots$.

Their ordinary homology is known as a Hopf ring, computed by Sinkinson [**Sin76**] following along the lines of Ravenel and Wilson's analysis [**RW77**] of $H\mathbb{F}_{p*}\underline{MU}_2$, and the result doesn't look very much like what we described above. Nonetheless, the primacy of Wilson spaces and their overlap with the Ando–Hopkins–Strickland results is encouraging. In light of our summary here, perhaps some connection would be uncovered if we could answer the following question:

Question 4.11. Is there a splitting principle for $\underline{BP\langle 2\rangle}_8$–classes?

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