AGT

Algebraic & Geometric Topology

Volume 22 (2022)

On homotopy groups of $E_C^{hG_{24}} \wedge A_1$

VIET-CUONG PHAM

msp

Algebraic & Geometric Topology 22 (2022) 3855–3938

On homotopy groups of $E_C^{hG_{24}} \wedge A_1$

VIET-CUONG PHAM

Let A_1 be any spectrum in a class of finite spectra whose mod 2 cohomology is isomorphic to a free module of rank one over the subalgebra $\mathcal{A}(1)$ of the Steenrod algebra. Let E_C be the second Morava E–theory associated to a universal deformation of the formal completion of the supersingular elliptic curve $C : y^2 + y = x^3$ defined over $\mathbb{F}_4$ and G_{24} a maximal finite subgroup of the automorphism group $\mathbb{S}_C$ of the formal completion of C. We compute the homotopy groups of $E_C^{hG_{24}} \wedge A_1$ by means of the homotopy fixed-point spectral sequence.

55Q10

Introduction

A central problem in stable homotopy theory is to understand the homotopy groups of the sphere spectrum localized at each prime p, $\pi_*(S^0_{(p)})$. A powerful tool for computing the latter is the Adams and Adams–Novikov spectral sequences, which allows one to compute $\pi_*(S^0_{(p)})$ at each stem $*$. Complementary to this method is the chromatic approach to stable homotopy theory, which aims at analyzing the latter in a large scale. In fact, the chromatic point of view offers a tool to analyze the global structure of the stable homotopy category, and hence that of $\pi_*(S^0_{(p)})$, in a systematic way by decomposing it into smaller pieces; see for example Ravenel [33] and Hovey

Published: 14 March 2023 DOI: 10.2140/agt.2022.22.3855

and Strickland [25]. Fundamental building blocks in the theory are the $K(n)$–local homotopy category, the Bousfield localization of the stable homotopy category at the Morava K–theories, which are defined at all primes and all natural numbers n; the prime is implicit in the notation of $K(n)$ and n is referred to as the chromatic level.

For this purpose, a general strategy is to study the homotopy type of the $K(n)$–localization of various finite spectra. A central result of the theory is the work of Devinatz and Hopkins [16], which expresses the $K(n)$–localization of a finite spectrum X as the continuous homotopy fixed-point spectrum

$$L_{K(n)}X \simeq E_n^{h\mathbb{G}_n} \wedge X,$$

where $\mathbb{G}_n$ is the extended Morava stabilizer group, which is profinite, and E_n is the n^{th} Morava E–theory.

The study of chromatic level one was a great success: the homotopy groups of $L_{K(1)}S^0$ have been completely computed at all primes and, at the prime 2, $L_{K(1)}S^0$ detects essentially the image of J, an infinite family of elements of $\pi_*(S^0)$. Chromatic level two has also been thoroughly investigated at odd primes. It started with the computation by Shimomura, Wang and Yabe [39; 36; 37; 38]. Later, Goerss, Henn, Mahowald and Rezk [21] proposed a conceptual framework to organize the $K(2)$–local homotopy category at the prime 3, in which the authors constructed a finite resolution of the $K(2)$–local sphere using higher real K–theories. See work of Goerss, Henn, Karamanov and Mahowald [20; 23; 19] for further investigations at $n=2$ and $p=3$ and Behrens [7] for an exposition at $p \geq 5$.

The situation of chromatic level two at the prime 2 turns out to be much more complicated and we are only beginning to understand it better. Considerable effort has recently been made to understand the $K(2)$–local homotopy category at the prime 2 by the community. In [11], Bobkova and Goerss established a finite resolution of a spectrum related to the $K(2)$–local sphere at the prime 2 analogous to that of [21], which realized an algebraic resolution of $\mathbb{S}_2^1$, a certain closed subgroup of the second Morava stabilizer group, constructed by Beaudry [5].

One reason why the latter is hard to deal with lies largely in the fact that the cohomological properties of the group $\mathbb{G}_2$ are much more complicated at the prime 2. However, one exciting feature of chromatic level 2 is its close relationship with the theory of elliptic curves and modular forms, see Section 1. At chromatic level 2 and at the prime 2, we can choose the Morava E–theory to be the Lubin–Tate theory associated to the formal group law of the elliptic curve $C : y^2 + y = x^3$ over $\mathbb{F}_4$. We denote by E_C and $\mathbb{G}_C$ the corresponding Morava E–theory and Morava stabilizer group. One of the main tools used to investigate the $K(2)$–local homotopy category is a finite resolution. There is a certain subgroup $\mathbb{S}_C^1$ of $\mathbb{G}_C$; let G_{24} be the automorphism group of C and C_6 be a cyclic subgroup of order 6 of G_{24} (see Section 1 for details).

Theorem 1 [11] *There is a resolution of $E_C^{h\mathbb{S}_C^1}$, in the $K(2)$–local homotopy category at the prime 2, of the form*

$$E_C^{h\mathbb{S}_C^1} \xrightarrow{\delta_0} \mathcal{E}_0 \xrightarrow{\delta_1} \mathcal{E}_1 \xrightarrow{\delta_2} \mathcal{E}_2 \xrightarrow{\delta_3} \mathcal{E}_3,$$

where $\mathcal{E}_0 = E_C^{hG_{24}}$, $\mathcal{E}_1 = \mathcal{E}_2 = E_C^{hC_6}$ and $\mathcal{E}_3 = \Sigma^{48} E_C^{hG_{24}}$.

This resolution is commonly called the topological duality resolution. The spectrum $E_C^{h\mathbb{S}_C^1}$ is used to build the spectrum $E_C^{h\mathbb{S}_C}$, where $\mathbb{S}_C$ is the Morava stabilizer group, via a certain cofiber sequence

$$E_C^{h\mathbb{S}_C} \to E_C^{h\mathbb{S}_C^1} \xrightarrow{1-\pi} E_C^{h\mathbb{S}_C^1},$$

and $E_C^{h\mathbb{S}_C}$ only differs from $L_{K(2)}S^0$ by the Galois action, ie there is a homotopy equivalence

$$L_{K(2)}S^0 \simeq (E_C^{h\mathbb{S}_C})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)}.$$

Thus, this theorem offers a useful instrument to study the homotopy type of $L_{K(2)}X$ for finite spectra X at the prime 2. In particular, it produces a spectral sequence, known as the topological duality spectral sequence, abbreviated by TDSS, converging to $\pi_*(E_C^{h\mathbb{S}_C^1} \wedge X)$:

$$\mathrm{E}_1^{p,q} \cong \pi_q(\mathcal{E}_p \wedge X) \Rightarrow \pi_{q-p}(E_C^{h\mathbb{S}_C^1} \wedge X). \tag{1}$$

By now, it should be clear that judicious choices of finite spectra become important. Main players in this paper are finite spectra constructed by Davis and Mahowald [13]. Let A_1 denote a class of finite spectra whose mod 2 cohomology is isomorphic, as a module over the subalgebra $\mathcal{A}(1)$ generated by Sq^1 and Sq^2 of the Steenrod algebra $\mathcal{A}$, to a free module of rank one on a class of degree 0. As shown in [13, Theorem 1.4(i)], the class A_1 contains four different homotopy types, which are distinguished by the structure of their mod 2 cohomology as modules over the Steenrod algebra. They are successively denoted by $A_1[00]$, $A_1[01]$, $A_1[10]$ and $A_1[11]$; see Definition 3.2.1. The spectra $A_1[01]$ and $A_1[10]$ are Spanier–Whitehead self-dual, ie $D(A_1[01]) \simeq \Sigma^{-6}A_1[01]$ and $D(A_1[10]) \simeq \Sigma^{-6}A_1[10]$; and the spectra $A_1[00]$

and $A_1[11]$ are Spanier–Whitehead dual to each other, ie $D(A_1[00]) \simeq \Sigma^{-6} A_1[11]$ (here $D(-)$ denotes the function spectra $F(-, S^0)$). By an abuse of language, we write A_1 to refer to any of these four spectra and refer to any of them as a version of A_1. The spectrum A_1 is constructed via three cofiber sequences starting from the sphere spectrum. First, let $V(0)$ be the mod 2 Moore spectrum, ie the cofiber of multiplication by 2 on the sphere. Next let Y be the cofiber of multiplication by η, the first Hopf element, on $V(0)$. Davis and Mahowald show that Y admits v_1–self-maps, $v_1 : \Sigma^2 Y \to Y$. Then A_1 is the cofiber of any of these v_1–self-maps of Y.

Here, we study the homotopy fixed-point spectral sequence, abbreviated by HFPSS, for $E_C^{hG_{24}} \wedge A_1$, which constitutes an important part of the E_1–term of the TDSS:

$$\mathrm{H}^*(G_{24}, (E_C)_*(A_1)) \Rightarrow \pi_*(E_C^{hG_{24}} \wedge A_1). \tag{2}$$

Here are qualitative versions of the main results of the paper; see Theorems 5.3.17 and 5.3.18 for more precise statements.

There are classes

$$\Delta^8 \in \mathrm{H}^0(G_{24}, (E_C)_{192}), \quad \bar{\kappa} \in \mathrm{H}^4(G_{24}, (E_C)_{24}), \quad \nu \in \mathrm{H}^1(G_{24}, (E_C)_4).$$

Theorem 2 *As a module over the ring $\mathbb{F}_4[\Delta^{\pm 8}, \bar{\kappa}, \nu]/(\nu\bar{\kappa})$, the E_∞–term of the HFPSS for $E_C^{hG_{24}} \wedge A_1[01]$ and $E_C^{hG_{24}} \wedge A_1[10]$ is a direct sum of 46 explicitly known cyclic modules.*

Theorem 3 *As a module over the ring $\mathbb{F}_4[\Delta^{\pm 8}, \bar{\kappa}, \nu]/(\nu\bar{\kappa})$, the E_∞–term of the HFPSS for $E_C^{hG_{24}} \wedge A_1[00]$ and $E_C^{hG_{24}} \wedge A_1[11]$ is a direct sum of 48 explicitly known cyclic modules.*

One of the interests in working with A_1 is that a sufficient understanding of the homotopy type of $L_{K(2)} A_1$ might allow us to determine the Gross–Hopkins duality formula for the $K(2)$–local homotopy category at the prime 2. In fact, the spectrum A_1 can be considered as an analog of the Toda–Smith complex $V(1)$ at the prime 3 and, as demonstrated in [19], computations of the homotopy groups of $L_{K(2)} V(1)$ allow one to characterize the Gross–Hopkins formula for the $K(2)$–local homotopy category at the prime 3.

One of the key ingredients, to this end, is a comparison between $\mathrm{tmf} \wedge A_1$ and $E_C^{hG_{24}} \wedge A_1$, where tmf denotes the connective spectrum of topological modular forms localized at the prime 2. In fact, there is a homotopy equivalence (Theorem 5.1.1)

$$(\Delta^8)^{-1} \mathrm{tmf} \wedge A_1 \simeq (E_C^{hG_{24}})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \wedge A_1,$$

where Δ^8 is the periodicity generator of $\pi_*\mathrm{tmf}$. Based on the latter, we first analyze the homotopy groups of $\mathrm{tmf} \wedge A_1$ by means of the Adams spectral sequence, abbreviated by ASS, then invert Δ^8 to get information about the homotopy groups of $E_C^{hG_{24}} \wedge A_1$.

We note that, in [10], Bhattacharya, Egger and Mahowald also discuss the E_2–term of the ASS for $\mathrm{tmf} \wedge A_1$. Our method is, however, different; the calculation is performed with a use of the Davis–Mahowald spectral sequence. A key technical result, Proposition 3.3.7, is the determination of a certain product in the E_2–term, which depends on the module structure of $H^*(A_1)$ over $\mathcal{A}$ — more precisely, on the action of Sq^4 on $H^*(A_1)$. The latter is different for different models of A_1. In turn, this results in differences in the $\bar{\kappa}$–nilpotence order of elements of $\pi_*(\mathrm{tmf} \wedge A_1)$; here $\bar{\kappa}$ is an element of $\pi_{20}(S^0)$.

Next, we summarize the contents of the paper. In Sections 1 and 2, we discuss some background and tools used in our computation. We recall Lubin–Tate theories and topological modular forms; in particular, we sketch a proof of the relationship between topological modular forms and the homotopy fixed-point spectrum $E_C^{hG_{24}}$. We give a generalization of the Davis–Mahowald spectral sequence, which is an important tool to analyze the cohomology of various Hopf algebras. In Section 3, we discuss the Davis–Mahowald spectral sequence for A_1 and obtain the E_2–term of the Adams spectral sequence for $\mathrm{tmf} \wedge A_1$. In Section 4, we study some differentials in the latter and then extract some suitable information about $\pi_*(\mathrm{tmf} \wedge A_1)$. In Section 5, we finally study the homotopy fixed-point spectral sequence for $E_C^{hG_{24}} \wedge A_1$. We emphasize that there are two different outcomes for the E_∞–term of the homotopy fixed-point spectral sequence, depending on the version of A_1; see Theorems 5.3.17 and 5.3.18.

Conventions and notation Unless otherwise stated, all spectra are localized at the prime 2. $H^*(X)$ and $H_*(X)$ denote the mod 2 cohomology and homology of the spectrum X, respectively. Given a Hopf algebra A over a field k and M an A–comodule, we will often abbreviate $\mathrm{Ext}_A^*(k, M)$ by $\mathrm{Ext}_A^*(M)$. In general, we will write C_f for the cofiber of a map $f : X \to Y$, except that we will write $V(0)$ for the Moore spectrum which is the cofiber of the multiplication by 2 on the sphere.

Acknowledgments I would like to thank my PhD advisor, Hans-Werner Henn, for suggesting this project and for guiding me through many stages of it and for carefully reading earlier drafts of this paper. Special thanks go to Irina Bobkova for many fruitful exchanges, to Agnès Beaudry and Paul Goerss for their interest in this work and helpful conversations, to Lennart Meier for answering many of my questions about topological modular forms. I would also like to thank John Rognes for making available his course notes on *The Adams spectral sequence* [35], from which I learned many details about the Davis–Mahowald spectral sequence. Last but not least, I am grateful to the referee for plenty of valuable suggestions that improved the paper in many aspects.

This work was partially supported by the ANR Project ChroK, ANR-16-CE40-0003.

1 Recollection on chromatic homotopy theory

1.1 Lubin–Tate theories

We recall some generalities on the deformation theory of formal group laws and Goerss–Hopkins–Miller theory. Let $\mathcal{FGL}$ be the category whose objects are pairs (k,Γ), where k is a perfect field of characteristic p and Γ is a formal group law over k, and morphisms between (k,Γ) and (k',Γ') are pairs (i,ϕ), where $i\colon k'\to k$ is a homomorphism of fields and $\phi\colon\Gamma\xrightarrow{\cong} i^*\Gamma'$ is a morphism of formal group laws.

Let $(k,\Gamma)\in\mathcal{FGL}$ with Γ of height n. A deformation of (k,Γ) to a complete local ring R with maximal ideal m is a pair (F,ι), where F is a formal group law over R and $\iota\colon k\to R/m$ is a map of fields such that $p^*F=\iota^*\Gamma$ with p the canonical projection $R\to R/m$. A $\star$–isomorphism ϕ between two deformations to R is an isomorphism between the underlying formal group laws which reduces to the identity over R/m, ie $\phi\equiv x \mod (m)$. This defines a functor Def_Γ, from the category $\mathfrak{Ring}_{c,l}$, which associates to every complete local ring R the set of $\star$–isomorphism classes of deformation of Γ over R. By Lubin–Tate deformation theory, Def_Γ is corepresentable; see [27, Theorem 3.1]. That is, there exists a complete local ring $E_{k,\Gamma}$ together with a deformation $(\widetilde{\Gamma},\iota)$ over $E_{k,\Gamma}$ which is a universal deformation of Γ, in the sense that the map

$$\mathrm{Hom}_{\mathfrak{Ring}_{c,l}}(E_{k,\Gamma},R)\to\mathrm{Def}_\Gamma(R)$$

sending f to $(f^*\widetilde{\Gamma},\bar{f}\circ\iota)$, where $\bar{f}$ is induced by f on the residue field. The ring $E_{k,\Gamma}$ is noncanonically isomorphic to $\mathbb{W}(k)[\![u_1,u_2,\dots,u_{n-1}]\!]$, where $\mathbb{W}(k)$ denotes the ring of Witt vectors on k.

Consider the graded ring $E_{k,\Gamma}[u^{\pm1}]$, where $|u_i|=0$ for $1\le i\le n-1$ and $|u|=-2$. The formal group law $u\widetilde{\Gamma}(u^{-1}x,u^{-1}y)$ is a formal group law of degree -2 (ie the coefficient of x^iy^j is in degree $2(i+j-1)$). Let MU be the complex cobordism spectrum. A famous theorem of Quillen asserts that the coefficient rings MU_* support the universal group law of degree -2. Thus, $u\widetilde{\Gamma}(u^{-1}x, u^{-1}y)$ is classified by a map of graded rings $MU_* \to E_{k,\Gamma}[u^{\pm 1}]$. Define a functor from the category of pointed spaces to that of graded abelian groups,

$$X \mapsto MU_*(X) \otimes_{MU_*} E_{k,\Gamma}[u^{\pm 1}].$$

The formal group $u\widetilde{\Gamma}(u^{-1}x, u^{-1}y)$ satisfies the Landweber exact functor criterion; see [34, 6.9]. By the Landweber exact functor theorem, the above functor is a homology functor. Thus, it is represented by a ring spectrum $E(k,\Gamma)$ with

$$(E(k,\Gamma))_* \cong \mathbb{W}(k)[\![u_1, u_2, \dots, u_{n-1}]\!][u^{\pm 1}].$$

The latter is known as an n^{th} Morava E–theory or Lubin–Tate theory.

The construction that associates to a formal group law (k,Γ) the Morava E–theory $E(k,\Gamma)$ defines a functor from $\mathcal{FGL}$ to Ho(Sp), the stable homotopy category. Let us denote by $\mathbb{G}(k,\Gamma)$ the automorphism group of the pair (k,Γ). We note that $\mathbb{G}(k,\Gamma)$ is a profinite group; see [18, Section 7.2]. By functoriality, the group $\mathbb{G}(k,\Gamma)$ acts on $E(k,\Gamma)$. This action is, however, defined only up to homotopy. The Goerss–Hopkins–Miller obstruction theory lifts this action to structured ring spectra.

Theorem 1.1.1 [22, Corollary 7.6] *The spectrum $E(k,\Gamma)$ has a unique structure of an E_∞–ring. Furthermore, $\mathbb{G}(k,\Gamma)$ acts on $E(k,\Gamma)$ via E_∞–ring maps.*

1.2 Topological modular forms

An astute choice of Morava E–theory or equivalently a choice of formal group law of height 2 will make the calculation easier. Let C be the supersingular elliptic curve over $\mathbb{F}_4$ given by the Weierstrass equation $y^2 + y = x^3$. Denote by F_C the formal completion of C at the origin. The latter is a formal group law of height 2. We abbreviate $E(\mathbb{F}_4, F_C)$ by E_C and $\mathbb{G}(\mathbb{F}_4, F_C)$ by $\mathbb{G}_C$. Let $\mathbb{S}_C$ denote the automorphism group of F_C. Let Gal denote the Galois group of $\mathbb{F}_4$ over $\mathbb{F}_2$. There is a short exact sequence

$$1 \to \mathbb{S}_C \to \mathbb{G}_C \to \mathrm{Gal} \to 1.$$

The image of $\mathbb{S}_C$ in $\mathbb{G}_C$ corresponds to the automorphisms of $(\mathbb{F}_4, F_C)$ fixing $\mathbb{F}_4$. Since F_C is defined over $\mathbb{F}_2$, Gal fixes F_C, the above short exact sequence splits, ie $\mathbb{G}_C \cong \mathbb{S}_C \rtimes \mathrm{Gal}$. The automorphism group of C has order 24 and these are all defined over $\mathbb{F}_4$; more precisely,

$$\mathrm{Aut}(C) = \mathrm{Aut}_{\mathbb{F}_4}(C) \cong \mathrm{SL}_2(\mathbb{Z}/3) \cong Q_8 \rtimes C_3 =: G_{24},$$

where Q_8 is the quaternion group and $C_3 = \langle \omega \rangle$ is a cyclic group of order 3; see [40, Appendix A (Proposition 1.2 and Exercise A.1)]. A representation of Q_8 is given by

$$Q_8 \cong \langle i, j \mid i^4 = 1, i^2 = j^2, iji^{-1} = j^{-1} \rangle.$$

The latter has eight elements $\{1, i, j, k, -1, -i, -j, -k\}$, where -1 denotes $i^2 = j^2$ and k the product ij. The group C_3 acts on Q_8 by permuting i, j and k:

$$\omega i \omega^2 = j, \quad \omega j \omega^2 = k.$$

The elements ω and i correspond to the automorphisms $\omega(x, y) = (\xi x, \xi^2 y)$ and $i(x, y) = (x + 1, y + x + \xi^2)$, respectively.

Since C is already defined over $\mathbb{F}_2$, Gal acts on $\mathrm{Aut}(C)$. Denote by G_{48} the semidirect product $G_{24} \rtimes \mathrm{Gal}$. Moreover, the automorphism group $\mathrm{Aut}(C)$ of C maps injectively to $\mathbb{S}_C$, and G_{48} maps injectively to $\mathbb{G}_C$. We view G_{24} and G_{48} as subgroups of $\mathbb{S}_C$ and $\mathbb{G}_C$, respectively.

The reasons for choosing the formal group law of the supersingular elliptic curve C are two-fold. First, the geometric origin of G_{48} allows one to have an explicit description of its action on $\pi_*(E_C)$; see [6] for more details. Thus, it allows us to adequately compute the E_2–term of various homotopy fixed-point spectral sequences. Second, this choice of the Morava E–theory enables us to compare the associated homotopy fixed-point spectrum with the spectrum of topological modular forms, hence providing us with more tools to understand the former.

Next, we recall the construction of the spectrum of topological modular forms and show its closed relationship with the homotopy fixed-point spectrum $E_C^{hG_{24}}$. Let $\mathcal{M}$ and $\mathcal{M}(3)$ be the noncompact moduli stack of elliptic curves and elliptic curves with a full level 3 structure over $\mathbb{Z}_{(2)}$, respectively. As functors of points on $\mathbb{Z}_{(2)}$–algebras, the former are described as follows. If R is a $\mathbb{Z}_{(2)}$–algebra, then:

- $\mathcal{M}(\mathrm{spec}(R))$ is the groupoid of $(E, p\colon E \to \mathrm{spec}\, R)$, elliptic curves over $\mathrm{spec}(R)$ and isomorphisms between them, ie an isomorphism between (E, p) and (E', p') consisting of two isomorphisms of schemes $(f\colon E \to E', g\colon R \to R)$ such that $g \circ p = p' \circ f$.
- $\mathcal{M}(3)(\mathrm{spec}(R))$ is the groupoid of (E, p, ϕ) consisting of an elliptic curve (E, p) over $\mathrm{spec}(R)$ with an isomorphism of group schemes $\phi\colon \mathbb{Z}/3 \times \mathbb{Z}/3 \to E[3]$ over $\mathrm{spec}(R)$, where $E[3]$ is the subscheme of 3–torsion points of E and

isomorphisms between them, ie $(f,g)\colon (E,p,\phi) \to (E',p',\phi')$ is an isomorphism if $(f,g)\colon (E,p) \to (E',p')$ is an isomorphism of elliptic curves over $\mathrm{spec}(R)$ and $f|_{E[3]} \circ \phi = \phi'$.

Theorem 1.2.1 (Goerss, Hopkins and Miller; see Behrens [8, Theorem 1.1]) *There is an E_∞–ring spectra-valued sheaf $\mathcal{O}^{\mathrm{top}}$ on the affine étale site $\mathrm{Aff}^{\text{ét}}_{\mathcal{M}}$ of $\mathcal{M}$ such that:*

(1) *The sheafification of $\pi_0\mathcal{O}^{\mathrm{top}}$ is the structure sheaf of $\mathcal{M}$.*

(2) *If $E\colon \mathrm{spec}(R) \to \mathcal{M}$ is an étale morphism, then $\mathcal{O}^{\mathrm{top}}(\mathrm{spec}(R))$ is a spectrum associated to the formal completion of E at its origin via the Landweber exact functor theorem.*

Remark 1.2.2 The spectra constructed by point (2) of this theorem are called elliptic spectra. They are even periodic spectra R whose formal group law on $\pi_0(R)$ is the completion of an elliptic curve. These are $E(2)$–local; see [4, Lemma 4.2].

Let $G := \mathrm{GL}_2(\mathbb{Z}/3)$ denote the automorphism group of the constant group scheme $\mathbb{Z}/3\times\mathbb{Z}/3$ over $\mathbb{Z}_{(2)}$. Then G acts on $\mathcal{M}(3)$ by precomposition with the level structure. The obvious forgetful functor gives rise to a finite étale morphism of stacks (because 3 is invertible in $\mathbb{Z}_{(2)}$),

$$\mathcal{M}(3) \to \mathcal{M}. \tag{3}$$

Thus, one can evaluate $\mathcal{O}^{\mathrm{top}}$ at $\mathcal{M}$ and $\mathcal{M}(3)$. Define

$$\mathrm{TMF} = \mathcal{O}^{\mathrm{top}}(\mathcal{M}) := \operatorname*{holim}_{U\in \mathrm{Aff}^{\text{ét}}_{\mathcal{M}}} \mathcal{O}^{\mathrm{top}}(U), \quad \mathrm{TMF}(3) = \mathcal{O}^{\mathrm{top}}(\mathcal{M}(3)) := \operatorname*{holim}_{U\in \mathrm{Aff}^{\text{ét}}_{\mathcal{M}(3)}} \mathcal{O}^{\mathrm{top}}(U).$$

These are known as nonperiodic versions of topological modular forms. The morphism of (3) is a Galois cover with Galois group G, or a G–torsor. As a consequence of the fact that $\mathcal{O}^{\mathrm{top}}$ satisfies descent, one obtains that

$$\mathrm{TMF} \simeq \mathrm{TMF}(3)^{hG}. \tag{4}$$

It is known that $\mathcal{M}(3)$ is affine over the ring $\mathbb{Z}_{(2)}[\zeta]$, where ζ is a primitive third root of unity; see [15, IV, Corollaire 2.9]. In particular, the only automorphism of an elliptic curve with full level 3 structure is the identity. Furthermore, up to isomorphism of elliptic curves with full level 3 structure, there is a unique supersingular elliptic curve with a full level 3 structure over $\mathbb{F}_4$. This follows from the fact that there is a unique supersingular elliptic curve over $\mathbb{F}_4$ up to isomorphism, and that the automorphism group of the supersingular elliptic curve $(C,\mathbb{F}_4)$ has order 48, which is equal to that of G, the automorphism group of $\mathbb{Z}/3\times\mathbb{Z}/3$. In other words, the fiber of the morphism $\mathcal{M}(3)\to\mathcal{M}$ over the supersingular locus of $\mathcal{M}$ is isomorphic to $\mathrm{spec}(\mathbb{F}_4)$; ie the square

$$\begin{array}{ccc} \mathrm{spec}(\mathbb{F}_4) & \longrightarrow & \mathcal{M}(3) \\ \downarrow & & \downarrow \\ \mathrm{spec}(\mathbb{F}_4)/\!/G_{48} & \longrightarrow & \mathcal{M} \end{array}$$

is a pullback of stacks, where the bottom is given by specifying a supersingular elliptic curve, for example C, and $\mathrm{spec}(\mathbb{F}_4)/\!/G_{48}$ is the quotient stack of $\mathrm{spec}(\mathbb{F}_4)$ by G_{48}, which acts on $\mathrm{spec}(\mathbb{F}_4)$ via the quotient $G_{48}\to\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)\cong C_2$. Therefore, by the construction of $\mathcal{O}^{\mathrm{top}}$, $L_{K(2)}\mathcal{O}^{\mathrm{top}}(\mathcal{M}(3))$ is the Lubin–Tate theory associated to the pair $(\mathbb{F}_4, F_C)$; see [8, Section 4]. This means that there is a homotopy equivalence

$$L_{K(2)}\mathrm{TMF}(3)\xrightarrow{\simeq} E_C. \tag{5}$$

Note that G can be identified with $\mathrm{Aut}(\mathbb{F}_4, C)=G_{48}$ so that the equivalence (5) is equivariant with respect to the action of G on the source and of G_{48} on the target, as follows. Suppose the map $\mathrm{spec}(\mathbb{F}_4)\to\mathcal{M}(3)$ specifies the elliptic curve C and a level 3 structure $\mathbb{Z}/3^{\times 2}\xrightarrow{\Gamma} C$. Then, for any $g\in G$, there is a unique $\phi(g)\in G_{48}$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{Z}/3^{\times 2} & \xrightarrow{\Gamma} & C \\ g\uparrow & & \uparrow\phi(g) \\ \mathbb{Z}/3^{\times 2} & \xrightarrow{\Gamma} & C \end{array}$$

Theorem 1.2.3 *There is a homotopy equivalence*

$$L_{K(2)}\mathrm{TMF}\simeq E_C^{hG_{48}}. \tag{6}$$

Proof Since an elliptic spectrum is $E(2)$–local, TMF(3) is $E(2)$–local, being a homotopy limit of $E(2)$–local spectra. Using the equivalence (4) and the fact that $K(2)$–localization commutes with homotopy limits in the category of $E(2)$–local spectra (see [25, Proposition 7.10(e)]), we obtain that

$$L_{K(2)}\mathrm{TMF}\cong L_{K(2)}(\mathrm{TMF}(3)^{hG})\cong (L_{K(2)}\mathrm{TMF}(3))^{hG}\cong E_C^{hG_{48}}. \qquad \square$$

A connective model of TMF In [17], a connective ring spectrum tmf was constructed together with a map of ring spectra tmf $\to$ TMF. There is an element $\Delta^8 \in \pi_{192}\mathrm{tmf}$ such that the latter map extends to a homotopy equivalence

$$[(\Delta^8)^{-1}]\mathrm{tmf} \simeq \mathrm{TMF}; \tag{7}$$

see [17]. The (co)homology of tmf, as a module over the Steenrod algebra $\mathcal{A}$ (see Section 2 for a recollection on the Steenrod algebra), was known by Hopkins and Mahowald; see [28; 34, Theorem 21.5]:

Theorem 1.2.4 *There is an isomorphism of modules over the Steenrod algebra*,

$$\mathrm{H}^*(\mathrm{tmf}) \cong \mathcal{A}/\!/\mathcal{A}(2),$$

where $\mathcal{A}(2)$ is the subalgebra of $\mathcal{A}$ generated by Sq^1, Sq^2 and Sq^4. Equivalently, there is an isomorphism of comodules over the dual $\mathcal{A}_*$ of Steenrod algebra

$$\mathrm{H}_*(\mathrm{tmf}) \cong \mathcal{A}_* \,\square_{\mathcal{A}(2)_*}\, \mathbb{F}_2,$$

where $\mathcal{A}(2)_*$ is the dual of $\mathcal{A}(2)$.

2 The Davis–Mahowald spectral sequence

We introduce a generalization of the Davis–Mahowald spectral sequence, which is an useful tool for analyzing Ext groups over various Hopf algebras. Initially, this spectral sequence was used by Davis and Mahowald in [14] to compute Ext groups over the subalgebra $\mathcal{A}(2)$ of the Steenrod algebra.

2.1 Construction of the Davis–Mahowald spectral sequence

Let k be a field of characteristic 2. We will later specialize to the case $k = \mathbb{F}_2$, the field of two elements. Let $(A, \Delta, \mu, \epsilon, \eta, \chi)$ be a commutative Hopf algebra over k, with Δ, μ, ϵ, η and χ the coproduct, product, counit, unit and conjugation, respectively.

Definition 2.1.1 An exterior coaugmented comodule over A is an A–comodule M together with a coaugmentation of A–comodules $k \to M$ having a chosen section s of k–vector space, which satisfies that

$$((\mathrm{Id} \otimes s) \circ \Delta_M)^2(V) = 0, \tag{8}$$

where Δ_M denotes the comultiplication of M and V is the kernel of s.

For M an exterior coaugmented A–comodule, let $E(M)$ be the exterior algebra generated by the kernel of s.

Lemma 2.1.2 *For M an exterior coaugmented A–comodule, $E(M)$ is a comodule algebra such that the natural inclusion $M \hookrightarrow E(M)$ is a map of A–comodules.*

Proof Let V be the kernel of the section s. As k–algebras,

$$E(M) \cong T(V)/\langle x \otimes x \mid x \in V \rangle,$$

where $T(V)$ is the tensor algebra over V. $T(V)$ has a unique structure as a comodule algebra such that the inclusion $M \cong k \oplus V \hookrightarrow T(V)$ is a map of A–comodules. It suffices to show that, if I denotes the ideal $\langle x \otimes x \mid x \in V \rangle$, then $\Delta_{T(V)}(I) \subset A \otimes I$. Equivalently, one needs to show that $\Delta_{T(V)}(x \otimes x) \in A \otimes I$ for $x \in V$. Indeed, this is a consequence of the commutativity of A and condition (8) of Definition 2.1.1 □

Let $\overline{M}$ be the cokernel of the coaugmentation $k \to M$ and $P(M)$ the polynomial algebra generated by $\overline{M}$.

Lemma 2.1.3 *$P(M)$ is an A–comodule algebra such that the inclusion $\overline{M} \hookrightarrow P(M)$ is a map of A–comodules.*

Proof As k–algebra,

$$P(M) \cong T(\overline{M})/\langle x \otimes y - y \otimes x \mid x, y \in \overline{M} \rangle.$$

As the free algebra on $\overline{M}$, $T(\overline{M})$ has a unique structure as an A–comodule algebra such that $\overline{M} \hookrightarrow P(\overline{M})$ is a map of A–comodules. In order to conclude, it suffices to show that $\Delta_{T(\overline{M})}(x \otimes y - y \otimes x) \in A \otimes \langle x \otimes y - y \otimes x \mid x, y \in \overline{M} \rangle$. This follows, in fact, from the commutativity of A. □

We write E and P for $E(M)$ and $P(M)$, if the underlying M is understood from the context. We introduce a grading on E and P by letting M and $\overline{M}$ have degree 1, respectively. Denote by E_i and $E_{\leq i}$ the subgroup constituting the elements of degree i and of degree not exceeding i of $E(M)$, respectively. Define P_i and $P_{\leq i}$ similarly for P. In the theory of Koszul duality, the polynomial algebra P is commonly referred to as the Koszul dual of the exterior algebra E. We will refer to (P, E) as the pair of Koszul duals associated to the exterior coaugmented A–comodule M. Let us recall the definition of the Koszul complex $(E \otimes P, d)$ associated to the pair (P, E), as follows:

(i) $(E \otimes P)_{-1} = k$.

(ii) $(E \otimes P)_m = E \otimes P_m$ for $m \geq 0$.

(iii) $d\colon k = (E \otimes P)_{-1} \to E = (E \otimes P)_0$ is the unit of E.

(iv) $d\left(\prod_{j=1}^n x_{i_j} \otimes z\right) = \sum_{t=1}^n \prod_{j \neq t} x_{i_j} \otimes p(x_{i_t})z$, where $x_{i_j} \in E_1$, $z \in P_m$ and p denotes the projection $M \to \overline{M}$.

Remark 2.1.4 In other words, $d\colon E_{\leq n} \otimes P_m \to E_{\leq n-1} \otimes P_{m+1}$ is the unique homomorphism making the diagram

$$\begin{array}{ccc} E_{\leq 1}^{\otimes n} \otimes P_{\leq m} & \xrightarrow{\left(\sum_\sigma (\mathrm{Id}^{\otimes(n-1)} \otimes p) \circ \sigma\right) \otimes \mathrm{Id}} & E_{\leq 1}^{\otimes(n-1)} \otimes P_1 \otimes P_m \\ \downarrow{\scriptstyle \mu \otimes \mathrm{Id}} & & \downarrow{\scriptstyle \mu \otimes \mu} \\ E_{\leq n} \otimes P_m & \xrightarrow{\quad d \quad} & E_{\leq n-1} \otimes P_{m+1} \end{array} \tag{9}$$

commute, where in the upper horizontal map the sum is taken over all cyclic permutations on n factors of E_1 in the tensor product $E_1^{\otimes n}$.

Proposition 2.1.5 *The complex $(E \otimes P, d)$ is an exact sequence of A–comodules. Furthermore, $(E \otimes P, d)$ has a structure of a differential graded algebra induced from the algebra structure of E and P.*

Proof Let $x_1, \dots, x_n$ be a basis of E_1. As a cochain complex over k, $(E \otimes P, d)$ is isomorphic to the tensor product of $(E(x_i) \otimes k[y_i], d_i)$, where $y_i = p(x_i)$ for $1 \leq i \leq n$. Here, each $(E(x_i) \otimes k[y_i], d_i)$ is the Koszul complex associated to the pair $(E(x_i), k[y_i])$. It is straightforward to see that the cochain complex $(E(x_i) \otimes k[y_i], d_i)$ is exact. Hence, $(E \otimes P, d)$ is exact by the Künneth theorem. This proves the first part.

Let us check that d is a map of A–comodules. In the diagram (9), the two vertical maps are ones of A–comodules because E and P are A–comodule algebras. In addition, they are surjective. It remains to check that the upper horizontal map is a map of A–comodules. Or, equivalently, each map $E_{\leq 1}^{\otimes n} \xrightarrow{(\mathrm{Id}^{\otimes(n-1)} \otimes p) \circ \sigma} E_{\leq 1}^{\otimes(n-1)} \otimes P_1$ is a map of A–comodules, where σ is a cyclic permutation on n elements. This is true because σ is a map of A–comodules as A is commutative and p is a map of A–comodules by definition. The second part follows.

Finally, it is straightforward from the formula of d in (iv) that d satisfies the Leibniz rule. □

Remark 2.1.6 Let $\mathcal{EC}_A$ be the category whose objects are exterior coaugmented comodules over A and morphisms are maps of comodules which commute with both the coaugmentation and its section. Then, we see that the association $M \mapsto (E(M) \otimes P(M), d)$ is a functor from $\mathcal{EC}_A$ to the category of differential graded comodule algebras.

This proposition allows us to construct a spectral sequence of algebras converging to $\mathrm{Ext}^s_A(k)$; see [32, Theorem A1.3.2] for example.

Proposition 2.1.7 (1) *There is a spectral sequence of algebras*

$$\mathrm{E}_1^{s,t} = \mathrm{Ext}^s_A(k, E \otimes P_t) \Rightarrow \mathrm{Ext}^{s+t}_A(k,k), \tag{10}$$

converging to $\mathrm{Ext}^s_A(k)$, *with* $d_r \colon E_r^{s,t} \to E_r^{s-r+1,t+r}$.

(2) *If N is an A–comodule, then there is a spectral sequence of modules over the previous one, converging to* $\mathrm{Ext}^s_A(N)$,

$$\mathrm{E}_1^{s,t} = \mathrm{Ext}^s_A(k, E \otimes P_t \otimes N) \Rightarrow \mathrm{Ext}^{s+t}_A(k,N).$$

Terminology We will call these spectral sequences the Davis–Mahowald spectral sequences, or DMSSs for short, associated to the almost graded A–module algebra E. The first grading s of the E_n–term is referred to as the cohomological grading or degree and the second grading t is referred to as the Davis–Mahowald grading or degree (or DM grading or degree for short).

With a view to carrying out explicit computations of products in $\mathrm{Ext}^*_A(k)$ and the action of $\mathrm{Ext}^*_A(k)$ on $\mathrm{Ext}^*_A(M)$, we recall a double complex from which the above spectral sequence is derived.

For each $t \geq 0$, let $(C^s(A, E \otimes P_t), d_v)_{s \geq 0}$ be the cobar complex whose cohomology is $\mathrm{Ext}^*_A(E \otimes P_t)$, ie

$$C^s(A, E \otimes P_t) = A^{\otimes s} \otimes E \otimes P_t$$

and $d_v \colon A^{\otimes s} \otimes E \otimes P_t \to A^{\otimes s+1} \otimes E \otimes P_t$ is given by

$$d_v(a_1 \otimes \cdots \otimes a_s \otimes m) = 1 \otimes a_1 \otimes \cdots \otimes a_s \otimes m + \sum_{i=1}^{s} a_1 \otimes \cdots \otimes a_{i-1} \otimes \Delta(a_i) \otimes \cdots \otimes a_s \otimes m \\ + a_1 \otimes \cdots \otimes a_s \otimes \Delta(m),$$

where $a_i \in A$ for $1 \le i \le s$ and $m \in E \otimes P_t$. We will abbreviate $a_1 \otimes \cdots \otimes a_s \otimes m$ by $[a_1|\cdots|a_s|m]$. By an abuse of notation, we will denote by d_v the differentials in the cobar complexes associated to $E \otimes P_t$ for different t. The fact that $d\colon E \otimes P_t \to E \otimes P_{t+1}$ is a map of A–comodules implies that the maps $d_h = \mathrm{Id}^{\otimes s} \otimes d\colon C^s(A, E \otimes P_t) \to C^s(A, E \otimes P_{t+1})$ assemble to give a map of cochain complexes

$$d_h\colon (C^s(A, E \otimes P_t), d_v)_{s \ge 0} \to (C^s(A, E \otimes P_{t+1}), d_v)_{s \ge 0}.$$

Finally, it is easily seen that the maps of cochain complexes assemble to form a double complex $(C^s(A, E \otimes P_t), d_v, d_h)_{s,t \ge 0}$:

$$\begin{array}{ccccccccc}
E & \xrightarrow{d_h} & E \otimes P_1 & \xrightarrow{d_h} & E \otimes P_2 & \xrightarrow{d_h} & E \otimes P_3 & \xrightarrow{d_h} & \cdots \\
\downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
A \otimes E & \xrightarrow{d_h} & A \otimes E \otimes P_1 & \xrightarrow{d_h} & A \otimes E \otimes P_2 & \xrightarrow{d_h} & A \otimes E \otimes P_3 & \xrightarrow{d_h} & \cdots \\
\downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
A^{\otimes 2} \otimes E & \xrightarrow{d_h} & A^{\otimes 2} \otimes E \otimes P_1 & \xrightarrow{d_h} & A^{\otimes 2} \otimes E \otimes P_2 & \xrightarrow{d_h} & A^{\otimes 2} \otimes E \otimes P_3 & \xrightarrow{d_h} & \cdots \\
\downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & \ddots & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We can see that the spectral sequence associated to the horizontal filtration has E_1–term isomorphic to $(A^s \otimes k, d_v)_{s \ge 0}$, which identifies with the cobar complex of the trivial A–comodule k. Thus this spectral sequence degenerates at the E_2–term and the $\mathrm{E}_\infty = \mathrm{E}_2$–term identifies with $\mathrm{Ext}_A^s(k)$. Since there are no possible extension problems, the cohomology of the total complex is isomorphic to $\mathrm{Ext}_A^s(k)$. Now, the spectral sequence associated to the vertical filtration has E_1–term isomorphic to $\mathrm{Ext}_A^s(E \otimes P_t)$. This spectral sequence is exactly the one appearing in Proposition 2.1.7.

Remark 2.1.8 The differential $d_1\colon \mathrm{Ext}_A^0(E \otimes P_t) \to \mathrm{Ext}_A^0(E \otimes P_{t+1})$ is the restriction of the derivation d of the Koszul complex on the A–primitives of $E \otimes P_t$.

2.2 Naturality of the Davis–Mahowald spectral sequence

We notice that the above construction is natural in pairs (A, M), where A is a commutative Hopf algebra and M is an exterior coaugmented left A–comodule. This allows us to compare Davis–Mahowald spectral sequences associated to different pairs (A, M).

Definition 2.2.1 Let (A, M) and (B, N) be such that A and B are commutative Hopf algebras and M and N are objects of $\mathcal{EC}_A$ and $\mathcal{EC}_B$, respectively. A morphism between (A, M) and (B, N) consists of $f_1\colon A \to B$ and $f_2\colon M \to N$, where f_1 is a map of Hopf algebras and f_2 is a morphism in $\mathcal{EC}_B$ with the B–comodule structure on M being induced from f_1.

Proposition 2.2.2 *A morphism between (A, M) and (B, N) induces a map between the associated Davis–Mahowald spectral sequences.*

Proof By Remark 2.1.6, the map $f_2\colon M \to N$ induces a map of cochain complexes of B–comodules $E(M) \otimes P(M) \to E(N) \otimes P(N)$. Together with f_1, one obtains a map of double complexes $(A^{\otimes s} \otimes E(M) \otimes P(M)_t) \to (B^{\otimes s} \otimes E(N) \otimes P(N)_t)$, and hence a map of Davis–Mahowald spectral sequences. □

Remark 2.2.3 Although we have only treated the ungraded situation so far, the construction carries over verbatim to the graded one. More precisely, suppose that A and E are graded algebras. We refer to this grading as the internal degree. We require the structural maps in the A–comodule structure of E to preserve the internal degree. Then we see that the Koszul dual P of E is internally graded and the Koszul complex is a graded cochain complex with respect to the internal degree. It follows that the associated DMSS is trigraded with the third grading associated to the internal degree and the differentials preserve the internal degree.

Let us present examples which are of the main interest in this paper. Recall that the Steenrod algebra $\mathcal{A}$ is generated by the Steenrod squares Sq^i for $i \geq 0$, subject to the Adem relations

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{i=0}^{\lfloor a/2 \rfloor} \binom{b-i-1}{a-2i} \mathrm{Sq}^{a+b-i} \mathrm{Sq}^i$$

for all $a, b > 0$ and $a < 2b$. Let $\mathcal{A}_*$ denote the dual of the Steenrod algebra. In [29], Milnor determines the Hopf algebra structure of $\mathcal{A}_*$. As a graded algebra, $\mathcal{A}_* = \mathbb{F}_2[\xi_i \mid i \geq 1]$, where ξ_i is in degree $|\xi_i| = 2^i - 1$. The coproduct is given by

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_i^{2^{k-i}} \otimes \xi_{k-i},$$

where $\xi_0 = 1$. Let us denote by ζ_i the conjugate of ξ_i. Then

$$\Delta(\zeta_k) = \sum_{i+j=k} \zeta_i \otimes \zeta_j^{2^i}. \tag{11}$$

A Hopf ideal of a Hopf algebra A is an ideal I such that $\Delta(I) \subset I \otimes A + A \otimes I$. If I is a Hopf ideal of A, then A/I inherits a structure of Hopf algebra from A such that the natural projection $A \to A/I$ is a map of Hopf algebras.

Example 2.2.4 Let $\mathcal{A}(n)_*$ be the quotient of $\mathcal{A}_*$ by the Hopf ideal I_n generated by $(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \dots, \zeta_{n+1}^2, \zeta_{n+2}, \dots)$. As an algebra,

$$\mathcal{A}(n)_* = \mathbb{F}_2[\zeta_1, \zeta_2, \dots, \zeta_{n+1}]/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \dots, \zeta_{n+1}^2).$$

It is dual to the subalgebra $\mathcal{A}(n) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^n} \rangle$ of the Steenrod algebra A. The canonical projection $\pi\colon \mathcal{A}(n)_* \to \mathcal{A}(n-1)_*$ induced by the inclusion $I_n \subset I_{n-1}$ of Hopf ideals is a map of Hopf algebras, and hence induces on $\mathcal{A}(n)_*$ a structure of a right $\mathcal{A}(n-1)_*$–comodule algebra,

$$(\mathrm{id} \otimes \pi)\Delta\colon \mathcal{A}(n)_* \to \mathcal{A}(n)_* \otimes \mathcal{A}(n)_* \to \mathcal{A}(n)_* \otimes \mathcal{A}(n-1)_*.$$

An easy computation shows that the group of primitives $\mathcal{A}(n)_* \,\square_{\mathcal{A}(n-1)_*} \mathbb{F}_2$ of this coaction is given by

$$\mathcal{A}(n)_* \,\square_{\mathcal{A}(n-1)_*} \mathbb{F}_2 = E(\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_{n+1}),$$

which is abstractly isomorphic to $E_n = E(x_1, \dots, x_{n+1})$, where x_i stands for $\zeta_i^{2^{n+1-i}}$. Here and elsewhere in this paper, $E(X)$ denotes the exterior algebra on the k–vector space spanned by the set X. We see that the algebra $E(x_1, x_2, \dots, x_{n+1})$ inherits a left $\mathcal{A}(n)_*$–comodule algebra structure from $\mathcal{A}(n)_*$, namely,

$$\Delta(x_k) = \sum_{i=0}^{k} \zeta_i^{2^{n+1-k}} \otimes x_{k-i} \quad \text{for } 1 \le k \le n+1,$$

where $x_0 = 1$ by convention. In particular, the subcomodule

$$M_n = \mathbb{F}_2\{x_0\} \oplus \mathbb{F}_2\{x_1, \dots, x_{n+1}\}$$

is an exterior coaugmented $\mathcal{A}(n)_*$–comodule, because $(\zeta_k^{2^{n+1-k}})^2 = 0 \in \mathcal{A}(n)_*$.

Example 2.2.5 Let $B(n)_*$ be the quotient of $\mathcal{A}_*$ by the Hopf ideal J_n generated by $(\zeta_1^{2^n}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \dots, \zeta_{n+1}^2, \zeta_{n+2}, \dots)$, so that

$$B(n)_* = \mathbb{F}_2[\zeta_1, \zeta_2, \dots, \zeta_{n+1}]/(\zeta_1^{2^n}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \dots, \zeta_{n+1}^2).$$

Similarly to Example 2.2.4, the projection $B(n)_* \to \mathcal{A}(n-1)_*$ induced by the inclusion of Hopf ideals $J_n \subset I_{n-1}$ defines a structure of a right $\mathcal{A}(-1)_*$–comodule algebra on $B(n)_*$. A calculation shows that

$$B(n)_* \,\square_{\mathcal{A}(n-1)_*} \mathbb{F}_2 = E(\zeta_2^{2^{n-1}}, \zeta_3^{2^{n-2}}, \dots, \zeta_{n+1}),$$

which is abstractly isomorphic to $F_n := E(x_2, \dots, x_{n+1})$. The notation is chosen to be coherent with that of Example 2.2.4. We see that F_n inherits a structure of a left $B(n)_*$–comodule algebra from that of $B(n)_*$, namely

$$\Delta(x_k) = \sum_{i=0, i\neq 1}^{k} \zeta_i^{2^{n+1-k}} \otimes x_{k-i} \quad \text{for } 2 \le k \le n+1,$$

where $x_0 = 1$. Since $(\zeta_k^{2^{n+1-k}})^2 = 0 \in B(n)_*$,

$$N_n = \mathbb{F}_2\{x_0\} \oplus \mathbb{F}_2\{x_2, \dots, x_{n+1}\}$$

is an exterior coaugmented $B(n)_*$–comodule.

Example 2.2.6 Recall that M_n is an exterior coaugmented $\mathcal{A}(n)_*$–comodule. Let $R(n)$ denote $P(M_n)$, the Koszul dual of E_n. In particular, it follows from Proposition 2.1.7 that, for any graded left $\mathcal{A}(n)_*$–comodule V, the DMSS converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(n)_*}(\mathbb{F}_2, V)$ has E_1–term isomorphic to

$$\mathrm{E}_1^{s,t,\sigma} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(E_n \otimes R(n)_\sigma \otimes V),$$

where s is the cohomological grading, t is the internal grading and σ is the Davis–Mahowald grading, which, recall, arises from the homogenous degree of the graded algebra $R(n)$. The change-of-rings isomorphism tells us that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(E_n \otimes R(n)_\sigma \otimes V) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(R(n)_\sigma \otimes V);$$

see [31, Appendix A1.3.13] for the change-of-rings isomorphism. This means that the problem of computing $\mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(-)$ can be reduced to two steps: first computing $\mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(-)$, then studying the corresponding Davis–Mahowald spectral sequence. We will demonstrate the efficiency of this method by carrying out explicit computations in the case $n = 2$.

Example 2.2.7 Similarly, for N_n, the exterior coaugmented $B(n)_*$–comodule, let $S(n)$ denote the Koszul dual of F_n. For any graded left $B(n)_*$–comodule V, the DMSS for $\mathrm{Ext}^{s+\sigma,t}_{B(n)_*}(V)$ has E_1–term isomorphic to

$$\mathrm{Ext}^{s,t}_{B(n)_*}(F_n \otimes S(n)_\sigma \otimes V) \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(n-1)_*}(S(n)_\sigma \otimes V),$$

again by the change-of-rings isomorphism.

Comparison of DMSS There is a morphism between $(\mathcal{A}(n)_*, M_n)$ and $(B(n)_*, N_n)$ given by the two projections

$$\begin{aligned} \mathcal{A}(n)_* &\to B(n)_*, & \zeta_i &\mapsto \zeta_i, \\ M_n &\to N_n, & x_1 &\mapsto 0, \quad x_i \mapsto x_i \quad \text{for } i = 0 \text{ and } i \geq 2. \end{aligned}$$

This induces a map of spectral sequences, for an $\mathcal{A}(n)_*$–comodule V,

$$\begin{array}{ccc} \mathrm{Ext}^{s,t}_{\mathcal{A}(n)_*}(E_n \otimes R(n)_\sigma \otimes V) & \longrightarrow & \mathrm{Ext}^{s,t}_{B(n)_*}(F_n \otimes S(n)_\sigma \otimes V) \\ \Downarrow & & \Downarrow \\ \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(n)_*}(V) & \longrightarrow & \mathrm{Ext}^{s+\sigma,t}_{B(n)_*}(V) \end{array}$$

This comparison allows us to transfer some computations in the former SS to the latter, which are simpler because all modules involved in the latter are smaller. This observation will be made concrete in Section 3.

3 The Davis–Mahowald spectral sequence for the $\mathcal{A}(2)_*$–comodule $\mathrm{H}_*(A_1)$

The goal of this section is to describe the structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ as a module over $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$. To achieve a part of this goal, we will study the DMSS

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_2 \otimes R(2)_\sigma \otimes A_1) \Rightarrow \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$$

as a spectral sequence of modules over the spectral sequence of algebras

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_2 \otimes R(2)_\sigma) \Rightarrow \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(2)_*}(\mathbb{F}_2).$$

We obtain then the structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ as a graded abelian group and a partial action of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ on it. However, there is an important action of an element of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ on some elements of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ that cannot be seen at the E_1–term of the DMSS. One way of understanding these exotic products is to carry out computations at the level of double complexes: find representatives of the cohomological classes in question in the double complexes from which the DMSS is derived and carry out products at that level. It turns out that a brute-force attack is messy. Instead, computations are simplified drastically by comparing the DMSS associated to $(\mathcal{A}(2)_*, M_2)$ to that of $(B(2)_*, N_2)$:

$$\begin{array}{ccc} \mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_n \otimes R(2)_\sigma \otimes \mathrm{H}_*(A_1)) & \longrightarrow & \mathrm{Ext}^{s,t}_{B(2)_*}\big(F_n \otimes S(2)_\sigma \otimes \mathrm{H}_*(A(1))\big) \\ \Downarrow & & \Downarrow \\ \mathrm{Ext}^{s+\sigma,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) & \longrightarrow & \mathrm{Ext}^{s+\sigma,t}_{B(2)_*}(\mathrm{H}_*(A_1)) \end{array}$$

3.1 Recollections on the Davis–Mahowald spectral sequence for the $\mathcal{A}(2)_*$–comodule $\mathbb{F}_2$

To fix notation, we recollect some information relevant for our purposes. This material was originally treated in [14] and reviewed in unpublished course notes of Rognes [35]. As we will specialize to the case $n = 2$, we will simplify the notation by writing R, R_σ, S and S_σ for $R(2)$, $R(2)_\sigma$, $S(2)$ and $S(2)_\sigma$ from Examples 2.2.4 and 2.2.5, respectively.

Recall that R is a homogenous graded polynomial algebra on three generators, say y_1, y_2 and y_3, and R_σ is its subspace of homogeneous elements of degree σ for $\sigma \geq 0$. Let us first explicitly give the coaction of $\mathcal{A}(2)_*$ on $R = \mathbb{F}_2[y_1, y_2, y_3]$ with $|y_1| = 4$, $|y_2| = 6$ and $|y_3| = 7$. From Example 2.2.6, we have

$$\Delta(y_1) = 1 \otimes y_1, \quad \Delta(y_2) = \xi_1^2 \otimes y_1 + 1 \otimes y_2, \quad \Delta(y_3) = \zeta_2 \otimes y_1 + \xi_1 \otimes y_2 + 1 \otimes y_3.$$

By the change-of-rings theorem, the E_1–term of the DMSS for $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ is isomorphic to $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}\big(\bigoplus_{\sigma \geq 0} R_\sigma\big)$. The coaction of $\mathcal{A}(1)_*$ on R_1 is induced from that of $\mathcal{A}(2)_*$ and hence is given by

$$\Delta(y_1) = 1 \otimes y_1, \quad \Delta(y_2) = \xi_1^2 \otimes y_1 + 1 \otimes y_2, \quad \Delta(y_3) = \zeta_2 \otimes y_1 + \xi_1 \otimes y_2 + 1 \otimes y_3.$$

In particular, y_1, y_2^2 and y_3^4 are $\mathcal{A}(1)_*$–primitives of R. Let R'_σ denote the $\mathcal{A}(1)_*$–subcomodule $\mathbb{F}_2\{y_1^i y_2^j y_3^k \mid i + j + k = \sigma, k \leq 3\}$ of R_σ. In particular, $R_i = R'_i$ for $1 \leq i \leq 3$.

Lemma 3.1.1 *As an $\mathcal{A}(1)_*$–comodule, R_σ can be decomposed as*

$$R_\sigma \cong \bigoplus_{\substack{i \equiv \sigma \ (\mathrm{mod}\ 4) \\ i \leq \sigma}} R'_i \otimes \mathbb{F}_2\{y_3^{\sigma - i}\}.$$

Therefore,

$$\bigoplus_{\sigma \geq 0} R_\sigma = \Big(\bigoplus_{\sigma \geq 0} R'_\sigma\Big) \otimes \mathbb{F}_2[y_3^4].$$

Proof If one views $\mathbb{F}_2\{y_3^{\sigma-i}\}$ as a subvector space of $R_{\sigma-i}$, then the product of R produces an isomorphism of vector spaces

$$\bigoplus_{\substack{i\equiv\sigma \pmod 4\\ i\le\sigma}} R'_i \otimes \mathbb{F}_2\{y_3^{\sigma-i}\} \xrightarrow{\cong} R_\sigma.$$

Since y_3^4 is an $\mathcal{A}(1)_*$–primitive of R_σ, this map is also a map of $\mathcal{A}(1)_*$–comodules. The lemma follows. □

Let us denote $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R'_\sigma)$ by G_σ, so that

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R) \cong \Big(\bigoplus_{\sigma\ge 0} G_\sigma\Big) \otimes \mathbb{F}_2[v_2^4],$$

where $v_2^4 \in \mathrm{Ext}^{0,24}_{\mathcal{A}(1)_*}(R_4)$ is represented by $y_3^4 \in R_4$. Determining the full multiplicative structure of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$ is quite involved. Instead, we will work modulo (v_2^4). This will suffice for us to obtain a set of algebra generators of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$. More precisely, since the product $R'_\sigma \otimes R'_\tau \to R_{\sigma+\tau}$ factors through $R'_{\sigma+\tau} \oplus (R_{\sigma+\tau-4} \otimes \mathbb{F}_2\{y_3^4\})$, we obtain a map

$$G_\sigma \otimes G_\tau \to G_{\sigma+\tau} \oplus (G_{\sigma+\tau-4} \otimes \mathbb{F}_2\{v_2^4\}).$$

We will analyze the map $G_\sigma \otimes G_\tau \to G_{\sigma+\tau}$ which is the composite

$$G_\sigma \otimes G_\tau \to G_{\sigma+\tau} \oplus (G_{\sigma+\tau-4} \otimes \mathbb{F}_2\{v_2^4\}) \to G_{\sigma+\tau},$$

where the second map is the projection on the first factor.

In what follows, we compute G_i for $i \ge 0$ as modules over G_0. For this, we decompose R'_i into smaller pieces, compute the Ext groups over $\mathcal{A}(1)_*$ of these pieces, then determine G_i via long exact sequences. Next, we study the pairings

$$G_\sigma \otimes G_\tau \to G_{\sigma+\tau},$$

which allows us to determine a set of algebra generators of the E_1–term. Finally, we compute d_1–differentials on this set of algebra generators. We do not intend to describe completely $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ but only a subalgebra in which we are interested.

Since y_1 is a primitive, multiplication by y_1 induces injections of $\mathcal{A}(1)_*$–comodules

$$\Sigma^4 R'_\sigma \to R'_{\sigma+1}.$$

Lemma 3.1.2 *There are short exact sequences of $\mathcal{A}(1)_*$–comodules*

(a) $0 \to \mathrm{H}_*(\Sigma^{12} C_\eta) \to R_2 \to \Sigma^8(\mathcal{A}(1)_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2) \to 0$, *where* $\eta\colon S^1 \to S^0$ *is the Hopf map and the map* $\mathrm{H}_*(\Sigma^{12} C_\eta) \to R_2$ *sends the generators of* $\mathrm{H}_{12}(\Sigma^{12} C_\eta)$ *and* $\mathrm{H}_{14}(\Sigma^{12} C_\eta)$ *to* y_2^2 *and* y_3^2, *respectively*;

(b) $0 \to \Sigma^4 R_1 \to R_2 \to \Sigma^{12} V_3 \to 0$, *where* $V_3 = \mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2)$.

Proof For part (a), the map $\Sigma^{12}\mathrm{H}_*(C_\eta) \to R_2$ described in the statement is a map of $\mathcal{A}(1)_*$–comodules. Its quotient is isomorphic to $\mathbb{F}_2\{y_1^2, y_1 y_2, y_1 y_3, y_2 y_3\}$, with the $\mathcal{A}(1)_*$–comodule structure given by

$$\begin{aligned}
\Delta(y_2 y_3) &= 1 \otimes y_2 y_3 + \xi_1^2 \otimes y_1 y_3 + \xi_2 \otimes y_1 y_2 + \zeta_2 \xi_1^2 \otimes y_1^2, \\
\Delta(y_1 y_3) &= 1 \otimes y_1 y_3 + \xi_1 \otimes y_1 y_2 + \zeta_2 \otimes y_1^2, \\
\Delta(y_1 y_2) &= 1 \otimes y_1 y_2 + \xi_1^2 \otimes y_1^2, \\
\Delta(y_1^2) &= 1 \otimes y_1^2.
\end{aligned}$$

We can check that this module is isomorphic to $\Sigma^8(\mathcal{A}(1)_* \square_{\mathcal{A}(0)_*} \mathbb{F}_2)$ as $\mathcal{A}(1)_*$–comodules.

For part (b), the quotient of R_2 by $\Sigma^4 R_1$ is isomorphic to $\mathbb{F}_2\{y_2^2, y_2 y_3, y_3^2\}$ with $\mathcal{A}(1)_*$–comodule structure given by

$$\Delta(y_2^2) = 1 \otimes y_2^2, \quad \Delta(y_2 y_3) = \xi_1 \otimes y_2^2 + 1 \otimes y_2 y_3, \quad \Delta(y_3^2) = \xi_1^2 \otimes y_2^2 + 1 \otimes y_3^2.$$

One can check that this quotient is isomorphic to $\Sigma^{12} V_3$. □

Lemma 3.1.3 *For every* $\sigma \geq 3$, *there is a short exact sequence of* $\mathcal{A}(1)_*$–*comodules*

$$0 \to \Sigma^4 R'_{\sigma-1} \xrightarrow{\times y_1} R'_\sigma \to \Sigma^{6\sigma} V_4 \to 0,$$

where V_4 *is* $\mathrm{H}_*(V(0) \wedge C_\eta)$.

Remark 3.1.4 The spectrum $V(0) \wedge C_\eta$ is homotopy equivalent to Y, introduced on page 3858 (see Section 3.2 for a presentation of $\mathrm{H}^*(Y)$).

Proof The quotient of R'_σ by $\Sigma^4 R'_{\sigma-1}$ is isomorphic to

$$\mathbb{F}_2\{y_2^\sigma, y_2^{\sigma-1} y_3, y_2^{\sigma-2} y_3^2, y_2^{\sigma-3} y_3^3\},$$

with $\mathcal{A}(1)_*$–comodule structure given by

$$\begin{aligned}
\Delta(y_2^{\sigma}) &= 1 \otimes y_2^{\sigma},\\
\Delta(y_2^{\sigma-1}y_3) &= \xi_1 \otimes y_2^{\sigma} + 1 \otimes y_2^{\sigma-1}y_3,\\
\Delta(y_2^{\sigma-2}y_3^2) &= \xi_1^2 \otimes y_2^{\sigma} + 1 \otimes y_2^{\sigma-2}y_3^2,\\
\Delta(y_2^{\sigma-3}y_3^3) &= \xi_1^3 \otimes y_2^{\sigma} + \xi_1^2 \otimes y_2^{\sigma-1}y_3 + \xi_1 \otimes y_2^{\sigma-2}y_3^2 + 1 \otimes y_2^{\sigma-3}y_3^3.
\end{aligned}$$

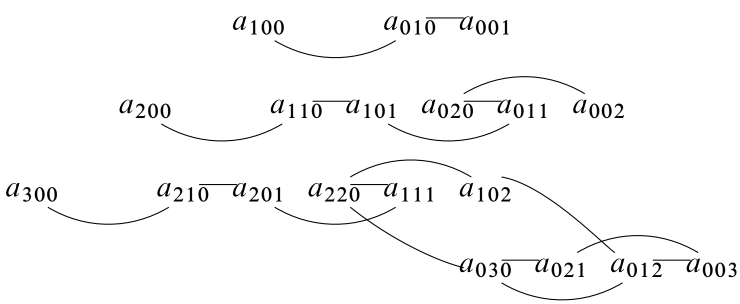
It can be easily seen that this quotient is isomorphic to $\Sigma^{6\sigma} V_4$. □

Remark 3.1.5 We describe above the comodule structure of R'_σ for the purpose of making explicit calculations of the Ext groups. For the sake of visualization, we present here the module structure of the duals $R_1^\vee$, $R_2^\vee$ and $R_3^\vee$ of R_1, R_2 and R_3, respectively, by their corresponding cell diagram. Let $\{a_{i,j,k} \mid i+j+k=\sigma\}$ be the dual basis to $\{y_1^i y_2^j y_3^k \mid i+j+k=\sigma\}$. In the following cell diagrams, the straight lines represent Sq^1 and the curved lines represent Sq^2:

$R_1^\vee$:

$R_2^\vee$:

$R_3^\vee$:



In the cell diagram of $R_3^\vee$, two curved lines from a_{220} to a_{102} and a_{030} means that $\mathrm{Sq}^2(a_{220}) = a_{102} + a_{030}$.

Remark 3.1.6 $R_1^\vee$ is the cohomology of the dual question mark complex, a key player in the $K(1)$–local homotopy theory; namely, its $K(1)$–localization represents the exotic element of the $K(1)$–local Picard group; see also Proposition 3.1.13 for its ko–homology, where ko is the connective real K–theory.

Next we describe the Ext groups of some $\mathcal{A}(1)_*$–comodules as basic steps towards computing G_σ. These calculations are elementary and classical.

Proposition 3.1.7 *There are classes* $h_0 \in \mathrm{Ext}^{1,1}$, $h_1 \in \mathrm{Ext}^{1,2}$, $v \in \mathrm{Ext}^{3,7}$ *and* $v_1^4 \in \mathrm{Ext}^{4,12}$ *such that there is an isomorphism of algebras*

$$G_0 := \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{F}_2) \cong \mathbb{F}_2[h_0, h_1, v, v_1^4]/(h_1^3, h_0h_1, h_1v, v^2 - h_0^2 v_1^4).$$

See for example [32, Theorem 3.1.25], and Figure 1.

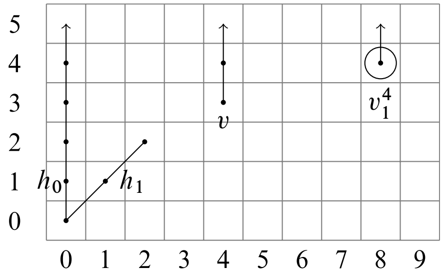


Figure 1: $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2,\mathbb{F}_2)$ in the range $0\le t-s\le 8$.

Lemma 3.1.8 *As a module over* $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathbb{F}_2)$, *we have* (see Figure 2):

(1) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0)))$ *is generated by* $1\in\mathrm{Ext}^{0,0}$ *and* $v_1\in\mathrm{Ext}^{1,3}$ *with the relations* $h_0 1 = v1 = vv_1 = 0$ *and* $h_1^2.1 = h_0v_1$.

(2) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(C_\eta))$ *is generated by* $1\in\mathrm{Ext}^{0,0}$, $v_1\in\mathrm{Ext}^{1,3}$, $v_1^2\in\mathrm{Ext}^{2,6}$ *and* $v_1^3\in\mathrm{Ext}^{3,9}$ *with* $h_1a=0$ *for* $a\in\{1,v_1,v_1^2,v_1^3\}$, $v1=h_0v_1^2$ *and* $vv_1=h_0v_1^3$.

(3) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0\cup_2 e^1\cup_\eta e^2))$ *is generated by* $1\in\mathrm{Ext}^{0,0}$, $v_1\in\mathrm{Ext}^{1,3}$, $a^1\in\mathrm{Ext}^{1,3}$, $v_1^2\in\mathrm{Ext}^{2,6}$ *and* $v_1^3\in\mathrm{Ext}^{3,9}$ *with* $h_0 1 = h_1 1 = h_1v_1 = h_0a^1 = va^1 = h_1v_1^2 = vv_1^2 = h_1v_1^3 = vv_1^3 = 0$ *and* $h_0v_1^2 = h_1^2a^1$.

(4) $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ *is generated by* $\{1,v_1,v_1^2,v_1^3\}$ *with* $h_0a=h_1a=va=0$ *for* $a\in\{1,v_1,v_1^2,v_1^3\}$.

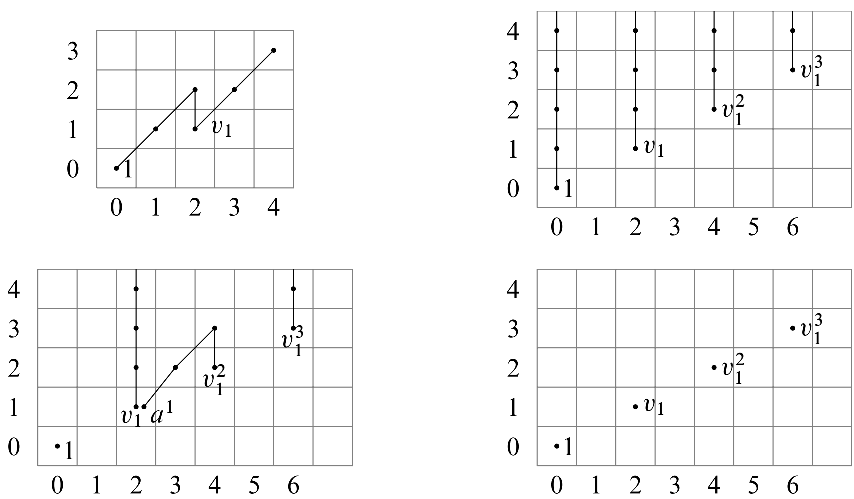


Figure 2: Clockwise from top left: $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0)))$ in the range $0\le t-s\le 4$ and $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(C_\eta))$, $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0\cup_2 e^1\cup_\eta e^2))$ and $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ in the range $0\le t-s\le 6$.

See [32, Theorem 3.1.27] for (1) and (4). The calculations for (2) and (3) are also elementary, so we omit the detail.

Remark 3.1.9 We use the same notation 1, v_1, v_1^2 and v_1^3 to denote certain generators of the above groups. This is justified by the fact that these generators have close relationships, which are described in the next lemma. The context will help avoid confusion.

Remark 3.1.10 The above Ext calculations also give us the homotopy groups of familiar spectra in $A(1)$–homotopy theory, namely those of ko (Proposition 3.1.7), $ko \wedge V(0)$ (Lemma 3.1.8(1)) and $ko \wedge C_\eta \simeq ku$ (Lemma 3.1.8(2)).

Consider cell inclusions $V(0) \to Y$ and $S^0 \cup_2 e^1 \cup_\eta e^2 \to Y$. The induced homomorphisms in Ext over $\mathcal{A}(1)_*$ are described as follows:

Lemma 3.1.11 (i) *The homomorphism* $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0))) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ *sends the classes* 1 *and* v_1 *to the nontrivial classes of the same name.*

(ii) *The homomorphism* $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ *sends the classes* 1, v_1, v_1^2 *and* v_1^3 *to the nontrivial classes of the same name.*

Proof For part (i), consider the short exact sequence of $\mathcal{A}(1)_*$–comodules

$$0 \to \mathrm{H}_*(V(0)) \to \mathrm{H}_*(Y) \to \mathrm{H}_*(\Sigma^2 V(0)) \to 0.$$

For degree reasons, the classes 1 and v_1 of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0)))$ do not belong to the image of the connecting homomorphism

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(\Sigma^2 V(0))) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(V(0))).$$

Therefore, they are sent to nontrivial classes of the same name in $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$. For part (ii), consider the short exact sequence of $\mathcal{A}(1)_*$–comodules

$$0 \to \mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2) \to \mathrm{H}_*(Y) \to \Sigma^3 \mathbb{F}_2 \to 0$$

and the resulting long exact sequence

$$\mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(\Sigma^3 \mathbb{F}_2)) \xrightarrow{\partial} \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2)) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y)).$$

For degree reasons, the classes 1, v_1^2 and v_1^3 of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(S^0 \cup_2 e^1 \cup_\eta e^2))$ are not in the image of the connecting homomorphism, and thus are sent to 1, v_1^2 and v_1^3 in

$\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$, respectively. Next, for degree reasons, the classes $h_0 v_1$ and $h_1 a^1$ are sent to $0 \in \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$. The only way for this to happen is that the connecting homomorphism sends $\Sigma^3 1 \in \mathrm{Ext}^{0,3}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \mathrm{H}_*(\Sigma^3\mathbb{F}_2))$ to the sum $v_1 + a^1$. It follows that v_1 is not in the image of the connecting homomorphism, and therefore is sent to $v_1 \in \mathrm{Ext}^{1,3}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ □

Lemma 3.1.12 $\mathrm{H}_*(Y)$ *has a structure of an* $\mathcal{A}(1)_*$*–comodule algebra. The resulting structure of an algebra on* $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ *is that of a polynomial algebra on the variable* v_1*.*

Proof It is not hard to see that $\mathrm{H}_*(Y)$ is isomorphic to $\mathcal{A}(1)_* \,\Box_{E(1)_*}\, \mathbb{F}_2$ as $\mathcal{A}(1)_*$–comodules, where $E(1)_*$ is the Hopf quotient of $\mathcal{A}(1)_*$ by the Hopf ideal (ζ_1), ie $E(1)_* \cong \mathbb{F}_2[\zeta_2]/(\zeta_2^2)$. In particular, $\mathrm{H}_*(Y)$ has the structure of an $\mathcal{A}(1)_*$–comodule algebra. As a consequence, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$ is an algebra and is furthermore isomorphic to $\mathrm{Ext}^{*,*}_{E(1)_*}(\mathbb{F}_2)$ by the change-of-rings isomorphism. It is well known that the latter is a polynomial algebra on one variable. □

We now compute $G_\sigma := \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R'_\sigma)$, where, as a reminder, R'_σ is a subcomodule of R_σ, the subcomodule of homogenous elements of degree σ of $R = \mathbb{F}_2[y_1, y_2, y_3]$. We denote by $\alpha_{s,t,\sigma}$ the nontrivial class of $\mathrm{Ext}^{s,s+t}_{\mathcal{A}(1)_*}(R'_\sigma)$ whenever there is a unique such one.

Proposition 3.1.13 *As a module over* G_0*,* $G_1 = \mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_1)$ *is generated by* $\alpha_{0,4,1} \in \mathrm{Ext}^{0,4}_{\mathcal{A}(1)_*}(R_1)$ *and* $\alpha_{1,8,1} \in \mathrm{Ext}^{1,9}_{\mathcal{A}(1)_*}(R_1)$ *with the relations* $h_1\alpha_{0,4,1} = 0$ *and* $v\alpha_{0,4,1} = h_0^2\alpha_{1,8,1}$*.*

Proof Consider the short exact sequence of $\mathcal{A}(1)_*$–comodules

$$0 \to \Sigma^4\mathbb{F}_2 \to R_1 \to \Sigma^6\mathrm{H}_*(V(0)) \to 0.$$

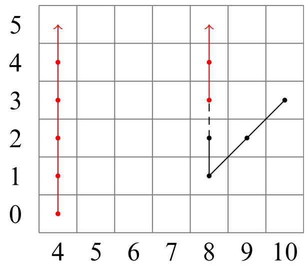
The connecting homomorphism

$$\partial\colon \mathrm{Ext}^{s,t-6}_{\mathcal{A}(1)_*}(V(0)) \to \mathrm{Ext}^{s+1,t-4}_{\mathcal{A}(1)_*}(\mathbb{F}_2)$$

of the resulting long exact sequence sends 1 to h_1 and v_1 to 0. The latter follows for degree reasons and the former from the map of short exact sequences of $\mathcal{A}(1)_*$–comodules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma^4\mathbb{F}_2 & \longrightarrow & R_1 & \longrightarrow & \Sigma^6\mathrm{H}_*(V(0)) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^4\mathbb{F}_2 & \longrightarrow & \mathrm{H}_*(\Sigma^4 C_\eta) & \longrightarrow & \Sigma^6\mathbb{F}_2 & \longrightarrow & 0 \end{array}$$

and the naturality of the connecting homomorphism. It follows that G_1 is v_1^4–periodic on the following generators:



The red part is the contribution of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\Sigma^4\mathbb{F}_2)$ and the black part the contribution of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(\Sigma^6\mathrm{H}_*(V(0)))$.

What remains to be established is the multiplication by h_0 on the generator of bidegree $(s,t-s)=(2,8)$. This is done by a similar consideration of the connecting homomorphism associated to the short exact sequence of $\mathcal{A}(1)_*$–comodules

$$0 \to \Sigma^4 C_\eta \to R_1 \to \Sigma^7\mathbb{F}_2 \to 0. \qquad \square$$

Proposition 3.1.14 *As a module over* G_0, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_2)=G_2$ *is generated by* $\alpha_{s,t,2}\in \mathrm{Ext}^{s,s+t}$, *where* $(s,t)\in\{(0,8),(0,12),(1,14),(2,16),(3,18)\}$ *with*

$$h_1\alpha_{s,t,2}=0, \quad v\alpha_{0,8,2}=h_0^3\alpha_{0,12,2},$$

$$v\alpha_{0,12,2}=h_0\alpha_{2,16,2}, \quad v\alpha_{1,14,2}=h_0\alpha_{3,18,2}, \quad v_1^4\alpha_{0,8,2}=h_0^2\alpha_{2,16,2}.$$

Proof The short exact sequence in Lemma 3.1.2(a) gives rise to the long exact sequence

$$\to \mathrm{Ext}^{s,t-12}_{\mathcal{A}(1)_*}(\mathrm{H}^*(C_\eta)) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_2) \to \mathrm{Ext}^{s,t-8}_{\mathcal{A}(0)_*}(\mathbb{F}_2) \to \mathrm{Ext}^{s+1,t-12}_{\mathcal{A}(1)_*}(\mathrm{H}^*(C_\eta)) \to .$$

Combining that $\mathrm{Ext}^{s,t}_{\mathcal{A}(0)_*}(\mathbb{F}_2)\cong\mathbb{F}_2[h_0]$ and the description of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}^*(C_\eta))$, we see that the connecting homomorphism is trivial for degree reasons; see Figure 3.

What remains is to establish the v_1^4–multiplication on the class $\alpha_{0,8,2}$ of bidegree $(0,8)$. Consider the long exact sequence associated to the short exact sequence in Lemma 3.1.2(b),

$$(12) \qquad \to \mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\Sigma^{12}V_3) \xrightarrow{\partial} \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_1) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_2) \to .$$

One can check that the class $\Sigma^4\alpha_{0,4,1}\in\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_1)$ is not in the image of ∂, and so is sent to $\alpha_{0,8,2}\in\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_2)$. For degree reasons, $v_1^4\Sigma^4\alpha_{0,4,1}$ is not in the image of ∂; thus, $v_1^4\alpha_{0,8,2}$ is nontrivial in G_2. $\square$

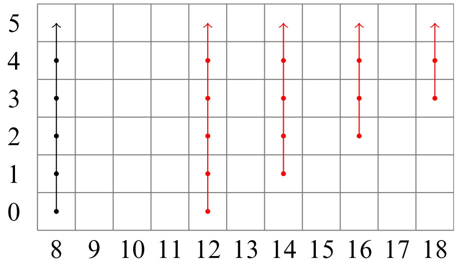


Figure 3: G_2. The red part is the contribution of $\mathrm{Ext}^{s,t}_{\mathcal{A}(0)_*}(\mathbb{F}_2, \mathbb{F}_2)$ and the black one of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(C_\eta))$.

Remark 3.1.15 We can make a complete calculation of the connecting homomorphism of (12), which results in the chart in Figure 4.

Lemma 3.1.16 *As a module over* G_0, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_3) = G_3$ *is generated by the elements* $\alpha_{s,t,3}$ *of* $\mathrm{Ext}^{s,s+t}$, *where* $(s,t) \in \{(0,12), (0,16), (0,18), (1,20), (2,22), (3,24)\}$, *with* $h_1\alpha_{s,t,3} = 0$, $v\alpha_{0,12,3} = h_0^3\alpha_{0,16,3}$, $v\alpha_{0,16,3} = h_0^2\alpha_{1,20,3}$, $v\alpha_{0,18,3} = h_0\alpha_{2,22,3}$, $v\alpha_{1,20,3} = h_0\alpha_{3,24,3}$, $v_1^4\alpha_{0,12,3} = h_0^3\alpha_{1,20,3}$ *and* $v_1^4\alpha_{0,16,3} = h_0\alpha_{3,24,3}$.

Proof The short exact sequence in Lemma 3.1.3 gives the long exact sequence

$$\to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_3) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{18} V_4) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_2) \to .$$

For degree reasons, the connecting homomorphism is trivial; hence, we obtain the additive structure of G_3 as in Figure 5. We need to establish the nontrivial h_0–multiplication on the generators $\{\alpha_{s,18+2s,3} \mid s \geq 0\}$. Taking the v_1^4–periodicity into account, we reduce to showing this property for the generators of

$$\alpha_{0,18,3}, \quad \alpha_{1,20,3}, \quad \alpha_{2,22,3}, \quad \alpha_{3,24,3}.$$

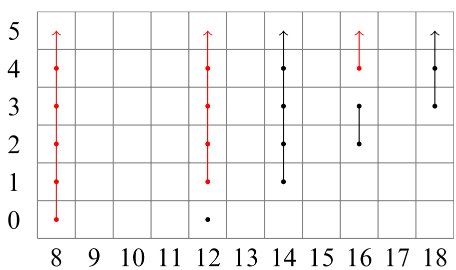


Figure 4: G_2. The red part is the contribution of G_1 and the black one of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(V_3)$.

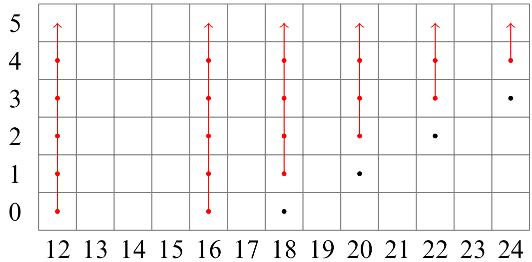


Figure 5: G_3. The red part is the contribution of G_2 and the black one of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(V_4)$.

For this, we can check that there are the short exact sequences

$$0 \to \Sigma^{18}\mathrm{H}_*(C_\eta) \to R_3 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta) \to 0$$

and

$$0 \to \Sigma^4 R_2 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta) \to \Sigma^{19}\mathrm{H}_*(C_\eta) \to 0,$$

where, as a $\mathcal{A}(1)_*$–subcomodule of R_3, $\Sigma^{18}\mathrm{H}_*(C_\eta)$ is equal to $\mathbb{F}_2\{y_1y_3^2 + y_2^3, y_2y_3^2\}$ and the map $\Sigma^4 R_2 \to R_3/\Sigma^{18}C_\eta$ is the composite $\Sigma^4 R_2 \xrightarrow{\times y_1} R_3 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta)$.

As a consequence, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$ sits in a long exact sequence

$$\to \mathrm{Ext}^{s-1,t}_{\mathcal{A}(1)_*}(\Sigma^{19}\mathrm{H}_*(C_\eta)) \xrightarrow{\partial} \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R_2) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta)) \to .$$

Since ∂ is G_0–linear, one only needs to compute ∂ on the two generators of

$$\mathrm{Ext}^{0,19}_{\mathcal{A}(1)_*}(\Sigma^{19}\mathrm{H}_*(C_\eta)) \quad \text{and} \quad \mathrm{Ext}^{1,21}_{\mathcal{A}(1)_*}(\mathbb{F}_2, \Sigma^{19}\mathrm{H}_*(C_\eta)).$$

Direct computations show that ∂ act nontrivially on these classes. It follows that ∂ is a monomorphism and so $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$ is v_1–free on the generators depicted in Figure 6.

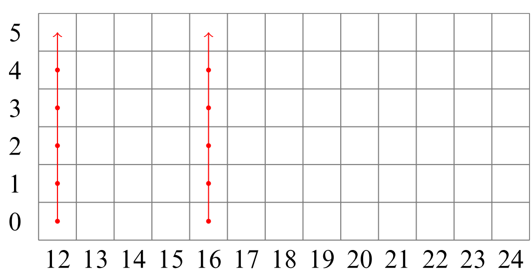


Figure 6: $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$.

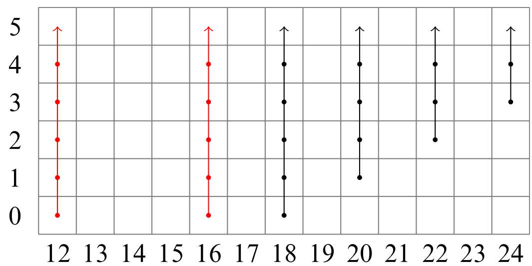


Figure 7: G_3. The red part is the contribution of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{R}_3/\Sigma^{18}\mathrm{H}_*(C_\eta))$ and the black one of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{18}\mathrm{H}_*(C_\eta))$.

It follows immediately from the exact sequence

$$0 \to \Sigma^{18}\mathrm{H}_*(C_\eta) \to R_3 \to R_3/\Sigma^{18}\mathrm{H}_*(C_\eta) \to 0$$

that $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R_3)$ is as depicted in Figure 7. In particular, missing h_0–extensions are established. □

Theorem 3.1.17 *As a module over G_0, we have* (see Figure 8):

(a) *For every $\sigma \geq 2$, $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R'_\sigma) = G_\sigma$ is generated by $\alpha_{s,t,\sigma} \in \mathrm{Ext}^{s,t+s}_{\mathcal{A}(1)_*}(R'_\sigma)$, where $(s,t) \in \{(0,4\sigma),(0,2j+4\sigma),(k,6\sigma+2k) \mid 2 \leq j \leq \sigma,\ 1 \leq k \leq 3\}$ with $h_1\alpha_{s,t,\sigma} = 0$.*

(b) *For all pairs of triples (s_1,t_1,σ_1) and (s_2,t_2,σ_2) with $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$, except for $(2,9,1)$ and $(3,10,1)$,*

$$\alpha_{s_1,t_1,\sigma_1}\alpha_{s_2,t_2,\sigma_2} = \alpha_{s_1+s_2,t_1+t_2,\sigma_1+\sigma_2}.$$

Proof (a) The statement for $\sigma = 2$ is Proposition 3.1.14. Let us prove the claim for $\sigma \geq 3$ by induction. The base case is Lemma 3.1.16.

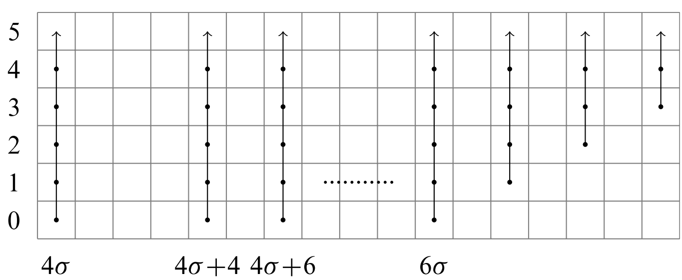


Figure 8: G_σ for $\sigma \geq 2$. There is an infinite tower of multiplication by h_0 in every even $t-s$ from $4\sigma+4$ to 6σ.

Suppose the claim is true for some $\sigma \geq 3$. The long exact sequence associated to the short exact sequence in Lemma 3.1.3 reads

$$\to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R'_{\sigma+1}) \to \mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{6\sigma+6} V_4) \to \mathrm{Ext}^{s+1,t}_{\mathcal{A}(1)_*}(\Sigma^4 R'_\sigma) \to .$$

Combining the additive structure of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^4 R'_\sigma)$ and that

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\Sigma^{6\sigma+6} V_4) \cong \Sigma^{6\sigma+6}\mathbb{F}_2[v_1],$$

we obtain the additive structure of $G_{\sigma+1}$ as described in the lemma because the connecting homomorphism vanishes for degree reasons. To establish the nontrivial h_0–multiplication on the generators $\{\alpha_{s,2s+6\sigma+6,\sigma+1} \mid s \geq 0\}$, we use the following identities:

(i) $G_{\sigma+1} \ni \alpha_{0,4,1}\alpha_{s,6\sigma+2s,\sigma} \neq 0$ for all $\sigma \geq 1$.

(ii) $\alpha_{1,8,1}\alpha_{s,2s+6\sigma-6,\sigma-1} = \alpha_{s+1,2s+6\sigma+2,\sigma}$ for all $\sigma \geq 2$.

(iii) $\alpha_{0,12,2}\alpha_{s,2s+6\sigma-6,\sigma-1} = \alpha_{s,2s+6\sigma+6,\sigma+1}$ for all $\sigma \geq 3$.

These identities are the content of part (b). For the sake of the presentation, we postpone the proof of (b); this is legitimate because, as we will see, the proof of (b) only uses the additive structure of the G_σ. Let us show how these identities allow us to conclude the proof of (a). Indeed, the classes $\alpha_{s,2s+6\sigma-6,\sigma-1}$ exist (ie are nontrivial) for all $\sigma \geq 3$ and $s \geq 0$. Therefore, we have that, for all $\sigma \geq 3$,

$$\begin{aligned} h_0\alpha_{s,2s+6\sigma+6,\sigma+1} \\ &= h_0\alpha_{0,12,2}\alpha_{s,2s+6\sigma-6,\sigma-1} && \text{(multiplying both sides of (iii) by } h_0\text{)} \\ &= \alpha_{0,4,1}\alpha_{1,8,1}\alpha_{s,2s+6\sigma-6,\sigma-1} && \text{(because of (i))} \\ &= \alpha_{0,4,1}\alpha_{s+1,2s+2+6\sigma,\sigma} && \text{(because of (ii))} \\ &\neq 0 && \text{(because of (i)).} \end{aligned}$$

(b) It follows from the long exact sequence in Ext associated to the short exact sequences in Lemmas 3.1.2(b) and 3.1.3 that

$$\alpha_{0,4,1}\alpha_{s,t,\sigma} = \alpha_{s,t+4,\sigma+1},$$

except for $(s,t,\sigma) = (2,9,1), (3,10,1)$ (see Remark 3.1.15 and the proof of part (a) of this proposition). It remains to prove that $\alpha_{s_1,t_1,\sigma_1}\alpha_{s_2,t_2,\sigma_2}$ is nontrivial for $(s_i,t_i,\sigma_i) \in \{(s_i, 6\sigma_i+2s_i, \sigma_i) \mid s_i = 0,1,2,3\}$ for $i = 1,2$.

Indeed, for every $\sigma, \tau \geq 1$, there is a commutative diagram of $\mathcal{A}(1)_*$–comodules

$$\begin{array}{ccccc}
R'_\sigma \otimes R'_\tau & \xrightarrow{\mu} & R_{\sigma+\tau} & \longrightarrow & R'_{\sigma+\tau} \\
\downarrow & & & & \downarrow \\
\mathrm{H}_*(\Sigma^{6\sigma} X_\sigma) \otimes \mathrm{H}_*(\Sigma^{6\tau} X_\tau) & & \xrightarrow{\mu} & & \mathrm{H}_*(\Sigma^{6\sigma+6\tau} X_{\sigma+\tau}) \\
\downarrow & & & & \downarrow \\
\mathrm{H}_*(\Sigma^{6\sigma} Y) \otimes \mathrm{H}_*(\Sigma^{6\tau} Y) & & \xrightarrow{\mu} & & \mathrm{H}_*(\Sigma^{6\sigma+6\tau} Y)
\end{array}$$

Let us explain the maps in this diagram. The spectrum X_σ is $V(0)$, $S^0 \cup_2 e^1 \cup_\eta e^2$ or Y if $\sigma = 1, 2$ or $\sigma > 2$, respectively; and in each case the map $R'_\sigma \to \mathrm{H}_*(\Sigma^{6\sigma} X_\sigma)$ is the projection appearing in the proof of Proposition 3.1.13, Lemma 3.1.2 or Lemma 3.1.3, respectively. The other vertical arrows are induced by the inclusions of X_σ into Y. The bottom horizontal arrow is the multiplication on $\mathrm{H}_*(Y)$, described in Lemma 3.1.12, and the middle one is induced by the latter. The second upper arrow is the projection on the factor $R'_{\sigma+\tau}$ of the decomposition in Lemma 3.1.1.

The induced homomorphisms in Ext over $\mathcal{A}(1)_*$ of all vertical arrows are studied in the proofs of Lemmas 3.1.13, 3.1.14, 3.1.17 and 3.1.11, which show that the classes $\alpha_{s,t,\sigma}$, where $\sigma \geq 1$ and $(s,t,\sigma) \in \{(s, 6\sigma + 2s, \sigma) \mid s = 0,1,2,3\}$, are sent nontrivially in a unique way to $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(\mathrm{H}_*(Y))$, and hence their products are nontrivial by Lemma 3.1.12. This proves (b). □

Remark 3.1.18 Let us summarize what has been done so far. First, Lemma 3.1.1 implies that

$$\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R) \cong \Big(\bigoplus_{i\geq 0} G_i\Big) \otimes \mathbb{F}_2[v_2^4],$$

where $v_2^4 \in \mathrm{Ext}^{4,28}(\mathbb{F}_2, R_4)$ is represented by y_3^4. Next, Theorem 3.1.17 describes completely the products between the G_i modulo the ideal generated by v_2^4. It is then straightforward to verify that $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$ is generated by the classes of

$$h_0,\ h_1,\ v,\ v_1^4,\ \alpha_{0,4,1},\ \alpha_{1,8,1},\ \alpha_{0,12,2},\ \alpha_{1,14,2},\ \alpha_{3,18,2},\ \alpha_{0,18,3},\ v_2^4. \tag{13}$$

Let us describe the subalgebra of primitives.

Corollary 3.1.19 *There is the isomorphism of graded algebras*

$$\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R) \cong \mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4, \alpha_{0,18,3}]/(\alpha_{0,18,3}^2 = \alpha_{0,12,2}^3 + \alpha_{0,4,1}^2 v_2^4).$$

Proof The algebra $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, R)$ is naturally identified with a subalgebra of $R = \mathbb{F}_2[y_1, y_2, y_3]$. Through this identification, $\alpha_{0,4,1}$, $\alpha_{0,12,2}$, v_2^4 and $\alpha_{0,18,3}$ identify with y_1, y_2^2, y_3^4 and $y_2^3 + y_1 y_3^2$, respectively. Thus the quotient in the statement is isomorphic to the subalgebra of $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(\mathbb{F}_2, R)$ generated by $\alpha_{0,4,1}$, $\alpha_{0,12,2}$, v_2^4 and $\alpha_{0,18,3}$. On the other hand, it follows from Remark 3.1.18 that $\alpha_{0,4,1}$, $\alpha_{0,12,2}$, v_2^4 and $\alpha_{0,18,3}$ generate the whole subalgebra of primitives of $\mathrm{Ext}^{*,*}_{\mathcal{A}(1)_*}(R)$. □

The differentials d_1 Since the DMSS for $\mathbb{F}_2$ is a spectral sequence of algebras, all d_1–differentials can be determined on the set of algebra generators of (13).

Proposition 3.1.20 *The differentials d_1 in the DMSS for A_1 is induced from*

(1) $d_1(h_0) = 0$,
(2) $d_1(h_1) = 0$,
(3) $d_1(\alpha_{0,4,1}) = 0$,
(4) $d_1(\alpha_{1,14,2}) = 0$,
(5) $d_1(\alpha_{0,18,3}) = 0$,
(6) $d_1(v_1^4) = 0$,
(7) $d_1(\alpha_{0,12,2}) = \alpha_{0,4,1}^3$,
(8) $d_1(\alpha_{1,8,1}) = h_0 \alpha_{0,4,1}^2$,
(9) $d_1(v) = h_0^3 \alpha_{0,4,1}$,
(10) $d_1(\alpha_{3,18,2}) = h_0^3 \alpha_{0,18,3}$,
(11) $d_1(v_2^4) = \alpha_{0,4,1} \alpha_{0,12,2}^2$.

Proof (1)–(2), (4) For degree reasons, there is no room for a nontrivial d_1–differential on h_0, h_1 or $\alpha_{1,14,2}$.

(3) It is easy to see that $\mathrm{Ext}^{1,4}_{\mathcal{A}(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$ is nontrivial and that $\alpha_{0,4,1}$ is the only class in the E_1–term that can contribute to it. Therefore $\alpha_{0,4,1}$ is a permanent cycle.

(5) We see that $h_0 \alpha_{0,18,3} = \alpha_{0,4,1} \alpha_{1,14,2}$. By the Leibniz rule, $h_0 d_1(\alpha_{0,18,3}) = 0$. As h_0 acts injectively on G_3, it follows that $d_1(\alpha_{0,18,3}) = 0$.

(6) Since $h_0^2 v_1^4 = v^2$, we have $h_0^2 d_1(v_1^4) = 2v d_1(v) = 0$. This follows because $d_1(v_1^4)$ takes values in $\mathrm{Ext}^{4,8}_{\mathcal{A}(1)_*}(\mathbb{F}_2, R_1)$ on which h_0 acts injectively.

(7) We have that $\alpha_{0,12,2}$ is represented by the $\mathcal{A}(2)$–primitive $[1|y_2^2] + [x_1|y_1^2] \in E \otimes R_2$. By Remark 2.1.8, $d_1(\alpha_{0,12,2})$ is represented by $d([1|y_2^2] + [x_1|y_1^2]) = [1|y_1^3] \in E \otimes R_3$, and hence is equal to $\alpha_{0,4,1}^3$.

(8) Because $\alpha_{0,4,1}\alpha_{1,8,1} = h_0\alpha_{0,12,2}$, the Leibniz rule implies that

$$\alpha_{0,4,1}d_1(\alpha_{1,8,1}) = h_0 d_1(\alpha_{0,12,2}) = h_0\alpha^3_{0,4,1}.$$

That $\alpha_{0,4,1}$ acts injectively on the E_1–term implies that $d_1(\alpha_{1,8,1}) = h_0\alpha^2_{0,4,1}$.

(9) The relation $\alpha_{0,4,1}v = h_0^2\alpha_{1,8,1}$ implies that

$$\alpha_{0,4,1}d_1(v) = h_0^2 d_1(\alpha_{1,8,1}) = h_0^3\alpha^2_{0,4,1}.$$

As $\alpha_{0,4,1}$ acts injectively on the E_1–term, we obtain that $d_1(v) = h_0^3\alpha_{0,4,1}$.

(10) The relation $v\alpha_{1,14,2} = h_0\alpha_{3,18,2}$ shows that

$$h_0 d_1(\alpha_{3,18,2}) = \alpha_{1,14,2}d_1(v) = \alpha_{1,14,2}h_0^3\alpha_{0,4,1} = h_0^4\alpha_{0,18,3}.$$

Therefore, $d_1(\alpha_{3,18,2}) = h_0^3\alpha_{0,18,3}$.

(11) We check that v_2^4 is represented by the $\mathcal{A}(2)$–primitive $[1|y_3^4]+[x_1|y_2^4]$ in $E\otimes R_4$. By Remark 2.1.8, $d_1(v_2^4)$ is represented by $[1|y_1y_2^4]$, and hence is equal to $\alpha_{0,4,1}\alpha^2_{0,12,2}$.

□

Remark 3.1.21 It turns out that the DMSS collapses at the E_2–term because there is no room for higher differentials. In particular, the classes $\alpha_{1,14,2}$, $\alpha_{0,4,1}$, $\alpha^2_{0,12,2}$, v_2^8 and $\alpha_{0,18,3}$ survive the spectral sequence, converging to elements of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2,\mathbb{F}_2)$ in appropriate bidegrees. Following [24], those elements are denoted by α, h_2, g, w_2 and β, respectively.

The differentials in Proposition 3.1.20 results in important information on $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$, and hence on $\pi_*(\mathrm{tmf})$. Among other things, the differential (11) implies that, in $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)}(\mathbb{F}_2)$, there is the relation $h_2g = 0$, and hence $\nu\bar{\kappa} = 0$ in $\pi_*(\mathrm{tmf})$, by sparseness, where ν and $\bar{\kappa}$ denote the elements in $\pi_*(\mathrm{tmf})$ which are detected by h_2 and g, respectively. In fact, both ν and $\bar{\kappa}$ have lifts in $\pi_*(S^0)$. Likewise, the differential (7) implies that $h_2^3 = 0$ in $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$.

Furthermore, h_2, g and w_2, β generate a subalgebra of $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2,\mathbb{F}_2)$, which is isomorphic to $\mathbb{F}_2[h_2,g,w_2,\beta]/(h_2^3,h_2g,\beta^4-g^3)$. The relation $\beta^4 = g^3$ is a consequence of the d_1–differential (7). Indeed, the relation $\alpha^2_{0,18,3} = \alpha^3_{0,12,2}+\alpha^2_{0,4,1}v_2^4$ implies the relation $\beta^4 - g^3 - h_2^4w_2 = 0$ in $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$. But $\alpha^4_{0,4,1}v_2^8$ gets hit by the differential

$$d_1(v_2^8\alpha_{0,4,1}\alpha_{0,12,2}) = v_2^8\alpha_{0,4,1}d_1(\alpha_{0,12,2}) = v_2^8\alpha^4_{0,4,1}.$$

Thus, the relation $\beta^4 = g^3 + h_2^4w_2$ becomes $\beta^4 = g^3$.

3.2 The Davis–Mahowald spectral sequence for A_1

The $\mathcal{A}(2)_*$–comodule structure of A_1 In [13], Davis and Mahowald constructed four finite spectra, whose mod 2 cohomology are isomorphic to a free module of rank one over the subalgebra $\mathcal{A}(1) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ of the Steenrod algebra $\mathcal{A}$. Let us review the construction of these spectra and their module structure over the subalgebra $\mathcal{A}(2) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4 \rangle$ of $\mathcal{A}$. Recall that Y is $V(0) \wedge C_\eta$. The $\mathcal{A}$–module structure of $\mathrm{H}^*(Y)$ is depicted in Figure 9. An element of $\mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$ can be represented by an $\mathcal{A}(1)$–module M sitting in a short exact sequence of $\mathcal{A}(1)$–modules

$$0 \to \mathrm{H}^*(\Sigma^3 Y) \to M \to \mathrm{H}^*(Y) \to 0.$$

It can be checked that M must be isomorphic either to $\mathrm{H}^*(\Sigma^3 Y) \oplus \mathrm{H}^*(Y)$ or to $\mathcal{A}(1)$ as an $\mathcal{A}(1)$–module. This means that

$$\mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \cong \mathbb{F}_2. \tag{14}$$

The $\mathcal{A}(1)$–module structure of $\mathcal{A}(1)$ is depicted in Figure 10. One can ask whether $\mathcal{A}(1)$ admits a structure of $\mathcal{A}(2)$–module. If such a structure exists, then, according to the Adem relations $\mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^2 = \mathrm{Sq}^4 \mathrm{Sq}^1 + \mathrm{Sq}^1 \mathrm{Sq}^4$, there must be a nontrivial action of Sq^4 on the nontrivial class of degree 1. It is straightforward to verify that the latter is the only constraint to putting an $\mathcal{A}(2)$–module structure on $\mathcal{A}(1)$. There are also possibilities for Sq^4 to act nontrivially on the classes of degree 0 and 2. These give in total four different $\mathcal{A}(2)$–module structures on A_1. In other words, the inclusion of Hopf algebras $\mathcal{A}(1) \hookrightarrow \mathcal{A}(2)$ induces a surjective homomorphism

$$\mathrm{Ext}^{1,3}_{\mathcal{A}(2)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \to \mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$$

whose kernel contains four elements. Therefore,

$$\mathrm{Ext}^{1,3}_{\mathcal{A}(2)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \cong \mathbb{F}_2^{\oplus 3}.$$

Next, one observes that restriction along $\mathcal{A}(2) \subset \mathcal{A}$ induces an isomorphism

$$\mathrm{Ext}^{1,3}_{\mathcal{A}}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)) \cong \mathrm{Ext}^{1,3}_{\mathcal{A}(2)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y)),$$

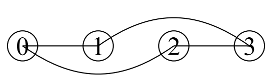


Figure 9: Diagram of $\mathrm{H}^*(Y)$. The straight lines represent Sq^1 and the curved lines represent Sq^2, the numbers represent the degree of the cell.

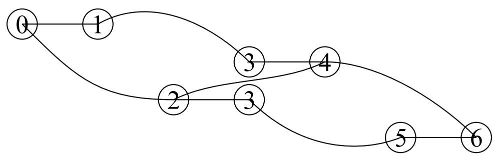


Figure 10: Diagram of $\mathcal{A}(1)$.

because, for any $\mathcal{A}$–module M sitting in a short exact sequence

$$0 \to \mathrm{H}^*(\Sigma^3 Y) \to M \to \mathrm{H}^*(Y) \to 0,$$

there cannot be any nontrivial Sq^k for $k \geq 8$ on M. It is proved in [13] that the four classes of $\mathrm{Ext}^{1,3}_{\mathcal{A}}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$ that are sent to the unique nontrivial class of $\mathrm{Ext}^{1,3}_{\mathcal{A}(1)}(\mathrm{H}^*(Y), \mathrm{H}^*(Y))$ are permanent cycles in the Adams spectral sequence and converge to four v_1–self-maps of Y, ie the maps $\Sigma^2 Y \to Y$ inducing isomorphisms in $K(1)$–homology theory. As a consequence, the cofibers of these v_1–self-maps realize the four different $\mathcal{A}$–module structures on $\mathcal{A}(1)$. We will write A_1 to refer to any of these four finite spectra.

Definition 3.2.1 [10] We define by $A_1[i, j]$ for $i, j \in \{0, 1\}$ the version of A_1 having the nontrivial Sq^4 on the generator of degree 0 (respectively 2) if and only if $i = 1$ (respectively $j = 1$). (See Figure 10.)

As $\mathbb{F}_2$–vector spaces,

$$\mathrm{H}_*(A_1[ij]) \cong \mathbb{F}_2\{a_0, a_1, a_2, a_3, \bar{a}_3, a_4, a_5, a_6\}, \tag{15}$$

where a_0, a_1, a_2, a_4, a_5 and a_6 are duals to the generators of degree 0, 1, 2, 4, 5 and 6 of $\mathrm{H}^*(A_1[ij])$, respectively, and a_3 and $\bar{a}_3$ are duals to the images of the generator of degree 0 by Sq^3 and $\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1$, respectively. By taking duals to the action of $\mathcal{A}(2)$ on $\mathrm{H}^*(A_1[ij])$, we obtain:

Proposition 3.2.2 *The left coaction of $\mathcal{A}(2)_*$ on $\mathrm{H}_*(A_1[ij])$ is given by*

$$\Delta(a_1) = [1|a_1] + [\xi_1|a_0],$$
$$\Delta(a_2) = [1|a_2] + [\xi_1^2|a_2]$$
$$\Delta(a_3) = [1|a_3] + [\xi_1|a_2] + [\xi_1^2|a_1] + [\xi_1^3|a_0],$$
$$\Delta(\bar{a}_3) = [1|\bar{a}_3] + [\xi_1^2|a_1] + [\xi_2|a_0],$$
$$\Delta(a_4) = [1|a_4] + [\xi_1|\bar{a}_3] + [\xi_1^2|a_2] + [\xi_1^3|a_1] + [\xi_2|a_1] + [\xi_2\xi_1|a_0] + \alpha_{i,j}[\xi_1^4|a_0],$$

$$\Delta(a_5) = [1|a_5] + [\xi_1^2|\bar{a}_3] + [\xi_1^2|a_3] + [\xi_2|a_2] + [\xi_1^4|a_1] + [\xi_2\xi_1^2|a_0],$$
$$\begin{aligned}\Delta(a_6) = {} & [1|a_6] + [\xi_1|a_5] + [\xi_1^2|a_4] + [\xi_1^3|\bar{a}_3] + [\xi_1^3|a_3] + [\xi_2|a_3] + [\xi_2\xi_1|a_2] \\ & + \beta_{i,j}[\xi_1^4|a_2] + [\xi_2\xi_1^2|a_1] + [\xi_1^5|a_1] + \gamma_{i,j}[\xi_1^6|a_0] + [\xi_2\xi_1^3|a_0] + \lambda_{i,j}[\xi_2^2|a_0],\end{aligned}$$

where

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } (i,j) \in \{(0,0),(0,1)\}, \\ 1 & \text{if } (i,j) \in \{(1,0),(1,1)\}, \end{cases} \qquad \gamma_{i,j} = 1 + \alpha_{i,j},$$
$$\beta_{i,j} = \begin{cases} 0 & \text{if } (i,j) \in \{(0,0),(1,0)\}, \\ 1 & \text{if } (i,j) \in \{(0,1),(1,1)\}, \end{cases} \qquad \lambda_{i,j} = \alpha_{i,j} + \beta_{i,j}.$$

Proof The proof is a straightforward translation from $\mathcal{A}(2)$–module structure to $\mathcal{A}(2)_*$–comodule structure using the formula of the duals of the Milnor basis in [29]. □

DMSS for A_1 In what follows, we will apply the shearing homomorphism to find primitives representing certain cohomology classes; see [2, Theorem 3.1]. In general, let C be a Hopf algebra with conjugation χ and B be a Hopf algebra quotient of C. Given a C–comodule M, consider the composite

$$C \otimes M \xrightarrow{\mathrm{id}\otimes\Delta} C \otimes C \otimes M \xrightarrow{\mathrm{id}\otimes\chi\otimes\mathrm{id}} C \otimes C \otimes M \xrightarrow{\mu\otimes\mathrm{id}} C \otimes M.$$

When restricting to $C \,\square_B\, M$, this composite factors through $(C \,\square_B\, k) \otimes M$, inducing the shearing isomorphism of C–comodules

$$\mathrm{Sh}\colon C \,\square_B\, M \to (C \,\square_B\, k) \otimes M,$$

where C coacts on $C \,\square_B\, M$ via the left factor and on $(C \,\square_B\, k) \otimes M$ diagonally. Combined with the change-of-rings isomorphism, we have the isomorphisms

$$\mathrm{Ext}_B^*(k, M) \cong \mathrm{Ext}_C^*(k, C \,\square_B\, M) \cong \mathrm{Ext}_C^*(k, (C \,\square_B\, k) \otimes M).$$

In particular, via these isomorphisms, a class $x \in \mathrm{Ext}_B^0(k, M)$ is sent to $\mathrm{Sh}(1 \otimes x)$.

Proposition 3.2.3 *The* E_1*–term of the Davis–Mahowald spectral sequence converging to* $\mathrm{Ext}_{\mathcal{A}(2)_*}^{s,t}(\mathrm{H}_*(A_1))$ *is given by*

$$\mathrm{E}_1^{s,\sigma,*} \cong \begin{cases} 0 & \text{if } s > 0, \\ R_\sigma & \text{if } s = 0. \end{cases}$$

As a module over $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4]$, $\mathrm{E}_1^{*,*,*}$ *is the free module of rank eight on the generators*

$$(16) \qquad 1, \quad y_3, \quad y_3^2, \quad y_3^3, \quad y_2, \quad y_2y_3, \quad y_2y_3^2, \quad y_2y_3^3.$$

Proof Indeed, $E_1^{s,\sigma,t}$ is equal to $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_\sigma \otimes \mathrm{H}_*(A_1))$ by definition. The coaction of $\mathcal{A}(1)_*$ on $R_\sigma \otimes \mathrm{H}_*(A_1)$ is the usual diagonal coaction on tensor products. In addition, $\mathrm{H}_*(A_1)$ is isomorphic to $\mathcal{A}(1)_*$ as $\mathcal{A}(1)_*$–comodules. By the change-of-rings isomorphism,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R_\sigma \otimes \mathrm{H}_*(A_1)) \cong \mathrm{Ext}^{s,t}_{\mathbb{F}_2}(R_\sigma) \cong R_\sigma. \tag{17}$$

The first part of the proposition follows.

For the second part, the action of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R)$ on $E_1^{s,t,\sigma}$,

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)_*}(R) \otimes \mathrm{Ext}^{s',t'}_{\mathcal{A}(1)_*}(R \otimes \mathrm{H}_*(A_1)) \to \mathrm{Ext}^{s+s',t+t'}_{\mathcal{A}(1)_*}(R \otimes \mathrm{H}_*(A_1)),$$

is induced by the multiplication on R,

$$R \otimes (R \otimes \mathrm{H}_*(A_1)) \to R \otimes \mathrm{H}_*(A_1).$$

Now let $r \in \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R) \subset R$ and $s \in R \cong \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes \mathrm{H}_*(A_1))$. By applying the shearing isomorphism, the class s is represented by a unique element of the form $s \otimes a_0 + \sum s_i \otimes a_i \in R \otimes \mathrm{H}_*(A_1)$, where the a_i are in positive degrees. The action of r on s is then represented by $rs \otimes a_0 + \sum rs_i \otimes a_i$, which represents $rs \in R \cong \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes A_1)$ via (17). In other words, the action of $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R)$ on $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes \mathrm{H}_*(A_1))$ is given by the multiplication of the polynomial algebra R. The proof follows from the fact that $\alpha_{0,4,1}, \alpha_{0,12,2}, v_2^4 \in \mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R)$ are represented by $y_1, y_2^2, y_3^4 \in R$, respectively. □

Let us analyze the differentials in this spectral sequence. As the d_r–differentials decrease s–filtration by $r-1$, ie $d_r : E_r^{s,\sigma,t} \to E_r^{s-r+1,\sigma+r,t}$ and $E_1^{s,\sigma,t} = 0$ if $s > 0$, the spectral sequence collapses at the E_2–term and there are no extension problems. Therefore,

$$E_2^{0,t,\sigma} \cong \mathrm{Ext}^{\sigma,t}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)).$$

We now turn our attention to the d_1–differentials. As all elements of the E_1–term are in $\mathrm{Ext}^{0,*}_{\mathcal{A}(1)_*}(R \otimes \mathrm{H}_*(A_1))$, we can apply the remark after Proposition 2.1.7. We have determined the d_1–differential on the classes $\alpha_{0,4,1}, \alpha_{0,12,2}$ and v_2^4 in Proposition 3.1.20. By the Leibniz rule, it remains to determine the d_1–differential on the classes of (16).

Proposition 3.2.4 *There are the d_1–differentials*

(1) $d_1(1) = 0$,

(2) $d_1(y_2) = 0$,

(3) $d_1(y_3) = 0$,

(4) $d_1(y_2 y_3) = 0$,

(5) $d_1(y_2 y_3^2) = 0$,

(6) $d_1(y_2 y_3^3) = 0$,

(7) $d_1(y_3^2) = \alpha_{0,4,1}^2 y_2$,

(8) $d_1(y_3^3) = \alpha_{0,4,1}^2 y_2 y_3$.

Proof Parts (1)–(4) follow from the sparseness of the E_1–term.

(5) The only nontrivial d_1–differential that $y_2 y_3^2$ can support is

$$d_1(y_2 y_3^2) = \alpha_{0,4,1}^2 \alpha_{0,12,2} 1.$$

However, by the Leibniz rule and Proposition 3.1.20(7),

$$d_1(\alpha_{0,4,1}^2 \alpha_{0,12,2} 1) = \alpha_{0,4,1}^2 d_1(\alpha_{0,12,2}) 1 = \alpha_{0,4,1}^5 1 \neq 0.$$

This means that $\alpha_{0,4,1}^2 \alpha_{0,12,2} 1$ is not a d_1–cycle, and so cannot be hit by a d_1–differential. Therefore, $y_2 y_3^2$ is a d_1–cycle.

(6) Similarly, a nontrivial d_1–differential on $y_2 y_3^3$ would be

$$d_1(y_2 y_3^3) = \alpha_{0,4,1}^2 \alpha_{0,12,2} y_3.$$

However,

$$d_1(\alpha_{0,4,1}^2 \alpha_{0,12,2} y_3) = \alpha_{0,4,1}^5 y_3 \neq 0$$

by the Leibniz rule. Thus, $y_2 y_3^3$ is a d_1–cycle.

(7)–(8) The class $\alpha_{0,4,1}$ in the DMSS for $\mathbb{F}_2$ represents h_2, the unique nontrivial class of $\mathrm{Ext}^{1,4}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$. By sparseness and parts (2) and (4), y_2 and $y_2 y_3$ represent nontrivial classes of $\mathrm{Ext}^{1,6}_{\mathcal{A}(1)_*}(\mathrm{H}_*(A_1))$ and $\mathrm{Ext}^{2,13}_{\mathcal{A}(1)_*}(\mathrm{H}_*(A_1))$, which we denote by the same names in this proof. It suffices to prove that $h_2^2 y_2 = 0$ and $h_2^2 y_2 y_3 = 0$ in $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ because the differentials in parts (7)–(8) are the only possibilities for the latter to occur. We will proceed using juggling formulas for Massey products; see [32, Appendix A1, Section 4]. The classes 1 and y_3 being permanent cycles by parts (1) and (3), they converge to classes in $\mathrm{Ext}^{0,0}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ and $\mathrm{Ext}^{1,6}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$, respectively. By sparseness of the E_1–term of the DMSS, $h_1 1 = h_1 y_3 = 0$. Hence the Massey product $\langle h_2, h_1, y_3^i \rangle$ with $i \in \{0, 1\}$ can be formed. In $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$, there is a well-known Massey product $\langle h_1, h_2, h_1\rangle$ that has zero indeterminacy and is equal to h_2^2. We have then that

$$h_2^2 y_3^i = \langle h_1, h_2, h_1\rangle y_3^i = h_1\langle h_2, h_1, y_3^i\rangle.$$

By sparseness of the DMSS, $\alpha_{0,4,1}^2 y_3^i$ survives the DMSS and so $h_2^2 y_3^i \neq 0$. It follows that $\langle h_2, h_1, y_3^i\rangle$ does not contain zero and, by sparseness of the DMSS, must be equal to $y_2 y_3^i$. The fact that $h_2^3 = 0 \in \mathrm{Ext}_{\mathcal{A}(2)_*}^{3,12}(\mathbb{F}_2)$ — see Remark 3.1.21 — allows us to apply the juggling formula

$$h_2^2 y_2 y_3^i = h_2^2\langle h_2, h_1, y_3^i\rangle = \langle h_2^2, h_2, h_1\rangle y_3^i.$$

However, the Massey product $\langle h_2^2, h_2, h_1\rangle$ lives in the group $\mathrm{Ext}_{\mathcal{A}(2)_*}^{3,14}(\mathbb{F}_2)$, which vanishes by Theorem 3.1.17. This concludes the proof of parts (7)–(8). □

$\mathbf{E_2}$–term of the Adams SS We describe $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathrm{H}_*(A_1))$ as a module over

$$\mathbb{F}_2[h_2, g, v_2^8]/(h_2^3, h_2 g) \subset \mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathbb{F}_2).$$

We recall that g is represented by $\alpha_{0,12,2}^2$ in the DMSS for $\mathbb{F}_2$. We will denote by $e[s,t]$ for $s, t \in \mathbb{N}$ the unique nontrivial class belonging to $\mathrm{Ext}_{\mathcal{A}(2)_*}^{s,s+t}(\mathrm{H}_*(A_1))$.

Theorem 3.2.5 *As a module over* $\mathbb{F}_2[h_2, g, v_2^8]/(h_2^3, h_2 g)$, $\mathrm{Ext}_{\mathcal{A}(2)_*}^{*,*}(\mathrm{H}_*(A_1))$ *is a direct sum of cyclic modules generated by the following elements*:

$e[0,0]$	$e[1,5]$	$e[1,6]$	$e[2,11]$	$e[3,15]$	$e[3,17]$	$e[4,21]$	$e[4,23]$
1	y_2	y_3	$y_2 y_3$	$y_2^3 + y_1 y_3^2$	$y_2 y_3^2$	$y_1 y_3^3 + y_2^3 y_3$	$y_2 y_3^3$
(0)	(h_2^2)	(0)	(h_2^2)	(h_2^2)	(0)	(h_2^2)	(0)

$e[6,30]$	$e[6,32]$	$e[7,36]$	$e[7,38]$
$y_2^6 + y_1^2 y_3^4$	$y_2^4 y_3^2 + y_1 y_2 y_3^4$	$y_2^6 y_3 + y_1^2 y_3^5$	$y_2^4 y_3^3 + y_1 y_2 y_3^5$
(h_2)	(h_2)	(h_2)	(h_2)

$e[8,42]$	$e[9,47]$	$e[9,48]$	$e[10,53]$
$y_2^6 y_3^2 + y_1^2 y_3^6 + y_1 y_2^3 y_3^4$	$y_2^7 y_3^2 + y_1^2 y_2 y_3^6$	$y_2^6 y_3^3 + y_1^2 y_3^7 + y_1 y_2^3 y_3^5$	$y_2^7 y_3^3 + y_1^2 y_2 y_3^7$
(h_2)	(h_2)	(h_2)	(h_2)

The second row in the table indicates a representative in the DMSS and the third row the annihilator ideal of the corresponding generator.

Proof Let R denote $\mathbb{F}_2[h_2, g, v_2^8]/(h_2^3, h_2 g)$ and A the set consisting of classes given in the second row of the table in the theorem. For each $a \in A$, let n_a be the h_2–nilpotency order of a, as given in the table. We will sketch the proof showing that

$$\bigoplus_{a \in A} R\{a\}/(h_2^{n_a}) = \mathrm{E}_2,$$

by first showing the inclusion of the left-hand side to the right-hand one, then showing the equality by comparing their Poincaré series with respect to the topological degree, which is the difference of the internal degree by the sum of the cohomological degree and the DM degree.

Let K, C and I denote the kernel, coker and image of $d_1 : \mathrm{E}_1 \to \mathrm{E}_1$, as a map of trigraded $\mathbb{F}_2$–vector spaces. The E_2–term of the DMSS is then isomorphic to the quotient K/I. Since h_2^3 and $h_2 g$ are trivial in $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ (see Remark 3.1.21), $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is a module over $\mathbb{F}_2[h_2, g, v_2^8]/(h_2^3, h_2 g)$. Moreover, using Propositions 3.1.20 and 3.2.4, it is straightforward to check that:

(1) The classes of A are d_1–cycles surviving to the E_2–term together with the corresponding nilpotency order of h_2.

(2) For an permanent cycle $x \in \mathrm{E}_1$, if x is not hit by a d_1–differential, then neither is $v_2^8 x$.

(3) For an permanent cycle $x \in \mathrm{E}_1$, if x is not divisible by $\alpha_{0,4,1}$, then $\alpha_{0,12,2}^2 x$ is not hit by a d_1–differential.

It follows that

$$U := \bigoplus_{a \in A} R\{a\}/(h_2^{n_a} a) \subset \mathrm{E}_2$$

and that

$$V := \bigoplus_{a \in A} R\{h_2^{n_a} a, h_2 g a\}/(g(h_2^{n_a} a) - h_2^{n_a - 1}(h_2 g a)) \subset I.$$

Consider these groups as graded $\mathbb{F}_2$–vector spaces regarding their topological degree. The Poincaré series associated to a graded $\mathbb{F}_2$–vector space M is denoted by $\chi_M(X)$. We have that

$$\chi_U(X) \ll \chi_{E_2}(X) \quad \text{and} \quad \chi_V(X) \ll \chi_I(X),$$

and, since d_1 decreases the topological degree by 1 and induces an isomorphism $C \cong I$, seen as nongraded $\mathbb{F}_2$–vector spaces,

$$\chi_V(X).X \ll \chi_C(X).$$

Since

$$\chi_{\mathrm{E}_1}(X) = \chi_C(X) + \chi_I(X) + \chi_{\mathrm{E}_2}(X),$$

we obtain that

$$\chi_U(X) + \chi_V(X) + \chi_V(X).X \ll \chi_{\mathrm{E}_1}(X).$$

On the other hand, an direct computation shows that the left-hand series is equal to $1/(1-X^3)(1-X^5)(1-X^6)$, which is equal to the right-hand series (see Proposition 3.2.3). This allows us to conclude that

$$\mathrm{E}_2 = \bigoplus_{a\in A} R\{a\}/(h_2^{n_a}a). \qquad \square$$

Remark 3.2.6 The entire DMSS for A_1 is quite messy. Nevertheless, we illustrate it by showing the differentials d_1 truncating the $\alpha_{0,4,1}$–tower of the first eight classes of A in Figure 11 (see also Figure 13 for the E_∞–term of the DMSS or the E_2–term of the ASS). In formulas,

$$\begin{aligned} d_1(y_2^2) &= \alpha_{0,4,1}^3 1, & d_1(y_3^2) &= \alpha_{0,4,1}^2 y_2, \\ d_1(y_2^2y_3) &= \alpha_{0,4,1}^2 y_3, & d_1(y_3^3) &= \alpha_{0,4,1}^2 y_2y_3, \\ d_1(y_2^2y_3^2) &= \alpha_{0,4,1}^2(y_2^3 + y_1y_3^2), & d_1(y_2^3y_3^2) &= \alpha_{0,4,1}^3 y_2y_3^2, \\ d_1(y_2^2y_3^3) &= \alpha_{0,4,1}^2(y_2^3y_3 + y_1y_3^3), & d_1(y_2^3y_3^3) &= \alpha_{0,4,1}^3 y_2y_3^3. \end{aligned}$$

3.3 Two products

We now study the product of $\alpha \in \mathrm{Ext}^{3,15}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ and $e[4,23] \in \mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$. This is a key result, which is the input in the study of d_2–differentials of the Adams spectral sequence in the next section. Recall that α is detected by $\alpha_{1,14,2}$ in the DMSS converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$.This product is not detected in the DMSS because α has σ–filtration 1 in the DMSS whereas all nontrivial groups in the E_∞–term of the DMSS converging to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ are in σ–filtration 0. Therefore, we need first to find a representative of α in the total cochain complex of the double complex $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R$ and that of $e[4,23]$ in $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R \otimes \mathrm{H}_*(A_1)$, then take the product at the level of cochain complexes, and finally check if this product is a coboundary. It is tedious to carry out this procedure because any representative of $e[4,23]$ contains many terms, and so it is not easy to check if the product is a coboundary. Here, by a term of $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R_*$ and $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R_* \otimes \mathrm{H}_*(A_1)$, we mean an element of the basis formed by the tensor products of a basis of $\mathcal{A}(2)_*$, E_2, R_* and $\mathrm{H}_*(A_1)$ chosen

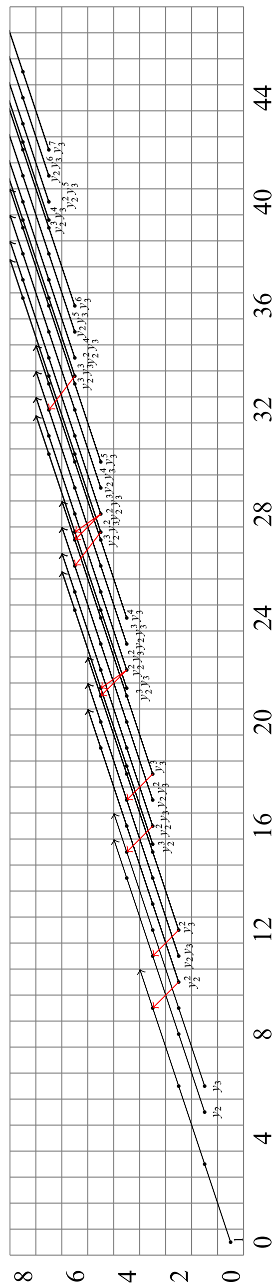


Figure 11: Black arrows represent the $\alpha_{0,4,1}$–tower of the classes $\{y_2^i y_3^j \mid 0 \le i \le 3,\ 0 \le j \le 7\}$, which generates the E_1–term as a module over $\mathbb{F}_2[\alpha_{0,4,1}, \alpha_{0,12,2}^2, v_2^8]$. Red arrows represent differential d_1. Double arrows starting from one class mean that the differential d_1 on that class hits the some of the targets.

to be the monomial basis and the basis of (15), respectively. We will use the same convention when working with $B(2)_*$, F_2 and S_* instead of $\mathcal{A}(2)_*$, E_2 and R_*. The following two lemmas simplify computations.

Lemma 3.3.1 *The product of* α *and* $e[4,23]$ *is equal either to* 0 *or to* $ge[3,15]$.

Proof This is because $ge[3,15]$ is the only nontrivial class in the appropriate bidegree. □

We recall from Section 2 that there is a map of pairs $(\mathcal{A}(2)_*, M_2) \to (B(2)_*, N_2)$ given by

$$\begin{gathered}\mathcal{A}(2)_* = \mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3]/(\zeta_1^8,\zeta_2^4,\zeta_3^2) \to B(2)_* = \mathbb{F}_2[\zeta_1,\zeta_2,\zeta_3]/(\zeta_1^4,\zeta_2^4,\zeta_3^2),\\ \zeta_i \mapsto \zeta_i \quad \text{for } i \in \{1,2,3\},\\ M_2 = \mathbb{F}_2\{x_0,x_1,x_2,x_3\} \to N_2 = \mathbb{F}_2\{x_0,x_2,x_3\},\\ x_1 \mapsto 0,\\ x_i \mapsto x_i \quad \text{for } i \in \{0,2,3\}.\end{gathered}$$

The induced map on the polynomial component of the associated pairs of Koszul duals is given by

$$R = \mathbb{F}_2[y_1,y_2,y_3] \to S = \mathbb{F}_2[y_2,y_3], \qquad y_1 \mapsto 0, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_3.$$

By an abuse of notation, we will denote by p these projection maps. The context will make it clear which map is referred to.

Lemma 3.3.2 *The map* $p_* = \mathrm{Ext}^{7,42}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) \to \mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1))$ *induced by the projection* $\mathcal{A}(2)_* \to B(2)_*$ *sends* $ge[3,15]$ *to a nontrivial element.*

Proof The projection $\mathcal{A}(2)_* \to B(2)_*$ induces a morphism of the DMSSs. The morphism of the E_1–terms reads

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(2)_*}(E_2 \otimes R \otimes \mathrm{H}_*(A_1)) \to \mathrm{Ext}^{s,t}_{B(2)_*}(F_2 \otimes S \otimes \mathrm{H}_*(A_1)).$$

By the change-of-rings isomorphism, this morphism identifies with the projection $p\colon R \to S$, which is surjective. The class $ge[3,15]$ is detected by $y_2^4(y_2^3 + y_1y_3^2) \in R^7$, which maps to $y_2^7 \in S^7$ via p. By naturality, y_2^7 is a permanent cycle in the target DMSS. The only class in the E_1–term which can support a differential hitting y_2^7 is y_3^6, which admits $v_2^4y_3^2$ as a lift in the source DMSS. We have

$$\begin{aligned}d_1(v_2^4y_3^2) &= d_1(v_2^4)y_3^2 + v_2^4d_1(y_3^2) = (\alpha_{0,4,1}\alpha_{0,12,2}^2)y_3^2 + v_2^4(\alpha_{0,4,1}y_2)\\ &= y_1y_2^4y_3^2 + y_3^4y_1y_2.\end{aligned}$$

This uses the Leibniz rule and Propositions 3.1.20(11) and 3.2.4(7). By naturality, the d_1–differential in the target DMSS is $p(y_1 y_2^4 y_3^2 + y_3^4 y_1 y_2)$, which is equal to 0. Therefore, the image of $ge[3,15]$ is nontrivial. □

Lemma 3.3.3 *The product of α and $e[4,23]$ is nontrivial, and hence equal to $ge[3,15]$ if and only if the product of $p_*(\alpha)$ and $p_*(e[4,23])$ is nontrivial.*

Proof The map $p\colon \mathcal{A}(2)_* \to B(2)_*$ induces the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{3,15}_{\mathcal{A}(2)_*}(\mathbb{F}_2) \otimes \mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) & \longrightarrow & \mathrm{Ext}^{7,42}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) \\ \downarrow p_* & & \downarrow p_* \\ \mathrm{Ext}^{3,15}_{B(2)_*}(\mathbb{F}_2) \otimes \mathrm{Ext}^{4,27}_{B(2)_*}(\mathrm{H}_*(A_1)) & \longrightarrow & \mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1)) \end{array}$$

where the horizontal maps are the respective multiplications. The result follows from the fact that $p_*(ge[3,15])$ is nontrivial by Lemma 3.3.2. □

Now let us compute the product of $p_*(\alpha)$ and $p_*(e[4,23])$.

Lemma 3.3.4 *In the total cochain complexes of*

$$B(2)_*^{\otimes *} \otimes F_2 \otimes S \quad \text{and} \quad B(2)_*^{\otimes *} \otimes F_2 \otimes S \otimes \mathrm{H}_*(A_1),$$

respectively,

(i) *$p_*(\alpha)$ is represented by* $[\xi_2|1|y_2^2] + [\xi_1^3|1|y_2^2] + [\xi_1|1|y_3^2] \in B(2) \otimes F_2 \otimes S^2$;

(ii) *$p_*(e[4,23])$ is represented by*

$$[1|y_2 y_3^3|a_0] + [1|y_2^2 y_3^2|a_1] + [1|y_2^3 y_3|a_2] + [1|y_2^4|a_3] \in F_2 \otimes S^4 \otimes \mathrm{H}_*(A_1).$$

Proof A direct computation shows that these elements are cocycles of the total differentials, which are not coboundaries. One way to prove that they represent the right classes is to prove that they lift to cocycles in the total cochain complexes of $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R$ and of $\mathcal{A}(2)_*^{\otimes *} \otimes E_2 \otimes R \otimes \mathrm{H}_*(A_1)$, respectively.

It is easy to check that

$$[\xi_2|1|y_2^2] + [\xi_1^3|1|y_2^2] + [\xi_1|1|y_3^2] + [\xi_2|x_1|y_1^2] + [\xi_1^3|x_1|y_1^2] + [\xi_1|x_2|y_1^2] + [1|1|y_1^2 y_3] \in (\mathcal{A}(2)_* \otimes E_2 \otimes R^2) \oplus (E_2 \otimes R^3)$$

is a lift for $[\xi_2|1|y_1^2] + [\xi_1^3|1|y_1^2] + [\xi_1|1|y_2^2]$.

For the other element, instead of finding a lift it suffices to show that p_* induces an isomorphism $\mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) \xrightarrow{\cong} \mathrm{Ext}^{4,27}_{B(2)_*}(\mathrm{H}_*(A_1))$, so that both are isomorphic to $\mathbb{F}_2$. This can be proved by a similar argument to that used in the proof of Lemma 3.3.2. Indeed, the nontrivial class of $\mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is detected by $y_2y_3^3$ in the DMSS. Via p_*, the latter is sent to $y_2y_3^3$, which is the unique nontrivial element of the E_1–term of the target DMSS in the appropriate tridegree. For degree reasons, $y_2y_3^3$ is not hit by any differential. Therefore, $y_2y_3^3$ survives the target DMSS and it follows that $\mathrm{Ext}^{4,27}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1)) \xrightarrow{\cong} \mathrm{Ext}^{4,27}_{B(2)_*}(\mathrm{H}_*(A_1)) \cong \mathbb{F}_2$. □

Set

$$\begin{aligned} M &= [\xi_2|1|y_2^2] + [\xi_1^3|1|y_2^2] + [\xi_1|1|y_3^2], \\ N &= [1|y_2y_3^3|a_0] + [1|y_2^2y_3^2|a_1] + [1|y_2^3y_3|a_2] + [1|y_2^4|a_3]. \end{aligned}$$

We need to show that MN, which is a (d_v+d_h)–cocycle, represents a nontrivial class in $\mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1))$. First, MN is an element in $B(2)_* \otimes F_2 \otimes S^6 \otimes \mathrm{H}_*(A_1)$ and $d_v(MN) = 0$. This means that MN represents a class in $\mathrm{Ext}^{1,42}_{B(2)_*}(F_2 \otimes S^6 \otimes \mathrm{H}_*(A_1))$, which is trivial because, by the change-of-rings theorem, $\mathrm{Ext}^{*,*}_{B(2)_*}(\mathbb{F}_2, F_2 \otimes S \otimes \mathrm{H}_*(A_1))$ is isomorphic to S, which is concentrated only in cohomological degree 0. There must be an element $P \in F_2 \otimes S^6 \otimes \mathrm{H}_*(A_1)$ such that $d_v(P) = MN$, and so $d_h(P)$ represents the same class in $\mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1))$ as MN does.

The following technical lemma is essential in proving Proposition 3.3.7, the key result of this section. We recall the values of $\lambda_{i,j}$ introduced in Proposition 3.2.2: $\lambda_{1,0} = \lambda_{0,1} = 1$ and $\lambda_{0,0} = \lambda_{1,1} = 0$.

Lemma 3.3.5 *If we express P in the monomial basis of $B(2) \otimes F_2 \otimes S^6 \otimes \mathrm{H}_*(A_1)$, then P contains the term $\lambda_{i,j}[1|x_2|y_2^6|a_0]$, ie*

$$P = \lambda_{i,j}[1|x_2|y_2^6|a_0] + \cdots .$$

Proof The product MN contains the term $[\xi_2|1|y_2^6|a_3]$. One can check that P must contain the term $[1|y_2^6|a_6]$, so that $d_v(P)$ contains the term $[\xi_2|1|y_2^6|a_3]$. Using the formula for the coaction of $\mathcal{A}(2)_*$ on a_6, one sees that $d_v(P)$ contains the term $\lambda_{i,j}[\xi_2^2|1|y_2^6|a_0]$, which is not a term of MN. In order to compensate for this term, P must contain the term $\lambda_{i,j}[1|x_2|y_2^6|a_0]$. □

Lemma 3.3.6 *A (d_h+d_v)–cycle in $F_2 \otimes S^7 \otimes A_1$ gives rise to a nontrivial class in $\mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1))$ if and only if it contains the term $[1|y_2^7|a_0]$.*

Proof It is shown in the proof of Lemma 3.3.2 that

$$\mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1)) \cong \mathbb{F}_2$$

and that this group arises from

$$\mathrm{Ext}^{0,42}_{B(2)}(F_2 \otimes S^7 \otimes \mathrm{H}_*(A_1)) \cong \mathbb{F}_2\{y_2^7\} \subset S^7.$$

Therefore, by the shearing homomorphism, the only element in $F_2 \otimes S^7 \otimes \mathrm{H}_*(A_1)$ that represents the nontrivial class of $\mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1))$ must contain the term $[1|y_2^7|a_0]$. □

Proposition 3.3.7 *The product* $\alpha e[4,23]$ *is equal to* $\lambda_{i,j} g e[3,15]$.

Proof Note $\alpha e[4,23]$ is nontrivial if and only if $d_h(P)$ represents a nontrivial class in $\mathrm{Ext}^{7,42}_{B(2)_*}(\mathrm{H}_*(A_1))$. Lemma 3.3.5 shows that $d_h(P)$ contains the term $\lambda_{i,j}[1|y_1^7|a_0]$. Hence, Lemma 3.3.6 concludes the proof. □

The product between $\beta \in \mathrm{Ext}^{3,18}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ and $e[3,15] \in \mathrm{Ext}^{3,18}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is easier because both have σ–filtration 0 in the Davis–Mahowald spectral sequence.

Proposition 3.3.8 $$\beta e[3,15] = e[6,30].$$

Proof The class β is represented by $y_2^3 + y_1 y_3^2$ in R^3 and $e[3,15]$ is represented by $[y_2^3 + y_1 y_3^2|a_0]$ in $R^3 \otimes A_1$. So the product $\beta e[3,15]$ is represented by $[y_2^6 + y_1^2 y_3^4|a_0]$, which represents $e[6,30]$ by Theorem 3.2.5. □

4 Partial study of the Adams spectral sequence for tmf $\wedge A_1$

In this section, we establish some differentials in the ASS for $\mathrm{tmf} \wedge A_1$ and a global structure of $\pi_*(\mathrm{tmf} \wedge A_1)$. This is essential information, allowing us to run the homotopy fixed-point spectral sequence in the next section.

Recall that the ASS for $\mathrm{tmf} \wedge A_1$ which has E_2–term isomorphic to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is a spectral sequence of modules over that for tmf, whose E_2–term is isomorphic to $\mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$. We first recollect some known properties of the ASS for tmf; see [24]. Recall that $\alpha, g, w_2, \beta \in \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ are detected by $\alpha_{1,14,2}$, $\alpha^2_{0,12,2}$, v_2^8 and $\alpha_{0,18,3}$ in the DMSS.

Theorem 4.0.1 (i) *The class* $g \in \mathrm{Ext}^{4,24}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$ *is a permanent cycle detecting the image of* $\bar{\kappa} \in \pi_{20}(S^0)$ *via the Hurewicz map* $S^0 \to \mathrm{tmf}$.

(ii) *There is the d_2–differential in the Adams spectral sequence for* tmf

$$d_2(w_2) = g\beta\alpha.$$

(iii) *There is the d_3–differential in the Adams spectral sequence for* tmf

$$d_3(w_2^2(v_2^4\eta)) = g^6.$$

(iv) *The class $\Delta^8 := w_2^4$ survives the Adams spectral sequence.*

Proposition 4.0.2 *In the ASS for* $\mathrm{tmf} \wedge A_1$*, there exists $\lambda \in \mathbb{F}_2$ such that the following statements are equivalent*:

(i) $d_2(w_2 e[4,23]) = \lambda g^2 e[6,30]$.

(ii) $d_2(w_2 e[9,48]) = \lambda g^4 e[3,15]$.

(iii) $d_2(w_2 e[10,53]) = \lambda g^5 e[0,0]$.

(iv) $d_2(w_2 e[7,38]) = \lambda g^4 e[1,5]$.

Proof We will prove that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). The charts of Figure 13 will make the proof easier to follow. First, we observe that all of the classes $e[4,23]$, $e[7,38]$, $e[9,48]$ and $e[10,53]$ are permanent cycles, by sparseness.

(i) $\Longrightarrow$ (ii) Suppose that

$$d_2(w_2 e[4,23]) = g^2 e[6,30].$$

Then

$$d_2(g^2 w_2 e[4,23]) = g^4 e[6,30],$$

by g–linearity. It follows that there is no room for a nontrivial differential on $w_2^2 e[3,15]$. In order words, $w_2^2 e[3,15]$ is a permanent cycle. Because of Theorem 4.0.1(iii), a g^k–multiple of $w_2^2 e[3,15]$ must be hit by a differential for some k less than 7. One can check that the only possibility is that

$$d_2(w_2^3 e[9,48]) = g^4 w_2^2 e[3,15].$$

Since w_2^2 is a d_2–cycle in the ASS for tmf, this differential implies that

$$d_2(w_2 e[9,48]) = g^4 e[3,15].$$

(ii) $\Longrightarrow$ (iii) Suppose that

$$d_2(w_2 e[9,48]) = g^4 e[3,15].$$

Then the class $w_2^2 e[0,0]$ is a permanent cycle, by sparseness. Again, a g^k–multiple of $w_2^2 e[0,0]$ for some k smaller than 7 must be hit by a differential. Inspection shows that the classes $w_2^3 e[10,53]$ and $w_2^4 e[1,5]$ are the only ones that have the appropriate bidegree to support such a differential. However, $w_2^4 e[1,5]$ is a permanent cycle, because w_2^4 and $e[1,5]$ are permanent cycles in their respective ASS. Thus,

$$d_2(w_2 e[10,53]) = g^5 e[0,0].$$

(iii) $\Longrightarrow$ (iv) Suppose that

$$d_2(w_2 e[10,53]) = g^5 e[0,0].$$

Then the class $w_2^2 e[1,5]$ is a permanent cycle, as there is no room for a nontrivial differential on it. Then $g^k w_2^2 e[1,5]$ must be hit by a differential for some k less than 7. Inspection shows that the only possibility is that

$$d_2(w_2^3 e[7,38]) = g^4 w_2^2 e[1,5].$$

As w_2^2 is a d_2–cycle, it follows that

$$d_2(w_2 e[7,38]) = g^4 e[1,5].$$

(iv) $\Longrightarrow$ (i) Suppose that

$$d_2(w_2 e[7,38]) = g^4 e[1,5].$$

By g–linearity,

$$d_2(g w_2 e[7,38]) = g^5 e[1,5].$$

Then, by sparseness, $w_2^2 e[6,30]$ is a permanent cycle. Then the class $g^k w_2^2 e[6,30]$ is hit by a differential for some k less than 7. Inspection shows that the only possibility is that

$$d_2(w_2^3 e[4,23]) = g^2 w_2^2 e[6,30].$$

Therefore,

$$d_2(w_2 e[4,23]) = g^2 e[6,30],$$

by w_2^2–linearity. □

Theorem 4.0.3 *In the Adams spectral sequence for* $\mathrm{tmf} \wedge A_1[ij]$, *there are the following differentials* d_2:

(i) $d_2(w_2 e[4,23]) = \lambda_{i,j} g^2 e[6,30]$.

(ii) $d_2(w_2 e[9,48]) = \lambda_{i,j} g^4 e[3,15]$.

(iii) $d_2(w_2 e[10,53]) = \lambda_{i,j} g^5 e[0,0]$.

(iv) $d_2(w_2 e[7,38]) = \lambda_{i,j} g^4 e[1,5]$.

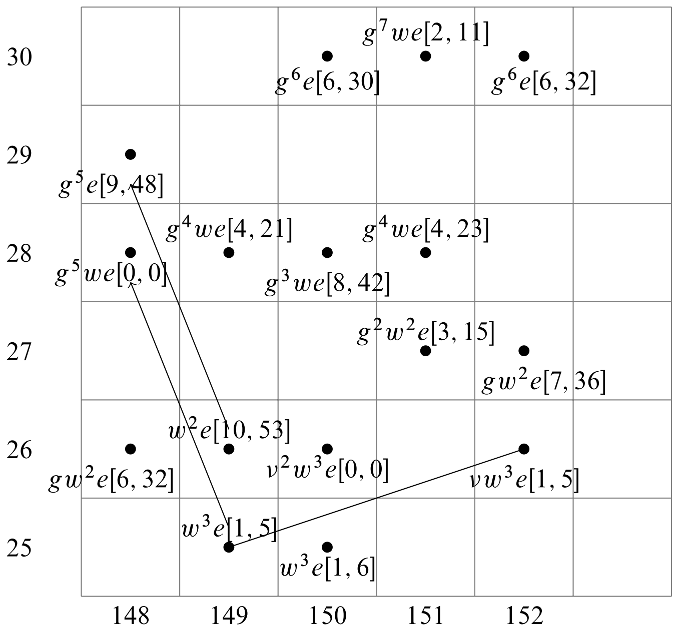


Figure 12: The Adams spectral sequence in the range $148 \le t-s \le 152$.

Proof By the Leibniz rule and Theorem 4.0.1(ii),

$$d_2(w_2e[4,23]) = d_2(w_2)e[4,23] = g\beta\alpha e[4,23] = \lambda_{i,j}g^2e[6,30],$$

where the last equality follows from Propositions 3.3.7 and 3.3.8. Thus, the theorem follows from Proposition 4.0.2. □

Remark 4.0.4 The homotopy group $\pi_*(\mathrm{tmf} \wedge A_1)$ is a module over $\pi_*(\mathrm{tmf})$; in particular, it is a module over $\mathbb{Z}[\bar{\kappa}]$. Since $\bar{\kappa}$ is detected by g in the ASS, Theorem 4.0.3 expresses a connection between the module structure of $\mathrm{H}_*(A_1)$ over the Steenrod algebra and the nilpotence order of the action of $\bar{\kappa}$ on $\pi_*(\mathrm{tmf} \wedge A_1)$. This is essential information in the determination of differentials in the HFPSS for $E_C^{hG_{24}} \wedge A_1$.

Proposition 4.0.5 *There are the d_3–differentials in the Adams spectral sequence for* $\mathrm{tmf} \wedge A_1$ (see Figure 12)

$$d_3(w_2^2e[10,53]) = g^5e[9,48] \quad \textit{and} \quad d_3(w_2^3e[1,5]) = g^5w_2e[0,0].$$

Proof By sparseness of the ASS for $\mathrm{tmf} \wedge A_1$ (see Figure 13), $e[9,48]$ and $we[0,0]$ are permanent cycles. Then $g^le[9,48]$ and $g^kwe[0,0]$ must be targets of some differentials

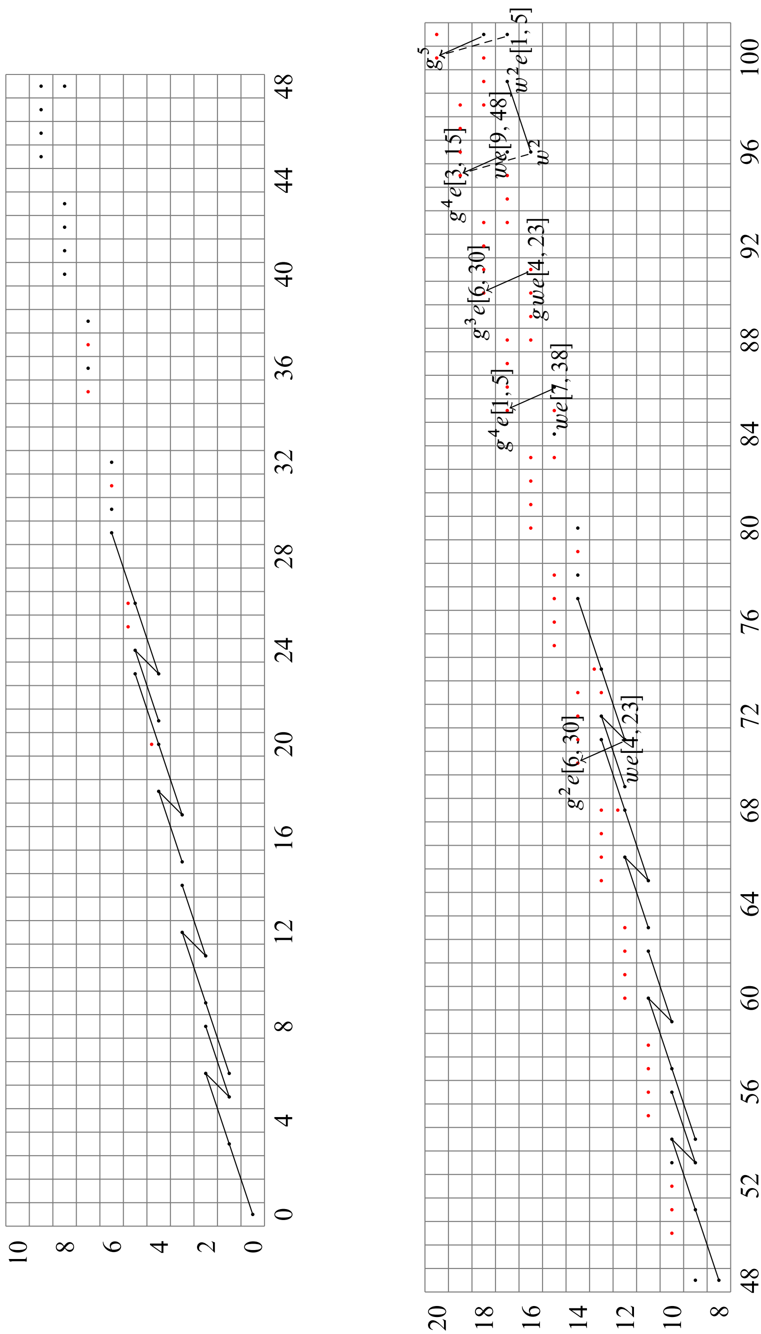


Figure 13: Adams spectral sequence for A_1 in the range $0 \le t-s \le 48$ (left) and $48 \le t-s \le 101$ (right). The arrows in bold are differentials for the models $A_1[10]$ and $A_1[01]$ and the dashed arrows for the models $A_1[00]$ and $A_1[11]$.

for some l and k less than 7. Inspection of the E_2–term shows that either

$$d_2(w_2^2 e[10,53]) = g^5 we[0,0] \quad \text{and} \quad d_4(w_2^3 e[1,5]) = g^5 e[9,48]$$

or

$$d_3(w_2^2 e[10,53]) = g^5 e[9,48] \quad \text{and} \quad d_3(w_2^3 e[1,5]) = g^5 w_2 e[0,0].$$

However, the former possibility is ruled out because of the Leibniz rule:

$$d_2(w_2^2 e[10,53]) = d_2(w_2^2) e[10,53] = 2w_2 d_2(w_2) e[10,53] = 0,$$

where the first equality follows from the fact that $e[10,53]$ is a permanent cycle, by spareness. □

Corollary 4.0.6 *The Toda bracket* $\langle \bar{\kappa}^5, e[9,48], \nu \rangle$, *where* $\bar{\kappa} \in \pi_*(S^0)$, $e[9,48] \in \pi_*(\mathrm{tmf} \wedge A_1)$ *and* $\nu \in [\mathrm{tmf} \wedge A_1, \mathrm{tmf} \wedge A_1]_*$, *can be formed and contains only multiples of* $\bar{\kappa}$ *in* $\pi_*(\mathrm{tmf} \wedge A_1)$.

For references on the Toda bracket, see [41; 26].

Proof In the E_4–term of the ASS, the Massey product $\langle g^5, e[9,48], \nu \rangle$ has cohomological filtration 27 and is equal to zero with zero indeterminacy. On the other hand, the corresponding Toda bracket can be formed with indeterminacy containing only multiples of $\bar{\kappa}$. We can check that all conditions of Moss's convergence theorem [30, Theorem 1.2] are met. It follows that the Toda bracket $\langle g^5, e[9,48], \nu \rangle$ contains an element detected in filtration 27 by 0, thus is a multiple of $\bar{\kappa}$. Therefore, this Toda bracket contains only multiples of $\bar{\kappa}$. □

Finally, we need to have control of the action of the class $\Delta^8 = w_2^4 \in \mathrm{Ext}_{\mathcal{A}(2)_*}^{32,224}(\mathbb{F}_2)$ on the E_∞–term of the ASS for $\mathrm{tmf} \wedge A_1$. This will allow us to compare $\pi_*(\mathrm{tmf} \wedge A_1)$ with $\pi_*(E_C^{hG_{24}} \wedge A_1)$ (see Corollary 5.1.3) and hence to discuss higher differentials in the HFPSS for $E_C^{hG_{24}} \wedge A_1$.

Proposition 4.0.7 *The class* w_2^4 *acts freely on the* E_∞*–term of the ASS for* $\mathrm{tmf} \wedge A_1$. *As a consequence, the element* $\Delta^8 \in \pi_{192}(\mathrm{tmf})$ *acts freely on the homotopy groups of* $\mathrm{tmf} \wedge A_1$.

Proof Using the description of the E_2–term of the ASS for $\mathrm{tmf} \wedge A_1$ in Theorem 3.2.5 and an elementary bidegree inspection, we can see that, if a class y is in an appropriate bidegree to support a differential hitting a class of the form $w_2^4 x$ for some class x, then

y is divisible by w_2^4. Knowing that w_2^4 is a permanent cycle in the ASS for tmf, we conclude that, if a class x survives the E_r–term, then the multiples of x by all powers of w_2^4 also survive that term. Therefore, the proposition follows by induction. □

Proposition 4.0.8 *For every element* $x \in \pi_*(\mathrm{tmf} \wedge A_1)$, *the element* $\Delta^8 x$ *is divisible by* $\bar{\kappa}$ *(resp.* ν*) if and only if* x *is divisible by* $\bar{\kappa}$ *(resp.* ν*).*

Proof The argument is similar to that used in the proof of Proposition 4.0.7. A bidegree inspection shows that, if a class $y \in \mathrm{Ext}^{*,*}_{\mathcal{A}(2)_*}(\mathrm{H}_*(A_1))$ is in an appropriate bidegree whose (exotic) product with g (resp. ν) might detect $\Delta^8 x$, then y is divisible by w_2^4. We conclude the proof by using the fact that the class w_2^4 acts freely on the ASS for $\mathrm{tmf} \wedge A_1$, by Proposition 4.0.7. □

5 The homotopy fixed-point spectral sequence for $E_C^{hG_{24}} \wedge A_1$

5.1 Preliminaries and recollection on cohomology of G_{24}

Theorem 5.1.1 *There is a homotopy equivalence*

$$[(\Delta^8)^{-1}]\mathrm{tmf} \wedge A_1 \simeq (E_C^{hG_{24}})^{h\mathrm{Gal}} \wedge A_1,$$

where Gal *denotes the Galois group* $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$.

Proof We have

$$\begin{aligned}
[(\Delta^8)^{-1}]\mathrm{tmf} \wedge A_1 &\simeq \mathrm{TMF} \wedge A_1 && \text{(by (7))}\\
&\simeq L_2(\mathrm{TMF}) \wedge A_1 && \text{(TMF is } E(2)\text{–local)}\\
&\simeq L_2(\mathrm{TMF} \wedge A_1) && (A_1 \text{ is a finite complex})\\
&\simeq L_{K(2)}(\mathrm{TMF}) \wedge A_1 &&\\
&\simeq (E_C^{hG_{24}})^{h\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \wedge A_1 && \text{(by (6))}.
\end{aligned}$$

The fourth equivalence is Lemma 7.2 of [25] applied to the $K(2)$–localization and A_1, which is a finite spectrum of type 2. □

Corollary 5.1.2 *There is a homotopy equivalence*

$$\mathrm{Gal}_+ \wedge [(\Delta^8)^{-1}]\mathrm{tmf} \wedge A_1 \simeq E_C^{hG_{24}} \wedge A_1.$$

Therefore,

$$\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} (\Delta^8)^{-1}(\pi_*(\mathrm{tmf} \wedge A_1)) \cong \pi_*(E_C^{hG_{24}} \wedge A_1).$$

Proof This is a consequence of Theorem 5.1.1 and [11, Lemma 1.37]. □

Let

$$\Theta\colon \mathbb{W}(\mathbb{F}_4)\otimes_{\mathbb{Z}_2}\pi_*(\mathrm{tmf}\wedge A_1)\to \pi_*(E_C^{hG_{24}}\wedge A_1) \tag{18}$$

be given by precomposing the isomorphism of Corollary 5.1.2 with the natural homomorphism $\pi_*(\mathrm{tmf}\wedge A_1)\to\pi_*([(\Delta^8)^{-1}]\mathrm{tmf}\wedge A_1)$. The following corollary recapitulates the relationship between $\pi_*(\mathrm{tmf}\wedge A_1)$ and $\pi_*(E_C^{hG_{24}}\wedge A_1)$.

Corollary 5.1.3 *The homomorphism Θ is injective. Moreover, it remains injective after quotienting out by the ideal of $\pi_*(S^0)$ generated by $(\bar{\kappa},\nu)$.*

Proof This follows from Theorem 5.1.1 and Propositions 4.0.7 and 4.0.8. □

We continue to recollect some necessary information about the HFPSS converging to $\pi_*(E_C^{hG_{24}})$:

$$\mathrm{H}^s(G_{24},(E_C)_t)\Rightarrow \pi_{t-s}(E_C^{hG_{24}}). \tag{19}$$

The elements $\eta\in\pi_1(S^0)$, $\nu\in\pi_3(S^0)$ and $\bar{\kappa}\in\pi_{20}(S^0)$ are sent nontrivially to elements of the same name in $\pi_*(E_C^{hG_{24}})$ via the Hurewicz map $S^0\to E_C^{hG_{24}}$. As the latter factors through the unit map of tmf, the element $\bar{\kappa}^6=0$ in $\pi_*(E_C^{hG_{24}})$ because $\bar{\kappa}^6=0$ in $\pi_*(\mathrm{tmf})$ (see [3, Section 8.3, page 36]). These elements are detected by $\eta\in\mathrm{H}^1(G_{24},(E_C)_2)$, $\nu\in\mathrm{H}^1(G_{24},(E_C)_4)$ and $\bar{\kappa}\in\mathrm{H}^4(G_{24},(E_C)_{24})$, respectively. Furthermore, there is a class $\Delta\in\mathrm{H}^0(G_{24},(E_C)_{24})$ such that Δ^8 is a permanent cycle detecting the periodicity of $E_C^{hG_{24}}$.

The HFPSS for $E_C^{hG_{24}}\wedge A_1$ is a spectral sequence of modules over that of (19),

$$\mathrm{H}^s(G_{24},(E_C)_tA_1)\Rightarrow \pi_{t-s}(E_C^{hG_{24}}\wedge A_1). \tag{20}$$

In Section 5.2, we will compute $\mathrm{H}^*(G_{24},(E_C)_*A_1)$ as a module over a certain subalgebra of $\mathrm{H}^*(G_{24},(E_C)_*)$. Let $\pi:(E_C)_*\to\mathbb{F}_4[u^{\pm1}]$ be the quotient of $(E_C)_*$ by the maximal ideal $(2,u_1)$. As the ideal $(2,u_1)$ is preserved by the action of $\mathbb{S}_C$, the ring $\mathbb{F}_4[u^{\pm1}]$ inherits an action of $\mathbb{S}_C$, and so of its subgroup G_{24}. We need the computation of the ring structure of $\mathrm{H}^*(G_{24},\mathbb{F}_4[u^{\pm1}])$, which is due to Hans-Werner Henn; see [6, Appendix A].

Proposition 5.1.4 *There are classes $z\in\mathrm{H}^4(G_{24},\mathbb{F}_4[u^{\pm1}]_0)$, $a\in\mathrm{H}^1(G_{24},\mathbb{F}_4[u^{\pm1}]_2)$, $b\in\mathrm{H}^1(G_{24},\mathbb{F}_4[u^{\pm1}]_4)$ and $v_2\in\mathrm{H}^0(G_{24},(\mathbb{F}_4[u^{\pm1}])_6)$ such that there is an isomorphism of graded algebras*

$$\mathrm{H}^*(G_{24},\mathbb{F}_4[u^{\pm1}])\cong\mathbb{F}_4[v_2^{\pm1},z,a,b]/(ab,b^3=v_2a^3).$$

Proposition 5.1.5 *The homomorphism of graded algebras*

$$\mathrm{H}^*(G_{24}, E^*_{C*}) \to \mathrm{H}(G_{24}, \mathbb{F}_4[u^{\pm 1}])$$

induced by the projection $(E_C)_* \to \mathbb{F}_4[u^{\pm 1}]$ *sends* η *to* a, ν *to* b, $\bar{\kappa}$ *to* $v_2^4 z$ *and* Δ *to* v_2^4.

5.2 On the cohomology groups $\mathrm{H}^*(G_{24}, (E_C)_*(A_1))$

We first determine $(E_C)_*(A_1)$ using the cofiber sequences through which A_1 are defined. The cofiber sequence $\Sigma S^0 \xrightarrow{\eta} S^0 \to C_\eta$ gives rise to a short exact sequence of E_C–homology,

$$0 \to (E_C)_* \to (E_C)_*(C_\eta) \to (E_C)_*(S^2) \to 0,$$

since $(E_C)_*$ is concentrated in even degrees. Hence, as an $(E_C)_*$–module,

$$(E_C)_*(C_\eta) \cong \mathbb{W}(\mathbb{F}_4)[\![u_1]\!][u^{\pm 1}]\{e_0, e_2\},$$

where e_0 is the image of $1 \in (E_C)_0$ and e_2 is a lift of $\Sigma^2 1 \in (E_C)_2(S^2)$. Next, the long exact sequence in E_C–homology associated to $C_\eta \xrightarrow{2} C_\eta \to Y$ is the short exact sequence of $(E_C)_*[G_{24}]$–modules

$$0 \to (E_C)_*(C_\eta) \xrightarrow{\times 2} (E_C)_*(C_\eta) \to (E_C)_*(Y) \to 0$$

since multiplication by 2 on $(E_C)_*(C_\eta) \cong \mathbb{W}(\mathbb{F}_4)[\![u_1]\!][u^{\pm 1}]\{e_0, e_2\}$ is injective. Therefore, as an E_*–module,

$$(E_C)_*(Y) \cong \mathbb{F}_4[\![u_1]\!][u^{\pm 1}]\{e_0, e_2\}.$$

Now A_1 is the cofiber of a v_1–self-map of Y, $\Sigma^2 Y \xrightarrow{v_1} Y \to A_1$. The following lemma describes the induced homomorphism in E_C–homology of these v_1–self-maps:

Lemma 5.2.1 *The image of the homomorphism* $(E_C)_*(v_1)$ *is* $(u_1 u^{-1})(E_C)_*(Y)$. *Therefore, as an* $(E_C)_*[G_{24}]$*–module,*

$$(E_C)_*(A_1) \cong (E_C)_*(C_\eta)/(2, u_1) \cong \mathbb{F}_4[u^{\pm 1}]\{e_0, e_2\}.$$

Proof Let $K(1)$ be the first Morava K–theory at the prime 2 such that $K(1)_* \cong \mathbb{F}_2[v_1^{\pm 1}]$, where $|v_1| = 2$, and let BP be the Brown–Peterson spectrum at the prime 2. There is a map of ring spectra $\mathrm{BP} \to K(1)$ that classifies the complex orientation of $K(1)$. Recall that the coefficient ring of BP is given by

$$\mathrm{BP}_* \cong \mathbb{Z}_{(2)}[v_1, v_2, \dots],$$

where $|v_i| = 2(2^i - 1)$; see [1, Part II]. The induced homomorphism of coefficient rings sends v_1 to v_1. Let $c\colon \mathrm{BP} \to E_C$ be the map of ring spectra that classifies the 2–typification of the formal group law of E_C. One can show that the 2–series of the latter has leading term $u_1 u^{-1} x^2$ modulo 2; see [6, Proposition 6.1.1]. This implies that the induced homomorphism $c_*\colon \mathrm{BP}_* \to (E_C)_*$ sends v_1 to $u_1 u^{-1}$ modulo 2. Similarly to the calculation of $(E_C)_*(Y)$,

$$\mathrm{BP}_*(Y) \cong \mathrm{BP}_*/2\{e_0, e_2\},$$

where e_0 and e_2 are chosen to be lifts of e_0 and e_2 via the map $c_*\colon \mathrm{BP}_*(Y) \to (E_C)_*(Y)$. It is also straightforward that

$$K(1)_*(Y) \cong \mathbb{F}_2[v_1^{\pm 1}]\{e_0, e_1, e_2, e_3\},$$

where $|e_i| = i$ and e_0 and e_2 can be chosen to be the images of e_0 and e_2 via the orientation map $\mathrm{BP} \to K(1)$. By definition, a v_1–self-map of Y induces an isomorphism of $K(1)_*$–modules on $K(1)$–homology. This means in particular that

$$K(1)_*(v_1)\begin{pmatrix} e_0 \\ e_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 v_1 & \epsilon_2 \\ \epsilon_3 v_1^2 & \epsilon_4 v_1 \end{pmatrix}\begin{pmatrix} e_0 \\ e_2 \end{pmatrix},$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{F}_2$ satisfy $\epsilon_1\epsilon_4 - \epsilon_2\epsilon_3 = 1$. The map $\mathrm{BP} \to K(1)$ gives rise to the commutative diagram

$$\begin{array}{ccc} \mathrm{BP}_*(\Sigma^2 Y) & \xrightarrow{\mathrm{BP}_*(v_1)} & \mathrm{BP}_*(Y) \\ \downarrow & & \downarrow \\ K(1)_*(\Sigma^2 Y) & \xrightarrow{K(1)_*(v_1)} & K(1)_*(Y) \end{array}$$

which forces, for degree reasons, that $\mathrm{BP}_*(v_1)$ is given by

$$\mathrm{BP}_*(v_1)\begin{pmatrix} e_0 \\ e_2 \end{pmatrix} = \begin{pmatrix} \epsilon_1 v_1 & \epsilon_2 \\ \epsilon_3 v_1^2 & \epsilon_4 v_1 \end{pmatrix}\begin{pmatrix} e_0 \\ e_2 \end{pmatrix}.$$

By taking into account the fact that $\mathrm{BP}_*(v_1)$ is a map of $\mathrm{BP}_*\mathrm{BP}$–comodules, ϵ_2 must be equal to 0, and hence $\epsilon_1 = \epsilon_4 = 1$. Using the orientation map $c\colon \mathrm{BP} \to E_C$, $(E_C)_*(v_1)$ is given by

$$(E_C)_*(v_1)\begin{pmatrix} e_0 \\ e_2 \end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ \epsilon_3 v_1^2 & v_1 \end{pmatrix}\begin{pmatrix} e_0 \\ e_2 \end{pmatrix}.$$

In particular, the image of $(E_C)_*(v_1)$ is the cyclic $(E_C)_*$submodule $(v_1)(E_C)_*(Y)$. □

The cohomology group $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm1}]\{e_i\})$ for $i \in \{0,2\}$ is free of rank one as a module over $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm1}])$. For $i \in \{0,2\}$, we choose the generators $e[0,i] \in \mathrm{H}^0(G_{24}, (\mathbb{F}_4[u^{\pm1}]\{e_i\})_i)$ of these modules.

Lemma 5.2.2 (a) *$E_{C*}(A_1)$ sits in the short exact sequence of G_{24}–modules*

$$0 \to \mathbb{F}_4[u^{\pm1}]\{e_0\} \to E_{C*}(A_1) \to \mathbb{F}_4[u^{\pm1}]\{e_2\} \to 0.$$

(b) *The induced connecting homomorphism of the above short exact sequence,*

$$\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm1}]\{e_2\}) \xrightarrow{\delta} \mathrm{H}^{*+1}(G_{24}, \mathbb{F}_4[u^{\pm1}]\{e_0\}),$$

is $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm1}])$–linear and sends $e[0,2]$ to $ae[0,0]$ up to a unit of $\mathbb{F}_4$, where $a \in \mathrm{H}^1(G_{24}, (\mathbb{F}_4[u^{\pm1}])_2)$.

Proof Part (a) is due to the fact that e_0 is G_{24}–invariant.

For part (b), since the ideal $(2, u_1)$ of $(E_C)_*$ is G_{24}–invariant, by taking the quotient by this ideal, we obtain the homomorphism of short exact sequences of G_{24}–modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (E_C)_*\{e_0\} & \longrightarrow & (E_C)_*(C_\eta) & \longrightarrow & (E_C)_*\{e_2\} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{F}_4[u^{\pm1}]\{e_0\} & \longrightarrow & (E_C)_*(A_1) & \longrightarrow & \mathbb{F}_4[u^{\pm1}]\{e_2\} & \longrightarrow & 0 \end{array}$$

Since the homomorphism $\mathrm{Ext}^{1,2}_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, \mathrm{BP}_*) \to \mathrm{H}^1(G_{24}, (E_C)_2)$ sends η to the class of the same name, the connecting homomorphism of the upper SES sends $e_2 \in \mathrm{H}^0(G_{24}, (E_C)_2\{e_2\})$ to $\eta e_0 \in \mathrm{H}^1(G_{24},)$. By naturality, the connecting homomorphism of the lower SES sends $e_2 \in \mathrm{H}^0(G_{24}, \mathbb{F}_4[u^{\pm1}]\{e_2\})$ to $ae_0 \in H^1(G_{24}, \mathbb{F}_4[u^{\pm1}]\{e_0\})$ because $\pi_* \colon \mathrm{H}^1(G_{24}, (E_C)_2) \to \mathrm{H}^1(G_{24}, \mathbb{F}_4[u^{\pm1}])$ sends η to a.

That δ is $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm1}])$–linear follows from the fact that the SES in (a) is a sequence of $\mathbb{F}_4[u^{\pm1}][G_{24}]$ and splits as a short exact sequence of $\mathbb{F}_4[u^{\pm1}]$–modules (see [12, Section V.3]). □

Using the description of $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm1}])$ and the long exact sequence associated to the short exact sequence of Lemma 5.2.2, we obtain the following description of $\mathrm{H}^*(G_{24}, (E_C)_*(A_1))$:

Proposition 5.2.3 *As a module over $\mathrm{H}^*(G_{24}, \mathbb{F}_4[u^{\pm}])$, there is an isomorphism*

$$\mathrm{H}^*(G_{24}, (E_C)_*(A_1)) \cong \mathbb{F}_4[v_2^{\pm1}, z, a, b]/(a, b^3)\{e[0,0], e[1,5]\},$$

where $e[0,0] \in \mathrm{H}^0(G_{24}, (E_C)_0(A_1))$ and $e[1,5] \in \mathrm{H}^1(G_{24}, (E_C)_6(A_1))$ (see Figure 14).

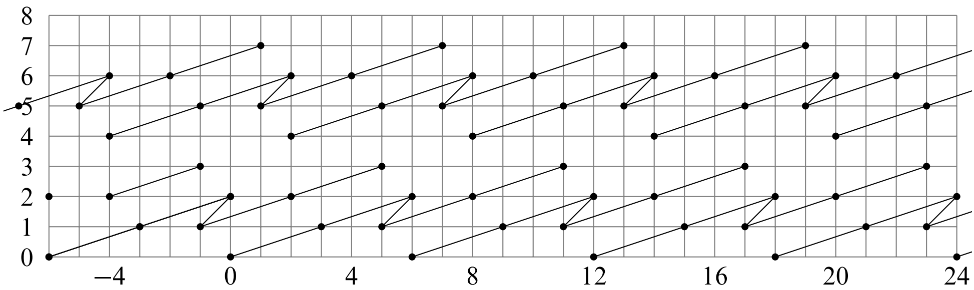


Figure 14: $\mathrm{H}^s(G_{24},(E_C)_t(A_1))$ depicted in the coordinate $(s,t-s)$.

The above proposition also gives the action of $\mathrm{H}^*(G_{24},(E_C)_*)$ on $\mathrm{H}^*(G_{24},(E_C)_*A_1)$. In fact, the action of E_{C*} on $E_{C*}(A_1)$ factors through $\mathbb{F}_4[u^{\pm 1}]$ via $E_{C*} \xrightarrow{\pi} \mathbb{F}_4[u^{\pm 1}]$. As a consequence, the action of $\mathrm{H}^*(G_{24},E_{C*})$ on $\mathrm{H}^*(G_{24},E_{C*}(A_1))$ factors through the induced homomorphism in cohomology of G_{24}. In particular, it follows from Proposition 5.1.5 that the classes Δ, $\bar{\kappa}$ and ν act on $\mathrm{H}^*(G_{24},E_{C*}(A_1))$ as v_2^4, $v_2^4 z$ and b do, respectively.

5.3 Differentials of the homotopy fixed-point spectral sequence for $E_C^{hG_{24}} \wedge A_1$

The HFPSS for $E_C^{hG_{24}} \wedge A_1$ has the following features. The spectrum $E_C \wedge A_1$ is a G_{24}–E_C–module in the sense that $E_C \wedge A_1$ is an E_C–module and the structure maps are G_{24}–equivariant. This guarantees that the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is a module over that for $E_C^{hG_{24}}$. In particular, all differentials are $\bar{\kappa}$–linear. This element plays a central role here: the group G_{24} is a group with periodic cohomology (see [12, Chapter VI, Theorem 9.5]) and the cohomological periodicity class $z \in H^4(G_{24},\mathbb{Z})$ is related to $\bar{\kappa} \in \mathrm{H}^4(G_{24},(E_C)_*)$ via the equation

$$\Delta z = \bar{\kappa}.$$

Since Δ is invertible in $\mathrm{H}^*(G_{24},(E_C)_*)$, $\bar{\kappa}$ plays the same role for cohomology of $(E_C)_*[G_{24}]$–modules as z for cohomology of $\mathbb{Z}[G_{24}]$–modules. This means that, if M is an $(E_C)_*[G_{24}]$, then multiplication by $\bar{\kappa}$ induces an homomorphism on $H^s(G_{24},M) \to H^{s+4}(G_{24},M)$, which is a surjection for $s \geq 0$ and a bijection for $s > 0$. We will define a cohomology class with this property a cohomological periodicity class. These features of $\bar{\kappa}$ induce more constrains on the HFPSS.

Definition 5.3.1 Let R be a ring spectrum and G be a finite group acting on R by maps of ring spectra. The pair (G,R) is said to be regular if G is a group with periodic cohomology and there exists a cohomological periodicity class $t \in \mathrm{H}^*(G, R_*)$ which is a permanent cycle in the HFPSS for R^{hG}.

Proposition 5.3.2 *Let* (G, R) *be a regular pair as in Definition 5.3.1 and* X *be a* G*–*R *spectrum. Suppose* $t \in \mathrm{H}^k(G, R_*)$ *is a cohomological periodicity class which is a permanent cycle in the HFPSS for* R^{hG}*. Then the* E_r*–term of the HFPSS for* X^{hG} *has the following properties*:

(i) *All classes of cohomological filtration at least* k *are divisible by* t*.*

(ii) *All classes of cohomological filtration at least* r *are* t*–free.*

Proof We will prove by induction on r that the E_r–term of the HFPSS for X^{hG} has the properties (i)–(ii). The E_2–term is isomorphic to $\mathrm{H}^*(G, \pi_*(X))$. We recall that the natural map from the cohomology to the Tate cohomology $\iota\colon \mathrm{H}^s(G, \pi_*(X)) \to \widehat{\mathrm{H}}^s(G, \pi_*(X))$ is an epimorphism and is an isomorphism when $s > 0$; see Chapter VI of [12]. Because G has periodic cohomology, we have

$$\widehat{\mathrm{H}}^*(G, \pi_* X) \cong \widehat{\mathrm{H}}^*(G, \pi_* X)[t^{-1}],$$

which means that the group $\widehat{\mathrm{H}}^*(G, \pi_* X)$ is t–free and is divisible by t. Since $\iota\colon \mathrm{H}^s(G, \pi_* X) \to \widehat{\mathrm{H}}^s(G, \pi_*(X))$ is an isomorphism when $s > 0$, all classes of positive cohomological degree of $\mathrm{H}^*(G, \pi_* X)$ are t–free.

Now suppose x is a class of $\mathrm{H}^s(G, \pi_* X)$ with $s \geq k$. Then the class $t^{-1}\iota(x) \in \widehat{\mathrm{H}}^{s-k}(G, \pi_* X)$ has a preimage $y \in \mathrm{H}^{s-k}(G, \pi_* X)$ (because $s - k \geq 0$), ie

$$\iota(y) = t^{-1}\iota(x).$$

This implies that

$$\iota(ty) = t\iota(y) = \iota(x),$$

and thus, since $s > 0$,

$$ty = x.$$

Thus, the E_2–term has the properties (i)–(ii). Suppose that the E_r–term satisfies (i)–(ii). Let $[x] \in \mathrm{E}_{r+1}$ be a nontrivial class represented by $x \in \mathrm{E}_r$. Suppose that x has its cohomological filtration $s \geq k$. By the induction hypothesis, there exists $y \in \mathrm{E}_r^{s-k,*}$ such that $ty = x$. We show that y is a d_r–cycle. Because x is a d_r–cycle, we have by t–linearity that $t d_r(y) = d_r(ty) = d_r(x) = 0$. However, the cohomological filtration of $d_r(y)$ is at least r, and so it is t–free by the induction hypothesis, and so $d_r(y) = 0$. Therefore, $[x]$ is divisible by t.

Now we prove that E_{r+1} has the property (ii). Suppose that $[x]$ is t–torsion and has cohomological filtration at least $r+1$. Without loss of generality, we can assume that $t[x]=0$. Then there exists $y \in E_r$ such that $d_r(y)=tx$. The cohomological filtration of y is at least $r+1+k-r=k+1$, and hence y is divisible by t, ie there exists $z \in E_r$ such that $tz=y$, and then, by t–linearity,

$$td_r(z)=d_r(tz)=d_r(y)=tx.$$

However, $d_r(z)-x$ has cohomological filtration at least $r+1$, so it must be t–free by hypothesis (ii), and hence is equal to zero, ie $[x]$ is trivial in E_{r+1}.

We conclude that the E_{r+1}–term satisfies (i)–(ii), thus finishing the proof by induction. □

The following corollary summarizes consequences on the structure of the HFPSS:

Corollary 5.3.3 *Let (G,R) be a regular pair and X be a G–R spectrum. Suppose $t \in H^k(G,R_*)$ is a cohomological periodicity class which is a permanent cycle in the HFPSS for R^{hG}. Then we have, in the HFPSS for X^{hG}:*

(1) *At the E_r–term, t–torsion classes are permanent cycles.*
(2) *Any t–free tower is truncated by at most one other t–free tower by the same differential. More precisely, if x is a class of cohomological filtration less than k, then there exists at most one class y of cohomological filtration less than k such that there exists a unique integer l and a unique integer r such that $d_r(t^m y)=t^{m+l}x$ for all nonnegative integers m. Moreover, all classes $t^i x$ for $i \in \{0,1,\dots,m-1\}$ survive the spectral sequence.*
(3) *Suppose some power of t is hit by a differential in the HFPSS for R^{hG}. Then any t–free tower consisting of permanent cycles is truncated by a unique t–free tower. Moreover, the HFPSS has a horizontal vanishing line.*
(4) *Every element of $\pi_*(X^{hG})$ that is detected in filtration at least k is divisible by $\bar{t}$, where $\bar{t}$ is an element of $\pi_*(R^{hG})$ detected by t.*

Proof Part (1) follows from Proposition 5.3.2(ii) because t–torsion classes in the E_r–term have cohomological degree less than r. Part (2) follows from parts (i)–(ii) of Proposition 5.3.2 because a t–tower can be hit by a differential only if it is t–free and then it becomes t–torsion in the next term. Part (3) follows from (2) and the fact that the HFPSS for X is a module over that for R. Part (4) follows from Proposition 5.3.2(i). □

Remark 5.3.4 This situation turns out to be common once the group in question is a group with periodic cohomology. For example, all finite subgroups of $\mathbb{G}_C$ have these properties.

We return to the HFPSS for $E_C^{hG_{24}} \wedge A_1$. We will call the set $\{\bar{\kappa}^l x \mid l \in \mathbb{N}\}$ associated to a class x in some page of the HFPSS the $\bar{\kappa}$–family of that class.

The following proposition gives us the horizontal vanishing line of the HFPSS for $E_C^{hG_{24}} \wedge A_1$:

Proposition 5.3.5 *The HFPSS for $E_C^{hG_{24}} \wedge A_1$ has a horizontal vanishing line of height 23, ie $\mathrm{E}_{24}^{s,t} = 0$ if $s > 23$. As a consequence, it collapses at the E_{24}–term.*

Proof As $\bar{\kappa}^6 = 0$ in $\pi_*(E_C^{hG_{24}})$, the class $\bar{\kappa}^6$ must be hit by a differential which is of length at most 23. This is because $\bar{\kappa}^6$ has cohomological filtration 24 and all even differentials are trivial. Hence $\bar{\kappa}^6$ is trivial in the E_{24}–term of the HFPSS for $E_C^{hG_{24}}$. Next, because the E_{24}–term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is a module over that for $E_C^{hG_{24}}$, the class $\bar{\kappa}^6$ acts trivially on the E_{24}–term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$. Since all classes which are not a multiple of $\bar{\kappa}$ have cohomological filtration at most 3, the HFPSS has the horizontal vanishing line of height 23. □

Proposition 5.3.6 *The following classes are permanent cycles*:

$$e[0,0], \quad e[1,5], \quad e[0,6], \quad e[1,11], \quad e[1,15], \quad e[1,17], \quad e[1,21], \quad e[1,23].$$

Proof Firstly, the class $e[0,0]$ is a permanent cycle because it detects the inclusion $S^0 \to A_1$ into the bottom cell of A_1. Next, we recapitulate, in the following table, the associated graded object with respect to the induced Adams filtration on the groups $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa})$ in the following stems (see Figure 13):

dim	6	15	17	21	23
value	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{F}_2$

By Corollary 5.1.3, the groups $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa})$ in these dimensions must have order twice as big as the respective groups. Inspection in the E_2–term of the HFPSS through dimensions from 0 to 23 and in cohomological filtration less than 4 show that the classes $e[0,6]$, $e[1,15]$, $e[1,21]$ and $e[1,23]$ are permanent cycles.

Note that the groups $\pi_0(\mathrm{tmf} \wedge A_1)$ and $\pi_6(\mathrm{tmf} \wedge A_1)$ are annihilated by η. This means that $e[0,0]$ and $e[0,6]$ detect two elements which are annihilated by η. It follows that the Toda brackets $\langle \nu, \eta, e[0,0]\rangle$ and $\langle \nu, \eta, e[0,6]\rangle$ can be formed with indeterminacy $\nu\pi_2(E_C^{hG_{24}} \wedge A_1)$ and $\nu\pi_8(E_C^{hG_{24}} \wedge A_1)$. By juggling,

$$\eta\langle \nu, \eta, e[0,0]\rangle = \langle \eta, \nu, \eta\rangle e[0,0] = \nu^2 e[0,0],$$
$$\eta\langle \nu, \eta, e[0,6]\rangle = \langle \eta, \nu, \eta\rangle e[0,6] = \nu^2 e[0,6].$$

Observe that $\nu^2 e[0,0]$ and $\nu^2 e[0,6]$ are nontrivial and are detected in cohomological filtration 2. Consequently, both $\langle \nu, \eta, e[0,0]\rangle$ and $\langle \nu, \eta, e[0,6]\rangle$ do not contain zero and are represented by classes in cohomological filtration at most 1. By sparseness, $e[1,5]$ and $e[1,11]$ are permanent cycles detecting $\langle \nu, \eta, e[0,0]\rangle$ and $\langle \nu, \eta, e[0,6]\rangle$, respectively.

The unique nontrivial element of $\pi_{11}(\mathrm{tmf} \wedge A_1)/(\bar{\kappa})$ is annihilated by ν^2. This implies that the class $\nu^2 e[1,11]$ is the target of some differential. Since $\pi_{17}(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa})$ has order at least equal to 4, the class $e[1,17]$ must be a permanent cycle representing the only element in stem 17 of $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa})$. □

Remark 5.3.7 As a memo-technique, we will attach to the above classes the names that retain better their homotopical meaning. The class $e[0,0]$ is the image of the generator of $\pi_0(S^0)$, so it can also be named 1. Next, in $\pi_*(C_\eta)$, if x denotes the generator of $\pi_0(C_\eta) \cong \mathbb{Z}$, then the Toda bracket $\langle \nu, \eta, x\rangle$ has the indeterminacy group $\nu\pi_2(C_\eta)$, which is divisible by 2. Thus, $\langle \nu, \eta, x\rangle$ is well-defined modulo 2. Via the cell inclusion $C_\eta \to A_1$, $\langle \nu, \eta, x\rangle$ is sent to $\langle \nu, \eta, 1\rangle$ which is well-defined. Historically, a choice of representative of $\langle \nu, \eta, x\rangle \in \pi_5(C_\eta)$ is denoted by w. Thus, $e[1,5] = \langle \nu, \eta, 1\rangle$ can be named w. The class $e[0,6]$ is represented by v_2 in the E_2–term. The other classes are products of these in the E_2–term of the HFPSS. Explicitly,

$$e[0,0] = 1, \quad e[1,5] = w, \quad e[0,6] = v_2, \quad e[1,11] = v_2 w,$$
$$e[1,15] = \nu v_2^2, \quad e[1,17] = v_2^2 w, \quad e[1,21] = \nu v_2^3, \quad e[1,23] = v_2^3 w.$$

d_3–differentials

Proposition 5.3.8 *As a module over $\mathbb{F}_4[\Delta^{\pm 1}, \bar{\kappa}, \nu]/(\nu^3)$, the term $\mathrm{E}_2 = \mathrm{E}_3$ is free on the generators*

(21) $$e[0,0], \ e[1,5], \ e[0,6], \ e[1,11], \ e[0,12], \ e[1,17], \ e[0,18], \ e[1,23].$$

Proposition 5.3.9 *The d_3–differential in the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is trivial on all of the generators of (21) with the exception of* (see Figure 15)

(1) $d_3(e[0,12]) = \nu^2 e[1,5]$,
(2) $d_3(e[0,18]) = \nu^2 e[1,11]$.

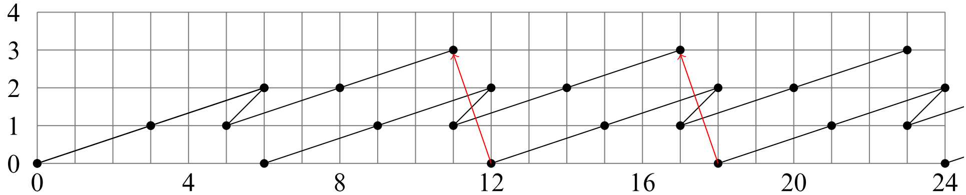


Figure 15: Differentials d_3.

Proof That $e[0,0]$, $e[1,5]$, $e[0,6]$, $e[1,11]$, $e[1,17]$ and $e[1,23]$ are d_3–cycles follows from Proposition 5.3.6. For the two other classes, the proof of Proposition 5.3.6 implies that the elements $\Theta(e[1,5])$ and $\Theta(e[2,11])$, where Θ is the comparison homomorphism from $\pi_*(\mathrm{tmf} \wedge A_1)$ and $\pi_*(E_C^{hG_{24}} \wedge A_1)$ in (18), are detected by $e[1,5]$ and $e[1,11]$, respectively. Moreover, the elements $e[1,5]$ and $e[2,11]$ are annihilated by ν^2 in $\pi_*(\mathrm{tmf} \wedge A_1)$. It follows that, in the HFPSS, the classes $\nu^2 e[1,5]$ and $\nu^2 e[1,11]$ must be hit by some differentials. The only possibilities are $d_3(e[0,12]) = \nu^2 e[1,5]$ and $d_3(e[0,18]) = \nu^2 e[1,11]$. □

Corollary 5.3.10 *As a module over* $\mathbb{F}_4[\Delta^{\pm 1}, \bar{\kappa}, \nu]/(\nu^3)$, *the term* $\mathrm{E}_4 = \mathrm{E}_5$ *is a direct sum of cyclic modules generated by the classes*

$$e[0,0],\quad e[1,5],\quad e[0,6],\quad e[1,11],\quad e[1,15],\quad e[1,17],\quad e[1,21],\quad e[1,23] \tag{22}$$

with the relations

$$\nu^2 e[1,5] = \nu^2 e[1,11] = \nu^2 e[1,15] = \nu^2 e[1,21] = 0. \tag{23}$$

Proof This is straightforward from Proposition 5.3.9 and from the fact that Δ, $\bar{\kappa}$ and ν are d_3–cycles in the HFPSS for $E_C^{hG_{24}}$. □

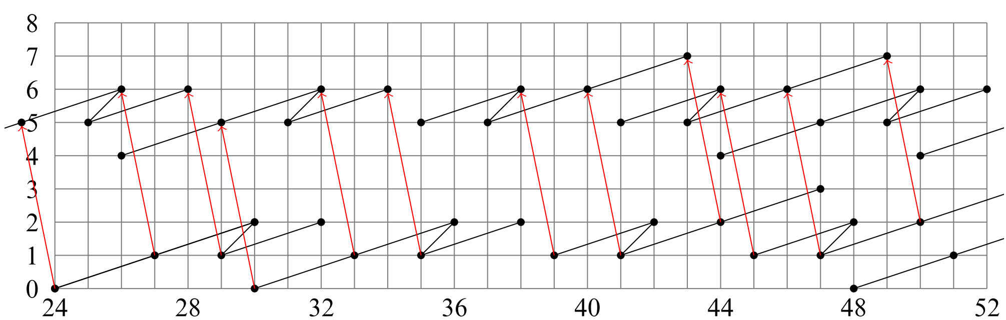


Figure 16: Differentials d_5.

d_5–differentials We need the d_5–differential, in the HFPSS for $E_C^{hG_{24}}$, $d_5(\Delta) = \bar{\kappa}\nu$ (see [3, Section 8.3]), depicted in Figure 16.

Proposition 5.3.11 *As a module over* $\mathbb{F}_4[(\Delta^8)^{\pm 1}, \bar{\kappa}, \nu]/(\bar{\kappa}\nu)$, $E_6 = E_7$ *is a direct sum of cyclic modules generated by the following classes for* $i \in 0, 2, 4, 6$ *with the respective annihilator ideal*:

generator *ideal*	$\Delta^i e[0,0]$ (ν^3)	$\Delta^i e[1,5]$ (ν^2)	$\Delta^i e[0,6]$ (ν^3)	$\Delta^i e[1,11]$ (ν^2)
generator *ideal*	$\Delta^i e[1,15]$ (ν^2)	$\Delta^i e[1,17]$ (ν^3)	$\Delta^i e[1,21]$ (ν^2)	$\Delta^i e[1,23]$ (ν^3)
generator *ideal*	$\Delta^i e[2,30]$ (ν)	$\Delta^i e[2,32]$ (ν)	$\Delta^i e[2,36]$ (ν)	$\Delta^i e[2,38]$ (ν)
generator *ideal*	$\Delta^i e[2,42]$ (ν)	$\Delta^i e[3,47]$ (ν)	$\Delta^i e[2,48]$ (ν)	$\Delta^i e[3,53]$ (ν)

Proof If x is a class in the E_5–term of the HFPSS for A_1, then, by the module structure of the latter over the HFPSS for S^0 and the Leibniz rule, for all $k \in \mathbb{Z}$,

$$d_5(\Delta^{2k}x) = d_5(\Delta^{2k})x + \Delta^{2k}d_5(x) = 2\Delta^k d_5(\Delta^k)x + \Delta^{2k}d_5(x) = \Delta^{2k}d_5(x).$$

This says in particular that the E_6–term is Δ^2–periodic. Next, if x is a d_5–cycle and is annihilated by ν^i, then $d_5(\Delta x) = \bar{\kappa}\nu x$ and $d_5(\Delta\nu^{i-1}x) = 0$. Together with the fact that all of the generators of (22) are permanent cycles (Proposition 5.3.6), it is straightforward to verify that the classes together with their annihilation ideal given in the statement of the proposition generate the E_6–term as a module over $\mathbb{F}_4[(\Delta^8)^{\pm 1}, \bar{\kappa}, \nu]/(\bar{\kappa}\nu)$. □

Remark 5.3.12 Since Δ^8 is a permanent cycle in the HFPSS for $E_C^{hG_{24}}$, the HFPSS for $E_C^{hG_{24}} \wedge A_1$ is linear with respect to Δ^8. Note that all $\bar{\kappa}$–free generators in the E_7–term are of the form $(\Delta^8)^k x$ where $k \in \mathbb{Z}$ and x is one of the generators listed in Proposition 5.3.11 (see Figure 17). Then, by Corollary 5.3.3, these free $\bar{\kappa}$–families pair up so that each nonpermanent $\bar{\kappa}$–family truncates one and only one permanent $\bar{\kappa}$–family. By Δ^8–linearity, among these 64 generators, only half of them are permanent cycles and the others support a differential. It reduces the problem into two steps: first identify all permanent $\bar{\kappa}$–families, then identify by which $\bar{\kappa}$–family they are truncated.

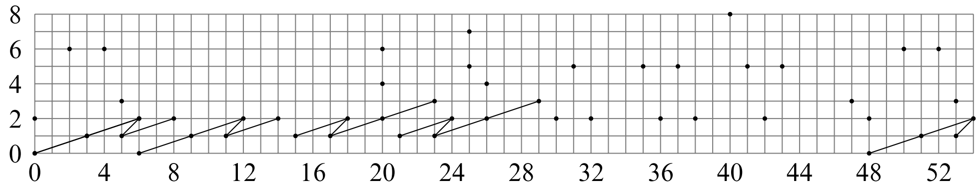


Figure 17: The E_7–term for $s \leq 8$ and $t-s \leq 54$.

Proposition 5.3.13 *The generators*

$$e[2,30], \quad e[2,32], \quad e[2,36], \quad e[2,38], \quad e[2,42], \quad e[3,47], \quad e[2,48], \quad e[3,53]$$

are permanent cycles.

Proof We give the proof for $e[2,30]$ and the other generators are proven in a similar manner. In the E_6–term, the Massey product $\langle \bar{\kappa}, \nu, \nu^2 e[0,0] \rangle$ can be formed. Since $d_5(\Delta) = \bar{\kappa}\nu$ and $\nu^3 e[0,0] = 0 \in E_5$, we see that

$$e[2,30] = \Delta \nu^2 e[0,0] \in \langle \bar{\kappa}, \nu, \nu^2 e[0,0] \rangle.$$

The indeterminacy consists of $\bar{\kappa} E_6^{-2,8} + E_6^{0,26} \nu^2 e[0,0]$, where $E_6^{-2,8}$ is in the E_6–term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$ and $E_6^{0,26}$ for $E_C^{hG_{24}}$. The latter are zero groups; hence, the indeterminacy is zero. Thus,

$$\langle \bar{\kappa}, \nu, \nu^2 e[0,0] \rangle = e[2,30].$$

At the level of the homotopy groups of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ one can form the corresponding Toda bracket $\langle \bar{\kappa}, \nu, \nu^2 e[0,0] \rangle$ because $\nu\bar{\kappa} = 0$ in $\pi_*(E_C^{hG_{24}})$ and inspection in $\pi_*(\mathrm{tmf} \wedge A_1)$ tells us that $\nu^3 e[0,0] = 0$. Furthermore, all hypotheses of Moss's convergence theorem are verified. Therefore, $e[2,30]$ is a permanent cycle representing the Toda bracket $\langle e[0,0], \nu^3, \bar{\kappa} \rangle$. For the sake of completeness, we record the Toda bracket expressions for the other elements:

$$\begin{aligned} \langle \bar{\kappa}, \nu, \nu e[1,5] \rangle &= e[2,32], \\ \langle \bar{\kappa}, \nu, \nu e[1,11] \rangle &= e[2,38], \\ \langle \bar{\kappa}, \nu, \nu^2 e[1,17] \rangle &= e[3,47], \end{aligned} \qquad \begin{aligned} \langle \bar{\kappa}, \nu, \nu^2 e[0,6] \rangle &= e[2,36], \\ \langle \bar{\kappa}, \nu, \nu e[1,15] \rangle &= e[2,42], \\ \langle \bar{\kappa}, \nu, \nu e[1,21] \rangle &= e[2,48], \\ \langle \bar{\kappa}, \nu, \nu^2 e[2,23] \rangle &= e[3,53]. \end{aligned}$$

□

We have already identified 16 out of 32 permanent cycles. The next 16 ones are not the same for different versions of A_1. The difference reflects the different behavior of the d_2–differential in the ASS for different models of A_1 (see Theorem 4.0.3).

Proposition 5.3.14 *In the HFPSSs for all four versions of A_1, the following 12 generators are permanent cycles:*

$$\Delta^2 e[0,0], \quad \Delta^2 e[1,5], \quad \Delta^2 e[0,6], \quad \Delta^2 e[1,11], \quad \Delta^2 e[1,15], \quad \Delta^2 e[1,17],$$
$$\Delta^2 e[1,21], \quad \Delta^2 e[2,30], \quad \Delta^2 e[2,32], \quad \Delta^2 e[2,36], \quad \Delta^2 e[2,42], \quad \Delta^2 e[3,47].$$

The remaining four permanent cycles for $A_1[00]$ and $A_1[11]$ are

$$\Delta^2 e[1,23], \quad \Delta^2 e[2,38], \quad \Delta^2 e[2,48], \quad \Delta^2 e[3,53],$$

whereas the remaining four permanent cycles for $A_1[10]$ and $A_1[01]$ are

$$\Delta^4 e[1,15], \quad \Delta^4 e[0,0], \quad \Delta^4 e[1,5], \quad \Delta^4 e[2,30].$$

Proof The associated graded object of the groups $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$, with respect to the Adams filtration, in the following stems are given in the following table:

stem	48	53	54	59	63	65	69	78	80	84	90	95
value	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$

In view of Corollaries 5.1.3 and 5.3.3, inspection in the E_7–term shows that the following 12 classes are permanent cycles in the HFPSSs for all four versions of A_1:

$$\Delta^2 e[0,0], \quad \Delta^2 e[1,5], \quad \Delta^2 e[0,6], \quad \Delta^2 e[1,11], \quad \Delta^2 e[1,15], \quad \Delta^2 e[1,17],$$
$$\Delta^2 e[1,21], \quad \Delta^2 e[2,30], \quad \Delta^2 e[2,32], \quad \Delta^2 e[2,36],, \quad \Delta^2 e[2,42], \quad \Delta^2 e[3,47].$$

Next, in the ASS for $\mathrm{tmf} \wedge A_1[00]$ and $\mathrm{tmf} \wedge A_1[11]$, there is no differential until stem 96. Again, inspection in the E_2–term (see Figure 13) shows that

$$\pi_{71}(\mathrm{tmf} \wedge A_1[00])/(\bar{\kappa}, \nu) = \pi_{71}(\mathrm{tmf} \wedge A_1[11])/(\bar{\kappa}, \nu) \cong \mathbb{F}_2,$$
$$\pi_{86}(\mathrm{tmf} \wedge A_1[00])/(\bar{\kappa}, \nu) = \pi_{86}(\mathrm{tmf} \wedge A_1[11])/(\bar{\kappa}, \nu) \cong \mathbb{F}_2.$$

It follows that the classes $\Delta^2 e[1,23]$ and $\Delta^2 e[2,38]$ are permanent cycles in the HFPSS for $E_C^{hG_{24}} \wedge A_1[00]$ and $E_C^{hG_{24}} \wedge A_1[11]$.

On the other hand, in the ASS for $\mathrm{tmf} \wedge A_1[10]$ and $\mathrm{tmf} \wedge A_1[01]$, Theorem 4.0.3 and g–linearity imply that $d_2(g^2 w_2 e[4,23]) = g^4 e[6,30]$ and $d_2(g^2 w_2 e[7,38]) = g^6 e[1,5]$. Hence, $w_2^2 e[3,15]$ and $w_2^2 e[6,30]$ survive to the E_∞–term, by sparseness. It then follows that $\Delta^4 e[1,15]$ and $\Delta^4 e[2,30]$ are permanent cycles in the HFPSS for $A_1[10]$ and $A_1[01]$.

For $A_1[00]$ and $A_1[11]$, the classes $w_2 e[9,48]$ and $w_2 e[10,53]$ do not support differentials, by Theorem 4.0.3, and hence persist to the E_∞–term, by sparseness. Neither

$\bar{\kappa}$ nor ν divides these classes. Lastly, both $w_2 e[9,48]$ and $w_2 e[10,53]$ are annihilated by ν. The only classes in the HFPSS that match those properties are $\Delta^2 e[2,48]$ and $\Delta^2 e[3,53]$, respectively. Thus, the latter are the last two of the 32 permanent cycles in the HFPSS for $A_1[00]$ and $A_1[11]$.

For $A_1[10]$ and $A_1[01]$, the classes $w_2 e[9,48]$ and $w_2 e[10,53]$ support nontrivial d_2–differentials. Thus $w_2^2 e[0,0]$ and $w_2^2 e[1,5]$ survive to the E_∞–term. For degree reasons, both $w_2^2 e[0,0]$ and $w_2^2 e[1,5]$ are not divisible either by $\bar{\kappa}$ or by ν, and moreover their multiples by ν are not divisible by $\bar{\kappa}$. In the HFPSS for $E_C^{hG_{24}} \wedge A_1[10]$ and $E_C^{hG_{24}} \wedge A_1[10]$, $\Delta^4 e[0,0]$ and $\Delta^4 e[1,5]$ are the only classes verifying the respective properties, and hence are permanent cycles. □

Remark 5.3.15 Having determined all permanent $\bar{\kappa}$–families, we consider differentials. We recall, from Remark 5.3.12, that each permanent $\bar{\kappa}$–family is truncated by one and only one nonpermanent $\bar{\kappa}$–family. We can proceed as follows: take a permanent cycle, say x; then locate all nonpermanent classes that can support a differential killing $\bar{\kappa}^n x$ for some $n \leq 6$. Precisely, one of the following situations will happen:

1. There is no ambiguity, ie there is only one generator that can support a differential killing $\bar{\kappa}^n x$ for some $n \leq 6$, so this differential occurs.
2. There are two generators that can support a differential killing multiples of x by different powers of $\bar{\kappa}$. In order to decide, we inspect the $\bar{\kappa}$–exponent of x using the ASS.
3. There are two generators that can support a differential killing the multiple of x by the same power of $\bar{\kappa}$. In this case, inspection on the $\bar{\kappa}$–exponent of x does not help. We will treat each of the particularity case by case. Some Toda brackets will be involved to resolve these cases.

A permanent cycle is said to be of type 1, 2 and 3, respectively, if its $\bar{\kappa}$–family is as in the situation (1), (2) and (3) above, respectively. The HFPSSs for different versions of A_1 do not behave in the same manner. It turns out the HFPSSs for the versions $A_1[10]$ and $A_1[01]$ behave in the same way and $A_1[00]$ and $A_1[11]$ in the same way. We will treat the HFPSS for $A_1[10]$ and $A_1[01]$ in detail and then point out the changes needed for $A_1[00]$ and $A_1[11]$.

Higher differentials for $A_1[01]$ and $A_1[10]$ The reader is invited to follow the discussion of the differentials using Figures 18 to 20.

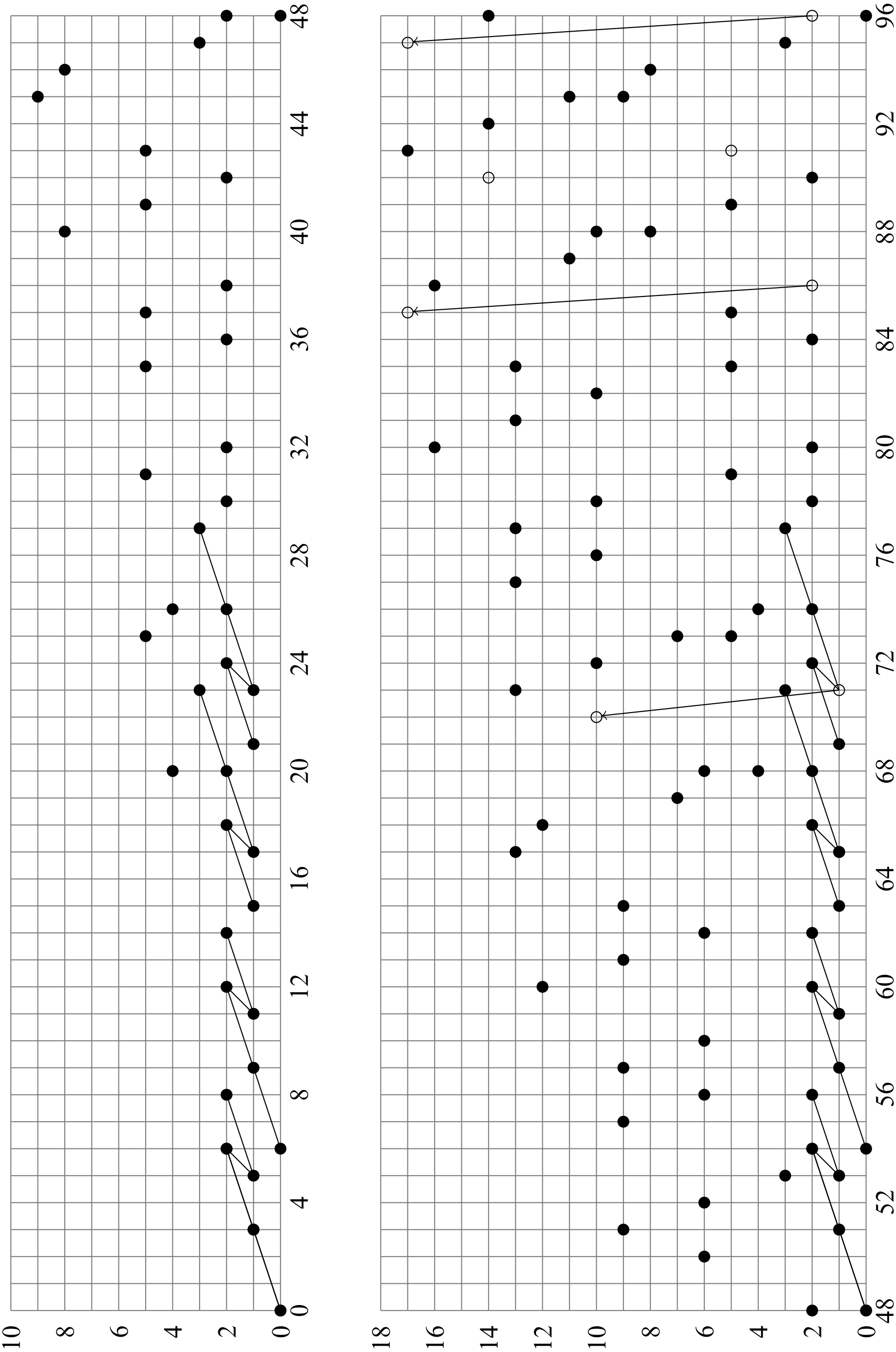


Figure 18: HFPSS for $A_1[10]$ and $A_1[01]$ from E_7–term with $0 \le t-s \le 48$ (left) and $48 \le t-s \le 96$ (right).

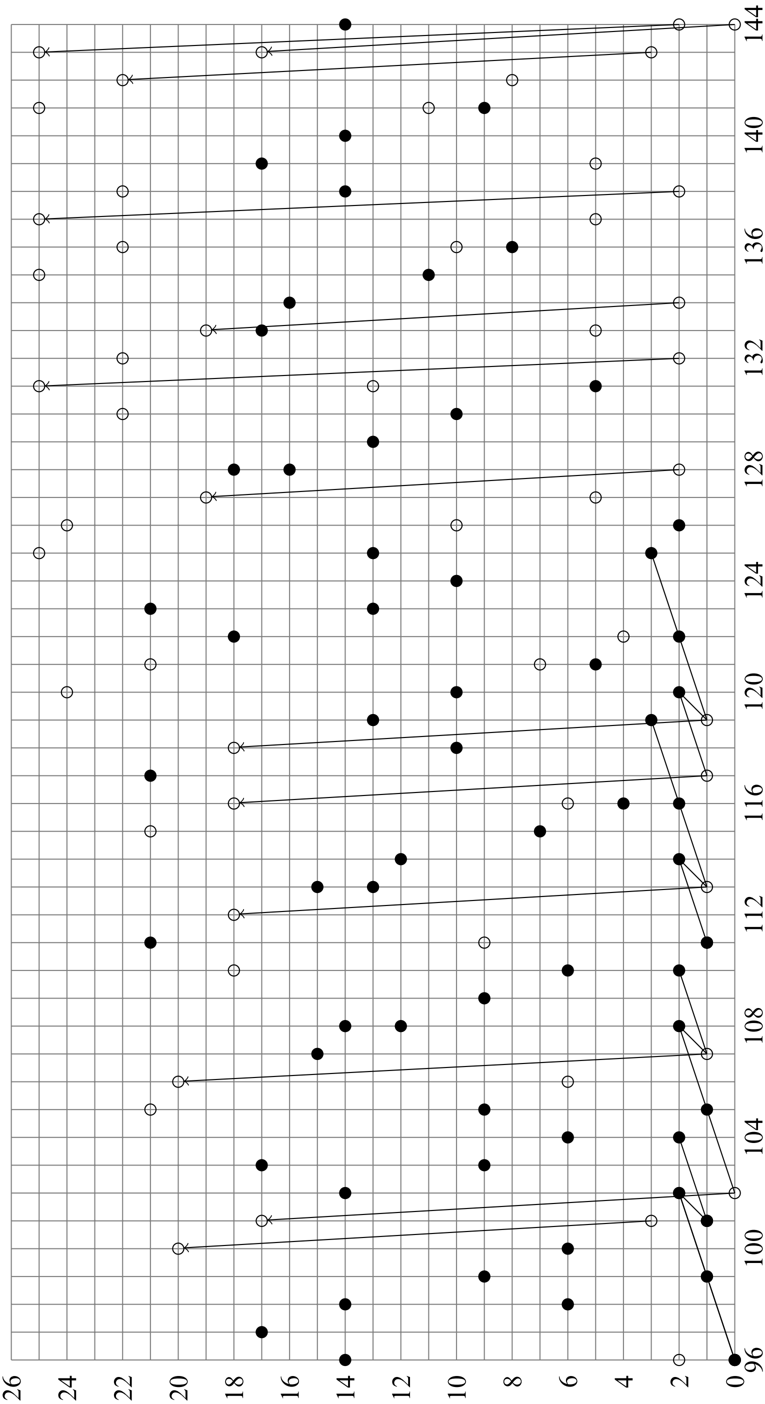


Figure 19: HFPSS for $A_1[10]$ and $A_1[01]$ from E_7–term with $96 \le t-s \le 144$.

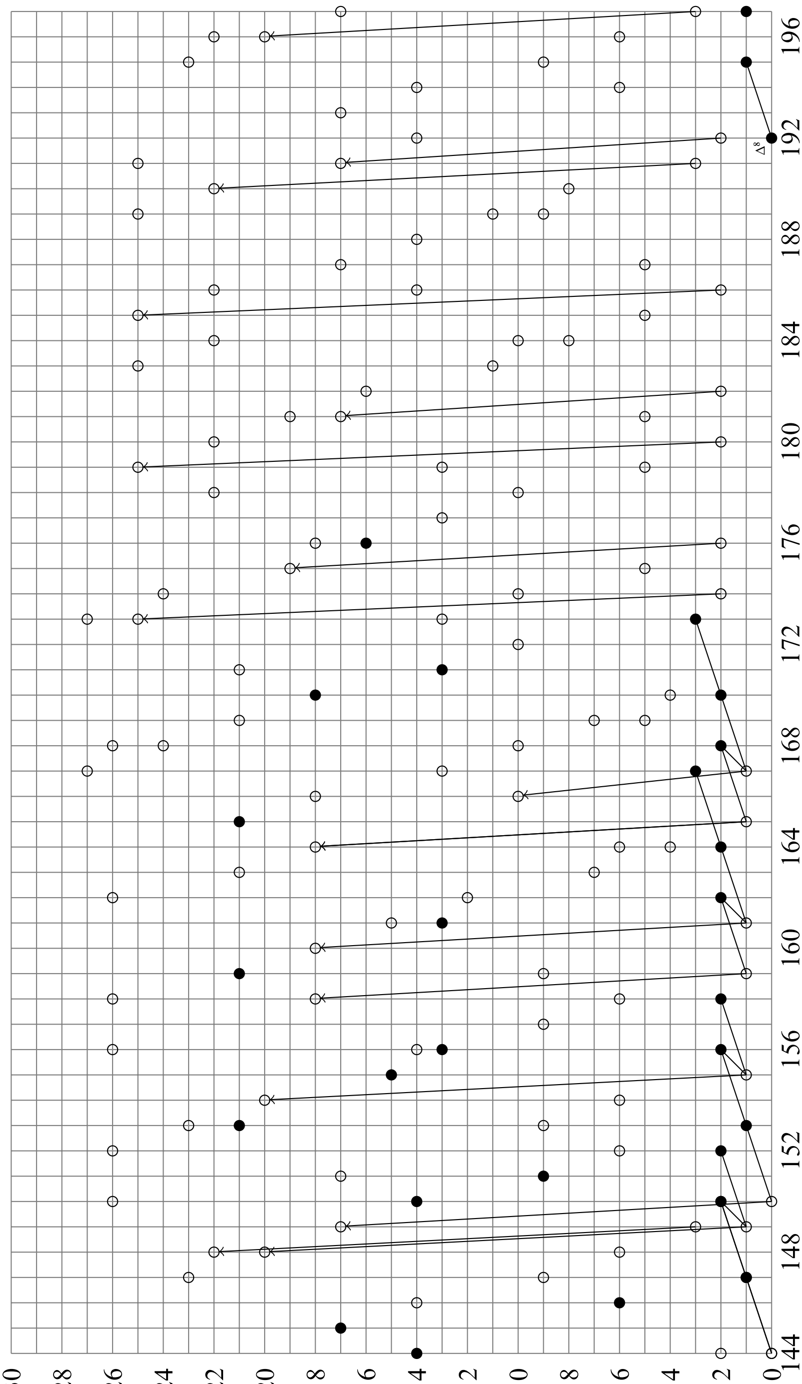


Figure 20: HFPSS for $A_1[10]$ and $A_1[01]$ from E_7–term with $144 \leq t - s \leq 197$.

Proposition 5.3.16 (A) *There are the d_9–differentials*

(1) $d_9(\Delta^2 e[1,23]) = \bar{\kappa}^2 e[2,30]$,

(2) $d_9(\Delta^6 e[1,23]) = \bar{\kappa}^2 \Delta^4 e[2,30]$.

(B) *There are the d_{15}–differentials*

(1) $d_{15}(\Delta^2 e[2,38]) = \bar{\kappa}^4 e[1,5]$,

(2) $d_{15}(\Delta^2 e[2,48]) = \bar{\kappa}^4 e[1,15]$,

(3) $d_{15}(\Delta^6 e[2,38]) = \bar{\kappa}^4 \Delta^4 e[1,5]$,

(4) $d_{15}(\Delta^6 e[2,48]) = \bar{\kappa}^4 \Delta^4 e[1,15]$.

(C) *There are the d_{17}–differentials*

(1) $d_{17}(\Delta^2 e[3,53]) = \bar{\kappa}^5 e[0,0]$,

(2) $d_{17}(\Delta^4 e[0,6]) = \bar{\kappa}^4 e[1,21]$,

(3) $d_{17}(\Delta^4 e[1,17]) = \bar{\kappa}^4 e[2,32]$,

(4) $d_{17}(\Delta^4 e[1,21]) = \bar{\kappa}^4 e[2,36]$,

(5) $d_{17}(\Delta^4 e[2,32]) = \bar{\kappa}^4 e[3,47]$,

(6) $d_{17}(\Delta^6 e[0,6]) = \bar{\kappa}^4 \Delta^2 e[1,21]$,

(7) $d_{17}(\Delta^6 e[1,17]) = \bar{\kappa}^4 \Delta^2 e[2,32]$,

(8) $d_{17}(\Delta^6 e[1,21]) = \bar{\kappa}^4 \Delta^2 e[2,36]$,

(9) $d_{17}(\Delta^6 e[2,32]) = \bar{\kappa}^4 \Delta^2 e[3,47]$,

(10) $d_{17}(\Delta^6 e[3,53]) = \bar{\kappa}^5 \Delta^4 e[0,0]$,

(11) $d_{17}(\Delta^4 e[1,23]) = \bar{\kappa}^4 e[2,38]$,

(12) $d_{17}(\Delta^4 e[2,38]) = \bar{\kappa}^4 e[3,53]$,

(13) $d_{17}(\Delta^6 e[0,0]) = \bar{\kappa}^4 \Delta^2 e[1,15]$,

(14) $d_{17}(\Delta^6 e[1,15]) = \bar{\kappa}^4 \Delta^2 e[2,30]$.

(D) *There are the d_{19}–differentials*

(1) $d_{19}(\Delta^4 e[1,11]) = \bar{\kappa}^5 e[0,6]$,

(2) $d_{19}(\Delta^4 e[3,47]) = \bar{\kappa}^5 e[2,42]$,

(3) $d_{19}(\Delta^6 e[1,11]) = \bar{\kappa}^5 \Delta^2 e[0,6]$,

(4) $d_{19}(\Delta^6 e[3,47]) = \bar{\kappa}^5 \Delta^2 e[2,42]$,

(5) $d_{19}(\Delta^6 e[1,5]) = \bar{\kappa}^5 \Delta^2 e[0,0]$,

(6) $d_{19}(\Delta^4 e[3,53]) = \bar{\kappa}^5 e[2,48]$.

(E) *There are the d_{23}–differentials*

(1) $d_{23}(\Delta^4 e[2,36]) = \bar{\kappa}^6 e[1,11]$,

(2) $d_{23}(\Delta^4 e[2,42]) = \bar{\kappa}^6 e[1,17]$,

(3) $d_{23}(\Delta^4 e[2,48]) = \bar{\kappa}^6 e[1,23]$,

(4) $d_{23}(\Delta^6 e[2,36]) = \bar{\kappa}^6 \Delta^2 e[1,11]$,

(5) $d_{23}(\Delta^6 e[2,42]) = \bar{\kappa}^6 \Delta^2 e[1,17]$,

(6) $d_{23}(\Delta^6 e[2,30]) = \bar{\kappa}^6 \Delta^2 e[1,5]$.

Proof (A) The classes $e[2,30]$ and $\Delta^4 e[2,30]$ are of type 1 and the only possibilities are

$$d_9(\Delta^2 e[1,23]) = \bar{\kappa}^2 e[2,30] \quad \text{and} \quad d_9(\Delta^6 e[1,23]) = \bar{\kappa}^2 \Delta^4 e[2,30],$$

respectively.

(B) All of the classes $e[1,5]$, $e[1,15]$, $\Delta^4 e[1,5]$ and $\Delta^4 e[1,15]$ are of type 1 and their $\bar{\kappa}$–family is truncated as indicated in the proposition.

(C)(1)–(10) All classes

$$\begin{array}{ccccc} e[0,0], & e[1,21], & e[2,32], & e[2,36], & e[3,47], \\ \Delta^2 e[1,21], & \Delta^2 e[2,32], & \Delta^2 e[2,36], & \Delta^2 e[3,47], & \Delta^4 e[0,0] \end{array}$$

are of type 1.

(C)(11) The class $e[2,38]$ is of type 2. The differentials that can truncate its $\bar{\kappa}$–family are

$$d_{17}(\Delta^4 e[1,23]) = \bar{\kappa}^4 e[2,38] \quad \text{and} \quad d_{25}(\Delta^6 e[1,15]) = \bar{\kappa}^6 e[2,38].$$

The latter cannot happen because the spectral sequence collapses at the E_{24}–term. Therefore,

$$d_{17}(\Delta^4 e[1,23]) = \bar{\kappa}^4 e[2,38].$$

(C)(12) The class $e[3,53]$ is of type 2. Its $\bar{\kappa}$–family can be truncated by

$$d_{17}(\Delta^4 e[2,38]) = \bar{\kappa}^4 e[3,53] \quad \text{or} \quad d_{25}(\Delta^6 e[2,30]) = \bar{\kappa}^6 e[3,53].$$

As above, there cannot be any d_{25}–differential in the spectral sequence. Hence,

$$d_{17}(\Delta^4 e[2,38]) = \bar{\kappa}^4 e[3,53].$$

(C)(13) The class $\Delta^2 e[1,15]$ is of type 3. In its $\bar{\kappa}$–family, only $\bar{\kappa}^4 \Delta^2 e[1,15]$ can be a target of differentials,

$$d_{17}(\Delta^6 e[0,0]) = \bar{\kappa}^4 \Delta^2 e[1,15] \quad \text{and} \quad d_{15}(\Delta^4 e[2,48]) = \bar{\kappa}^4 \Delta^2 e[1,15].$$

However, if

$$d_{15}(\Delta^4 e[2,48]) = \bar{\kappa}^4 \Delta^2 e[1,15],$$

then the only class that can truncate the $\bar{\kappa}$–family of $e[1,23]$ is $\Delta^6 e[0,0]$ and by a d_{25}–differential

$$d_{25}(\Delta^6 e[0,0]) = \bar{\kappa}^6 e[1,23].$$

This contradicts the fact that the spectral sequence collapses at the E_{24}–term. Thus,

$$d_{17}(\Delta^6 e[0,0]) = \bar{\kappa}^4 \Delta^2 e[1,15].$$

(C)(14) $\Delta^2 e[2,30]$ is of type 2. Its $\bar{\kappa}$–family can be truncated by a d_9–differential on $\Delta^4 e[1,23]$ or by a d_{17}–differential on $\Delta^6 e[1,15]$. However, the former possibility cannot occur because of part (11). Therefore,

$$d_{17}(\Delta^6 e[1,15]) = \bar{\kappa}^4 \Delta^2 e[2,30].$$

(D)(1)–(4) All of the classes

$$e[0,6], \quad e[2,42], \quad \Delta^2 e[0,6], \quad \Delta^2 e[2,42]$$

are of type 1.

(D)(5) The class $\Delta^2 e[0,0]$ is of type 3 and its $\bar{\kappa}$–family can be truncated by either

$$d_{17}(\Delta^4 e[3,53]) = \bar{\kappa}^5 \Delta^2 e[0,0] \quad \text{or} \quad d_{19}(\Delta^6 e[1,5]) = \bar{\kappa}^5 \Delta^2 e[0,0].$$

Suppose that the former happened. This would leave us with the differential

$$d_{21}(\Delta^6 e[1,5]) = \bar{\kappa}^5 e[2,48].$$

It would imply the Massey product in the E_{22}–term

$$\langle \bar{\kappa}^5, e[2,48], \nu \rangle = \nu \Delta^6 e[1,5]$$

with zero indeterminacy in the E_{22}–term. We see that the Toda bracket $\langle \bar{\kappa}^5, e[2,48], \nu \rangle$ could then be formed because

$$\bar{\kappa}^4 e[2,48] = \nu e[2,48] = 0 \in \pi_*(E_C^{hG_{24}} \wedge A_1).$$

We check that all conditions of Moss's convergence theorem [30, Theorem 1.2] are met, and so the Toda bracket $\langle \bar{\kappa}^5, e[2,48], \nu \rangle$ would contain an element represented by $\nu \Delta^6 e[1,5]$. This contradicts Corollary 4.0.6. This contradiction proves that

$$d_{19}(\Delta^6 e[1,5]) = \bar{\kappa}^5 \Delta^2 e[0,0].$$

(D)(6) The class $e[2,48]$ is of type 2 and its $\bar{\kappa}$–family is truncated by either

$$d_{19}(\Delta^4 e[3,53]) = \bar{\kappa}^5 e[2,48] \quad \text{or} \quad d_{21}(\Delta^6 e[1,5]) = \bar{\kappa}^5 e[2,48].$$

However, part (D)(5) rules out the latter.

(E)(1)–(5) All of the classes

$$e[1,11], \quad e[1,17], \quad e[1,23], \quad \Delta^2 e[1,11], \quad \Delta^2 e[1,17]$$

are of type 1.

(E)(6) The class $\Delta^2 e[1,5]$ is of type 2. The two possibilities are

$$d_{15}(\Delta^4 e[2,38]) = \bar{\kappa}^4 \Delta^2 e[1,5] \quad \text{and} \quad d_{23}(\Delta^6 e[2,30]) = \bar{\kappa}^6 \Delta^2 e[1,5].$$

However, part (C)(12) rules out the former because the class $\Delta^4 e[2,38]$ must pair up with the class $e[3,38]$, by the differential

$$d_{17}(\Delta^4 e[2,38]) = \bar{\kappa}^4 e[3,53]. \qquad \square$$

The above differentials from d_9 to d_{23}, together with the $\bar{\kappa}$– and Δ^8–linearity exhaust all differentials. In Theorems 5.3.17 and 5.3.18, we write e_{t-s} for the permanent cycle $e[s,t-s]$ in bidegree (s,t) listed in Proposition 5.3.11, for the sake of presentation.

Theorem 5.3.17 *As a module over* $\mathbb{F}_4[\Delta^{\pm 8},\bar{\kappa},\nu]/(\bar{\kappa}\nu)$, *the* E_∞*–term of the HFPSS for* $E_C^{hG_{24}} \wedge A_1$ *for* $A_1 = A_1[10]$ *and* $A_1[01]$ *is a direct sum of cyclic modules generated by the following elements and with the respective annihilator ideal*:

$(0,0)$ e_0 $(\bar{\kappa}^5,\nu^3)$	$(1,5)$ e_5 $(\bar{\kappa}^4,\nu^2)$	$(0,6)$ e_6 $(\bar{\kappa}^5,\nu^3)$	$(1,11)$ e_{11} $(\bar{\kappa}^6,\nu^2)$	$(1,15)$ e_{15} $(\bar{\kappa}^4,\nu^2)$	$(1,17)$ e_{17} $(\bar{\kappa}^6,\nu^3)$	$(1,21)$ e_{21} $(\bar{\kappa}^4,\nu^2)$	$(1,23)$ e_{23} $(\bar{\kappa}^6,\nu^3)$
$(2,30)$ e_{30} $(\bar{\kappa}^2,\nu)$	$(2,32)$ e_{32} $(\bar{\kappa}^4,\nu)$	$(2,36)$ e_{36} $(\bar{\kappa}^4,\nu)$	$(2,38)$ e_{38} $(\bar{\kappa}^4,\nu)$	$(2,42)$ e_{42} $(\bar{\kappa}^5,\nu)$	$(3,47)$ e_{47} $(\bar{\kappa}^4,\nu)$	$(2,48)$ e_{48} $(\bar{\kappa}^5,\nu)$	$(3,53)$ e_{53} $(\bar{\kappa}^4,\nu)$
$(0,48)$ $\Delta^2 e_0$ $(\bar{\kappa}^5,\nu^3)$	$(1,53)$ $\Delta^2 e_5$ $(\bar{\kappa}^6,\nu^2)$	$(0,54)$ $\Delta^2 e_6$ $(\bar{\kappa}^5,\nu^3)$	$(1,59)$ $\Delta^2 e_{11}$ $(\bar{\kappa}^6,\nu^2)$	$(1,63)$ $\Delta^2 e_{15}$ $(\bar{\kappa}^4,\nu^2)$	$(1,65)$ $\Delta^2 e_{17}$ $(\bar{\kappa}^6,\nu^3)$	$(1,69)$ $\Delta^2 e_{21}$ $(\bar{\kappa}^4,\nu^2)$	$(2,74)$ $\Delta^2 \nu e_{23}$ $(\bar{\kappa},\nu^2)$
$(2,78)$ $\Delta^2 e_{30}$ $(\bar{\kappa}^4,\nu)$	$(2,80)$ $\Delta^2 e_{32}$ $(\bar{\kappa}^4,\nu)$	$(2,84)$ $\Delta^2 e_{36}$ $(\bar{\kappa}^4,\nu)$	$(2,90)$ $\Delta^2 e_{42}$ $(\bar{\kappa}^5,\nu)$	$(3,95)$ $\Delta^2 e_{47}$ $(\bar{\kappa}^4,\nu)$	$(0,96)$ $\Delta^4 e_0$ $(\bar{\kappa}^5,\nu^3)$	$(1,101)$ $\Delta^4 e_5$ $(\bar{\kappa}^4,\nu^2)$	$(1,105)$ $\Delta^4 \nu e_6$ $(\bar{\kappa},\nu^2)$
$(2,110)$ $\Delta^4 \nu e_{11}$ $(\bar{\kappa},\nu)$	$(1,111)$ $\Delta^4 e_{15}$ $(\bar{\kappa}^4,\nu^2)$	$(2,116)$ $\Delta^4 \nu e_{17}$ $(\bar{\kappa},\nu^2)$	$(2,120)$ $\Delta^4 \nu e_{21}$ $(\bar{\kappa},\nu)$	$(2,122)$ $\Delta^4 \nu e_{23}$ $(\bar{\kappa},\nu^2)$	$(2,126)$ $(\Delta^4 e_{30})$ $(\bar{\kappa}^2,\nu)$	$(1,147)$ $\Delta^6 \nu e_0$ $(\bar{\kappa},\nu^2)$	$(2,152)$ $\Delta^6 \nu e_5$ $(\bar{\kappa},\nu)$
$(1,153)$ $\Delta^6 \nu e_6$ $(\bar{\kappa},\nu^2)$	$(2,158)$ $\Delta^6 \nu e_{11}$ $(\bar{\kappa},\nu)$	$(2,162)$ $\Delta^6 \nu e_{15}$ $(\bar{\kappa},\nu)$	$(2,164)$ $\Delta^6 \nu e_{17}$ $(\bar{\kappa},\nu^2)$	$(2,168)$ $\Delta^6 \nu e_{21}$ $(\bar{\kappa},\nu)$	$(2,170)$ $\Delta^6 \nu e_{23}$ $(\bar{\kappa},\nu^2)$		

The case of $A_1[00]$ and $A_1[11]$ The analysis of the HFPSS for $A_1[00]$ and $A_1[11]$ can be done in the same manner as for $A_1[10]$ and $A_1[01]$. All differentials are identical except for eight ones. These differences reflect the different behavior between the ASS for $A_1[10]$ and $A_1[01]$ and that for $A_1[11]$ and $A_1[00]$. Below are all the changes whose justifications are based on similar considerations as in the proof of Proposition 5.3.16; see Figures 21 to 23:

- $d_{17}(\Delta^4 e[1,15]) = \bar{\kappa}^4 e[2,30]$ instead of $d_9(\Delta^2 e[1,23]) = \bar{\kappa}^2 e[2,30]$.
- $d_{17}(\Delta^6 e[1,23]) = \bar{\kappa}^4 \Delta^2 e[2,38]$ instead of $d_9(\Delta^6 e[1,23]) = \bar{\kappa}^2 \Delta^4 e[2,30]$.

- $d_{17}(\Delta^4 e[0,0]) = \bar\kappa^4 e[1,15]$ instead of $d_{15}(\Delta^2 e[2,48]) = \bar\kappa^4 e[1,15]$.
- $d_{17}(\Delta^6 e[2,38]) = \bar\kappa^4 \Delta^2 e[3,53]$ instead of $d_{15}(\Delta^6 e[2,38]) = \bar\kappa^4 \Delta^2 e[1,5]$.
- $d_{19}(\Delta^4 e[1,5]) = \bar\kappa^5 e[0,0]$ instead of $d_{17}(\Delta^2 e[3,53]) = \bar\kappa^5 e[0,0]$.
- $d_{19}(\Delta^6 e[3,53]) = \bar\kappa^5 \Delta^2 e[2,48]$ instead of $d_{17}(\Delta^6 e[3,53]) = \bar\kappa^5 \Delta^4 e[0,0]$.
- $d_{23}(\Delta^6 e[2,48]) = \bar\kappa^6 \Delta^2 e[1,23]$ instead of $d_{15}(\Delta^6 e[2,48]) = \bar\kappa^4 \Delta^4 e[1,15]$.
- $d_{23}(\Delta^4 e[2,30]) = \bar\kappa^6 e[1,5]$ instead of $d_{15}(\Delta^2 e[2,38]) = \bar\kappa^4 e[1,5]$.

Theorem 5.3.18 *As a module over $\mathbb{F}_4[\Delta^{\pm 8}, \bar\kappa, \nu]/(\bar\kappa\nu)$, the* E_∞*–term of the HFPSS for $E_C^{hG_{24}} \wedge A_1$ for $A_1 = A_1[00]$ and $A_1[11]$ is a direct sum of cyclic modules generated by the following elements and with the respective annihilator ideals*:

$(0,0)$ e_0 $(\bar\kappa^5,\nu^3)$	$(1,5)$ e_5 $(\bar\kappa^6,\nu^2)$	$(0,6)$ e_6 $(\bar\kappa^5,\nu^3)$	$(1,11)$ e_{11} $(\bar\kappa^6,\nu^2)$	$(1,15)$ e_{15} $(\bar\kappa^4,\nu^2)$	$(1,17)$ e_{17} $(\bar\kappa^6,\nu^3)$	$(1,21)$ e_{21} $(\bar\kappa^4,\nu^2)$	$(1,23)$ e_{23} $(\bar\kappa^6,\nu^3)$
$(2,30)$ e_{30} $(\bar\kappa^4,\nu)$	$(2,32)$ e_{32} $(\bar\kappa^4,\nu)$	$(2,36)$ e_{36} $(\bar\kappa^4,\nu)$	$(2,38)$ e_{38} $(\bar\kappa^4,\nu)$	$(2,42)$ e_{42} $(\bar\kappa^5,\nu)$	$(3,47)$ e_{47} $(\bar\kappa^4,\nu)$	$(2,48)$ e_{48} $(\bar\kappa^5,\nu)$	$(3,53)$ e_{53} $(\bar\kappa^4,\nu)$
$(0,48)$ $\Delta^2 e_0$ $(\bar\kappa^5,\nu^3)$	$(1,53)$ $\Delta^2 e_5$ $(\bar\kappa^6,\nu^2)$	$(0,54)$ $\Delta^2 e_6$ $(\bar\kappa^5,\nu^3)$	$(1,59)$ $\Delta^2 e_{11}$ $(\bar\kappa^6,\nu^2)$	$(1,63)$ $\Delta^2 e_{15}$ $(\bar\kappa^4,\nu^2)$	$(1,65)$ $\Delta^2 e_{17}$ $(\bar\kappa^6,\nu^3)$	$(1,69)$ $\Delta^2 e_{21}$ $(\bar\kappa^4,\nu^2)$	$(1,71)$ $\Delta^2 e_{23}$ $(\bar\kappa^6,\nu^3)$
$(2,78)$ $\Delta^2 e_{30}$ $(\bar\kappa^4,\nu)$	$(2,80)$ $\Delta^2 e_{32}$ $(\bar\kappa^4,\nu)$	$(2,84)$ $\Delta^2 e_{36}$ $(\bar\kappa^4,\nu)$	$(2,86)$ $\Delta^2 e_{38}$ $(\bar\kappa^4,\nu)$	$(2,90)$ $\Delta^2 e_{42}$ $(\bar\kappa^5,\nu)$	$(3,95)$ $\Delta^2 e_{47}$ $(\bar\kappa^4,\nu)$	$(2,96)$ $\Delta^2 e_{48}$ $(\bar\kappa^5,\nu)$	$(3,101)$ $\Delta^2 e_{53}$ $(\bar\kappa^4,\nu)$
$(1,99)$ $\Delta^4 \nu e_0$ $(\bar\kappa,\nu^2)$	$(2,104)$ $\Delta^4 \nu e_5$ $(\bar\kappa,\nu)$	$(1,105)$ $\Delta^4 \nu e_6$ $(\bar\kappa,\nu^2)$	$(2,110)$ $\Delta^4 \nu e_{11}$ $(\bar\kappa,\nu)$	$(2,114)$ $\Delta^4 \nu e_{15}$ $(\bar\kappa,\nu)$	$(2,116)$ $\Delta^4 \nu e_{17}$ $(\bar\kappa,\nu^2)$	$(2,120)$ $\Delta^4 \nu e_{21}$ $(\bar\kappa,\nu)$	$(2,122)$ $\Delta^4 \nu e_{23}$ $(\bar\kappa,\nu^2)$
$(1,147)$ $\Delta^6 \nu e_0$ $(\bar\kappa,\nu^2)$	$(2,152)$ $\Delta^6 \nu e_5$ $(\bar\kappa,\nu)$	$(1,153)$ $\Delta^6 \nu e_6$ $(\bar\kappa,\nu^2)$	$(2,158)$ $\Delta^6 \nu e_{11}$ $(\bar\kappa,\nu)$	$(2,162)$ $\Delta^6 \nu e_{15}$ $(\bar\kappa,\nu)$	$(2,164)$ $\Delta^6 \nu e_{17}$ $(\bar\kappa,\nu^2)$	$(2,168)$ $\Delta^6 \nu e_{21}$ $(\bar\kappa,\nu)$	$(2,170)$ $\Delta^6 \nu e_{23}$ $(\bar\kappa,\nu^2)$

Remark 5.3.19 We emphasize that the relations given in Theorems 5.3.17 and 5.3.18 are only the relations in the E_∞–term. In fact, we can see by sparseness that the annihilator exponents of $\bar\kappa$ are still true in $\pi_*(E_C^{hG_{24}} \wedge A_1)$. In contrast, there are exotic extensions by ν, ie multiplications by ν that are not detected in the E_∞–term. These

can be determined by two different methods: by using the Tate spectral sequence as in [9, Section 2.3] or by computing the Gross–Hopkins dual of $E_C^{hG_{24}} \wedge A_1$; however, we do not discuss this point here.

Using the structure of the E_∞–term, we can read off the action of the ideal $(\bar{\kappa}, \nu)$ on $\pi_*(E_C^{hG_{24}} \wedge A_1)$. From this, we obtain the following corollary:

Theorem 5.3.20 (a) *The map*

$$\Theta' \colon \mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_k(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu) \to \pi_k(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa}, \nu),$$

induced by Θ *in* (18), *is an isomorphism for* $k \geq 0$, *independent of the version of* A_1.

(b) *The map*

$$\Theta \colon \mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_k(\mathrm{tmf} \wedge A_1) \to \pi_k(E_C^{hG_{24}} \wedge A_1)$$

is also an isomorphism for $k \geq 0$, *independent of the version of* A_1.

(c) *Multiplication by* Δ^8 *induces isomorphisms*

$$\pi_k(\mathrm{tmf} \wedge A_1) \to \pi_{k+192}(\mathrm{tmf} \wedge A_1),$$
$$\pi_k(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu) \to \pi_{k+192}(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$$

for $k \geq 0$.

Proof For part (a), Corollary 5.1.3 asserts that Θ' is injective. To show that the latter is surjective, it suffices to show that its source and target have the same order. The order of the target can be seen from Theorems 5.3.17 and 5.3.18; in particular, it has order 0 or 4 in all stems, except for the stems 48 and 53 modulo 192, in which it has order 8. The remaining part of the proof is an inspection of the ASS for $\mathrm{tmf} \wedge A_1$, together with the fact that Θ is injective, by Corollary 5.1.3, and is linear with respect to $\bar{\kappa}$ and ν, to show that $\mathbb{W} \otimes_{\mathbb{Z}_2} \pi_*(\mathrm{tmf} \wedge A_1)$ has the same order as $\pi_*(E_C^{hG_{24}} \wedge A_1)$, in nonnegative stems. Because of the dependence of the structure of $\pi_*(E_C^{hG_{24}} \wedge A_1)$ on the version of A_1, we consider them separately; we only give a detailed treatment for $A_1[00]$ and $A_1[11]$ and claim that the treatment for $A_1[01]$ and $A_1[10]$ is completely similar. For the remaining part of the proof, A_1 will be $A_1[00]$ or $A_1[11]$.

By sparseness and Theorem 4.0.3(i), all classes $w_2^l e[i,j]$ for $l = 0, 1$ and $e[i,j]$, the classes in the table of Theorem 3.2.5, survive to the E_∞–term of the ASS for $\mathrm{tmf} \wedge A_1$.

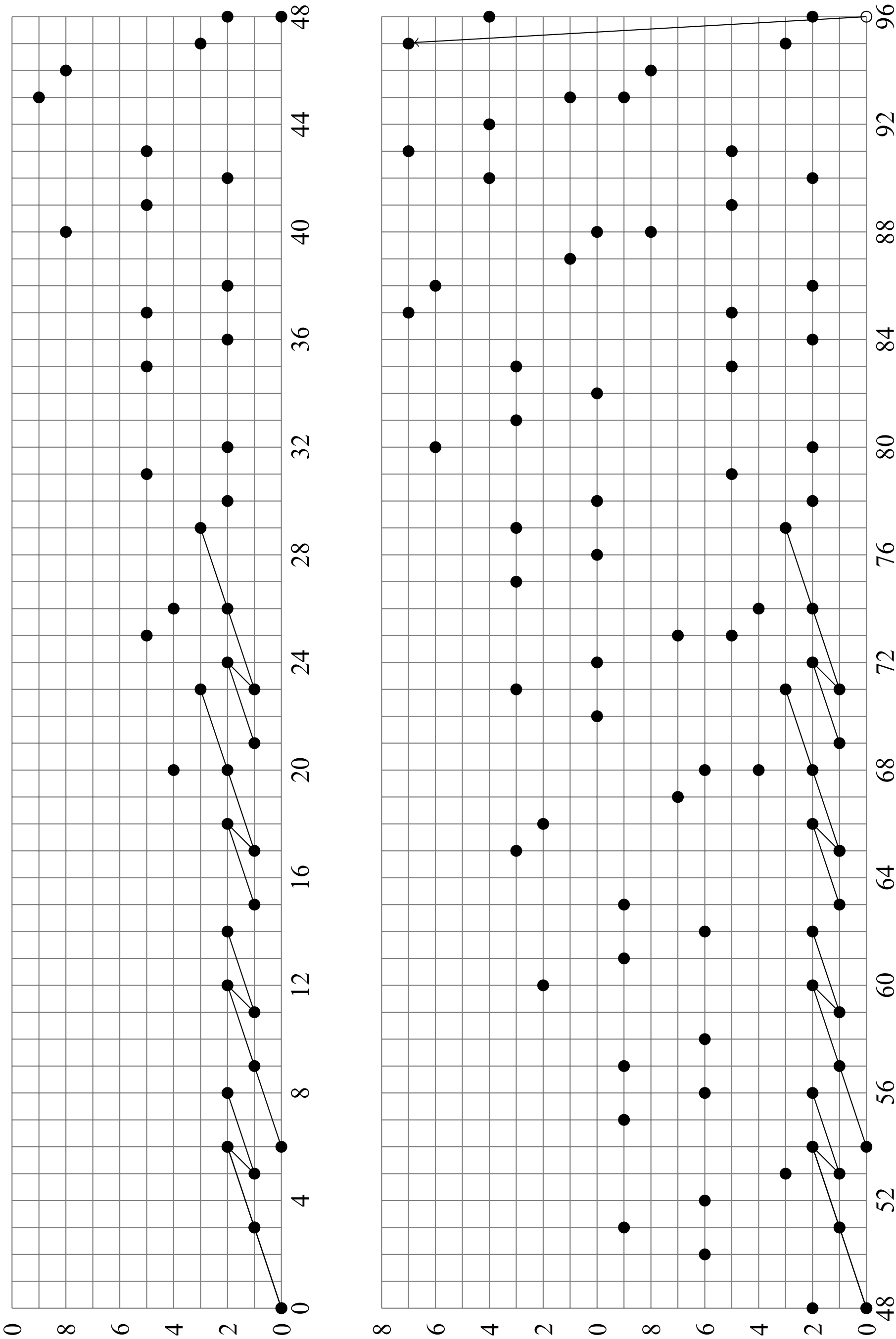


Figure 21: HFPSS for $A_1[00]$ and $A_1[11]$ from E_7–term with $0 \le t - s \le 48$ (left) and $48 \le t - s \le 96$ (right).

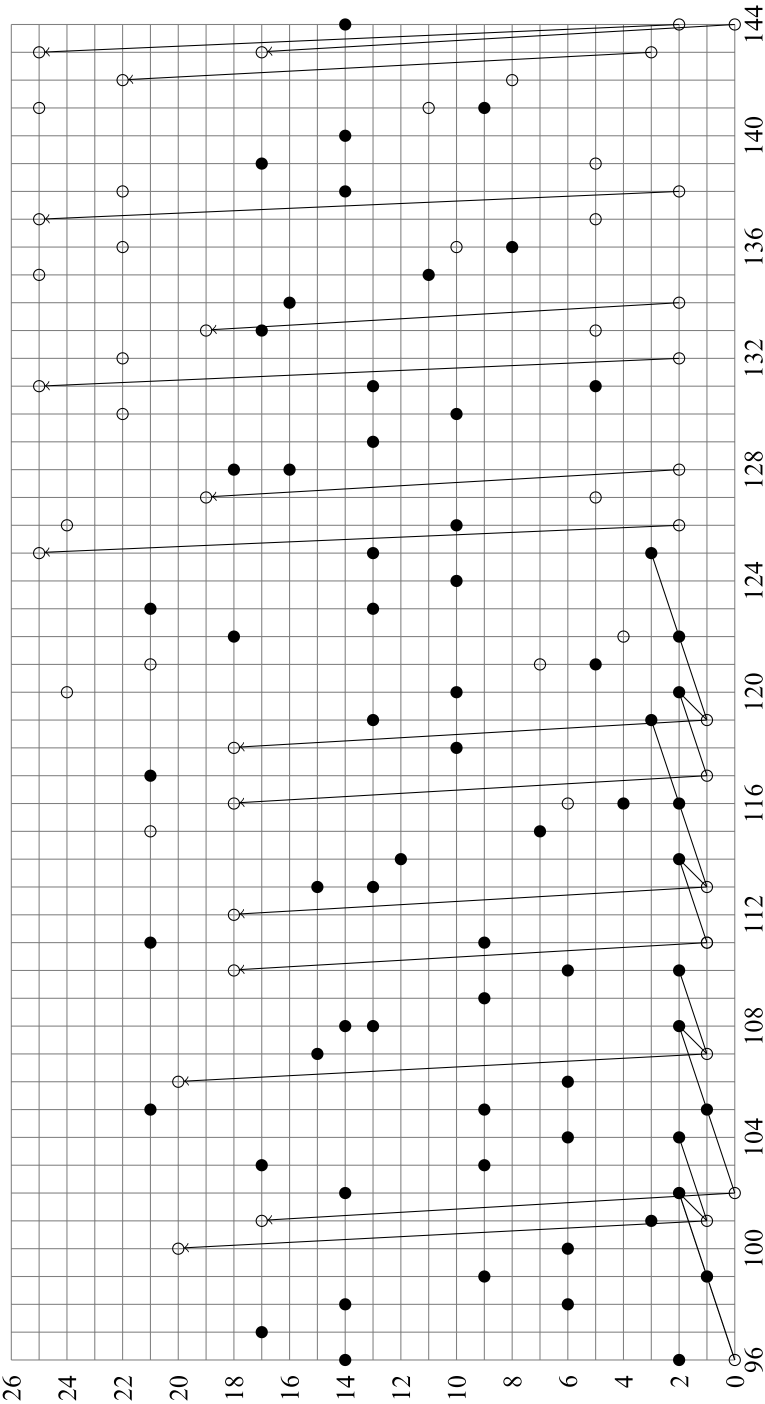


Figure 22: HFPSS for $A_1[00]$ and $A_1[11]$ from E_7–term with $96 \le t - s \le 144$.

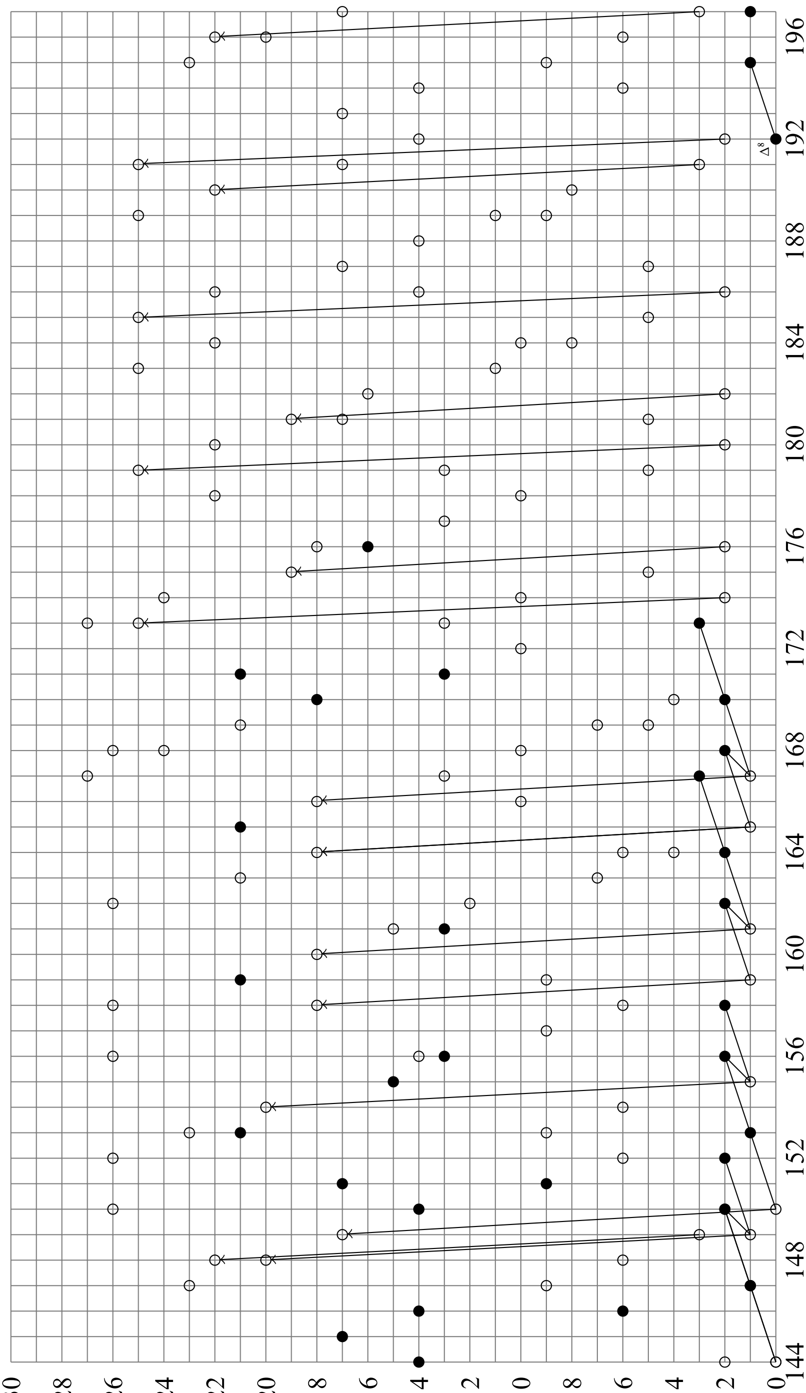


Figure 23: HFPSS for $A_1[00]$ and $A_1[11]$ from E_7–term with $144 \le t - s \le 197$.

Moreover, for degree reasons, these classes must converge to nontrivial elements of $\pi_*(\mathrm{tmf}\wedge A_1)/(\bar\kappa,\nu)$ in the appropriate stems. Therefore, $\mathbb{W}\otimes_{\mathbb{Z}_2}\pi_*(\mathrm{tmf}\wedge A_1)/(\bar\kappa,\nu)$ has the same order as $\pi_*(E_C^{hG_{24}}\wedge A_1)$ up to stem 96 and in stem 101.

All of the classes

$$\begin{gathered} w_2^2e[0,0],\quad w_2^2e[1,5],\quad w_2^2e[1,6],\quad w_2^2e[2,11],\\ w_2^2e[3,15],\quad w_2^2e[3,17],\quad w_2^2e[4,21],\quad w_2^2e[4,23] \end{gathered}$$

are d_2–cycles in the ASS and the d_3–differentials on them can only hit g–multiple classes. Thus, by ν–linearity and the fact that $g\nu=0$ in $\mathrm{Ext}^{5,28}_{\mathcal{A}(2)_*}(\mathbb{F}_2)$, the classes

$$\begin{gathered} \nu w_2^2e[0,0],\quad \nu w_2^2e[1,5],\quad \nu w_2^2e[1,6],\quad \nu w_2^2e[2,11],\\ \nu w_2^2e[3,15],\quad \nu w_2^2e[3,17],\quad \nu w_2^2e[4,21],\quad \nu w_2^2e[4,23] \end{gathered}$$

are d_3–cycles and hence survive to the E_∞–term, by sparseness. As above, these classes must converge to nontrivial elements of $\pi_*(\mathrm{tmf}\wedge A_1)/(\bar\kappa,\nu)$ in the appropriate stems. It follows that $\mathbb{W}\otimes_{\mathbb{Z}_2}\pi_*(\mathrm{tmf}\wedge A_1)/(\bar\kappa,\nu)$ has the same order as $\pi_*(E_C^{hG_{24}}\wedge A_1)$ for stems from 96 to 144.

Consider the classes

$$\begin{gathered} \nu w_2^3e[0,0],\quad \nu w_2^3e[1,5],\quad \nu w_2^3e[1,6],\quad \nu w_2^3e[2,11],\\ \nu w_2^3e[3,15],\quad \nu w_2^3e[3,17],\quad \nu w_2^3e[4,21],\quad \nu w_2^3e[4,23]. \end{gathered} \tag{24}$$

As above, these classes survive to the E_4–term of the ASS for $\mathrm{tmf}\wedge A_1$. By sparseness, $\nu w_2^3e[4,23]$ survives to the E_∞–term and converges to a nontrivial element of $\pi_{170}(\mathrm{tmf}\wedge A_1)/(\bar\kappa,\nu)$. By sparseness, the other classes can only support d_4–differentials hitting the classes

$$g^7e[1,6],\ g^7e[2,11],\ g^6e[6,32],\ g^7e[3,17],\ g^7e[4,21],\ g^7e[4,23],\ g^6e[9,47],$$

respectively. However, the class

$$g^ke[i,j]\quad\text{for}\ (i,j)\in\{(1,6),(2,11),(6,32),(3,17),(4,21),(4,23),(9,47)\}$$

is killed by a differential for a certain integer k less than 7, and hence $g^7e[i,j]$ for these (i,j) is killed by a differential on a certain g–multiple class. This means that

$$\nu w_2^3e[0,0],\quad \nu w_2^3e[1,5],\quad \nu w_2^3e[2,11],\quad \nu w_2^3e[3,15],\quad \nu w_2^3e[3,17]$$

survive to the E_∞–term, hence, as above, to nontrivial elements of $\pi_*(\mathrm{tmf}\wedge A_1)/(\bar\kappa,\nu)$. Next, the map Θ sends $e[6,32]$ and $e[9,47]$ to $e[2,32]$ and $e[3,47]$, respectively. The latter are both annihilated by $\bar{\kappa}^4$, so that $g^4 e[6,32]$ and $g^4 e[9,47]$ are hit by certain differentials in the ASS; hence, $g^6 e[6,32]$ and $g^6 e[9,47]$ are hit by differentials supported on g–multiple classes. As above, this implies that $\nu w_2^3 e[1,6]$ and $\nu w_2^3 e[4,23]$ survive to nontrivial elements of $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$. In total, we have proved that all classes of (24) converge to nontrivial elements of $\pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$; as a consequence, $\mathbb{W} \otimes_{\mathbb{Z}_2} \pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu)$ has the same order as $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa}, \nu)$ in stems from 144 to 192.

Together with the fact that $\pi_*(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa}, \nu)$ is Δ^8–periodic, we conclude that Θ' is a surjection, and hence is an isomorphism.

For part (b), there is a commutative diagram

$$\begin{array}{ccc}
\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(\mathrm{tmf} \wedge A_1) & \xrightarrow{\Theta} & \pi_*(E_C^{hG_{24}} \wedge A_1) \\
\downarrow & & \downarrow \\
\mathbb{W}(\mathbb{F}_4) \otimes_{\mathbb{Z}_2} \pi_*(\mathrm{tmf} \wedge A_1)/(\bar{\kappa}, \nu) & \xrightarrow{\Theta'} & \pi_*(E_C^{hG_{24}} \wedge A_1)/(\bar{\kappa}, \nu)
\end{array}$$

Part (b) then follows from part (a) and the fact that $\pi_*(\mathrm{tmf} \wedge A_1)$ is bounded below.

Part (c) follows from parts (a)–(b) and the fact that Δ^8 is invertible in $\pi_*(E_C^{hG_{24}})$. □

Figures 18 to 20 represent the HFPSS for $E_C^{hG_{24}} \wedge A_1[10]$ and $E_C^{hG_{24}} \wedge A_1[01]$ from the E_7–term on. Each black dot • represents a class generating a group $\mathbb{F}_4$ which survives to the E_∞–term. Each circle ∘ represent a class which either is hit by a differential or supports a differential higher than d_5. We only represent the differentials on generators listed in Proposition 5.3.11 but not those generated by $\bar{\kappa}$–linearity.

Figures 21 to 23 represent the HFPSS for $E_C^{hG_{24}} \wedge A_1[00]$ and $E_C^{hG_{24}} \wedge A_1[11]$ from the E_7–term on. Each black dot • represents a class generating a group $\mathbb{F}_4$ which survives to the E_∞–term. Each circle ∘ represent a class which either is hit by a differential or supports a differential higher than d_5. We only represent the differentials on generators listed in Proposition 5.3.11 but not those generated by $\bar{\kappa}$–linearity.

References

[1] **J F Adams**, *Stable homotopy and generalised homology*, University of Chicago Press (1974) MR Zbl

[2] **D W Anderson**, **E H Brown, Jr**, **F P Peterson**, *Pin cobordism and related topics*, Comment. Math. Helv. 44 (1969) 462–468 MR Zbl

[3] **T Bauer**, *Computation of the homotopy of the spectrum* `tmf`, from "Groups, homotopy and configuration spaces" (N Iwase, T Kohno, R Levi, D Tamaki, J Wu, editors), Geom. Topol. Monogr. 13, Geom. Topol. Publ., Coventry (2008) 11–40 MR Zbl

[4] **T Bauer**, *Bousfield localization and the Hasse square*, from "Topological modular forms" (C L Douglas, J Francis, A G Henriques, M A Hill, editors), Mathematical Surveys and Monographs 201, Amer. Math. Soc., Providence, RI (2014) 79–88 MR Zbl

[5] **A Beaudry**, *The algebraic duality resolution at $p = 2$*, Algebr. Geom. Topol. 15 (2015) 3653–3705 MR Zbl

[6] **A Beaudry**, *Towards the homotopy of the $K(2)$–local Moore spectrum at $p = 2$*, Adv. Math. 306 (2017) 722–788 MR Zbl

[7] **M Behrens**, *The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited*, Adv. Math. 230 (2012) 458–492 MR Zbl

[8] **M Behrens**, *The construction of tmf*, from "Topological modular forms" (C L Douglas, J Francis, A G Henriques, M A Hill, editors), Mathematical Surveys and Monographs 201, Amer. Math. Soc., Providence, RI (2014) 131–188 MR Zbl

[9] **M Behrens**, **K Ormsby**, *On the homotopy of $Q(3)$ and $Q(5)$ at the prime 2*, Algebr. Geom. Topol. 16 (2016) 2459–2534 MR Zbl

[10] **P Bhattacharya**, **P Egger**, **M Mahowald**, *On the periodic v_2–self-map of A_1*, Algebr. Geom. Topol. 17 (2017) 657–692 MR

[11] **I Bobkova**, **P G Goerss**, *Topological resolutions in $K(2)$–local homotopy theory at the prime 2*, J. Topol. 11 (2018) 918–957 MR Zbl

[12] **K S Brown**, *Cohomology of groups*, Graduate Texts in Math. 87, Springer (1982) MR Zbl

[13] **D M Davis**, **M Mahowald**, *v_1– and v_2–periodicity in stable homotopy theory*, Amer. J. Math. 103 (1981) 615–659 MR Zbl

[14] **D M Davis**, **M Mahowald**, *Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces*, from "Current trends in algebraic topology, I" (R M Kane, S O Kochman, P S Selick, V P Snaith, editors), CMS Conf. Proc. 2, Amer. Math. Soc., Providence, RI (1982) 297–342 MR Zbl

[15] **P Deligne**, **M Rapoport**, *Les schémas de modules de courbes elliptiques*, from "Modular functions of one variable, II" (P Deligne, W Kuyk, editors), Lecture Notes in Math. 349, Springer (1973) 143–316 MR Zbl

[16] **E S Devinatz**, **M J Hopkins**, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology 43 (2004) 1–47 MR Zbl

[17] **C L Douglas**, **J Francis**, **A G Henriques**, **M A Hill** (editors), *Topological modular forms*, Mathematical Surveys and Monographs 201, Amer. Math. Soc., Providence, RI (2014) MR

[18] **P G Goerss**, *Quasi-coherent sheaves on the moduli stack of formal groups*, preprint (2008) arXiv 0802.0996

[19] **P G Goerss**, **H-W Henn**, *The Brown–Comenetz dual of the $K(2)$–local sphere at the prime 3*, Adv. Math. 288 (2016) 648–678 MR Zbl

[20] **P Goerss**, **H-W Henn**, **M Mahowald**, *The homotopy of $L_2V(1)$ for the prime 3*, from "Categorical decomposition techniques in algebraic topology" (G Arone, J Hubbuck, R Levi, M Weiss, editors), Progr. Math. 215, Birkhäuser, Basel (2004) 125–151 MR Zbl

[21] **P Goerss**, **H-W Henn**, **M Mahowald**, **C Rezk**, *A resolution of the $K(2)$–local sphere at the prime 3*, Ann. of Math. 162 (2005) 777–822 MR

[22] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from "Structured ring spectra" (A Baker, B Richter, editors), London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 MR Zbl

[23] **H-W Henn**, **N Karamanov**, **M Mahowald**, *The homotopy of the $K(2)$–local Moore spectrum at the prime 3 revisited*, Math. Z. 275 (2013) 953–1004 MR Zbl

[24] **A G Henriques**, *The homotopy groups of tmf and of its localizations*, from "Topological modular forms" (C L Douglas, J Francis, A G Henriques, M A Hill, editors), Mathematical Surveys and Monographs 201, Amer. Math. Soc., Providence, RI (2014) 189–205 MR Zbl

[25] **M Hovey**, **N P Strickland**, *Morava K–theories and localisation*, Mem. Amer. Math. Soc. 666, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[26] **S O Kochman**, *Stable homotopy groups of spheres: a computer-assisted approach*, Lecture Notes in Math. 1423, Springer (1990) MR Zbl

[27] **J Lubin**, **J Tate**, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France 94 (1966) 49–59 MR Zbl

[28] **A Mathew**, *The homology of tmf*, Homology Homotopy Appl. 18 (2016) 1–29 MR Zbl

[29] **J Milnor**, *The Steenrod algebra and its dual*, Ann. of Math. 67 (1958) 150–171 MR Zbl

[30] **R M F Moss**, *Secondary compositions and the Adams spectral sequence*, Math. Z. 115 (1970) 283–310 MR Zbl

[31] **D C Ravenel**, *Localization with respect to certain periodic homology theories*, Amer. J. Math. 106 (1984) 351–414 MR Zbl

[32] **D C Ravenel**, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics 121, Academic, Orlando, FL (1986) MR Zbl

[33] **D C Ravenel**, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies 128, Princeton Univ. Press (1992) MR Zbl

[34] **C Rezk**, *Notes on the Hopkins–Miller theorem*, from "Homotopy theory via algebraic geometry and group representations" (M Mahowald, S Priddy, editors), Contemp. Math. 220, Amer. Math. Soc., Providence, RI (1998) 313–366 MR Zbl

[35] **J Rognes**, *The Adams spectral sequence* (2012) Available at https://folk.uio.no/rognes/papers/notes.050612.pdf

[36] **K Shimomura**, *The homotopy groups of the L_2–localized Toda–Smith spectrum $V(1)$ at the prime 3*, Trans. Amer. Math. Soc. 349 (1997) 1821–1850 MR Zbl

[37] **K Shimomura**, *The homotopy groups of the L_2–localized mod 3 Moore spectrum*, J. Math. Soc. Japan 52 (2000) 65–90 MR Zbl

[38] **K Shimomura**, **X Wang**, *The homotopy groups $\pi_*(L_2 S^0)$ at the prime 3*, Topology 41 (2002) 1183–1198 MR Zbl

[39] **K Shimomura**, **A Yabe**, *The homotopy groups $\pi_*(L_2 S^0)$*, Topology 34 (1995) 261–289 MR Zbl

[40] **J H Silverman**, *The arithmetic of elliptic curves*, 2nd edition, Graduate Texts in Math. 106, Springer (2009) MR Zbl

[41] **H Toda**, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies 49, Princeton Univ. Press (1962) MR Zbl

MPIM Bonn
Bonn, Germany

vcpham@math.unistra.fr

Received: 14 December 2020 Revised: 11 May 2021

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2022 is US \$605/year for the electronic version, and \$880/year (+\$65, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 22 Issue 8 (pages 3533–4008) 2022
