HIGHER HOMOTOPY CATEGORIES, HIGHER DERIVATORS, AND K-THEORY

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Abstract. For every ∞-category $\mathcal{C}$, there is a homotopy n-category $\mathrm{h}_n\mathcal{C}$ and a canonical functor $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$. We study these higher homotopy categories, especially in connection with the existence and preservation of (co)limits, by introducing a higher categorical notion of weak colimit. Based on the idea of the homotopy n-category, we introduce the notion of an n-derivator and study the main examples arising from ∞-categories. Following the work of Maltsiniotis and Garkusha, we define K-theory for ∞-derivators and prove that the canonical comparison map from the Waldhausen K-theory of $\mathcal{C}$ to the K-theory of the associated n-derivator $\mathbb{D}_{\mathcal{C}}^{(n)}$ is $(n+1)$-connected. We also prove that this comparison map identifies derivator K-theory of ∞-derivators in terms of a universal property. Moreover, using the canonical structure of higher weak pushouts in the homotopy n-category, we define also a K-theory space $K(\mathrm{h}_n\mathcal{C}, \mathrm{can})$ associated to $\mathrm{h}_n\mathcal{C}$. We prove that the canonical comparison map from the Waldhausen K-theory of $\mathcal{C}$ to $K(\mathrm{h}_n\mathcal{C}, \mathrm{can})$ is n-connected.

Contents

1. Introduction

It has been long understood in homotopy theory that the homotopy category is only a crude invariant of a much richer homotopy–theoretic structure. The problem of finding a suitable formalism for this additional structure, one that encodes homotopy–theoretic extensions of ordinary categorical notions, led to several foundational approaches, each with its own distinctive advantages and characteristics. The theory of ∞-categories (or quasi–categories) [19, 20, 21, 4] is one of several successful approaches to develop useful foundations for the study of homotopy theories and it has led to groundbreaking new perspectives and results in the field.

Even though passing to the homotopy category neglects this additional homotopy–theoretic structure, the general problem of understanding how much information this process retains still poses interesting questions in specific contexts. This has inspired many important developments, for example, in the context of rigidity theorems for homotopy theories [11, 33, 31, 29], derived/homotopical Morita and tilting theory (see [34] for a nice survey), or in connection with K-theory regarded as an invariant of homotopy theories [8, 27, 32, 23].

The theory of derivators, first introduced and developed by Grothendieck [15], Heller [18] and Franke [11], is a different foundational approach, based on the idea of considering the homotopy categories of all diagram categories as a remedy to the defects of the homotopy category (see also [17]). By supplementing the homotopy category with the network of all these (homotopy) categories, it is possible to encode the collection of homotopy (co)limit functors and general homotopy Kan extensions as an enhancement of the homotopy category. This approach provides a different, lower (=2–) categorical formalism for expressing homotopy-theoretic universal properties (see [5] and [35, 7] for some interesting applications). Moreover, Maltsiniotis [23] defined K-theory in the context of derivators with a view towards partially recovering Waldhausen K-theory from the derivator. The K-theory of derivators and its comparison with Waldhausen K-theory has been studied extensively in [6, 13, 14, 23, 25, 26]. In the context of the theory of derivators, the question about the information retained by the homotopy category is then upgraded to the analogous question for the derivator. However, the classical theory of derivators still does not provide in general a faithful representation of a homotopy theory, even though it is possible in certain cases to recover in a non–canonical way the homotopy theory from the derivator (see [30]).

The purpose of this paper is to extend these ideas on the comparison between homotopy theories and homotopy categories or derivators to n-categories (=$(n,1)$-categories), where the ordinary homotopy category is now replaced by the homotopy n-category of an ∞-category. More specifically:

(a) *Higher homotopy categories.* Using the definition of the higher homotopy categories by Lurie [21], we consider the tower of homotopy n-categories $\{\mathrm{h}_n\mathcal{C}\}_{n\geq 1}$ associated to an ∞-category $\mathcal{C}$, and analyse the properties of the comparison maps $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$. (Sections 2–3)

(b) *Higher derivators.* We introduce a higher categorical notion of a derivator which takes values in n-categories. Then we develop the basic theory of higher derivators with a special emphasis on the examples which arise from ∞-categories. (Section 4)

(c) *K-theory of higher derivators.* We extend the definition of derivator K-theory by Maltsiniotis [23] and Garkusha [13, 14] to n-derivators and study the comparison map from Waldhausen K-theory. Our main result shows that the comparison map is $(n+1)$-connected (Theorem 5.5). Moreover, following [25], we prove that this comparison map has a universal property (Theorem 5.13).

(d) *K-theory of homotopy n-categories.* In analogy with the K-theory of triangulated categories [27], we introduce K-theory for n-categories equipped with distinguished squares. In the case of a homotopy n-category, we study the comparison map from Waldhausen K-theory and prove that it is n-connected (Theorem 6.5).

(a) **Higher homotopy categories.** Every ∞-category $\mathcal{C}$ has an associated homotopy n-category $\mathrm{h}_n\mathcal{C}$ and a canonical functor $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$. The construction of the homotopy n-category and its properties are studied in [21]. We review this construction and its properties in Section 2. Intuitively, for $n \geq 1$, $\mathrm{h}_n\mathcal{C}$ is an ∞-category with the same objects as $\mathcal{C}$ and whose mapping spaces are the appropriate Postnikov truncations of the mapping spaces in $\mathcal{C}$. For $n = 1$, the homotopy category $\mathrm{h}_1\mathcal{C}$ is the ordinary homotopy category of $\mathcal{C}$. The collection of homotopy n-categories defines a tower of ∞-categories:

$$\begin{array}{ccccccccc} & & \mathcal{C} & & & & & & \\ & & \downarrow{\scriptstyle \gamma_n} & \searrow^{\gamma_{n-1}} & & & & \searrow^{\gamma_1} & \\ \cdots & \longrightarrow & \mathrm{h}_n\mathcal{C} & \longrightarrow & \mathrm{h}_{n-1}\mathcal{C} & \longrightarrow & \cdots & \longrightarrow & \mathrm{h}_1\mathcal{C} \end{array}$$

which approximates $\mathcal{C}$. One of the defects of the homotopy category $\mathrm{h}_1\mathcal{C}$, which is essentially what the theory of derivators tries to rectify, is that it does not in general inherit (co)limits from $\mathcal{C}$. As a general rule, if $\mathcal{C}$ admits (co)limits, then $\mathrm{h}_1\mathcal{C}$ admits only weak (co)limits – which may or may not be induced from $\mathcal{C}$. (We recall that a weak colimit of a diagram is a cocone on the diagram which admits a morphism to every other such cocone, but this morphism may not be unique in general.) Do the higher homotopy categories have *better* inheritance properties for (co)limits, and in what sense? This question is closely related to the problem of understanding how much information $\mathrm{h}_n\mathcal{C}$ retains from $\mathcal{C}$. We introduce a higher categorical version of weak (co)limit in order to address this question. In analogy with ordinary weak (co)limits, a higher weak colimit is a cocone for which the mapping spaces to other cocones are highly connected, but not necessarily contractible. The relative strength of the weak colimit is measured by how highly connected these mapping spaces are; the connectivity of these mapping spaces is called the order of the weak colimit. The properties of higher weak (co)limits are discussed in Section 3. For any simplicial set K, there is a canonical functor

$$\Phi_n^K \colon \mathrm{h}_n(\mathcal{C}^K) \to \mathrm{h}_n(\mathcal{C})^K$$

which is usually not an equivalence. The properties of this functor are relevant for understanding the interaction of K-colimits in $\mathcal{C}$ and in $\mathrm{h}_n\mathcal{C}$. One of our conclusions (Corollary 3.17) is the following:

$$\Phi_n^K \text{ induces an equivalence: } \mathrm{h}_{n-\dim(K)}(\mathcal{C}^K) \simeq \mathrm{h}_{n-\dim(K)}\big(\mathrm{h}_n(\mathcal{C})^K\big).$$

Moreover, in connection with higher weak colimits in $\mathrm{h}_n\mathcal{C}$, we also conclude (Corollary 3.23):

> Suppose that $\mathcal{C}$ admits finite colimits. Then $\mathrm{h}_n\mathcal{C}$ admits finite coproducts and weak pushouts of order $n-1$. In addition, the functor $\gamma_n \colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ preserves coproducts and sends pushouts in $\mathcal{C}$ to weak pushouts of order $n-1$.

We will explore the connections of higher weak (co)limits with adjoint functor theorems and Brown representability in future joint work with H. K. Nguyen and C. Schrade (continuing our work in [28]).

(b) **Higher derivators.** The main example of a (pre)derivator is given by the 2-functor which sends a small category I to the homotopy category $\mathrm{h}_1(\mathcal{C}^{N(I)})$, for suitable choices of an ∞-category $\mathcal{C}$. Using homotopy n-categories instead, we may consider more generally the example of the enriched functor which sends a simplicial set K to the homotopy n-category $\mathrm{h}_n(\mathcal{C}^K)$. Following the axiomatic definition of ordinary derivators, we introduce the definition of an n-derivator which encapsulates the salient, abstract properties of this example. The basic definitions and properties of (left, right, pointed, stable) n-(pre)derivators, $1 \leq n \leq \infty$, are discussed in Section 4. For any ∞-category $\mathcal{C}$, there is an associated n-prederivator

$$\mathbb{D}^{(n)}_{\mathcal{C}} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat_n}, \quad K \mapsto \mathrm{h}_n(\mathcal{C}^K)$$

where Dia denotes a category of diagram shapes and $\mathsf{Cat_n}$ is the ∞-category of n-categories. These assemble to define a tower of ∞-prederivators:

$$\begin{array}{ccccccc} & & \mathbb{D}^{(\infty)}_{\mathcal{C}} & & & & \\ & & \downarrow & \searrow & & \searrow & \\ \cdots \longrightarrow & & \mathbb{D}^{(n)}_{\mathcal{C}} & \longrightarrow & \mathbb{D}^{(n-1)}_{\mathcal{C}} & \longrightarrow \cdots \longrightarrow & \mathbb{D}^{(1)}_{\mathcal{C}} \end{array}$$

which approximates $\mathbb{D}^{(\infty)}_{\mathcal{C}} \colon K \mapsto \mathcal{C}^K$. The n-prederivator $\mathbb{D}^{(n)}_{\mathcal{C}}$ is an n-derivator if certain homotopy Kan extensions exist in $\mathcal{C}$ and the corresponding base change transformations satisfy the Beck–Chevalley condition. We prove the following fact which can be used to obtain many examples of n-derivators (Proposition 4.17 and Theorem 4.18):

$\mathcal{C}$ admits limits and colimits indexed by diagrams in Dia
if and only if $\mathbb{D}^{(n)}_{\mathcal{C}}$ is an n-derivator.

The motivation for higher derivators is to bridge the gap between ∞-categories and derivators by introducing a hierarchy of intermediate notions, a different one for each categorical level, starting with ordinary derivators. For any fixed $1 \leq n < \infty$, the theory of n-derivators is still not a faithful representation of homotopy theories; it dwells in an $(n+1)$-categorical context in the same way the classical theory of derivators involves a 2-categorical context. In this respect, our approach using n-derivators remains close to the original idea of a derivator, and differs from other recent perspectives on (pre)derivators in which (pre)derivators are reconstructed into a model for the theory of ∞-categories [12, 3, 25]. We will address the problem of comparing suitable nice classes of ∞-categories with n-derivators in future joint work with D.–C. Cisinski.

(c) **K-theory of higher derivators.** K-theory for (pointed, left) derivators was introduced by Maltsiniotis [23] and Garkusha [13, 14]. The basic feature of a derivator that allows this definition of K-theory is that there is a natural notion of cocartesian square for a derivator. The motivation for introducing this K-theory is connected with the problem of recovering Waldhausen K-theory from the derivator. Maltsiniotis [23] conjectured that derivator K-theory satisfies *additivity*, *localization*, and that it *agrees* with Quillen K-theory for exact categories. Cisinski and Neeman [6] proved additivity for the derivator K-theory of stable derivators. In joint work with Muro [26], we proved that localization and agreement with Quillen K-theory cannot both hold. On the other hand, Muro [24] proved that agreement with Waldhausen K-theory holds for K_0 and K_1 (see also [23, Section 6]), and Garkusha [14] obtained further positive results in the case of abelian categories. In Section 5, after a short review of the K-theory of ∞-categories, we define derivator K-theory for general (pointed, left) ∞-derivators. For any pointed ∞-category $\mathcal{C}$ with finite colimits, there is a comparison map to derivator K-theory,

$$\mu_n \colon K(\mathcal{C}) \to K(\mathbb{D}^{(n)}_{\mathcal{C}}),$$

and these comparison maps define a tower of derivator K-theories:

$$\begin{array}{ccccccc} & & K(\mathcal{C}) & & & & \\ & & \downarrow{\scriptstyle \mu_n} & \searrow^{\mu_{n-1}} & & \searrow^{\mu_1} & \\ \cdots & \longrightarrow & K(\mathbb{D}_{\mathcal{C}}^{(n)}) & \longrightarrow & K(\mathbb{D}_{\mathcal{C}}^{(n-1)}) & \longrightarrow \cdots \longrightarrow & K(\mathbb{D}_{\mathcal{C}}^{(1)}) \end{array}$$

which approximates $K(\mathcal{C})$. Our main result on the comparison map μ_n is the following connectivity estimate (Theorem 5.5):

$$\mu_n \text{ is } (n+1)\text{-connected.}$$

We believe that this connectivity estimate is best possible in general (Remarks 5.7 and 5.8). Following the ideas of [25], we also consider the Waldhausen K-theory $K^{W,\mathrm{Ob}}(\mathbb{D})$ of a general (pointed, left) ∞-derivator $\mathbb{D}$. This K-theory always agrees with Waldhausen K-theory (Propositon 5.10), but it is not invariant under equivalences of derivators in general. Similarly to the case of ordinary derivators treated in [25], we prove that the comparison map to derivator K-theory,

$$\mu \colon K^{W,\mathrm{Ob}}(\mathbb{D}) \to K(\mathbb{D}),$$

identifies derivator K-theory in terms of a universal property (Theorem 5.13):

μ is the initial natural transformation to a functor
which is invariant under equivalences of ∞-derivators.

(d) **K-theory of homotopy n-categories.** The motivation for introducing K-theory for homotopy n-categories is to identify the part of Waldhausen K-theory which may be reconstructed from the homotopy n-category. As a basic instance of this phenomenon, we recall that $K_0(\mathcal{C})$ can be recovered from the triangulated category $\mathrm{h}_1\mathcal{C}$ for any stable ∞-category $\mathcal{C}$. The main feature of the homotopy n-category that is needed for our definition of K-theory is the collection of higher weak pushouts which come from pushouts in $\mathcal{C}$. We revisit the properties of homotopy n-categories in Section 6 and discuss possible axiomatizations of these properties. We consider a general notion of K-theory for pointed n-categories with distinguished squares – this is a higher categorical, but more elementary, version of Neeman's K-theory of categories with squares [27]. For a pointed ∞-category $\mathcal{C}$ with finite colimits, we consider the K-theory $K(\mathrm{h}_n\mathcal{C}, \mathrm{can})$ associated to $\mathrm{h}_n\mathcal{C}$ with the canonical structure of higher weak pushouts as distinguished squares. For every $n \geq 1$, there is a comparison map

$$\rho_n \colon K(\mathcal{C}) \to K(\mathrm{h}_n\mathcal{C}, \mathrm{can})$$

and these assemble to define a tower of K-theories:

$$\begin{array}{ccccccc} & & K(\mathcal{C}) & & & & \\ & & \downarrow{\scriptstyle \rho_n} & \searrow^{\rho_{n-1}} & & \searrow^{\rho_1} & \\ \cdots & \longrightarrow & K(\mathrm{h}_n\mathcal{C}, \mathrm{can}) & \longrightarrow & K(\mathrm{h}_{n-1}\mathcal{C}, \mathrm{can}) & \longrightarrow \cdots \longrightarrow & K(\mathrm{h}_1\mathcal{C}, \mathrm{can}) \end{array}$$

which approximates $K(\mathcal{C})$. Our main result on the comparison map ρ_n is the following connectivity estimate (Theorem 6.5):

$$\rho_n \text{ is } n\text{-connected.}$$

This connectivity estimate is best possible in general (Remark 6.10). Let $P_n X$ denote the Postnikov n-truncation of a topological space X, that is, the homotopy groups of $P_n X$ vanish in degrees $> n$ and the canonical map $X \to P_n X$ is $(n+1)$-connected. Based on the connectivity estimate above, we conclude (Corollary 6.11):

$$P_{n-1}K(\mathcal{C}) \text{ depends only on } (\mathrm{h}_n\mathcal{C}, \mathrm{can}).$$

This confirms a recent conjecture of Antieau [1, Conjecture 8.36] in the case of connective K-theory.

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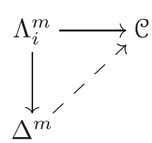
2. Higher homotopy categories

2.1. **n-categories.** We recall the definition and basic properties of n-categories following [21, 2.3.4]. Let $\mathcal{C}$ be an ∞-category and let $n \geq -1$ be an integer. $\mathcal{C}$ is an *n-category* if it satisfies the following conditions:

(1) Given $f, f' : \Delta^n \to \mathcal{C}$, if f and f' are homotopic relative to $\partial\Delta^n$, then $f = f'$.
(We recall that the notion of homotopy employed here means that the two maps are homotopic via equivalences in $\mathcal{C}$.)

(2) Given $f, f' : \Delta^m \to \mathcal{C}$, for $m > n$, if $f_{|\partial\Delta^m} = f'_{|\partial\Delta^m}$, then $f = f'$.

These conditions say that $\mathcal{C}$ has no morphisms in degrees $> n$ and any two morphisms in degree n agree if they are equivalent. The conditions can be equivalently expressed as follows: $\mathcal{C}$ is an n-category, $n \geq 1$, if for every diagram



where $m > n$ and $0 < i < m$, there exists a *unique* dotted arrow which makes the diagram commutative [21, Proposition 2.3.4.9]. Using an inductive argument (see [21, Proposition 2.3.4.7]), it can also be shown that conditions (1) and (2) are equivalent to:

(3) Given a simplicial set K and maps $f, f' : K \to \mathcal{C}$ such that $f_{|\mathrm{sk}_n(K)}$ and $f'_{|\mathrm{sk}_n(K)}$ are homotopic relative to $\mathrm{sk}_{n-1}(K)$, then $f = f'$.

An important immediate consequence of (3) is that for every n-category $\mathcal{C}$, the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ is again an n-category for any simplicial set K [21, Corollary 2.3.4.8].

Example 2.1. The only (-1)-categories up to isomorphism are $\varnothing$ and Δ^0. An ∞-category is a 0-category if and only if it is isomorphic to (the nerve of) a poset. 1-categories are up to isomorphism (nerves of) ordinary categories. See [21, Examples 2.3.4.2-2.3.4.3, Proposition 2.3.4.5].

The property of being an n-category is not invariant under equivalences of ∞-categories. The following proposition gives a characterization of the invariant property that an ∞-category is equivalent to an n-category. We recall that an

∞-groupoid (= Kan complex) X is *n-truncated*, where $n \geq -1$, if it has vanishing homotopy groups in degrees $> n$. We say that X is *(-2)-truncated* if it is contractible.

Proposition 2.2. *Let $\mathcal{C}$ be an ∞-category and let $n \geq -1$ be an integer. Then $\mathcal{C}$ is equivalent to an n-category if and only if $\mathrm{Map}_{\mathcal{C}}(x, y)$ is $(n-1)$-truncated for all $x, y \in \mathcal{C}$.*

Proof. See [21, Proposition 2.3.4.18]. □

2.2. **Higher homotopy categories.** Let $\mathcal{C}$ be an ∞-category and let $n \geq 1$ be an integer. We recall from [21] the construction of the homotopy n-category $\mathrm{h}_n\mathcal{C}$ of $\mathcal{C}$. Given a simplicial set K, we denote by $[K, \mathcal{C}]_n$ the set of maps

$$\mathrm{sk}_n(K) \to \mathcal{C}$$

which extend to $\mathrm{sk}_{n+1}(K)$. Two elements $f, g \in [K, \mathcal{C}]_n$ are called equivalent, denoted $f \sim g$, if the maps $f, g : \mathrm{sk}_n(K) \to \mathcal{C}$ are homotopic relative to $\mathrm{sk}_{n-1}(K)$. The equivalence classes of such maps for $K = \Delta^m$ define the m-simplices of a simplicial set $\mathrm{h}_n\mathcal{C}$, i.e.,

$$(\mathrm{h}_n\mathcal{C})_m : = [\Delta^m, \mathcal{C}]_n / \sim .$$

Clearly an m-simplex of $\mathcal{C}$ defines an m-simplex in $\mathrm{h}_n\mathcal{C}$, so we have a canonical map $\gamma_n \colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$. Note that this map is a bijection in simplicial degrees $< n$ and surjective in degrees n and $n+1$.

The following proposition summarises some of the basic properties of this construction.

Proposition 2.3. *Let $\mathcal{C}$ be an ∞-category and $n \geq 1$.*

(a) The set of maps $K \to \mathrm{h}_n\mathcal{C}$ is in natural bijection with the set $[K, \mathcal{C}]_n / \sim$.
(b) $\mathrm{h}_n\mathcal{C}$ is an n-category. In particular, it is an ∞-category.
(c) $\mathcal{C}$ is an n-category if and only if the map $\gamma_n \colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ is an isomorphism.
(d) Let $\mathcal{D}$ be an n-category. Then the restriction functor along $\gamma_n \colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$,

$$\gamma_n^* \colon \mathrm{Fun}(\mathrm{h}_n\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

is an isomorphism.

Proof. See [21, Proposition 2.3.4.12]. □

Example 2.4. For $n = 1$, the 1-category $\mathrm{h}_1\mathcal{C}$ is isomorphic to the (nerve of the) usual homotopy category of $\mathcal{C}$.

Remark 2.5. For an ∞-category $\mathcal{C}$, the homotopy 0-category $\mathrm{h}_0\mathcal{C}$ can be described in the following way. For $x, y \in \mathcal{C}$, we write $x \leq y$ if $\mathrm{Map}_{\mathcal{C}}(x, y)$ is non-empty. This defines a reflexive and transitive relation. We say that two objects x and y are equivalent if $x \leq y$ and $y \leq x$. Then the relation $\leq$ descends to a partial order on the set of equivalence classes of objects in $\mathcal{C}$. The homotopy 0-category $\mathrm{h}_0\mathcal{C}$ is isomorphic to the nerve of this poset. We will usually ignore the case $n = 0$ and restrict to homotopy n-categories for $n \geq 1$.

Proposition 2.6. *The functor $\gamma_n \colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ is a categorical fibration between ∞-categories. In addition, for every lifting problem*

$$\begin{array}{ccc} \partial\Delta^m & \xrightarrow{u} & \mathcal{C} \\ \downarrow & \nearrow & \downarrow{\scriptstyle \gamma_n} \\ \Delta^m & \xrightarrow[\sigma]{} & \mathrm{h}_n\mathcal{C} \end{array}$$

where $m \leq n+1$ (resp. $m < n$), there is a (unique) filler $\Delta^m \to \mathcal{C}$ which makes the diagram commutative.

Proof. Clearly for any object c in $\mathcal{C}$ and any equivalence $f \colon c \to c'$ in $\mathrm{h}_n\mathcal{C}$, we may find a lift $\tilde{f} \colon c \to c'$ of f in $\mathcal{C}$ (uniquely if $n > 1$), which is again an equivalence. Then we need to show that γ_n is an inner fibration. Consider a lifting problem

$$\begin{array}{ccc} \Lambda^m_i & \xrightarrow{u} & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle \gamma_n} \\ \Delta^m & \longrightarrow & \mathrm{h}_n\mathcal{C} \end{array}$$

where $0 < i < m$. For $m \leq n$, there is a diagonal filler $\Delta^m \to \mathcal{C}$ because γ_n is a bijection in simplicial degrees $< n$ and surjective on n-simplices. For $m > n$, any map $v \colon \Delta^m \to \mathcal{C}$ which extends u defines a diagonal filler for the diagram. To see this, note that v is unique up to homotopy relative to Λ^m_i ($\supseteq \mathrm{sk}_{n-1}\Delta^m$), and then recall that the n-category $\mathrm{h}_n\mathcal{C}$ satisfies condition (3). Therefore γ_n is an inner fibration and this completes the proof of the first claim.

The second claim for $m \leq n$ follows again from the fact that γ_n is bijective in simplicial degrees $< n$ and surjective in degree n. For $m = n+1$, we may find a map $\sigma' \colon \Delta^{n+1} \to \mathcal{C}$ such that $\gamma_n \sigma' = \sigma$, since γ_n is surjective on $(n+1)$-simplices. The maps

$$u, \sigma'_{|\partial\Delta^{n+1}} \colon \partial\Delta^{n+1} \to \mathcal{C}$$

become equal after postcomposition with γ_n. By Proposition 2.3(a), this means that they are homotopic relative to $\mathrm{sk}_{n-1}(\partial\Delta^{n+1})$. Using standard arguments, we may extend this homotopy to a homotopy (relative to $\mathrm{sk}_{n-1}(\Delta^{n+1})$) between σ' and a map $v \colon \Delta^{n+1} \to \mathcal{C}$ such that $v_{|\partial\Delta^{n+1}} = u$. Moreover, since σ' and v are homotopic relative to $\mathrm{sk}_{n-1}(\Delta^{n+1})$, it follows that $\gamma_n v = \gamma_n \sigma' = \sigma$, and therefore v defines a diagonal filler for the diagram. □

Proposition 2.7. *Let $\mathcal{C}$ be an ∞-category and let $n \geq 1$ be an integer. There is a (non-canonical) map*

$$\epsilon \colon \mathrm{sk}_{n+1}\mathrm{h}_n\mathcal{C} \to \mathrm{sk}_{n+1}\mathcal{C}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{sk}_{n+1}\mathcal{C} & \longrightarrow & \mathcal{C} \\ {\scriptstyle \epsilon}\uparrow & & \downarrow{\scriptstyle \gamma_n} \\ \mathrm{sk}_{n+1}\mathrm{h}_n\mathcal{C} & \longrightarrow & \mathrm{h}_n\mathcal{C} \end{array}$$

where the horizontal maps are the canonical inclusions.

Proof. We have a diagram as follows:

$$\begin{array}{ccccccc}
\mathrm{sk}_{n-1}\mathcal{C} & \longrightarrow & \mathrm{sk}_n\mathcal{C} & \longrightarrow & \mathrm{sk}_{n+1}\mathcal{C} & \longrightarrow & \mathcal{C} \\
\downarrow \cong & & \twoheaddownarrow & & \twoheaddownarrow & & \downarrow \gamma_n \\
\mathrm{sk}_{n-1}\mathrm{h}_n\mathcal{C} & \longrightarrow & \mathrm{sk}_n\mathrm{h}_n\mathcal{C} & \longrightarrow & \mathrm{sk}_{n+1}\mathrm{h}_n\mathcal{C} & \longrightarrow & \mathrm{h}_n\mathcal{C}.
\end{array}$$

We may choose a section $\epsilon'\colon \mathrm{sk}_n\mathrm{h}_n\mathcal{C} \to \mathrm{sk}_n\mathcal{C}$ – uniquely up to equivalence. We claim that this section can be extended further to a section ϵ as required. Let $[\sigma']$ be an $(n+1)$-simplex in $\mathrm{h}_n\mathcal{C}$, represented by:

$$\begin{array}{ccc}
\mathrm{sk}_n\Delta^{n+1} & \xrightarrow{\sigma'} & \mathcal{C}. \\
\downarrow & \nearrow_{\sigma} & \\
\Delta^{n+1} & &
\end{array}$$

Using Proposition 2.3(a) for $K = \mathrm{sk}_n(\Delta^{n+1})$, the composition

$$\mathrm{sk}_n\Delta^{n+1} \xrightarrow{\sigma'} \mathrm{sk}_n\mathcal{C} \xrightarrow{\mathrm{sk}_n(\gamma_n)} \mathrm{sk}_n\mathrm{h}_n\mathcal{C} \xrightarrow{\epsilon'} \mathrm{sk}_n\mathcal{C} \to \mathcal{C}$$

is homotopic to σ' relative to $\mathrm{sk}_{n-1}\Delta^{n+1}$. Let $H\colon \mathrm{sk}_n(\Delta^{n+1}) \times \Delta^1 \to \mathcal{C}$ be such a homotopy and consider the map:

$$\Delta^{n+1} \times \{0\} \cup_{\mathrm{sk}_n(\Delta^{n+1})\times\{0\}} \mathrm{sk}_n(\Delta^{n+1}) \times \Delta^1 \xrightarrow{\sigma\cup H} \mathcal{C}.$$

This map extends to a homotopy $H'\colon \Delta^{n+1} \times \Delta^1 \to \mathcal{C}$, which restricts to an $(n+1)$-simplex $\tau' = H'_{|\Delta^{n+1}\times\{1\}}\colon \Delta^{n+1} \to \mathcal{C}$. We set $\epsilon([\sigma'])\colon = \tau'$. Repeating this process for each $[\sigma']$, we obtain the required extension $\epsilon\colon \mathrm{sk}_{n+1}\mathrm{h}_n\mathcal{C} \to \mathrm{sk}_{n+1}\mathcal{C}$. □

Example 2.8. Let $\mathcal{C}$ be an ∞-category. The functor $\gamma_1\colon \mathcal{C} \to \mathrm{h}_1\mathcal{C}$ is bijective on objects, so there is a unique section $\mathrm{sk}_0\mathrm{h}_1\mathcal{C} \to \mathrm{sk}_0\mathcal{C}$. By making choices of morphisms, one from each homotopy class, this map extends to a section $\mathrm{sk}_1\mathrm{h}_1\mathcal{C} \to \mathrm{sk}_1\mathcal{C}$. The last map extends further to a section $\mathrm{sk}_2\mathrm{h}_1\mathcal{C} \to \mathrm{sk}_2\mathcal{C}$ by making (non-canonical) choices of homotopies for compositions.

The functor $\mathrm{h}_n(-)$ preserves categorical equivalences of ∞-categories. Using Proposition 2.3, it follows that there is a tower of ∞-categories:

$$\mathcal{C} \to \cdots \to \mathrm{h}_n\mathcal{C} \to \mathrm{h}_{n-1}\mathcal{C} \to \cdots \to \mathrm{h}_1\mathcal{C}.$$

By construction, the canonical map

$$\mathcal{C} \longrightarrow \lim(\cdots \to \mathrm{h}_n\mathcal{C} \to \mathrm{h}_{n-1}\mathcal{C} \to \cdots \to \mathrm{h}_1\mathcal{C})$$

is an isomorphism, and by Proposition 2.6, this inverse limit defines also a homotopy limit in the Joyal model structure.

Example 2.9. As a consequence of Proposition 2.2, an ∞-groupoid X is categorically equivalent to an n-category if and only if it is n-truncated. For example, a Kan complex is equivalent to a 0-category if and only if it is homotopically discrete and to an 1-category if and only if it is equivalent to the nerve of a groupoid. Given an ∞-groupoid X, the canonical tower of ∞-groupoids:

$$X \to \cdots \mathrm{h}_nX \to \mathrm{h}_{n-1}X \to \cdots \to \mathrm{h}_1X \to \pi_0X$$

models the Postnikov tower of X and the map $X \to \mathrm{h}_n X$ is $(n+1)$-connected (i.e., for every $x \in X$, the map $\pi_k(X, x) \to \pi_k(\mathrm{h}_n X, x)$ is a bijection for $k \leq n$ and surjective for $k = n+1$.)

Remark 2.10. An n-category $\mathcal{C}$ is n-truncated in the ∞-category of ∞-categories, that is, the ∞-groupoid $\mathrm{Map}(K, \mathcal{C})$ is n-truncated for any ∞-category K (see [21, 5.5.6] for the definition and properties of truncated objects in an ∞-category). To see this, recall that $\mathrm{Fun}(K, \mathcal{C})$ is again an n-category and then apply Proposition 2.2. But an n-truncated ∞-category $\mathcal{C}$ is not equivalent to an n-category in general, so the analogue of Example 2.9 fails for general ∞-categories. An ∞-category $\mathcal{C}$ is n-truncated if and only if $\mathcal{C}$ is equivalent to an $(n+1)$-category and the maximal Kan subgroupoid $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ is n-truncated. Indeed, given an n-truncated ∞-category $\mathcal{C}$, then $\mathcal{C}^{\simeq} \simeq \mathrm{Map}(\Delta^0, \mathcal{C})$ is n-truncated. Moreover, since n-truncated objects are closed under limits, it follows that $\mathrm{Map}_{\mathcal{C}}(x, y)$ is n-truncated for every $x, y \in \mathcal{C}$, using the fact that there is a pullback in the ∞-category of ∞-categories:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(x,y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C}. \end{array}$$

Conversely, if $\mathcal{C}$ is an $(n+1)$-category and $\mathcal{C}^{\simeq}$ is n-truncated, then it is possible to show that $\mathrm{Map}(\Delta^k, \mathcal{C})$ is n-truncated by induction on $k \geq 0$, from which it follows that $\mathcal{C}$ is n-truncated. (I am grateful to Hoang Kim Nguyen for interesting discussions related to this remark.)

Let Cat_∞ denote the category of ∞-categories, regarded as enriched in ∞-categories, and let $\mathsf{Cat_n}$ denote the full subcategory of Cat_∞ which is spanned by n-categories.

Proposition 2.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $n \geq 1$ be an integer.*

(a) The natural map $\mathrm{h}_n(\mathcal{C} \times \mathcal{D}) \xrightarrow{\cong} \mathrm{h}_n\mathcal{C} \times \mathrm{h}_n\mathcal{D}$ is an isomorphism.

(b) There is a functor

$$\mathrm{h}_n^{\mathcal{C},\mathcal{D}} \colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{h}_n\mathcal{C}, \mathrm{h}_n\mathcal{D})$$

which is natural in $\mathcal{C}$ and $\mathcal{D}$. In particular, $\mathrm{h}_n \colon \mathsf{Cat}_\infty \to \mathsf{Cat_n}$ is an enriched functor.

Proof. (a) follows directly from the definition of h_n. For (b), we define the functor $\mathrm{h}_n^{\mathcal{C},\mathcal{D}}$ as follows: a k-simplex $F \colon \mathcal{C} \times \Delta^k \to \mathcal{D}$ is sent to the composite

$$\mathrm{h}_n\mathcal{C} \times \Delta^k \cong \mathrm{h}_n\mathcal{C} \times \mathrm{h}_n\Delta^k \cong \mathrm{h}_n(\mathcal{C} \times \Delta^k) \xrightarrow{\mathrm{h}_n F} \mathrm{h}_n\mathcal{D}.$$

The functor $\mathrm{h}_n^{\mathcal{C},\mathcal{D}}$ is natural in $\mathcal{C}$ and $\mathcal{D}$ and turns h_n into an enriched functor. □

3. Higher weak colimits

3.1. **Basic definitions and properties.** It is well known that homotopy categories do not admit small (co)limits in general, even when the underlying ∞-category has small (co)limits. On the other hand, if the ∞-category $\mathcal{C}$ admits, for example, pushouts (resp. coproducts), then the homotopy category $\mathrm{h}_1\mathcal{C}$ admits weak pushouts (resp. coproducts), which are induced from pushouts (resp. coproducts) in $\mathcal{C}$. Moreover, if $\mathcal{C}$ admits small colimits, then $\mathrm{h}_1\mathcal{C}$ admits small weak colimits – which may or may not be induced from $\mathcal{C}$. These observations suggest the following questions: does $\mathrm{h}_n\mathcal{C}$, $n > 1$, have in some sense more or *better* (co)limits than the homotopy category, and how do these compare with (co)limits in $\mathcal{C}$?

We introduce a notion of higher weak (co)limit in the context of ∞-categories which is both a higher categorical version of the classical notion of weak (co)limit and a weak version of the higher categorical notion of (co)limit. We will restrict to higher weak colimits as the corresponding definitions and results about higher weak limits are obtained dually.

We begin with the definition of a weakly initial object. We fix $t \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Definition 3.1. An object x of an ∞-category $\mathcal{C}$ is *weakly initial of order t* if the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is $(t-1)$-connected for every object $y \in \mathcal{C}$.

Example 3.2. If $\mathcal{C}$ is an ordinary category, a weakly initial object $x \in \mathcal{C}$ of order 0 is a weakly initial object in the classical sense. For a general n-category $\mathcal{C}$, a weakly initial object of order n is an initial object.

Proposition 3.3. *Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$. The following are equivalent:*

(1) *x is weakly initial in $\mathcal{C}$ of order t.*
(2) *x is weakly initial in $\mathrm{h}_n\mathcal{C}$ of order t for $n > t$.*
(3) *x is initial in $\mathrm{h}_t\mathcal{C}$.*

These imply:

(4) *x is initial in $\mathrm{h}_n\mathcal{C}$ for $n < t$.*

Proof. This follows from the fact that the functor $\gamma_n \colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ restricts to an n-connected map $\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathrm{h}_n\mathcal{C}}(x, y)$ for every $x, y \in \mathcal{C}$. □

Proposition 3.4. *Let $\mathcal{C}$ be an ∞-category and let $t > 0$. The full subcategory $\mathcal{C}'$ of $\mathcal{C}$ which is spanned by the weakly initial objects of order t is either empty or a t-connected ∞-groupoid.*

Proof. Suppose that the full subcategory $\mathcal{C}'$ is non-empty. Then the mapping spaces of $\mathcal{C}'$ are $(t-1)$-connected, where $t - 1 \geq 0$. It follows that every morphism in $\mathcal{C}'$ is an equivalence, therefore $\mathcal{C}'$ is an ∞-groupoid. □

Remark 3.5. The full subcategory $\mathcal{C}'$ of weakly initial objects in $\mathcal{C}$ of order 0 is not an ∞-groupoid in general. In this case, we only have that $\mathrm{Map}_{\mathcal{C}}(x, y)$ is non-empty for every $x, y \in \mathcal{C}'$.

Definition 3.6. Let $\mathcal{C}$ be an ∞-category, K a simplicial set, and let $F \colon K \to \mathcal{C}$ be a K-diagram in $\mathcal{C}$. A weakly initial object $G \in \mathcal{C}_{F/}$ of order t is called a *weak colimit of F of order t*.

Example 3.7. If $\mathcal{C}$ is an n-category and $G \colon K^{\triangleright} \to \mathcal{C}$ is a weak colimit of $F = G_{|K} \colon K \to \mathcal{C}$ order $t \geq n$, then G is a colimit diagram. This follows from Example 3.2 using the fact that $\mathcal{C}_{F/}$ is again an n-category (see [21, Corollary 2.3.4.10]). In particular, a weak colimit of order ∞ is a colimit diagram. If $\mathcal{C}$ is an ordinary category, then a weak colimit of order 0 is a weak colimit diagram in the classical sense.

Remark 3.8. There is an important difference between weak colimits of order 0 and weak colimits of order > 0: as a consequence of Proposition 3.4, any two weak colimits of F of order > 0 are equivalent. In particular, if $G \in \mathcal{C}_{F/}$ is a weak colimit of order $t > 0$ and $G' \in \mathcal{C}_{F/}$ is a weak colimit of order > 0, then G' is also a weak colimit of order t.

The following proposition gives an alternative characterization of higher weak colimits following the analogous characterization for colimits in [21, Lemma 4.2.4.3].

Proposition 3.9. *Let $\mathcal{C}$ be an ∞-category, K a simplicial set, and let $G\colon K^{\triangleright} \to \mathcal{C}$ be a diagram with cone object $x \in \mathcal{C}$. Then G is a weak colimit of $F = G_{|K}$ of order t if and only if the canonical restriction map:*

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{Map}_{\mathcal{C}^{K^{\triangleright}}}(G, c_y) \to \mathrm{Map}_{\mathcal{C}^{K}}(F, c_y)$$

is t-connected for every $y \in \mathcal{C}$, where c_y denotes the constant K-diagram at $y \in \mathcal{C}$.

Proof. The fiber of the restriction map over $F'\colon K^{\triangleright} \to \mathcal{C}$ with cone object y is identified with $\mathrm{Map}_{\mathcal{C}_{F/}}(G, F')$ (see the proof of [21, Lemma 4.2.4.3]). □

The basic rules for the manipulation of higher weak colimits can be established similarly as for colimits. The following procedure shows that higher weak colimits can be computed iteratively, exactly like colimits, but with the difference that the order of the weak colimit may decrease with each iteration.

Proposition 3.10. *Let $\mathcal{C}$ be an ∞-category and let $K = K_1 \cup_{K_0} K_2$ be a simplicial set where $K_0 \subseteq K_1$ is a simplicial subset. Let $F\colon K \to \mathcal{C}$ be a diagram and denote its restrictions by $F_i\colon = F_{|K_i}$, $i = 0, 1, 2$. Suppose that $G_i\colon K_i^{\triangleright} \to \mathcal{C}$ is a weak colimit of F_i of order t_i, $i = 0, 1, 2$.*

- *(a) There are morphisms $G_0 \to G_{1|K_0^{\triangleright}}$ and $G_0 \to G_{2|K_0^{\triangleright}}$ in $\mathcal{C}_{F_0/}$. These together with G_1 and G_2 determine a diagram in $\mathcal{C}$ as follows,*

  $$H\colon K_1^{\triangleright} \cup_{K_0 * \Delta^{\{1\}}} (K_0 * \Delta^1) \cup_{K_0 * \Delta^{\{0\}}} (K_0 * \Delta^1) \cup_{K_0 * \Delta^{\{1\}}} K_2^{\triangleright} \to \mathcal{C}.$$

- *(b) Let $H_{\ulcorner}\colon \Delta^1 \cup_{\Delta^0} \Delta^1 \to \mathcal{C}$ be the restriction of H to the cone objects. Suppose that $H'\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ is a weak pushout of $H_{\ulcorner}$ of order k. Then H' determines a cocone $G\colon K^{\triangleright} \to \mathcal{C}$ over F, with the same cone object as H', which is a weak colimit of F of order $\ell\colon = \min(k, t_1, t_0 - 1, t_2)$.*

Proof. (a) is clear by the properties of higher weak colimits. For (b), we first explain the construction of the cocone $G\colon K^{\triangleright} \to \mathcal{C}$. The functor H' is represented by a diagram

$$\begin{array}{ccc} x_0 & \xrightarrow{u} & x_1 \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ x_2 & \xrightarrow[g]{} & y \end{array}$$

where x_i is the cone object of G_i, and the morphisms u and v are given respectively by the morphisms $G_0 \to G_{1|K_0^{\triangleright}}$ and $G_0 \to G_{2|K_0^{\triangleright}}$ in $\mathcal{C}_{F_0/}$. The morphisms f and g produce two new cocones essentially uniquely: $G_1'\colon K_1^{\triangleright} \to \mathcal{C}$ over F_1, and $G_2'\colon K_2^{\triangleright} \to \mathcal{C}$ over F_2, with common cone object y. The restrictions $G'_{1|K_0^{\triangleright}}$ and $G'_{2|K_0^{\triangleright}}$ are equivalent as cocones over F_0. We may then extend $G'_{2|K_0^{\triangleright}}$ in an essentially unique way to a new cocone $G_1''\colon K_1^{\triangleright} \to \mathcal{C}$ over F_1, which is equivalent to G_1'. The resulting cocones

$$G'_{2|K_0^{\triangleright}}\colon K_0^{\triangleright} \to \mathcal{C},\ G_1''\colon K_1^{\triangleright} \to \mathcal{C}, \text{and } G_2'\colon K_2^{\triangleright} \to \mathcal{C}$$

assemble to define the required cocone $G\colon K^{\triangleright} \to \mathcal{C}$. Then the claim in (b) is shown by applying Proposition 3.9, first for weak pushouts and then for K_i-diagrams, and using the equivalence of ∞-categories: $\mathcal{C}^K \simeq \mathcal{C}^{K_1} \times_{\mathcal{C}^{K_0}} \mathcal{C}^{K_2}$. □

Example 3.11. Let $\mathcal{C}$ be an ordinary category that admits small coproducts and weak pushouts. By Proposition 3.10, every diagram $F\colon K \to \mathcal{C}$ where K is 1-dimensional admits a weak colimit (of order 0). Now suppose that $F\colon I \to \mathcal{C}$ is a diagram where I is an arbitrary ordinary small category. Since $\mathcal{C}$ is an ordinary category, a cocone $G\colon I^{\triangleright} \to \mathcal{C}$ over F is determined uniquely by its restriction to a cocone $G'\colon (\mathrm{sk}_1 I)^{\triangleright} \to \mathcal{C}$ over $F_{|\mathrm{sk}_1 I}$, and similarly for morphisms between cocones. As a consequence, we may deduce the well known fact that $\mathcal{C}$ has weak I-colimits.

Example 3.12. Let $T \subset \Delta^1 \times \Delta^2$ be the full subcategory spanned by the objects $(0,i)$, for $i = 0,1,2$, and $(1,0)$. Let $\mathcal{C}$ be an ∞-category with weak pushouts of order t and let $F\colon T \to \mathcal{C}$ be a T-diagram in $\mathcal{C}$. Write $T = T_1 \cup_{T_0} T_2$ where T_1 is spanned by $(0,i)$, $i = 0,1$, and $(1,0)$, T_2 is spanned by $(0,i)$, $i = 1,2$, and $T_0 = \{(0,1)\}$. Using Proposition 3.10, we may compute a weak colimit of F of order t in terms of iterated weak pushouts of order t.

3.2. **Homotopy categories and (co)limits.** Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. By the universal property of $\mathrm{h}_n(-)$, the functor

$$\mathrm{Fun}(K,\mathcal{C}) \to \mathrm{Fun}(K,\mathrm{h}_n\mathcal{C}),$$

which is given by composition with $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$, factors canonically through the homotopy n-category:

$$\Phi_n^K : \mathrm{h}_n\mathrm{Fun}(K,\mathcal{C}) \to \mathrm{Fun}(K,\mathrm{h}_n\mathcal{C}). \tag{3.13}$$

The comparison between K-colimits in $\mathcal{C}$ and in $\mathrm{h}_n\mathcal{C}$ is essentially a question about the properties of the functor Φ_n^K. Note that for $n=1$, Φ_1^K is simply the canonical functor of ordinary categories: $\mathrm{h}_1(\mathcal{C}^K) \to \mathrm{h}_1(\mathcal{C})^K$.

Lemma 3.14. *Let $\mathcal{C}$ be an ∞-category, K a finite dimensional simplicial set of dimension $d > 0$, and let $n \geq 1$ be an integer. The functor*

$$\Phi_n^K : \mathrm{h}_n\mathrm{Fun}(K,\mathcal{C}) \to \mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})$$

satisfies the following:

(a) *Φ_n^K is a bijection in simplicial degrees $< n-d$.*
(b) *Φ_n^K is surjective in simplicial degree $n-d$; it identifies $(n-d)$-simplices which are homotopic relative to the $(n-1)$-skeleton of $\Delta^{n-d} \times K$.*
(c) *Φ_n^K is surjective in simplicial degree $n-d+1$.*

Proof. The m-simplices of $\mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})$ are equivalence classes of maps

$$\mathrm{sk}_n(\Delta^m \times K) \to \mathcal{C}$$

that extend to $\mathrm{sk}_{n+1}(\Delta^m \times K)$. On the other hand, the m-simplices of $\mathrm{h}_n\mathrm{Fun}(K,\mathcal{C})$ are equivalence classes of maps

$$\mathrm{sk}_n(\Delta^m) \times K \to \mathcal{C}$$

that extend to $\mathrm{sk}_{n+1}(\Delta^m) \times K$. The functor Φ_n^K is induced by the canonical map

$$\mathrm{sk}_n(\Delta^m \times K) \to \mathrm{sk}_n(\Delta^m) \times K$$

which is an isomorphism if $d \leq \max(n-m, 0)$. This shows that the map Φ_n^K is surjective in simplicial degrees $\leq n-d+1$. Similarly, the map

$$\mathrm{sk}_{n-1}(\Delta^m \times K) \to \mathrm{sk}_{n-1}(\Delta^m) \times K$$

is an isomorphism if $d \leq \max(n-m-1, 0)$, so the two equivalence relations agree for $m < n-d$. □

Remark 3.15. The case $d = 0$ is both special and essentially trivial, since the functor Φ_n^K is an isomorphism in this case.

Proposition 3.16. *Let $\mathcal{C}$ be an ∞-category, K a finite dimensional simplicial set of dimension $d > 0$, and let $n \geq 1$ be an integer. Then for every lifting problem*

$$\begin{array}{ccc} \partial\Delta^m & \xrightarrow{u} & \mathrm{h}_n(\mathcal{C}^K) \\ \downarrow & \nearrow & \downarrow{\scriptstyle \Phi_n^K} \\ \Delta^m & \xrightarrow[\sigma]{} & (\mathrm{h}_n\mathcal{C})^K \end{array}$$

where $m \leq n-d+1$ (resp. $m < n-d$), there is a (unique) filler $\Delta^m \to \mathrm{h}_n(\mathcal{C}^K)$ which makes the diagram commutative.

Proof. The case $m < n-d+1$ is a direct consequence of Lemma 3.14. For $m = n-d+1 \leq n$, Lemma 3.14 shows that there is a lift $\tau\colon \Delta^m \to \mathrm{h}_n(\mathcal{C}^K)$ of σ, represented by a map $\tau'\colon \Delta^m \times K \to \mathcal{C}$. Since the maps

$$u, \gamma_n \circ \tau'_{|\partial\Delta^m}\colon \partial\Delta^m \to \mathrm{h}_n(\mathcal{C}^K)$$

become equal after composition with Φ_n^K, it follows that they correspond to the equivalence classes of maps

$$\widetilde{u}, \tau'_{|\partial\Delta^m \times K}\colon \partial\Delta^m \times K \to \mathcal{C}$$

which are homotopic relative to $\mathrm{sk}_{n-1}(\partial\Delta^m \times K)$. Let $H\colon (\partial\Delta^m \times K) \times J \to \mathcal{C}$ be a homotopy from $\tau'_{|\partial\Delta^m \times K}$ to $\widetilde{u}$, where $J = N(0 \rightleftarrows 1)$ denotes the Joyal interval object. Using standard arguments, this homotopy can be extended to a homotopy

$$H'\colon (\Delta^m \times K) \times J \to \mathcal{C}$$

from $\tau' = H'_{|\Delta^m \times K \times \{0\}}$ which is constant on $\mathrm{sk}_{n-1}(\Delta^m \times K)$. Then the map

$$\widetilde{\tau}\colon = H'_{|\Delta^m \times K \times \{1\}}\colon \Delta^m \times K \to \mathcal{C}$$

extends $\widetilde{u}$ and its equivalence class in $\mathrm{h}_n(\mathcal{C})^K_m$ is equal to σ, so it defines a filler as required. □

As a consequence of Proposition 3.16, we obtain the following result about the comparison between the n-categories $\mathrm{h}_n(\mathcal{C}^K)$ and $\mathrm{h}_n(\mathcal{C})^K$.

Corollary 3.17. *Let $\mathcal{C}$ be an ∞-category, K a finite dimensional simplicial set of dimension $d > 0$, and let $n \geq d$ be an integer. The functor $\Phi_n^K\colon \mathrm{h}_n(\mathcal{C}^K) \to (\mathrm{h}_n\mathcal{C})^K$ is essentially surjective and for every pair of objects F, G in $\mathrm{h}_n(\mathcal{C}^K)$, the induced map between mapping spaces*

$$\mathrm{Map}_{\mathrm{h}_n(\mathcal{C}^K)}(F, G) \longrightarrow \mathrm{Map}_{(\mathrm{h}_n\mathcal{C})^K}(\Phi_n^K(F), \Phi_n^K(G))$$

is $(n-d)$-connected. As a consequence, Φ_n^K induces an equivalence:

$$\mathrm{h}_{n-d}\big(\mathrm{Fun}(K, \mathcal{C})\big) \simeq \mathrm{h}_{n-d}\big(\mathrm{Fun}(K, \mathrm{h}_n\mathcal{C})\big). \tag{3.18}$$

Now assume that $\mathcal{C}$ is an ∞-category which admits K-colimits, where K is a simplicial set. We have colimit-functors:

$$\begin{array}{ccc} \mathrm{Fun}(K,\mathcal{C}) & \xrightarrow{\mathrm{colim}_K} & \mathcal{C} \\ {\scriptstyle \gamma_n}\downarrow & & \downarrow{\scriptstyle \gamma_n} \\ \mathrm{h}_n\mathrm{Fun}(K,\mathcal{C}) & \xrightarrow{\mathrm{h}_n(\mathrm{colim}_K)} & \mathrm{h}_n\mathcal{C}. \end{array}$$

Assuming also that $\mathcal{C}$ and K are as in Corollary 3.17, and passing to the homotopy $(n-d)$-categories as in (3.18), we obtain the following corollary.

Corollary 3.19. *Let $\mathcal{C}$ be an ∞-category, K a finite dimensional simplicial set of dimension $d > 0$, and let $n \geq d$ be an integer. Suppose that $\mathcal{C}$ admits K-colimits. Then there is an adjoint pair*

$$\mathrm{h}_{n-d}(\mathrm{colim}_K)\colon \mathrm{h}_{n-d}\big(\mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})\big) \rightleftarrows \mathrm{h}_{n-d}(\mathcal{C})\colon \mathrm{h}_{n-d}(c)$$

where c denotes the constant K-diagram functor.

Remark 3.20. Corollary 3.19 produces a *truncated K-colimit functor* for $\mathrm{h}_n\mathcal{C}$:

$$\mathrm{h}_{n-d}(\mathrm{colim}_K)\colon \mathrm{h}_{n-d}\big(\mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})\big) \to \mathrm{h}_{n-d}(\mathcal{C}).$$

According to Proposition 2.7, there is a (non-canonical) section

$$\epsilon\colon \mathrm{sk}_{n-d+1}\mathrm{h}_{n-d}(\mathcal{C}^K) \to \mathrm{sk}_{n-d+1}(\mathcal{C}^K).$$

Using this section ϵ, we may consider the *partial K-colimit functor* for $\mathrm{h}_n\mathcal{C}$ (which depends on $\mathcal{C}$ and ϵ):

$$\mathrm{sk}_{n-d+1}\big(\mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})\big) \to \mathrm{h}_n(\mathcal{C})$$

that is defined by the following diagram

$$\begin{array}{ccccccc} \mathrm{sk}_{n-d+1}\big(\mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})\big) & \longrightarrow & \mathrm{sk}_{n-d+1}(\mathcal{C}) & \longrightarrow & \mathcal{C} & \xrightarrow{\gamma_n} & \mathrm{h}_n(\mathcal{C}) \\ \downarrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{colim}_K} & & \uparrow{\scriptstyle \mathrm{h}_n(\mathrm{colim}_K)} \\ \mathrm{sk}_{n-d+1}\big(\mathrm{h}_{n-d}\mathrm{Fun}(K,\mathrm{h}_n\mathcal{C})\big) & & & & & & \\ \downarrow{\scriptstyle (3.18)} & & & & & & \\ \mathrm{sk}_{n-d+1}\big(\mathrm{h}_{n-d}(\mathcal{C}^K)\big) & \xrightarrow{\epsilon} & \mathrm{sk}_{n-d+1}\big(\mathcal{C}^K\big) & \longrightarrow & \mathcal{C}^K & \longrightarrow & \mathrm{h}_n(\mathcal{C}^K). \end{array}$$

Proposition 3.21. *Let $\mathcal{C}$ be an ∞-category with weak K-colimits of order k, where K is a simplicial set of dimension $d > 0$, and let $n \geq 1$ be an integer. Then $\mathrm{h}_n\mathcal{C}$ has weak K-colimits of order $\ell = \min(n-d, k)$. Moreover, the functor $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ sends weak K-colimits of order k to weak K-colimits of order ℓ.*

Proof. We may assume that $n \geq d$ and therefore the functor $\mathcal{C}^K \to (\mathrm{h}_n\mathcal{C})^K$ is surjective on objects. Then it suffices to prove the second claim. Let $G\colon K^{\triangleright} \to \mathcal{C}$ be a weak colimit of $F = G_{|K}$ of order k with cone object $x \in \mathcal{C}$. We claim that the canonical map

$$\mathrm{Map}_{(\mathrm{h}_n\mathcal{C})^{K^{\triangleright}}}(G, c_y) \to \mathrm{Map}_{(\mathrm{h}_n\mathcal{C})^K}(F, c_y)$$

is ℓ-connected for every $y \in \mathcal{C}$. Note that there is a k-connected map:

$$\mathrm{Map}_{(\mathrm{h}_n\mathcal{C})^{K^{\triangleright}}}(G, c_y) \simeq \mathrm{Map}_{\mathrm{h}_n\mathcal{C}}(x,y) \simeq \mathrm{Map}_{\mathrm{h}_n(\mathcal{C}^{K^{\triangleright}})}(G, c_y) \to \mathrm{Map}_{\mathrm{h}_n(\mathcal{C}^K)}(F, c_y).$$

Hence it suffices to show that the canonical map

$$\mathrm{Map}_{\mathrm{h}_n(\mathcal{C}^K)}(F, c_y) \to \mathrm{Map}_{(\mathrm{h}_n\mathcal{C})^K}(F, c_y)$$

is $(n-d)$-connected. This follows from Corollary 3.17. (Alternatively, note that the last map is identified with the canonical map from the $(n-1)$-truncation of a K^{op}-limit of ∞-groupoids to the K^{op}-limit of the $(n-1)$-truncations of the ∞-groupoids:

$$\mathrm{h}_{n-1}\big(\lim_{K^{\mathrm{op}}} \mathrm{Map}_{\mathcal{C}}(F(-), y)\big) \to \lim_{K^{\mathrm{op}}} \mathrm{h}_{n-1}\big(\mathrm{Map}_{\mathcal{C}}(F(-), y)\big).$$

An inductive argument on d shows that the map is $(n-d)$-connected.) □

Remark 3.22. A different proof of Proposition 3.21 is also possible using elementary lifting arguments based on Proposition 3.16.

Corollary 3.23. *Let $\mathcal{C}$ be an ∞-category which admits finite colimits.*

(a) The homotopy n-category $\mathrm{h}_n\mathcal{C}$ admits finite coproducts and weak pushouts of order $n-1$. Moreover, the functor $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ preserves coproducts and sends pushouts in $\mathcal{C}$ to weak pushouts of order $n-1$.

(b) Suppose that $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ preserves finite colimits. Then $\mathcal{C}$ is equivalent to an n-category.

Proof. (a) $\mathrm{h}_n\mathcal{C}$ admits finite coproducts by Remark 3.15. The existence and preservation of higher weak pushouts is a consequence of Proposition 3.21. (b) is a consequence of [28, Corollary 3.3.5]. □

Example 3.24. Let $\mathcal{C}$ be an ∞-category which has pushouts and let K denote the (nerve of the) "corner" category $\ulcorner\colon = \Delta^1 \cup_{\Delta^0} \Delta^1$. The functor

$$\Phi_1^K\colon \mathrm{h}_1(\mathcal{C}^K) \to (\mathrm{h}_1\mathcal{C})^K$$

is surjective on objects and full. By Corollary 3.17, the pushout-functor on $\mathcal{C}$ induces a truncated pushout-functor:

$$\mathrm{h}_0(\mathrm{colim}_K)\colon \mathrm{h}_0\big(\mathrm{Fun}(K, \mathrm{h}_1\mathcal{C})\big) \to \mathrm{h}_0(\mathcal{C})$$

which is a left adjoint to the constant diagram functor. Furthermore we have a map as follows,

$$\mathrm{sk}_0\big(\mathrm{Fun}(K, \mathrm{h}_1\mathcal{C})\big) \to \mathrm{h}_1(\mathcal{C}) \tag{3.25}$$

which sends $F\colon K \to \mathrm{h}_1\mathcal{C}$ to the pushout of a choice of a lift $\widetilde{F}\colon K \to \mathcal{C}$. This is simply regarded as a map from the *set* of 0-simplices. Moreover, (3.25) extends further to the 1-skeleton, but this involves non-canonical choices which are not unique even up to homotopy. As explained in Remark 3.20, an extension of this type can be obtained from a section $\epsilon\colon \mathrm{sk}_1\mathrm{h}_0(\mathcal{C}^K) \to \mathrm{sk}_1(\mathcal{C}^K)$. The fact that this process cannot be continued to higher dimensional skeleta relates to the non-functoriality of the weak pushouts in the homotopy category.

More generally, for $n \geq 1$, there is a left adjoint truncated pushout-functor:

$$\mathrm{h}_{n-1}\big(\mathrm{Fun}(K, \mathrm{h}_n\mathcal{C})\big) \to \mathrm{h}_{n-1}(\mathcal{C})$$

and partial pushout-functors:

$$\mathrm{sk}_n\big(\mathrm{Fun}(K, \mathrm{h}_n\mathcal{C})\big) \to \mathrm{h}_n(\mathcal{C})$$

which define weak pushouts of order $n-1$.

4. Higher derivators

4.1. **Basic definitions and properties.** We recall that Cat_∞ denotes the category of ∞-categories, regarded as enriched in ∞-categories. Let Dia denote a full subcategory of Cat_∞ which has the following properties:

(Dia 0) Dia contains the (nerves of) finite posets.
(Dia 1) Dia is closed under finite coproducts and under pullbacks along an inner fibration.
(Dia 2) For every $\mathsf{X} \in \mathsf{Dia}$ and $x \in \mathsf{X}$, the ∞-category $\mathsf{X}_{/x}$ is in Dia.
(Dia 3) Dia is closed under passing to the opposite ∞-category.

The main examples of such subcategories of Cat_∞ are the following:

(i) The full subcategory of (nerves of) finite posets.
(ii) The full subcategory of (ordinary) finite direct categores $\mathcal{D}ir_f$. (We recall that an ordinary category C is called *finite direct* if its nerve is a finite simplicial set.)
(iii) The full subcategory $\mathsf{Cat_n} \subset \mathsf{Cat}_\infty$ of n-categories for any $n \geq 1$.
(iv) Cat_∞.

We denote by $\mathsf{Dia}^{\mathrm{op}}$ the opposite category taken 1-categorically, that is, the enrichment of $\mathsf{Dia}^{\mathrm{op}}$ is given by:

$$\underline{\mathrm{Hom}}_{\mathsf{Dia}^{\mathrm{op}}}(\mathsf{X}, \mathsf{Y}) = \underline{\mathrm{Hom}}_{\mathsf{Dia}}(\mathsf{Y}, \mathsf{X}) = \mathrm{Fun}(\mathsf{Y}, \mathsf{X}).$$

Definition 4.1. An ∞-*prederivator* with domain Dia is an enriched functor

$$\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty.$$

An ∞-prederivator $\mathbb{D}$ with domain Dia is an n-*prederivator* if it factors through the inclusion $\mathsf{Cat_n} \subset \mathsf{Cat}_\infty$, that is, $\mathbb{D}$ is an enriched functor

$$\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat_n}.$$

A *strict* morphism of ∞-prederivators is a natural transformation $F \colon \mathbb{D} \to \mathbb{D}'$ between enriched functors. Thus, we obtain a category of ∞-prederivators, denoted by PreDer_∞, which is enriched in ∞-categories. For any $n \geq 1$, there is a full subcategory $\mathsf{PreDer}_n \subset \mathsf{PreDer}_\infty$ spanned by the n-prederivators. In the same way that the classical theory of (pre)derivators is founded on a ((2,2)=)2-categorical context, the general theory of ∞-prederivators involves an $(\infty, 2)$-categorical context. We point out that it would be natural to consider also non-strict morphisms (aka. pseudonatural transformations) between ∞-prederivators, because these employ non-trivial 2-morphisms in Cat_∞, but these will not be needed in this paper.

Notation. Let $\mathbb{D}$ be an ∞-prederivator with domain Dia and let $u \colon \mathsf{X} \to \mathsf{Y}$ be a functor in Dia. We will often denote the induced functor $\mathbb{D}(u) \colon \mathbb{D}(\mathsf{Y}) \to \mathbb{D}(\mathsf{X})$ by u^*. Moreover, if $i_{\mathsf{Y},y} \colon \Delta^0 \to \mathsf{Y}$ is the inclusion of the object $y \in \mathsf{Y}$ and $F \in \mathbb{D}(\mathsf{Y})$, we will often denote the object $\mathbb{D}(i_{\mathsf{Y},y})(F)$ in $\mathbb{D}(\Delta^0)$ by F_y.

Example 4.2. An 1-prederivator with domain $\mathsf{Cat_1}$ is a prederivator in the usual sense [23, 17].

Example 4.3. Let $\mathcal{C}$ be an ∞-category. There is an associated ∞-prederivator (with domain Dia) defined by $\mathbb{D}_{\mathcal{C}}^{(\infty)} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty,\ \mathsf{X} \mapsto \mathrm{Fun}(\mathsf{X}, \mathcal{C})$. Moreover, for any $n \geq 1$,

$$\mathbb{D}_{\mathcal{C}}^{(n)} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat_n}, \quad \mathsf{X} \mapsto \mathrm{h}_n\big(\mathrm{Fun}(\mathsf{X}, \mathcal{C})\big)$$

defines an n-prederivator.

Definition 4.4. An ∞-prederivator $\mathbb{D}\colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is a *left ∞-derivator* if it satisfies the following properties:

(Der 1) For every pair of ∞-categories X and Y in Dia, the functor induced by the inclusions of the factors to the coproduct $\mathsf{X} \sqcup \mathsf{Y}$,

$$\mathbb{D}(\mathsf{X} \sqcup \mathsf{Y}) \to \mathbb{D}(\mathsf{X}) \times \mathbb{D}(\mathsf{Y}),$$

is an equivalence. Moreover, $\mathbb{D}(\varnothing)$ is the final ∞-category Δ^0.

(Der 2) For every ∞-category X in Dia, the functor

$$(i^*_{\mathsf{X},x} = \mathbb{D}(i_{\mathsf{X},x}))_{x \in \mathsf{X}} \colon \mathbb{D}(\mathsf{X}) \to \prod_{x \in \mathsf{X}} \mathbb{D}(\Delta^0)$$

is conservative, i.e., it detects equivalences. We recall that $i_{\mathsf{X},x}\colon \Delta^0 \to \mathsf{X}$ is the functor that corresponds to the object $x \in \mathsf{X}$.

(Der 3) For every morphism $u\colon \mathsf{X} \to \mathsf{Y}$ in Dia, the functor $u^* = \mathbb{D}(u)\colon \mathbb{D}(\mathsf{Y}) \to \mathbb{D}(\mathsf{X})$ admits a left adjoint:

$$u_!\colon \mathbb{D}(\mathsf{X}) \to \mathbb{D}(\mathsf{Y}).$$

(Der 4) Given $u\colon \mathsf{X} \to \mathsf{Y}$ in Dia and $y \in \mathsf{Y}$, consider the following pullback diagram in Dia,

$$\begin{array}{ccc} u_{/y} & \xrightarrow{j_{u/y}} & \mathsf{X} \\ {\scriptstyle p_{u/y}}\downarrow & & \downarrow{\scriptstyle u} \\ \mathsf{Y}_{/y} & \xrightarrow[q_{\mathsf{Y}/y}]{} & \mathsf{Y}. \end{array}$$

Then the canonical base change natural transformation:

$$c_{u,y}\colon (p_{u/y})_! j^*_{u,y} \longrightarrow q^*_{\mathsf{Y}/y} u_!$$

is a natural equivalence of functors.

We define right ∞-derivators dually.

Definition 4.5. An ∞-prederivator $\mathbb{D}\colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is a *right ∞-derivator* if it satisfies (Der1)–(Der2) as stated above together with the following dual versions of (Der3)–(Der4):

(Der3)* For every morphism $u\colon \mathsf{X} \to \mathsf{Y}$ in Dia, the functor $u^* = \mathbb{D}(u)\colon \mathbb{D}(\mathsf{Y}) \to \mathbb{D}(\mathsf{X})$ admits a right adjoint:

$$u_*\colon \mathbb{D}(\mathsf{X}) \to \mathbb{D}(\mathsf{Y}).$$

(Der4)* Given $u\colon \mathsf{X} \to \mathsf{Y}$ in Dia and $y \in \mathsf{Y}$, consider the following pullback diagram in Dia,

$$\begin{array}{ccc} u_{y/} & \xrightarrow{j_{y/u}} & \mathsf{X} \\ {\scriptstyle p_{y/u}}\downarrow & & \downarrow{\scriptstyle u} \\ \mathsf{Y}_{y/} & \xrightarrow[q_{y/\mathsf{Y}}]{} & \mathsf{Y}. \end{array}$$

Then the canonical base change natural transformation:

$$c'_{u,y} \colon q^*_{y/\mathsf{Y}} u_* \longrightarrow (p_{y/u})_* j^*_{y/u}$$

is a natural equivalence of functors.

Example 4.6. Let $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ be a left ∞-derivator. Then the ∞-prederivator

$$\mathbb{D}(-^{\mathrm{op}})^{\mathrm{op}} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty,\ \mathsf{X} \mapsto \mathbb{D}(\mathsf{X}^{\mathrm{op}})^{\mathrm{op}}$$

is a right ∞-derivator.

Definition 4.7. An ∞-prederivator $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is an ∞-*derivator* if it is both a left and a right ∞-derivator.

We also specialize these definitions to n-prederivators as follows.

Definition 4.8. An ∞-prederivator $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is a (left, right) n-*derivator* if it is an n-prederivator and a (left, right) ∞-derivator.

Example 4.9. A (left, right) 1-derivator with domain Cat_1 is a (left, right) derivator in the usual sense [23, 17].

Example 4.10. Let $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\mathsf{n}$ be an n-prederivator, where $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. For any $k < n$, there is an associated k-prederivator:

$$\mathrm{h}_k\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\mathsf{n} \xrightarrow{\mathrm{h}_k} \mathsf{Cat}_k.$$

If $\mathbb{D}$ is a left (right) n-derivator, then $\mathrm{h}_k\mathbb{D}$ is a left (right) k-derivator.

The axioms of Definition 4.4 have the following consequence which is a useful strong version of (Der4) and (Der4)* and identifies a larger class of squares for which the base change transformations are equivalences. Similar results are known for ∞-categories and for ordinary (1-)derivators (see, for example, [4] and [22, 17]).

Proposition 4.11. *Let $\mathbb{D}$ be an ∞-derivator with domain* Dia. *Consider a pullback square in* Dia*:*

$$\begin{array}{ccc} \mathsf{Z} & \xrightarrow{j} & \mathsf{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ \mathsf{W} & \xrightarrow[q]{} & \mathsf{Y}. \end{array}$$

(1) *The canonical base change transformation:*

$$p_! j^* \longrightarrow q^* u_!$$

is a natural equivalence if u is a cocartesian fibration or if q is a cartesian fibration.

(2) *The canonical base change transformation:*

$$q^* u_* \longrightarrow p_* j^*$$

is a natural equivalence if u is a cartesian fibration or if q is a cocartesian fibration.

Proof. Using the duality $\mathbb{D} \mapsto \mathbb{D}(-^{\mathrm{op}})^{\mathrm{op}}$, it suffices to prove only (1). Suppose that u is a cocartesian fibration. Applying (Der 2) and the naturality properties of base change transformations, it suffices to prove the claim only in the case where $W = \Delta^0$:

$$\begin{array}{ccc} \mathsf{X}_y & \xrightarrow{j} & \mathsf{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ \Delta^0 & \xrightarrow[i_y]{} & \mathsf{Y}. \end{array}$$

Using the following factorization of this square and applying (Der4), it suffices to prove the claim for the left square in the following diagram

$$\begin{array}{ccccc} \mathsf{X}_y & \xrightarrow{j} & \mathsf{X}_{/y} & \xrightarrow{j_{u,y}} & \mathsf{X} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_{u/y}}\downarrow & & \downarrow{\scriptstyle u} \\ \Delta^0 & \xrightarrow[i_{(y=y)}]{} & \mathsf{Y}_{/y} & \xrightarrow[q_{\mathsf{Y},y}]{} & \mathsf{Y}. \end{array}$$

Note that the horizontal functors in the left square admit left adjoints

$$\ell_1 \colon \mathsf{X}_{/y} \to \mathsf{X}_y$$

$$\ell_2 \colon \mathsf{Y}_{/y} \to \Delta^0$$

because u is a cocartesian fibration and $\mathsf{Y}_{/y}$ has a terminal object $(y = y)$. Thus, after applying $\mathbb{D}$, we obtain pairs of adjoint functors (j^*, ℓ_1^*) and $(i^*_{(y=y)}, \ell_2^*)$. Therefore the base change transformation between compositions of left adjoints:

$$p_! j^* \longrightarrow i^*_{(y=y)} (p_{u/y})_! \tag{4.12}$$

is conjugate to the natural equivalence

$$\mathrm{id} \colon p^*_{u/y} (\ell_2)^* \simeq (\ell_1)^* p^*$$

which then implies that (4.12) is an equivalence, as claimed.

Suppose now that q is a cartesian fibration. We show that the conjugate base change transformation

$$u^* q_* \to j_* p^* \tag{4.13}$$

is a natural equivalence. Again, by (Der2), it suffices to restrict to the case where $X = \Delta^0$:

$$\begin{array}{ccc} \mathsf{Z} & \xrightarrow{p} & \mathsf{W} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle q} \\ \Delta^0 & \xrightarrow[u]{} & \mathsf{Y}. \end{array}$$

The proof of (4.13) in this case is obtained similarly by dualizing the arguments in the previous proof. □

As in the theory of ordinary derivators, there is an additional axiom which is very useful in practice. Before we state this axiom, we recall that as in the classical case, for every ∞-prederivator $\mathbb{D}$ and $\mathsf{X}, \mathsf{Y} \in \mathsf{Dia}$, there is an *underlying* $(\mathsf{X}-)$*diagram functor*:

$$\mathrm{dia}_{\mathsf{X},\mathsf{Y}} \colon \mathbb{D}(\mathsf{X} \times \mathsf{Y}) \to \mathrm{Fun}(\mathsf{X}, \mathbb{D}(\mathsf{Y}))$$

which is the adjoint of the following composition:

$$\mathsf{X} \cong \mathrm{Fun}(\Delta^0, \mathsf{X}) \xrightarrow{(-\times \mathsf{Y})} \mathrm{Fun}(\mathsf{Y}, \mathsf{X} \times \mathsf{Y}) \xrightarrow{\mathbb{D}} \mathrm{Fun}(\mathbb{D}(\mathsf{X} \times \mathsf{Y}), \mathbb{D}(\mathsf{Y})).$$

We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is *n–full* if it restricts to $(n-1)$-connected maps between the mapping spaces. For example, a full functor between ordinary categories is 1-full in this sense.

Definition 4.14. Let $\mathbb{D}$ be an ∞-prederivator and let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. We say that $\mathbb{D}$ is *n-strong* if the following axiom is satisfied:

(Der 5_n) For every $\mathsf{X} \in \mathsf{Dia}$, the underlying diagram functor

$$\mathrm{dia}_{\Delta^1, \mathsf{X}}\colon \mathbb{D}(\Delta^1 \times \mathsf{X}) \to \mathrm{Fun}(\Delta^1, \mathbb{D}(\mathsf{X}))$$

is n-full and essentially surjective.

Remark 4.15. Similarly to the case of ordinary (pre)derivators, we may also consider a stronger form of the last axiom (cf. [18]) which states that the underlying diagram functor

$$\mathrm{dia}_{\mathcal{I}, \mathsf{X}}\colon \mathbb{D}(\mathcal{I} \times \mathsf{X}) \to \mathrm{Fun}(\mathcal{I}, \mathbb{D}(\mathsf{X}))$$

is n-full and essentially surjective for every $\mathcal{I}$ in Dia which is equivalent in the Joyal model structure to a finite 1-dimensional simplicial set.

Following the definitions of pointed and stable (1-)derivators [23, 17], we also define pointed and stable ∞-(pre)derivators as follows.

Definition 4.16. An ∞-prederivator $\mathbb{D}\colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is called *pointed* if it lifts to the ∞-category $\mathsf{Cat}_{\infty,*}$ of pointed ∞-categories and functors which preserve zero objects. An ∞-derivator $\mathbb{D}\colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ is called *stable* if it is pointed and the associated 1-derivator $\mathrm{h}_1\mathbb{D}$ is stable.

4.2. **Examples of n-derivators.** The examples of n-derivators we are mainly interested in are those where the underlying prederivator arises from an ∞-category as in Example 4.3. Let $\mathsf{Dia} \subset \mathsf{Cat}_\infty$ be a fixed full subcategory satifying the conditions (Dia 0)–(Dia 3).

Proposition 4.17. *Let $\mathcal{C}$ be an ∞-category and let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.*

(a) The n-prederivator $\mathbb{D}^{(n)}_{\mathcal{C}}$ satisfies (Der 1), (Der 2) and (Der 5_n). Moreover, the underlying diagram functor

$$\mathrm{dia}_{\mathcal{I}, \mathsf{X}}\colon \mathbb{D}^{(n)}_{\mathcal{C}}(\mathcal{I} \times \mathsf{X}) \to \mathrm{Fun}(\mathcal{I}, \mathbb{D}^{(n)}_{\mathcal{C}}(\mathsf{X}))$$

is n-full and essentially surjective for every $\mathcal{I}$ in Dia which is equivalent (in the Joyal model structure) to an 1-dimensional simplicial set.

(b) Suppose that $\mathcal{C}$ admits X-colimits (resp. X-limits) for any $\mathsf{X} \in \mathsf{Dia}$. Then $\mathbb{D}^{(n)}_{\mathcal{C}}$ satisfies (Der 3) and (Der 4) (resp. (Der 3) and (Der 4)*). As a consequence, $\mathbb{D}^{(n)}_{\mathcal{C}}$ is a left (resp. right) n-derivator which is n-strong.*

Proof. (a) (Der 1) is obvious. (Der 2) says that the equivalences in $\mathrm{h}_n(\mathcal{C}^{\mathsf{X}})$, $\mathsf{X} \in \mathsf{Dia}$, are given pointwise, which holds by a theorem of Joyal [19, Chapter 5]. (Der 5_n) follows from Corollary 3.17. The second claim also follows from Corollary 3.17 because we may replace $\mathcal{I}$ by an 1-dimensional simplicial set. (b) It suffices to show that (Der 3) and (Der 4) hold for the ∞-prederivator $\mathbb{D}^{(\infty)}_{\mathcal{C}}$, that is, it suffices to show that $\mathcal{C}$ admits Kan extensions along functors in Dia and that Kan extensions are given pointwise by the usual formulas as axiomatized in (Der 4). These claims are established in [4, 6.4.7] (see also [21, 4.3.2-4.3.3]). □

It is often possible to reduce statements about ∞-derivators to corresponding known statements about 1-derivators. This happens, for example, in the case of statements which involve the detection of equivalences. The following proposition shows another instance of this phenomenon and it produces many examples of n-derivators from known examples of 1-derivators.

Theorem 4.18. *Let $\mathcal{C}$ be an ∞-category. The following are equivalent:*

(1) $\mathcal{C}$ admits X-colimits and X-limits for every $\mathsf{X} \in \mathsf{Dia}$.
(2) $\mathbb{D}^{(\infty)}_{\mathcal{C}}$ is an ∞-derivator.
(3) $\mathbb{D}_{\mathcal{C}} = \mathbb{D}^{(1)}_{\mathcal{C}}$ is an 1-derivator.

Proof. (1) $\Rightarrow$ (2) follows from Proposition 4.17 and (2) $\Rightarrow$ (3) is obvious. We prove (3) $\Rightarrow$ (1). We restrict to showing that $\mathcal{C}$ admits X-colimits as the case of X-limits can be treated similarly by duality. Let $F\colon \mathsf{X} \to \mathcal{C}$ be an X-diagram and let $G\colon \mathsf{X}^{\triangleright} \to \mathcal{C}$ be the diagram which is the image of $F \in \mathbb{D}_{\mathcal{C}}(\mathsf{X})$ under

$$u_!\colon \mathbb{D}_{\mathcal{C}}(\mathsf{X}) \longrightarrow \mathbb{D}_{\mathcal{C}}(\mathsf{X}^{\triangleright})$$

where $u\colon \mathsf{X} \to \mathsf{X}^{\triangleright}$ is the canonical inclusion. As a consequence of (Der 4), the functor $u_!$ is fully faithful because u is so. In particular, there is a canonical isomorphism $F \cong u^* u_!(F)$ and therefore we may assume that G is an extension of the diagram F. We claim that G is a colimit diagram in $\mathcal{C}$ for the functor F. For this, it suffices to prove that for each sieve between finite posets $v\colon \mathsf{Y}' \to \mathsf{Y}$ in Dia, the canonical map

$$\begin{array}{c} \mathrm{Map}(\mathsf{Y}, \mathrm{Map}_{\mathcal{C}^{\mathsf{X}^{\triangleright}}}(G, c_y)) \\ \big\downarrow \\ \mathrm{Map}(\mathsf{Y}', \mathrm{Map}_{\mathcal{C}^{\mathsf{X}^{\triangleright}}}(G, c_y)) \times_{\mathrm{Map}(\mathsf{Y}', \mathrm{Map}_{\mathcal{C}^{\mathsf{X}}}(F, c_y))} \mathrm{Map}(\mathsf{Y}, \mathrm{Map}_{\mathcal{C}^{\mathsf{X}}}(F, c_y)) \end{array} \tag{4.19}$$

is a π_0-isomorphism, where c_y denotes the constant functor at $y \in \mathcal{C}$ in the respective functor ∞-category. We will do this by expressing π_0 of the domain and the target of this map (4.19) in terms of morphism sets in the (1-)category $\mathbb{D}_{\mathcal{C}}(\mathsf{X}^{\triangleright} \times \mathsf{Y})$ and then using the derivator properties of $\mathbb{D}_{\mathcal{C}}$.

First, note that we have a canonical isomorphism of morphism sets:

$$\begin{aligned} \mathbb{D}_{\mathcal{C}}(\mathsf{X}^{\triangleright} \times \mathsf{Y})(\pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G), c_y) &= \pi_0\big(\mathrm{Map}_{\mathcal{C}^{\mathsf{X}^{\triangleright} \times \mathsf{Y}}}(\pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G), c_y)\big) \\ &\cong \pi_0\big(\mathrm{Map}(\mathsf{Y}, \mathrm{Map}_{\mathcal{C}^{\mathsf{X}^{\triangleright}}}(G, c_y))\big) \end{aligned} \tag{4.20}$$

where $\pi_{\mathsf{X}^{\triangleright},\mathsf{Y}}\colon \mathsf{X}^{\triangleright} \times \mathsf{Y} \to \mathsf{X}$ denotes the projection functor. Similarly, we have canonical isomorphisms

$$\begin{aligned} \mathbb{D}_{\mathcal{C}}(\mathsf{X} \times \mathsf{Y})(\pi^*_{\mathsf{X},\mathsf{Y}}(F), c_y) &\cong \pi_0\big(\mathrm{Map}(\mathsf{Y}, \mathrm{Map}_{\mathcal{C}^{\mathsf{X}}}(F, c_y))\big) \\ \mathbb{D}_{\mathcal{C}}(\mathsf{X} \times \mathsf{Y}^{'})(\pi^*_{\mathsf{X},\mathsf{Y}'}(F), c_y) &\cong \pi_0\big(\mathrm{Map}(\mathsf{Y}^{'}, \mathrm{Map}_{\mathcal{C}^{\mathsf{X}}}(F, c_y))\big) \end{aligned} \tag{4.21}$$

where $\pi_{\mathsf{X},\mathsf{Y}}$ and $\pi_{\mathsf{X},\mathsf{Y}'}$ denote again the projection functors.

Then consider the following pullback diagram in Dia:

$$\begin{array}{ccc} \mathsf{X} \times \mathsf{Y} & \xrightarrow{\pi_{\mathsf{X},\mathsf{Y}}} & \mathsf{X} \\ {\scriptstyle u\times 1}\downarrow & & \downarrow{\scriptstyle u} \\ \mathsf{X}^{\triangleright} \times \mathsf{Y} & \xrightarrow[\pi_{\mathsf{X}^{\triangleright},\mathsf{Y}}]{} & \mathsf{X}^{\triangleright}. \end{array}$$

Since the horizontal functors are cartesian fibrations, it follows from (Der 4) and Proposition 4.11(1) that the canonical base change morphism in $\mathbb{D}_{\mathcal{C}}(\mathsf{X}^{\triangleright} \times \mathsf{Y})$:

$$\pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G) = \pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}} u_!(F) \xleftarrow{\cong} (u \times 1)_! \pi^*_{\mathsf{X},\mathsf{Y}}(F) \tag{4.22}$$

is an isomorphism.

Let $i\colon \mathsf{A} = \mathsf{X}^{\triangleright} \times \mathsf{Y}' \cup_{\mathsf{X}\times\mathsf{Y}'} \mathsf{X} \times \mathsf{Y} \subset \mathsf{X}^{\triangleright} \times \mathsf{Y}$ denote the full subcategory. This is again in Dia by (Dia 1) and (Dia 3) because it can be described as the pullback of a functor $\mathsf{X}^{\triangleright} \times \mathsf{Y} \to \Delta^1 \times \Delta^1$ along the 'upper corner' inclusion $\Delta^1 \cup_{\Delta^0} \Delta^1 \to \Delta^1 \times \Delta^1$. Note that using (4.21), we can identify the set of components of the target of (4.19) canonically with the morphism set:

$$\begin{aligned} \mathbb{D}_{\mathcal{C}}(\mathsf{A})(i^*\pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G), c_y) &= \mathbb{D}_{\mathcal{C}}(\mathsf{A})(i^*\pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}} u_!(F), c_y) \\ &\cong \pi_0(\text{TARGET OF}(4.19)) \end{aligned} \tag{4.23}$$

and that the map (4.19) on π_0 agrees using the identifications (4.20) and (4.23) with the map defined by the restriction functor

$$i^*\colon \mathbb{D}_{\mathcal{C}}(\mathsf{X}^{\triangleright} \times \mathsf{Y}) \to \mathbb{D}_{\mathcal{C}}(\mathsf{A})$$

Since i is full, it follows from (Der 4) that the unit transformation $1 \to i^* i_!$ of the adjunction $(i_!, i^*)$ is a natural isomorphism. Therefore it suffices to show that the counit morphism

$$i_! i^*\big(\pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G)\big) \to \pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G) \tag{4.24}$$

is an isomorphism.

Consider the following pullback diagram in Dia:

$$\begin{array}{ccc} \mathsf{X} \times \mathsf{Y} & \xrightarrow{\pi_{\mathsf{X},\mathsf{Y}}} & \mathsf{X} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle u} \\ \mathsf{A} & \xrightarrow[q=\pi_{\mathsf{X}^{\triangleright},\mathsf{Y}} i]{} & \mathsf{X}^{\triangleright}. \end{array}$$

The bottom functor q is a cartesian fibration because it is the composition of cartesian fibrations. Therefore it follows from (Der 4) and Proposition 4.11(1) that the canonical base change morphism in $\mathbb{D}_{\mathcal{C}}(\mathsf{A})$:

$$q^*(G) = q^* u_!(F) \xleftarrow{\cong} j_! \pi^*_{\mathsf{X},\mathsf{Y}}(F) \tag{4.25}$$

is an isomorphism. As a consequence of (4.22) and (4.25), we obtain canonical isomorphisms as follows,

$$\begin{aligned} \pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G) = \pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}} u_!(F) &\cong (u \times 1)_! \pi^*_{\mathsf{X},\mathsf{Y}}(F) \\ &\cong i_! j_! \pi^*_{\mathsf{X},\mathsf{Y}}(F) \cong i_! q^* u_!(F) \\ &= i_! i^* \pi^*_{\mathsf{X}^{\triangleright},\mathsf{Y}}(G). \end{aligned}$$

This implies that (4.24) is an isomorphism and therefore, using the adjunction $(i_!, i^*)$ as explained above, it follows that the map (4.19) is a π_0-isomorphism. □

Remark 4.26. We point out a significant simplification of the proof of Theorem 4.18 in the case where Dia is large enough so that it can detect equivalences of ∞-groupoids, i.e., in the case where the following holds: a map of ∞-groupoids $X \to Z$ is an equivalence if and only if

$$\pi_0(\mathrm{Map}(\mathsf{Y}, X)) \longrightarrow \pi_0(\mathrm{Map}(\mathsf{Y}, Z))$$

is an isomorphism for every $\mathsf{Y} \in \mathsf{Dia}$. This happens, for example, when Dia contains all posets – and not just the finite ones. Assuming that Dia is large enough in this sense, then we may restrict to the case $\mathsf{Y}' = \varnothing$ in the proof above, in which case the proof becomes more immediate.

5. K-theory of higher derivators

5.1. **Recollections.** We first recall the ∞-categorical version of Waldhausen's $\mathrm{S}_\bullet$-construction [37, 2]. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. For every $n \geq 0$, let $\mathrm{Ar}[n]$ denote the (nerve of the) category of morphisms of the poset $[n]$. The ∞-category $\mathrm{S}_n\mathcal{C}$ is the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{C})$ spanned by the objects $F\colon \mathrm{Ar}[n] \to \mathcal{C}$ such that:

(i) $F(i \to i)$ is a zero object for all $i \in [n]$.
(ii) For every $i \leq j \leq k$, the following diagram in $\mathcal{C}$,

$$\begin{array}{ccc} F(i \to j) & \longrightarrow & F(i \to k) \\ \downarrow & & \downarrow \\ F(j \to j) & \longrightarrow & F(j \to k) \end{array}$$

is a pushout.

The construction is clearly functorial in $[n]$, $n \geq 0$, and $\mathrm{S}_\bullet\mathcal{C}$ defines a simplicial object of pointed ∞-categories, which is functorial in $\mathcal{C}$ with respect to functors which preserve zero objects and finite colimits. We denote by $\mathrm{S}_\bullet^\simeq\mathcal{C}$ the associated simplicial object of pointed ∞-groupoids, which is obtained by passing pointwise to the maximal ∞-subgroupoids of $\mathrm{S}_\bullet\mathcal{C}$. For $n \geq 1$, the ∞-groupoid $\mathrm{S}_n^\simeq\mathcal{C}$ is equivalent to $\mathrm{Map}(\Delta^{n-1}, \mathcal{C})$. Moreover, we have $\mathrm{S}_0^\simeq\mathcal{C} \simeq \Delta^0$ and we may regard the geometric realization $|\mathrm{S}_\bullet^\simeq\mathcal{C}|$ as canonically pointed by a zero object in $\mathcal{C}$. The Waldhausen K-theory of $\mathcal{C}$ is defined to be the space:

$$K(\mathcal{C})\colon = \Omega|S_\bullet^\simeq\mathcal{C}|.$$

If $\mathcal{C}$ arises from a good Waldhausen category, this definition of K-theory agrees up to homotopy equivalence with the Waldhausen K-theory of the corresponding Waldhausen category (see [2]). The definition of K-theory is functorial with respect to functors $F\colon \mathcal{C} \to \mathcal{C}'$ which preserve zero objects and finite colimits.

Following [37, Lemma 1.4.1] and [25, Proposition 4.2.1], we consider also the following simpler model for Waldhausen K-theory. Restricting to the objects of $\mathrm{S}_\bullet\mathcal{C}$ pointwise, we obtain a simplicial set

$$\mathsf{s}_\bullet\mathcal{C}\colon \Delta^{\mathrm{op}} \to \mathsf{Set}, \quad [n] \mapsto \mathsf{s}_n\mathcal{C}\colon = (\mathrm{S}_n\mathcal{C})_0.$$

There is a canonical comparison map, given by the inclusion of objects,

$$\iota\colon \Omega|\mathsf{s}_\bullet\mathcal{C}| \longrightarrow \Omega|\mathrm{S}_\bullet^\simeq\mathcal{C}| = K(\mathcal{C}).$$

Proposition 5.1. *The comparison map ι is a weak equivalence.*

Proof. The proof is essentially the same as the proof of [37, Lemma 1.4.1 and Corollary] (see also [25, Proposition 4.2.1]). □

5.2. **Derivator K-theory.** We extend the definition of derivator K-theory of Maltsiniotis [23] and Garkusha [13, 14] to general pointed left ∞-derivators. As in the case of ordinary derivators, this definition is based on the following intrinsic notion of cocartesian square.

Let $i \colon \ulcorner = \Delta^1 \cup_{\Delta^0} \Delta^1 \to \square = \Delta^1 \times \Delta^1$ denote the 'upper corner' inclusion. For any left ∞-derivator $\mathbb{D}$, we have an adjunction:

$$i_! \colon \mathbb{D}(\ulcorner) \rightleftarrows \mathbb{D}(\square) \colon i^*.$$

Definition 5.2. Let $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ be a left ∞-derivator with domain Dia. An object $F \in \mathbb{D}(\square)$ is called *cocartesian* if the canonical morphism

$$i_! i^*(F) \to F$$

is an equivalence in $\mathbb{D}(\square)$.

Let $\mathbb{D}$ be a pointed left ∞-derivator (with domain Dia). Before we define the K-theory of $\mathbb{D}$, we first need to introduce some more notation: for every $0 \leq i \leq j \leq k \leq n$, we denote by $i_{i,j,k} \colon \square \to \mathrm{Ar}[n]$ the inclusion of the following square in $\mathrm{Ar}[n]$:

$$\begin{array}{ccc} (i \to j) & \longrightarrow & (i \to k) \\ \downarrow & & \downarrow \\ (j \to j) & \longrightarrow & (j \to k). \end{array}$$

We define $\mathrm{S}_n\mathbb{D}$ to be the full subcategory of $\mathbb{D}(\mathrm{Ar}[n])$ which is spanned by the objects $F \in \mathbb{D}(\mathrm{Ar}[n])$ such that:

(i) $F_{(i \to i)}$ is a zero object for all $i \in [n]$.
(ii) For every $i \leq j \leq k$, the object $i^*_{i,j,k}(F) \in \mathbb{D}(\square)$, which may be depicted as follows:

$$\begin{array}{ccc} F_{(i \to j)} & \longrightarrow & F_{(i \to k)} \\ \downarrow & & \downarrow \\ F_{(j \to j)} & \longrightarrow & F_{(j \to k)}, \end{array}$$

is cocartesian in $\mathbb{D}(\square)$.

The assignment $[n] \mapsto \mathrm{S}_n\mathbb{D}$ defines a simplicial object of pointed ∞-categories. Moreover, it is natural with respect to strict morphisms between pointed left ∞-derivators which preserve the zero objects and cocartesian squares.

Let $\mathrm{S}^{\simeq}_{\bullet}\mathbb{D}$ denote the simplicial object of pointed ∞-groupoids, which is obtained by passing pointwise to the maximal ∞-subgroupoids of $\mathrm{S}_{\bullet}\mathbb{D}$. We have $\mathrm{S}^{\simeq}_0\mathbb{D} \simeq \Delta^0$ and we regard the geometric realization $|\mathrm{S}^{\simeq}_{\bullet}\mathbb{D}|$ as based at a zero object of $\mathbb{D}(\Delta^0)$.

Definition 5.3. Let $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ be a pointed left ∞-derivator with domain Dia. The *derivator K-theory* of $\mathbb{D}$ is defined to be the space:

$$K(\mathbb{D}) \colon = \Omega |\mathrm{S}^{\simeq}_{\bullet}\mathbb{D}|.$$

We note that the definition of derivator K-theory is functorial with respect to strict morphisms $F\colon \mathbb{D} \to \mathbb{D}'$ which preserve the zero objects and cocartesian squares. Moreover, derivator K-theory is invariant under those strict morphisms which are pointwise equivalences of ∞-categories.

Remark 5.4. Waldhausen's Additivity Theorem [37] establishes one of the fundamental properties of Waldhausen K-theory. The analogue of this theorem has been established for the derivator K-theory of stable 1-derivators by Cisinski–Neeman [6], confirming one of Maltsiniotis' conjectures in [23]. It would be interesting to know if the additivity theorem holds more generally for the derivator K-theory of pointed left ∞-derivators.

5.3. **Comparison with Waldhausen K-theory.** Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. Applying the homotopy n-category functor pointwise to the simplicial object $[k] \mapsto \mathrm{S}_k^{\simeq}\mathcal{C}$, we obtain a new simplicial object of (pointed) ∞-groupoids,

$$\mathrm{h}_n\mathrm{S}_\bullet^{\simeq}\mathcal{C} : \Delta^{\mathrm{op}} \to \mathsf{Grpd}_\infty,\ [k] \mapsto \mathrm{h}_n\big(\mathrm{S}_k^{\simeq}\mathcal{C}\big),$$

and there is a canonical comparison map:

$$\mathrm{S}_\bullet^{\simeq}\mathcal{C} \longrightarrow \mathrm{h}_n\big(\mathrm{S}_\bullet^{\simeq}\mathcal{C}\big).$$

Let $\mathbb{D}_{\mathcal{C}}^{(n)}$ be the pointed left n-derivator associated to $\mathcal{C}$ with domain $\mathcal{D}ir_f$ (see Proposition 4.17). As a consequence of the natural identification

$$\mathrm{h}_n\big(\mathrm{S}_\bullet^{\simeq}\mathcal{C}\big) \simeq \mathrm{S}_\bullet^{\simeq}\mathbb{D}_{\mathcal{C}}^{(n)},$$

we obtain a canonical comparison map from Waldhausen to derivator K-theory:

$$\mu_n : K(\mathcal{C}) \to K(\mathbb{D}_{\mathcal{C}}^{(n)}).$$

In addition, the natural morphisms of simplicial objects $\mathrm{h}_n\big(\mathrm{S}_\bullet^{\simeq}\mathcal{C}\big) \to \mathrm{h}_{n-1}\big(\mathrm{S}_\bullet^{\simeq}\mathcal{C}\big)$, for $n > 1$, define a tower of derivator K-theories for $\mathcal{C}$ which is compatible with the comparison maps μ_n:

$$\begin{array}{ccccccc} & K(\mathcal{C}) & & & & & \\ & \downarrow{\scriptstyle \mu_n} & \searrow{\scriptstyle \mu_{n-1}} & & & \searrow{\scriptstyle \mu_1} & \\ \cdots \longrightarrow & K(\mathbb{D}_{\mathcal{C}}^{(n)}) & \longrightarrow & K(\mathbb{D}_{\mathcal{C}}^{(n-1)}) & \longrightarrow \cdots \longrightarrow & & K(\mathbb{D}_{\mathcal{C}}^{(1)}). \end{array}$$

In the case of ordinary derivators, Maltsiniotis conjectured in [23] that the comparison map μ_1 is a weak equivalence for exact categories. The comparison map μ_1 was subsequently studied in [14, 24, 25, 26]. It is known (see [26]) that μ_1 is not a weak equivalence for general $\mathcal{C}$, and moreover, that it will fail to be a weak equivalence even for exact categories *if* derivator K-theory satisfies *localization* – a property which was also conjectured by Maltsiniotis [23].

We prove a general result about the connectivity of the comparison map μ_n for general $\mathcal{C}$ and $n \geq 1$. The connectivity estimate is also a small improvement of the known estimate for $n = 1$ that was shown by Muro [24].

Theorem 5.5. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then the comparison map $\mu_n : K(\mathcal{C}) \to K(\mathbb{D}_{\mathcal{C}}^{(n)})$ is $(n+1)$-connected.*

We will need the following useful elementary fact about simplicial spaces.

Lemma 5.6. *Let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a map of simplicial spaces. Suppose that f_k is $(m-k)$-connected for every $k \geq 0$. Then the map $||f_\bullet|| \colon ||X_\bullet|| \to ||Y_\bullet||$ is m-connected. (Here $||-||$ denotes the fat geometric realization of a simplicial space.)*

Proof. See [10, Lemma 2.4]. □

Proof. (of Theorem 5.5) By Lemma 5.6, it suffices to show that the map of ∞-groupoids

$$\mathrm{S}_k^{\simeq}\mathcal{C} \to \mathrm{h}_n\big(\mathrm{S}_k^{\simeq}\mathcal{C}\big)$$

is $(n+2-k)$-connected for all $k \geq 0$. This holds since the map is an equivalence for $k = 0$ and $(n+1)$-connected for $k > 0$. □

Remark 5.7. The main result of [26, Theorem 1.2] shows that the comparison map μ_1 is not a π_3-isomorphism in general. (In addition, a closer inspection of the proofs in [26] also shows that the map μ_1 will not be 3-connected if derivator K-theory satisfies localization.) It seems likely that the connectivity estimate of Theorem 5.5 is best possible in general.

Remark 5.8. By [14, Theorem 7.1], the comparison map μ_1 is π_*-split injective in the case where $\mathcal{C}$ is the bounded derived category of an abelian category. In fact, it is shown [14] that there is a retraction map to μ_1. As a consequence, the comparison map μ_n also admits a retraction in these cases for all $n \geq 1$. Related to this, an interesting problem suggested by B. Antieau is whether μ_n is a weak equivalence when $\mathcal{C}$ is a stable ∞-category which admits a bounded t-structure.

5.4. **Waldhausen K-theory of derivators.** Waldhausen K-theory for pointed left 1-derivators was defined in [25] and it was shown that it agrees with Waldhausen K-theory for all well–behaved Waldhausen categories [25, Theorem 4.3.1]. We consider an analogous definition of K-theory for general pointed left ∞-derivators.

Let $\mathbb{D}$ be a pointed left ∞-derivator with domain Dia. Let $\mathrm{S}_{\bullet\bullet}\mathbb{D}$ be the bisimplicial set whose set of (n,m)-simplices $\mathrm{S}_{n,m}\mathbb{D}$ is the set of objects

$$F \in \mathrm{Ob}\big(\mathbb{D}(\Delta^m \times \mathrm{Ar}[n])\big)$$

such that:

(1) for every $j \colon [0] \to [m]$ the object $(j \times \mathrm{id})^*(F) \in \mathrm{Ob}\big(\mathbb{D}(\mathrm{Ar}[n])\big)$ is in $\mathrm{S}_n\mathbb{D}$.
(2) the underlying diagram functor associated to F

$$\mathrm{dia}_{\Delta^m,\mathrm{Ar}[n]}(F) \colon \Delta^m \to \mathbb{D}(\mathrm{Ar}[n])$$

takes values in equivalences.

The bisimplicial operators of $\mathrm{S}_{\bullet\bullet}\mathbb{D}$ are again defined using the structure of the underlying ∞-prederivator. Moreover, it is easy to see that the construction is functorial in $\mathbb{D}$ with respect to strict morphisms which preserve the zero objects and cocartesian squares. Since $\mathrm{S}_{0,m}\mathbb{D} \simeq \Delta^0$, we may regard the geometric realization $|\mathrm{S}_{\bullet\bullet}\mathbb{D}|$ as based at a zero object of $\mathbb{D}(\Delta^0)$.

Definition 5.9. Let $\mathbb{D} \colon \mathsf{Dia}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ be a pointed left ∞-derivator with domain Dia. The *Waldhausen K-theory* of $\mathbb{D}$ is defined to be the space:

$$K^W(\mathbb{D}) := \Omega|\mathrm{S}_{\bullet\bullet}\mathbb{D}|.$$

Following [25], we consider also the analogue of the $\mathsf{s}_\bullet$-construction in this context. Restricting to the objects of $\mathrm{S}_\bullet\mathbb{D}$ pointwise, we obtain a simplicial set

$$\mathsf{s}_\bullet\mathbb{D}\colon \Delta^{\mathrm{op}} \to \mathsf{Set}, \quad [n] \mapsto \mathsf{s}_n\mathbb{D}\colon = \mathrm{S}_{n,0}\mathbb{D} = (\mathrm{S}_n\mathbb{D})_0.$$

There is a canonical comparison map, given by the inclusion of objects,

$$\iota\colon \Omega|\mathsf{s}_\bullet\mathbb{D}| \longrightarrow \Omega|\mathrm{S}_{\bullet\bullet}\mathbb{D}| = K^W(\mathbb{D}).$$

This is the analogue of the comparison map in Proposition 5.1 for pointed left ∞-derivators.

Proposition 5.10. *(a) The comparison map ι is a weak equivalence. (b) Let $\mathcal{C}$ be a pointed ∞-category with finite colimits and let $n \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. There is a commutative diagram of weak equivalences:*

$$\begin{array}{ccc}
\Omega|\mathsf{s}_\bullet\mathcal{C}| & = & \Omega|\,\mathrm{Ob}\,\mathrm{S}_\bullet\mathbb{D}^{(n)}_{\mathcal{C}}| \\
\downarrow\simeq & & \downarrow\simeq \\
K(\mathcal{C}) = \Omega|\mathrm{S}^{\simeq}_\bullet\mathcal{C}| & \xrightarrow[\simeq]{} & \Omega|\mathrm{S}_{\bullet\bullet}\mathbb{D}^{(n)}_{\mathcal{C}}| = K^W(\mathbb{D}^{(n)}_{\mathcal{C}}).
\end{array}$$

Proof. (a) The proof is essentially the same as the proof of [25, Proposition 4.2.1] (see also [37, Lemma 1.4.1 and Corollary]). (b) The bottom map is a weak equivalence (independently of n!) because we have $(\mathrm{S}^{\simeq}_k\mathcal{C})_m = \mathrm{S}_{k,m}\mathbb{D}^{(n)}_{\mathcal{C}}$. The left vertical map is a weak equivalence by Proposition 5.1. The result follows. □

5.5. **Universal property of derivator K-theory.** The comparison maps $\{\mu_n\}$ from Waldhausen K-theory to derivator K-theory can be defined more generally for pointed left ∞-derivators. Given a pointed left ∞-derivator $\mathbb{D}$ (with domain Dia), the underlying diagram functors define a bisimplicial map as follows,

$$\mathrm{dia}_{\Delta^m,\mathrm{Ar}[n]}\colon \mathrm{S}_{n,m}\mathbb{D} \to (\mathrm{S}^{\simeq}_n\mathbb{D})_m$$

which after passing to the geometric realization and taking loop spaces defines a comparison map

$$\mu\colon \Omega|\mathsf{s}_\bullet\mathbb{D}| \overset{5.10}{\simeq} K^W(\mathbb{D}) \to K(\mathbb{D}).$$

A universal property of this comparison map in the case of 1-derivators was shown in [25, Theorem 5.2.2]. More specifically, it was shown that it is homotopically initial among all natural transformations from Waldhausen K-theory to a functor which is invariant under (pointwise) equivalences of pointed left derivators. The proof of this universal property extends similarly to our present ∞-categorical context.

Let Der denote the (ordinary) category of pointed left ∞-derivators and strict morphisms which preserve zero objects and cocartesian squares. It will be convenient to work with the simpler model for the Waldhausen K-theory of derivators using the $\mathsf{s}_\bullet$-construction. This will be denoted by

$$K^{W,\mathrm{Ob}}\colon \mathsf{Der} \to \mathrm{Top}, \quad \mathbb{D} \mapsto \Omega|\mathsf{s}_\bullet\mathbb{D}|,$$

where Top denotes the ordinary category of topological spaces. Then we may regard the comparison map μ as a natural transformation between functors $K^{W,\mathrm{Ob}} \to K$ defined on Der.

Definition 5.11. The *category* $\mathcal{E}$ *of invariant approximations* to Waldhausen K-theory is the full subcategory of the comma category $K^{W,\mathrm{Ob}} \downarrow \mathrm{Top}^{\mathsf{Der}}$ spanned by the objects $\eta\colon K^{W,\mathrm{Ob}} \to F$ such that $F\colon \mathsf{Der} \to \mathrm{Top}$ sends pointwise equivalences in Der to weak equivalences. A morphism in $\mathcal{E}$

$$\begin{array}{ccc} & K^{W,\mathrm{Ob}} & \\ {}^{\eta}\swarrow & & \searrow^{\eta'} \\ F & \xrightarrow[u]{} & F' \end{array}$$

is a *weak equivalence* if the components of u are weak equivalences.

Remark 5.12. The category $\mathcal{E}$ (denoted by $\mathtt{App}$ in [25]) is not even locally small in general as defined in Definition 5.11. This set–theoretical issue can be addressed by restricting to suitable small subcategories of Der as was done in [25].

We recall from [9] that an object $x \in \mathcal{C}$ in an (ordinary) category with weak equivalences $(\mathcal{C}, \mathcal{W})$ (satisfying in addition the "2-out-of-6" property) is *homotopically initial* if there are functors $F_0, F_1 : \mathcal{C} \to \mathcal{C}$ which preserve the weak equivalences and a natural transformation $\phi\colon F_0 \Rightarrow F_1$ such that:

(i) F_0 is naturally weakly equivalent to the constant functor at $x \in \mathcal{C}$.
(ii) F_1 is naturally weakly equivalent to the identity functor on $\mathcal{C}$.
(iii) $\phi_x : F_0(x) \to F_1(x)$ is a weak equivalence.

A homotopically initial object in $(\mathcal{C}, \mathcal{W})$ defines an initial object in the associated ∞-category.

Theorem 5.13. *The object* $(\mu\colon K^{W,\mathrm{Ob}} \to K) \in \mathcal{E}$ *is homotopically initial.*

Proof. (Sketch) The proof is similar to [25, Theorem 5.2.2] so we only sketch the details. Given $\mathbb{D} \in \mathsf{Der}$ and $m \geq 0$, let $\mathbb{D}^{\simeq}_m$ denote the ∞-prederivator whose value at $\mathsf{X} \in \mathsf{Dia}$ is the full subcategory $\mathrm{Fun}_{\simeq}(\Delta^m, \mathbb{D}(\mathsf{X})) \subset \mathrm{Fun}(\Delta^m, \mathbb{D}(\mathsf{X}))$ spanned by the functors $\Delta^m \to \mathbb{D}(\mathsf{X})$ which take values in equivalences. This ∞-prederivator is pointwise equivalent to $\mathbb{D}$ and therefore also a pointed left ∞-derivator. Varying $m \geq 0$, we obtain a simplicial object $(\mathbb{D}^{\simeq}_m)_{m\geq 0}$ in Der with $\mathbb{D}^{\simeq}_0 = \mathbb{D}$.

For the proof of the theorem, it suffices to note that every object in $\mathcal{E}$,

$$(\eta_{\mathbb{D}}\colon K^{W,\mathrm{Ob}}(\mathbb{D}) \to F(\mathbb{D}))_{\mathbb{D}\in\mathsf{Der}}$$

is naturally weakly equivalent (as object in $\mathcal{E}$) to the composite

$$\big(K^{W,\mathrm{Ob}}(\mathbb{D}) \to ||K^{W,\mathrm{Ob}}(\mathbb{D}^{\simeq}_\bullet)|| \xrightarrow{\phi} ||F(\mathbb{D}^{\simeq}_\bullet)||\big)_{\mathbb{D}\in\mathsf{Der}}.$$

Moreover, the first map above defines a natural transformation which is canonically identified with μ. As a result, we have constructed a zigzag of natural transformations from the constant endofunctor at μ to $\mathrm{id}_{\mathcal{E}}$ satisfying (i)–(iii). □

6. K-theory of homotopy n-categories

6.1. **Revisiting the properties of homotopy n-categories.** Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. Then the associated homotopy n-category $\mathrm{h}_n\mathcal{C}$ satisfies the following:

(a) $\mathrm{h}_n\mathcal{C}$ is a pointed n-category.
(b) The suspension functor $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ induces a functor $\Sigma\colon \mathrm{h}_n\mathcal{C} \to \mathrm{h}_n\mathcal{C}$. This is an equivalence if and only if $\mathcal{C}$ is stable (see [20, Corollary 1.4.2.27]).

(c) $\mathrm{h}_n\mathcal{C}$ admits finite coproducts and weak pushouts of order $n-1$. Moreover, these are preserved by the functor $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$ (Proposition 3.21).

(d) For every $x \in \mathrm{h}_n\mathcal{C}$, there is a natural weak pushout of order $n-1$,

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma x. \end{array}$$

Assuming that $\mathcal{C}$ is a stable ∞-category, then the adjoint equivalence $(\Sigma_{\mathcal{C}}, \Omega_{\mathcal{C}})$ induces an adjoint equivalence $\Sigma\colon \mathrm{h}_n\mathcal{C} \rightleftarrows \mathrm{h}_n\mathcal{C}\colon \Omega$. Moreover, by duality, $\mathrm{h}_n\mathcal{C}$ also satisfies in this case the following dual versions of (c)–(d):

(c)′ $\mathrm{h}_n\mathcal{C}$ admits finite products and weak pullbacks of order $n-1$. Moreover, these are preserved by the functor $\gamma_n\colon \mathcal{C} \to \mathrm{h}_n\mathcal{C}$.

(d)′ For every $x \in \mathrm{h}_n\mathcal{C}$, there is a natural weak pullback of order $n-1$

$$\begin{array}{ccc} \Omega x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & x. \end{array}$$

In addition, if $\mathcal{C}$ is a stable ∞-category, $\mathrm{h}_n\mathcal{C}$ satisfies the following property:

(e) A square in $\mathrm{h}_n\mathcal{C}$ is a weak pushout of order $n-1$ if and only if it is a weak pullback of order $n-1$.

Note that if $n > 1$, weak pushouts (resp. weak pullbacks) of order $n-1$ are unique up to (non-canonical) equivalence. This observation can be used to deduce (e) for $n > 1$. The validity of (e) for $n = 1$ can be verified by a direct argument.

An attempt towards an axiomatization of the properties (a)–(d) would naturally lead to considering triples

$$(\mathcal{D}, \Sigma\colon \mathcal{D} \to \mathcal{D}, \sigma\colon \mathcal{D} \to \mathcal{D}^{\square})$$

where:

(1) $\mathcal{D}$ is a pointed n-category.

(2) $\Sigma\colon \mathcal{D} \to \mathcal{D}$ is an endofunctor.

(3) $\mathcal{D}$ admits finite coproducts and weak pushouts of order $n-1$.

(4) The functor σ sends an object $x \in \mathcal{D}$ to a weak pushout of order $n-1$ which has the following form:

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma x. \end{array}$$

Specializing to the stable context, it would be natural to require in addition:

(2)′ $\Sigma\colon \mathcal{D} \to \mathcal{D}$ is an equivalence.

(3)′ $\mathcal{D}$ admits finite products and weak pullbacks of order $n-1$.

(5) A square in $\mathcal{D}$ is a weak pushout of order $n-1$ if and only if it is a weak pullback of order $n-1$.

Antieau [1, Conjecture 8.30] has recently conjectured that there is a good theory of *stable n-categories*, $1 \leq n \leq \infty$, satisfying the following properties:

(i) Stable n-categories, exact functors and natural transformations form an $(n,2)$-category with a forgetful functor to $\mathsf{Cat_n}$.
(ii) For each $n \geq k$, the homotopy k-category functor defines a functor from stable n-categories to stable k-categories.
(iii) For $n = \infty$, the theory recovers the theory of stable ∞-categories, exact functors, and natural transformations.
(iv) For $n = 1$, the theory recovers the theory of triangulated categories, exact functors, and natural transformations.

It seems reasonable to take properties (1)–(5) (incl. $(2)'$–$(3)'$) as a minimal basis for such a notion of stable n-category. Firstly, these properties are preserved after passing to lower homotopy categories (Proposition 3.21). Moreover, for any pointed n-category $\mathcal{C}$ which satisfies these properties for $n > 2$, the associated homotopy category $\mathrm{h}_1\mathcal{C}$ can be equipped with a canonical triangulated structure (the proof is essentially the same as for stable ∞-categories in [20, 1.1.2], using the properties of weak pushouts of order $n-1>1$ instead of actual pushouts). It would also be interesting to relate the properties (1)–(5) to the notion of an n-angulated category in the sense of [16]. Finally, for $n = \infty$, these properties characterize stable ∞-categories.

On the other hand, concerning the case $n = 1$, the notion of a triangulated structure includes more structure than what is required in (1)–(5). This could be regarded either as a singularity that arises at the lowest level of coherence – since weak pushouts (of order 0) are not unique up to equivalence, they do not yield canonical connecting "boundary" maps, not even up to homotopy, and therefore they do not suffice for defining distinguished triangles –, or it may in fact be desirable to consider additional structure in the form of fixing choices of higher weak colimits and stipulate their properties. We will not attempt to give an axiomatic definition of a stable n-category in this paper, but we will suggest to consider a triple satisfying (1)–(4) as a basic invariant of any good notion of a stable n-category.

6.2. **K-theory of pointed n-categories with distinguished squares.** We define K-theory for certain n-categories equipped with distinguished squares that are meant to play the role of pushout squares in the definition of Waldhausen K-theory. In the case of ordinary categories, this notion of a category with distinguished squares and its K-theory correspond to a more basic version of Neeman's K-theory of a category with squares as defined in [27, Sections 5–7].

The main example we are interested in is the homotopy n-category $\mathrm{h}_n\mathcal{C}$ of a pointed ∞-category $\mathcal{C}$ which admits finite colimits, equipped with the squares which come from pushout squares in $\mathcal{C}$ as the distinguished squares. The purpose of introducing K-theory for $\mathrm{h}_n\mathcal{C}$ is in order to identify a part of Waldhausen K-theory $K(\mathcal{C})$ which may be recovered from $\mathrm{h}_n\mathcal{C}$, regarded as an n-category with distinguished squares. In particular, our main result (Theorem 6.5) generalizes the well–known fact that $K_0(\mathcal{C})$ can be recovered from $\mathrm{h}_1\mathcal{C}$, regarded as a category equipped with those squares which arise from pushouts in $\mathcal{C}$.

Definition 6.1. A *pointed n-category with distinguished squares*, $n \geq 1$, is a pair $(\mathcal{C}, \mathcal{T})$ where $\mathcal{C}$ is a pointed n-category and $\mathcal{T}$ is a collection of weak pushout squares in $\mathcal{C}$ of order $n-1$ which contains the constant squares at a zero object.

An *exact functor* $F\colon (\mathcal{C}, \mathcal{T}) \to (\mathcal{C}', \mathcal{T}')$ between pointed n-categories with distinguished squares is a functor $F\colon \mathcal{C} \to \mathcal{C}'$ which preserves zero objects and distinguished squares.

Definition 6.2. Let $\mathcal{C}$ be a pointed ∞-category with finite colimits and let $n \geq 1$. We define the *canonical structure of distinguished squares in* $\mathrm{h}_n\mathcal{C}$ to be the collection of squares in $\mathrm{h}_n\mathcal{C}$ which are equivalent in $\mathrm{h}_n\mathcal{C}$ to the image of a pushout square in $\mathcal{C}$. By Proposition 3.21, these are weak pushout squares of order $n-1$. For $n > 1$, they are precisely the weak pushouts of order $n-1$. We will denote this pointed n-category with distinguished squares by $(\mathrm{h}_n\mathcal{C}, \mathrm{can})$.

Remark 6.3. The canonical structure on $\mathrm{h}_n\mathcal{C}$ for $n > 1$ is described in terms of an intrinsic property of $\mathrm{h}_n\mathcal{C}$ and therefore depends only on $\mathrm{h}_n\mathcal{C}$. For $n = 1$, the canonical structure is an additional structure on $\mathrm{h}_1\mathcal{C}$ that is canonically induced from $\mathcal{C}$.

Let $(\mathcal{C}, \mathcal{T})$ be a pointed n-category with distinguished squares. Let $\mathrm{S}_q(\mathcal{C}, \mathcal{T})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[q], \mathcal{C})$ which is spanned by $F \in \mathrm{Fun}(\mathrm{Ar}[q], \mathcal{C})$ such that:

(i) $F(i \to i)$ is a zero object for all $i \in [q]$.
(ii) For every $0 \leq i \leq j \leq k \leq m \leq q$, the following diagram in $\mathcal{C}$,

$$\begin{array}{ccc} F(i \to k) & \longrightarrow & F(i \to m) \\ \downarrow & & \downarrow \\ F(j \to k) & \longrightarrow & F(j \to m) \end{array}$$

is a distinguished square in $(\mathcal{C}, \mathcal{T})$.

Note that the construction is functorial in $q \geq 0$ and $\mathrm{S}_\bullet(\mathcal{C}, \mathcal{T})$ defines a simplicial object of pointed n-categories. Moreover, this is functorial with respect to exact functors between pointed n-categories with distinguished squares. We denote by $\mathrm{S}^{\simeq}_\bullet(\mathcal{C}, \mathcal{T})$ the associated simplicial object of ∞-groupoids which is obtained by passing to the maximal ∞-subgroupoids pointwise. We have $\mathrm{S}^{\simeq}_0(\mathcal{C}, \mathcal{T}) \simeq \Delta^0$ and therefore we may regard the geometric realization $|\mathrm{S}^{\simeq}_\bullet(\mathcal{C}, \mathcal{T})|$ as based at a zero object of $\mathcal{C}$.

Definition 6.4. Let $(\mathcal{C}, \mathcal{T})$ be a pointed n-category with distinguished squares. The *K-theory of* $(\mathcal{C}, \mathcal{T})$ is defined to be the space:

$$K(\mathcal{C}, \mathcal{T})\colon = \Omega|S^{\simeq}_\bullet(\mathcal{C}, \mathcal{T})|.$$

6.3. **Comparison with Waldhausen K-theory.** Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. By Proposition 3.21, it follows that passing from $\mathrm{S}_\bullet\mathcal{C}$ to the homotopy n-category pointwise defines map of simplicial objects,

$$\mathrm{S}_\bullet\mathcal{C} \to \mathrm{S}_\bullet(\mathrm{h}_n\mathcal{C}, \mathrm{can}),$$

and therefore also a comparison map between K-theory spaces

$$\rho_n\colon K(\mathcal{C}) \to K(\mathrm{h}_n\mathcal{C}, \mathrm{can}).$$

Note that this comparison map factors canonically through the comparison map $\mu_n\colon K(\mathcal{C}) \to K(\mathbb{D}^{(n)}_{\mathcal{C}})$ (Subsection 5.3). By Proposition 3.21, the canonical functors $\gamma_{n-1}\colon \mathrm{h}_n\mathcal{C} \to \mathrm{h}_{n-1}\mathcal{C}$, $n > 1$, define exact functors $(\mathrm{h}_n\mathcal{C}, \mathrm{can}) \to (\mathrm{h}_{n-1}\mathcal{C}, \mathrm{can})$.

Therefore, we obtain a tower of K-theories for $\mathcal{C}$ which is compatible with the comparison maps ρ_n:

$$\begin{array}{ccccccc} & K(\mathcal{C}) & & & & & \\ & \downarrow{\scriptstyle \rho_n} & \searrow{\scriptstyle \rho_{n-1}} & & & \searrow{\scriptstyle \rho_1} & \\ \cdots \longrightarrow & K(\mathrm{h}_n\mathcal{C}, \mathrm{can}) & \longrightarrow & K(\mathrm{h}_{n-1}\mathcal{C}, \mathrm{can}) & \longrightarrow \cdots \longrightarrow & & K(\mathrm{h}_1\mathcal{C}, \mathrm{can}). \end{array}$$

The next result gives a general connectivity estimate for the comparison map ρ_n.

Theorem 6.5. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. For $n \geq 1$, the comparison map $\rho_n \colon K(\mathcal{C}) \to K(\mathrm{h}_n\mathcal{C}, \mathrm{can})$ is n-connected.*

Proof. We write $\mathrm{h}_n\mathcal{C}$ for $(\mathrm{h}_n\mathcal{C}, \mathrm{can})$ when the canonical structure is understood from the context in order to simplify the notation. By Lemma 5.6, it suffices to show that the map of ∞-groupoids

$$\mathrm{S}_q^{\simeq}\mathcal{C} \to \mathrm{S}_q^{\simeq}\mathrm{h}_n\mathcal{C} \tag{6.6}$$

is $(n+1-q)$-connected for every $q \geq 0$. The claim is obvious for $q = 0$. For $q = 1$, the map is $(n+1)$-connected. For $q = 2$ and $n = 1$, the map is 0-connected by definition. This completes the proof for $n = 1$ and we may now restrict to the case $n > 1$. We show that for every $q > 1$, the map (6.6) is $(n-1)$-connected. We consider the diagram

$$\begin{array}{ccccc} \mathrm{S}_q^{\simeq}\mathcal{C} & & \longrightarrow & & \mathrm{S}_q^{\simeq}\mathrm{h}_n\mathcal{C} \\ \downarrow{\scriptstyle \simeq} & & & & \downarrow \\ (\mathcal{C}^{\Delta^{q-1}})^{\simeq} & \longrightarrow & \mathrm{h}_n(\mathcal{C}^{\Delta^{q-1}})^{\simeq} & \longrightarrow & \big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-1}}\big)^{\simeq} \end{array} \tag{6.7}$$

where the vertical maps are given by restriction along the inclusion map of posets $[q-1] \subseteq \mathrm{Ar}[q],\ j \mapsto (0 \to j+1)$. The left vertical map in (6.7) is an equivalence. The lower left map is $(n+1)$-connected (Example 2.9). The lower right map in (6.7) is n-connected by Corollary 3.17 since we may replace Δ^{q-1} by its spine which is 1-dimensional. Thus, it suffices to show that the right vertical map in (6.7) is n-connected for any $q > 0$. This claim is obvious for $q = 1$. For $q > 1$, we proceed by induction and consider the following diagram

$$\begin{array}{ccc} \mathrm{S}_q^{\simeq}\mathrm{h}_n\mathcal{C} & \xrightarrow{d_q} & \mathrm{S}_{q-1}^{\simeq}\mathrm{h}_n\mathcal{C} \\ \downarrow & & \downarrow \\ \big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-1}}\big)^{\simeq} & \xrightarrow[d_q]{} & \big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-2}}\big)^{\simeq}. \end{array}$$

For $0 \leq k \leq q$, let $T_k^q \subseteq \mathrm{Ar}[q]$ be the full subcategory which contains the subposet $\mathrm{Ar}[q-1] \subseteq \mathrm{Ar}[q]$ and the elements $\{(j \to q) | \ 0 \leq j \leq k\}$. Moreover, let $\mathcal{T}_k^q$ denote the full subcategory of $\mathrm{Map}(T_k^q, \mathrm{h}_n\mathcal{C})$ which satisfies properties (i)–(ii) (Subsection 6.2) restricted to T_k^q. In other words, this is the full subcategory which is spanned by the image of $\mathrm{S}_q^{\simeq}\mathrm{h}_n\mathcal{C}$ under the restriction functor $\mathrm{Map}(\mathrm{Ar}[q], \mathrm{h}_n\mathcal{C}) \to \mathrm{Map}(T_k^q, \mathrm{h}_n\mathcal{C})$.

Then we may factorize the square above as the composition of the following composite square

$$\begin{array}{ccccccccc}
\mathrm{S}_q^{\simeq}\mathrm{h}_n\mathcal{C} = \mathcal{T}_q^q & \xrightarrow{\simeq} & \mathcal{T}_{q-1}^q & \longrightarrow & \cdots & \longrightarrow & \mathcal{T}_1^q & \longrightarrow & \mathcal{T}_0^q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-1}}\big)^{\simeq} & = & \big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-1}}\big)^{\simeq} & = & \cdots & = & \cdots & = & \big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-1}}\big)^{\simeq}
\end{array}$$

followed by the pullback square

$$\begin{array}{ccc}
\mathcal{T}_0^q & \longrightarrow & \mathrm{S}_{q-1}^{\simeq}\mathrm{h}_n\mathcal{C} \\
\downarrow & & \downarrow \\
\big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-1}}\big)^{\simeq} & \xrightarrow[d_q]{} & \big((\mathrm{h}_n\mathcal{C})^{\Delta^{q-2}}\big)^{\simeq}.
\end{array}$$

We note that all the horizontal and vertical maps in these diagrams are given by the respective restriction functors. We claim that each map $\mathcal{T}_k^q \to \mathcal{T}_{k-1}^q$ is n-connected, for any $1 \leq k < q$, from which the required result follows. To see this, we consider the pullback of ∞-groupoids

$$\begin{array}{ccc}
\mathcal{T}_k^q & \longrightarrow & \big(\mathrm{h}_n(\mathcal{C})^{\square,\mathrm{can}}\big)^{\simeq} \\
\downarrow & & \downarrow \\
\mathcal{T}_{k-1}^q & \longrightarrow & \big(\mathrm{h}_n(\mathcal{C})^{\ulcorner}\big)^{\simeq}
\end{array} \tag{6.8}$$

where the bottom map is given by the restriction to the subposet of $\mathcal{T}_{k-1}^q$

$$\begin{array}{ccc}
(k-1 \to q-1) & \longrightarrow & (k-1 \to q) \\
\downarrow & & \\
(k \to q-1) & &
\end{array}$$

and $\big(\mathrm{h}_n(\mathcal{C})^{\square,\mathrm{can}}\big)^{\simeq} \subset \big(\mathrm{h}_n(\mathcal{C})^{\square}\big)^{\simeq}$ is the full ∞-subgroupoid that is spanned by the weak pushouts of order $n-1$. The right vertical map in (6.8) is given by restriction along the upper corner inclusion in $\Delta^1 \times \Delta^1$. The fiber of this map at $F \in \big(\mathrm{h}_n(\mathcal{C})^{\ulcorner}\big)^{\simeq}$ is exactly the ∞-groupoid of weak colimits of F of order $n-1$. Since $\mathrm{h}_n\mathcal{C}$ admits weak pushouts of order $n-1 > 0$, it follows that the fibers of the right vertical map in (6.8) are $(n-1)$-connected. This means that the vertical maps in (6.8) are n-connected and the result follows. □

Example 6.9. Theorem 6.5 for $n = 1$ shows that the map $\rho_1 \colon K(\mathcal{C}) \to K(\mathrm{h}_1\mathcal{C}, \mathrm{can})$ is 1-connected. In particular, this recovers the well–known fact that $K_0(\mathcal{C})$ can be obtained from $\mathrm{h}_1\mathcal{C}$ equipped with the canonical structure of distinguished squares.

Remark 6.10. The connectivity estimate in Theorem 6.5 is best possible in general. Indeed, for $n = 1$ and $\mathcal{E}$ an exact category, the comparison map ρ_1 for the ∞-category associated to the Waldhausen category of bounded chain complexes in $\mathcal{E}$ factors through the comparison map to Neeman's K-theory of the triangulated category $D^b(\mathcal{E})$, that is, we have maps

$$\rho_1 \colon K(\mathcal{E}) \xrightarrow{\beta\alpha} K({}^dD^b(\mathcal{E})) \to K(D^b(\mathcal{E}), \mathrm{can})$$

where the last map is induced by the forgetful map of simplicial objects ${}^{d}\mathrm{S}_\bullet D^b(\mathcal{E}) \to \mathrm{S}_\bullet(D^b(\mathcal{E}), \mathrm{can})$. (We refer to [27] for a nice overview of the K-theory of triangulated categories and details about the comparison maps α and β.) The map induced on K_1 by $\beta\alpha$ is not injective in general by [27, Section 11, Proposition 1], see [36, Sections 2 and 5].

We write $P_n X$ for the Postnikov n-section of a topological space X, that is, the canonical map $X \to P_n X$ is $(n+1)$-connected – this agrees with the homotopy n-category of an ∞-groupoid. Theorem 6.5 implies that the functor $\mathcal{C} \mapsto P_{n-1}K(\mathcal{C})$ descends to a functor defined for $(\mathrm{h}_n\mathcal{C}, \mathrm{can})$. The following immediate corollary confirms a conjecture of Antieau in the case of connective K-theory [1, Conjecture 8.36].

Corollary 6.11. *Let $\mathcal{C}$ and $\mathcal{C}'$ be stable ∞-categories such that there is an equivalence $(\mathrm{h}_n\mathcal{C}, \mathrm{can}) \simeq (\mathrm{h}_n\mathcal{C}', \mathrm{can})$, as pointed n-categories with distinguished squares. Then there is a weak equivalence*

$$P_{n-1}K(\mathcal{C}) \simeq P_{n-1}K(\mathcal{C}').$$

Remark 6.12. As explained in Remark 6.3, the canonical structure is invariant under equivalences for $n > 1$, that is, an equivalence $\mathrm{h}_n\mathcal{C} \simeq \mathrm{h}_n\mathcal{C}'$ for $n > 1$ is automatically an equivalence of pointed n-categories with distinguished squares. For $n = 1$, note that an equivalence $\mathrm{h}_1\mathcal{C} \overset{\Delta}{\simeq} \mathrm{h}_1\mathcal{C}'$, as triangulated categories, clearly also preserves the canonical structures of distinguished squares.

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