$\pi_*(ko)$ and some of $\pi_*(tmf)$ at the prime 2

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Acknowledgements

I would like to thank Zhouli Xu for suggesting and helping me through the calculation of $\pi_*(tmf)$ up to stem 13; it is his patience and encouragement which kept my heart light.

I must also thank Mark Mahowald, though I have never met him, for telling Zhouli Xu that calculating $\pi_*(tmf)$ was a rite of passage for computational homotopy theorists.

I thank Peter May for suggesting the calculation of $\pi_*(ko)$ using the Adams spectral sequence, and his suggestions for me to try a longer calculation. I also thank Peter and Zhouli for emphasizing that my other thesis project on height 3 formal group laws was too complicated to be finished on time, and encouraged me to put together this expository project as my thesis instead.

I thank Eric Peterson and Mike Hill for helping me with an alternative proof of how $H_*(ko) \simeq (A//A(1))^*$. The proof I ended up using is an expanded version of an outline of John Rognes in his unpublished notes.

1 Pep talk

This thesis is meant to fill a gap in the literature, for those starting out on their first larger calculations (ones which are too large for normal paper), and to emphasize some crucial differential finding tricks often left unpublished, or abandoned shivering in a footnote or side comment. This exposition will be repetitive, there will often be multiple proofs of the same things – the aim is to teach methods used by active topologists by showing them in action.

We introduce the May spectral sequence via two examples of its application, toward understanding $\text{Ext}_{A(n)}(F_2, F_2)$, a crucial figure in computing the stable homotopy groups of spheres. We do the examples of $A(1)$ and $A(2)$, the latter only in a range.
Further, we hope to illuminate the philosophy behind the May spectral sequence, and show the fun of presenting elements in the May spectral sequence as Massey products.

The May spectral sequence is the systematic destruction of regularity. We find any relations, propagate them, and remove them from the picture. We will be like Michelangelo: starting from the very regular block of marble, and chipping away carefully until our beast emerges. All that can support a differential will support a differential.

It is crucial to emphasize that the calculations we exposit are not difficult, they are only complicated.

This thesis will assume that the reader is familiar with the basics of spectral sequences, and Massey products. For a discussion of the definition of Massey products and their indeterminancy, the author recommends O’Neill [4].

2 Set up of the Exercise

The May spectral sequence converges to the $E_2$ page of the Adams spectral sequence, so let’s talk about the Adams spectral sequence. We work with mod-2 coefficients. The standard Adams spectral sequence says:

$$\text{Ext}_{A^*}^\ast(H_\ast(X), H_\ast(Y)) \Rightarrow [X, Y]_2^\wedge$$

Where here $\text{Ext}$ means comodule $\text{Ext}$. We plug in $X = S^0$ and $Y = M$, where $M$ is some spectrum, to get:

$$\text{Ext}_{A^*}^\ast(H_\ast(S^0), H_\ast(M)) \Rightarrow \pi_\ast(M)\hat{\wedge}$$

Note that $H_\ast(S^0) \simeq \mathbb{F}_2$. We plug in $ko$ for $M$ and use the isomorphism (which we prove in section 7) $H_\ast(ko) \simeq A//A(1)$.

$$\text{Ext}_{A^*}^\ast(\mathbb{F}_2, A//A(1)) \Rightarrow \pi_\ast(ko)\hat{\wedge}$$

Using the change of rings theorem, we slip in:

$$\text{Ext}_{A(1)^{\ast}}^\ast(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_\ast(ko)\hat{\wedge}$$

3 Why use the May Spectral Sequence?

The calculation of the homotopy ring of connective $KO$ using the Adams or May spectral sequence begins with the homotopy groups of $ko$, and reveals the ring structure and Toda bracket presentations of the generators.
Note that we need that

\[ H^* (ko) \simeq A_\langle A(1) \rangle \]

where \( A(1) \) is the submodule of the Steenrod algebra generated by \( Sq^1, Sq^2 \), and \( A_\langle A(1) \rangle \) represents the Hopf-algebra quotient \( A \otimes_{A(1)} \mathbb{F}_2 \).

The only way to prove this isomorphism is by using the cofibration \( \Sigma ko \xrightarrow{\eta} ko \xrightarrow{\partial} ko \), which requires you to know the homotopy groups of \( ko \). There is no way around it! Nevertheless, calculating \( \text{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \) is a good exercise, because it immediately collapses to \( \pi_* (ko) \).

There are two ways we can go about calculating \( \text{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \). First, we could try to take an \( A(1) \)-resolution of \( \mathbb{F}_2 \), and then use this resolution to write down the Adams spectral sequence \( E_2 \) page. This is very doable, we find a periodicity in the resolution very quickly, it appears as the cell structure of \( RP^n \) with a Joker on the end (this may be seen in Bob Bruner’s artwork on resolutions of \( A(1) \) [3]. The disadvantage of this hands on approach is the lack of ability to generalize. If we want to compute \( \pi_* (tmf) \), for which a key ingredient is \( HF_2^* (tmf) = A_\langle A(2) \rangle \), resolution with copies of \( A(2) \) by hand is practically impossible and at the very least hellish.

We often like to break up complex computations into bite sized pieces. Instead of calculating an \( A(2) \)-resolution of \( \mathbb{F}_2 \) to get directly to the Adams spectral sequence, we can use the May spectral sequence.

The May spectral sequence is an entirely algebraic method of calculation, even the differentials are completely algebraic.

\[ E_1^{*,*,*} = \mathbb{F}_2 [h_{ij} | i \geq 1, j \geq 0] \Rightarrow \text{Ext}_{A(1)}^{*,*,*} (\mathbb{F}_2, \mathbb{F}_2) \]

Here we care only about \( A(1) \), so our \( E_1 \) page will mercifully be a subalgebra of \( \mathbb{F}_2 [h_{ij} | i \geq 1, j \geq 0] \).

Truly, we use this paper as a “first hit,” an exposure to ideas that will allow the reader to calculate \( \pi_* (tmf) \). We will start with the May spectral sequence to stem 13, and then compute the only Adams differential in this range.

4 Calculation of \( \text{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \) using May Spectral Sequence

We begin given that the \( E_1 \) page of the May spectral sequence is:

\[ E_1^{*,*,*} = \mathbb{F}_2 [h_{10}, h_{11}, h_{20}] \]
where \( h_{ij} \) corresponds to \( \xi^{2^j}_i \) in the cobar complex, and the degree in \((t - s, s, u)\) coordinates is \( |h_{ij}| = (2^j(2^i - 1) - 1, 1, 1) \). We are further given that the only \( d_1 \) differential is:

\[
d_1(h_{20}) = h_{10}h_{11}
\]

As short hand, we write \( h_{in} =: h_n \). We propagate the \( E_1 \) page, and draw in the \( d_1 \)'s. We see immediately that our \( d_1 \) differential has both \( h_0 \) and \( h_1 \) ladders. The \( E_1 \) page is a bit awful to behold in its full glory without witnessing it being drawn in real time, so we do not picture it here. After propagating the differential, we flip to the \( E_2 \) page:

Let’s look for \( d_2 \)'s. We needn’t look long. The only one which could possibly exist is on the only remaining generator of May degree 2, \( h_{20}^2 \).

**Theorem 1.** We have a Massey product presentation of \( h_{20}^2 =: b_{20} \).

\[
h_{20}^2 = \langle h_{10}, h_{11}, h_{10}, h_{11} \rangle
\]

**Proof.**

We see 0 on the bottom row because \( h_{10}h_{20} + h_{10}h_{20} = 0 \) mod 2, and \( h_{11}h_{20} + h_{11}h_{20} = 0 \) mod 2 as well. So, 0 kills them both! Yippee!

**Theorem 2.** \( d_2(h_{20}^2) = h_1^2 + h_0^2h_2 \)
Proof. We may take the higher differential of $h_{20}^2$ by treating it as a 4-fold Massey product. This is May’s original proof although he did not include it in his thesis.

One of the main tricks we will use to calculate differentials is using the higher Leibnitz formula. Let’s talk about the 3-fold version first.

\[ d_2(\langle a_1, a_2, a_3 \rangle) = a_1 c + ba_3 \]

where $a_1 a_2 = b$ and $a_2 a_3 = c$.

We are ready for the 4-fold version:

\[ d_2(\langle a, b, c, d \rangle) = x \cdot d + y \cdot a \]

where $\langle a, b, c \rangle$ is detected by $x \in E_r$ page of the May SS, and $\langle b, c, d \rangle$ is detected by $y \in E_r$ page of the May SS.

In our case, the 3-fold Massey products that appear on the Adams spectral sequence obey the rule $\langle h_i, x, h_i \rangle = h_i+1 x$:

\[ \langle h_0, h_1, h_0 \rangle = h_1^2 \]
\[ \langle h_1, h_0, h_1 \rangle = h_0 h_2 \]

And thus, $d(\langle h_{10}, h_{11}, h_{10}, h_{11} \rangle) = h_1^3 + h_0^2 h_2$, note that they are in the same degree.

\[ \square \]

Remark. We use the derivation of this differential as a discussion of the 3 different ways to shed light on differentials: the higher Liebnitz rule, the cobar complex, and Nakamura’s lemma. They are parts of a whole. The first way is the most intuitive: we use the Massey product decomposition of an element, this is guaranteed to work because there is a Massey product presentation of any element on the May spectral sequence.

May understood the determination of what the differential of a Massey product should be, and his proof of this higher Leibnitz rule uses hypothesis: the vanishing of the elements that could obstruct the proof (the condition is given in Theorem 4.5 pg. 565) as Corollary 4.6 \[?\]. Yet, it works far more generally.

After he had the heuristics for what we know the differential must be, May skirted the technicality in the general theory of differentials of Massey products by embedding in the bar construction. This makes the differential Massey product method perfectly rigorous.

The third way is to use Nakamura’s lemma, which is of the form

\[ Sq^{i+1}(d_r(a)) = d_{2r}(Sq^r(a)) \]
These three methods reinforce each other. There is an indeterminacy subtlety in Nakamura’s lemma, but since we have 3 weights in the May spectral sequence, the indeterminacy is negligible.

We include an alternative proof of $d_2(h_{20}^2) = h_1^3 + h_0^2 h_2$ using Nakamura’s lemma.

Proof.

$$d_2(Sq^0(h_{20})) = Sq^1(d_1(h_{20}))$$

$$= Sq^1(h_0 h_1)$$

$$= Sq^1(h_0)Sq^0(h_1) + Sq^0(h_0)Sq^1(h_1)$$

$$= h_1 h_1^2 + h_0^2 h_2$$

This uses that $Sq^0(h_{ij}) = h_{ij}^2$, and $Sq^1(h_{ij}) = h_{i,j+1}$ (page 7, [2]).

So, we have proven this differential:

and using the $h_{11}$ ladder, we propagate the differential.

Flipping to page 3, we see that there cannot be any more differentials, so the spectral sequence collapses on the 3rd page.
Since the $E_\infty$ page of the May SS is the $E_2$ page of the Adams SS up to hidden extensions, we check for hidden extensions. This is a highly merciful task because there are no dots in higher filtrations! Therefore, there cannot possibly be hidden extensions.

Now that we have the Adams $E_2$ page, we look to see if there are any Adams differentials, for grading reasons we see that there are none, so the Adams SS collapses on the 2nd page. So, reading off the resultant homotopy groups:

$$\mathbb{Z}, \mathbb{C}_2, \mathbb{C}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$$

We are thinking of $2 := h_0$, $\eta := h_1$, $\alpha := h_0 b_{20}$, and $\beta := b_{20}^2$. These are now elements living in the homotopy group of $ko$, where we are thinking of $h_0$ as “multiplication by 2 in $\pi_*(ko)$.” Thus, just by looking at the below picture, we have $\mathbb{Z}_2^2[\eta, \alpha, \beta]/(\eta^3, 2\eta, \alpha^2 - 4\beta, \alpha \eta)$. 
Wow! We really did just get that ring structure with very little work. Now, you’ll notice we’ve also written in the Toda bracket presentations of some of the elements, because they are fun! Let’s see how we derived them. We started with the Massey product presentations of the corresponding elements:

\[
\langle h_0^2, h_1, h_1^2 \rangle = b_0 b_{20} \quad \langle h_1, h_1^2, h_1^2 \rangle = b_2^2
\]

Note that when we take the Massey product, we only consider the differentials of highest degree.

Now, how do we get from the Massey product to the Toda bracket? We apply the May convergence theorem (also known as Moss’s Theorem), which we state in the 3-fold setting:

**Theorem (May’s convergence theorem)** [Theorem 4.1, page 561[1]].

Given:

- \( a, b, c \) are permanent cycles in May \( E_r \) page, and \( a \Rightarrow \alpha, \quad b \Rightarrow \beta, \quad c \Rightarrow \gamma \), where \( \alpha, \beta, \gamma \in \pi_* \).
- \( \langle a, b, c \rangle \) is defined in May \( E_r \)
- \( \langle \alpha, \beta, \gamma \rangle \) is defined in Adams \( E_2 \) (which is May \( E_\infty \) up to extensions)
- No “crossing differentials”.

\[ \Rightarrow \exists \text{ one element in } \langle a, b, c \rangle \text{ which is a permanent cycle, and it converges to an element in } \langle \alpha, \beta, \gamma \rangle \text{ Adams } E_2. \]

5 Calculation of \( \text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \) to stem 12 using May Spectral Sequence

As adventurers, we must humble ourselves – the case of \( A(1) \) was the first little step. When I became confused in the \( tmf \) case I went back to the \( ko \) case to get more familiar with the basic mechanics before carrying on with the game.

We next turn to the case of topological modular forms, whose intricacies will from the very outset require of us much greater attention and care. We go out far enough to see something interesting.
Just like the $A(1)$ case, we use the Adam’s grading $(t - s, s, u)$, so we grade $|h_{ij}| := (2^i(2^j - 1)) - 1, 1, 1)$. Note that $h_n := h_{1n}$. We take the convention of increasing the May weight by $r$ when we apply the differential $r$.

We take as given that

$$E_1^{*,**,*} = \mathbb{F}_2[h_0, h_1, h_2, h_{20}, h_{21}, h_{30}]$$

Now, though you may balk at tables of gradings, I emphasize that it is extremely helpful to keep them near your bedside as you compute.

Our first page of generators thus looks like this:

<table>
<thead>
<tr>
<th></th>
<th>$t - s$</th>
<th>$s$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_{20}$</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_2$</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_{21}$</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_{30}$</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We are given as $d_1$:

$$d_1(h_n) = 0$$
$$d_1(h_{20}) = h_0 h_1$$
$$d_1(h_{21}) = h_1 h_2$$
$$d_1(h_{30}) = h_0 h_{21} + h_2 h_{20}$$

We use a computer program, *Ext Chart* by Eric Peterson, to propagate the $d_1$ differentials via the Leibnitz rule, because it is a task perfect for a computer.

Looking over his program output, we see that $h_1 h_{30} + h_{21} h_{20}$ survives. Indeed, we check and see that the differentials of the individual components ($h_1 h_{30}$ and $h_{21} h_{20}$) both kill $h_{20} h_1 h_{21} + h_2 h_{21}$. Since they both kill the same thing, and we are working in the magical mod 2 world, they add and stay as a cycle. We name $x_7 := h_1 h_{30} + h_{21} h_{20}$ because it is in stem 7.

We take the perspective that if $c$ kills $a + b$, then $a = b$. So via $d_1$ differentials, we get that $x_7 h_0 = h_2 b_{20}$ (thus $h_2 b_{20}^2 = b_{20} x_7 h_0$) and $h_0 h_{21} = h_2 h_{20}$.

When we flip the page to $E_2$, we see:
The first thing we do on the $E_2$ page is uncover the Massey product presentations of our generators, their true form, because this is how we computed the $d_2$ differentials in the case of $A(1)$.

<table>
<thead>
<tr>
<th></th>
<th>$t - s$</th>
<th>$s$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$h_2$</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$b_{20}$</td>
<td>4</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$b_{21}$</td>
<td>10</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$b_{30}$</td>
<td>12</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$x_7$</td>
<td>7</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

The Massey product for $b_{30}$ must be a matrix Massey product because $h_0$ and $h_2$ pair nontrivially so we cannot do the usual square trick. Instead,
we must pair to get \( h_0 h_2 + h_2 h_0 \) such that they cancel (we are living in mod 2 land).

We use these Massey product representations to derive the \( d_2 \)'s on the generators, using the higher Leibnitz rule as we did before:

\[
\begin{align*}
\langle h_1, h_1 \rangle & \left( \langle h_0, h_1 \rangle, \langle h_0, h_1 \rangle \right) = h_{30}^2 \\
\langle h_0, h_0 \rangle & = 0 \\
\langle h_0, h_0 \rangle & = 0 \\
\langle h_0, h_0 \rangle & = 0
\end{align*}
\]

To derive the differential of \( b_{30} \), we use that by the definition of \( x_7 \),
\[
x_7^2 = h_1^2 b_{30} + b_{20} b_{21}.
\]

\[
0 = d_2(x_7^2) := h_1^2 d_2(b_{30}) + b_{20} d_2(b_{21}) + b_{21} d_2(b_{20})
\]

\[
\Rightarrow h_1^2 d_2(b_{30}) = b_{20} (h_1^3) + b_{21} (h_1^3 + h_0^2 h_2)
\]

\[
= b_{21} h_1^3
\]

The other two elements in the sum do not appear in the \( E_2 \) page. When we look at the picture below, we may using “\( h_1^2 \) division” to get that \( d_2(b_{30}) = h_1 b_{21} \). That is, \( d_2(b_{30}) = 0 \Rightarrow d_2(h_1^2 b_{31}) = 0 \) which is false.
And thus:

$$d_2(b_{30}) = h_1 b_{21}$$

After all of this work, we have our differentials. We propagate these differentials using the Leibnitz rule, for example, lets look at $b_{20}x_7$, the lone dot in (11, 4):

$$d_2(b_{20}x_7) = x_7 h_1^3 + h_0^2 x_7 + b_{20} h_2^2 h_0$$

$$= x_7 h_1^3$$

because when we multiply out the other two using that $x_7 := h_1 h_{30} + h_{21} h_{20}$, we see that they died via $d_1$'s.
Taking a head count, we see that all squares of the generators of $E_2$ are generators on the $E_3$ page. We also find that we have some special generators in view. We have in sight the families $w_1 := b_{20}^2, \alpha := h_1 b_{30}$ (the bottom of the tower in stem 12) and $c_0 = x_7 h_1$. Further, there are no higher differentials because the gradings of possible differential pairs don’t have the appropriate/compatible May gradings. Thus, this range of the spectral sequence collapses on $E_3$.

Why did we stop at 13? Besides the fact that illustrating large spectral sequences is a pain, the range we illustrated is just enough to see an Adams differential! We address this in the next section.

Just for fun, let’s uncover some of the surviving elements’ true forms! They all have Massey product presentations.

We see that $b_{20}^2 = \langle h_1, h_1^2, h_1, h_1^2 \rangle$ (since $b_{20}$ kills $h_1^3$), but can we find a representation of $b_{20}^2$ using the Massey product for $b_{20}$?

$$w_1 = b_{20}^2 = b_{20}(h_0, h_1, h_0, h_1)$$
$$= \langle (h_1, h_2), (h_1^2, h_0^2), (h_1, h_2), (h_1^2, h_0^2) \rangle$$

Now, let’s do $c_0$:

$$c_0 = x_7 h_1$$
$$= \langle h_2, h_1, h_0, h_1 \rangle h_1$$
$$= \langle (h_1, h_2, h_1), h_0, h_1 \rangle$$
$$= \langle h_2, h_1, h_0, h_1^2 \rangle$$

**Remark.** Note that the symmetric Massey products are nonzero, this is because we are considering the Massey products as living on the $E_\infty$ page.

Immediately to the right of the range shown (in stem 14) is the element $d_0 := x_7^2 = \langle c_0, h_0, h, h_2 \rangle$.

**Remark.** Unrelated to this discussion, it is perhaps useful to note that $Sq^0$ commutes with Massey products. That is:

$$Sq^0(\langle a, b, c \rangle) = \langle Sq^0(a), Sq^0(b), Sq^0(c) \rangle$$

This is because, if $d_2(e) = ab$, then by Nakamura’s lemma, $Sq^0(d_2(e)) = d_1(Sq^0(e)) = Sq^0(ab) = Sq^0(a)Sq^0(b)$. Thus, we may intimately apply Nakamura’s lemma inside of Massey products. It is important to note that the $Sq^0$ case is special, we cannot apply this in the case of $Sq^4$ because $Sq^4(ab) \neq Sq^4(a)Sq^4(b)$. 

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**Remark.** If we wish to avoid Massey products, one thing humans do better than computers, we can entirely rely on a computer program and Nakamura’s lemma to compute the May spectral sequence converging to $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ in this range. Here is how we avoid all of the work (in a way that presently only works for small ranges due to the limitations of existing computer programs and the fact that we need more tricks past the 20 or 30th stem):

Start on the $E_1$ page, take the $d_1$’s of the generators $h_0, h_1, h_2, h_{20}, h_{21}, h_{30}$. Applying Nakamura’s lemma ($Sq^1(d_1(a)) = d_2(Sq^0(a))$ up to some indeterminacy), we get the $d_2$’s of $h_0^2, h_1^2, h_2^2, h_{20}^2, h_{21}^2, h_{30}^2$ on the $E_2$ page.

After propagating these differentials via the Leibnitz rule, we wish to deduce the differentials on the rest of the generators, in our case and our range, the only generator in $E_2$ that is not a square of a generator on $E_1$ is $x_7$.

We derived earlier the equality $x_7 h_0 = h_2 b_{20}$, and via the Leibnitz rule, we get: $d_2(h_2 b_{20}) = h_3^2 h_2 + b_0^2 h_3^2$, where the first term is zero since $h_3^2 h_2$ is dead. So, we have: $d_2(h_0 x_7) = h_0^2 h_2$.

Then, we apply “$h_0$ divisibility”: that is, $d_2(x_7) = 0 \implies d_2(h_0 x_7) = 0$ which is false. We get this implication because the differential is $h_0$ linear.

There are no more generators in this range, and no more differentials on future pages because the May weights don’t match up, so we are done.

### 6 On the cell structure of $tmf$

Now we are out to stem 13, we can see some interesting phenomena. We will show that the following differential exists on the Adams $E_2$ page:
This is done using the 4-spectral sequence method – an amped up commutative diagram argument. We will show that the differential
\[ d_{12} : 8[12] \rightarrow Ph_2[0] \]
on the Atiyah-Hirzebruch Spectral Sequence (AHSS) implies
\[ d_2 : h_0^3[12] \rightarrow Ph_2[0] \]
on the Adams Spectral sequence (ASS).

**Step 1.** This method begins with the creation of a finite cell complex \( X \), a skeleton of \( tmf \). Then, any Adams differential we find in \( X \) in this range, we may push forward to \( tmf \).

First of all, we can show what the cell diagram of \( tmf \) is up to dimension 12.

\[ 0 \to 8 \to 12 \]

This can be done (assuming \( HF_2(tmf) \simeq A//A(2) \)) by looking at \( A//A(2) \) up to degree 12 and seeing that it is the above, thinking of \( \sigma \) as \( Sq^8 \), \( \nu \) as \( Sq^4 \), and so on.

**Step 2.** Now we begin the argument by going up one side: the cell-ASS collapses on the \( E_2 \) page (there are no differentials up to stem 13). The cell-ASS is the cell-wise ASS - different cells have their own ASS, and this is the direct sum of these ASS.

\[ \pi_*(X) \rightarrow \oplus \pi_*(S^n) \]

\[ \oplus \text{Ext}(S^n) \rightarrow \text{cell-ASS} \]
Step 3. We exploit a relationship between 3-fold Toda brackets and differentials in the AHSS of certain 3-cell complexes developed in Section 6 of the 61-sphere paper [6].

\[ \oplus \pi_\ast(X) \]

\[ \oplus \pi_\ast(S^n) \]

In particular, we use Lemma 6.1 to deduce our differential. Our 3-cell complex satisfies the 2 hypotheses of the lemma: \( 8\nu = 0 \) in \( \pi_3 \) because \( \pi_3(X) = \mathbb{Z}/8 \). The second hypothesis is also satisfied: \( 8\pi_{11}(X) = 0 \). This is because \( \pi_{11} = \mathbb{Z}/8 \), so \( 8\pi_{11} = 0 \). Further, \( \pi_6 \) is generated by \( h_2^2 = \nu^2 \), thus \( \sigma\pi_6 = \sigma\nu^2 = 0 \) (though not pictured, it is indeed the case that \( \sigma\nu^2 = 0 \)).

Applying Lemma 6.1 to the cell complex:

We get:

\[ d_{12}(8[12]) = (8, \nu, \sigma)[0] \]

**Theorem 2.** \( (8, \nu, \sigma) = \{Ph_2\} \) up to indeterminacy.
Proof.

\[ \langle 8, \nu, \sigma \rangle = \langle 2, 4\nu, \sigma \rangle = \langle 2, \eta^3, \sigma \rangle = \langle 2, \eta, \eta^2\sigma \rangle = \langle 2, \eta, \nu^3 + \eta\epsilon \rangle = \langle 2, \eta, \nu^3 \rangle + \langle 2, \eta, \eta\epsilon \rangle = \langle 2, \eta, \nu \rangle^2 + \langle 2, \eta, \eta\epsilon \rangle = 0 + Ph_2. \]

The last bracket’s well-definedness follows from the May convergence theorem on the following Massey product \( \langle h_0, h_1, h_1c_0 \rangle = Ph_2 \). This Massey product is well defined because \( x_7 b_20 \) (in 11, 4) kills \( h_1^2c_0 \) in the May SS.

Note that \( \nu^3 = \eta^2\sigma + \eta\epsilon \) is a relation in the homotopy groups of spheres.

\[ \square \]

**Step 4.** On the other side, we do the algebraic AHSS. This will simply give us the name of the element we are attempting to kill. The AAHSS is “coning off pieces” process, i.e., going from shifted copies of \( \text{Ext}(S) \) to \( \text{Ext}(X) \).

We see that \( X: \)

```
0
\sigma
8
\nu
0
```

\[ \pi^*(X) \]

\[ \text{Ext}(X) \]

\[ \oplus \pi^*(S^n) \]

\[ \oplus \text{Ext}(S^n) \]

\[ \text{cell-ASS} \]

\[ \text{AHSS} \]

\[ \text{ASS} \]
matches the cell structure of \( tmf \) quite explicitly by coning off the fellows \( \sigma \) and \( \nu \) successively.

Here’s the first 12 stems of the stable homotopy groups of spheres:

We cone off \( \sigma \), and by this we mean that we place a shifted copy of the stable stems such that the \( d_1 \) on the shifted copy of 1 kills the generator \( \sigma \). We use \( 1[8] \) to denote the copy of 1 coming from the cell in dimension 8.
We cone off $\nu \sigma$, here $\nu$ is $h_2$.

This construction using the cell structure gives us the name of the source and the target of the differential we are attempting to deduce: $h_0^3[12] \rightarrow Ph_2[0]$.

**Step 5.** After we complete the AAHSS, we see that $Ph_2[0]$ is alive and well in $\text{Ext}(X)$. However, we know that $Ph_2[0]$ does not survive in $\pi_*(X)$. 

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because of the $d_{12}$ killing it in the AHSS. So, something must kill $Ph_2[0]$ in the Adams spectral sequence.

![Diagram: \( \pi_\ast(X) \rightarrow \text{Ext}(X) \rightarrow \oplus \pi_\ast(S^n) \rightarrow \oplus \text{Ext}(S^n) \rightarrow \text{cell-ASS} \rightarrow \text{AHSS} \rightarrow \text{ASS} \rightarrow \pi_\ast(X) \)]

We use a filtration argument. Note $Ph_2[0]$ is in stem 11, filtration 5, and we see in stem 12, there is only the $h_0$ tower supported on $h_0b_{30}$ (i.e. $h_0^3[12]$). So, there is no other option for the source element which kills $Ph_2[0]$ – it must be $h_0^3[12]$. Thus,

$$d_2 : h_0^3[12] \rightarrow Ph_2[0]$$

exists on the Adams spectral sequence for $X$, and we push it forward to exist on the Adams spectral sequence for $tmf$. Tada! Our first Adam's differential!

7 Proof that $A\!/A(1) \simeq H_\ast(ko; \mathbb{F}_2)$

This proof is an expansion of a proof in an unpublished paper of John Rognes, which was obtained by Dominic Culver.

Let’s orient ourselves before we dive in. We will assume the following two things:

1. The cofiber sequence $\Sigma ko \rightarrow ko \rightarrow ku \rightarrow \Sigma^2 ko$. We define $d := c\partial$.
2. $H^\ast(ku; \mathbb{F}_2) \simeq A\!/E(1)$, where $E(1)$ is the ideal generated by $Sq^1$ and $[Sq^1, Sq^2] =: Q_1$.

Our proof will consist of the following steps:

1. Lemma 1: Show $\text{im } d^\ast \simeq A(Sq^2)$.
2. Lemma 2: Show $H^\ast ko \simeq A\!/E_1/(\text{im } d^\ast)$

Remark. We can think of this cofiber sequence as coming from the sequence $S^1 \rightarrow S^0 \rightarrow C\eta \rightarrow S^2$ by smashing with $ko$. Where $\eta$ is the Hopf map, and the map from $ko$ to $ku$ is the induced map of complexification of bundles.
Lemma 1. The composite map $c\partial : ku \to \Sigma^2 ku$ induces right multiplication by $Sq^2$ in cohomology. This composite map is more honestly presented as $\Sigma^2 c \circ \partial$. For it is $ku \xrightarrow{\partial} \Sigma^2 ko$ composed with $\Sigma^2 ko \xrightarrow{\Sigma^2 c} \Sigma^2 ku$.

Proof. We look at the map from $C\eta \to ku$, which is the “smash with ko” map. This is nonzero.

$$H^2(\Sigma^2 C\eta) \xleftarrow{(c\partial)^*} H^*(\Sigma^2 ku) \simeq H^2(\Sigma^2 ko) \otimes H^2(\Sigma^2 C\eta)$$

The induced map on cohomology must also be nonzero, as the map behaves like a projection onto one factor $(c\partial)^* : 1 \otimes x \mapsto x$. Thus the generator of $H^2(\Sigma^2 ku)$, which we call $\Sigma^2 1$, pulls back to the generator of $H^2(\Sigma^2 C\eta)$.

Now, we examine the generator $\Sigma^2 1$ of $H^2(\Sigma^2 C\eta)$ as we pull it back to $H^2(C\eta)$. We pull it back over the map

$$C\eta \to S^2 \to \Sigma^2 C\eta$$

We see that the generator $\Sigma^2 1$ pulls back over this map

$$C\eta \xrightarrow{\alpha} S^2 \xrightarrow{\beta} \Sigma^2 C\eta$$

to the nonzero class in degree 2, the only such nonzero class in $H^2(C\eta)$ is $Sq^2$. 

\[ \begin{tikzcd}[column sep=3em] ku \arrow{r}{\Sigma^2 c \circ \partial} & \Sigma^2 ku \\
C\eta \arrow{u} \arrow{r}[swap]{\beta \circ \alpha} & \Sigma^2 C\eta \\
S^2 \arrow{u}{\alpha} \arrow{ru}[swap]{\beta} & \\
\end{tikzcd} \]
Since \( C\eta \hookrightarrow ku \) is an injection, the map \( \Sigma^2 c \circ \partial = d \) acts by sending \( d^*(\Sigma^2 1) = Sq^2 \). So, \( \text{im}(d^*) \simeq (Sq^2) \).

Next, we wish to show that \( A\!/A(1) \) is contained in \( H^*(ko; \mathbb{F}_2) \). Finally, we will show that \( H^*(ko; \mathbb{F}_2) \simeq A\!/A(1) \) using an induction argument on degrees.

**Lemma 2.** \( A\!/A(1) \) is isomorphic to \( H^*(ko) \) as an \( A \)-module.

**Step 1.** \( A\!/A(1) \) injects into \( H^*(ko) \) as an \( A \)-module.

**Proof.** Examine the exact sequence induced on cohomology by \( \eta \), \( c \), \( \partial \):

\[
\begin{array}{c}
\Sigma^2 H^*(ko) \overset{\partial^*}{\longrightarrow} H^*(ku) \overset{c^*}{\longrightarrow} H^*(ko) \overset{\eta^*}{\longrightarrow} \Sigma H^*(ko)
\end{array}
\]

We are further interested in the map \( d = \Sigma^2 c \circ \partial : ku \to \Sigma^2 ku \), so we substitute \( H^*ku \simeq A\!/E(1) \) and add in the map \( d^* : \Sigma^2 A\!/E(1) \to \Sigma^2 A\!/E(1) \):

\[
\begin{array}{c}
\Sigma^2 H^* ko \overset{\partial^*}{\longrightarrow} A\!/E(1) \overset{c^*}{\longrightarrow} H^* ko \overset{\eta^*}{\longrightarrow} \Sigma H^* ko
\end{array}
\]

\[
\begin{array}{c}
\Sigma^2 A\!/E(1) \overset{d^*}{\longrightarrow} A\!/E(1) \overset{c^*}{\longrightarrow} H^* ko \overset{\eta^*}{\longrightarrow} \Sigma H^* ko
\end{array}
\]

Since \( A \) is a free \( A(1) \)-module, the map \( d^* : \Sigma^2 A\!/E(1) \to A\!/E(1) \) - which by Lemma 1 sends \( 1 \mapsto Sq^2(1) \) - is induced up from the \( A(1) \)-module homomorphism \( \Sigma^2 A(1) \to A(1) \) which is also right multiplication by \( Sq^2 \).

Let’s examine \( A(1) \!/E(1) \) by writing it out in full form, that is,

\[
A(Sq^1, Sq^2) / A(Sq^1, SQ^2) = A(Sq^2)
\]

Now, note that this module has only 2 nonzero elements, 1 and \( Sq^2 \). Let us convince ourselves of this fact: Since \( E(1) \) acts freely on \( A(1) \) from the right, \( A(1) \) has \( \text{dim} \) 8, \( E(1) \) has \( \text{dim} \) 4, so \( A(1) \!/E(1) \) has \( \text{dim} \) 2.

Applying the map \( d' : \Sigma^2 A(1) \!/E(1) \to A(1) \!/E(1) \), that is, \( \Sigma^2 \{ 1, Sq^2 \} \to \{ 1, Sq^2 \} \), which sends \( \Sigma^2 1 \mapsto Sq^2(1) \), we see that the image of this map is \( Sq^2(1) \), and the kernel is \( Sq^2(\Sigma^2 1) \) (since \( Sq^2(\Sigma^2 1) \to Sq^2 \Sigma^2 = 0 \)). That is,

\[
\Sigma^2 \text{im}(d') = \ker(d')
\]
Therefore, 
\[ \Sigma^2 \text{im}(d^*) = \ker(d^*) \]

Recall that we defined \( d^* \) as \( (\Sigma^2 c \circ \partial)^* \). Thus, using the diagram above, and the simple fact that for any maps \( i, j: \ker(i) \subseteq \ker(i \circ j) \), and \( \text{im}(i \circ j) \subseteq \text{im}(j) \), we see:

\[ \ker(\Sigma^2 c^*) \subseteq \ker((\Sigma^2 c \circ \partial)^*) = \Sigma^2 \text{im}(\partial^*) = \ker(c^*) \]

We may rewrite this, substituting \( d^* \) for \( (\Sigma^2 c \circ \partial)^* \):

\[ \ker(\Sigma^2 c^*) \subseteq \ker(d^*) = \Sigma^2 \text{im}(d^*) \subseteq \text{im}(\partial^*) = \ker(c^*) \]

From this chain of inclusions, we get \( \text{im}(d^*) \cong \ker(c^*) \). By Noether’s isomorphism theorem,

\[ \text{im}(c^*) \cong A//E(1)/\ker(c^*) \cong A//E(1)/\text{im}(d^*) \]

By Lemma 1, \( \text{im}(d^*) = A(Sq^2) \), and we see that \( \text{im}(c^*) \cong A//E(1)/(Sq^2) \cong A//A(1) \). Thus, 

\[ A//A(1) \cong \text{im}(c^*) \subseteq H^* ko \]

**Step 2.** \( A//A(1) \) is isomorphic to \( H^* ko \) as an \( A \)-module.

**Proof.** We construct the following map of exact sequences. We know by Step 1 that the vertical maps are injective:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Sigma^2 A//A(1) & \rightarrow & A//E(1) & \rightarrow & A//A(1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Sigma H^* ko & \rightarrow & \Sigma^2 A//A(1) & \rightarrow & A//E(1) & \rightarrow & A//A(1) & \rightarrow & 0 \\
\end{array}
\]

We see that \( \eta^* \) being zero in degree \( k \) forces \( A//A(1) \) and \( H^* ko \) to be isomorphic in degree \( k \). This is because \( \eta^* \) being zero forces \( c^* \) to be surjective, and thus forces \( b \circ a \) to be surjective. Thus, \( b \) must be surjective in degree \( k \), and \( b \) is injective by assumption, so \( b: A//A(1) \rightarrow H^* ko \) is an isomorphism in degree \( k \).

Our proof will be an inductive proof that \( \eta^* \) is the zero map. We start our induction very simply: since \( ko \) is connective, we see that the map \( \eta^* \) must be zero in degrees \( \leq 0 \). In degree 1, we look at \( \eta^*: H^1 ko \rightarrow \Sigma H^1 ko \). We know, given the group structure of \( \pi_k (ko) \), that \( H^1 ko = \mathbb{Z}_2^\infty \). Further, just glancing at the first stem of the ASS for \( ko \), we see that \( 2\eta^* = 0 \). There is only one map from \( \mathbb{Z}_2^\infty \rightarrow \mathbb{Z}_2^\infty \) that when doubled is equal to zero, and that is the zero map. Thus \( \eta^* \) must be zero.

We take as the inductive hypothesis that \( \eta^* \) is zero in degrees \( \leq n \). We wish to show that this implies that \( \eta^* \) is zero in degree \( n + 1 \).
We have the following chain of reasoning:

\[ \eta^* = 0 \text{ in } \deg \leq n \]
\[ \implies \ c^* \text{ is surjective in } \deg \leq n \]
\[ \implies \ b \text{ is surjective (and thus is an isomorphism) in } \deg \leq n \]
\[ \implies \ \Sigma^2 b \text{ is an isomorphism in } \deg \leq n + 2 \]
\[ \implies \ \partial^* \text{ is injective in } \deg \leq n + 2; \text{ since } e \text{ is injective} \]
\[ \implies \ \Sigma \eta^* \text{ is zero in } \deg \leq n + 2 \]
\[ \implies \ \eta^* \text{ is zero in } \deg \leq n + 1. \]

This concludes the inductive argument.

References


