different connected components of \(M_{H/G}\). Moreover, the connected components may have different dimensions, as well as the same dimension.

REFERENCES


\[ \text{Z}_2\text{-equivariant James Construction} \]

by

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Summary. In this paper we investigate a relation between the James space \(J(X)\) with an action of \(\mathbb{Z}_2\) and the \(\mathbb{Z}_2\)-space \(\Omega^V S^V X\) for \(V\) the nontrivial one-dimensional representation of the group \(\mathbb{Z}_2\). After defining the action on \(J(X)\) we prove that the \(\mathbb{Z}_2\)-spaces \(J(X)\) and \(\Omega^V S^V X\) are weak \(\mathbb{Z}_2\)-homotopically equivalent. As an application of the result we get a construction of a \(\mathbb{Z}_2\)-EHP sequence.

Introduction. In this paper by \((X, q)\) we denote arcwise-connected and compact topological \(\mathbb{Z}_2\)-space \(X\) with chosen nondegenerated base point \(x_0 \in X\) and a continuous map \(q : X \rightarrow \mathbb{R}_+\) such that

\[
\begin{align*}
(i) & \quad q^{-1}(0) = x_0, \\
(ii) & \quad q(g \cdot x) = q(x) \quad \text{for all } g \in \mathbb{Z}_2 \text{ and } x \in X.
\end{align*}
\]

In Section 1 of this paper we formulate definitions of spaces \(\Gamma^V(X, q)\) and \(\Omega^V \Gamma^V(X, q)\) and give some their properties. Moreover, we construct a fiber which fiber is the space \((\Omega^V \Gamma^V(X, q))^{\mathbb{Z}_2}\).

Section 2 contains definition of an action of the group \(\mathbb{Z}_2\) on the space \(J(X)\). We also define a continuous \(\mathbb{Z}_2\)-equivariant map \(\lambda : J(X) \rightarrow \Omega^V \Gamma^V(X, q)\). In addition we construct a quasi-fibering which fiber is the space \(J(X)\)\(^{\mathbb{Z}_2}\).

In Section 3 we formulate and prove the main result of this paper. Namely, we prove that the map \(\lambda\) is a weak \(\mathbb{Z}_2\)-homotopic equivalence.

As an application, in Section 4 we give a construction of \(\mathbb{Z}_2\)-EHP sequence for a case not known before.

1. Spaces \(\Gamma^V(X, q), S^V X, \Omega^V \Gamma^V(X, q), \Omega^V S^V X\) and their properties. Let us define the spaces \(S^V X\) and \(\Gamma^V(X, q)\) in the following way

\[
S^V X = S^V \wedge X \text{ and } \Gamma^V(X, q) = V \times X / \{(v, x) : \|v\| \geq q(x)\}.
\]
We define the action of the group $\mathbb{Z}_2$ on the spaces $\Omega^V(X,q)$ and $S^V$ by the formula $g \cdot (v,x) = [(g \cdot v, g \cdot x)]$. The space $\Omega^V(X,q)$ is called $\mathbb{Z}_2$-Moore suspension on the $\mathbb{Z}_2$-space $(X, q)$. It is easy to see that the above spaces are $\mathbb{Z}_2$-homeomorphic.

Next we define the space $\Omega^* \Omega^V(X,q)$ as follows

$$\Omega^* \Omega^V(X,q) = \{(r,f) \in \mathbb{R}_+ \times \overline{F}(V, \Omega^V(X,q)) : v \in V \forall v \mid \|v\| \geq r \Rightarrow f(v) = x_0\}.$$ 

We consider the space $\Omega^* \Omega^V(X,q)$ as $\mathbb{Z}_2$-space with the action of $\mathbb{Z}_2$ given by the formula $g \cdot (r,f) = (r, g \cdot f)$. The space $\Omega^* \Omega^V(X,q)$ is called the $\mathbb{Z}_2$-space of the Moore loops on the $\mathbb{Z}_2$-Moore suspension of the $\mathbb{Z}_2$-space $(X, q)$.

**Lemma 1.1.** $\Omega^* \Omega^V(X,q)$ is homotopically equivalent to $\Omega^SV^V X$.

We omit easy proof of this Lemma. Let us introduce a structure of a manifold on the space $\Omega^* \Omega^V(X,q)$. For $(r,f)$ and $(s,g) \in \Omega^* \Omega^V(X,q)$ we define the multiplication by $(r,f) \cdot (s,g) = (r+s,h)$, where

$$h(v) = \begin{cases} f(v + s \cdot w) & \text{for } v \in (-(r+s) \cdot w, (r+s) \cdot w) \\ g(v - r \cdot w) & \text{for } v \notin (-(r+s) \cdot w, (r+s) \cdot w) \\ x_0 & \text{for } v = (-(r+s) \cdot w, (r+s) \cdot w) \end{cases}.$$ 

Next we construct a fiberation with fiber equal to $(\Omega^* \Omega^V(X,q))^\mathbb{Z}_2$. First we define the $\mathbb{Z}_2$-space of Moore paths

$$P^V(\Omega^V(X,q)) = \{(r,f) \in \mathbb{R}_+ \times \overline{F}(V, \Omega^V(X,q)) : \exists x, \overline{x} \in \Omega^V(X,q) \forall v \in V \text{ or } (v) \cdot \|v\| \geq r \Rightarrow f(x) = \overline{x}, \text{ and } (v) \cdot \|v\| \leq -r \Rightarrow f(x) = \overline{x}\}.$$ 

The group $\mathbb{Z}_2$ acts on the space $P^V(\Omega^V(X,q))$ by multiplication $g \cdot (r,f) = (r,g \cdot f)$. The equation $\overline{p}_1(r,f) = f(r \cdot w)$ defines a continuous map

$$\overline{p}_1 : (P^V(\Omega^V(X,q))^\mathbb{Z}_2) \rightarrow \Omega^V(X,q).$$ 

Note that $\overline{p}_1^{-1}(x_0) = (\Omega^V(\Omega^V(X,q))^\mathbb{Z}_2)$.

**Lemma 1.2.** The map $\overline{p}_1 : (P^V(\Omega^V(X,q))^\mathbb{Z}_2) \rightarrow \Omega^V(X,q)$ defined as above is a fiberation.

**Proof.** is standard and left to the reader. An easy computation shows that the space $(P^V(\Omega^V(X,q))^\mathbb{Z}_2)$ has the same homotopy type as the space $X^\mathbb{Z}_2$. We define a homeomorphism $\epsilon : \Omega^V(X,q) \rightarrow \Omega^V(X,q)$ by $\epsilon([(v,x)])= [(v,g \cdot x)]$, where $g$ is the nontrivial element of $\mathbb{Z}_2$. The map $p_1 = \epsilon \circ \overline{p}_1 : (P^V(\Omega^V(X,q))^\mathbb{Z}_2) \rightarrow \Omega^V(X,q)$ is a fiberation.

We see that the exact homotopy sequence for the fiberation can be written in the following form

$$\cdots \rightarrow \pi_n((\Omega^* \Omega^V(X,q))^\mathbb{Z}_2) \rightarrow \pi_n(X^\mathbb{Z}_2) \rightarrow$$

2. $\mathbb{Z}_2$-equivariant James construction and its properties. We use the definition of James construction given in [4]. The James construction on the space $(X, q)$ will be denoted by $J(X)$. The elements of the space $J(X)$ will be written in the form $x_1 \ldots x_k$. We define an action of the group $\mathbb{Z}_2$ on the space $J(X)$ as follows

$$g \cdot (x_1 \ldots x_k) = \begin{cases} x_1 \ldots x_k & \text{for } g = 0 \in \mathbb{Z}_2 \\ (g \cdot x_k) \ldots (g \cdot x_1) & \text{for } g = 1 \in \mathbb{Z}_2 \end{cases}$$ 

The space $J(X)$ with this action of $\mathbb{Z}_2$ is called $\mathbb{Z}_2$-equivariant James construction on the $\mathbb{Z}_2$-space $(X, q)$.

Let us define a continuous $\mathbb{Z}_2$-equivariant map $\lambda : X \rightarrow \Omega^* \Omega^V(X,q)$ by formula $\lambda(x) = (q(x), \lambda_2(\cdot q))$, where $\lambda_2(r) = [(r, x)]$. The map $\lambda$ extends to a continuous $\mathbb{Z}_2$-equivariant map $\lambda : J(X) \rightarrow \Omega^* \Omega^V(X,q)$ defined by $\lambda(x_1 \ldots x_k) = \lambda(x_1) \ldots \lambda(x_k)$.

We shall construct a quasifibering, which fiber is the space $J(X)^{\mathbb{Z}_2}$. Form a map $f : X \times J(X)^{\mathbb{Z}_2} \rightarrow J(X)^{\mathbb{Z}_2}$ by formula

$$f(x, x_1 \ldots x_k) = x \cdot x_1 \ldots x_k \cdot (g \cdot x),$$ 

where $g$ is the nontrivial element of $\mathbb{Z}_2$. We define the cone $C^V(X,q)$ in the following way

$$C^V(X,q) = \{(v,x) \in X \times V : \|v\| \leq q(x)\} \cup \{(v, x) \in V \times X : \|v\| = q(x) \text{ and } \|v\| \leq 0\}.$$ 

We are given an injection $X \times J(X)^{\mathbb{Z}_2} \rightarrow C^V(X,q) \times J(X)^{\mathbb{Z}_2}$ defined by $(x,y) \rightarrow ([q(x) \cdot w, x], y)$. Let $E^V = (C^V(X,q) \times J(X)^{\mathbb{Z}_2}) \cup f J(X)^{\mathbb{Z}_2}$ be the cylinder of the map $f$. Now let us define a map $p : E^V \rightarrow \Omega^V(X,q)$ as follows

$$p([(v,x), y]) = \|v(x)\| \text{ and } p(y) = x_0.$$ 

By the James Lemma in [5] it follows that the map $p$ is a quasifibering. We have obtained the following exact sequence of the quasifibering

$$\cdots \rightarrow \pi_n(J(X)^{\mathbb{Z}_2}) \rightarrow \pi_n(E^V) \rightarrow \pi_n(\Omega^V(X,q)) \rightarrow \pi_{n-1} J(X)^{\mathbb{Z}_2} \rightarrow \cdots$$

On the space $C^V(X,q) \times J(X)^{\mathbb{Z}_2}$ we define the following relation

$$([q(x) \cdot w, x], y) \sim (x_0, x \cdot y \cdot (g \cdot x)),$$

where $g$ is the nontrivial element of $\mathbb{Z}_2$. Note that $E^V = C^V(X,q) \times J(X)^{\mathbb{Z}_2} / \sim$.

Let us introduce the following filtration $E_n^V$ in the space $E^V$:

$$E_n^V = C^V(X,q) \times J_n(X) / \sim \text{ for } n = 1, 2, 3, \ldots$$
THEOREM 2.1 (Milnor Theorem). Let $Z$ and $X$ be topological spaces with filtrations $Z_n$ and $Y_n$ respectively and $f : Y \to Z$ be a map preserving the filtrations. If

a) for all $n \geq 1$ the pairs $(Z_{n+1}, Z_n)$ and $(Y_{n+1}, Y_n)$ are cofibering,

b) for all $n \geq 1$ the maps $f_n = f|_{Z_n} : Z_n \to Y_n$ are homotopy equivalences,

then the map $f : Z \to Y$ is a homotopy equivalence.

For the proof of the above theorem see for instance [5].

LEMMA 2.1. The space $E^V$ has the same homotopy type as the space $X^{Z_2}$.

Proof. Let $g$ be the nontrivial element of the group $Z_2$. We define a continuous map $h : E^V \to (P^* V)^{Z_2}$ as follows

$$h((v,x), x_1, \ldots, x_k) = \left( \sum_{i=1}^{n} \frac{q(x_i)}{2} + q(x) + \text{or}(v) \cdot \|x\|, \right)$$

$$\gamma_1(v,x) \ast \lambda(x_1, \ldots, x_k) + \gamma_2(v,x),$$

where $\gamma_1(v,x) : V \to \Gamma^V(X,q)$ is given by

$$|(-v,x)|$$

for $\text{or}(t) \cdot \|t\| \leq -(q(x) + \text{or}(v) \cdot \|v\|)$

and where $\gamma_2(v,x) : V \to \Gamma^V(X,q)$ is defined by

$$|(v,g \cdot x)|$$

for $(q(x) + \text{or}(v) \cdot \|v\|) \leq \text{or}(t) \cdot \|t\| \leq 0$

and

$$\gamma_2(v,x) = \left( |((\|t\| - q(x)) \cdot w, g \cdot x)|, \right.$$  

$$x_0$$

for $0 \leq \text{or}(t) \cdot \|t\| \leq 0$

and

$$(v,g \cdot x)$$

for $(q(x) + \text{or}(v) \cdot \|v\|) \leq \text{or}(t) \cdot \|t\| \leq 0$

and

$$\gamma_2(v,x) = \left( |((\|t\| - q(x)) \cdot w, g \cdot x)|, \right.$$  

$$x_0$$

for $0 \leq \text{or}(t) \cdot \|t\| \leq 0$

Let $\Pi : \{0\} \times X^{Z_2} \to X^{Z_2}$ be the projection. We denote the coordinates of the map $h$ as $h(y) = (s(y), t(y))$. We define a map $d : E^V \to X^{Z_2}$ in the following way $d(y) = \Pi(t(y)(0))$. Let us apply Theorem 2.1. to the following case

$$Z = E^V, \quad Y = X^{Z_2}, \quad Z_n = P_n^V \quad \text{and} \quad f = d.$$

It is easy to see that for $n \geq 1$

a) the pairs $(Z_{n+1}, Z_n)$ and $(Y_{n+1}, Y_n)$ are cofibering,

b) the map $d_n = d|_{Z_n} : Z_n \to Y_n$ is a homotopy equivalence.

By Theorem 2.1 it follows that the map $d$ is a homotopy equivalence. \hfill \square

Using Lemma 2.1 we simplify the exact sequence of the quasi-fibering $p : E^V \to \Gamma^V(X,q)$. We obtain the following exact sequence

$$\cdots \to \pi_n(\mathcal{J}(X)^{Z_2}) \to \pi_n(\mathcal{J}(X)) \to \pi_n(\Delta^V \Gamma^V(X,q)) \to \pi_{n-1}(\mathcal{J}(X)^{Z_2}) \to \cdots$$

In our notation the Theorem 2.2 has the following form

THEOREM 2.2 (James Theorem). The map $\lambda : \mathcal{J}(X) \to \Omega^* \text{V} \Gamma^V(X,q)$ is a weak homotopy equivalence.

The proof of the theorem can be found in [4]. A stronger version of the theorem is due to [9].

3. $Z_2$-equivariant James Theorem. In this section we formulate and prove the main theorem of the paper.

THEOREM 3.1 ($Z_2$-equivariant James Theorem). For every finite $Z_2$-CW complex $K$ the map

$$\lambda_* : [K, \mathcal{J}(X)] \to [K, \Omega^* \text{V} \Gamma^V(X,q)]$$

is an isomorphism.

Proof. An easy computation shows that the map $h : E^V \to (P^* \text{V} \Gamma^V(X,q))^{Z_2}$ defined in Lemma 2.1 is a homotopy equivalence. It follows immediately that $h_{|\mathcal{J}(X)}^{Z_2} = \lambda^{Z_2}$.

We shall prove that the constructed exact sequences of fibering and quasi-fibering are isomorphic. We have the following commutative diagram

$$\begin{array}{ccc}
J(X)^{Z_2} & \xrightarrow{\lambda^{Z_2}} & (\Omega^* \text{V} \Gamma^V(X,q))^{Z_2} \\
\downarrow \phi & & \downarrow \phi \\
E^V & \xrightarrow{h} & (P^* \text{V} \Gamma^V(X,q))^{Z_2} \\
\downarrow \pi & & \downarrow \pi \\
\text{V} \Gamma^V(X,q) & \xrightarrow{h_{|\mathcal{J}(X)}} & \Gamma^V(X,q)
\end{array}$$

By the Five Lemma the map

$$\lambda^{Z_2} : \pi_n(\mathcal{J}(X)^{Z_2}) \to \pi_n((\Omega^* \text{V} \Gamma^V(X,q))^{Z_2})$$

is an isomorphism for any $n \in \mathbb{N}$. From this it follows that the maps $\lambda_{|\mathcal{J}(X)}$ and $\lambda_{|\mathcal{J}(X)}^{Z_2}$ are isomorphisms. Using the above and Proposition 11.2 in [1] we complete our proof.
4. Application of $\mathbb{Z}_2$-equivariant James Theorem. In this section we use notation introduced in [1], where the exact sequence of $\mathbb{Z}_2$-pair $(X, A)$ was constructed. This sequence is of the following form

\[ \cdots \longrightarrow \pi_{p,q}(r+2, s+1) \longrightarrow \pi_{p,q}(r+1, s+1) \longrightarrow \pi_{p,q-1}(r+2, s+1) \longrightarrow \cdots \]

Let us fix indexes $r$ and $s$ and write the above exact sequence for the $\mathbb{Z}_2$-pair $(\mathcal{J}(\Sigma^r), \Sigma^r)$

\[ \cdots \longrightarrow \pi_{p,q}(\Sigma^r) \longrightarrow \pi_{p,q}(\mathcal{J}(\Sigma^r)) \longrightarrow \pi_{p,q-1}(\Sigma^r) \longrightarrow \cdots \]

The spaces $\Omega^V f(V(X, q))$ and $\Omega^{1,0} \Sigma^{1,0}$ are $\mathbb{Z}_2$-homotopically equivalent. Denote by $\gamma$ a homotopy equivalence between these spaces. By the properties of the exact sequence of pair $(X, A)$ and Theorem 3.1 we deduce that the above sequence is equivalent to the sequence

\[ \cdots \longrightarrow \pi_{p,q}(\Sigma^r) \longrightarrow \pi_{p+1,q}(\Sigma^{r+1}) \longrightarrow \pi_{p+1,q-1}(\Sigma^{r+1}) \longrightarrow \cdots \]

If $\alpha_*$ is the isomorphism of the groups $\pi_{p,q}(\Sigma^{1,0} \Sigma^{1,0} \Sigma^{r,s})$ and $\pi_{p+1,q}(\Sigma^{r+1})$, then $\Sigma^{1,0} = \alpha_* \circ \gamma_* \circ \lambda_* \circ \alpha_*$. It can be verified that the groups $\pi_{p,q}(\mathcal{J}(\Sigma^r), \Sigma^r)$ and $\pi_{p,q-1}(\mathcal{J}(\Sigma^r, \Sigma^r))$ are isomorphic for $p + q < 3 \cdot (r + s) - 2$ and $q < r + s - 1$. Denote by $\sigma_*$ this isomorphism.

We are given an action of $\mathbb{Z}_2$ on the space $\Sigma^r$ by

\[ g \cdot (x \wedge y) = \begin{cases} (g \cdot y) \wedge (g \cdot x) & \text{if } g \in \mathbb{Z}_2^2, \\ x \wedge y & \text{if } g = 0 \in \mathbb{Z}_2^2. \end{cases} \]

**Proposition 4.1.** Assume that $p + q < 3 \cdot (r + s) - 2$ and $q < r + s - 1$. The simplest form of the exact sequence for the pair $(\mathcal{J}(\Sigma^r), \Sigma^r)$ is the following

\[ \cdots \longrightarrow \pi_{p,q}(\Sigma^r) \longrightarrow \pi_{p+1,q}(\Sigma^{r+1}) \longrightarrow \pi_{p+1,q-1}(\Sigma^{r+1}) \longrightarrow \cdots \]

**Proof.** Using the isomorphism $\sigma_*$ we simplify the sequence $(*)$

\[ \cdots \longrightarrow \pi_{p,q}(\Sigma^r) \longrightarrow \pi_{p+1,q}(\Sigma^{r+1}) \longrightarrow \pi_{p+1,q-1}(\Sigma^{r+1}) \longrightarrow \cdots \]

It can be proved that the space $\Sigma^{r,s} \wedge \Sigma^{r,s}$ with action $(**)$ of $\mathbb{Z}_2$ is $\mathbb{Z}_2$-homeomorphic to the space $\Sigma^{r+s}$. Applying the above remark and Theorem 3.1 we conclude that the group $\pi_{p,q}(\mathcal{J}(\Sigma^{r,s} \wedge \Sigma^{r,s}))$ is isomorphic to $\pi_{p+1,q}(\Sigma^{r+s} \wedge \Sigma^{r+s})$. Denote the isomorphism by $\eta$.

Finally we get

\[ \cdots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \longrightarrow \pi_{p+1,q}(\Sigma^{r+s}) \longrightarrow \pi_{p+1,q-1}(\Sigma^{r+s}) \longrightarrow \cdots \]

**Notations.**

- $V$ — a one-dimensional nontrivial representation of the group $\mathbb{Z}_2$,
- $0 \in V$ — a vector of the representation $V$,
- $\text{or}(v) = \text{sgn}(k)$, where $v = k \cdot w$,
- $S^V = V \cup \{\infty\}$ — one-point compactification of the representation $V$ with a chosen base point $x_0 = \infty$,
- $F(A, B)$ — the space of all continuous maps from a topological space $A$ into a topological space $B$, considered with compact open topology,
- $F(A, B)$ — the space of all these continuous maps from the topological space $A$ into the topological space $B$, which preserve base points, considered with compact-open topology,
- $G$ — is any group,
- $[Z, T]_G$ — the set of $G$-homotopy classes of $G$-equivariant maps from a $G$-space $Z$ to a $G$-space $T$ preserving base points,
- $X^G$ — the set of all fixed points of the action of the group $G$ on the $G$-space $X$,
- $f^H : X^H \longrightarrow Y^H$ — the restriction of the $G$-equivariant map $f : X \longrightarrow Y$ to the set of fixed points of the action of the subgroup $H$ of the group $G$.

Given $\mathbb{Z}_2$-spaces $A$ and $B$ we define the following action of the group $\mathbb{Z}_2$ on the space $\mathcal{F}(A, B)$: $(g \cdot f)(a) = g \cdot f(g \cdot a)$, where $f \in \mathcal{F}(A, B)$, $g \in \mathbb{Z}_2$ and $a \in A$.

We introduce the action of the group $\mathbb{Z}_2$ on the space $F(A, B)$ in analogous way.
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Skein Modules of 3-Manifolds
by
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Summary. It is natural to try to place the new polynomial invariants of links in algebraic topology (e.g. to try to interpret them using homology or homotopy groups). However, one can think that these new polynomial invariants are byproducts of a new more delicate algebraic invariant of 3-manifolds which measures the obstruction to isotopy of links (which are homotopic). We propose such an algebraic invariant based on skein theory introduced by Conway (1969) and developed by Giller (1982) as well as Lickorish and Millett (1987).

Let \( M \) be an oriented 3-manifold and \( R \) a commutative ring with 1. For \( r_0, \ldots, r_{k-1} \in R \) we define the \( k \)th skein module \( S_k(M; R)(r_0, \ldots, r_{k-1}) \) as follows:

Let \( \mathcal{L}(M) \) be the set of all ambient isotopy classes of oriented links in \( M \). Let \( M(\mathcal{L}, R) \) be a free \( R \)-module generated by \( \mathcal{L}(M) \) and \( S_{\mathcal{L}(M)}(r_0, \ldots, r_{k-1}) \) the submodule generated by linear skein expressions \( r_0 L_0 + r_1 L_1 + \ldots + r_{k-1} L_{k-1} \), where \( L_0, L_1, \ldots, L_{k-1} \) are classes of links identical except the parts shown in Fig. 1.

\[
\begin{align*}
L_0 & \quad L_1 & \quad L_2 & \quad \ldots & \quad L_{k-1} \\
\end{align*}
\]

**Fig. 1**

**DEFINITION 1.** The \( R \)-module

\[
S_k(M; R)(r_0, \ldots, r_{k-1}) = M(\mathcal{L}, R) / S_{\mathcal{L}(M)}(r_0, \ldots, r_{k-1})
\]

is called the \( k \)th skein module of \( M \).

**EXAMPLE 2.**
(a) \( S_k(M; R)(0, \ldots, 0) = M(\mathcal{L}, R) \),