EQUIVARIANT COHOMOLOGY AND THE SUPER RECIPROCAL PLANE OF A HYPERPLANE ARRANGEMENT

SOPHIE KRIZ

Abstract. In this paper, we investigate certain graded-commutative rings which are related to the reciprocal plane compactification of the coordinate ring of a complement of a hyperplane arrangement. We give a presentation of these rings by generators and defining relations. Our presentation was recently used by Holler and I. Kriz [7] to calculate the $\mathbb{Z}$-graded coefficients of localizations of ordinary $RO((\mathbb{Z}/p)^n)$-graded equivariant cohomology at a given set of representation spheres. We also give an interpretation of these rings in terms of superschemes, which can be used to further illuminate their structure.

1. Introduction

G-equivariant generalized homology and cohomology theory for a compact lie group G is best behaved when the (co)-homology groups are graded by elements of the real representation ring $RO(G)$. In this case (see Lewis, May, Steinberger [13] for background), the theory enjoys many of the properties of non-equivariant (co)-homology, for example, Spanier-Whitehead duality. Explicit calculations of equivariant cohomology groups, however, are much harder than in the non-equivariant case. A telling example is the case of "ordinary" G-equivariant cohomology theories, defined by Lewis, May and McClure [12]. These theories satisfy a "dimension axiom" in the sense that the $\mathbb{Z}$-graded part of their coefficients (i.e. (co)-homology of a point) are zero except in dimension 0 for all (closed) subgroups of G.

However, calculation of the $RO(G)$-graded coefficients of these "ordinary" G-equivariant cohomology theories has been an open problem since the 1980s, and these groups carry some deep information. For example, for the "constant" $\underline{\mathbb{Z}}$ Mackey functor coefficients, (which means that restrictions to subgroups are identities), a partial calculation of the $RO(G)$-graded coefficients for $G = \mathbb{Z}/8$ was a key ingredient in the solution by Hill, Hopkins and Ravenel [6] of the Kervaire invariant 1 problem.

The algebraic calculations made in the present paper are relevant to the ordinary $RO(G)$-graded (co)homology theory with constant $\underline{\mathbb{Z}/p}$ coefficients for $G = (\mathbb{Z}/p)^n$. We denote this theory by $H\underline{\mathbb{Z}/p}_{(\mathbb{Z}/p)^n}$. In the paper [8], Holler and I. Kriz calculated the "positive" part of these coefficients, meaning the groups

$$(1) \qquad H\underline{\mathbb{Z}/p}^V_{(\mathbb{Z}/p)^n}(*)$$

with V an actual (not virtual) representation for $p = 2$. A key ingredient in this calculation was the geometric fixed point ring

$$(2) \qquad (\Phi^{(\mathbb{Z}/p)^n} H\underline{\mathbb{Z}/p})_*,$$

which is the localization of the full $RO((\mathbb{Z}/p)^n)$-graded coefficient ring by inverting the inclusions $S^0 \to S^\alpha$ for all non-trivial irreducible representations α (see Tom Dieck [20] and [13], chapter 11, Def. 9.7).

Holler and I. Kriz [8] calculated the ring (2) for $p = 2$ by hand using a spectral sequence, and commented that the rings seemed to have an unusual algebraic structure, and asked about its geometric significance. They also did not know how to complete the same computation for $p > 2$, where the structure seemed much more complicated.

Answering these algebraic questions is the main purpose of the present paper. Using our main theorem (Theorem 2 below), Holler and I. Kriz [7] then generalized their calculations of the geometric fixed point coefficient ring (2) to $p > 2$, and also answered the following more general question:

What is the structure of the $\mathbb{Z}$-graded coefficient ring R_S of the $(\mathbb{Z}/p)^n$-fixed point specctrum given by localizing $H\underline{\mathbb{Z}/p}_{(\mathbb{Z}/p)^n}$ by inverting the maps $S^0 \to S^\alpha$ for a given set S of irreducible $(\mathbb{Z}/p)^n$-representations?

Symbolically, we may write

$$(3) \qquad R_S = ((\bigwedge_{i=1}^m S^{\infty\alpha_i}) \wedge H\underline{\mathbb{Z}/p})_*^{(\mathbb{Z}/p)^n}$$

where $S = \{\alpha_1, \dots, \alpha_m\}$.

Then, in particular, the geometric fixed point coefficient ring (2) is equal to R_S where

$$S = \{\alpha_1, \dots, \alpha_{p^n-1}\}$$

consists of all non-trivial irreducible representations of $(\mathbb{Z}/p)^n$.

The contribution of the present paper was essential to 1. better understanding the algebraic structure for the case of $\mathbb{Z}/2$, which was necessary for considering the case of an arbitrary set S, and 2. finding a graded-commutative analog, which is relevant for the case of $\mathbb{Z}/p$-coefficients for $p > 2$ (since in that case, $\mathbb{Z}/p$-valued cohomology forms a graded-commutative, and not a commutative ring), which give a *candidate* for the R_S. The algebraic computations of this present paper then gives sufficient control on the structure of this ring to show that the candidate is correct by a counting argument [7].

What kind of algebra are we talking about? In [8] Theorem 2, Holler and I. Kriz proved that

$$\begin{gathered}(\Phi^{(\mathbb{Z}/2)^n} H\mathbb{Z}/2)_* = \\ \mathbb{Z}/2[t_\alpha | \alpha \in (\mathbb{Z}/2)^n \setminus \{0\}]/(t_\alpha t_\beta + t_\alpha t_\gamma + t_\beta t_\gamma | \alpha + \beta + \gamma = 0).\end{gathered} \tag{4}$$

Where t_α are in degree 1. They proved this by counting the dimension of the submodule of homogeneous elements of a given degree and matching it with a spectral sequence. But what do these relations mean?

Consider the affine space

$$\mathbb{A}^n_{\mathbb{F}_2} = \operatorname{Spec}(\mathbb{F}_2[x_1, \dots, x_n]).$$

Then the elements z_α, can be identified with non-zero linear combinations of the coordinates $x_1, \dots, x_n$ with coefficients in $\mathbb{F}_2$. Such linear combinations can, in turn, be identified with equations of hyperplanes through the origin in $\mathbb{A}^n_{\mathbb{F}_2}$. (All possible rational hyperplanes, as it turns out.) If we remove these hyperplanes from $\mathbb{A}^n_{\mathbb{F}_2}$, we obtain an affine variety with coordinate ring

$$\Big(\prod_{\alpha \in (\mathbb{Z}/2)^n \setminus \{0\}} z_\alpha^{-1}\Big)\mathbb{F}_2[x_1, \dots, x_n]. \tag{5}$$

I showed that the ring (4) is isomorphic to the subring of the ring (5) generated by the elements $t_\alpha = z_\alpha^{-1}$. This result turned out to be known (for example, [16], Theorem 4). In fact, the affine variety with coordinate ring (4) is known as the *reciprocal plane* of the hyperplane arrangement $\{z_\alpha\}$ (see [3]).

The reciprocal plane can, of course, be considered over any field, and the likely reason this significance of the ring (4) was not noticed before is that the focus of the previous work was mostly on characteristic 0: certainly not on the arrangement of *all* rational hyperplanes over a finite field. This interpretation, then, begged the question as to what happens if we remove just some *subset* S of hyperplanes from $\mathbb{A}^n_{\mathbb{F}_2}$? What is the *topological* significance of the reciprocal plane in that case? Using the known presentation [16] I rediscovered, Holler and I. Kriz subsequently proved that those rings are isomorphic to the rings (3).

The real story, and the main contribution of the present paper, however, is for $p > 2$. From the point of view of algebraic geometry, there is no difference: As we already mentioned, the reciprocal plane construction is independent of characteristic.

In algebraic topology, however, when we are dealing with characteristic $p \neq 2$, coefficient rings become *graded-commutative*, i.e.

$$xy = (-1)^{|x||y|}yx$$

where $|x|$ denotes the degree of x. So to solve the structure of the rings (2), (3) for $p > 2$, it was necessary to discover the *appropriate graded-commutative analogue* of the reciprocal plane, and to prove structure results analogous to [16]. This is the main result of the present paper.

Very briefly, we consider the ring

$$\mathbb{F}_p[x_1, \dots, x_n] \otimes \Lambda_{\mathbb{F}_p}[dx_1, \dots, dx_n]$$

where Λ denotes the exterior algebra. In this ring, invert a set of linear combinations z_α of the elements x_α. The right ring turns out to be the subring generated by $t_\alpha = z_\alpha^{-1}$ and $u_\alpha = z_\alpha^{-1}dz_\alpha$. Topologically, the element t_α has degree 2 and the element u_α has degree 1, corresponding to the fact that we are dealing with complex, not real, representations for $p > 2$. I determine the structure of these subrings in a way analogous to (but more complicated than) the commutative case. Holler and I. Kriz [7] then used my structure theorems to prove that these rings are isomorphic to the rings (3) for $p > 2$. This is the main topological application of the results of the present paper. The very striking geometric interpretation of the reciprocal planes begs the question what is the appropriate analogue of this interpretation in the graded-commutative case. The Spec of a graded-commutative ring is a *superscheme* (for a survey, see [21]). In section 6, I develop the superscheme analog of some of the known geometric structures associated with the reciprocal plane, which correspond to my algebraic generalization to graded-commutative rings. (Again, the algebraic geometry side of the story is independent of characteristic).

The present paper is organized as follows: In the next section, I give precise statements of the algebraic results of this paper, which amount to finding a presentation of the rings in question, in Theorem 1 in the commutative case and Theorem 3 in the graded-commutative case. Essentially, the proof is by describing an explicit algorithm of reducing a given relation to the relations in my presentation, which I do not think was known before. In Section 3, as a warm-up, I give an explicit proof of Theorem 1 which can be generalized to the graded-commutative case. In Section 4, I use this method to prove Theorem 3. In Section 5, I prove that the relation ideals I, K are also generated by the relation polynomials $P_L, P_{L,S}$ where the L's are restricted to "minimal" relations. The commutative case is particularly simple. This was also proved in [16] in the commutative case by less explicit methods. In Section 6, I discuss the geometric interpretation, including the construction of the superscheme corresponding to the graded-commutative case (Theorem 3).

Acknowledgement: I am most thankful to J.P. May for comments and encouragement.

2. Statement of the results

Following Terao [19], consider an n-dimensional affine space $\mathbb{A}^n_F$ over a field F. Let $z_1, \dots, z_m$ be non-zero linear combinations of the coordinates $x_1, \dots, x_n$ with coefficients in F. We can think of the z_i's as equations of hyperplanes in $\mathbb{A}^n_F$. Then the coordinates $t_i = z_i^{-1}$ define a morphism of affine varieties

$$\pi : \mathbb{A}^n_F \setminus Z(z_1 \dots z_m) \to \mathbb{A}^m_F$$

where $ZI = Z(I)$ is the set of zeros of an ideal I. The morphism π is an embedding if the z_j's linearly span the x_i's. Consider the Zariski closure of $\mathrm{Im}(\pi)$. As we shall see, this variety is a cone, so we can speak of the corresponding projective variety. This construction, called the reciprocal plane, has been studied extensively (see [16, 14, 9, 15, 18, 17, 10, 11]). For a survey, see [3].

To understand this construction better, we must describe it algebraically, which will also bring us closer to the motivation of the present paper. Let

$$R = z_1^{-1} \dots z_m^{-1} F[x_1, \dots, x_n] = F[x_1, \dots, x_n][z_1^{-1}, \dots, z_n^{-1}].$$

Then we have a homomorphism of rings

$$h : F[t_1, \dots, t_m] \to R$$

with $h(t_i) = z_i^{-1}$ (which is, of course, not onto). Consider the ideal $I = \mathrm{Ker}(h)$. Denote $\mathcal{A} = \{z_1, \dots, z_m\}$, and put

$$R_{\mathcal{A},\mathbb{A}^n_F} = F[t_1, \dots, t_m]/I.$$

Then $\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})$ is, by definition, the Zariski closure of $\mathrm{Im}(\pi)$. Also by the homomorphism theorem, $R_{\mathcal{A},\mathbb{A}^n_F}$ is a subring of R. Observe that

I is a prime ideal (therefore a radical) since R is an integral domain, and hence so are its subrings. Further, if the z_i's generate the x_j's, then

$$R = (t_1 \cdot \dots \cdot t_m)^{-1} R_{\mathcal{A},\mathbb{A}^n_F}.$$

Thus, in particular, in this case π is an open embedding of the hyperplane arrangement complement into the Zariski closure of its image.

The ideal I is non-zero when there are linear dependencies among the hyperplane equations z_i. Suppose, then,

$$L = a_1 z_{i_1} + \cdots + a_k z_{i_k} = 0 \in F[x_1, \dots, x_n] \tag{6}$$

where $a_1, \dots, a_k \in F$ are not 0, and

$$1 \leq i_1 < \cdots < i_k \leq m.$$

So, in R, we have $\frac{a}{t_{i_1}} + \cdots + \frac{a_k}{t_{i_k}} = 0$ where $k > 1$ (where, in the rest of this paper, we indentify $t_j = z_j^{-1}$). Thus,

(7)
$$\frac{a_1 t_{i_2} \dots t_{i_k} + \cdots + a_j t_{i_1} \dots \widehat{t_{i_j}} \dots t_{i_k} + \cdots + a_k t_{i_1} \dots t_{i_{k-1}}}{t_{i_1} \dots t_{i_k}} = 0 \in R,$$

where the hat means an omitted term.

Hence, the numerator P_L of the left hand side of (7) is in I.

Theorem 1. *([16], [3], (5.3)) Let $\mathcal{Z}$ be the set of all linear relations L among the hyperplane equations z_i. Then*

$$I = (P_L(t_1, \dots, t_m) | L \in \mathcal{Z}), \tag{8}$$

or in other words,

$$R_{\mathcal{A},\mathbb{A}^n_F} = F[t_1, \dots, t_m]/(P_L(t_1, \dots, t_m) | L \in \mathcal{Z}).$$

Corollary 2. *([7, 8]) For $p = 2$, the $\mathbb{Z}$-graded coefficient ring (6) of the constant $\underline{\mathbb{Z}/2}$-Mackey functor ordinary $(\mathbb{Z}/2)^n$-equivariant cohomology spectrum with the inclusion $S^0 \to S^{\alpha_i}$ inverted where α_i are real irreducible representations corresponding to the hyperplanes z_i is*

$$R_S = R_{\mathcal{A},\mathbb{A}^n_{\mathbb{F}_2}}.$$

This was proved in [8] using a direct method for the case of *all* $2^n - 1$ rational hyperplanes through the origin in $\mathbb{A}^n_{\mathbb{F}_2}$. The authors of [8] asked about the algebraic interpretation of this ring. I found the above interpretation and proved Theorem 1 by describing an explicit algorithm for reducing relations. Using this and the commutative case of Theorem 8 below then led to the proof of the general case of Corollary 2 in [7].

Both Theorem 1 and the commutative case of Theorem 8 turned out to be known ([16], Theorem 4). The reason the connection with [8], and Corollary 2 of [7] were not noticed before is probably that the focus of [16] and other previous work was not on the case of all hyperplanes over a finite field.

While the work of [16] (and hence Theorem 1 and the commutative case of Theorem 8) work over any field, in algebraic topology, those calculations are not relevant in characteristic $p > 2$, where the relevant rings are graded-commutative. In fact, [8] was written entirely in characteristic 2 because the graded-commutative analog of the ring $R_{\mathcal{A},\mathbb{A}_F^n}$ was not known. I found this ring algebrically by looking for a "graded-commutative" analog of the geometric structures described above, and my reduction algorithm. I then proved a graded-commutative analog of Theorem 1, (and the corresponding part of Theorem 8), which is the main result of the present paper.

For the graded-commutative case, consider

$$\Omega = F[x_1, \dots, x_n] \otimes \Lambda[dx_1, \dots, dx_n]$$

where Λ denotes the exterior algebra over the field F. Then the non-zero F-linear combinations z_i of the x_i's are in the center of Ω. Now consider

$$T = z_1^{-1} \dots z_m^{-1}\Omega \supset \Omega.$$

This is the graded-commutative analog of the ring R. We are interested in the subring $T_{\mathcal{A},\mathbb{A}_F^n}$ of T generated by $z_1^{-1}, \dots, z_m^{-1}, z_1^{-1}dz_1, \dots, z_m^{-1}dz_m$. Put $t_i = z_i^{-1}$and $u_i = z_i^{-1}dz_i$. Then we have a canonical homomorphism of rings

$$\psi : \Xi = F[t_1, \dots, t_m] \otimes \Lambda[u_1, \dots, u_m] \to T.$$

Let $K = \mathrm{Ker}(\psi)$. Note that $I \subsetneq K$. Thus, we have

$$T_{\mathcal{A},\mathbb{A}_F^n} = \Xi/K.$$

We want to find the generators of the ideal K. If L is again the left hand side of (6), then

$$dL = a_{i_1}dz_{i_1} + \dots + a_{i_k}dz_{i_k} = 0 \in T.$$

If we multiply

$$\begin{gathered} P_L = a_{i_1}t_{i_2} \dots t_{i_k} + a_{i_2}t_{i_1}\widehat{t_{i_2}} \dots t_{i_k} + \\ \dots + a_{i_j}t_{i_1} \dots \widehat{t_{i_j}} \dots t_{i_k} + \dots + a_{i_k}t_{i_1} \dots t_{i_{k-1}} \end{gathered}$$

by $dz_{j_1} \dots dz_{j_l}$ where

$$S = \{j_1 < \dots < j_l\} \subseteq \{i_1, \dots, i_k\}, \tag{9}$$

some monomial summands can be expressed in terms of the u_j's. If a monomial summand does not contain t_{j_s} but does contain dz_{j_s}, then use $dL = a_{i_1}dz_{i_1} + \cdots + a_{i_k}dz_{i_k}$ to eliminate dz_{j_s}. Explicitly, let

$$P_{L,S} = P_L dz_{j_1} \dots dz_{j_l}$$
$$-\sum_{s=1}^{l} t_{i_1} \dots \widehat{t_{j_s}} \dots t_{i_k} dz_{j_1} \dots \widehat{dz_{j_s}} dL \dots dz_{j_l}.$$

We have $P_{L,S} \in \Xi$. Note that, by definition, $P_{L,\emptyset} = P_L$. Our main result is

Theorem 3. *Let $\mathcal{Y}$ be the set of all pairs (L,S) where L is a linear relation among hyperplanes equations as in (6), and S is a subset of the index set as in (9). Then*

$$K = (P_{L,S} \mid (L,S) \in \mathcal{Y}).$$

In other words,

$$T_{\mathcal{A},\mathbb{A}^n_F} = \Xi/(P_{L,S} \mid (L,S) \in \mathcal{Y}).$$

This algebraic Theorem, along with Theorem 8 below was used in [7] to prove the following result:

Corollary 4. *([7]) For $p > 2$, the $\mathbb{Z}$-graded coefficient ring (6) of the constant $\underline{\mathbb{Z}/p}$-Mackey functor ordinary $(\mathbb{Z}/p)^n$-equivariant cohomology spectrum with inclusions $S^0 \to S^{\alpha_i}$ inverted where α_i are complex irreducible representations corresponding to the hyperplanes z_i is*

$$R_S = T_{\mathcal{A},\mathbb{A}^n_F}.$$

Since the commutative algebra methods of [16] are at present not available for graded-commutative rings, our proof of Theorem 3 is elementary and in fact is an elaboration of my algorithm used to prove Theorem 1. As a warm-up, I also include my original elmentary proof of Theorem 1, which I then generalized to the graded-commutativee case. Some brief notes on the interpretation of the graded-commutative result in algebraic geometry are given in Section 6 below.

Example: Let $L = z_1 + z_2 + z_3 = 0 \in \Omega$. Then we have

$$P_L = P_{L,\emptyset} = \frac{z_1 + z_2 + z_3}{z_1 z_2 z_3} = t_2 t_3 + t_1 t_3 + t_1 t_2.$$

Now to compute $P_{L,\{2\}}$, write

$$P_L dz_2 = t_2 t_3 dz_2 + t_1 t_2 dz_2 + t_1 t_3 dz_2 = u_2 t_3 + t_1 u_2 + t_1 t_3 dz_2. \tag{10}$$

Now use

$$dL = dz_1 + dz_2 + dz_3 = 0 \tag{11}$$

to express $dz_2 = -dz_1 - dz_3$, which we use to conclude

$$t_2t_3dz_2 = -t_1t_3(dz_1 + dz_3) = u_1t_3 + u_3t_1.$$

Substituting this into (10) gives the relation

$$P_{L,\{2\}} = u_2(t_1 + t_3) - u_1t_3 - u_3t_1.$$

To calculate $P_{L,\{1,2\}}$, we start with the expression

$$\begin{aligned} p_Ldz_1dz_2 &= t_2t_3dz_1dz_2 + t_1t_2dz_1dz_2 + t_1t_3dz_1dz_2 = \\ &= t_2t_3dz_1dz_2 + u_1u_2 + t_1t_3dz_1dz_3. \end{aligned}$$

Using (11) again, we get

$$t_2t_3dz_1dz_2 = t_2t_3(-dz_2 - dz_3)dz_2 = t_2t_3dz_2dz_3 = u_2u_3$$

and

$$t_1t_3dz_1dz_2 = t_1t_3dz_1(-dz_1 - dz_3) = u_3u_1.$$

Thus, we obtain the relation

$$P_{L,\{1,2\}} = u_1u_2 + u_2u_3 + u_3u_1.$$

The reader should keep in mind that the above derivation of examples of the relations $P_{L,S}$ is used simply to explain our definition of these relations. Nevertheless, they illustrate the fact that $P_{L,S}$ is a relation in $t_1^{-1}\dots t_m^{-1}\Xi$ which is contained in Ξ, and thus is valid in Ξ.

3. The commutative case

The purpose of this section is to prove Theorem 1.

Lemma 5. *The relation ideal I of Theorem 1 satisfies*

$$\begin{gathered} I = \{q = p_1L_1 + \dots + p_NL_N| \\ p_i \in t_1^{-1}\dots t_m^{-1}F[t_1,\dots,t_m], q \in F[t_1,\dots,t_m]\} \end{gathered} \tag{12}$$

Proof. Consider the diagram

(13)

$$\begin{tikzcd} F[t_1,\dots,t_m] \arrow[r, "\pi"] \arrow[d] & R & F[x_1,\dots,x_n] \arrow[l] \\ F[t_1,\dots,t_m,t_1^{-1},\dots,t_m^{-1}] \arrow[r, "="] & t_1^{-1}\dots t_m^{-1}F[t_1,\dots,t_m] \arrow[u, "\widetilde{\pi}"] & F[z_1,\dots,z_m] \arrow[l] \arrow[u, "\overline{\pi}"] \end{tikzcd}$$

We know $ker(\overline{\pi}) = (L_1,\dots,L_N) \subseteq F[z_1,\dots,z_m]$. Therefore we know that

$$\begin{aligned} &ker(\widetilde{\pi}) = z_1^{-1},\dots,z_m^{-1}ker(\overline{\pi}) = (L_1,\dots,L_N) \\ &\subseteq z_1^{-1}\dots z_m^{-1}F[z_1,\dots,z_m] = t_1^{-1}\dots t_m^{-1}F[t_1,\dots,t_m] \end{aligned}$$

by exactness of localization.

$\square$

Proof of Theorem 1. Let J be the ideal in I which is generated by all the P_L's. Then we want J to equal I. We shall perform induction on m (the number of the variables z_j). If $m = 0$, then $R_{\mathcal{A},\mathbb{A}^n_F} = F$ and there is nothing to prove. Let

$$q = p_1 L_1 + \cdots + p_N L_N \in I \tag{14}$$

be as in Lemma 5. We want to show that $q \in J$. We may assume $L_1, \ldots, L_N$ are in reduced row echelon form where the order of columns corresponds to the order of variables $z_m, \ldots z_1$. Let $\mathcal{A}' = \{z_1, \ldots, z_{m-1}\}$ and define an ideal $I' \subset F[t_1, \ldots, t_{m-1}]$ by

$$R_{\mathcal{A}',\mathbb{A}^n_F} = F[t_1, \ldots, t_{m-1}]/I'.$$

If the first pivot of our RREF is not in the first column, there is no relation $L_1, \ldots, L_N$ with $a_1 \neq 0$ involving $z_m = t_m^{-1}$. In this case, by construction, we have a homomorphism of rings

$$R_{\mathcal{A}',\mathbb{A}^n_F} \to R_{\mathcal{A},\mathbb{A}^n_F}.$$

Now, we may write each $p_1, \ldots, p_N$ as a Laurent polynomial in the variable t_m. Since $L_1, \ldots, L_N$ do not involve z_m, by Lemma 5, the coefficients q_i of q at t_m^i are in $F[t_1, \ldots, t_{m-1}]$, and are 0 for $i < 0$. Thus, by Lemma 5, each q_i maps to 0 in $R_{\mathcal{A}',\mathbb{A}^n_F}$, and thus, by the induction hypothesis, is an $F[t_1, \ldots, t_{m-1}]$-linear combination of the elements P_L where L runs through all linear relations among $z_1, \ldots, z_{m-1}$. Thus, $q \in J$ and we are done. Thus, assume that the RREF of the relations $L_1, \ldots, L_N$, as described above, has a pivot in the first column, corresponding to z_m.

Now write $p_i \in F[t_1, \ldots, t_{m-1}, t_1^{-1}, \ldots, t_{m-1}^{-1}][t_m, t_m^{-1}]$. Denote by

$$p_{i,k} \in t_1^{-1} \ldots t_{m-1}^{-1} F[t_1, \ldots, t_{m-1}]$$

the coefficients of every power t_m^k, $k \in \mathbb{Z}$.

Without loss of generality,

$$\begin{gathered} L_1 = a_1 z_m + \ldots, \\ a_1 \neq 0 \in F \end{gathered} \tag{15}$$

has more than two non-zero terms. Otherwise, the number of nonzero terms in L would be exactly 2 (since we did not allow $z_m = 0$). But if the number of non-zero terms in L_1 is exactly 2, then z_m is a non-zero multiple of some z_i, $i < m$. Therefore, if we put $\mathcal{A}' = \{z_1, \ldots, z_{m-1}\}$, $R_{\mathcal{A},\mathbb{A}^n_F} = R_{\mathcal{A}',\mathbb{A}^n_F}$, and our statement follows from the induction hypothesis.

Suppose, therefore, that (15) has at least 3 non-zero terms. Now consider the highest $k \in \mathbb{Z}$ such that $p_{i,k} \neq 0$ for some i. Let L_1' be the linear combination obtained from L_1 by omitting the $a_1 z_m$ summand.

Case 1: $k > 0$. Then

$$\overline{q} = p_{1,k} L_1' + p_{2,k} L_2 + \cdots + p_{N,k} L_N \in F[t_1, \ldots, t_{m-1}]. \tag{16}$$

Note: we do not, of course, claim that (16) is a relation among the chosen z_j's. However, since the relations are in reduced row echelon form, and L_1 has at least three terms, the relation L_1' only introduces a linear relation among at least two of the non-pivot variables of the relationns $L_2, \ldots, L_N$. Therefore, there exist some non-zero linear combinations $z_2, \ldots, z_m$ of some other parameters $y_1, \ldots, y_{n'}$, which satisfy the relations $L_1', L_2, \ldots, L_N$. Since our induction is only on the number m of the hyperplanes, the induction hypothesis applies. By Lemma 5, $\overline{q} \in I'$. By the induction hypothesis, (16) is a linear combination

$$w_1 P_{L_1'} + w_2 P_{L_2} + \cdots + w_N P_{L_N}, w_i \in F[t_1, \ldots, t_{m-1}].$$

Then subtracting

$$w_1 t_m^{k-1} P_{L_1} + w_2 t_m^k P_{L_2} + \cdots + w_N t_m^k P_{L_N}$$

from (14), we obtain an element $q' \in F[t_1, \ldots, t_m]$ which differs from L by an element of J, and for which the number $k \in \mathbb{Z}$ is lower. Thus,we are reduced to:

Case 2: $k \leq 0$. Then consider the lowest $l \leq 0$ for which there exists an i with $p_{i,l} \neq 0$. Then, $p_{1,l} = 0$, since $p_{1,l} z_m^{-l+1}$ has nothing to cancel out against in (14) (since all the other powers of z_m are $\leq -l$. Thus, since $-l + 1 > 0$, this contradicts $q \in F[t_1, \ldots, t_m]$.) Thus,

$$q' = p_{2,l} L_2 + \cdots + p_{N,l} L_N \in F[t_1, \ldots, t_m],$$

and thus, by Lemma 5, we have $q' \in I'$. Thus, by the induction hypothesis applied to $p_{i,\ell}$, q' is a linear combination

$$w_2 P_{L_2} + \cdots + w_N P_{L_N}, w_2, \ldots, w_N \in F[t_1, \ldots, t_{m-1}].$$

If $k = l = 0$, we are done. If $l < 0$, we must have

$$-p_{1,\ell+1} = a_1^{-1}(p_{2,\ell} L_2 + \cdots + p_{N,\ell} L_N)$$

for cancellation, so

$$\begin{aligned}
&(p_{2,l} L_2 + \cdots + p_{N,l} L_N) z_m^{-l} + \\
&p_{1,l+1} L_1 z_m^{-\ell-1} + a_1^{-1}(p_{2,l} L_2 + \cdots + p_{N,l} L_N) L_1' z_m^{-l-1} \\
&= 0 \in t_1^{-1} \ldots t_m^{-1} F[t_1 \ldots t_m]
\end{aligned}$$

can be subtracted from (14), thus increasing l without violating $k \leq 0$. Thus, by repeating this process, we are done.

$\square$

4. The odd case

In this section, we prove Theorem 3.

Lemma 6. *The relation ideal K of Theorem 3 is given by*

$$K = \{q = p_1L_1 + \cdots + p_NL_N + r_1dL_1 + \cdots + r_NdL_N \\ \mid q \in \Xi, p_i, r_i \in \Xi[t_1^{-1} \dots t_m^{-1}]\}.$$

Proof. Denote $Y = F[z_1 \dots z_m] \otimes \Lambda[dz_1 \dots dz_m]$. The analog of diagram (13) is

$$\begin{tikzcd} \Xi \arrow[r] \arrow[d] & T = z_1^{-1} \dots z_m^{-1}\Omega & \Omega \arrow[l] \\ \Xi[t_1^{-1} \dots t_m^{-1}] \arrow[r, "="] & z_1^{-1} \dots z_m^{-1}Y \arrow[u, "\psi"] & Y \arrow[l] \arrow[u, "\overline{\psi}"'] \end{tikzcd} \tag{17}$$

where $\overline{\psi}$ is the canonical map. Then

$$\mathrm{Ker}(\overline{\psi}) = (L_1, \dots, L_N, dL_1, \dots, dL_N). \tag{18}$$

Now certainly $\mathrm{Ker}(\psi) \supseteq z_1^{-1} \dots z_m^{-1}(L_1, \dots, L_N, dL_1, \dots, dL_N)$. To prove the converse, note that exactness of localization works the same here as in the commutative case. If $\psi(x) = 0$, $x \in z_1^{-1} \dots z_m^{-1}Y$, then $(z_1, \dots, z_m)^N\psi(x) = \psi((z_1, \dots, z_m)^N x) = 0$. Without loss of generality (by increasing N if necessary), we may then also assume $x \in Y$, $\bar{\psi}(x) = 0$. $\square$

It is useful to note here that in general, for a commutative ring R, and a multiplicative set $S \subseteq R$

$$S^{-1}(\Lambda_R[u_1, \dots, u_m]) = \Lambda_{S^{-1}R}[u_1, \dots, u_m]. \tag{19}$$

Proof of Theorem 3: The reader is encouraged to follow along the corresponding steps of the proof of Theorem 1, which we shall mimic. Let

$$M \subseteq K \tag{20}$$

be the ideal generated by the elements $P_{L,S}$. Again, we prove the statement by induction by m, the number of hyperplanes z_j. For $m = 0$, again $T_{\mathcal{A},\mathbb{A}^n_F} = F$, so there is nothing to prove. For $m > 0$, again, put

$$\mathcal{A}' = \{z_1, \dots, z_{m-1}\},$$
$$\Xi_0 = F[t_1, \dots, t_{m-1}] \otimes \Lambda[u_1, \dots, u_{m-1}],$$
$$T_{\mathcal{A}',\mathbb{A}^n_F} = \Xi_0/K'.$$

First we again put $L_1, \dots, L_N$ in reduced row echelon form so that the columns correspond to $z_m, z_{m-1}, \dots, z_1$. Then let

$$w = p_1 L_1 + \cdots + p_N L_N + q_1 dL_1 + \cdots + q_N dL_N \in \Xi \tag{21}$$

and $p_j, q_j \in \Xi[t_1^{-1}, \dots t_m^{-1}]$. Again, we may assume that the first column (corresponding to z_m) has a pivot: Otherwise, consider again the canonical homomorphism of rings

$$T_{\mathcal{A}', \mathbb{A}^n_F} \to T_{\mathcal{A}, \mathbb{A}^n_F}.$$

Again, each coefficients $w_{i,0}$ and $w_{i,1}$ of w at t_m^i and $t_m^i u_m$, respectively are in Ξ' by Lemma 6, and hence are linear combinations of $P_{L,S}$ where L are the relations among $z_1, \dots, z_{m-1}$ by the induction hypothesis. Thus $w \in M$ and we are done. Again, we may also assume that L_1 has at least 3 terms: 1 term is excluded by $z_1 \neq 0$, and 2 terms would give $T_{\mathcal{A}, \mathbb{A}^n_F} = R_{\mathcal{A}', \mathbb{A}^n_F}$, and our statement would follow from the induction hypothesis.

Then

$$p_i = \sum_{k \in \mathbb{Z}} (p_{i,k} t_m^k + \overline{p_{i,k}} t_m^k u_m)$$

and

$$q_i = \sum_{k \in \mathbb{Z}} (q_{i,k} t_m^k + \overline{q_{i,k}} t_m^k u_m)$$

with $p_{i,k}, \overline{p_{i,k}}, , q_{i,k}, \overline{q_{i,k}} \in \Xi_0[t_1^{-1}, \dots, t_{m-1}^{-1}]$. Let L_1' be L_1 with the z_m term removed. Consider the highest k for which at least one of the polynomials $p_{i,k}, \overline{p_{i,k}}, , q_{i,k}, \overline{q_{i,k}}$ is non-zero. Let us distinguish two cases. (Note that $dz_m = t_m^{-1} u_m$, so the omitted terms have lower total power of t_m).

Case 1:

Suppose $k > 0$. By maximality of k,

$$p_{1,k} L_1' + p_{2,k} L_2 + \cdots + p_{N,k} L_N + q_{1,k} dL_1' + \cdots + q_{N,k} dL_N \in \Xi_0. \tag{22}$$

By the induction hypothesis this is a linear combination of $P_{L_1', S}, P_{L_i, S}, i > 1$. Similarly for

$$\overline{p_{1,k}} L_1' + \overline{p_{2,k}} L_2 \cdots + \overline{p_{N,k}} L_N + \overline{q_{1,k}} dL_1' + \overline{q_{2,k}} dL_2 \cdots + \overline{q_{N,k}} dL_N \in \Xi_0. \tag{23}$$

(This time, we are also using $u_m^2 = 0$.) Subtracting the corresponding linear combinations of $P_{L_1,S}, P_{L_i,S}$ from

$$p_{1,k} L_1 + \cdots + p_{N,k} L_N + q_{1,k} dL_1 + \cdots + q_{N,k} dL_N$$

or

$$\overline{p_{1,k}} L_1 + \cdots + \overline{p_{N,k}} + \overline{q_{1,k}} dL_1 + \cdots + \overline{q_{N,k}} dL_N,$$

we decrease k. Thus, again, we are reduced to

Case 2:

Suppose $k \leq 0$. Then consider the lowest ℓ for which at least one of the polynomials $p_{i,k}, \overline{p_{i,k}}, , q_{i,k}, \overline{q_{i,k}}$ is non-zero. Let

$$\begin{aligned}\omega &= p_{1,\ell}L_1 + \cdots + p_{N,\ell}L_N\\ \rho &= q_{1,\ell}dL_1 + \cdots + q_{N,\ell}dL_N\\ \overline{\omega} &= \overline{p_{1,\ell}}L_1 + \cdots + \overline{p_{N,\ell}}L_N\\ \overline{\rho} &= \overline{q_{1,\ell}}dL_1 + \cdots + \overline{q_{N,\ell}}dL_N\end{aligned} \tag{24}$$

and denote by $\omega', \rho', \overline{\omega}', \overline{\rho}'$ the linear combinations obtained by replacing L_1, dL_1 by L_1', dL_1' in (24). Then by minimality of ℓ,

$$\omega + \rho + \overline{\omega} + \overline{\rho} = \omega' + \rho' + \overline{\omega}' + \overline{\rho}'. \tag{25}$$

(The extra terms on the left hand side of (25) have nothing to cancel against so their sum must be 0.) Therefore, if $k = \ell = 0$, the right hand side of (25) (which is equal to (21)) is in $\Xi_0 + u_m\Xi_0$, and the statment follows from the induction hypothesis and the lemma following this proof.

If $\ell < 0$, for cancellation, we must have

$$a_1 p_{1,\ell+1} = -\rho' - \omega'$$
$$a_1(q_{1,\ell+1}u_m + \overline{p_{1,\ell+1}}u_m) = -\overline{\rho}' - \overline{\omega}',$$

using the convention (15) for the definition of $0 \neq a_1 \in F$. Thus, adding to (21)

$$\begin{aligned}&0 = -(\omega + \rho + \overline{\omega} + \overline{\rho})z_m^{-\ell}\\ &-(p_{1,\ell+1} + q_{1,\ell+1}u_m + \overline{p_{1,\ell+1}}u_m)z_m^{-\ell-1}L_1\\ &-a_1^{-1}(\rho + \omega + \overline{\omega} + \overline{\rho})z_m^{-\ell-1}L_1',\end{aligned}$$

which is a $t_1^{-1}\dots t_m^{-1}\Xi$-linear combination of the elements L_i and dL_i, increases ℓ, without increasing k.

□

Lemma 7. *Assuming the statement of Theorem 3 holds with m replaced by $m-1$, and $L_1, \dots, L_M$ are in reduced row echelon form in the order of columns $z_m, z_{m-1}, \dots, z_1$ with a pivot in the first column and assume L_1 has at least 3 non-zero terms. Then, we have*

$$(\Xi_0 + u_m\Xi_0) \cap \mathrm{Ker}(\psi) \subseteq M + (\Xi_0 \cap \mathrm{Ker}(\psi)) \tag{26}$$

(see (20)).

Proof. We want to rephrase the Lemma to say

$$\Xi_0 + (P_{L,S}) \supseteq u_m\Xi_0 \cap \psi^{-1}(\psi\Xi_0). \tag{27}$$

This is possible because if $a \in \Xi_0$ and $b \in u_m\Xi_0$, and $\psi(a+b) = 0$ then $-\psi(a) = \psi(b) \in \psi\Xi_0$. Then $b \in u_m\Xi_0 \cap \psi^{-1}(\psi\Xi_0)$ so if (27) holds, then

$b \in \Xi_0 + (P_{L,S})$. In other words, $b = c+$ linear combinations of $P_{L,S}$ with $c \in \Xi_0$. Then $a+b = a+c+$linear combination of $P_{L,S}$, $a+c \in \Xi_0$, $\psi(a+c) = 0$ (since $\psi(P_{L,S}) = 0$). Thus, proving the statement (26) is reduced to proving the statement (27).

To prove (27), let $\overline{R} = \{p \in \Xi_0 | \psi(u_m p) \in \psi(\Xi_0)\}$. (Note that $u_m \overline{R} = u_m \Xi_0 \cap \psi^{-1}(\psi \Xi_0)$.) Let

$$Y_0 = F[z_1 \dots z_{m-1}] \otimes \Lambda[dz_1 \dots dz_{m-1}].$$

Put $Q = \{p \in z_1^{-1} \dots z_{m-1}^{-1} Y_0 | \psi(pu_m) \in \psi(z_1^{-1} \dots z_{m-1}^{-1} Y_0)\}$. We have

$$z_1^{-1} \dots z_m^{-1} Y = t_1^{-1} \dots t_m^{-1} \Xi$$

and

$$z_1^{-1} \dots z_{m-1}^{-1} Y_0 = t_1^{-1} \dots t_{m-1}^{-1} \Xi_0. \tag{28}$$

Then $\overline{R} \subseteq Q \cap \Xi_0$. So

$$\psi(pu_m) = \psi(p)\psi(u_m) = \psi(p)\frac{\psi(dL_1')}{\psi(L_1')} \in \psi(L_1')^{-1} T. \tag{29}$$

Note that $\psi(L_1')$ is a non-zero linear combination of the x_i's which is not an F-multiple by any of the z_j's by our assumption on L_1. Then $u_m Y_0$ goes to $(\psi(L_1'))^{-1}\Omega$ by ψ and $u_m(z_1^{-1} \dots z_{m-1}^{-1} Y_0)$ goes to T by ψ. Let $\overline{Q} = \{p \in Y_0 | \psi(pu_m) \in \psi(Y_0)\}$. So $Q = z_1^{-1} \dots z_{m-1}^{-1}\overline{Q}$. By (29), $\bar{Q}$ is the ideal of all $p \in Y_0$ such that

$$\psi(L_1') | \psi(p)\psi(dL_1') \in T. \tag{30}$$

But since L_1' is not an F-multiple of any of the z_j's, by (19), (30) is equivalent to

$$\bar{\psi}(L_1') | \bar{\psi}(p)\bar{\psi}(dL_1') \in \Omega.$$

Also, $\psi(L_1') \in \Omega$ is a regular element: let

$$\mu = \frac{\bar{\psi}(p)\bar{\psi}(dL_1')}{\bar{\psi}(L_1')} \in \Omega.$$

We may now assume that the z_j's linearly span the x_i's (since otherwise we could replace the x_i's by the span of z_j's), so

$$\bar{\psi}|_{Y_0} : Y_0 \to \Omega$$

is onto. Let $\psi(\bar{\mu}) = \mu$, $\bar{\mu} \in Y_0$. Thus, by (18), we have

$$p dL_1' \in \bar{\mu} L_1' + (L_2, \dots, L_N, dL_2, \dots, dL_N).$$

Now note that in Y_0, L_1' is not a linear combination of $L_2, \dots, L_n$, (since $z_m \neq 0$), so by basic properties of polynomial and exterior algebras, writing

$$Y_0 = F[L_1', \gamma_2, \dots, \gamma_{m-1}] \otimes \Lambda_F[dL_1', d\gamma_2, \dots, d\gamma_{m-1}],$$

we see that $dL_1'|\bar{\mu} \in Y_0$. Also, the kernel of multiplication by dL_1' is (dL_1'), so

$$p \in (L_1', L_2, \dots, L_N, dL_1', dL_2, \dots, dL_N) \subseteq Y_0.$$

Thus, we proved

$$\bar{Q} = (L_1', L_2, \dots, L_N, dL_1', dL_2, \dots, dL_N) \subseteq Y_0.$$

By exactness of localization,

$$Q = (L_1', L_2, \dots, L_N, dL_1', dL_2, \dots, dL_N) \subseteq z_1^{-1} \dots z_{m-1}^{-1} Y_0.$$

Now we need to prove

$$u_m(Q \cap \Xi_0) \subseteq \Xi_0 + M$$

(see (20)). Let $v \in Q \cap \Xi_0$. Then we have

$$v \in p_1 L_1' + p_2 dL_1' + (L_2, \dots, L_N, dL_2, \dots, dL_N) \subseteq z_1^{-1} \dots z_{m-1}^{-1} Y_0$$

with $p_1, p_2 \in \Xi_0[t_1^{-1}, \dots, t_{m-1}^{-1}]$. Let

$$L_1' = \sum_{j=2}^{k} a_j z_{i_j} \tag{31}$$

for $1 \leq i_2 < \dots, i_k < m$.

Note that L_1' is linearly independent over F of $L_2, \dots, L_N$ (since $z_m \neq 0$). Now subtract a linear combination of $L_2, \dots, L_N$ from (31) so that the right hand side $\widetilde{L_1}'$ has the fewest possible non-zero terms. Without loss of generality, thus, $\widetilde{L_1}' = L_1'$ and $z_{i_2}, \dots, z_{i_k}, L_2, \dots, L_N$ are linearly independent over F. Then, choosing a basis for $F\{z_1, \dots, z_{m-1}\}$ containing $\{z_{i_2}, \dots, z_{i_k}, L_2, \dots, L_N\}$ and writing elements of $t_1^{-1} \dots t_{m-1}^{-1} \Xi_0$ using this basis, we see that the elements $p_1 L_1', p_2 dL_1'$ must be in Ξ_0, if we absorb any monomials from

$$(L_2, \dots, L_N, dL_2, \dots, dL_N) \tag{32}$$

containing dL_1' into $p_2 dL_1'$ and any remaining monomials from (32) containing L_1' into $p_1 L_1'$. This means that (recalling that $z_{i_2}, \dots, z_{i_k}$ are the summands of L_1' with non-zero coefficients), then p_1 is a Ξ_0-multiple of $t_{i_2} \dots, t_{i_k}$, and p_2 is a Ξ_0-multiple of $t_{i_2} \dots t_{i_k} dz_{j_2} \dots dz_{j_l}$ for some $\{j_2 < \dots < j_l\} \subset \{i_2 < \dots < i_k\}$.

Now $u_m t_{i_2} \dots t_{i_k} L_1'$ can be eliminated by $P_{L_1, \{m\}}$. For

$$S = \{j_1 < \dots < j_l\} \subseteq \{i_2, \dots, i_k\},$$

$$u_m t_{i_2} \dots t_{i_k} dz_{j_1} \dots dz_{j_l} dL_1'$$

can be eliminated by $u_m P_{L_1, S\cup\{m\}}$. This is because

$$\begin{aligned}
&u_m P_{L_1,S\cup\{m\}} = \\
&u_m P_{L_1} dz_{j_1}\dots dz_{j_l} dz_m \\
&-\sum_{q=1}^{l} u_m t_{i_2}\dots \widehat{t_{j_q}} \dots t_{i_k} dz_{j_1}\dots dL_1 \widehat{dz_{j_q}} \dots dz_{j_l} dz_m \\
&- u_m t_{i_2}\dots t_{j_k} dz_{j_1}\dots dz_{j_l} dL_1' \\
&= -u_m t_{i_2}\dots t_{j_k} dz_{j_1}\dots dz_{j_l} dL_1'
\end{aligned}$$

□

Some concrete examples of the eliminations we used in the conclusion of the proof of Lemma 7 are shown below.

Example 1: $z_m + z_1 + z_2 = L_1, m > 2$,then $L_1' = z_1 + z_2$, so $u_m t_1 t_2 (z_1 + z_2) = t_m dz_m t_1 t_2 (z_1 + z_2) = u_m(t_1 + t_2) = u_m t_1 + u_m t_2$. And $P_{L_1,m} = u_m t_1 + u_m t_2 - t_1 u_2 - t_2 u_1$.

Example 2: $S = \emptyset$ so

$$u_m t_1 t_2 dL_1' = u_m u_1 t_2 + u_m u_2 t_1.$$

This is elimenated by

$$u_m P_{L_1\cup\{m\}} = -u_m t u_2 - u_m t_2 u_1$$

Example 3: $S = 1$ so

$$u_m t_1 t_2 dz_1 (dz_1 + dz_2) = u_m u_1 u_2$$

is eliminated by

$$u_m P_{L_1,\{1,m\}} = u_m u_1 u_2.$$

5. Minimality

Let

$$L = a_1 z_{i_1} + \dots + a_k z_{i_k}$$

where

$$1 \le i_1 < \dots < i_k \le m,$$
$$a_i \ne 0 \in F.$$

Then put

$$|L| := \{i_1, \dots, i_k\}. \tag{33}$$

Call L *minimal* if there do not exist relations L_1, L_2 such that

$$L_1 + L_2 = L$$

$$|L_1|, |L_2| \subsetneq |L|.$$

Define shuffle permutations as follows: for sets of natural numbers

$$S_1 = \{i_1 < \cdots < i_k\}$$

$$S_2 = \{j_1 < \cdots < j_l\}$$

$$S_1 \cap S_2 = \emptyset,$$

denote by σ_{S_1,S_2} the permutation which puts the sequence

$$(i_1, \ldots, i_k, j_1, \ldots, j_l)$$

in increasing order. Also define for $S = \{i_1 < \cdots < i_k\}$:

$$t_S := t_{i_1} \ldots t_{i_k}$$

$$u_S := u_{i_1} \ldots u_{i_k}$$

$$dz_S := dz_{i_1} \ldots dz_{i_k}$$

Theorem 8.

$$I = (P_L | L \textit{ is a minimal relation})$$

$$K = (P_{L,S} | L \textit{ is a minimal relation and } S \subseteq |L|)$$

(For the case of I, see [16]*, Theorem 4.)*

Let L be as in (33), $S \subseteq |L|$. Put

$$Q_{L,S} := t_{|L|} dL dz_S.$$

So obviously, $Q_{L,S} \in K$.

Lemma 9. $Q_{L,S} \in (P_{L,T} | T \subseteq |L|)$

Proof. If $S \neq \emptyset$, let $i \in S$. Then $Q_{L,S} = u_i P_{L,S}$. On the other hand,

$$Q_{L,\emptyset} = u_{i_1} P_{L,\emptyset} - t_{i_1} P_{L,\{i_1\}}.$$

In the first summand the surviving term is the term of $P_{L,\emptyset}$ which omits t_{i_1}. In the second summand the surviving terms are the "error terms" of the summand of P_L which omits t_{i_1}. All remaining terms cancel. □

Proof of Theorem 8: Even case : Suppose we know

$$L_1 + L_2 = L$$

$$|L_1|, |L_2| \subsetneq |L|.$$

Then

$$P_L = t_{L \setminus L_1} P_{L_1} + t_{L \setminus L_2} P_{L_2}.$$

Odd case : If L is not minimal we know $L_1+L_2=L$ and $|L_1|,|L_2|\subsetneq |L|$. Based on the even case, the first guess for $P_{L,S}$ could be

$$P_{L_1,S_1}u_{S\setminus S_1}t_{|L|\setminus(|L_1|\cup S)}sign(\sigma_{S,S\setminus S_1})$$
$$+P_{L_2,S_2}u_{S\setminus S_2}t_{|L|\setminus(|L_2|\cup S)}sign(\sigma_{S_2,S\setminus S_2}),$$

$$S_1=S\cap|L_1|, S_2=S\cap|L_2|.$$

The terms that match are those when we omit t_i from t_L with $i\in |L|\setminus S$ or $i\in|L_1|\cap|L_2|\cap S$. The terms which do not match are for $i\in(|L_1|\cap S)\setminus|L_2|$ or $(|L_2|\cap S)\setminus|L_1|$. For $i\in(|L_1|\cap S)\setminus|L_2|$, the term missing in our first guess is

$$u_{S\setminus S_2\setminus\{i\}}Q_{L_2,S_2}t_{|L|\setminus(|L_2|\cup S)}sign(\sigma_{S\setminus S_2\setminus\{i\},\{i\}})sign(\sigma_{S\setminus S_2,S_2}).$$

Symmetrically for $|L_2|\cap S\setminus|L_1|$. Thus we have

$$\begin{aligned}&P_{L,S}=P_{L_1,S_1}u_{S\setminus S_1}t_{|L|\setminus(|L_1|\cup S)}sign(\sigma_{S,S\setminus S_1})\\&+P_{L_2,S_2}u_{S\setminus S_2}t_{|L|\setminus(|L_2|\cup S)}sign(\sigma_{S_2,S\setminus S_2})\\&+\textstyle\sum_{i\in S\setminus S_2}u_{S\setminus S_2\setminus\{i\}}Q_{L_2,S_2}t_{|L|\setminus(|L_2|\cup S)}\\&sign(\sigma_{S\setminus S_2\setminus\{i\},\{i\}})sign(\sigma_{S\setminus S_2,S_2})\\&+\textstyle\sum_{i\in S\setminus S_1}u_{S\setminus S_1\setminus\{i\}}Q_{L_1,S_1}t_{|L|\setminus(|L_1|\cup S)}\\&sign(\sigma_{S\setminus S_1\setminus\{i\},\{i\}})sign(\sigma_{S\setminus S_1,S_1}).\end{aligned}$$

Use Lemma 9. □

6. THE GEOMETRIC INTERPRETATION

Since the well known paper by W. Fulton and R. MacPherson [4], compactifications of configuration spaces, and complements of hyperplane arrangements [2], became an important topic of algebraic geometry. For a good survey, see [3]. Our geometric interpretation is related to a compactification known as the *reciprocal plane* [3], Section 5.1, and its super analog.

Let us assume the z_j's linearly span the vector space $\mathbb{A}^n_F$ (otherwise, we can replace $x_1,\dots,x_n$ by a basis of the span of $z_1,\dots,z_m$). Denote

$$\mathcal{A}=\{z_1,\dots,z_m\},\mathcal{A}_S=\{z_i|i\in S\}.$$

Let $R_{\mathcal{A},\mathbb{A}^n_F}=F[t_1,\dots,t_m]/I$ (see Theorem 1). We can then similarly write $R_{\mathcal{A},W}$ where $\mathcal{A}$ is a set of vectors spanning the dual of an F-vector space W. A stratification of $\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})$ can be described as follows. Recall that we have a canonical embedding

$$\mathbb{A}^n_F\setminus Z(z_1\dots z_m)\subseteq\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F}).\tag{34}$$

Call a vector subspace $V \subseteq \mathbb{A}^n_F$ *special* if $V = Z(\mathcal{A}_S)$ for some $S \subseteq \{1, \dots, m\}$. (Note: S can be empty.) Put also

$$S_V = \{i \in \{1, \dots, m\} | V \subseteq Z(z_i)\}.$$

(Note [3] that the sets of i's for which the z_i's are linearly independent are the independet sets of a matroid. Then the sets S_V are precisely what is called the *flats* of this matroid.) For a scheme X, denote by $|X|$ the underlying topological space.

Theorem 10. *([16], Remark 6) For $V \subseteq \mathbb{A}^n_F$ special, there is a canonical embedding*

$$\mathrm{Spec}(R_{\mathcal{A}_{S_V}, \mathbb{A}^n_F/V}) \to \mathrm{Spec}(R_{\mathcal{A}, \mathbb{A}^n_F}). \tag{35}$$

Composing (35) with

$$\mathbb{A}^n_F/V \setminus \bigcup_{i \in S} Z(z_i) \subseteq \mathrm{Spec}(R_{\mathcal{A}_{S_V}, \mathbb{A}^n_F/V}),$$

(see (34)), induces a decomposition of sets (not topological spaces),

$$|\mathrm{Spec}(R_{\mathcal{A}, \mathbb{A}^n_F})| = \coprod_{V \subseteq \mathbb{A}^n_F \ special} |(\mathbb{A}^n_F/V) \setminus \bigcup_{i \in S_V} Z(z_i)|. \tag{36}$$

Proof. We have

$$R_{\mathcal{A}, \mathbb{A}^n_F}/(t_i | i \notin S_V) = R_{\mathcal{A}_S, \mathbb{A}^n_F/V},$$

which gives the maps (35). (The point is that there is no linear relation between the z_i's in which all but one term would have $i \in S_V$. Thus, all the relations P_L where L contains a term not in S_V are in $(t_i | i \notin S_V)$.)

To prove (36), first note that the images of the inclusions of the components of the right hand side of (36) are clearly disjoint since they correspond to imposing relations t_i with $i \notin S_V$ for some special vector subspace V, and inverting all other t_i's. Thus, our task is to show that the canonical map from the right hand side to the left hand side of (36) is onto. To this end, let $Q \in \mathrm{Spec}(R_{\mathcal{A}, \mathbb{A}^n_F})$ and let

$$S = \{j \in \{1, \dots, m\} | Q \in (t_j)\}.$$

Let

$$V = \bigcap_{j \in S} Z(z_j).$$

We want to prove that $S = S_V$. The fact that $S \subseteq S_V$ is automatic. Suppose $j \in S_V \setminus S$. Then $z_j = a_1 z_{j_1} + \dots a_k z_{j_k}$ with $j_1 < \dots < j_k \in S$, $a_1, \dots, a_k \neq 0 \in F$. Let

$$L = z_j - a_1 z_{j_1} - \dots - a_k z_{j_k}.$$

By assumption, $Q \in (t_j)$. But in $R_{\mathcal{A},\mathbb{A}^n_F}/(t_j)$, P_L is a non-zero multiple of

$$t_{j_1} \cdot \dots \cdot t_{j_k}.$$

This implies $Q \in (t_{j_i})$ for some $i = 1, \dots, k$. Contradiction. □

Theorem 10 suggests that $\mathrm{Spec}(R_{\mathcal{A},\mathbb{A}^n_F})$ should have a compactification where on the right hand side of (36) we replace each

$$(\mathbb{A}^n_F/V) \setminus \bigcup_{i \in S_V} Z(z_i)$$

with the corresponding affine space $(\mathbb{A}^n_F/V)$. In fact, there is such a compactification $X_{\mathbb{A}^n_F,\mathcal{A}}$ and it can be described as the Zariski closure of the image of the embedding

$$\mathbb{A}^n_F \setminus Z(z_1 \dots z_m) \xrightarrow{(z_1,\dots,z_m)} \prod_{i=1}^m \mathbb{P}^1_F. \tag{37}$$

In the terminology of [3], this is an example of what is called a *toric compactification*. It was also studied, from a different point of view, in [1]. Note that while (37) resembles superficially the formula for the De Concini-Procesi wonderful compactification [2], (37) is in fact quite different. While the wonderful compactification uses projections to (typically) higher-dimensional projective spaces, (37) uses inclusions of the affine coordinates z_i into $\mathbb{P}^1_F$.

The projective variety $X_{\mathbb{A}^n_F,\mathcal{A}}$ is covered by a system of affine open sets, closed under intersection,

$$U_{V,T} = \mathrm{Spec} \prod_{j \in T} z_j^{-1} F[t_i, z_j | i \notin S_V, j \in S_V]/(\frac{P_L}{t_{S_V \cap |L|}})$$

where V runs through special subspaces of $\mathbb{A}^n_F$, L runs through all linear relations among the z_i's, and T is any subset of S_V. The following fact follows from the definitions:

Lemma 11. *We have*

$$U_{V,T} \bigcap U_{V',T'} = U_{W,T \cup T' \cup (S_V - S_{V'}) \cup (S_{V'} - S_V)}$$

where

$$V + V' \subseteq W = \bigcap_{i \in S_V \cap S_{V'}} Z(z_i)$$

so

$$S_V \bigcap S_{V'} = S_W.$$

□

It follows from Theorem 10 that $|U_{V,T}|$ are open subsets covering $X_{\mathbb{A}^n_F,\mathcal{A}}$. To show the affine schemes $U_{V,T}$ are reduced (their coordinate rings have no nilpotent elements), we have the following generalization of Theorem 1:

Theorem 12. *Let V be a special subspace of $\mathbb{A}^n_F$. The kernel of the homomorphism of rings*

$$F[t_i, z_j | i \notin S_V, j \in S_V] \to \prod_{i \notin S_V} z_i^{-1} F[z_1, \dots, z_m]/(\mathcal{Z}_V)$$

given by $t_i \mapsto z_i^{-1}$, where $\mathcal{Z}_V$ is the set of all linear relations among the z_i's, $i \in S_V$, is

$$(\frac{P_L}{t_{S_V \cap |L|}}).$$

Proof. Note that by the proof of Theorem 10, any linear relation among the z_i's which involves a z_i for $i \notin S_V$ involves at least two of them. Therefore, we can repeat the induction in Section 3 with $\{1, \dots, m\}$ replaced by $\{1, \dots, m\} \setminus S_V$. □

We also have a similar analog of Theorem 3:

Theorem 13. *Let V be a special subspace of $\mathbb{A}^n_F$. The kernel of the homomorphism of rings*

$$\begin{array}{c} F[t_i, z_j | i \notin S_V, j \in S_V] \otimes \Lambda[u_i, dz_j | i \notin S_V, j \in S_V] \\ \downarrow \\ \prod_{i \notin S_V} z_i^{-1} F[z_1, \dots, z_m] \otimes \Lambda[dz_i, \dots, dz_m]/(\mathcal{Y}_V) \end{array}$$

given by $t_i \mapsto z_i^{-1}$, $u_i \mapsto z_i^{-1} dz_i$, where $\mathcal{Y}_V = \mathcal{Z}_V \cup \{dL | L \in \mathcal{Z}_V\}$, is

$$(\frac{P_{L,S}}{t_{S_V \cap |L|}})$$

where L runs through the linear relations among the z_i's and $S \subseteq |L|$.

Accordingly, we have a superscheme analog $\widetilde{X}_{\mathbb{A}^n_F,\mathcal{A}}$ of $X_{\mathbb{A}^n_F,\mathcal{A}}$. Here by a superscheme, we mean a locally ringed space by $\mathbb{Z}/2$-graded commutative rings which is locally isomorphic to Spec of a $\mathbb{Z}/2$-graded commutative ring (see e.g. [21]). $\widetilde{X}_{\mathbb{A}^n_F,\mathcal{A}}$ is covered by super-affine open subsets

$$\begin{array}{c} \widetilde{U}_{V,T} = \mathrm{Spec} \prod_{j \in T} z_j^{-1} F[t_i, z_j | i \notin S_V, j \in S_V] \\ \otimes \Lambda[u_i, dz_j || i \notin S_V, j \in S_V]/(\frac{P_{L,S}}{t_{T \cap |L|}}). \end{array}$$

We clearly have

$$|\widetilde{U}_{V,T}| = |U_{V,T}|$$

and for $|U_{V',T'}| \subseteq |U_{V,T}|$, $\widetilde{U}_{V',T'}$ is a complement of the zero set of an (even) principal ideal in $\widetilde{U}_{V,T}$. Therefore, $\widetilde{X}_{\mathbb{A}^n_F,\mathcal{A}}$ can be defined as the colimit of the $\widetilde{U}_{V,T}$'s in the category of superschemes.

References

[1] F.Ardila, A.Boocher: The closure of a linear space in a product of lines, *J. Algebraic Combin.* 42 (2016) 199-235

[2] C. De Concini, C. Procesi: Wonderful models of subspace arrangments, *Selecta Math.* (N.S) 1.(1995), 3, 459-494

[3] G. Denham: Toric and tropical compactifications of hyperplane complements, *Ann. Sci. Toulouse Math.* (6) 23 (2014) 2, 297-333

[4] W. Fulton, R. MacPherson: A compactification of configuration spaces, *Ann. of Math.* (2) 139 (1994), no. 1, 183-225

[5] J.P.C.Greenlees: *Adams spectral sequences in equivariant topology*, Thesis, Cambridge University (1985)

[6] M. A. Hill, M. J. Hopkins, D. C. Ravenel: On the nonexistence of elements of Kervaire invariant one, *Ann. of Math.* (2) 184 (2016), no. 1, 1-262

[7] J.Holler, I. Kriz: The coefficients of $(\mathbb{Z}/p)^n$ equivariant geometric fixed points of $H\mathbb{Z}/p$, arXiv:2002.05284

[8] J.Holler, I.Kriz: On $RO(G)$-graded equivariant "ordinary" cohomology where G is a power of $\mathbb{Z}/2$, *Algebr. Geom. Topol.* 17 (2017), no. 2, 741-763

[9] H. Horiuchi and H. Terao: The Poincaré series of the algebra of rational functions which are regular outside hyperplanes, *J. Algebra* 266 (2003), no. 1, 169-179

[10] J. Huh and E. Katz: Log-concavity of characteristic polynomials and the Bergman fan of matroids, *Math. Ann.* 354 (2012), no. 3, 1103-1116

[11] M. Lenz: The f-vector of a representable-matroid complex is strictly log-concave, *Adv. in Appl. Math.* 51 (2013) 543-545

[12] G. Lewis, J. P. May, J. McClure: Ordinary $RO(G)$-graded cohomology, *Bull. Amer. Math. Soc.* 4 (1981), no. 2, 208-212

[13] L.G. Lewis, J.P. May, M. Steinberger, J.E. McClure: *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, 1213 (1986)

[14] E. Looijenga: Compactifications defined by arrangements. I. The ball quotient case, *Duke Math. J.* 118 (2003), no. 1, 151-187

[15] A. Postnikov: Permutohedra, associahedra, and beyond, *Int. Math. Res. Not.* (2009), no. 6, 1026-1106

[16] N. J. Proudfoot and D. Speyer: A broken circuit ring, *Beiträge Algebra Geom.* 47 (2006), no. 1, 161-166

[17] R. Sanyal, B. Sturmfels, and C. Vinzant: The entropic discriminant, *Adv. Math.*, to appear. arXiv:1108.2925

[18] H. Schenck and S O. Tohaneanu: The Orlik-Terao algebra and 2-formality, *Math. Res. Lett.* 16 (2009), no. 1, 171-182

[19] H. Terao: Algebras generated by reciprocals of linear forms, *J. Algebra* 250 (2002), no. 2, 549-558
[20] T. Dieck: Bordism of G-manifolds and integrality theorems, *Topology* 9 (1970) 345-358
[21] D. Westra: Superschemes, www.mat.univie.ac.at/∼westra/superschemes.pdf