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ON THE C_p-EQUIVARIANT DUAL STEENROD ALGEBRA

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ABSTRACT. We compute the C_p-equivariant dual Steenrod algebras associated to the constant Mackey functors $\underline{\mathbb{F}}_p$ and $\underline{\mathbb{Z}}_{(p)}$, as $\mathrm{H}\underline{\mathbb{Z}}_{(p)}$-modules. The C_p-spectrum $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ is not a direct sum of $RO(C_p)$-graded suspensions of $\mathrm{H}\underline{\mathbb{F}}_p$ when p is odd, in contrast with the classical and C_2-equivariant dual Steenrod algebras.

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INTRODUCTION

For over a decade, since the Hill-Hopkins-Ravenel solution of the Kervaire invariant one problem [HHR16], there has been great success in using exotic homotopy theories, like C_{2^n}-equivariant homotopy theory and motivic homotopy theory, to study classical homotopy theory at the prime 2. A key foundational input to many of these applications is the computation of the appropriate version of the dual Steenrod algebra, $\mathrm{H}\underline{\mathbb{F}}_2 \wedge \mathrm{H}\underline{\mathbb{F}}_2$, which was carried out by Hu-Kriz [HK01] in C_2-equivariant homotopy theory and by Voevodsky [Voe03] in motivic homotopy theory. One of the major obstacles to carrying out a similar program at odd primes is that we do not understand the structure of the dual Steenrod algebra in C_p-equivariant homotopy theory. The purpose of this paper is to make some progress towards this goal.

To motivate the statement of our main result, recall that we have the following description of the classical, p-local dual Steenrod algebra as a $\mathbb{Z}_{(p)}$-algebra[1]

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[1]We learned this fact from John Rognes. One proof is to base change the equivalence $\mathrm{BP} \wedge_{S^0[v_1,\dots]} S^0 \simeq \mathrm{H}\mathbb{Z}_{(p)}$ to $\mathrm{H}\mathbb{Z}_{(p)}$ and use that the Hurewicz image of the v_i's are pt_i, mod decomposables.

$$\mathrm{H}\mathbb{Z}_{(p)} \wedge \mathrm{H}\mathbb{Z}_{(p)} \simeq \mathrm{H}\mathbb{Z}_{(p)} \wedge \bigwedge_i \mathrm{cofib}\left(\Sigma^{|t_i|} S^0[t_i] \xrightarrow{\cdot pt_i} S^0[t_i]\right).$$

Here the tensor product is taken over the sphere spectrum, $S^0[x]$ denotes the free $\mathbb{E}_1$-algebra on a class x, and the classes t_i live in degree $2p^i - 2$. Modding out by p causes each of the above cofibers to split into two classes related by a Bockstein; modding out by p once more introduces the class τ_0 and recovers Milnor's computation of $\mathcal{A}_* = \pi_*(\mathrm{H}\mathbb{F}_p \wedge \mathrm{H}\mathbb{F}_p)$, as an $\mathbb{F}_p$-algebra.

In the C_p-equivariant case our description involves a similar decomposition but is more complicated in two ways:

- Rather than extending the class t_i to a map from $S^0[t_i]$ using the multiplication on $\mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z}$, we will want to choose as generators a mixture of *ordinary powers* of t_i and of *norms*, $N(t_i)$, of t_i.
- Rather than modding out by the relation '$pt_i = 0$' we will need to enforce the relation that '$\theta t_i = 0$', where θ is an equivariant lift of p to an element in nontrivial $RO(C_p)$-degree. We will then *also* need to enforce the relation $pN(t_i) = 0$.

To make this precise, we will assume that the reader is comfortable with equivariant stable homotopy theory as used, for example, in [HHR16], and introduce the following conventions, in force throughout the paper:

- We will use ϱ to denote the real regular representation of C_p.
- We will use λ^k to denote the representation of C_p on $\mathbb{R}^2 = \mathbb{C}$ where the generator acts by $e^{2\pi i k/p}$. When $k = 1$ we abbreviate $\lambda^1 = \lambda$.
- We denote by $\theta : S^{\lambda-2} \to S^0$ the map of C_p-spectra arising from the degree p map $S^\lambda \to S^2$. We'll denote the cofiber of θ by $C\theta$. Note that the underlying nonequivariant spectrum of $C\theta$ is the Moore space $M(p)$.
- If X is a spectrum, we will denote by $N(X)$ the C_p-equivariant Hill-Hopkins-Ravenel norm of X, which is a C_p-equivariant refinement of the ordinary spectrum $X^{\wedge p}$.
- We denote by $\mathrm{H}\underline{\mathbb{Z}}$ and $\mathrm{H}\underline{\mathbb{F}}_p$ the C_p-equivariant Eilenberg-MacLane spectra associated to the constant Mackey functors at $\mathbb{Z}$ and $\mathbb{F}_p$, respectively.
- We use $\pi_\star X$ to denote the $RO(C_p)$-graded homotopy groups of a C_p spectrum, so that, when $\star = V - W$ is a virtual representation, $\pi_{V-W} X = \pi_0 \mathrm{Map}_{\mathsf{Sp}^{C_p}}(S^{V-W}, X)$.
- The degree k map
$$u_{\lambda - \lambda^k} : S^\lambda \to S^{\lambda^k}$$
is a p-local equivalence when $(k,p) = 1$. When working with any p-local C_p-spectrum, *we will use these equivalences to redefine $\pi_\star$ to indicate grading over $RO(C_p)/(\lambda - \lambda^k : (k,p) = 1)$*. We will also write expressions such as
$$S^\varrho_{(p)} = S^{1 + \frac{p-1}{2}\lambda}_{(p)},$$
where equality is meant to indicate 'equivalent via the aforementioned identification'.

Now we can give a somewhat ad-hoc description of the equivariant refinements of the building blocks in $\mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z}$.

Construction. Let x be a formal variable in an $RO(C_p)$-grading $|x|$. Define a C_p-spectrum as follows:

$$T_\theta(x) := \Sigma^{|x|}C\theta \oplus \Sigma^{2|x|}C\theta \oplus \cdots \oplus \Sigma^{(p-1)|x|}C\theta \oplus \Sigma^{|x|\varrho}M(p),$$

where $M(p)$ is the mod p Moore spectrum. Denote by $N(x) : S^{|x|\varrho} \to T_\theta(x)$ the inclusion of the bottom cell of $\Sigma^{|x|\varrho}M(p)$.

Now suppose that R is a C_p-ring spectrum equipped with a norm $N(R) \to R$. If we have a class $x \in \pi_\star R$ such that $\theta x = 0$, it follows that $p \cdot N(x) = 0$ (see the proof of Lemma 4.4), so we may produce a map

$$S^0 \oplus (S^0[Nx] \wedge T_\theta(x)) \to R,$$

which only depends on the choice of the nullhomotopy witnessing $\theta x = 0$.

We can now state our main theorem.

Theorem A. *There are equivariant refinements*

$$t_{G,i} : S^{2p^{i-1}\varrho-\lambda} \to \mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)}$$

of the nonequivariant classes $t_i \in \pi_*(\mathrm{H}\mathbb{Z}_{(p)} \wedge \mathrm{H}\mathbb{Z}_{(p)})$ which satisfy the relation $\theta t_{G,i} = 0$. For any choice of witness for these relations, the resulting map

$$\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \bigwedge_{i\geq 1} \left(S^0 \oplus (S^0[Nt_{G,i}] \wedge T_\theta(t_{G,i}))\right) \longrightarrow \mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)}$$

is an equivalence.

As an immediate corollary we have:

Corollary. *With notation as above, we have*

$$\underline{\mathbb{F}}_p \wedge \underline{\mathbb{F}}_p \simeq \Lambda(\tau_0) \wedge_{\underline{\mathbb{F}}_p} \mathrm{H}\underline{\mathbb{F}}_p \wedge \bigwedge_{i\geq 1} \left(S^0 \oplus (S^0[Nt_{G,i}] \wedge T_\theta(t_{G,i}))\right),$$

where τ_0 is dual to the Bockstein, in degree 1 and $\Lambda(\tau_0) = \mathrm{H}\underline{\mathbb{F}}_p \oplus \Sigma\mathrm{H}\underline{\mathbb{F}}_p$. In particular, since $\mathrm{H}\underline{\mathbb{F}}_p \wedge C\theta$ is indecomposable at odd primes, the spectrum $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ is not *a direct sum of $RO(C_p)$-graded suspensions of $\mathrm{H}\underline{\mathbb{F}}_p$ at odd primes.*

Remark. When $p = 2$ we have an accidental splitting $\mathrm{H}\underline{\mathbb{F}}_2 \wedge C\theta \simeq \Sigma^{\sigma-1}\mathrm{H}\underline{\mathbb{F}}_2 \oplus \Sigma^\sigma \mathrm{H}\underline{\mathbb{F}}_2$, where σ is the sign representation.

Remark. One can show that $\mathrm{H}\underline{\mathbb{F}}_p \wedge C\theta \wedge C\theta$ splits as $(\mathrm{H}\underline{\mathbb{F}}_p \wedge C\theta) \oplus (\mathrm{H}\underline{\mathbb{F}}_p \wedge \Sigma^{\lambda-1}C\theta)$. It follows that $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ splits as a direct sum of cell complexes with at most 2 cells.

Our result raises a few natural questions which would be interesting to investigate.

Question 1. When specialized to $p = 2$, how does our basis compare to the Hu-Kriz basis?

Question 2. Is it possible to profitably study the $\mathrm{H}\underline{\mathbb{F}}_p$-based Adams spectral sequence using this decomposition? Since $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ is not flat over $\mathrm{H}\underline{\mathbb{F}}_p$, one would be forced to start with the E_1-term. But this is not an unprecedented situation (e.g. Mahowald had great success with the ko-based Adams spectral sequence).

Question 3. Can one describe the multiplication on $\pi_\star \mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ in terms of our decomposition?

Relation to other work. As we mentioned before, we were very much motivated by the description of the C_2-equivariant dual Steenrod algebra given by Hu-Kriz [HK01]. That said, our generators are slightly different than the Hu-Kriz generators when we specialize to $p = 2$. For example, the generator t_1 lives in degree $2\rho_{C_2} - \lambda = 2$, whereas the Hu-Kriz generator ξ_1 lives in degree $\rho = 1 + \sigma$.[2] Hill and Hopkins have also obtained a presentation of the C_{2^n}-dual Steenrod algebra, using quotients of BP**R** and its norms, which is similar in style to the one obtained here.

At odd primes, Caruso [Car99] studied the C_p-equivariant Steenrod algebra, $\pi_\star \mathrm{map}(\mathrm{H}\underline{\mathbb{F}}_p, \mathrm{H}\underline{\mathbb{F}}_p)$, essentially by comparing with the Borel equivariant Steenrod algebra and the geometric fixed point Steenrod algebra, and was able to compute the ranks of the integer-graded stems. There is also work of Oruç [Oru89] computing the dual Steenrod algebra for the Eilenberg-MacLane spectra associated to *Mackey fields* (which does not include $\underline{\mathbb{F}}_p$).

In the *Borel* equivariant setting, the dual Steenrod algebra is given by the action Hopf algebroid for the coaction of the classical dual Steenrod algebra on $H^*(\mathrm{B}C_p)$ (see [Gre88]).

There is also related work from the first and second authors. The first author produced a splitting of $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ in [San19] using the symmetric power filtration. The summands in that splitting were roughly given by the homology of classifying spaces, and were much larger than the summands produced here. The second author and Jeremy Hahn showed [HW20] that $\mathrm{H}\underline{\mathbb{F}}_p$ can be obtained as a Thom spectrum on $\Omega^\lambda S^{\lambda+1}$. The Thom isomorphism then reduces the study of the dual Steenrod algebra to the computation of the homology of $\Omega^\lambda S^{\lambda+1}$. Understanding the relationship between this picture and the one in this article is work in progress.

1. Outline of the proof

To motivate our method of proof, let's first revisit the classical story. We are interested in where the classes $t_i \in \pi_*(\mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z})$ come from, and why they are annihilated by p.

Recall that the homology of $\mathbb{C}P^\infty$ is a divided power algebra

$$H_*(\mathbb{C}P^\infty) = \Gamma_{\mathbb{Z}}\{\beta_1\},$$

where β_1 is dual to the first Chern class c_1. Write $\beta_{(i)} := \gamma_{p^i}(\beta_1)$. Since $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, we have a map of spectra

$$\mathbb{C}P^\infty_+ \to \Sigma^2\mathbb{Z}$$

and hence a homology suspension map

$$\sigma : H_*(\mathbb{C}P^\infty) \to \pi_{*-2}(\mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z})$$

which annihilates elements decomposable with respect to the product structure on $H_*(\mathbb{C}P^\infty)$. We can take[3] $t_i := \sigma(\beta_{(i)})$. The relation $pt_i = 0$ follows from the fact that $p\beta_{(i)}$ is, up to a p-local unit, decomposable as $\beta_{(i-1)}^p$ in $H_*(\mathbb{C}P^\infty)$.

In the equivariant case, we will proceed similarly.

Step 1. Compute the homology of $K(\underline{\mathbb{Z}}, \lambda)$ and use the homology suspension to define classes in $\pi_\star(\mathrm{H}\underline{\mathbb{Z}} \wedge \mathrm{H}\underline{\mathbb{Z}})$.

[2]In this low degree, it seems likely that, modulo decomposables, we have $u_\sigma \xi_1 = t_1$ and that ξ_1 is recovered from $\hat{t}_1$ by restricting along $\mathrm{H}\underline{\mathbb{F}}_2 \wedge S^{1+\sigma} \to \mathrm{H}\underline{\mathbb{F}}_2 \wedge \Sigma^2 C\theta$.

[3]Depending on ones preferences, this might be the *conjugate* of the generator you want; but we are only really concerned with these classes modulo decomposables.

Step 2. Use information about the product structure on the homologies of $K(\underline{\mathbb{Z}}, \lambda)$ and $K(\underline{\mathbb{Z}}, 2)$ to deduce relations for these classes, and hence produce the map described in Theorem A.

Step 3. Verify that the map in Theorem A is an equivalence by proving that it is an underlying equivalence and an equivalence on geometric fixed points.

The first step is carried out in §2 and §3 by identifying $K(\underline{\mathbb{Z}}, \lambda)$ with an equivariant version of $\mathbb{C}P^\infty$ and then specializing a computation due to Lewis [Lew88], which we review in our context. The second step is carried out in §4. The third and final step is carried out in §6 using a lemma proven in §5 that allows us to check that the map on geometric fixed points is an equivalence by just verifying that the source and target have the same dimensions in each degree.

2. Homology of $\mathrm{B}_{C_p} S^1$

Recall that we have the C_p-space $\mathrm{B}_{C_p} S^1$ classifying equivariant principal S^1-bundles. The following lemmas give two useful ways of thinking about this space.

Lemma 2.1. *The complex projective space $\mathbb{P}(\mathbb{C}[z])$ is a model for $\mathrm{B}_{C_p} S^1$, where the generator of C_p acts on $\mathbb{C}[z]$ through ring maps by $z \mapsto e^{2\pi i/p} z$. Here $\mathbb{C}[z]$ is the ordinary polynomial ring over $\mathbb{C}$, and the projective space $\mathbb{P}(\mathbb{C}[z]) = (\mathbb{C}[z] - \{0\})/(\mathbb{C}^\times)$ inherits an action in the evident way.*

Lemma 2.2. *The space $\mathrm{B}_{C_p} S^1$ is a model for $K(\underline{\mathbb{Z}}, \lambda)$.*

Proof. The map

$$\mathbb{P}(\mathbb{C}[z]) \to \mathrm{SP}^\infty(S^\lambda)$$

to the infinite symmetric product, which sends a polynomial $f(z)$ to its set of roots (with multiplicity), is an equivariant homeomorphism. The group-completion of the latter is a model for $K(\underline{\mathbb{Z}}, \lambda)$ by the equivariant Dold-Thom theorem [LF97]. But $\mathrm{SP}^\infty(S^\lambda)$ is already group-complete: the monoid of connected components of the fixed points is $\mathbb{N}/p = \mathbb{Z}/p$. $\square$

Remark 2.3. The reader may object that the definition of $\mathrm{B}_{C_p} S^1$ makes no reference to λ, so how does $\mathrm{B}_{C_p} S^1$ know about this representation rather than λ^k for some k coprime to p? The answer is that, in fact, each of the Eilenberg-MacLane spaces $K(\underline{\mathbb{Z}}, \lambda^k)$ coincide for such k: we have an equivalence of $\mathrm{H}\underline{\mathbb{Z}}$-modules

$$\Sigma^\lambda \mathrm{H}\underline{\mathbb{Z}} \simeq \Sigma^{\lambda^k} \mathrm{H}\underline{\mathbb{Z}}$$

whenever $(k, p) = 1$. This follows from the computations in [FL04, Proposition 9.2], for example.

The filtration of $\mathbb{C}[z]$ by the subspaces $\mathbb{C}[z]_{\leq n}$ of polynomials of degree at most n gives a filtration of $\mathrm{B}_{C_p} S^1$.

Lemma 2.4. *There is a canonical equivalence*

$$\mathrm{gr}_k \mathrm{B}_{C_p} S^1, \cong S^{V_k},$$

where $V_k = \bigoplus_{0 \leq i \leq k-1} \lambda^{i-k}$.

Proof. This follows from a more general observation. If L is a one-dimensional complex representation, and V is an arbitrary complex representation, then the function assigning to a linear map its graph,

$$\mathrm{Hom}_{\mathbb{C}}(L, V) \longrightarrow \mathbb{P}(V \oplus L) - \mathbb{P}(V),$$

is an equivariant homeomorphism. So it induces an equivalence on one-point compactifications

$$S^{L^\vee \otimes V} \cong \mathbb{P}(V \oplus L)/\mathbb{P}(V).$$

□

Proposition 2.5 now follows from [Lew88, Proposition 3.1].

Proposition 2.5 (Lewis)**.** *The above filtration on* $\mathrm{B}_{C_p}S^1$ *splits after tensoring with* $\mathrm{H}\underline{\mathbb{Z}}$*, giving an equivalence*

$$\mathrm{H}\underline{\mathbb{Z}} \wedge \mathrm{B}_{C_p}S^1_+ \simeq \mathrm{H}\underline{\mathbb{Z}}\{e_0, e_1, \dots\},$$

where

$$|e_k| = \bigoplus_{0 \le i \le k-1} \lambda^{i-k}.$$

In particular, for $i \geq 1$ *we have* $|e_{p^i}| = 2p^{i-1}\varrho$.

We will also need some information about the multiplicative structure on homology.

Lemma 2.6. *Writing* $x \doteq y$ *to mean that* $x = \alpha y$ *for some* $\alpha \in \mathbb{Z}_{(p)}^\times$*, we have*

$$e_1^p \doteq \theta e_p, \text{ and } e_{p^i}^p \doteq p e_{p^{i+1}} \text{ for } i \geq 1.$$

Proof. Using the model for $\mathrm{B}_{C_p}S^1$ given by $\mathbb{P}(\mathbb{C}[z])$, we see that, in fact, $\mathbb{P}(\mathbb{C}[z])$ has the structure of a filtered monoid. It follows that the product in homology respects the filtration by the classes $\{e_i\}$. Thus, for $i \geq 0$, we have

$$e_{p^i}^p = \sum_{j \le p^{i+1}} c_{i,j} e_j,$$

where the coefficients lie in $\pi_\star \mathrm{H}\underline{\mathbb{Z}}$. When $j < p^{i+1}$ we see that the virtual representations $|c_{i,j}|$ have positive virtual dimension and their fixed points also have positive virtual dimension. The homotopy of $\mathrm{H}\underline{\mathbb{Z}}$ vanishes in these degrees (see, e.g., [FL04, Theorem 8.1(iv)]), so we must have

$$e_{p^i}^p = c_{i,p^{i+1}} e_{p^{i+1}},$$

where $|c_{0,p}| = \lambda - 2$ and $|c_{i,p^{i+1}}| = 0$ when $i \geq 1$. In both cases, the restriction map on $\pi_\star \mathrm{H}\underline{\mathbb{Z}}$ is injective in this degree, so the result follows from the nonequivariant calculation. □

3. Suspending classes

We begin with some generalities. If X is any C_p-spectrum, we have the counit

$$\Sigma^\infty_{C_p,+}\Omega^\infty_{C_p} X \to X$$

which induces a map

$$\sigma : \mathrm{H}\underline{\mathbb{Z}} \wedge \Sigma^\infty_{C_p,+}\Omega^\infty_{C_p} X \to \mathrm{H}\underline{\mathbb{Z}} \wedge X,$$

called the homology suspension. Just as in the classical case, it follows from the equivariant Snaith splitting [LMSM86, §VII.5] that σ annihilates decomposable elements in $\pi_*(\mathrm{H}\underline{\mathbb{Z}} \wedge \Sigma^\infty_{C_p,+}\Omega^\infty_{C_p} X)$.

Construction 3.1. For $i \geq 1$, we define

$$t_{G,i} : S^{2p^{i-1}\varrho - \lambda} \to \mathrm{H}\underline{\mathbb{Z}} \wedge \mathrm{H}\underline{\mathbb{Z}}$$

as the homology suspension of the element $e_{p^i} \in \pi_{2p^{i-1}\varrho}(\mathrm{H}\underline{\mathbb{Z}} \wedge \mathrm{B}_{C_p} S^1_+)$. Here we use the identification

$$\mathrm{B}_{C_p} S^1 \simeq K(\underline{\mathbb{Z}}, \lambda) = \Omega^\infty \Sigma^\lambda \mathrm{H}\underline{\mathbb{Z}}.$$

4. Two relations in homology

We begin with a brief review of norms, transfers, and restrictions.

Remark 4.1 (Transfer and restriction). Given a nonequivariant equivalence $(S^V)^e \cong S^n$, we define

$$\mathrm{res} : \pi_V X \to \pi_n X^e, \quad (x : S^V \to X) \mapsto (S^n \cong (S^V)^e \to X)$$

and

$$\mathrm{tr}_V : \pi_n X^e \to \pi_V X, \quad (y : S^n \to X^e) \mapsto (S^V \to C_{p+} \wedge S^V \cong C_{p+} \wedge S^n \to C_{p+} \wedge X \to X).$$

For example, when $V = \lambda - 2$ and $X = S^0$, then $\mathrm{tr}_{\lambda-2}(1) = \theta$.

Changing the equivalence $(S^V)^e \cong S^n$ has the effect of altering these classes by ± 1; in our case the representations in question have canonical orientations so this will not be a concern. Given a map $X \wedge Y \to Y$ we have a relation:

$$\mathrm{tr}(x \otimes \mathrm{res}(y)) = \mathrm{tr}(x) \otimes y.$$

Remark 4.2 (Norms). If a C_p-spectrum X has a map $N(X) \to X$, then, given an underlying class $x : S^n \to X^e$, we may define a **norm** by the composite

$$Nx : N(S^n) = S^{n\varrho} \to N(X) \to X.$$

The underlying nonequivariant class is given by $\mathrm{res}(Nx) = \prod_{g \in C_p} (gx) \in \pi_{pn} X^e$.

Our goal in this section is to prove the following two lemmas.

Lemma 4.3. *The classes $t_{G,i} \in \pi_{2p^{i-1}\varrho - \lambda}(\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)})$ satisfy $\theta t_{G,i} = 0$.*

Lemma 4.4. *The classes $N(t_i) \in \pi_{(2p^i - 2)\varrho}(\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)})$ satisfy $pN(t_i) = 0$.*

In fact, the second relation follows from the first.

Proof of Lemma 4.4 *assuming Lemma* 4.3. Since $p = \mathrm{tr}(1)$, the class $pN(t_i)$ is the transfer of the class $\mathrm{res}(t_{G,i})^p$ into degree $(2p^i - 2)\varrho$. Notice that $(2p^i - 2)\varrho - |t_{G,i}^p| = \lambda - 2$ (after identifying the λ^k suspensions with λ for $(k, p) = 1$), and the transfer of 1 into this degree is θ, so we have

$$pN(t_i) = \theta t_{G,i}^p = 0.$$

□

Proof of Lemma 4.3. By Lemma 2.6, we have $e_1^p \doteq \theta e_p$ so that $\theta t_{G,1} = \sigma(\theta e_p) = 0$, since σ annihilates decomposables. For the remaining classes, consider the commutative diagram

$$\begin{tikzcd}
\Sigma^\infty_{C_p,+} K(\underline{\mathbb{Z}}, \lambda)_+ \arrow[r] \arrow[d, "{[\theta]}"'] & \Sigma^\lambda \mathrm{H}\underline{\mathbb{Z}} \arrow[d, "\theta"] \\
\Sigma^\infty_{C_p,+} K(\underline{\mathbb{Z}}, 2)_+ \arrow[r] & \Sigma^2 \mathrm{H}\underline{\mathbb{Z}}
\end{tikzcd},$$

where $[\theta] = \Omega^\infty(\theta)$. Thus, to show that $\theta t_{G,i} = 0$ for $i \geq 2$, it is enough to show that $[\theta]_* e_{p^i}$ is decomposable in $\pi_\star(\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge K(\underline{\mathbb{Z}}, 2)_+)$ for $i \geq 2$.

Write

$$\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge K(\underline{\mathbb{Z}}, 2)_+ = \mathrm{H}\underline{\mathbb{Z}}_{(p)}\{\gamma_i(\beta_1)\},$$

where the elements $\gamma_i(\beta_1)$ are the standard module generators of $H_*(\mathbb{C}P^\infty; \mathbb{Z})$, and write $\beta_{(i)} = \gamma_{p^i}\beta_1$. To show that $[\theta]_*(e_{p^i})$ is decomposable for $i \geq 2$, it is enough to establish the following two claims:

(a) $[\theta]_*(e_{p^i}) \doteq \frac{p^{i-1}\theta}{u_\lambda^{p^{i-1}(p-1)-1}}\beta_{(i)}$, and
(b) $\beta_{(i-1)}^p \doteq p\beta_{(i)}$.

Claim (b) is just the classical computation of the product in homology for $H_*(\mathbb{C}P^\infty, \mathbb{Z})$. For claim (a), let ι_λ denote the fundamental class in cohomology for $K(\underline{\mathbb{Z}}, \lambda)$ and ι_2 the same for $K(\underline{\mathbb{Z}}, 2)$. Then we have $[\theta]^*(\iota_2) = \theta\iota_\lambda$ by design, and hence

$$[\theta]^*(\iota_2^j) = \theta^j \iota_\lambda^j.$$

The map on homology is now determined by the relation

$$\langle [\theta]_* e_{p^i}, \iota_2^j \rangle = \theta^j \langle e_{p^i}, \iota_\lambda^j \rangle \in \pi_\star \mathrm{H}\underline{\mathbb{Z}}_{(p)}.$$

Since θ^j is a transferred class, the value above is also a transfer, and hence determined by its restriction to an underlying class. But $\mathrm{res}([\theta]) = [p]$ and we clearly have $[p]_*(\mathrm{res}(e_{p^i})) = p^i\beta_{(i)}$, which agrees with the restriction of $\frac{p^{i-1}\theta}{u_\lambda^{p^{i-1}(p-1)-1}}\beta_{(i)}$. This completes the proof. $\square$

5. Digression: Detecting equivalences nonequivariantly

The goal of this section is to establish a criterion for detecting equivalences of $\underline{\mathbb{Z}}$-modules. We recall that

$$\mathrm{H}\underline{\mathbb{Z}}^{\Phi C_p} \simeq \mathrm{H}\mathbb{F}_p[b],$$

where the class b in degree 2 arises from taking the geometric fixed points of the Thom class $u_\lambda : S^\lambda \to \Sigma^2\underline{\mathbb{Z}}$.

Proposition 5.1. *Let $f : M \to N$ be a map of $\mathrm{H}\underline{\mathbb{Z}}$-modules which are bounded below (on underlying and fixed points). Assume the following conditions are satisfied:*

(i) *f is an underlying equivalence.*
(ii) *$\pi_j M^{\Phi C_p}$ and $\pi_j N^{\Phi C_p}$ are finite dimensional of the same rank, for all j.*
(iii) *$\pi_* M^{\Phi C_p}$ and $\pi_* N^{\Phi C_p}$ are graded-free $\mathbb{F}_p[b]$-modules.*

Then f is an equivalence.

We will deduce this proposition from the following one, which relates geometric and Tate fixed points.

Proposition 5.2. *Let M be an $\mathrm{H}\underline{\mathbb{Z}}$-module which is both bounded above and below (on underlying and fixed point spectra). Then the natural map*

$$M^{\Phi C_p}[b^{-1}] \to M^{tC_p}$$

is an equivalence.

Proof of Proposition 5.1 *assuming Proposition* 5.2. By assumption (i), it is enough to check that $f^{\Phi C_p}$ is an equivalence; by assumption (ii), it is enough to check that $\pi_*(f^{\Phi C_p})$ is an injection; and by assumption (iii) it is enough to check that $\pi_*(f^{\Phi C_p})[b^{-1}]$ is an injection.

Again by (i), the map f^{tC_p} is an equivalence. So, from the diagram

$$\begin{tikzcd} M^{\Phi C_p}[b^{-1}] \arrow[r] \arrow[d] & N^{\Phi C_p}[b^{-1}] \arrow[d] \\ M^{tC_p} \arrow[r, "\simeq"'] & N^{tC_p} \end{tikzcd}$$

we see that it is enough to check that the vertical maps are injective on homotopy. More generally, we show that whenever X is a bounded below $\mathrm{H}\underline{\mathbb{Z}}$-module, the map

$$\pi_* X^{\Phi C_p}[b^{-1}] \to \pi_* X^{tC_p}$$

is injective. Indeed, by Proposition 5.2 and the fact that the Tate construction commutes with limits of Postnikov towers (see, e.g., [NS18, I.2.6]), we have

$$\lim_n \left((\tau_{\leq n} X)^{\Phi C_p}[b^{-1}]\right) \xrightarrow{\simeq} \lim_n (\tau_{\leq n} X)^{tC_p} \simeq X^{tC_p}.$$

Therefore, we need only check that

$$\pi_* X^{\Phi C_p}[b^{-1}] \to \pi_* \lim_n \left((\tau_{\leq n} X)^{\Phi C_p}[b^{-1}]\right)$$

is injective. Since the maps $X^{\Phi C_p} \to (\tau_{\leq n} X)^{\Phi C_p}$ have increasingly connective fibers, we can replace the left hand side by $(\lim_n \pi_*(\tau_{\leq n} X)^{\Phi C_p})[b^{-1}]$ and reduce to showing that

$$(\lim_n \pi_*(\tau_{\leq n} X)^{\Phi C_p})[b^{-1}] \to \lim_n \pi_* \left((\tau_{\leq n} X)^{\Phi C_p}[b^{-1}]\right)$$

is injective. Finally, this reduces to showing that the kernel of

$$\lim_n \pi_*(\tau_{\leq n} X)^{\Phi C_p} \to \lim_n \pi_* \left((\tau_{\leq n} X)^{\Phi C_p}[b^{-1}]\right)$$

consists of elements annihilated by a power of b. This is clear because, for each j, the system $\{\pi_j(\tau_{\leq n} X)^{\Phi C_p}\}_n$ is eventually constant. □

Proof of Proposition 5.2. Let $\mathcal{E}$ denote the full subcategory of $\mathrm{H}\underline{\mathbb{Z}}$-modules M for which

$$M^{\Phi C_p}[b^{-1}] \to M^{tC_p}$$

is an equivalence. Then $\mathcal{E}$ is stable, closed under retracts, and closed under suspending by representation spheres.

The map $M^{\Phi C_p}[b^{-1}] \to M^{tC_p}$ is one of $\mathrm{H}\underline{\mathbb{Z}}^{\Phi C_p} = \mathrm{H}\mathbb{F}_p[b]$-modules, and hence one of $\mathrm{H}\mathbb{F}_p$-modules, so it must be a retract of

$$(M/p)^{\Phi C_p}[b^{-1}] = M^{\Phi C_p}[b^{-1}]/p \to M^{tC_p}/p = (M/p)^{tC_p}.$$

Thus $M/p \in \mathcal{E}$ if and only if $M \in \mathcal{E}$. So, by replacing M with M/p and considering the Postnikov tower, we are reduced to proving the proposition in the case where $M \in \mathsf{Mod}_{\underline{\mathbb{Z}}}^{\heartsuit}$ is a Mackey functor which is a module over $\underline{\mathbb{F}}_p$.

In particular, M^e is an $\mathbb{F}_p[C_p]$-module. Let γ denote the generator of C_p so that $\mathbb{F}_p[C_p] = \mathbb{F}_p[\gamma]/(1-\gamma)^p$. Let $F_j M \subseteq M$ be the sub-Mackey functor generated by $(1-\gamma)^j M^e \subseteq M^e$. This is a finite filtration with associated graded pieces given by Mackey functors with trivial underlying action. So, since $\mathcal{E}$ is a thick subcategory, we are reduced to the case when M is a discrete $\mathrm{H}\underline{\mathbb{F}}_p$-module with trivial underlying action.

For the next reduction we recall some notation. If N is any Mackey functor, denote by N_{C_p} the Mackey functor $N \otimes C_{p+}$ and, if A is an abelian group, denote by $\underline{A}_{\mathrm{tr}}$ the Mackey functor whose transfer map is the identity on A and whose restriction map is multiplication by p. We also recall that the transfer extends to a map of Mackey functors $\mathrm{tr} : N_{C_p} \to N$.

Now consider the two exact sequences

$$0 \to \mathrm{im}(\mathrm{tr}) \to M \to M/\mathrm{im}(tr) \to 0$$

$$0 \to \ker(\mathrm{tr}) \to \underline{M}^e_{\mathrm{tr}} \to \mathrm{im}(\mathrm{tr}) \to 0.$$

Notice both $M/\mathrm{im}(tr)$ and $\ker(\mathrm{tr})$ vanish on $\{e\}$. If N is any Mackey functor with $N^e = 0$, then $N \in \mathcal{E}$ since then $N = N^{\Phi C_p}$ is bounded above and hence $N^{\Phi C_p}[b^{-1}] = 0$. Thus, from the exact sequences above, we are reduced to the case where M is of the form $\underline{V}_{\mathrm{tr}}$ for an $\mathbb{F}_p$-vector space V (with trivial action). Now recall that $\mathrm{H}(\underline{\mathbb{F}}_p)_{\mathrm{tr}} = \Sigma^{2-\lambda}\mathrm{H}\underline{\mathbb{F}}_p$ and hence $\mathrm{H}\underline{V}_{\mathrm{tr}} = \Sigma^{2-\lambda}\mathrm{H}\underline{V}$. So we are reduced to showing that $\mathrm{H}\underline{V}$ lies in $\mathcal{E}$, where V is an $\mathbb{F}_p$-vector space with trivial action. This certainly holds for $V = \mathbb{F}_p$, and in general we have

$$\mathrm{H}\underline{V}^{\Phi C_p} \simeq \mathrm{H}\underline{\mathbb{F}}_p^{\Phi C_p} \wedge_{\mathrm{H}\mathbb{F}_p} V,$$

since geometric fixed points commutes with colimits, and

$$\mathrm{H}V^{tC_p} \simeq \mathrm{H}\mathbb{F}_p^{tC_p} \wedge_{\mathrm{H}\mathbb{F}_p} \mathrm{H}V$$

by direct calculation. (Notice this holds even when V is infinite-dimensional.) This completes the proof. □

6. Proof of the main theorem

We are now ready to prove the main theorem. Recall that we have constructed classes

$$t_{G,i} \in \pi_{2p^{i-1}\varrho-\lambda}(\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)}),$$

and shown that $\theta t_{G,i} = 0$ and $pN(t_i) = 0$. With notation as in the introduction, let

$$X_i = \left(S^0 \oplus (S^0[Nt_{G,i}] \wedge T_\theta(t_{G,i}))\right)$$

and

$$X = \bigwedge_{i\geq 1} \left(S^0 \oplus (S^0[Nt_{G,i}] \wedge T_\theta(t_{G,i}))\right).$$

Then, choosing nullhomotopies which witness $\theta t_{G,i} = 0$ (end hence, by restriction, witnesses for $pt_i = 0$), we get a map:

$$f : \mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \bigwedge_{i\geq 1} \left(S^0 \oplus (S^0[Nt_{G,i}] \wedge T_\theta(t_{G,i}))\right) \longrightarrow \mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)}.$$

The main theorem is then the statement:

Theorem 6.1. *The map f is an equivalence.*

Proof. Combine Proposition 5.1 with the two lemmas below. □

Lemma 6.2. *The map f^e is an underlying equivalence.*

Proof. We denote the inclusion of the summand $\Sigma^{i|x|}C\theta$ by

$$x^{i-1}\hat{x} : \Sigma^{i|x|}C\theta \to T_\theta(x),$$

the restriction of $\hat{x}$ to the bottom cell by x, and the inclusion of the final summand by $\widehat{Nx}$. We denote by

$$Nx : S^{|x|\varrho} \to T_\theta(x)$$

the restriction of $\widehat{Nx}$ to the bottom cell of the mod p Moore spectrum.

First observe that, by our construction in the proof of Lemma 4.4, the map $\widehat{N(t_{G,i})}$ restricts to the map $t_i^{p-1}\hat{t}_i$, since the nullhomotopy witnessing $pN(t_i) = 0$ was chosen to restrict to the nullhomotopy chosen for pt_i^p that came from the already chosen nullhomotopy of pt_i. The upshot is that the map

$$S^0 \oplus S^0[Nt_{G,i}] \wedge T_\theta(t_{G,i}) \to \mathrm{H}\underline{\mathbb{Z}} \wedge \mathrm{H}\underline{\mathbb{Z}}$$

restricts on underlying spectra to the map

$$S^0[t_i]/(pt_i) \to \mathrm{H}\mathbb{Z} \wedge \mathrm{H}\mathbb{Z}$$

obtained just from the relation $pt_i = 0$ and extended via the multiplicative structure.

In particular, on mod p homology f^e induces a ring map

$$\mathbb{F}_p[t_i] \otimes \Lambda(x_i) \to \mathbb{F}_p[\xi_i] \otimes \Lambda(\tau_i).$$

We know that t_i maps to ξ_i and that $\beta x_i = t_i$, so that $\beta(f_*^e(x_i)) = \xi_i$. Modulo decomposables, τ_i is the only element whose Bockstein is ξ_i. So x_i must map to τ_i, mod decomposables. It follows that f^e is a mod p equivalence, and hence an equivalence. $\square$

Lemma 6.3. *$\pi_*(\mathrm{H}\underline{\mathbb{Z}} \wedge X)^{\Phi C_p}$ and $\pi_*(\mathrm{H}\underline{\mathbb{Z}} \wedge \underline{\mathbb{Z}})^{\Phi C_p}$ are free $\mathbb{F}_p[b]$-modules, finite-dimensional in each degree, and isomorphic as graded vector spaces over $\mathbb{F}_p$.*

Proof. If Y is any C_p-spectrum, then

$$(\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge Y)^{\Phi C_p} = \mathrm{H}\mathbb{F}_p[b] \wedge Y^{\Phi C_p} \simeq \mathrm{H}\mathbb{F}_p[b] \wedge_{\mathrm{H}\mathbb{F}_p} (\mathrm{H}\mathbb{F}_p \wedge Y^{\Phi C_p})$$

is a free $\mathrm{H}\mathbb{F}_p[b]$-module. Applying this in the cases $Y = X$ and $Y = \mathrm{H}\underline{\mathbb{Z}}$, we see that each is a free $\mathrm{H}\mathbb{F}_p[b]$, evidently finite-dimensional in each degree. So it suffices to prove that

$$\pi_*(\mathrm{H}\mathbb{F}_p \wedge X^{\Phi C_p}) \cong \pi_*(\mathrm{H}\mathbb{F}_p \wedge (\mathrm{H}\mathbb{F}_p[b]))$$

as graded vector spaces. Notice that we can write, *as graded vector spaces*,

$$\pi_*(\mathrm{H}\mathbb{F}_p \wedge X_i^{\Phi C_p}) \cong \mathbb{F}_p[d_{i-1}, \xi_i] \otimes_{\mathbb{F}_p} \Lambda(\sigma_{i-1}, \tau_i)/(d_{i-1}^p, d_{i-1}\tau_i, d_{i-1}^{p-1}\sigma_{i-1}, \sigma_{i-1}\tau_i),$$

where $|\sigma_{i-1}| = 2p^{i-1} - 1$ and $|d_{i-1}| = 2p^{i-1}$. Indeed, $\hat{t}_i$, on geometric fixed points, gives rise to two classes; one we are calling d_{i-1} and the other we are calling σ_{i-1}. Similarly, $\widehat{N(t_{G,i})}$, on geometric fixed points, gives rise to two classes: one we are calling ξ_i and the other τ_i, in their usual degrees. The relations are the ones needed to ensure that the monomials not arising from geometric fixed points of elements in X_i are omitted.

It follows that we have an isomorphism of graded vector spaces

$$\begin{aligned}\pi_*(\mathrm{H}\mathbb{F}_p \wedge X^{\Phi C_p}) \cong{}& \mathbb{F}_p[\xi_n : n \geq 1] \otimes_{\mathbb{F}_p} \mathbb{F}_p[d_i : i \geq 0] \\ &\otimes_{\mathbb{F}_p} \Lambda(\sigma_j, \tau_k : j \geq 0, k \geq 1)/(d_i^p, d_{i-1}\tau_i, d_i^{p-1}\sigma_i, \sigma_{i-1}\tau_i).\end{aligned}$$

We are trying to show that this is isomorphic, as a graded vector space to

$$\pi_*(\mathrm{H}\mathbb{F}_p \wedge \mathrm{H}\mathbb{F}_p[b]) \cong \mathbb{F}_p[\xi_n : n \geq 1] \otimes_{\mathbb{F}_p} \Lambda(\tau_i : i \geq 0) \otimes_{\mathbb{F}_p} \mathbb{F}_p[b].$$

We may regard each vector space as a module over $\mathbb{F}_p[\xi_n : n \geq 1]$ in the evident way, and hence reduce to showing that the two vector spaces

$$V = \Lambda(\tau_i : i \geq 0) \otimes_{\mathbb{F}_p} \mathbb{F}_p[b]$$

and

$$W = \mathbb{F}_p[d_i : i \geq 0] \otimes_{\mathbb{F}_p} \Lambda(\sigma_j, \tau_k : j \geq 0, k \geq 1)/(d_i^p, d_{i-1}\tau_i, d_i^{p-1}\sigma_i, \sigma_{i-1}\tau_i)$$

are isomorphic. (Here recall that $|\sigma_i| = |\tau_i| = 2p^i - 1$, $|b| = 2$, and $|d_i| = 2p^i$.)

Let I range over sequences $(a_0, a_1, \dots)$ with $0 \leq a_i \leq p-2$, J range over sequences $(\varepsilon_0, \varepsilon_1, \dots)$ with $\varepsilon_i \in \{0,1\}$, K range over sequences $(\kappa_0, \kappa_1, \dots)$ with $\kappa_i \in \{0,1\}$, and let K' range over sequences $(\kappa'_0, \kappa'_1, \dots)$ with $\kappa'_i \in \{0,1\}$. We impose the following requirements on these sequences:

- Each sequence has finite support.
- $\kappa'_i \leq \kappa_i$. (So K' is otained from K by changing some subset of 1s to 0s).
- $J \cdot K = I \cdot K = (0, 0, \dots)$. That is: I and K have disjoint support and J and K have disjoint support.

Then V has a basis of monomials

$$M_{I,J,K,K'} = (\prod_{i \geq 0} b^{a_i p^i})\tau_J(\prod_{i \geq 0} b^{\kappa_i (p-1)p^i})\tau_{K'}$$

and W has a basis of monomials

$$N_{I,J,K,K'} = d_I \sigma_J (\prod_{i \geq 0} d_i^{(\kappa_i - \kappa'_i)(p-1)})\tau_{K'[1]},$$

where $K'[1] = (0, \kappa'_0, \kappa'_1, \dots)$. These have the same number of basis elements in each dimension, so $V \cong W$. $\square$

We end with a question, a satisfying answer to which would yield a proof of the main theorem which avoids the use of Proposition 5.1.

Question 4. The geometric fixed points of $\mathrm{H}\underline{\mathbb{Z}}_{(p)} \wedge \mathrm{H}\underline{\mathbb{Z}}_{(p)}$ are given by $(\mathrm{H}\mathbb{F}_p \wedge \mathrm{H}\mathbb{F}_p)[b, \overline{b}]$, where $\overline{b}$ is the conjugate of b, a class in degree 2. It is possible to understand what happens to the generators $t_{G,i}$ and $\widehat{N(t_{G,i})}$ upon taking geometric fixed points. One is left with trying to understand the remaining class hit by $\hat{t}_i$ on geometric fixed points. We don't know what this should be. One guess that seems consistent with computations is that this class is given, up to conjugating the τ_i and modding out by (b), by

$$\tau_{i-1} + \overline{b}^{p^{i-1} - p^{i-2}}\tau_{i-2} + \cdots + \overline{b}^{p^{i-1}-1}\tau_0.$$

It would be useful for computations to sort out what actually occurs.

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