Pergamon

Topology Vol. 34, No. 2, pp. 261-289, 1995

0040-9383/95 $9.50 + 0.00

0040-9383(94)00032-8

THE HOMOTOPY GROUPS $\pi_*(L_2S^0)$

KATSUMI SHIMOMURA and ATSUKO YABE

(*Received* 13 *April* 1993; *in revised form* 18 *May* 1994)

1. INTRODUCTION

To determine the homotopy groups $\pi_*(S^0)$ of spheres is one of the main problems in homotopy theory, and several methods have been found to reach it. One of them is the one using a generalized Adams spectral sequence based on a ring spectrum E. In many examples, it converges to the homotopy groups of the localized spheres with respect to E, not to those of the unlocalized one. Consider the Brown–Peterson spectrum BP at a prime p. Then the localized sphere with respect to BP is the one localized at the prime p. We call the spectral sequence based on it *the Adams–Novikov spectral sequence*. It converges to the homotopy groups of the p-localized spheres and its E_2-term is expressed by the Ext group, which is computable object. For odd prime p, it seems more powerful than the Adams spectral sequence that is based on the Eilenberg–MacLane spectrum. When we use the spectral sequence, we have to compute the E_2-term. The E_2-term $E_2^{s,t}$ of the Adams–Novikov spectral sequence based on BP for the sphere is computed for $t < p^3q$ by Ravenel, and for $s < 3$ by Miller, Ravenel and Wilson using the chromatic spectral sequences. The E_1-term of the chromatic spectral sequence does not only converge to the E_2-term of the Adams–Novikov spectral sequence but also is itself the E_2-term of the Adams–Novikov spectral sequence for computing homotopy groups of a spectrum whose existence is shown by Ravenel [12].

Now we fix a prime $p > 3$. The E_1-term of the chromatic spectral sequence is denoted by $H^tM_n^s$ and converging to $H^{s+t}N_n^0$ (see section 4). In this paper, we study it for the Johnson–Wilson spectrum $E(2)$ whose homotopy group $E(2)_*$ is the polynomial ring $\mathbf{Z}_{(p)}[v_1, v_2, v_2^{-1}]$ on the generators of BP_*. Here note that $E(n)$ is not proved to be a ring spectrum. Since $E(n)$ is a spectrum representing the homology theory $E(n)_*(X) = E(n)_* \otimes_{BP_*} BP_*(X)$, we can construct a generalized Adams spectral sequence based on $E(n)$ similarly to the original one. So we use $E(2)$ here. Then we may put $N_0^0 = E(2)_*$, $N_1^0 = E(2)_*/(p)$ and $M_n^s = v_{n+s}^{-1}N_n^s$ $(v_0 = p)$ for the E_1-term of the chromatic spectral sequence. Moreover, $M_n^s = 0$ if $n + s > 2$. So far, for this case, the E_1-terms of the chromatic spectral sequence are computed for all n and s but $n = 0$ and $s = 2$. Here we obtain this case $(n, s) = (0, 2)$ (Theorem 2.3). This module $H^*M_0^2$ seems to have many applications. One of them is the one for the Greek letter elements in the stable homotopy groups of spheres. As is remarked in [6], it gives complete information on products of α's and β's and decomposability of the γ's. Let L_2 denote the Bousfield localization functor with respect to the spectrum $E(2)$ [2, 10]. Before our computation, we only know about the homotopy groups $\pi_*(L_2M)$ for the mod p Moore spectrum M. Our computation on $H^*M_0^2$ gives rise to the homotopy groups of $\pi_*(L_2S^0)$ (Theorem 2.4) by the mod p Bockstein spectral sequence. This has also much information on the products of the homotopy elements. In fact, the localization map $\eta: S^0 \to L_2S^0$ induces the homomorphism $\eta_*: \pi_*(S^0) \to \pi_*(L_2S^0)$, by which we can tell some information. On this map, we have the relating exact sequence $H^{*-3}N_0^3 \xrightarrow{Gr} H^*BP_* \xrightarrow{\eta_*} H^*E(2)_* \to H^{*-2}N_0^3$ of the E_2-terms given in [3], where $N_0^3 = BP_*/(p^\infty, v_1^\infty, v_2^\infty)$, and Gr denotes the universal Greek letter map introduced in [6]. Thus the map η_* maps the elements of $\pi_*(S^0)$ whose filtration degree is less than 3 monomorphically to those of $\pi_*(L_2S^0)$. Furthermore, M. Hopkins and D. Ravenel show that L_2X is homotopic to $L_2S^0 \wedge X$ (cf. [13]). So our computation will be a grip to understand the L_2-localization. These applications will be discussed in the forthcoming papers.

2. STATEMENT OF RESULTS

Let $E(2)$ denote the Johnson–Wilson spectrum at a prime $p > 3$ with the homotopy groups $E(2)_* = \mathbf{Z}_{(p)}[v_1, v_2, v_2^{-1}]$. Then it is known (cf. [2, 10]) that we have the Adams–Novikov spectral sequence converging to $\pi_*(L_2S^0)$ with the E_2-term

$$H^{s,t}E(2)_* = \mathrm{Ext}^{s,t}_{E(2)_*(E(2))}(E(2)_*, E(2)_*).$$

Here L_2 denotes the Bousfield localization functor with respect to $E(2)$. Consider the comodules N_n^i and M_n^i for $i + n \le 2$ such that $M_n^i = v_{n+i}^{-1}N_n^i$, $N_n^{2-n} = M_n^{2-n}$ and

$$N_0^0 = E(2)_*$$
$$N_1^0 = E(2)_*/(p), \qquad N_0^1 = E(2)_*/(p^\infty)$$
$$N_2^0 = E(2)_*/(p, v_1), \qquad N_1^1 = E(2)_*/(p, v_1^\infty), \qquad N_0^2 = E(2)_*/(p^\infty, v_1^\infty).$$

Then we have the chromatic spectral sequence converging to our target $H^{s,t}E(2)_*$ with the E_1-term $H^tM_0^s$. The E_1-terms for $s < 2$ are determined in [6]. In order to determine $H^*M_0^2$, we have the v_1- and the mod p-Bockstein spectral sequences coverging to $H^*M_n^{2-n}$ with the E_1-term $H^tM_{n+1}^{1-n}$ for $n = 0, 1$. Ravenel shows the following result.

THEOREM 2.1 (Ravenel [9]). $H^*M_2^0 = F_p[v_2, v_2^{-1}]\{1, h_0, h_1, g_0, g_1, h_0g_1\} \otimes E(\zeta)$.

Here F_p denotes the prime field of characteristic p, which is identified with $\mathbf{Z}/p$, $R\{x\}$ denotes the R-module generated by x, and $E(x)$ the exterior algebra generated by x. By the v_1-Bockstein spectral sequence, we compute $H^*M_1^1$ from Theorem 2.1. For simplicity, we denote a cocycle by its leading term. Put

$$X = F_p[v_1]\{v_2^{sp^n}/v_1^{a_n}: n \ge 0, s \in \mathbf{Z} - p\mathbf{Z}\}$$
$$X_\infty = F_p\{1/v_1^j: j > 0\} \cong F_p[v_1, v_1^{-1}]/F_p[v_1]$$
$$Y_0 = F_p[v_1]\{v_2^m h_0/v_1^{2+A_n}: m \in \mathbf{Z}(0),\ n = v_p(m)\}$$
$$Y_1 = F_p[v_1]\{v_2^m h_0/v_1^{2+A'_n}: m \in \mathbf{Z}(2),\ n = v_p(m)\}$$
$$Y = F_p[v_1]\{v_2^{tp-1}h_1/v_1^{p-1}: t \in \mathbf{Z}\}$$
$$Y_\infty = F_p\{h_0/v_1^j: j > 0\} \cong F_p[v_1, v_1^{-1}]/F_p[v_1] \text{ and}$$
$$G = F_p[v_1]\{v_2^{sp^n-(p^{n-1}-1)/(p-1)}g_1/v_1^{a_n}, v_2^s g_0/v_1: n \ge 1, s + 1 \in \mathbf{Z} - p\mathbf{Z}\}.$$

Here $v_p(m)$ denotes the maximal power of p dividing m, the integers a_n, A_n and A'_n are given by

$$a_0 = 1, \quad a_n = p^n + p^{n-1} - 1$$
$$A_n = (p+1)(p^n - 1)/(p-1)$$
$$A'_n = (p+1)(p^{n+1} - p^n + (p^n - 1)/(p-1))$$

and we use the subsets of integers

$$\mathbf{Z}(0) = \{m\colon m = sp^n \text{ with } p \nmid s(s+1)\}$$
$$\mathbf{Z}(2) = \{m\colon m = (sp^2 - 1)p^n\}.$$

Then we have the structure of $H^*M_1^1$ shown as follows.

THEOREM 2.2. (Miller *et al.* [6], Shimomura and Tamura [16] and Shimomura [14]).

$$H^*M_1^1 = (X \oplus X_\infty \oplus Y_0 \oplus Y_1 \oplus Y \oplus Y_\infty \oplus G) \otimes E(\zeta).$$

In this paper, we use the mod p-Bockstein spectral sequence, and obtain the following theorem.

THEOREM 2.3. *The module $H^*M_0^2$ is isomorphic to*

$$(X_\infty^\infty \oplus Y_{\infty,C}^\infty \oplus G_0^\infty) \otimes E(\zeta) \oplus X^\infty \oplus X\zeta_C^\infty \oplus Y_{0,C}^\infty \oplus Y_{1,C}^\infty \oplus Y_C^\infty \oplus G^\infty.$$

Here the modules are defined by

$$X^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n}/p^{i+1}v_1^j\colon n \geq 0,\ s \in \mathbf{Z} - p\mathbf{Z},\ i \geq 0,$$
$$j \geq 1 \text{ with } p^i | j \leq a_{n-i} \text{ and either } p^{i+1} \nmid j \text{ or } a_{n-i-1} < j\}$$
$$X_\infty^\infty = \mathbf{Z}_{(p)}\{1/p^{i+1}v_1^j\colon i = v_p(j) \geq 0\}$$

for dimension 0,

$$X\zeta_C^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n}\zeta/p^{i+1}v_1^j\colon s \in \mathbf{Z} - p\mathbf{Z},\ j > 0,\ p^i | j \leq a_{n-i}$$
$$\text{either } p^{i+1} \nmid j \text{ or } j > a_{n-i-1}, \text{ and } p^{i+1} | j \text{ if } p^{k+1} | j \text{ for } s = tp^{k+1} - 1 \text{ with } k \geq 0\}$$
$$Y_{0,C}^\infty = \mathbf{Z}_{(p)}\{v_2^{sp^n}h_0/p^{i+1}v_1^{kp^i+1}\colon p \nmid s(s+1), \text{ for } k = 0,\ i = n, \text{ and for } k > 0,$$
$$kp^i + 1 \leq A_{n-i} + 2,\ kp^i + 1 > a_{n-i} \text{ if } p \nmid k, \text{ and } > A_{n-i-1} + 2 \text{ otherwise}\}$$
$$Y_{1,C}^\infty = \mathbf{Z}_{(p)}\{v_2^{(tp^2-1)p^n}h_0/p^l v_1^{kp^i+1}\colon l = n+1 \text{ if } k = 0; \text{ for } k > 0 \text{ with } kp^i > a_{n-i},$$
$$l = i > 0 \text{ for } p^{n+2} - p^n < kp^i < p^{n+2} - p^n + A_{n-i+1} + 2 \text{ and}$$
$$p^{n+2} - p^n + A_{n-i} + 2 \leq kp^i \text{ if } p | k$$
$$l = i + 1 \text{ for } i = 0 \text{ and } p \nmid (k + p^{n-i}), \text{ for } kp^i = (p^2 - 1)p^n \text{ or}$$
$$\text{for } kp^i < p^{n+2} - p^n,\ p \nmid (k + p^{n-i}) \text{ and } 0 < i \leq n$$
$$l = n + 2 \text{ for } i = n,\ k \leq p^2 - 1,\ p | (k+1) \text{ and } k \neq p^2 - p - 1; \text{ and}$$
$$l = n + 3 \text{ if } i = n \text{ and } k = p^2 - p - 1\}$$
$$Y_C^\infty = \mathbf{Z}_{(p)}\{v_2^{tp-1}h_1/p^l v_1^j\colon l = 1 \text{ if } j < p - 1, \text{ and } l = 2 \text{ if } p | t \text{ and } j = p - 1\}$$
$$Y_{\infty,C}^\infty = \mathbf{Q}/\mathbf{Z}_{(p)} \text{ generated by the set } \{h_0/p^j v_1\colon j > 0\}$$

for dimension 1, and

$$G^{\infty} = G_C^{\infty} \oplus Y\zeta_C^{\infty}$$
$$Y\zeta_C^{\infty} = (Y_{0,C}^{\infty,G} \oplus Y_{1,C}^{\infty,G}) \otimes \mathbf{Z}_{(p)}\{\zeta\}$$

for

$$\begin{aligned} Y_{0,C}^{\infty,G} &= \mathbf{Z}_{(p)}\{v_2^{sp^n} h_0/p^{i+1} v_1^{kp^{i+1}+1}\colon p \nmid s(s+1), \\ &\qquad k \neq 0,\ A_{n-i-1} + 1 < kp^{i+1} \le A_{n-i} + 1 \text{ for } i \ge 0\} \\ Y_{1,C}^{\infty,G} &= \mathbf{Z}_{(p)}\{v_2^{(tp^2-1)p^n} h_0/p^{i+1} v_1^{kp^{i+1}+1}\colon k \neq 0, \\ &\qquad p^{n+2} - p^n + A_{n-i-1} + 1 < kp^{i+1} \le p^{n+2} - p^n + A_{n-i} + 1 \text{ for } i \ge 0\} \\ G_C^{\infty} &= \mathbf{Z}_{(p)}\{v_2^{sp^n} g_0/p^{n+1} v_1,\ v_2^{sp^n - (p^{n-1}-1)/(p-1)} g_1/p^l v_1^j\colon \\ &\qquad p \nmid (s+1),\ 0 < j \le a_n,\ p^{i+1} \nmid (j + A_{n-i-1} + 1) \text{ if } s = up^i \in \mathbf{Z}(0), \\ &\qquad p^i \nmid (j + A_{n-i} + 1) \text{ if } s = up^i \in \mathbf{Z}(2), \text{ and } l = i+1 \text{ if } n = 0 \text{ and } v_p(s) = i; \\ &\qquad l = i + 1 \text{ if } n \ge 1 \text{ and } v_p(j + A_{n-1} + 1) = i\} \\ G_0^{\infty} &= \mathbf{Q}/\mathbf{Z}_{(p)} \text{ generated by the set } \{g_0/p^j v_1\colon j > 0\}. \end{aligned}$$

As a corollary of this theorem, we have the E_2-term $H^*E(2)_*$ of the Adams–Novikov spectral sequence by the chromatic spectral sequence. Furthermore it collapses since $E_2^s = 0$ for $s > 4$, and so the E_2-term is isomorphic to the homotopy groups of L_2S^0. Thus we have our main theorem.

THEOREM 2.4. *$\pi_*(L_2S^0)$ is isomorphic to $H^*E(2)_*$, which is isomorphic to*

$$\begin{gathered} \mathbf{Z}_{(p)} \oplus \mathbf{Z}_{(p)}\{v_1^{sp^i}/p^{i+1}\colon i \ge 0,\ s \ge 0,\ p \nmid s\} \oplus X^{\infty} \\ \oplus Y_{0,C}^{\infty} \oplus Y_{1,C}^{\infty} \oplus Y_C^{\infty} \oplus X\zeta_C^{\infty} \oplus (X_{\infty}^{\infty} \otimes \mathbf{Z}_{(p)}\{\zeta\}) \\ \oplus Y\zeta_C^{\infty} \oplus G_C^{\infty} \oplus (Y_{\infty,C}^{\infty} \otimes \mathbf{Z}_{(p)}\{\zeta\}) \oplus G_0^{\infty} \\ \oplus (G_0^{\infty} \otimes \mathbf{Z}_{(p)}\{\zeta\}). \end{gathered}$$

The degrees of the elements are read off from Theorem 10.1 as follows. Here a homotopy element $\xi \in \pi_*(L_2S^0)$ has *degree* r if $\xi \in \pi_r(L_2S^0)$, and we denote $|\xi| = r$. Then all elements in the first factor are 0. If we identify the elements in the theorem with the corresponding homotopy elements under the isomorphism, we have degrees:

$$\begin{aligned} |v_1^j/p^{i+1}| &= jq - 1 \\ |v_2^m/p^i v_1^j| &= m(p+1)q - jq - 2 \\ |v_2^m h_0/p^i v_1^j| &= m(p+1)q + q - jq - 3 \\ |v_2^{tp-1} h_1/p^i v_1^j| &= tp(p+1)q - q - jq - 3 \\ |v_2^m g_0/p^i v_1^j| &= m(p+1)q + q - jq - 4 \\ |v_2^m g_1/p^i v_1^j| &= m(p+1)q - q - jq - 4 \end{aligned}$$

and for the elements of the form $z \otimes \zeta$,

$$|z \otimes \zeta| = |z| - 1.$$

3. HOPF ALGEBROIDS

Let E be a ring spectrum, and denote $E_* = E_*(S^0)$. If the homology $E_*(E)$ of E is flat over E_*, then the pair $(E_*, E_*(E))$ becomes a Hopf algebroid in the usual way (cf. [1, 11]), and we can do homological algebra in the category of $E_*(E)$-comodules (cf. [11, A1]).

Among such spectra E, at each prime number p, we have the Brown–Peterson spectrum BP and the Johnson–Wilson spectrum $E(n)$ for a nonnegative integer n. Here we note that, although we do not know whether or not $E(n)$ is a ring spectrum, we have the Hopf algebroid $(E(n)_*, (E(n)_*(E(n)))$ whose structure is induced from that of $BP_*(BP)$, since $E(n)_*(X) = E(n)_* \otimes_{BP_*} BP_*(X)$ for any spectrum X. Here the action of BP_* to $E(n)_*$ is given by sending v_k ($k > n$) to 0, in which v_k is the Hazewinkel's generator of the coefficient rings $E(0)_* = \mathbf{Q}$,

$$BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad E(n)_* = \mathbf{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}] \tag{3.1}$$

for $n > 0$ (cf. [11]). Their self-homologies are

$$\begin{aligned} BP_*(BP) &= BP_*[t_1, t_2, \dots] \\ E(n)_*(E(n)) &= E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*. \end{aligned} \tag{3.2}$$

We obtain the formulae of the structure maps of the Hopf algebroids associated to these spectra by [7] (cf. [11]). The structure of the Hopf algebroid associated to $E(n)$ is induced from that of BP_*. So we give here the formulae for BP. The left unit $\eta_L: BP_* \to BP_*(BP)$ is the inclusion $BP_* \subset BP_*(BP)$. Then $BP_*(BP)$ is a left BP_*-module by η_L. The right unit $\eta_R: BP_* \to BP_*(BP)$, which also gives $BP_*(BP)$ a right BP_*-module structure, sustains Landweber's formula

$$\eta_R(v_n) \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \tag{3.3}$$

mod I_{n-1} for the prime ideal $I_n = (p, v_1, \dots, v_{n-1})$ of BP_*. We also have

$$\begin{aligned} \eta_R(v_1) &= v_1 + pt_1 \\ \eta_R(v_2) &= v_2 + v_1t_1^p + pt_2 - t_1(v_1 + pt_1)^p(p+1)v_1^p t_1 \\ &\quad - p^{-1}((v_1 + pt_1)^p - v_1^p) \\ &\equiv v_2 + v_1t_1^p + pt_2 - (p+1)v_1^p t_1 \mod (p^2) \\ \eta_R(v_3) &\equiv v_3 + v_2t_1^{p^2} + v_1t_2^p - t_1\eta_R(v_2)^p + v_1^2 V \mod (p, v_1^{p^2}) \\ &\equiv v_3 + v_2t_1^{p^2} + pt_3 - v_2^p t_1 \mod (p^2, v_1) \end{aligned} \tag{3.4}$$

where we use the same notation V as that of [16] defined by

$$pv_1V = v_2^p + v_1^p t_1^{p^2} - v_1^{p^2} t_1^p - (v_2 + v_1t_1^p - v_1^p t_1)^p. \tag{3.5}$$

For the diagonal $\Delta: BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$, we have

$$\begin{aligned} \Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1 \\ \Delta(t_2) &= t_2 \otimes 1 + 1 \otimes t_2 + t_1 \otimes t_1^p + v_1 T \\ \Delta(t_3) &\equiv t_3 \otimes 1 + 1 \otimes t_3 + g + v_2 T^p \mod (p, v_1) \end{aligned} \tag{3.6}$$

where $g, T \in BP_*(BP) \otimes_{BP_*} BP_*(BP)$ denote the elements

$$\begin{aligned} g &= t_1 \otimes t_2^p + t_2 \otimes t_1^{p^2} \\ pT &= t_1^p \otimes 1 + 1 \otimes t_1^p - (t \otimes 1 + 1 \otimes t_1)^p. \end{aligned} \tag{3.7}$$

Turn now to the structure of $E(n)_*(E(n))$. Noticing that $E(n)_*(BP) = E(n)_* \otimes_{BP_*} BP_*(BP) = E(n)_*[t_1, t_2, \dots]$, we have

$$E(n)_*(E(n)) = E(n)_*(BP) \otimes_{BP_*} E(n)_*$$
$$= E(n)_*[t_1, t_2, \dots]/(\eta_R(v_i)\colon i > n).$$

In this paper, we consider only for the case $n = 2$. Then the formula on v_3 in (3.4) gives rise to the relation in $E(2)_*(E(2))$:

$$v_2 t_1^{p^2} \equiv t_1 \eta_R(v_2)^p - v_1 t_2^p - v_1^2 V \mod (p, v_1^{p^2}). \tag{3.8}$$

Furthermore, we have the following result.

(3.9) (Shimomura and Tamura [16, Lemma 3.2]). *In the* $E(2)_*(E(2))$, *we have the relations*

$$v_2 t_n^{p^2} \equiv v_2^{p^n} t_n - v_1 t_{n+1}^p \mod (p, v_1^2) \text{ for } n > 0$$

and

$$v_2^p T^{p^2} \equiv v_2^{p^2} T \mod (p, v_1).$$

Let (A, Γ) denote one of the Hopf algebroids $(BP_*, BP_*(BP))$ and $(E(2)_*, E(2)_*(E(2)))$. Then the Ext group

$$H^*M = \mathrm{Ext}_\Gamma^*(A, M)$$

of a comodule M can be computed by the homology of the cobar complex $(\Omega_\Gamma^* M, d_*)$. It is shown in [5] that there is an isomorphism

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, M) \xrightarrow{\cong} \mathrm{Ext}_{E(2)_*(E(2))}(E(2)_*, E(2)_* \otimes_{BP_*} M)$$

for a v_2-local $BP_*(BP)$-comodule M. Thus there would be no confusion if we write H^*M for those Ext groups, as long as we consider v_2-local comodules. A cobar complex of a comodule M is a pair $(\Omega_\Gamma^{*,*}M, d_*)$ of graded $\mathbf{Z}_{(p)}$-modules

$$\Omega_\Gamma^{s,*} M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (s \text{ copies of } \Gamma)$$

for $s \geq 0$ and the differentials $d_s\colon \Omega_\Gamma^{s,*}M \to \Omega_\Gamma^{s+1,*}M$ in the sense $d_{s+1}d_s = 0$ for $s \geq 0$ which are defined inductively by

$$d_0(m) = \psi(m) - m \otimes 1$$
$$d_1(m \otimes x) = \psi(m) \otimes x - m \otimes \Delta(x) + m \otimes x \otimes 1 \tag{3.10}$$
$$d_s(m \otimes x \otimes x_{s-1}) = d_1(m \otimes x) \otimes x_{s-1} - m \otimes x \otimes d_{s-1}(x_{s-1})$$

for $m \in M$, $x \in \Gamma$ and $x_s \in \Omega_\Gamma^s A = \Gamma \otimes_A \cdots \otimes_A \Gamma$ (s copies). Here $\psi\colon M \to M \otimes_A \Gamma$ is the comodule structure of M.

We note that in the following sections, we mainly treat comodules with structure maps induced from η_R, and so the comodule structure ψ is computable by using the formulae (3.3) and (3.4). We further use the notation η_R for such a structure map ψ. For example, by (3.4) and (3.10), we obtain the following lemma.

LEMMA 3.11. *In the cobar complex* $\Omega_\Gamma^1 A$, *we have* $d_0(v_2^p) \equiv v_1^p t_1^{p^2} - v_1^{p^2} t_1^p - pv_1 V + p^2(v_2 + v_1 t_1^p - v_1^p t_1)^{p-1}(t_2 - v_1^p t_1) \mod (p^3)$. *In particular,* $d_0(v_2^p) \equiv pv_1 v_2^{p-1} t_1^p + \binom{p}{2} v_1^2 v_2^{p-2} t_1^{2p} + p^2 v_2^{p-1} t_2 - p^2 v_1 v_2^{p-2} t_1^p t_2 + (p^2/2) v_1^2 v_2^{p-3} t_1^{2p} t_2 \mod (p^3, v_1^3)$.

We have more formulae on the differential shown immediately from the definition (3.10).

(3.12)(Shimomura and Tamura [16]). *For any $u, v \in A$ and $x, y \in \Gamma$, we have*

$$d_0(uv) = d_0(u)\eta_R(v) + ud_0(v)$$
$$d_1(xy) = d_1(x)\Delta(y) + (x \otimes 1 + 1 \otimes x)d_1(y) - x \otimes y - y \otimes x$$
$$d_1(uy) = d_0(u) \otimes y + ud_1(y)$$
$$d_1(x\eta_R(v)) = d_1(x)(1 \otimes \eta_R(v)) - x \otimes d_0(v).$$

Note that once we give an element in $\Omega_\Gamma^2 A$, then we have the corresponding elements in $\Omega_\Gamma^2 A/I$ for any ideal I of A. In this case we use the same notation for those elements. First define the elements g_0 and $g_1 \in \Omega_\Gamma^2 A$:

$$g_0 = v_2^{-p}g \quad \text{and} \quad g_1 = v_2^{-p^2-1}g^p = v_2^{-1}g_0^p \tag{3.13}$$

for g given in (3.7). The element ζ_2 in $\Omega_\Gamma^1 A$ is defined in [6] by

$$\zeta_2 = v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p}) - v_2^{-p-1}v_3t_1^p.$$

LEMMA 3.14. *In the cobar complex $\Omega_\Gamma^2 A/(p, v_1)$, we have*

$$d_1(t_1t_2) = v_2t_1 \otimes \zeta_2 - v_2g_0 - 2t_1 \otimes t_2 - t_1^2 \otimes t_1^p$$
$$d_1(t_1^pt_2) = v_2\zeta_2 \otimes t_1^p - v_2^2g_1 - 2t_2 \otimes t_1^p - t_1 \otimes t_1^{2p}$$
$$d_1(t_1t_2^p) = v_2^p\zeta_2 \otimes t_1 - v_2^pg_0 - 2t_2^p \otimes t_1 - t_1^p \otimes t_1^{p^2+1}.$$

This follows from a direct calculation by definition with the help of (3.6) and (3.12).

4. THE CHROMATIC SPECTRAL SEQUENCE

In this section we also consider the ring spectra BP and $E(2)$, the Brown–Peterson and the Johnson–Wilson spectra, respectively, and denote those spectra by E. Then we have the Adams–Novikov spectral sequence converging to the homotopy groups of $\pi_*(L_EX)$ of the Bousfield localization of X with respect to E if X is connected (cf. [1, 2]). Note that $L_{BP}X = X$ for a connected p-local spectrum X. The E_2-term is $H^*E_*(X) = \mathrm{Ext}_{E_*(E)}^*(E_*, E_*(X))$. By virtue of the Landweber filtration theorem [4], the E_2-term can be lead by computing H^*E_*/I_k's. Here I_k denotes the invariant prime ideal $(p, v_1, \dots, v_{k-1})$ of E_*, and $k \le 2$ if $E = E(2)$. The Ext groups H^*E_*/I_k's are also the E_2-term of the Adams–Novikov spectral sequence for computing the homotopy $\pi_*(L_EV(k-1))$ of the Toda–Smith spectra $V(k-1)$ when they exist. Miller *et al.* [6] introduced the chromatic spectral sequence for computing the Ext groups H^*E_*/I_k's.

We now give the definition of the chromatic spectral sequence. Put first $N_k^0 = E_*/I_k$ and inductively suppose that N_k^s is defined. Then define $M_k^s = v_{k+s}^{-1}N_k^s$, which has the comodule structure induced from that of N_k^s by [5]. Now N_k^{s+1} is the cokernel of the inclusion $N_k^s \subset M_k^s$, which also has the induced comodule structure. In other words, we have the short exact sequence of comodules

$$0 \to N_k^s \xrightarrow{\subset} M_k^s \to N_k^{s+1} \to 0. \tag{4.1}$$

As usual, we will denote an element ξ of M_k^s by a linear combination of the elements of the form

$$x/v_k^{e_k} \cdots v_{k+s-1}^{e_{k+s-1}} \quad (v_0 = p)$$

for $e_i > 0$ with $k \le i \le k+s-1$ and for $x \in M^0_{k+s}$. Furthermore, the element $x/v_k^{e_k} \cdots v_{k+s-1}^{e_{k+s-1}}$ is killed by $v_i^{e_i}$ for each i.

Applying the functor $H^* -$ to the exact sequence (4.1), we have the long exact sequence

$$0 \to H^0 N_k^s \to H^0 M_k^s \to H^0 N_k^{s+1} \xrightarrow{\delta_0} H^1 N_k^s \to H^1 M_k^s$$
$$\to H^1 N_k^{s+1} \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-1}} H^n N_k^s \to H^n M_k^s \to H^n N_k^{s+1} \to \cdots \tag{4.2}$$

for each nonnegative integers k and s. These exact sequences are the exact couple that gives rise to the chromatic spectral sequence. The E_1-term is $E_1^{s,t} = H^t M_k^s$ and the abutting module is the desired Ext group $H^{s+t} N_k^0 = H^{s+t} E_*/I_k$. To compute the E_1-term, Miller *et al.* [6] further introduced the Bockstein spectral sequence that is defined by the exact couple obtained by applying the functor $H^* -$ to the short exact sequence

$$0 \to M_{k+1}^{s-1} \xrightarrow{\varphi} M_k^s \xrightarrow{v_k} M_k^s \to 0$$

where φ is the comodule map defined by $\varphi(x) = x/v_k$. The Bockstein spectral sequence has the E_1-term $H^* M_{k+1}^{s-1}$ and abuts to $H^* M_k^s$. Thus we can compute inductively the E_1-term of the chromatic spectral sequence. When we work on the Bockstein spectral sequence, $(k, s) = (0, 2)$ in our case, we mainly use the following result.

LEMMA 4.3 (Miller *et al.* [6, Remark 3.11]). *Consider a map of exact couples*

$$\begin{array}{ccccccccccc} 0 \to \cdots & \xrightarrow{\delta_{t-1}} & H^t M_1^1 & \xrightarrow{\varphi} & B^t & \xrightarrow{p} & B^t & \xrightarrow{\delta_t} & H^{t+1} M_1^1 & \xrightarrow{\varphi_*} & \cdots \\ & & \downarrow = & & \downarrow f & & \downarrow f & & \downarrow = & & \\ 0 \to \cdots & \xrightarrow{\delta_{t-1}} & H^t M_1^1 & \xrightarrow{\varphi} & H^t M_0^2 & \xrightarrow{p} & H^t M_0^2 & \xrightarrow{\delta_t} & H^{t+1} M_1^1 & \xrightarrow{\varphi_*} & \cdots \end{array}$$

If B^t is p-torsion, then f is an isomorphism.

The first step of the induction is Morava's theorem.

(4.4) (Ravenel [9]). *If $p > 2$, then $H^* M_1^0 = F_p[v_1, v_1^{-1}] \otimes E(t_1)$. If $p > 3$, then $H^* M_2^0 = F_p[v_2, v_2^{-1}]\{1, t_1, t_1^p, g_0, g_1, g_0 t_1^p\} \otimes E(\zeta_2)$.*

Here $E(x)$ and $F\{b_i\}$ denote the exterior algebra over the generators $\{x\}$ and the F-vector space with basis $\{b_i\}$, respectively, in which F denotes a field.

Turn to the second step.

(4.5) (Miller *et al.* [6]). *If $p > 2$, then $H^t M_0^1 = 0$ for $t > 1$, $H^1 M_0^1 = \mathbf{Q}/\mathbf{Z}_{(p)}$ whose subgroup of order p^j is generated by*

$$y_{1,j} = -\sum_{k>0} \frac{(-1)^k v_1^{-k} t_1^k}{k p^{j+1-k}}$$

and

$$H^0 M_0^1 = \mathbf{Q}/\mathbf{Z}_{(p)} \oplus \sum_{i \ge 0, (p,s)=1} (\mathbf{Z}/p^{i+1}) \langle v_1^{sp^i}/p^{i+1} \rangle.$$

Here $G\langle x \rangle$ denotes the group isomorphic to G whose generator is x.

For stating the results on $H^*M_1^1$, we introduce some more elements. From here on we work on $E(2)$ not on BP, and the Hopf algebroid (A, Γ) denotes $(E(2)_*, E(2)_*(E(2)))$. The elements $x_n \in \Omega_\Gamma^0 A$ are inductively defined by

$$x_0 = v_2, \quad x_1 = v_2^p, \quad x_2 = x_1^p - v_1^{p^2-1}v_2^{p^2-p+1}$$
$$x_n = x_{n-1}^p - 2v_1^{a_n-p}v_2^{p^n-p^{n-1}+1} \quad \text{for } n > 2 \tag{4.6}$$

for the integer a_i with $a_0 = 1$ and

$$a_n = p^n + p^{n-1} - 1 \tag{4.7}$$

for $n > 0$. For the differential d_0 of the cobar complex, we have the following result.

(4.8) (Miller *et al.* [6]). mod $(p, v_1^{2+a_i})$,

$$\begin{aligned} d_0(x_i) &\equiv v_1 t_1^p, \quad i = 0 \\ &\equiv v_1^p v_2^{p-1}(t_1 + v_1(v_2^{-1}(t_2 - t_1^{p+1}) - \zeta_2)), \quad i = 1 \\ &\equiv 2v_1^{a_i}v_2^{(p-1)p^{i-1}}\sigma_{i-1}, \quad i > 1. \end{aligned}$$

The above element σ_n is given by

$$\sigma_n = t_1 - \tfrac{1}{2}v_1\zeta_2^{p^n}.$$

The element ζ_2 satisfies the following.

(4.9) (Miller *et al.* [6]). *$\zeta_2 = v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p})$ is homologous to $\zeta_2^{p^i}$ for $i \geq 0$ in $\Omega_\Gamma^1 A/(p, v_1)$.*

(4.10) (Shimomura [15]). *We have a cocycle ζ in each cobar complex $\Omega_\Gamma^1 A/(p^{i+1}, v_1^{mp^i})$ such that ζ is homologous to ζ_2 in $\Omega_\Gamma^1 A/(p, v_1)$.*

By virtue of this, we will use the notation ζ for a cocycle of $\Omega_\Gamma^{1,0} A/(p^{i+1}, v_1^{mp^i})$ such that ζ is homologous to ζ_2 in $\Omega_\Gamma^1 A/(p, v_1)$ including ζ^{p^i}. We also use the notation

$$\sigma = t_1 - \tfrac{1}{2}v_1\zeta. \tag{4.11}$$

Then σ is homologous to σ_n for any n in $\Omega_\Gamma^1 A/(p, v_1)$.

Divide the set $\mathbf{Z} - p\mathbf{Z}$ of integers into three parts:

$$\begin{aligned} \mathbf{Z}_0 &= \{s\colon s \in \mathbf{Z} \text{ with } p \nmid s(s+1)\} \\ \mathbf{Z}_1 &= \{sp - 1\colon s \in \mathbf{Z} \text{ with } p \nmid s\} \\ \mathbf{Z}_2 &= \{sp^2 - 1\colon s \in \mathbf{Z}\} \end{aligned} \tag{4.12}$$

and $\mathbf{Z} - \{0\}$ into

$$\mathbf{Z}(i) = \{m\colon m = sp^n \text{ with } n \geq 0 \text{ and } s \in \mathbf{Z}_i\} \tag{4.13}$$

for $i = 0, 1$ and 2. We then introduce the element $y_m = v_2^m t_1 + v_1 \bar{y}_m$ of $\Omega_\Gamma^1 A$ for $m \in \mathbf{Z}(0) \cup \mathbf{Z}(2)$ defined in [16] such that

$$d_1(y_m) \equiv -s_m v_1^{A(m)} v_2^{e(m)} g_1 \quad \bmod (p, v_1^{A(m)+1}) \tag{4.14}$$

for $g_1 = v_2^{-p^2-1}(t_1^p \otimes t_2^{p^2} + t_2^p \otimes t_1^{p^3})$ in (3.13). Here s_m for $m = sp^n$ with $p \nmid s$ equals

$$\begin{aligned} &\binom{s+1}{2} \quad \text{if } n = 0 \text{ and } m \in \mathbf{Z}(0) \\ &\frac{(-1)^n}{2}\binom{s+1}{2} \quad \text{if } n > 0 \text{ and } m \in \mathbf{Z}(0) \\ &1 \quad \text{if } n = 0 \text{ and } m \in \mathbf{Z}(2) \\ &\frac{(-1)^n}{4} \quad \text{if } n > 0 \text{ and } m \in \mathbf{Z}(2). \end{aligned} \tag{4.15}$$

We define the integers $e(m)$ and $A(m)$ for $m = sp^n$ with $p \nmid s$ by

$$\begin{aligned} e(m) &= m - (p^n - 1)/(p - 1) \quad \text{if } m \in \mathbf{Z}(0) \\ e(m) &= m - p^n(p - 1) - (p^n - 1)/(p - 1) \quad \text{if } m \in \mathbf{Z}(2) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} A_n &= (p+1)(p^n - 1)/(p - 1) \\ A_n' &= (p+1)(p^{n+1} - p^n + (p^n - 1)/(p - 1)) \\ A(m) &= A_n + 2 \quad \text{if } m = sp^n \text{ and } m \in \mathbf{Z}(0) \\ &= A_n' + 2 \quad \text{if } m = sp^n \text{ and } m \in \mathbf{Z}(2) \\ &= \infty \quad \text{if } m = 0. \end{aligned} \tag{4.17}$$

Now we define inductively the elements y_m for $m \in \mathbf{Z}(0) \cup \mathbf{Z}(2)$ [16]. Let $m = sp^n$ with $p \nmid s$. Then, for $s \in \mathbf{Z}_0$, we put

$$\begin{aligned} y_s = v_2^s t_1 &+ s v_1 v_2^{s-1}(t_1^{p+1} - t_2) + \frac{s-1}{2} v_1 v_2^s \zeta \\ &+ \binom{s}{2} v_1^2 v_2^{s-2} t_1^p (t_1^{p+1} - t_2 + v_2\zeta) + s v_1^2 v_2^{s-1-p^2} t_3^p \\ y_{sp} = v_2^{sp}\sigma &- \frac{s}{2} v_1^2 Z_s \end{aligned} \tag{4.18}$$

and, for $s = tp^2 - 1 \in \mathbf{Z}_2$,

$$\begin{aligned} y_s &= W_t^p + v_1^{p^2-p-2} v_2^{s+1} X \\ 2v_1^p y_{sp} &= v_1 y_s^p - d_0(v_2^{sp+1}) + v_1^{p^3+2} W_{tp^2-p}. \end{aligned} \tag{4.19}$$

Once y_m for $p \mid m$ is defined, y_{mp} is given by

$$v_1^p y_{mp} = v_1 y_m^p - d_0(v_2^{mp+1}) + s_m v_1^{pA(m)-p+2} W_{e(m)}. \tag{4.20}$$

Here the elements W_s, Z_s and X are defined in [16] so that they satisfy

$$\begin{aligned} d_1(W_s) &\equiv v_1^{p-1} v_2^{sp} g_1^p - \frac{s-1}{2} v_1^{p+1} v_2^{sp-1} g_1 \mod (p, v_1^{p+2}) \\ d_1(Z_s) &\equiv v_1^{p-2} v_2^{sp-p} t_1^{p^2} \otimes \sigma - \frac{(s+1)}{2} v_1^{p+1} v_2^{sp-1} g_1 \mod (p, v_1^{p+2}) \\ d_1(X) &\equiv -v_1^2 g_1^{p^2} - v_1^{p+3} v_2^{-p} g_1 \mod (p, v_1^{p+4}) \end{aligned} \tag{4.21}$$

and

$$W_s = -v_2^{sp-p}(V + v_1^{p-1}v_2^{-p^3}t_3^{p^2} - \frac{s-1}{2}v_1^p(v_2^{-1}(t_1^{p+1}-t_2)^p(2-v_1v_2^{-1}t_1^p) + v_2^{p-1}\zeta^p))$$

$$Z_s \equiv W_s \mod (p, v_1^{p-2}). \tag{4.22}$$

We need other cocycles G_n of $\Omega_\Gamma^2 A/(p, v_1^{a_n})$ for $n \geq 0$ introduced in [14] such that

$$G_0 \equiv g_0 \quad \text{and} \quad G_n \equiv v_2^{-(p^{n-1}-1)/(p-1)}g_1 \mod (p, v_1) \text{ for } n > 0 \tag{4.23}$$

where the elements g_0 and g_1 are given in (3.13). We prepare some notation here:

$k(1)_* = F_p[v_1]$

$K(1)_* = v_1^{-1}k(1)_* = F_p[v_1, v_1^{-1}]$

$k(1)_*\{x/v_1^j\colon x \in \Lambda\}$ denotes the direct sum of the cyclic $k(1)_*$-modules isomorphic to $k(1)_*/(v_1^j)$ generated by x/v_1^j, and

$k(1)_*\{x/v_1^\infty\colon x \in \Lambda\}$ denotes the direct sum of the modules isomorphic to $K(1)_*/k(1)_*$ with F_p-basis $\{x/v_1^j\colon j > 0\}$.

Now consider the following $k(1)_*$-modules:

$$\begin{aligned}
X &= k(1)_*\{x_n^s/v_1^{a_n}\colon n \geq 0,\ s \in \mathbf{Z} - p\mathbf{Z}\}\\
X_\infty &= k(1)_*\{1/v_1^\infty\}\\
Y_0 &= k(1)_*\{y_m/v_1^{A_n}\colon m \in \mathbf{Z}(0),\ n = \nu_p(m)\}\\
Y_1 &= k(1)_*\{y_m/v_1^{A_n'}\colon m \in \mathbf{Z}(2),\ n = \nu_p(m)\}\\
Y &= k(1)_*\{v_2^{tp}V/v_1^{p-1}\colon t \in \mathbf{Z}\}\\
Y_\infty &= k(1)_*\{t_1/v_1^\infty\}\\
G &= k(1)_*\{x_n^s G_n/v_1^{a_n}\colon n \geq 0,\ s+1 \in \mathbf{Z} - p\mathbf{Z}\}.
\end{aligned}$$

Here $\nu_p(m)$ denotes the maximal power of p dividing m. Then we have the structure of $H^*M_1^1$ obtained in [6, 16, 14]:

$$H^*M_1^1 = (X \oplus X_\infty \oplus Y_0 \oplus Y_1 \oplus Y \oplus Y_\infty \oplus G) \otimes E(\zeta). \tag{4.24}$$

We will end this section with rewriting the element y_m as follows.

LEMMA 4.25. *Let s and n be integers with $n > 0$. Then we have*

$$y_{sp^n} \equiv v_2^{sp^n}(t_1 - \tfrac{1}{2}v_1\zeta) + \frac{s}{2}v_1^{1+p^{n-1}}v_2^{(s-1)p^n}V^{p^{n-1}} \tag{4.26}$$

$\mod (p, v_1^{(p-1)p^{n-1}+1})$ *and moreover*

$$2v_1^{p^n-1}y_{(tp^2-1)p^n} \equiv -v_2^{(t-1)p^{n+2}}V^{p^{n+1}} \tag{4.27}$$

$\mod (p, v_1^{p^n-1+F(n)})$ *for $F(n) = p^{n+2} - p^{n+1} - 3p^n + 1$, up to homology.*

Remark. We can define y_m so that the congruence (4.26) holds $\mod (p, v_1^{p^n})$ after replacing $v_2^{sp^n}$ by x_n^s.

Proof. We first prove (4.26). By the definition of the elements (4.18) and (4.22),

$$y_{sp} = v_2^{sp}\sigma - \frac{s}{2}(v_1^2 Z_s)$$

if $s \in \mathbf{Z}_0$ and

$$Z_s \equiv W_s \equiv -v_2^{sp-p}V \quad \text{mod } (p, v_1^{p-2}).$$

Therefore we have the case for $n = 1$ and $s \in \mathbf{Z}_0$.

Next suppose that $s = tp^2 - 1$. Then the definitions (4.19) and (4.22) give

$$\begin{aligned} y_s &\equiv W_t^p \quad \text{mod}(p, v_1^p) \\ &\equiv -v_2^{tp^2-p^2}V^p \quad \text{mod } (p, v_1^p) \\ &\equiv v_2^{tp^2-p}t_1^{p^2} \quad \text{mod}(p, v_1^p) \end{aligned}$$

which is congruent to $v_2^{tp^2-1}t_1 - v_1 v_2^{tp^2-p-1}t_2^p \,\text{mod}(p, v_1^2)$ by (3.8). Thus we have

$$y_s^p \equiv v_2^{(tp^2-1)p}t_1^p - v_1^p v_2^{(tp^2-p-1)p}t_2^{p^2}$$

mod (p, v_1^{2p}). Furthermore, the formula (3.4) gives us

$$\begin{aligned} d_0(v_2^{1+(tp^2-1)p}) &\equiv (v_2 + v_1 t_1^p - v_1^p t_1)(v_2^{(tp^2-1)p} - v_1^p v_2^{(tp^2-2)p}t_1^{p^2}) - v_2^{1+(tp^2-1)p} \\ &\equiv v_1 v_2^{(tp^2-1)p}t_1^p - v_1^p v_2^{(tp^2-1)p}t_1 \\ &\quad - v_1^p v_2^{(tp^2-2)p+1}t_1^{p^2} - v_1^{p+1}v_2^{(tp^2-2)p}t_1^{p^2+p} \end{aligned}$$

mod (p, v_1^{2p}) which turns out to be congruent, again by (3.8), to

$$v_1 v_2^{(tp^2-1)p}t_1^p - 2v_1^p v_2^{(tp^2-1)p}t_1 + v_1^{p+1}v_2^{(tp^2-2)p}t_2^p + v_1^{p+2}v_2^{(tp^2-2)p}V - v_1^{p+1}v_2^{(tp^2-2)p}t_1^{p^2+p}.$$

Put these into the congruence

$$2v_1^p y_{sp} \equiv v_1 y_s^p - d_0(v_2^{(tp^2-1)p+1}) \quad \text{mod } (p, v_1^{2p})$$

of (4.19), and we have the same result as the case for $s \in \mathbf{Z}_0$.

Since $d_0(v_2^{mp+1}) \equiv v_2^{mp}(v_1 t_1^p - v_1^p t_1)\,\text{mod}(p, v_1^{p^{n+1}})$ for $m = sp^n$ with $n \geq 2$, we have

$$v_1^p y_{mp} \equiv v_1 v_2^{mp}(t_1^p - \tfrac{1}{2}v_1^p \zeta^p) + \frac{s}{2}v_1^{1+p+p^n}v_2^{mp-p^{n+1}}V^{p^n} - v_2^{mp}(v_1 t_1^p - v_1^p t_1)$$

mod $(p, v_1^{(p-1)p^n+p+1})$ by the definition (4.20) under the inductive hypothesis. Thus the case for $n \geq 2$ immediately follows from the induction on n.

Now turn to (4.27). By the definitions (4.19) and (4.22), we have

$$y_s \equiv -v_2^{(t-1)p^2}V^p \tag{4.28}$$

mod $(p, v_1^{p^2-p-2})$ for $n = 0$, and $2v_1^{p-1}y_{sp} \equiv -v_2^{(t-1)p^3}V^{p^2}\,\text{mod}(p, v_1^{p^3-p^2-2p})$ for $n = 1$ up to homology. Suppose that

$$2v_1^{p^n-1}y_{(tp^2-1)p^n} \equiv -v_2^{(t-1)p^{n+2}}V^{p^{n+1}} \tag{4.29}$$

mod $(p, v_1^{p^n-1+F(n)})$ for $F(n) = p^{n+2} - p^{n+1} - 3p^n + 1$ up to homology. Then the definition (4.20) leads us to

$$2v_1^{p^{n+1}-1}y_{(tp^2-1)p^{n+1}} \equiv 2v_1^{p^{n+1}-p}y_{(tp^2-1)p^n}^p$$

mod $(p, v_1^{p^{n+1}-1+F(n+1)})$ up to homology. Hence we have the desired congruence. Q.E.D.

COROLLARY 4.30. *Let m be an integer in $\mathbf{Z}(0)\cup\mathbf{Z}(2)$ with $p|m$. In the complex $\Omega_\Gamma A/(p, v_1^2)$, $t_1\otimes y_m$ is homologous to $-\frac{1}{2}v_1y_m\otimes\zeta$.*

Proof. By (4.26), we see that $t_1\otimes y_m = t_1\otimes v_2^m(t_1-\frac{1}{2}v_1\zeta)$. Since $p|m$, the first term is $t_1\otimes v_2^mt_1 = v_2^mt_1\otimes t_1$ in our complex, which equals $d_0(-\frac{1}{2}v_2^mt_1^2)$. Thus $t_1\otimes y_m$ is homologous to $-\frac{1}{2}v_1v_2^mt_1\otimes\zeta$. Q.E.D.

5. THE COKERNEL OF δ_0

In this section, we consider the connecting homomorphism $\delta_0: H^0M_0^2\to H^1M_1^1$ associated to the short exact sequence $0\to M_0^2\to M_1^1\xrightarrow{p} M_1^1\to 0$. The image of δ_0 is given in [6, Prop. 6.9]. First we rewrite the result by using the bases of $H^1M_1^1$. By Lemma 4.25, we obtain the following lemma.

LEMMA 5.1. *The connecting homomorphism $\delta_0: H^0M_0^2\to H^1M_1^1$ sends an element $x_{i+k}^s/p^{i+1}v_1^j$ for $s\in\mathbf{Z}-p\mathbf{Z}$ and $j=mp^i$ with $0\le i\le k$ and $1\le m\le a_{k-i}$ to*

$$-y_s/v_1^2+(s-1)v_2^s\zeta/2v_1 \quad \textit{if } k=0$$
$$-my_{sp^{i+1}}/v_1^{j+1}-mx_{i+1}^s\zeta/2v_1^j-sv_2^{sp^{i+1}-p}V/v_1^{j-1}+\cdots \quad \textit{if } k=1$$
$$-my_{sp^{i+2}}/v_1^{j+1}-mx_{i+2}^s\zeta/2v_1^j+sy_{sp^{i+2}-1}/v_1^{j-p}+sv_2^{sp^{i+2}-p-1}V/v_1^{j-p-2}+\cdots \quad \textit{if } k=2$$
$$-my_{sp^{i+k}}/v_1^{j+1}-mx_{i+k}^s\zeta/2v_1^j+2sy_{(sp^{i+2}-1)p^{k-2}}/v_1^{j-a_{k-1}}+\cdots \quad \textit{if } k>2.$$

For the generator $h_0/v_1^j\in H^1M_1^1$, which is represented by t_1/v_1^j, we also have Lemma 5.2.

LEMMA 5.2. *Put $j=kp^l>0$ for k with $p\nmid k$. Then,*

$$\delta_0(1/p^{l+1}v_1^j) = -kh_0/v_1^{j+1}.$$

Proof. This follows immediately from the definition of δ_0 and the formula

$$d_0(v_1^j)\equiv jpv_1^{j-1}t_1 \mod (p^{l+2}). \qquad \text{Q.E.D.}$$

We read off the following proposition on the cokernel of δ_0 from Lemma 5.1 and the structures of $H^0M_0^2$ given in [6] and $H^1M_1^1$ in (4.24). We prepare the following notation:

$$\mathbf{Z}_{(p)}\{x/p^iv_1^j\colon x\in\Lambda,\ j\in J,\ i\in I\}$$

denotes the direct sum of the cyclic $\mathbf{Z}_{(p)}$-modules isomorphic to $\mathbf{Z}/(p^i)$ generated by the elements $x/p^iv_1^j$ subject to the conditions $x\in\Lambda$, $j\in J$, and $i\in I$.

$X^\infty = \mathbf{Z}_{(p)}\{x_n^s/p^{i+1}v_1^j\colon n\ge 0,\ s\in\mathbf{Z}-p\mathbf{Z},\ i\ge 0,\ j\ge 1$, such that $p^i|j\le a_{n-i}$ and either $p^{i+1}\nmid j$ or $a_{n-i-1}<j\}$; and

$$X_\infty^\infty = \mathbf{Z}_{(p)}\{1/p^{i+1}v_1^j\colon i=v_p(j)\ge 0\}. \tag{5.3}$$

Under these notations, we have the following result.

(5.4) (Miller *et al.* [6, Theorem 6.1]). $H^0M_0^2 = X^\infty\oplus X_\infty^\infty$.

We also have

$$H^1M_1^1 = (X\oplus X_\infty)\otimes F_p\{\zeta\}\oplus Y_0\oplus Y_1\oplus Y_\infty\oplus Y \tag{5.5}$$

by (4.24). Recall the notation (4.12) and (4.13):

$$\mathbf{Z}_0 = \{s\colon s\in\mathbf{Z} \text{ with } p\nmid s(s+1)\}$$

$$\mathbf{Z}_1 = \{sp-1\colon s\in\mathbf{Z} \text{ with } p\nmid s\}$$

$$\mathbf{Z}_2 = \{sp^2-1\colon s\in\mathbf{Z}\}$$

$$\mathbf{Z}(i) = \{m\colon m = sp^n \text{ with } n\ge 0 \text{ and } s\in\mathbf{Z}_i\} \quad (i\in\{0,1,2\}).$$

We further define a subset of $\mathbf{Z}(2)$ by

$$\mathbf{Z}_2^i = \{tp^{i+2}-1\colon t\in\mathbf{Z}-p\mathbf{Z}\} \tag{5.6}$$

for each nonnegative integer i. Then $\bigcup_i \mathbf{Z}_2^i = \mathbf{Z}_2$, which is a disjoint union.

PROPOSITION 5.7. *The cokernel of $\delta_0\colon H^0M_0^2 \to H^1M_1^1$ is a vector space spanned by the bases represented by the cocycles:*

(i) t_1/v_1 *and* ζ/v_1^j *for* $j\ge 1$;
(ii) y_{sp^n}/v_1^j *for* $s\in\mathbf{Z}_0\cup\mathbf{Z}_2$, $n\ge 0$, $j\le A(sp^n)$ *such that* $j=1$ *or* $j-1>a_{n-i}$ *if* $p^i|(j-1)$, *subject to* $p^{k+1}\nmid(j+a_{n+1})$ *or* $j>a_{n+2}-a_{n+1}$ *if* $s\in\mathbf{Z}_2^k$;
(iii) $x_n^s\zeta/v_1^j$ *for* $s\in\mathbf{Z}-p\mathbf{Z}$, $n\ge 0$ *and* $1\le j\le a_n$ *such that* $j>a_{n-l}$ *if* $p^l|j$ *for either* $s\in\mathbf{Z}_1$ *or* $s\in\mathbf{Z}_2^k$ *and* $p^{k+1}|j$;
(iv) $v_2^{sp}V/v_1^j$ *for* $s\in\mathbf{Z}$ *and* $1\le j\le p-1$ *with a condition that* $p|(s+1)$ *if* $j=p-1$.

Proof. By Lemma 5.1, the leading terms of δ_0-images of the generators $x_n^s/p^{i+1}v_1^j$ of $H^0M_0^2$ are rewritten to be:

(1) y_{sp^n}/v_1^{j+1}, $s\in\mathbf{Z}_0\cup\mathbf{Z}_2$, $n\ge 0$, $j\ge 1$ and $j\le a_{n-i}$ if $p^{i+1}\nmid j$.
(2) $x_n^s\zeta/v_1^j$, $s\in\mathbf{Z}_1$,$n\ge 0$, $j\ge 1$, and $j\le a_{n-i}$ if $p^{i+1}\nmid j$.
(3) $x_n^s\zeta/v_1^j$, $s\in\mathbf{Z}_2$, $n\ge 0$, $j\ge 1$, and $j\le a_{n-i}$ if $p^{i+1}\nmid j$, subject to $p^{k+1}|j$ if $s\in\mathbf{Z}_2^k$.
(4) $v_2^{sp-p}V/v_1^{p-1}$, $s\in\mathbf{Z}-p\mathbf{Z}$.
(5) y_{sp^n}/v_1^j, $s\in\mathbf{Z}_2$, $n\ge 0$, $j\ge 1$, and $p^{k+1}|(j+a_{n+1})\le a_{n+2}$ if $s\in\mathbf{Z}_2^k$.

Here we make a note on the elements of (3). Since $s\in\mathbf{Z}_2^k$, we may put $s=tp^{k+2}-1$ for t with $p\nmid t$. Lemma 5.1 says that if $p^{k+1}|(j+a_{n+1}+1)\le a_{n+2}$,

$$\delta_0(x_{n+2+k}^t/p^{k+1}v_1^{j+a_{n+1}+1}) = \varepsilon t y_{(tp^{k+2}-1)p^n}/v_1^{j+1} + \cdots$$

where $\varepsilon=1$ if $n=0$, and $=2$ otherwise. Thus we have the elements in both of (1) and (5). Therefore, the second terms of the last two equations of Lemma 5.1 turn into the leading terms and give the elements of (3).

The cokernel of δ_0 is expressed by the generators of $H^1M_1^1$ other than those that appear above as the leading terms. For the generator of the form $v_2^{tp}V/v_1^j$, $v_2^{tp}\ V/v_1^{p-1}$ with $p\nmid(t+1)$ die in the cokernel by (4). Thus we have (iv).

Considering the negative statements, we deduce (ii) from (1) and (5), and (iii) from (2) and (3).

Lemma 5.2 gives the first half of (i), and the second half follows from Lemma 5.1.

Q.E.D.

6. SOME LEMMAS ON δ_1

Consider the short exact sequence

$$0\to M_1^1 \xrightarrow{\varphi} M_0^2 \xrightarrow{p} M_0^2 \to 0$$

and apply the functor $H^* -$ to it to obtain the Bockstein spectral sequence. The differentials d_r of the spectral sequence depend on the computation of the connecting homomorphism $\delta_t: H^tM_0^2 \to H^{t+1}M_1^1$.

As we have remarked in (4.10), we have a cocycle ζ in each cobar complex $\Omega_\Gamma^1 A/(p^{i+1}, v_1^{mp^i})$. So we have Lemma 6.1.

Lemma 6.1. *Let* $x \in H^{t-1}M_0^2$, and $\delta_t: H^tM_0^2 \to H^{t+1}M_1^1$, *the connecting homomorphism. Then*

$$\delta_t(x \otimes \zeta) = \delta_{t-1}(x) \otimes \zeta.$$

By the definition of δ_t, we obtain the following lemma.

Lemma 6.2. *Let w be an element of $E(2)_*$ such that $d_0(w) \equiv 0 \bmod (p^{i+1}, v_1^j)$. Then we have*

$$w\delta_t(x/p^iv_1^j) = \delta_t(wx/p^iv_1^j)$$

for $x/p^iv_1^j \in H^tM_0^2$.

Proof. This follows from the definition of δ_t and the calculation

$$d_1(wx/p^{i+1}v_1^j) = wd_1(x/p^{i+1}v_1^j). \qquad \text{Q.E.D.}$$

Since $d_0(v_1^{p^i}) \equiv 0 \bmod (p^{i+1}, v_1^j)$ even if $p^i < j$, we have the following corollary.

Corollary 6.3. *Let $\delta_t: H^tM_0^2 \to H^{t+1}M_1^1$ be the connecting homomorphism. Then we have*

$$v_1^{p^i}\delta_t(x/p^iv_1^j) = \delta_t(x/p^iv_1^{j-p^i})$$

for $x/p^iv_1^j \in H^tM_0^2$. *In particular,*

$$v_1^p\delta_t(x/pv_1^j) = \delta_t(x/pv_1^{j-p})$$

for $x/pv_1^j \in H^tM_0^2$.

In order to state the next corollary, we define integers $p(i, j)$ depending on integers i and j by

$$p(i,j) = \min\{n \in \mathbf{Z}: p^i \mid n \text{ and } j \le n\}.$$

Corollary 6.4. *For a cocycle $x/p^iv_1^j \in H^tM_0^2$, suppose that*

$$\delta_t(x/p^iv_1^j) = \sum \xi/v_1^k \neq 0$$

for the elements $\xi/v_1^k \in H^{t+1}M_1^1$ with $\xi/v_1 \neq 0$. Then $k \le p(i,j)$.

Proof. Corollary 6.3 implies that $v_1^{p(i,j)}\delta_t(x/p^iv_1^j) = 0$ since $j \le p(i,j)$. Thus we have $\xi/v_1^{k-p(i,j)} = 0$. If $k > p(i,j)$, then $\xi/v_1^{k-p(i,j)} \neq 0$, which is a contradiction. Q.E.D.

For the next section we introduce some more elements defined by

$$pV_n = v_1^{p^n}t_1^{p^{n+1}} - v_1^{p^{n+1}}t_1^{p^n} - d_0(v_2^{p^n}) \tag{6.5}$$

for $n \ge 1$, in which the right-hand side is divisible by p since $\eta_R(v_2) \equiv v_2 + v_1t_1^p - v_1^pt_1 \bmod (p, v_1)$. Then we have Lemma 6.6.

LEMMA 6.6. *In the complex* $\Omega_\Gamma^* A$,

$$V_{n+1} \equiv v_1^{p^n} V^{p^n} \mod (p)$$
$$\equiv -v_1^{p^n} v_2^{(p-1)p^n} t_1^{p^{n+1}} \mod (p, v_1)$$
$$d_1(V_n) \equiv 0 \mod (p^n, v_1^{p^n}).$$

Here V *is the element of* (3.5).

Proof. The first two are seen by direct calculations and the definition of V. We deduce the last one by the definition (6.5) and the following facts: $d_1 d_0 = 0$, $d_1(v_1^{p^n} t_1^{p^{n+1}}) \equiv 0$ and $d_1(v_1^{p^{n+1}} t_1^{p^n}) \equiv 0 \mod (p^{n+1}, v_1^{p^n})$, the map $p: \Omega_\Gamma^2 A/(p^n, v_1^{p^n}) \to \Omega_\Gamma^2 A/(p^{n+1}, v_1^{p^n})$ is monomorphic and $(p^{n+1}, v_1^{p^n})$ is an invariant ideal. Q.E.D.

Noticing that $d_0(v_2^{p^{n+2}}) \equiv v_1^{p^{n+2}} t_1^{p^{n+3}} - pV_{n+2} \mod (v_1^{p^{n+3}})$ by the definition (6.5), we obtain Lemma 6.7.

LEMMA 6.7. *Let* u, n *and* $l \geq 0$ *be integers such that* $p \nmid u$ *and* $n, l \geq 0$. *Then in the complex* $\Omega_\Gamma^1 A$,

$$d_0(v_2^{up^{n+l+2}}) \equiv -up^l v_2^{(up^{l-1}-1)p^{n+3}} V_{n+3}$$
$$\equiv up^l v_2^{(up^l-1)p^{n+2}} (v_1^{p^{n+2}} t_1^{p^{n+3}} - pV_{n+2})$$

$\mod (p^{l+2}, p^{l+1} v_1^{p^{n+2}+p^{n+1}}, v_1^{2p^{n+2}})$.

We have some relations on the elements V_n's.

LEMMA 6.8. *In the cobar complex* $\Omega_\Gamma^2 A/(p, v_1^2)$, $2t_1 \otimes V$ *is homologous to*

$$-v_1 v_2^p g_1 - v_1 v_2^{p-1} \zeta \otimes t_1^p.$$

Proof. By the definition of V, $\mod (p, v_1^2)$,

$$2t_1 \otimes V \equiv 2t_1 \otimes (-v_2^{p-1} t_1^p) + v_1 v_2^{p-2} t_1 \otimes t_1^{2p}$$
$$\equiv -2v_2^{p-1} t_1 \otimes t_1^p + 2v_1 v_2^{p-2} t_1^{p+1} \otimes t_1^p + v_1 v_2^{p-2} t_1 \otimes t_1^{2p}.$$

Direct computation by the formulae (3.6) and (3.10) brings us

$$d_1(-2v_2^{p-1} t_2) \equiv 2v_1 v_2^{p-2} t_1^p \otimes t_2 + 2v_2^{p-1} t_1 \otimes t_1^p + 2v_1 v_2^{p-1} T \mod (p, v_1^2).$$

The last term is homologous to $-2v_1 v_2^p g_1$ by (3.6), (3.9) and (3.13). We also see the following by Lemma 3.14:

$$d_1(v_1 v_2^{p-2} t_1^p t_2) \equiv v_1 v_2^{p-2} (v_2 \zeta \otimes t_1^p - v_2^2 g_1 - 2t_2 \otimes t_1^p - t_1 \otimes t_1^{2p})$$

$\mod (p, v_1^2)$. These together with the equation $t_1^p \otimes t_2 + t_1^{p+1} \otimes t_1^p - t_2 \otimes t_1^p = v_2^2 g_1 - v_2 \zeta \otimes t_1^p$ show

$$2t_1 \otimes V \equiv -v_1 v_2^p g_1 - v_1 v_2^{p-1} \zeta \otimes t_1^p \mod (p, v_1^2)$$

up to homology. Q.E.D.

LEMMA 6.9. *In the cobar complex* $\Omega_\Gamma^2 A/(p, v_1^2)$, $2t_1 \otimes V^p$ *is homologous to*

$$v_1 v_2^{p^2-1} (g_0 - \zeta \otimes t_1).$$

Proof. We see that $-2t_1\otimes V^p = 2v_2^{p^2-1}t_1\otimes t_1 - 2v_1v_2^{p^2-2}t_1^{p+1}\otimes t_1 - 2v_1v_2^{p^2-p-1}t_1 \otimes t_2^p$ by (3.5), (3.8) and (3.4). We also compute

$$d_1(v_2^{p^2-1}t_1^2) = -v_1v_2^{p^2-2}t_1^p\otimes t_1^2 - 2v_2^{p^2-1}t_1\otimes t_1$$
$$d_1(v_1v_2^{p^2-p-1}t_1t_2^p) = -v_1v_2^{p^2-p-1}(t_1^{p+1}\otimes t_1^{p^2} + t_1^p\otimes t_1^{p^2+1} + t_1\otimes t_2^p + t_2^p\otimes t_1)$$

by (3.4), (3.6) and (3.10). Since $v_1t_1^{p^2} = v_1v_2^{p-1}t_1$, these computations show that $2t_1\otimes V^p$ is homologous to $-v_1v_2^{p^2-p-1}(t_1^p\otimes t_1^{p^2+1} + 2t_2^p\otimes t_1)$. Thus Lemma 3.14 implies the lemma.

Q.E.D.

For the last lemma of this section, we prepare the following.

LEMMA 6.10 (Shimomura and Yabe [17, Lemma 3.4]). *We have the elements $w(s,n)\in v_1^{-1}\Gamma$ such that $v_1^{a_n}w(s,n)\in\Gamma$, and*

$$d_1(w(s,0)) \equiv 2v_2^{sp-p}V\otimes\sigma - v_1v_2^{sp}g_1 \mod (p,v_1^2)$$
$$d_1(w(s,n)) \equiv 2v_2^{(s-1)p^{n+1}}V^{p^n}\otimes\sigma - \frac{(-1)^n}{2}v_1^{p^{n-1}+A_{n-1}+2}v_2^{e(n,sp)}g_1 \quad (n\ge 1)$$

mod $(p,v_1^{p^{n-1}+A_{n-1}+3})$. *Here $a_n = p^n + p^{n-1} - 1$ and $e(n,s) = sp^n - (p^n-1)/(p-1)$.*

LEMMA 6.11. *In the cobar complex $\Omega_\Gamma^2 A/(p,v_1^{p^{n-2}+A_{n-2}+3})$, $2t_1\otimes v_2^{(s-1)p^n}V^{p^{n-1}}$ for $n\ge 2$ is homologous to*

$$-2v_1^{p^{n-2}}\zeta\otimes y_{(sp^2-1)p^{n-2}} - \frac{(-1)^{n-1}}{2}v_1^{p^{n-2}+A_{n-2}+2}x_n^sG_n.$$

Proof. Since $2t_1\otimes v_2^{(s-1)p^n}V^{p^{n-1}} = 2\sigma\otimes v_2^{(s-1)p^n}V^{p^{n-1}} + v_1\zeta\otimes v_2^{(s-1)p^n}V^{p^{n-1}}$, Lemma 4.25 gives

$$2t_1\otimes v_2^{(s-1)p^n}V^{p^{n-1}} = 2\sigma\otimes v_2^{(s-1)p^n}V^{p^{n-1}} - 2v_1\zeta\otimes v_1^{p^{n-2}}y_{(sp^2-1)p^{n-2}}.$$

Now apply Lemma 6.10 to it, and we have the result by the property (4.23) of G_n.

Q.E.D.

7. COMPUTATION OF δ_1

We will compute the δ_1-image of the generators of $H^1M_1^1$. Let x/v_1^j denote one of the generators of the cokernel of $\delta_{t-1}: H^{t-1}M_0^2 \to H^tM_1^1$ that are given in Proposition 5.7. Then $x(j,1) = x/pv_1^j$ gives a nonzero element of $H^tM_0^2$. Suppose inductively that there exists a nonzero element $x(j,l)\in H^1M_0^2$ such that $px(j,l) = x(j,l-1)$. If $\delta_t(x(j,l)) = 0$, then there exists a cochain ρ such that $d_t(x(j,l)/p) = d_t(\rho)$. Put now

$$x(j,l+1) = x(j,l)/p - \rho$$

and we see that $x(j,l+1)$ is a cocycle and

$$px(j,l+1) = x(j,l).$$

Thus this proceeds until we have an integer i (maybe infinity) such that

$$\delta_t(x(j,i)) \ne 0.$$

We will find such an integer i for each generator x/v_1^j of the cokernel.

We first consider the elements of X of (4.24). Notice that an element $x_n^s\zeta\otimes\zeta/v_1^j$ is bounded by $-\frac{1}{2}x_n^s\zeta^2/v_1^j$ as long as $j\le a_n$. By Lemmas 5.1 and 6.1, we then have the following result.

PROPOSITION 7.1. *The connecting homomorphism $\delta_1: H^1M_0^2\to H^2M_1^1$ sends an element $x_{i+k}^s\zeta(j,i+1)=x_{i+k}^s\zeta/p^{i+1}v_1^j$ for $s\in\mathbf{Z}-p\mathbf{Z}$, $0\le i\le k$ and $1\le m\le a_{k-i}$ for $j=mp^i$ to*

$$-y_s\otimes\zeta/v_1^2 \quad \text{if } k=0$$

$$-my_{sp^{i+1}}\otimes\zeta/v_1^{j+1}-sv_2^{sp^{i+1}-p}V\otimes\zeta/v_1^{j-1}+\cdots \quad \text{if } k=1$$

$$-my_{sp^{i+2}}\otimes\zeta/v_1^{j+1}+sy_{sp^{i+2}-1}\otimes\zeta/v_1^{j-p}+sv_2^{sp^{i+2}-p-1}V\otimes\zeta/v_1^{j-p-2}+\cdots \quad \text{if } k=2$$

$$-my_{sp^{i+k}}\otimes\zeta/v_1^{j+1}+2sy_{(sp^{i+2}-1)p^{k-2}}\otimes\zeta/v_1^{j-a_{k-1}}+\cdots \quad \text{if } k>2.$$

Next we study the elements in Y of (4.24).

PROPOSITION 7.2. *For each integer t, we have the cocycles $y'_{tp}(j,1)=v_2^{tp}V/pv_1^j$ and $y'_{tp}(p-1,2)=v_2^{tp}V/p^2v_1^{p-1}-\frac{1}{2}v_2^{(t-1)p+1}t_1^{2p^2}/p^2v_1$ and*

$$\delta_1(y'_{tp}(j,1))=\frac{j+1}{2}(x_1^{t+1}G_1/v_1^j+v_2^{tp}V\otimes\zeta/v_1^j)+\cdots$$

$$\delta_1(y'_{tp}(p-1,2))=\tfrac{1}{2}(x_1^{t+1}G_1/v_1^{p-1}+v_2^{tp}V\otimes\zeta/v_1^{p-1})+\cdots$$

Proof. We get the congruences $d_0(v_2^{tp})\equiv tv_2^{tp-p}d_0(v_2^p)$ and $d_0(v_2^p)\equiv -pv_1V \bmod (p^2,v_1^p)$ by Lemma 3.11. These congruences lead us to the equation $v_2^{tp}V/pv_1^j=-d_0(v_2^{(t+1)p})/(t+1)p^2v_1^{j+1}$ in our cobar complex $\Omega_\Gamma M_0^2$ if $j+1\le p$. Therefore the definition of the connecting homomorphism shows that

$$\delta_1(v_2^{tp}V/pv_1^j)=\varphi^{-1}(\xi_j)$$

for $\xi_j=d_1(-d_0(v_2^{(t+1)p})/(t+1)p^3v_1^{j+1})$. Put $t+1=sp^u$ for s and u with $p\nmid s$ and $u\ge 0$. Then ξ_j equals $-d_1(v_1^{kp^{u+2}-j-1}d_0(v_2^{(t+1)p}))/(t+1)p^3v_1^{kp^{u+2}}$, and we have

$$\xi_j=-(j+1)v_2^{tp}t_1\otimes V/pv_1^{j+1}\quad (-v_2^{tp}t_1\otimes t_1^{p^2}/pv_1 \text{ if } j=p-1)$$

by the equation $d_1d_0=0$. Now applying Lemma 6.8, we have the case for $j<p-1$ by definition of the elements. The above computation is applied for the case $j=p-1$ by setting $\xi_{p-1}=d_1(-d_0(v_2^{(t+1)p})/(t+1)p^4v_1^p)$, and we have

$$\xi_{p-1}=(t_1/pv_1^p)\otimes v_2^{tp}V-(t_1/p^2v_1)\otimes v_2^{tp}t_1^{p^2}.$$

We also compute $d_1(v_2^{(t-1)p+1}t_1^{2p^2}/p^2v_1)=-(\eta_R(v_2^{(t-1)p+1})t_1/pv_1^2)\otimes t_1^{2p^2}+(v_2^{(t-1)p}t_2/pv_1)\otimes t_1^{2p^2}-2(v_2^{1-p}t_1^{p^2}/p^2v_1)\otimes v_2^{tp}t_1^{p^2}-v_2^{(t-1)p+1}T^p\Delta(t_1^{p^2})/pv_1$ by (3.4). Since the last congruence of (3.4) also gives $v_2^{1-p}t_1^{p^2}/p^2v_1=t_1/p^2v_1-v_2^{-p}t_3/pv_1$, the second term of ξ_{p-1} is homologous to an element X/pv_1^2 for some $X\in BP_*/(p,v_1)$, which is denoted by $\cdots$ in the proposition.

Q.E.D.

For the case $j=0$, we also have Proposition 7.3.

PROPOSITION 7.3 (Shimomura and Yabe [17, Prop. 4.4]). *For each $sp^n\in\mathbf{Z}(0)\cup\mathbf{Z}(2)$ with $n\ge 0$ and $p\nmid s$, we have a cocycle $y_{sp^n}(1,n+1)$ such that*

$$\delta_1(y_{sp^n}(1,n+1))=\frac{s}{2}(x_0^{sp^n}G_0/v_1-y_{sp^n}\otimes\zeta/v_1).$$

We introduce here elements:

$$\sigma_{j,l} = y_{j,l} - \tfrac{1}{2}\zeta/p^l v_1^{j-1}$$

for positive integers j and l. Here

$$y_{j,l} = \sum_{k>0} \binom{k+j-2}{k-1} \frac{(-1)^{k-1} t_1^k}{kp^{l+1-k} v_1^{j-1+k}}$$

is given in [6] to satisfy

$$p^{l-1} y_{j,l} = t_1/pv_1^j, \quad \text{and } d_1(y_{j,l}) = 0$$

for any j and l. Then we see that

$$p^{l-1}\sigma_{j,l} = \sigma/pv_1^j,$$

and the following lemma holds.

LEMMA 7.4. $d_1(\sigma_{kp^i+1,l}) = 0$ *if* $l \le i+1$, *and* $= \tfrac{1}{2}kt_1 \otimes \zeta/pv_1^{kp^i+1}$ *if* $l = i+2$.

Proof. Note that $d_1(y_{j,l}) = 0$ and $d_1(\zeta) \equiv 0 \bmod J$ for any ideal $J = (p^l, v_1^m)$ by the convention on ζ. We further see that $d_0(1/p^l v_1^{kp^i}) = -kt_1/p^{l-i-1} v_1^{kp^i+1}$ in $\Omega^1 M_0^2$ if $l \le i+2$. Thus the lemma follows. Q.E.D.

For a while we abuse the notation: For an integer $s \in \mathbf{Z} - p\mathbf{Z}$, $y_{sp^n}(j,l)$ denotes a cocycle whose leading term is $v_2^{sp^n} t_1/p^l v_1^j$, if such a cocycle exists. It would be possible that the cocycle is a coboundary.

PROPOSITION 7.5. *Let s, n, i and k be integers with $s \in \mathbf{Z} - p\mathbf{Z}$, $k \ge 1$, $n \ge i \ge 0$ and $kp^i < A_{n-i} + 2$ and put $m = sp^n$. Then we have cocycles $y_m(kp^i+1,l)$ for $0 \le l \le i+1$, whose δ_1-images are given by $\delta_1(y_m(kp^i+1,l)) = 0$ for $l \le i$ and*

$$\delta_1(y_m(kp^i+1,i+1)) = \frac{k}{2} y_m \otimes \zeta/v_1^{kp^i+1} + s\lambda_{n-i} x_{n-i}^{sp^i} G_{n-i}/v_1^{kp^i - A_{n-i-1}-1}$$

$$+ \left(-\frac{ks}{2}\lambda_n x_n^s G_n/v_1^{k-A_{n-1}-1} \quad \text{if } i = 0 \right) + \cdots$$

where $\lambda_n = (-1)^n/4$ for $n > 1$, $= -\tfrac{1}{2}$ for $n = 1$, and $\cdots$ denotes an element killed by $v_1^{k-2p^{n-1}}$.

Proof. For $i = 0$, we see that $y_m(k+1,1) = \varphi(y_m/v_1^{k+1})$ is a cocycle, and consider $\omega(m;k+1,2) = \eta_R(v_2^m)\sigma_{k+1,2} - \tfrac{1}{2}d_0(v_2^m)/p^3 v_1^k$. Then $p\omega(m;k+1,2) = y_m(k+1,1)$ for $k \le (p-1)p^{n-1}$ with $n = v_p(m)$ by Lemma 4.25, and $\delta_1(y_m(k+1,1)) = \varphi^{-1}(d_1(\omega(m;k+1,2)))$. Use (3.12) and Lemma 7.4 to get

$$d_1(\eta_R(v_2^m)\sigma_{k+1,2}) = \frac{k}{2} t_1 \otimes \zeta\eta_R(v_2^m)/pv_1^{k+1} - \sigma_{k+1,2} \otimes d_0(v_2^m)$$

$$d_1(d_0(v_2^m)/p^3 v_1^k) = -kt_1 \otimes d_0(v_2^m)/p^2 v_1^{k+1} + \binom{k+1}{2} t_1^2 \otimes d_0(v_2^m)/pv_1^{k+2}.$$

Lemma 6.7 says that $d_0(v_2^m) \equiv -spv_2^{m-p^n} V_n \bmod (p^2, v_1^{p^n})$ for $n = v_p(m)$, in which $V_n \equiv v_1^{p^{n-1}} V^{p^{n-1}} \bmod (p)$ by Lemma 6.6. Substituting this to the above equations gives the following:

$$d_1(\omega(m; k+1, 2)) = \frac{k}{2} v_2^m t_1 \otimes \zeta / p v_1^{k+1} + s v_2^{m-p^n} \sigma \otimes V^{p^{n-1}} / p v_1^{k+1-p^{n-1}}$$

$$- \frac{sk}{2} v_2^{m-p^n} t_1 \otimes V^{p^{n-1}} / p v_1^{k+1-p^{n-1}}$$

which is homologous to $\frac{1}{2}ky_m \otimes \zeta/pv_1^{k+1} + s(1-k/2)v_2^{m-p^n}\sigma \otimes V^{p^{n-1}}/pv_1^{k+1-p^{n-1}}$ by Lemma 4.25, noticing that $v_2^m\zeta \otimes \zeta/pv_1^{k+1}$ is homologous to zero. This gives $\delta_1(y_m(k+1,1))$ for $k \leq (p-1)p^{n-1}$. For $k > (p-1)p^{n-1}$ with $k > p$ or $n > 1$, use Corollary 6.3 to find the leading term. In fact, we have an integer e such that $0 < k - ep \leq (p-1)p^{n-1}$ and consider $v_1^{ep}\delta_1(y_m(k+1,1)) = \delta_1(y_m(k+1-ep,1))$. A direct calculation also gives the case $n=1$ and $k=p$. Thus we have the case $i=0$. For the additional case for $m \in \mathbf{Z}(2)$ is also seen in the same manner.

Now turn to the case $i > 0$. Assume that $kp^i + 1 \leq A_{n-i} + 2$ and put $y_m(kp^i+1, l) = \eta_R(v_2^m)\sigma_{kp^i+1,l}$ for $l > 0$. Then we obtain that $d_1(y_m(kp^i+1,l)) = 0$ for $l \leq i$. We further see that $y_m(kp^i+1, i+1)$ is also a cocycle for $kp^i < a_{n-i}$, if we put $y_m(kp^i+1,i+1) = \eta_R(x_{n-i}^{sp^i})\sigma_{kp^i+1,i+1}$.

Inductively, we have a cocycle $y_m(kp^i+1,i+1)$ for $kp^i+1 \leq A_{n-i}$ such that $py_m(kp^i+1,i+1) = y_m(kp^i+1,i)$ and $v_1^{ap^i}y_m(kp^i+1,i+1) = y_m((k-a)p^i+1,i+1)$ for a with $A_{n-i-1}+2 < (k-a)p^i+1 \leq p^{n-i}+1$. Then to tell the leading term of $\delta_1(y_m(kp^i+1,i+1))$ suffices to show the one only for $kp^i \leq p^{n-i}$ by virtue of Corollary 6.3. Now compute $d_1(\eta_R(v_2^m)\sigma_{kp^i+1,i+2})$ for $kp^i \leq p^{n-i}$ by (3.12) and Lemma 7.4 as above, and obtain $\frac{1}{2}kv_2^m t_1 \otimes \zeta/pv_1^{kp^i+1} + sv_2^{m-p^{n-i}}\sigma \otimes V^{p^{n-i-1}}/pv_1^{kp^i-p^{n-i-1}+1} + \xi/pv_1$. The proposition then follows from Lemmas 4.25 and 6.10. Q.E.D.

PROPOSITION 7.6. *Let t and j be integers with $1 \leq j \leq p^2+1$. Then there exist cocycles $y_{tp^2-1}(j,l)$ for $1 \leq l \leq i+1$ such that*

$$\delta_1(y_{tp^2-1}(j,1)) = \frac{j}{2} y_{tp^2-1} \otimes \zeta/v_1^j + \cdots$$

$$\delta_1(y_{tp^2-1}(p^2,1)) = x_0^{(tp-1)p}G_0/v_1 \quad (p^2 \nmid t)$$

$$= y_{(tp-1)p} \otimes \zeta/v_1 \quad (p^2 | t)$$

$$\delta_1(y_{tp^2-1}(kp,2)) = \frac{k+1}{2} y_{tp^2-1} \otimes \zeta/v_1^{kp}$$

$$\delta_1(y_{tp^2-1}(p^2-p,3)) = \tfrac{1}{2} y_{tp^2-1} \otimes \zeta/v_1^{p^2-p}.$$

Proof. First consider the case $j = p^2$, where we put

$$y_{tp^2-1}(p^2,1) = d_0(x_2^t)/tp^2v_1^{p^2+p}$$

and obtain the desired equation from (4.8) and Proposition 7.3.

By the definition (6.5), we see that $d_0(v_2^{tp^2}) \equiv -tpv_2^{(t-1)p^2}V_2 \bmod (p^2, v_1^{p^2})$. So if we put

$$y_{tp^2-1}(kp^i-p; i+1) = d_0(v_2^{tp^2})/tp^{i+2}v_1^{kp^i} \quad (+\, v_2^{(t+p-2)p}t_2/pv_1 \text{ if } i=2)$$

then $p^i y_{tp^2-1}(kp^i-p, i+1) = d_0(v_2^{tp^2})/tp^2v_1^{kp^i} = -v_2^{(t-1)p^2}V^p/pv_1^{kp^i-p} = y_{tp^2-1}/pv_1^{kp^i-p}$ by Lemma 6.6 and (4.28). Besides, if $i < 2$,

$$d_1(y_{tp^2-1}(kp^i-p, i+1)/p) = -kt_1 \otimes d_0(v_2^{tp^2})/tp^2v_1^{kp^i+1}$$

which equals $kt_1 \otimes v_2^{(t-1)p^2}V^p/pv_1^{kp^i-p+1}$. Use now Lemma 6.9 to obtain the proposition, noticing that some element bounds the element corresponding to g_0 in Lemma 6.9, which can be read off from the structure of $H^2M_1^1$ of (4.24). Comparing degrees, the structure of $H^2M_1^1$ also induces the other fact that the δ_1-image has no lower term. A similar argument also shows the case $i = 2$. Q.E.D.

Thus we have computed the δ_1-images of the elements associated to the generators y_m/v_1^j of $Y_{1,C}$ with $p \nmid m$ or $p \nmid (j+1)$.

LEMMA 7.7. *Let n, t, k and i be integers such that $n, k, i > 0$ and $kp^i + 1 \le A_n' + 2 = p^{n+2} - p^n + A_n + 2$. Then $y_{(tp^2-1)p^n}(kp^i+1,1)$ is redefined to be a sum of $\frac{1}{2}d_0(x_{n+2}^t)/tp^2v_1^{kp^i+p^{n+1}+p^n}$ and $-y_{(tp-1)p^{n+1}}(kp^i+p^n-p^{n+2}+1,2)$.*

Proof. In this proof, we put $m = (tp-1)p^{n+1}$ and $jp^i = kp^i + p^n - p^{n+2}$. We read off that $y_m(jp^i+1,2) = y_{(tp-1)p^{n+1}}(kp^i+p^n-p^{n+2}+1,2)$ is a cocycle from Proposition 7.5, since $kp^i + 1 \le p^{n+2} - p^2 + A_n + 2$. Furthermore, $y_m(jp^i+1,2)$ has the leading term $\eta_R(v_2^m)\sigma_{jp^i+1,2}$ by the proof of Proposition 7.5. We compute

$$d_0(x_{n+2}^t) \equiv 2tv_1^{a_{n+2}}v_2^{(tp-1)p^{n+1}}\sigma \mod(p^{l+1}, v_1^{2+a_{n+2}})$$
$$\equiv 2tpv_1^{a_{n+1}}v_2^{(tp^2-1)p^n}\sigma \mod(p^{l+2}, v_1^{2+a_{n+1}})$$

by (4.8) and the binomial theorem, where $l = v_p(t)$.

Put now

$$\xi = \tfrac{1}{2}d_0(x_{n+2}^t)/tp^2v_1^{kp^i+p^{n+1}+p^n} - y_m(jp^i+1,2).$$

Then the above statements say that $p\xi = 0$, ξ is a cocycle and ξ has the leading term $v_2^{(tp^2-1)p^n}\sigma/pv_1^{kp^i+1}$. These properties are those of $y_{(tp^2-1)p^n}(kp^i+1,1)$, and so we redefine $y_{(tp^2-1)p^n}(kp^i+1,1) = \xi$. Q.E.D.

PROPOSITION 7.8. *Consider an integer $m = (tp^2-1)p^n \in \mathbf{Z}(2)$ for $t, n \in \mathbf{Z}$ with $n > 0$. If i and k are positive integers with $kp^i < p^{n+2} - p^n + A_{n-i+1} + 2$, then we have cocycles $y_m(kp^i+1,l)$ for $0 \le l \le i$, whose δ_1-images are given by $\delta_1(y_m(kp^i+1,l)) = 0$ for $l < i$ and*

$$\delta_1(y_m(kp^i+1,i)) = -\lambda_n x_{n-i+1}^{(tp-1)p^i}G_{n-i+1}/v_1^{kp^i-p^{n+2}+p^n-A_{n-i}-1} + \cdots.$$

Furthermore, if $kp^i \le p^{n+2} - p^n$, then we have more cocycles such that

$$\delta_1(y_m(kp^i+1,i+1)) = \frac{k+p^{n-i}}{2}y_m \otimes \zeta/v_1^{kp^i+1} + \cdots$$

$$\delta_1(y_m(p^{n+2}-p^n+1,n+1)) = \tfrac{1}{2}x_0^{(tp-1)p^{n+1}}G_0/v_1 \quad (p^{n+2} \nmid t)$$
$$= \tfrac{1}{2}y_{(tp-1)p^{n+1}} \otimes \zeta/v_1 \quad (p^{n+2} | t)$$

$$\delta_1(y_m(kp^{n+1}-p^n+1,n+2)) = \frac{k+1}{2}y_m \otimes \zeta/v_1^{kp^{n+1}-p^{n+1}+1} + \cdots$$

$$\delta_1(y_m(p^{n+2}-p^{n+1}-p^n+1,n+3)) = \tfrac{1}{2}y_m \otimes \zeta/v_1^{p^{n+2}-p^{n+1}-p^n+1}.$$

In the above equations, $\cdots$ denotes a lower term.

Proof. For the case $kp^i \le p^{n+2} - p^n$, we put

$$y_m(kp^i+1,l) = \tfrac{1}{2}d_0(v_2^{tp^{n+2}})/tp^{l+1}v_1^{kp^i+p^{n+1}+p^n}$$

if $kp^i \neq p^{n+2} - p^n$, and

$$y_m(p^{n+2} - p^n + 1, l) = \tfrac{1}{2} d_0(x_{n+2}^t)/tp^{l+1} v_1^{p^{n+2}+p^{n+1}}.$$

This is guaranteed by Lemmas 4.25 and 6.7. Note that this element y may differ from the one y in Proposition 7.5 and we will denote the latter by $\tilde{y}$. Thus if $kp^i < p^{n+2} - p^n$, we see that $\delta_1(y_m(kp^i + 1, l)) = 0$ for $l \leq i$ as we have seen above. For $l = i + 1$, we deduce the results from (3.12) and Lemmas 6.7 and 6.11 using the formula $d_0(1/p^{i+2} v_1^{kp^i+p^{n+1}+p^n}) = -(k + p^{n-i})t_1/pv_1^{kp^i+p^{n+1}+p^n+1}$. Even in the case $kp^i = p^{n+2} - p^n$, we have the same results as above by (4.8) and Proposition 7.3. Furthermore, a similar computation gives the case $i = n$ and $p|(k + 1)$ and the case $i = n$ and $p^2|(k + p + 1)$. Note that the last condition $i = n$ and $p^2|(k + p + 1)$ is equivalent to $k = p^2 - p - 1$.

Turn now to the case $kp^i > p^{n+2} - p^n$. Then Lemma 7.7 enables us to define

$$y_m(kp^i + 1, l) = \tfrac{1}{2} d_0(v_2^{tp^{n+2}})/tp^{l+1} v_1^{kp^i+p^{n+1}+p^n} - \tilde{y}_{(tp-1)p^{n+1}}(kp^i + p^n - p^{n+2} + 1, l + 1)$$

where $\tilde{y}$ denotes the element y in Proposition 7.5 as we noted above. We use the notation $\tilde{y}$ here in order to distinguish these y's appearing in both of Propositions 7.5 and 7.8. Then the first term is a cocycle for $l \leq i$ and mapped to $\tfrac{1}{2}k\, y_{(tp-1)p^{n+1}} \otimes \zeta/pv_1^{kp^i+p^n-p^{n+2}+1}$ for $l = i + 1$ by d_1 as we have seen above. For the second term, use Proposition 7.5 to see that it is a cocycle for $l \leq i$, and

$$\begin{aligned}\delta_1(y_m(kp^i + 1, i)) &= \frac{k}{2} y_{(tp-1)p^{n+1}} \otimes \zeta/v_1^{kp^i+p^n-p^{n+2}+1} - \frac{k}{2} y_{(tp-1)p^{n+1}} \otimes \zeta/v_1^{kp^i+p^n-p^{n+2}+1} \\ &\quad - s\lambda_{n-i} x_{n-i+1}^{(tp-1)p^i} G_{n-i+1}/v_1^{kp^i-p^{n+1}-p^n-A_{n-i}-1}\end{aligned}$$

as desired. Q.E.D.

8. $H^1 M_0^2$

Let $\delta_t : H^t M_0^2 \to H^{t+1} M_1^1$ be the connecting homomorphism. Then we introduce some notation:

For a submodule M of $H^{t+1}M_1^1$, M_C (resp. M_I) denotes the intersection of M and cokernel (resp. image) of δ_1 up to isomorphism. M^∞ denotes the submodule of $H^{t+1}M_0^2$ consisting of $x \in H^{t+1}M_0^2$ such that $p^n x \in \varphi(M)$ for some n.

We also denote

$$X\zeta = X \otimes \mathbf{Z}_{(p)}\{\zeta\} \quad \text{and} \quad X_\infty\zeta = X_\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}.$$

Then Proposition 5.7 gives

$X\zeta_C = F_p\{x_n^s\zeta/v_1^j : s \in \mathbf{Z} - p\mathbf{Z},\ n \geq 0,\ 1 \leq j \leq a_n$ such that

$\qquad j > a_{n-i}$ if $p^i | j$ for either $s \in \mathbf{Z}_1$, or $s \in \mathbf{Z}_2^k$ and $p^{k+1} | j\}$

$X_\infty\zeta_C = X_\infty\zeta$

$Y_{0,C} = F_p\{y_{sp^n}/v_1^j : s \in \mathbf{Z}_0,\ n \geq 0,\ j \leq A_n + 2$, such that

$\qquad j = 1$ or $j - 1 > a_{n-i}$ if $p^i | (j - 1)\}$

$Y_{1,C} = F_p\{y_{sp^n}/v_1^j : s \in \mathbf{Z}_2^k,\ k \geq 0,\ n \geq 0,\ j \leq A'_n + 2$, such that

$\qquad j = 1, j - 1 > a_{n-i}$ if $p^i | (j - 1)$ and $p^{k+1} \nmid (j + a_{n+1})$, or $j > a_{n+2} - a_{n+1}\}$

$Y_C = F_p\{v_2^{sp} V/v_1^j : s \in \mathbf{Z},\ 1 \leq j \leq p - 1,\ p | (s + 1)$ if $j = p - 1\}$

$Y_{\infty,C} = F_p\{t_1/v_1\}$.

Since $X\zeta_C^\infty \subset X^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}$, we have

$$X\zeta_C^\infty = \mathbf{Z}_{(p)}\{x_n^s\zeta/p^{i+1}v_1^j : s \in \mathbf{Z} - p\mathbf{Z}, j > 0, p^i | j \le a_{n-i}$$
$$\text{either } p^{i+1} \nmid j \text{ or } j > a_{n-i-1}, \text{ and}$$
$$p^{i+1}|j \text{ if either } s \in \mathbf{Z}_1 \text{ or } s \in \mathbf{Z}_2^k \text{ and } p^{k+1}|j\}$$

by Proposition 7.1. We also see that

$$X_\infty\zeta_C^\infty = X_\infty^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}.$$

We have more modules:

$$Y_{0,C}^\infty = \mathbf{Z}_{(p)}\{y_{sp^n}(kp^i + 1, i + 1) : y_{sp^n}/v_1^{kp^i+1} \in Y_{0,C},$$
$$\text{for } k = 0,\ i = n, \text{ and}$$
$$\text{for } k > 0,\ kp^i + 1 \le A_{n-i} + 2,$$
$$kp^i + 1 > a_{n-i} \text{ if } p \nmid k, \text{ and } > A_{n-i-1} + 2 \text{ otherwise}\}.$$

$$Y_{1,C}^\infty = \mathbf{Z}_{(p)}\{y_{(tp^2-1)p^n}(kp^i + 1, l) : y_{(tp^2-1)p^n}/v_1^{kp^i+1} \in Y_{1,C},$$
$$l = n + 1 \text{ if } k = 0;$$
$$\text{For } k > 0,$$
$$l = i > 0 \text{ for } p^{n+2} - p^n < kp^i < p^{n+2} - p^n + A_{n-i+1} + 2 \text{ and}$$
$$p^{n+2} - p^n + A_{n-i} + 2 \le kp^i \text{ if } p|k;$$
$$l = i + 1 \text{ for } i = 0 \text{ and } p \nmid (k + p^{n-i}), \text{ or}$$
$$\text{for } kp^i \le p^{n+2} - p^n,\ p \nmid (k + p^{n-i}) \text{ and } 0 < i \le n;$$
$$l = n + 2 \text{ for } i = n,\ k \le p^2 - 1,\ p|(k + 1) \text{ and } k \ne p^2 - p - 1; \text{ and}$$
$$l = n + 3 \text{ if } i = n \text{ and } k = p^2 - p - 1\}.$$

$$Y_C^\infty = \mathbf{Z}_{(p)}\{y'_{tp}(j, l) : v_2^{tp}V/v_1^j \in Y_C,$$
$$l = 1 \text{ if } j < p - 1, \text{ and } l = 2 \text{ if } j = p - 1\}.$$

$$Y_{\infty,C}^\infty = \mathbf{Z}_{(p)}\{y_{1,\infty}\} \cong \mathbf{Q}/\mathbf{Z}_{(p)}.$$

Moreover, by Propositions 7.2, 7.3, 7.5, 7.6 and 7.8, we divide $Y_{0,C}^\infty$ and $Y_{1,C}^\infty$ into two submodules, respectively:

$$Y_{0,C}^{\infty;G} = \mathbf{Z}_{(p)}\{y_{sp^n}(kp^{i+1} + 1, i + 1) : y_{sp^n}/v_1^{kp^{i+1}+1} \in Y_{0,C},\ k \ne 0,$$
$$A_{n-i-1} + 1 < kp^{i+1} \le A_{n-i} + 1 \quad \text{for } i \ge 0\}.$$

$$Y_{0,C}^{\infty;Y} = \{0\} \cup Y_{0,C}^\infty - Y_{0,C}^{\infty;G}.$$

$$Y_{1,C}^{\infty;G} = \mathbf{Z}_{(p)}\{y_{sp^n}(kp^{i+1} + 1, i + 1) : y_{sp^n}/v_1^{kp^{i+1}+1} \in Y_{1,C},\ k \ne 0,$$
$$p^{n+2} - p^n + A_{n-i-1} + 1 < kp^{i+1} \le p^{n+2} - p^n + A_{n-i} + 1 \quad \text{for } i \ge 0\}.$$

$$Y_{1,C}^{\infty;Y} = \{0\} \cup Y_{1,C}^\infty - Y_{1,C}^{\infty;G}.$$

We summarize the results of the previous section as follows.

PROPOSITION 8.1. *The connecting homomorphism δ_1 sends an element $y/p^iv_1^j$ of $Y_{0,C}^{\infty;Y} \oplus Y_{1,C}^{\infty;Y} \oplus Y_C^\infty$ to the element $y \otimes \zeta/v_1^j$ of $H^2M_1^1$. An element $y/p^iv_1^j$ of $Y_{0,C}^{\infty;G} \oplus Y_{1,C}^{\infty;G}$*

is mapped to an element of $G \subset H^2M_1^1$ *by* δ_1. *Furthermore,* X^∞ *and* X_∞^∞ *are sent to* $Y_{0,I} \oplus Y_{1,I} \oplus Y_I$ *and* Y_∞, *respectively, and* $Y_{\infty,C}^\infty$ *to* 0.

Now we have the following theorem.

THEOREM 8.2. $H^1M_0^2$ *is a* $\mathbf{Z}_{(p)}$*-module isomorphic to*

$$Y_{0,C}^\infty \oplus Y_{1,C}^\infty \oplus Y_C^\infty \oplus Y_{\infty,C}^\infty \oplus X\zeta_C^\infty \oplus (X_\infty^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}). \tag{8.3}$$

Proof. We will prove this by Lemma 4.3. Let B^1 be the module (8.3) and the map $f: B^1 \to H^1M_0^2$ the inclusion. Since the cokernel of δ_0 is isomorphic to the image of φ, φ induces the map $\varphi: H^1M_1^1 \to B^1$ by the definition of the modules M_C. It is easy to see that $pB^1 \subset B^1$. Thus we have the commutative diagram of Lemma 4.3.

Now it is sufficient to show that the sequence including B^1 is exact. It follows from the exact couple of the Bockstein spectral sequence that the sequence $H^1M_1^1 \to B^1 \to B^1$ is exact. To see that the sequence $B^1 \to B^1 \xrightarrow{\delta_1} H^2M_1^1$ is exact, we assume that a linear combination $\sum \xi$ of the elements of B^1 maps to zero by δ_1. If $\delta_1(\xi) = 0$, then there exists $\xi' \in B^1$ such that $\xi = p\xi'$ by the definition of B^1. Furthermore, if the sum of ξ's with $\delta_1(\xi) \neq 0$ is null, then there is some nontrivial relation between these elements, which is a contradiction to Proposition 8.1. In fact, the generators in $H^2M_1^1$ are linearly independent. Thus the linear combination does not have a term ξ such that $\delta_1(\xi) \neq 0$, and so it is in the image of p. Q.E.D.

9. $H^2M_0^2$

In order to state the structure of $H^2M_0^2$, we divide the module G into two parts: one is G_C and the other is G_I. Propositions 7.3, 7.5, 7.6 and 7.8 show that

$$\begin{aligned} G_C = \mathbf{Z}_{(p)}\{& x_0^{sp^k} G_0/v_1, x_n^s G_n/v_1^j \colon k \geq 0,\ n > 0,\ s+1 \in \mathbf{Z} - p\mathbf{Z}, \\ & 1 \leq j \leq a_n, \text{ and for } n > 0, \\ & p^{i+1} \nmid (j + A_{n-i-1} + 1) \text{ if } s = up^i \in \mathbf{Z}(0), \text{ or} \\ & p^i \nmid (j + A_{n-i} + 1) \text{ if } s = up^i \in \mathbf{Z}(2) \text{ and } i > 0\}. \end{aligned}$$

Now we compute the connecting homomorphism $\delta_2: H^2M_0^2 \to H^3M_1^1$.

PROPOSITION 9.1 (Shimomura and Yabe [17, Prop. 4.1 and 4.3]).

$$\delta_2(x_1^s G_1/pv_1^j) = -\frac{j+1}{2} x_1^s G_1 \otimes \zeta/v_1^j$$

$$\delta_2(x_1^s G_1/p^2 v_1^{p-1}) = -\tfrac{1}{2} x_1^s G_1 \otimes \zeta/v_1^{p-1}.$$

In order to generalize the results of [17, Prop. 4.1], we redefine the generators $x_n^s G_n/v_1^j$ of $H^2M_1^1$ as for the generator y_m/v_1^j in Section 7.

Recall that the generator $x_n^s G_n/v_1^j$ with $n > 0$ is characterized by the two conditions: $v_1^{j-1} x_n^s G_n/v_1^j = v_2^{sp^n - (p^{n-1}-1)/(p-1)} g_1/v_1$ and $d_1(x_n^s G_n/v_1^j) = 0$. Put now

$$x_n^s G_n(j, 1) = \frac{\lambda_n'}{u} d_0(v_2^{sp^n}) \otimes \sigma_{j+A_{n-1}+2, e+2}$$

for $s = up^e$ with $p \nmid u$, where $\lambda_n' = (-1)^{n+1}4$ if $n > 1$ and $=2$ if $n = 1$. Then we have Lemma 9.2.

LEMMA 9.2. *Let n, s and j be integers such that $n > 0$, $0 < j \leq a_n$ and $p|s(j + A_{n-1} + 1)$. Then the element $x_n^s G_n(j, 1)$ satisfies the following: $px_n^s G_n(j, 1)$ and $v_1^{j-1} x_n^s G_n(j, 1)$ are homologous to zero and $v_2^{sp^n - (p^{n-1}-1)/(p-1)} g_1/pv_1$, respectively, and $x_n^s G_n(j, 1)$ is a cocycle in the cobar complex $\Omega^2 M_0^2$. Therefore, we have a generator $x_n^s G_n/v_1^j$ of $H^2 M_1^1$ such that $\varphi(x_n^s G_n/v_1^j) = x_n^s G_n(j, 1)$.*

Proof. Since $x_n^s G_n/v_1^j$ is a generator, we have $j \leq a_n$, and so $j + A_{n-1} + 2 \leq A_n + 1$. Note that

$$d_0(v_2^{sp^n}) \equiv -sv_1^{p^n} v_2^{sp^n - p^{n+1}} V^{p^n} - psv_1^{p^{n-1}} v_2^{sp^n - p^n} V^{p^{n-1}} \tag{9.3}$$

mod $(p^{e+2}, p^{e+1} v_1^{p^n + p^{n-1}}, v_1^{2p^n})$ for $e = v_p(s)$ by Lemma 6.7. Then we see that $px_n^s G_n(j, 1)$ is homologous to zero by Lemma 6.10, and $v_1^{j-1} x_n^s G_n(j, 1) = \lambda'_n v_2^{(s-1)p^n} V^{p^{n-1}} \otimes \sigma / pv_1^{p^{n-2} + A_{n-2} + 3}$ is homologous to $v_2^{sp^n - (p^{n-1}-1)/(p-1)} g_1/pv_1$ by Lemma 6.10. We also see that $x_n^s G_n(j, 1)$ is a cocycle by Lemma 7.4. Q.E.D.

LEMMA 9.4. *Suppose that $x_n^s G_n(j, l)$ is a cocycle, $kp^i = j + A_{n-1} + 1 \leq A_n + 1$ and $l \leq i + 1$. Then $\delta_2(x_n^s G_n(j, l)) = \lambda x_n^s G_n \otimes \zeta / v_1^j$.*

Proof. Since $H^3 M_1^1$ is generated by the elements $\gamma/v_1^a = x_m^t G_m \otimes \zeta / v_1^a$ with $p \nmid (t + 1)$ and $a < p^{m-1}(p + 1)$, we may put

$$\delta_2(x_n^s G_n(j, l)) = \sum k_\gamma \gamma / v_1^a$$

for $k_\gamma \in F_p$. In the summation, we see that $a \leq p^{n+1}$ by Corollary 6.4 since $j < p^{n+1}$. Furthermore the above equation is homogeneous, and so the internal degree of γ/v_1^a is the same as that of $x_n^s G_n(j, l)$. As is stated in [14, (4.3.3)], $|x_n^s G_n/v_1^j| = (sp^n - (p^{n-1} - 1)/(p - 1))(p + 1) - 1 - j$. Thus we have an equation and an inequality

$$sp^n(p + 1) - kp^i = (tp^m - (p^{m-1} - 1)/(p - 1))(p + 1) - 1 - a$$
$$0 < a < \min\{p^{n+1}, p^{m-1}(p + 1)\}$$

since $kp^i = j + 1 + (p + 1)(p^n - 1)/(p - 1)$. Here we note that $a \neq p^{n+1}$. In fact, if so, we deduce that $i = n$ and $k = 1$ and the equation does not hold even if we consider it modulo p. Now we solve these. First suppose that $m \geq n + 1$. Then $a < p^{n+1}$. Note that $A_{n-1} + 1 < kp^i \leq A_n + 1$, and the above equations give us

$$\left(tp^m - sp^n - \frac{(p^{m-1} - p^{n-1})}{(p - 1)}\right)(p + 1) < a < p^{n+1}$$
$$0 < a < \left(tp^m - sp^n - \frac{(p^{m-1} - p^n)}{(p - 1)}\right)(p + 1) - 1.$$

This gives $tp^m = sp^n + (p^{m-1} - p^n)/(p - 1) + p^n$ and then deduce the contradiction $a > p^{n+1}$. Next consider the case that $m \leq n$. Then $a < p^{m-1}(p + 1)$ and similarly to the above, we have inequalities

$$\left(tp^m - sp^n + \frac{(p^{n-1} - p^{m-1})}{(p - 1)}\right)(p + 1) < a < p^{m-1}(p + 1)$$
$$0 < a < \left(tp^m - sp^n + \frac{(p^n - p^{m-1})}{(p - 1)}\right)(p + 1) - 1.$$

If $m = n$, then we have the trivial solution: $t = s$ and $a = j$. For the case $m < n$, we obtain that $t = sp^{n-m} - (p^{n-m} - 1)/(p-1) + \alpha$ and $a = (-(p^n - 1)/(p-1) + p^{m+1} + \alpha p^m)(p+1) - 1 + kp^i$ for $0 \leq \alpha < p^{n-m-1}$. We further see that the inequality $a > 0$ indicates $\alpha \geq p^{n-m-1}$. This is a contradiction. Therefore we have no solution in this case, either. Hence the above summation has only a term $k_\gamma \gamma / v_1^a = \lambda x_n^s G_n \otimes \zeta / v_1^j$. Q.E.D.

Now we have the generalization of [17, Prop. 4.1].

PROPOSITION 9.5. *Let n, s, i, j and k be integers such that $n > 1$, $i \geq 0$, $j, k > 0$, $p \nmid (s+1)$, $j \leq a_n$ and $kp^i = j + A_{n-1} + 1$. Then we have cocycles $x_n^s G_n(j, l)$ for $0 < l \leq i+1$, and*

$$\delta_2(x_n^s G_n(j, i+1)) = -\frac{k}{2} x_n^s G_n \otimes \zeta / v_1^j.$$

Proof. We show first that $x_n^s G_n(j, l)$ is a cocycle for $0 < l \leq i+1$, inductively. For $l = 1$, it is trivial since $x_n^s G_n(j, 1) = \varphi(x_n^s G_n / v_1^j)$. Assume now that $x_n^s G_n(j, l)$ is a cocycle for $l \leq i$. Lemma 9.4 says that

$$\delta_2(x_n^s G_n(j, l)) = \lambda x_n^s G_n \otimes \zeta / v_1^j \tag{9.6}$$

for some $\lambda \in F_p$. By virtue of Lemma 6.10, we may put

$$\begin{aligned} x_n^s G_n(j, l) &= \lambda_n' v_2^{(s-1)p^n} V_n \otimes \sigma_{kp^i+1, l+1} \\ &= \frac{\lambda_n'}{s} d_0(v_2^{sp^n}) \otimes \sigma_{kp^i+1, l+1} \quad \text{(by Lemma 6.7)} \end{aligned} \tag{9.7}$$

for $kp^i \leq p^n$, and for $kp^i > p^n$, some lower terms would be added. Here $\lambda_n' = (-1)^{n+1} 4$ for $n > 1$. Lemma 7.4 tells us that $x_n^s G_n(j, l+1)$ is a cocycle for $j \leq p^n$. For $j > p^n$, use Corollary 6.3 to find that λ in (9.6) is null, which means that $x_n^s G_n(j, l+1)$ is a cocycle.

We compute $d_2(x_n^s G_n(j, i+2)) = -\frac{1}{2} k \lambda_n' v_2^{(s-1)p^n} V_n \otimes t_1 \otimes \zeta / p v_1^{kp^i+1}$ for a small value j by Lemma 7.4. Then the proposition follows from the definition of δ_2 and (9.7) for a small value of j. For a larger value of j, again use Corollary 6.3, and we have $\lambda = -\frac{1}{2}k$ in (9.6) by comparing internal degrees. Q.E.D.

We also have in [17, Prop. 4.4 and Lemma 4.5] the following:

$$\delta_1(y_{sp^n} / p^{n+1} v_1) = \frac{s}{2} v_2^{sp^n} (g_0 - t_1 \otimes \zeta) / v_1$$

and

$$\delta_2(y_m \otimes \zeta / p^{n+1} v_1) = \frac{s}{2} x_0^m G_0 \otimes \zeta / v_1.$$

Putting these together, we obtain Proposition 9.8.

PROPOSITION 9.8. *For the integers s and n with $p \nmid s(s+1)$, we have*

$$\delta_2(x_0^{sp^n} G_0 / p^{n+1} v_1) = \frac{s}{2} x_0^{sp^n} G_0 \otimes \zeta / v_1.$$

In fact, the first equation gives $x_0^{sp^n} G_0 / p v_1 = y_{sp^n} \otimes \zeta / p v_1$ and then use the second one.

COROLLARY 9.9. *For any $j > 0$, we have*

$$\delta_2(G_0/p^j v_1) = 0.$$

Proof. Suppose that there exists a positive integer j such that $\delta_2(G_0/p^j v_1) \neq 0$. By Lemma 6.2 and Proposition 9.8, we see that $v_2^{p^j}\delta_2(G_0/p^j v_1) = 0$. Since v_2^p acts monomorphically on the submodule $\langle x_0^s G_0/v_1 : s+1 \in \mathbf{Z} - p\mathbf{Z}\rangle$ of $H^2M_1^1$, the above two statements produce a contradiction. Q.E.D.

Now define

$$\begin{aligned} G_C^\infty &= \mathbf{Z}_{(p)}\{x_n^s G_n(j,l) : x_n^s G_n/v_1^j \in G_C - \{G_0/v_1\},\\ &\quad l = i+1 \text{ if } n = 0 \text{ and } v_p(s) = i;\\ &\quad l = i+1 \text{ if } n \geq 1 \text{ and } v_p(j + A_{n-1} + 1) = i\}\\ G_0^\infty &= \mathbf{Z}_{(p)}\{G_0/p^j v_1 : j > 0\}. \end{aligned}$$

From Propositions 7.5 and 7.8, we further have $Y\zeta_C = ((Y_{0,C} \oplus Y_{1,C}) \otimes F_p\{\zeta\})_C$ that is an F_p-vector space over the basis

$$\begin{aligned} \{ y_{sp^n} \otimes \zeta/v_1^{kp^{i+1}+1} &: y_{sp^n}/v_1^{kp^{i+1}+1} \in Y_{0,C} \oplus Y_{1,C},\ k = 0, \text{ or}\\ p \nmid s &\text{ and } A_{n-i-1} + 2 < kp^{i+1} + 1 \leq A_{n-i} + 2 \text{ if } s \in \mathbf{Z}_0, \text{ and}\\ kp^{i+1} &> p^{n+2} - p^n \text{ for } i > 0 \text{ and } kp > A_{n-1} + 1 \text{ for } i = 0 \text{ if } s \in \mathbf{Z}_2\}. \end{aligned}$$

Note that we have an isomorphism.

Remark 9.10. $Y\zeta_C \cong G_I$ as F_p-vector spaces.

In fact, the correspondence can be read off from Propositions 7.5 and 7.8. Note also that $Y\zeta_C$ produces the submodule $(Y_{0,C}^{\infty;G} \oplus Y_{1,C}^{\infty;G}) \otimes \mathbf{Z}_{(p)}\{\zeta\}$ of $H^2M_0^2$. Thus we introduce another notation:

$$Y\zeta_C^\infty = (Y_{0,C}^{\infty;G} \oplus Y_{1,C}^{\infty;G}) \otimes \mathbf{Z}_{(p)}\{\zeta\}.$$

Now we have the following result.

THEOREM 9.11.

$$H^2M_0^2 = Y\zeta_C^\infty \oplus G_C^\infty \oplus (Y_{\infty,C}^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}) \oplus G_0^\infty.$$

Proof. First we study the cokernel of δ_1. By the results of Section 6, we see that the submodule of $H^2M_1^1$ of the form $M \otimes \mathbf{Z}_{(p)}\{\zeta\}$ is in the image of δ_1 except for $Y\zeta_C$ and $Y_\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}$. For the submodule G, $G = G_C \oplus G_I$ and G_I is in the image of δ_1. Now the theorem follows in the same way as the proof of Theorem 8.2. Q.E.D.

Note that δ_2 maps $Y\zeta_C^\infty$ isomorphically to $G_I \otimes \mathbf{Z}_{(p)}\{\zeta\}$, which is deduced from Lemma 6.1 and Propositions 7.5 and 7.8, and G_C^∞ to $(G_C - \{G_0/v_1\}) \otimes F_p\{\zeta\}$. Besides, $\delta_2(Y_\infty^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}) = 0$. Therefore we have Lemma 9.12.

LEMMA 9.12. *The cokernel of δ_2 is the submodule generated by $G_0 \otimes \zeta/v_1$.*

Using Lemma 4.3, the following is now a corollary of this lemma and Corollary 9.9.

THEOREM 9.13. *The module* $H^3M_0^2$ *is isomorphic to* $\mathbf{Q}/\mathbf{Z}_{(p)}$ *generated by* $G_0 \otimes \zeta/p^j v_1$.

Summarizing these, we have the following result.

THEOREM 9.14. *The module* $H^*M_0^2$ *is isomorphic to*

$$(X_\infty^\infty \oplus Y_{\infty,C}^\infty \oplus G_0^\infty) \otimes E(\zeta) \oplus X^\infty \oplus X\zeta_C^\infty \oplus Y_{0,C}^\infty \oplus Y_{1,C}^\infty \oplus Y_C^\infty \oplus G^\infty.$$

Here $G^\infty = G_C^\infty \oplus Y\zeta^\infty$.

Since we see that $Y\zeta_C$ is isomorphic to G_I, the notation G^∞ is reasonable.

10. $\pi_*(L_2S^0)$

Consider the Adams–Novikov spectral sequence based on $E(2)$ converging to the homotopy groups $\pi_*(L_2S^0)$ of the Bousfield localization of the sphere S^0 ([1, 2], cf. [10]). Then the E_2-term of the spectral sequence is

$$H^{s,t}A = \mathrm{Ext}_\Gamma^{s,t}(A, A)$$

where (A, Γ) denotes the Hopf algebroid $(E(2)_*, E(2)_* E(2))$ associated to the spectrum $E(2)$. We have the long exact sequence (4.2)

$$0 \to H^0N_0^0 \to H^0M_0^0 \to H^0N_0^1 \xrightarrow{\delta_0'} H^1N_0^0 \to \cdots$$

$$\to H^tN_0^0 \to H^tM_0^0 \to H^tN_0^1 \xrightarrow{\delta_t'} H^{t+1}N_0^0 \to \cdots$$

and

$$0 \to H^0N_0^1 \to H^0M_0^1 \to H^0M_0^2 \xrightarrow{\delta_0} H^1N_0^1 \to \cdots$$

$$\to H^tN_0^1 \to H^tM_0^1 \to H^tM_0^2 \xrightarrow{\delta_t} H^{t+1}N_0^1 \to \cdots$$

In these long exact sequences, $H^*N_0^0 = H^*A$, and the modules $H^*M_0^0$, $H^*M_0^1$ and $H^*M_0^2$ are known now. Since $H^tM_0^1 = 0$ for $t > 1$, $\delta_t\colon H^tM_0^2 \to H^{t+1}N_0^1$ is isomorphic for $t > 1$ and epimorphic for $t = 1$. The kernel of δ_1 is $Y_{\infty,C}^\infty$, since $H^1M_0^1 = Y_{\infty,C}^\infty$ by (4.5). This further means that the map $H^1M_0^1 \to H^1M_0^2$ in the above sequence is a monomorphism, and we have the exact sequences

$$0 \to H^0N_0^1 \to H^0M_0^1 \xrightarrow{f} H^0M_0^2 \xrightarrow{\delta_0} H^1N_0^1 \to 0$$

and

$$0 \to H^1M_0^1 \to H^1M_0^2 \xrightarrow{\delta_1} H^2N_0^1 \to 0.$$

By the structures (4.5) and Theorem 9.14, we see that

$$\mathrm{Ker} f = \mathbf{Z}_{(p)}\{v_1^{sp^i}/p^{i+1}; i \geq 0,\ s \geq 0,\ p \nmid s\} \oplus \mathbf{Q}/\mathbf{Z}_{(p)}$$

$$\mathrm{Im} f = X_\infty^\infty.$$

Furthermore $H^tM_0^0 = 0$ if $t > 0$, and $= \mathbf{Q}$ at the internal degree 0 if $t = 0$. Therefore we have the following theorem.

THEOREM 10.1. *The E_2-term E_2^s of the Adams–Novikov spectral sequence for $\pi_*(L_2S^0)$ is given by*

(0) $E_2^0 \cong \mathbf{Z}_{(p)}$,

(1) $E_2^1 \cong \mathbf{Z}_{(p)}\{v_1^{sp^i}/p^{i+1}: i \geq 0,\ s \geq 0,\ p \nmid s\}$,

(2) $E_2^2 \cong X^\infty$,

(3) $E_2^3 \cong Y_{0,C}^\infty \oplus Y_{1,C}^\infty \oplus Y_C^\infty \oplus X\zeta_C^\infty \oplus (X_\infty^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\})$,

(4) $E_2^4 \cong Y\zeta_C^\infty \oplus G_C^\infty \oplus (Y_{\infty,C}^\infty \otimes \mathbf{Z}_{(p)}\{\zeta\}) \oplus G_0^\infty$,

(5) $E_2^5 \cong G_0^\infty \cong \mathbf{Q}/\mathbf{Z}_{(p)}$, *and*

(6) $E_2^t = 0$ *for* $t > 5$.

Since the prime p is greater than 3, the Adams–Novikov spectral sequence for $\pi_*(L_2S^0)$ collapses from the E_2-term and so Theorem 10.1 gives the structure of the homotopy groups as well.

REFERENCES

1. J. F. ADAMS: *Stable homotopy and generalised homology*, University of Chicago Press, Chicago (1974).
2. A. K. BOUSFIELD: The localization of spectra with respect to homology, *Topology* **18** (1979), 257–281.
3. M. HIKIDA and K. SHIMOMURA: An exact sequence related to Adams–Novikov E_2-terms of a cofibering, to appear in *J. Math. Soc. Japan.*
4. P. S. LANDWEBER: Associated prime ideals and Hopf algebras, *J. Pure Appl. Algebra* **3** (1973), 175–179.
5. H. R. MILLER and D. C. RAVENEL: Morava stabilizer algebras and the localization of Novikov's E_2-term, *Duke Math. J.* **44** (1977), 433–447.
6. H. R. MILLER, D. C. RAVENEL and W. S. WILSON: Periodic phenomena in Adams–Novikov spectral sequence, *Ann. Math.* **106** (1977), 469–516.
7. D. G. QUILLEN: On the formal group laws of unoriented and complex cobordism theory, *Bull. AMS* **75** (1969), 1293–1298.
8. D. C. RAVENEL: The structure of BP_*BP modulo an invariant prime ideal, *Topology* **15** (1976), 149–153.
9. D. C. RAVENEL: The cohomology of the Morava stabilizer algebras, *Math. Z.* **152** (1977), 287–297.
10. D. C. RAVENEL: Localization with respect to certain periodic homology theories, *Amer. J. Math.* **106** (1984), 351–414.
11. D. C. RAVENEL: *Complex cobordism and stable homotopy groups of spheres*, Academic Press, New York (1986).
12. D. C. RAVENEL: The geometric realization of the chromatic resolution, in *Algebraic topology and algebraic K-theory*, W. Browder, Ed., *Ann. Math. Studies* **113**, Princeton Univ. Press, Princeton, NJ (1987), pp. 168–179.
13. D. C. RAVENEL: *Nilpotence and periodicity in stable homotopy theory*, Ann. of Math. Studies **128**, Princeton Univ. Press, Princeton, NJ (1992).
14. K. SHIMOMURA: On the Adams–Novikov spectral sequence and products of β-elements, *Hiroshima Math. J.* **16** (1986), 209–224.
15. K. SHIMOMURA: Non-triviality of some products of β-elements in the stable homotopy of spheres, *Hiroshima Math. J.* **17** (1987), 349–353.
16. K. SHIMOMURA and H. TAMURA: Non-triviality of some compositions of β-elements in the stable homotopy of the Moore spaces, *Hiroshima Math. J.* **16** (1986), 121–133.
17. K. SHIMOMURA and A. YABE: On the chromatic E_1-term $H^*M_0^2$, Topology and Representation Theory, *Contemporary Math.* **158** (1994), 217–228.

Faculty of Education
Tottori University
Tottori, 680
Japan

Yazu High School
Kôge, Yazu-Gun
Tottori-Pref., 680-04
Japan