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Jan Stienstra

American Journal of Mathematics, Vol. 109, No. 5 (Oct., 1987), 907-925.

Stable URL:
http://links.jstor.org/sici?sici=0002-9327%28198710%29109%3A5%3C907%3AFGLAFA%3E2.0.CO%3B2-B

American Journal of Mathematics is currently published by The Johns Hopkins University Press.

FORMAL GROUP LAWS ARISING FROM ALGEBRAIC VARIETIES

By Jan Stienstra

Introduction. In this note we show how one can sometimes very easily determine explicit logarithms for the formal groups of Artin and Mazur [1]. Preliminaries on formal groups and the Artin-Mazur construction are recalled in Sections 1 and 2. We shall prove, in Sections 3, 4, 5:

Theorem 1. *Let K be a noetherian ring. Let $F_1, \ldots, F_r$ be a regular sequence of homogeneous polynomials in $K[T_0, \ldots, T_N]$ and let X be the subscheme of $\mathbf{P}_K^N$ defined by the ideal $(F_1, \ldots, F_r)$. Put $d_i = \deg F_i$ and $d = \Sigma d_i$. Assume X is flat over K and $d_i \geq d - N \geq 1$ for all i. Then $H^{N-r}(X, \hat{\mathbf{G}}_{mX})$ (see Section 2) is a formal group over K of dimension $n = \binom{d-1}{N}$.*

Assume moreover that K is flat over $\mathbf{Z}$. Put

$$J = \{i = (i_0, \ldots, i_N) \in \mathbf{Z}^{N+1} | i_0, \ldots, i_N \geq 1, i_0 + \cdots + i_N = d\}.$$

Then there is a formal group law for $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$ whose logarithm $\ell(\tau)$ is the n-tuple $(\ell_i(\tau))_{i \in J}$ of power series in the n-tuple of variables $\tau = (\tau_i)_{i \in J}$ given by

$$\ell_i(\tau) = \sum_{m \geq 1} \sum_{j \in J} m^{-1} \beta_{m,i,j} \tau_j^m$$

$$\beta_{m,i,j} = \text{coefficient of } T_0^{mj_0 - i_0} \cdot \ldots \cdot T_N^{mj_N - i_N} \quad \text{in} \quad (F_1 \cdot \ldots \cdot F_r)^{m-1}. \qquad \square$$

Theorem 2. *Let K be a noetherian ring. Let F be a homogeneous polynomial in $K[T_0, \ldots, T_N]$ of degree $2d > 2N$ and let X be the double covering of $\mathbf{P}_K^N$ defined by the equation $W^2 = F$ (where W is a new variable of weight d).*

Manuscript received 16 March 1986.
American Journal of Mathematics 109 (1987), 907–925.

Then $H^N(X, \hat{\mathbf{G}}_{m,X})$ is a formal group over K of dimension $n = \binom{d-1}{N}$. Assume that K is flat over $\mathbf{Z}$ and let J be as in Theorem 1. Then there is a formal group law for $H^N(X, \hat{\mathbf{G}}_{m,X})$ with logarithm $\ell(\tau) = (\ell_i(\tau_j)_{j\in J})_{i\in J}$ given by

$$\ell_i(\tau) = \sum_{m\geq 1}\sum_{j\in J} m^{-1}\beta_{m,i,j}\,\tau_j^m$$

$$\beta_{m,i,j} = \begin{cases} 0 & \text{if } m \text{ is even} \\ \text{coefficient of } T_0^{mj_0-i_0}\cdot\ldots\cdot T_N^{mj_N-i_N} \text{ in } F^{(m-1)/2}. & \\ & \text{if } m \text{ is odd.} \end{cases} \qquad \square$$

Moreover, in both cases, if $\varphi\colon K \to K'$ is a surjective homomorphism of rings and $X' = X \times_{\operatorname{spec} K} \operatorname{spec} K'$, one obtains a formal group law for $H^a(X', \hat{\mathbf{G}}_{m,X'})$ ($a = N - r$ resp. N) by applying φ to the coefficients of the formal group law for $H^a(X, \hat{\mathbf{G}}_{m,X})$.

The formal group laws we find in the above theorems are so-called curvilinear formal group laws, i.e. the power series expansions of their logarithm contain no monomials involving more than one variable. The logarithm can be written as

$$\ell(\tau) = \sum_{m\geq 1} m^{-1}\beta_m\tau^m$$

in which $\ell(\tau)$ is the vector with coordinates $\ell_i(\tau)$, indexed by the elements of the set J, τ^m is the vector with coordinates $\tau_j^m (j \in J)$ and β_m is the $n \times n$-matrix with entries $\beta_{m,i,j}(i, j \in J)$ as in Theorem 1 resp. 2.

Examples are presented in (4.13) and (5.5). Arithmetic applications are given in [10].

In the *appendix* we re-interpret the expression for the logarithm $\ell(\tau)$ of Theorem 2 in terms of integrals of holomorphic differential forms of degree N on (the smooth part of) the complex analytic space associated with X and the embedding of K into $\mathbf{C}$.

For instance, for the formal Brauer group, $H^2(X, \hat{\mathbf{G}}_{m,X})$, of the $K3$-surface

$$X\colon W^2 = T_0^6 + T_1^6 + T_2^6$$

we get from Theorem 2 the logarithm

$$\ell(\tau) = \sum_{k\geq 0} \frac{(3k)!}{(k!)^3} \frac{\tau^{6k+1}}{6k+1}.$$

For sufficiently small complex values of τ the considerations in the appendix yield

$$\ell(\tau) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma(\tau)} \frac{dt_1 \wedge dt_2}{2\sqrt{1 + t_1^6 + t_2^6}}$$

where $\Gamma(\tau)$ is an appropriate 2-chain on $X(\mathbf{C})$ with boundary

$$\partial\Gamma(\tau) = \{(w, t_1, t_2) \in \mathbf{C}^3 \mid w^2 = 1 + t_1^6 + t_2^6,\ |t_1| < |t_2| = 1/2,\ t_1 t_2 w^{-1} = \tau\};$$

$\sqrt{1 + t_1^6 + t_2^6}$ must be taken so that on $\partial\Gamma(\tau)$ its argument is the same as the argument of $t_1 t_2 \tau^{-1}$.

Some characteristic features of integrals like the one above are discussed in (A4) in the appendix.

While the formal groups of Artin and Mazur generalize the formal Picard group, the integral expressions for $\ell(\tau)$ are generalizations of the classical (*hyper-*)*elliptic integrals and the essence of Theorem 2 is an addition theorem for such integrals* (*be it in general not an algebraic addition law*).

Acknowledgement. I want to thank the Université de Paris-Sud at Orsay for its hospitality and support during the spring semester of 1985, when most of the research for this paper was done.

1. In this section we recall some preliminaries on formal groups; references are [5, 6, 8, 12]. By formal group we shall mean smooth commutative formal group. The rings and algebras in this paper are all commutative and associative and all rings have a unit element.

1.1. Fix a ring K and an integer $n \geq 1$. Let $\mathcal{N}il\mathcal{A}lgs_K$ denote the category of *nil-K-algebras*, i.e. of K-algebras in which every element is nilpotent. *Formal affine n-space over K* is defined to be the functor $\mathbf{A}_K^n$:

$\mathcal{N}il\mathcal{A}lg\mathcal{s}_K \to \mathcal{S}ets$ which assigns to a nil-K-algebra A the set $A \times \cdots \times A$ (n factors) and to a morphism f the map $f \times \cdots \times f$. An *n-dimensional formal group over K* is a functor $G\colon \mathcal{N}il\mathcal{A}lg\mathcal{s}_K \to \mathcal{A}b.groups$ whose underlying set valued functor admits a functorial bijection onto $\mathbf{A}_K^n$. A functorial bijection $G \rightrightarrows \mathbf{A}_K^n$ is called a *coordinatization* of the formal group G.

1.2. Having chosen a coordinatization $c\colon G \rightrightarrows \mathbf{A}_K^n$ one can describe the formal group G by an n-dimensional *formal group law* $L(\xi, \eta)$ over K: there exists an n-tuple $L = (L_1, \ldots, L_n)$ of formal power series with coefficients in K in two n-tuples of variables $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$ such that for every nil-K-algebra A and for all elements $\alpha, \beta \in G(A)$:

$$c(\alpha \boxplus \beta) = L(c(\alpha), c(\beta));$$

here $\boxplus$ denotes the group structure on G. The group axioms for G correspond to the following identities for $L(\xi, \eta)$:

$$L(L(\xi, \eta), \zeta) = L(\xi, L(\eta, \zeta)), \qquad L(\xi, O) = \xi,$$

$$L(\xi, \eta) = L(\eta, \xi), \qquad L(\xi, \eta) \equiv \xi + \eta \mod \deg \geq 2;$$

here ζ is another n-tuple of variables and $O = (0, \ldots, 0)$.

1.3. If the canonical map $K \to K \otimes \mathbf{Q}$ is injective, every n-dimensional formal group law $L(\xi, \eta)$ over K determines an n-tuple $\ell(\tau)$ of power series in one n-tuple of variables with coefficients in $K \otimes \mathbf{Q}$ such that

$$\ell(\tau) \equiv \tau \mod \deg \geq 2$$

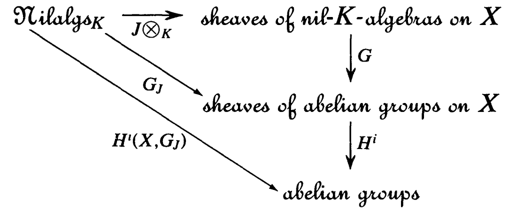
$$L(\xi, \eta) = \ell^{-1}(\ell(\xi) + \ell(\eta)).$$

One calls $\ell(\tau)$ the *logarithm* of the formal group law $L(\xi, \eta)$.

1.4. The simplest examples are the 1-dim *additive formal group* $\hat{\mathbf{G}}_a$ and the 1-dim *multiplicative formal group* $\hat{\mathbf{G}}_m$, both defined over $\mathbf{Z}$ (and hence over any ring). $\hat{\mathbf{G}}_a$ admits a coordinatization in which the formal group law is $L(\xi, \eta) = \xi + \eta$ and the logarithm is $\ell(\tau) = \tau$. $\hat{\mathbf{G}}_m$ admits a coordinatization in which the formal group law is $L(\xi, \eta) = \xi + \eta - \xi\eta$ and the logarithm is $\ell(\tau) = -\log(1 - \tau) = \Sigma_{n \geq 1} n^{-1}\tau^n$.

2. In this section we recall the definition of the Artin-Mazur functors [1].

2.1. Let K be a ring, G a formal group over K, X a scheme over K, J a sheaf of K-algebras on X and i an integer ≥ 0. In this situation one can construct the following commutative diagram of categories and functors



where $J \otimes_K$ assigns to a nil-K-algebra A the sheaf $J \otimes_K A$ associated with the pre-sheaf (open U) $\mapsto \Gamma(U, J) \otimes_K A$; the functor G assigns to a sheaf $\mathfrak{A}$ of nil-K-algebras the sheaf of abelian groups $G(\mathfrak{A})$ defined by $\Gamma(U, G(\mathfrak{A})) = G(\Gamma(U, \mathfrak{A}))$ for every open $U \subset X$; the functor H^i is taking i-th cohomology. The functors G_J and $H^i(X, G_J)$ are defined by commutativity of the diagram.

2.2. We are mainly interested in using (2.1) with $G = \hat{\mathbf{G}}_m$ and $J = \mathcal{O}_X$, the structure sheaf on X. We write $\hat{\mathbf{G}}_{m,X}$ instead of $\hat{\mathbf{G}}_{m,\mathcal{O}_X}$. The functors

$$H^i(X, \hat{\mathbf{G}}_{m,X}) \colon \mathcal{N}ilalgs_K \to \text{abelian groups}$$

are the *Artin-Mazur functors*, introduced in [1] and denoted Φ^i in op.cit. $H^1(X, \hat{\mathbf{G}}_{m,X})$ and $H^2(X, \hat{\mathbf{G}}_{m,X})$ are usually called the *formal Picard group* and the *formal Brauer group*, respectively (at least if they are formal groups). We refer to [1] for a discussion of necessary and sufficient conditions for these functors to be formal groups (i.e. pro-representable and formally smooth). For instance, the vanishing of $H^{i-1}(X, \mathcal{O}_X)$ and $H^{i+1}(X, \mathcal{O}_X)$ in the situations of Theorems 1 and 2 implies that $H^i(X, \hat{\mathbf{G}}_{m,X})$ is a formal group (cf. op.cit. II.4). However, instead of invoking this general result, we shall explicitly construct coordinatizations.

3.

3.1. Theorem. *Let K be a ring. Let $f \colon X \to Y$ be a morphism of schemes over K with X flat over K. Assume that for every ideal I of K*

$$f_*(\mathcal{O}_X \otimes_K (K/I)) = (f_*\mathcal{O}_X) \otimes_K (K/I)$$

$$R^j f_*(\mathcal{O}_X \otimes_K (K/I)) = 0 \quad \textit{for} \quad j \geq 1.$$

Then there is a functorial isomorphism, for every $i \geq 0$,

$$H^i(X, \hat{\mathbf{G}}_{m,X}) \simeq H^i(Y, \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X})$$

The proof of Theorem (3.1) *is presented in* (3.2)–(3.7). We refer to [4] for general facts and further references on algebraic geometry.

3.2. First we prove $R^j f_* \hat{\mathbf{G}}_{m,X}(A) = 0$ for every $j \geq 1$ and every nil-K-algebra A. From this and the Leray spectral sequence

$$E_2^{ij} = H^i(Y, R^j f_* \hat{\mathbf{G}}_{m,X}(A)) \Rightarrow H^*(X, \hat{\mathbf{G}}_{m,X}(A))$$

one can then conclude

$$H^i(X, \hat{\mathbf{G}}_{m,X}) \simeq H^i(Y, f_* \hat{\mathbf{G}}_{m,X}) \quad \text{for every} \quad i \geq 0.$$

3.3. Formation of tensor products, sheafification and taking cohomology commute with inductive limits. Hence

$$R^j f_* \hat{\mathbf{G}}_{m,X}(\varinjlim_n A_n) = \varinjlim_n R^j f_* \hat{\mathbf{G}}_{m,X}(A_n)$$

for every inductive system $\{A_n\}$ of nil-K-algebras. Since every nil-K-algebra is the inductive limit of its finitely generated sub-algebras, the problem of proving $R^j f_* \hat{\mathbf{G}}_{m,X}(A) = 0$ for arbitrary nil-K-algebras A reduces to proving this for finitely generated ones.

3.4. To make further reductions we need *small extensions*. A small extension is a surjective homomorphism $A \twoheadrightarrow \bar{A}$ of nil-K-algebras with kernel generated, as a K-module, by a single nonzero element ϵ with the property $\epsilon A = 0$.

3.5. Lemma. *For every finitely generated nil-K-algebra A there exists a finite sequence of small extensions*

$$A = A_0 \twoheadrightarrow A_1 \twoheadrightarrow \cdots \twoheadrightarrow A_q \twoheadrightarrow A_{q+1} = 0.$$

Proof. Every finitely generated nil-K-algebra A is a quotient of an algebra of the form $(x_1, \ldots, x_n)K[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^m$ for some n and m. For the latter kind of algebras there is clearly such a finite sequence of small extensions. This can be pushed down to A. □

3.6. Since X is flat over K, the functor $\mathcal{O}_X \otimes_K -$ is exact. Also the functor $\hat{\mathbf{G}}_m$ is exact. Thus, if $A \twoheadrightarrow \bar{A}$ is a small extension with kernel $K\epsilon$, one has a short exact sequence of sheaves of abelian groups on X

$$0 \to \hat{\mathbf{G}}_{m,X}(K\epsilon) \to \hat{\mathbf{G}}_{m,X}(A) \to \hat{\mathbf{G}}_{m,X}(\bar{A}) \to 0,$$

and a long exact sequence of sheaves of abelian groups on Y

$$\cdots \to R^j f_* \hat{\mathbf{G}}_{m,X}(K\epsilon) \to R^j f_* \hat{\mathbf{G}}_{m,X}(A) \to R^j f_* \hat{\mathbf{G}}_{m,X}(\bar{A}) \to \cdots .$$

Such a sequence enables one to proceed by induction along small extensions. Thus proving $R^j f_* \hat{\mathbf{G}}_{m,X}(A) = 0$ for a finitely generated nil-K-algebra A is reduced to proving it for $A = K\epsilon$, $\epsilon^2 = 0$.

Because of $\epsilon^2 = 0$ the sheaf $\hat{\mathbf{G}}_{m,X}(K\epsilon)$ is isomorphic to the sheaf of additive groups $\mathcal{O}_X \otimes_K (K/I)$, where $I \subset K$ is the annihilator of ϵ. Therefore the vanishing of $R^j f_* \hat{\mathbf{G}}_{m,X}(K\epsilon)$ is a consequence of the hypothesis $R^j f_*(\mathcal{O}_X \otimes_K (K/I)) = 0$ for $j \geq 1$.

3.7. As remarked in (3.2) we now have a functorial isomorphism

$$H^i(X, \hat{\mathbf{G}}_{m,X}) \simeq H^i(Y, f_* \hat{\mathbf{G}}_{m,X}) \quad \text{for all} \quad i \geq 0.$$

To complete the proof of (3.1) we must show

$$f_* \hat{\mathbf{G}}_{m,X} \simeq \hat{\mathbf{G}}_m(f_*(\mathcal{O}_X \otimes_K -)) \simeq \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X}.$$

The first isomorphism is obvious. Thus we are left with proving $f_*(\mathcal{O}_X \otimes_K A) \simeq (f_*\mathcal{O}_X) \otimes_K A$ for all nil-K-algebras A. One notices that there is map $\leftarrow$. To prove that this is an isomorphism one reduces, with arguments as in (3.3)–(3.6), to the case $A = K\epsilon$, $\epsilon^2 = 0$, in which case it becomes the hypothesis $f_*(\mathcal{O}_X \otimes_K (K/I)) = (f_*\mathcal{O}_X) \otimes_K (K/I)$ assumed for (3.1). The proof of (3.1) is now complete. □

We mention three applications of (3.1).

3.8. *Computation.* Let K be a ring, $f\colon X \to Y$ an affine morphism of schemes over K with X noetherian and flat over K. Then the hypotheses of (3.1) are satisfied. Whence $H^i(X, \hat{\mathbf{G}}_{m,X}) \simeq H^i(Y, \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X})$. This can be used for computations if the cohomology of Y and $f_*\mathcal{O}_X$ are easily computable; e.g. $Y = \mathbf{P}^N$, $f_*\mathcal{O}_X = \mathcal{O}_{\mathbf{P}^N}(-d)$; see Sections 4 and 5.

3.9. *Specialization.* Let $\phi\colon K \to \bar{K}$ be a surjective homomorphism of rings. Let Y be a flat noetherian scheme over $S = \operatorname{Spec} K$. Put $X = Y \times_S \operatorname{Spec} \bar{K}$. This is an affine morphism. But, since X is not flat over S, the hypotheses of (3.1) are not satisfied. However, as X is flat over $\bar{S} = \operatorname{Spec} \bar{K}$, the proof of (3.1), in particular (3.6), works as long as one restricts to nil-$\bar{K}$-algebras.

Note that $f_*(\mathcal{O}_X \otimes_K A) = \mathcal{O}_Y \otimes_K A$ if A is a nil-$\bar{K}$-algebra. Thus we get a functorial isomorphism

$$H^i(X, \hat{\mathbf{G}}_{m,X}) \simeq H^i(Y, \hat{\mathbf{G}}_{m,Y}) \quad \text{on} \quad \mathfrak{Nilalgs}_{\bar{K}}.$$

So, if $H^i(Y, \hat{\mathbf{G}}_{m,Y})$ is a formal group over K, $H^i(X, \hat{\mathbf{G}}_{m,X})$ is a formal group over $\bar{K}$; a coordinatization restricts to a coordinatization and a formal group law for $H^i(X, \hat{\mathbf{G}}_{m,X})$ is obtained by applying ϕ to the coefficients of a formal group law for $H^i(Y, \hat{\mathbf{G}}_{m,Y})$.

3.10. *Desingularization.* Another situation in which (3.1) is useful, is a resolution of singularities $f\colon X \to Y$ with $R^jf_*\mathcal{O}_X = 0$ for $j \geq 1$, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and K a field. Then $H^i(X, \hat{\mathbf{G}}_{m,X}) \simeq H^i(Y, \hat{\mathbf{G}}_{m,Y})$ for all $i \geq 0$. One needs this possibility of passing from a singular model to a smooth one without changing $H^i\hat{\mathbf{G}}_m$, if one wants to exploit the connection of the Artin-Mazur formal groups with zeta-functions. This connection runs via the theory of the De Rham-Witt complex and crystalline cohomology, which has only been sufficiently worked out for smooth projective varieties. We will not digress on these matters here. Examples can be found in [9, 10].

4.

4.1. In this section we prove Theorem 1. Recall the situation: K is a noetherian ring, $F_1, \ldots, F_r$ is a regular sequence of homogeneous polynomials in $K[T_0, \ldots, T_N]$ (i.e. for $i = 1, \ldots, r$ the image of F_i in $K[T_0, \ldots, T_N]/(F_1, \ldots, F_{i-1})$ is not a zero divisor), X is the subscheme of $\mathbf{P}^N_K$ defined by the ideal $(F_1, \ldots, F_r)$. We assume that X is flat over K. Moreover, let $d_i = \deg F_i$, $d = \Sigma\, d_i$ and assume $d_i \geq d - N \geq 1$ for all i.

We want to show that $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$ is a formal group over K of dimension $\binom{d-1}{N}$, and give a formal group law for it.

4.2. The inclusion $f\colon X \to \mathbf{P}_K^N$ induces, according to (3.8), an isomorphism

$$H^{N-r}(X, \hat{\mathbf{G}}_{m,X}) \simeq H^{N-r}(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X}). \tag{4.2.1}$$

4.3. In order to get a hold on the right-hand side of (4.2.1) we construct a kind of Koszul resolution (cf. [4]). For a subset ρ of $\{1, \ldots, r\}$ put $F_\rho = \Pi_{i\in\rho} F_i$ and let $\tilde{F}_\rho$ be the corresponding sheaf of ideals on $\mathbf{P}_K^N$. Associated with $\tilde{F}_\rho$ we have the functor (see Section 2)

$$\hat{\mathbf{G}}_{m,\tilde{F}_\rho}\colon \mathcal{N}\mathit{ilalgs}_K \to \textit{sheaves of abelian groups on } \mathbf{P}_K^N,$$

which we shall henceforth denote as $\hat{\mathbf{G}}_{m,\rho}$, to keep the notation simple. For two subsets ρ, ρ' of $\{1, \ldots, r\}$ we define a homomorphism $\partial_{\rho,\rho'}\colon \hat{\mathbf{G}}_{m,\rho} \to \hat{\mathbf{G}}_{m,\rho'}$ as follows. If $\rho = \{i_1 < i_2 < \cdots < i_t\}$ and $\rho' = \rho\backslash\{i_k\}$, we let $\partial_{\rho,\rho'}$ be $(-1)^k$ times (in the sense of the group structure) the homomorphism induced by the inclusion $\tilde{F}_\rho \subset \tilde{F}_{\rho'}$. In all other cases (i.e. $\rho' \not\subset \rho$ or $\#(\rho\backslash\rho') \neq 1$) we put $\partial_{\rho,\rho'} = 0$. Then we get a complex of functors with values in the category of sheaves of abelian groups on $\mathbf{P}_K^N$

$$0 \longrightarrow \bigoplus_{\#\rho=r} \hat{\mathbf{G}}_{m,\rho} \xrightarrow[\partial_r]{} \bigoplus_{\#\rho=r-1} \hat{\mathbf{G}}_{m,\rho} \xrightarrow[\partial_{r-1}]{} \cdots \tag{4.3.1}$$
$$\cdots \xrightarrow[\partial_2]{} \bigoplus_{\#\rho=1} \hat{\mathbf{G}}_{m,\rho} \xrightarrow[\partial_1]{} \hat{\mathbf{G}}_{m,\mathbf{P}_K^N} \xrightarrow[\partial_0]{} \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X} \longrightarrow 0,$$

where the direct sums are taken over all subsets of $\{1, \ldots, r\}$ of the indicated cardinality; ∂_t is given by the matrix $(\partial_{\rho,\rho'})$ if $t \geq 1$; ∂_0 is the obvious map.

4.4. LEMMA. (4.3.1) *is an exact sequence.*

Proof. Let $\mathcal{C}^\cdot(A)$ denote (4.3.1) evaluated at the nil-K-algebra A. One has to show that each complex $\mathcal{C}^\cdot(A)$ is in fact an exact sequence. By a limit argument one reduces to finitely generated A, and then one proceeds by induction along small extensions. A small extension $A \twoheadrightarrow \bar{A}$ with kernel $K\epsilon$ leads to a short exact sequence of complexes $0 \to \mathcal{C}^\cdot(K\epsilon) \to \mathcal{C}^\cdot(A) \to \mathcal{C}^\cdot(\bar{A}) \to 0$. The corresponding long exact sequence of homology groups shows that $\mathcal{C}^\cdot(A)$ is exact if $\mathcal{C}^\cdot(K\epsilon)$ and $\mathcal{C}^\cdot(\bar{A})$ are exact. This reduces the question to proving that the complexes $\mathcal{C}^\cdot(K\epsilon)$ with $\epsilon^2 = 0$ are exact. Because of $\epsilon^2 = 0$, $\hat{\mathbf{G}}_{m,\rho}(K\epsilon)$ is equal to the additive group of $\tilde{F}_\rho \otimes_K K\epsilon$. Hence $\mathcal{C}^\cdot(K\epsilon)$ is just (a form of) the Koszul resolution [4]

$$0 \longrightarrow \bigoplus_{\#\rho=r} \tilde{F}_\rho \xrightarrow[\partial_r]{} \bigoplus_{\#\rho=r-1} \tilde{F}_\rho \longrightarrow \cdots \longrightarrow \bigoplus_{\#\rho=1} \tilde{F}_\rho \xrightarrow[\partial_1]{} \mathcal{O}_{\mathbf{P}_K^N} \xrightarrow[\partial_0]{} f_*\mathcal{O}_X \longrightarrow 0$$

of $f_*\mathcal{O}_X$, tensored with $K\epsilon$. The Koszul resolution is an exact sequence and its terms are flat over K. So $\mathcal{C}^\cdot(K\epsilon)$ is indeed an exact sequence. □

4.5. Lemma. *$H^i(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,\rho}) = 0$ for every proper subset ρ of $\{1, \ldots, r\}$ and for $i = 1, \ldots, N$.*

Proof. The standard limit argument and induction along small extensions reduce the question to proving $H^i(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,\rho}(K\epsilon)) = 0$, with $\epsilon^2 = 0$. Given $K\epsilon$ with $\epsilon^2 = 0$ let $I \subset K$ be the annihilator of ϵ and $\bar{K} = K/I$ ($\simeq K\epsilon$ as a K-module). Then $H^i(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,\rho}(K\epsilon)) \simeq H^i(\mathbf{P}_K^N, \mathcal{O}_{\mathbf{P}_K^N}(-\Sigma_{j\in\rho} d_j)) = 0$, because $\Sigma_{j\in\rho} d_j \leq N$ (cf. [4]). □

4.6. Putting (4.2)–(4.5) together we get a functorial isomorphism

$$H^{N-r}(X, \hat{\mathbf{G}}_{m,X}) \simeq H^N(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,\tilde{F}}), \tag{4.6.1}$$

where $\tilde{F} = \tilde{F}_{\{1,..r\}}$ is the ideal sheaf on $\mathbf{P}_K^N$ associated with the ideal of $K[T_0 \ldots, T_N]$ which is generated by the polynomial $F \overset{def}{=} F_1 \cdot \ldots \cdot F_r$. Note: deg $F = d$ and hence $\tilde{F} \simeq \mathcal{O}_{\mathbf{P}_K^N}(-d)$.

Thus we have translated the question of coordinatizing $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$ to the question of coordinatizing $H^N(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,\tilde{F}})$. The latter problem will be solved by Čech cocycle computations.

Let $\mathcal{U} = \{U_0, \ldots, U_N\}$ be the standard affine open covering of $\mathbf{P}_K^N$; i.e. U_i is the open subset where the homogeneous coordinate T_i is invertible.

4.7. Lemma. *$H^N(\mathbf{P}_K^N, \hat{\mathbf{G}}_{m,\tilde{F}}) \simeq \check{H}^N(\mathcal{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$.*

Proof. It suffices to check $H^i(U, \hat{\mathbf{G}}_{m,\tilde{F}}(A)) = 0$ for all $i \geq 1$, all affine open $U \subset \mathbf{P}_K^N$ and all nil-K-algebras A. The standard arguments reduce this to $H^i(U, \tilde{F} \otimes_K K\epsilon) = 0$ with $\epsilon^2 = 0$, which is a well-known result about the cohomology of coherent sheaves [4]. □

4.8. We fix for $\hat{\mathbf{G}}_m$ the coordinatization corresponding to the formal group law $\xi + \eta - \xi\eta$ (cf. (1.4)). Using this coordinatization we identify, for every nil-K-algebra A, the sheaf $\hat{\mathbf{G}}_{m,\tilde{F}}(A)$ with the sheaf $\tilde{F} \otimes_K A$ on which the group structure $\boxplus$ is given by the formal group law. Let $n = \binom{d-1}{N}$ and $J = \{(i_0, \ldots, i_N) \in \mathbf{Z}^{N+1} \mid \text{all } i_j \geq 1 \text{ and } i_0 + \cdots + i_N = d\}$. We define a functorial map

$$\jmath\colon \mathbf{A}^n_K \to \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$$

by setting, for a nil-K-algebra A and an element $(a_i)_{i\in J}$ of $A \times \cdots \times A$ (n factors, indexed by the elements of J), $\jmath((a_i)_{i\in J})$ = the cohomology class of the Čech N-cocycle given by $\boxplus_{i\in J} FT^{-i} \otimes a_i$ on $U_0 \cap \cdots \cap U_N$; here $T^{-i} = T_0^{-i_0} \cdot \ldots \cdot T_N^{-i_N}$ for $i = (i_0, \ldots, i_N) \in J$.

4.9. Theorem. *The map $\jmath$ is a functorial bijection.*

Proof. It suffices to prove this on the category of finitely generated nil-K-algebras, because from there it extends by inductive limits. For finitely generated nil-K-algebras, one proceeds by induction along small extensions. The induction starts with the observation that for algebras of the type $K\epsilon$ with $\epsilon^2 = 0$ the bijectivity of the map

$$(K\epsilon)^n \xrightarrow[\jmath]{} \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m\tilde{F}}(K\epsilon)) = \check{H}^N(\mathfrak{U}, \tilde{F} \otimes_K K\epsilon)$$

is a well-known fact about the cohomology of $\mathcal{O}_{\mathbf{P}^N}(-d)$. To continue the induction take a small extension $A \twoheadrightarrow \bar{A}$ with kernel $K\epsilon$ such that the result holds for $\bar{A}$. Then one has a short exact sequence

$$0 \to \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}}(K\epsilon)) \to \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}}(A)) \to \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}}(\bar{A})) \to 0,$$

where the left-hand 0 is due to the vanishing of $H^{N-1}(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(-d))$. Now let $\alpha \in \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}}(A))$ be given. Let $\bar{\alpha}$ be its image in $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}}(\bar{A}))$. Then there is, by the induction hypothesis, a unique $(\bar{a}_i)_{i\in J} \in \bar{A}^n$ such that $\bar{\alpha}$ is represented by the Čech N-cocycle $\{\boxplus_{i\in J} FT^{-i} \otimes \bar{a}_i\}$. Choose $a_i' \in A$ lifting $\bar{a}_i$ and let α' be the cohomology class of the cocycle $\{\boxplus_{i\in J} FT^{-i} \otimes a_i'\}$. Then $\alpha - \alpha'$ lies in $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}}(K\epsilon))$ and can therefore be represented by a cocycle $\{\boxplus_{i\in J} FT^{-i} \otimes k_i\epsilon\}$ with $k_i \in K$. So α is the cohomology class of the cocycle $\{\boxplus_{i\in J} (FT^{-i} \otimes a_i' + FT^{-i} \otimes k_i\epsilon - (FT^{-i})^2 \otimes a_i' k_i \epsilon)\}$. Since $\epsilon A = 0$, this cocycle is actually $\{\boxplus_{i\in J} FT^{-i} \otimes (a_i' + k_i\epsilon)\}$. This proves the surjectivity of $\jmath$ for A.

The argument for the injectivity of $\jmath$ is equally simple, and left to the reader. □

4.10. Corollary. *Setting* $c = j^{-1}$ *one obtains a coordinatization of* $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$. *Thus the functor* $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$ *is a* $\binom{d-1}{N}$*-dimensional formal group over K. The same is true for the functor* $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$, *which as a group valued functor is isomorphic to* $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$. □

4.11. We want to describe the formal group law corresponding to the coordinatization of $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$ given in (4.8). The same group law is associated with the coordinatization of $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$ which one obtains by composing the coordinatization of $\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,\tilde{F}})$ with the isomorphism of group valued functors given in (4.6.1) and (4.7). The situation of (4.1) can always be realized as a specialization of a similar situation in which K is flat over $\mathbf{Z}$. By (3.9) this specialization carries over to the formal group $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$. Also the coordinatizations commute with specialization. So to describe the formal group law we may henceforth assume that $K \to K \otimes \mathbf{Q}$ is injective. This enables us to give the formal group law by means of its logarithm.

In the arguments of (4.8)–(4.10) one can replace $\hat{\mathbf{G}}_m$ and its group law $\xi + \eta - \xi\eta$ by the additive formal group $\hat{\mathbf{G}}_a$ and its group law $\xi + \eta$. One then obtains a coordinatization

$$\gamma : \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{a,\tilde{F}}) \xrightarrow{\sim} \mathbf{A}_K^n$$

such that for every nil-K-algebra A and every element $(a_i)_{i\in J}$ in A^n $\gamma^{-1}((a_i)_{i\in J})$ is the cohomology class of the Čech N-cocycle $\{\Sigma_{i\in J} FT^{-i} \otimes a_i\}$. The corresponding formal group law is obviously the standard n-dimensional additive formal group law $(\xi_i + \eta_i)_{i\in J}$.

Let us now restrict to the category of nil-algebras over $K \otimes \mathbf{Q}$. There one has an isomorphism of formal groups $\hat{\mathbf{G}}_m \xrightarrow{\simeq} \hat{\mathbf{G}}_a$ given, in terms of the coordinatizations corresponding with the formal group laws $\xi + \eta - \xi\eta$ and $\xi + \eta$ respectively, by the power series $-\log(1 - u) = \Sigma_{m\geq 1} m^{-1}u^m$. This induces an isomorphism of formal groups over $K \otimes \mathbf{Q}$

$$\check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,F}) \simeq \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{a,F}).$$

The logarithm we are looking for is the n-tuple of power series which expresses this isomorphism in terms of the given coordinates; i.e. it is the n-tuple $(\ell_i(\tau))_{i\in J}$ of power series in the n-tuple of variables $\tau = (\tau_j)_{j\in J}$ determined by the cohomology class of the cocycle $\{\Sigma_{i\in J} FT^{-i} \otimes \ell_i(\tau)\}$ = the cohomology class of the cocycle $\{\Sigma_{j\in J} \Sigma_{m\geq 1} F^m T^{-mj} \otimes \tau_j^m/m\}$. Hence

$$\ell_i(\tau) = \sum_{m \geq 1} \sum_{j \in J} m^{-1} \beta_{m,i,j} \tau_j^m$$

with

$$\beta_{m,i,j} = \text{the coefficient of } T_0^{mj_0 - i_0} \cdot \ldots \cdot T_N^{mj_N - i_N} \quad \text{in} \quad F^{m-1}.$$

This concludes the proof of Theorem 1.

4.12. *Remark.* In order to obtain the above result we made three times a choice of coordinates. Firstly, there are the defining equations $F_1, \ldots, F_r$ for X. Secondly, there is the choice of the Čech N-cocycles $\{FT^{-i}\}$, $i \in J$, to represent a basis of $H^N(\mathbf{P}_K^N, \tilde{F})$. Thirdly, there is the coordinatization of $\hat{\mathbf{G}}_m$ corresponding to the formal group law $\xi + \eta - \xi\eta$. One can easily see from the presented algorithm how a change of these coordinate choices affects the formal group law for $H^{N-r}(X, \hat{\mathbf{G}}_{m,X})$.

4.13. *Example.* Take for X the Fermat hypersurface of degree d in $\mathbf{P}_{\mathbf{Z}}^N$:

$$T_0^d + \cdots + T_N^d = 0.$$

If $d = N + 1$, $H^{N-1}(X, \hat{\mathbf{G}}_{m,X})$ is a formal group over $\mathbf{Z}$ of dimension 1 with logarithm

$$\ell(\tau) = \sum_{m \geq 0} \frac{(md)!}{(m!)^d} \frac{\tau^{md+1}}{md + 1}.$$

This example of a 1-dim formal group law over $\mathbf{Z}$ is (essentially) one of the examples which Honda obtained in [7] by a completely different method.

If $d = N + 2$, $H^{N-1}(X, \hat{\mathbf{G}}_{m,X})$ is the direct sum of $N + 1$ copies of the 1-dim formal group law with logarithm

$$\sum_{m \geq 0} \frac{(md)!}{(m!)^N (2m)!} \frac{\tau^{md+1}}{md + 1}.$$

5.

5.1. In this section we prove Theorem 2. Recall the situation: K is a noetherian ring, F is a homogeneous polynomial of degree $2d$ over K in

$N+1$ variables $T_0, \ldots, T_N$ and X is the double covering of $\mathbf{P}_K^N$ given by the equation $W^2 = F$. We want to show that $H^N(X, \hat{\mathbf{G}}_{m,X})$ is a $\binom{d-1}{N}$-dimensional formal group over K and describe a formal group law for it.

5.2. The method is in this case almost the same as in Section 4. We shall therefore only indicate the relevant new details. Using the morphism $f: X \to \mathbf{P}_K^N$ and Čech cohomology with respect to the standard open covering $\mathfrak{U}$ of $\mathbf{P}_K^N$ one obtains, as in (4.6.1) and (4.7), a functorial isomorphism

$$H^N(X, \hat{\mathbf{G}}_{m,X}) \simeq \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X}).$$

The sheaf $f_*\mathcal{O}_X$ is isomorphic to $\mathcal{O}_{\mathbf{P}_K^N} \oplus \mathcal{O}_{\mathbf{P}_K^N}(-d)$. So $\check{H}^N(\mathfrak{U}, f_*\mathcal{O}_X)$ is a free K-module of rank $n = \binom{d-1}{N}$. To represent a basis we take the Čech N-cocycles given by WT^{-i} on $U_0 \cap \cdots \cap U_N$, where i ranges over the set $J = \{i = (i_0, \ldots, i_N) \in \mathbf{Z}^{N+1} \mid i_0, \ldots, i_N \geq 1, i_0 + \cdots + i_N = d\}$ and $T^{-i} = T_0^{-i_0} \cdot \ldots \cdot T_N^{-i_N}$ (cf. (4.8)).

Then, as in (4.8)-(4.9), we get a functorial bijection

$$\jmath: \mathbf{A}_K^1 \rightrightarrows \check{H}^N(\mathfrak{U}, \hat{\mathbf{G}}_{m,f_*\mathcal{O}_X})$$

by defining for a nil-K-algebra A and an element $(a_i)_{i\in J}$ of A^n $\jmath((a_i)_{i\in J}) =$ the cohomology class of the Čech N-cocycle given by $\boxplus_{i\in J} WT^{-i} \otimes a_i$ on $U_0 \cap \cdots \cap U_N$.

As in (4.10) we conclude that $H^N(X, \hat{\mathbf{G}}_{m,X})$ is a formal group over K of dimension $\binom{d-1}{N}$.

5.3. In order to describe the formal group law for $H^N(X, \hat{\mathbf{G}}_{m,X})$ which corresponds to the above coordinatization, one first argues as in (4.11) that it suffices to do this under the additional hypothesis that $K \to K \otimes \mathbf{Q}$ is injective, and then one determines $\ell(\tau) = (\ell_i((\tau_j)_{j\in J}))_{i\in J}$ exactly as in (4.11). So: the cohomology class of the cocycle $\{\Sigma_{i\in J} WT^{-i} \otimes \ell_i(\tau)\} =$ the cohomology class of the cocycle $\{\Sigma_{m\geq 1} \Sigma_{j\in J} W^m T^{-mj} \otimes \tau_j^m/m\}$. For even m $\{W^m T^{-mj} \otimes \tau_j^m/m\}$ is a coboundary because $W^m = F^{m/2}$ is in $K[T_0, \ldots, T_N]$. For odd m the cohomology class of $\{W^m T^{-mj} \otimes \tau_j^m/m\}$ is equal to the class of $\{\Sigma_{i\in J} WT^{-i} \otimes \beta_{m,i,j}\tau_j^m/m\}$ with

$$\beta_{m,i,j} = \begin{cases} \text{the coefficient of } T_0^{mj_0-i_0} \cdot \ldots \cdot T_N^{mj_N-i_N} \text{ in } F^{(m-1)/2} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Thus we get

$$\ell_i(\tau) = \sum_{m \geq 1} \sum_{j \in J} m^{-1} \beta_{m,i,j} \tau_j^m.$$

This completes the proof of Theorem 2.

5.4. *Remark.* See (4.12).

5.5. *Example.* Take the $K3$-surface with Néron-Sevéri lattice of rank 20 and discriminant 4. It can be constructed from the surface with equation

$$W^2 = T_0 T_1 T_2 (T_0 - T_1)(T_1 - T_2)(T_2 - T_0),$$

which has rational singularities, by taking the minimal resolution (cf. [9]). The logarithm for the corresponding formal group over $\mathbf{Z}$ is

$$\ell(\tau) = \sum_{m \geq 0} (-1)^m \frac{(3m)!}{(m!)^3} \frac{\tau^{4m+1}}{4m+1}$$

(see (5.3), (3.10)).

To illustrate the remarks (5.4) and (4.12) replace the coordinates T_0, T_1, T_2 on $\mathbf{P}^2_{\mathbf{Z}}$ by $T_0' = T_0$, $T_1' = T_1 - T_2$, $T_2' = T_2 - T_0$. The equation of the surface becomes

$$W^2 = -T_0' T_1' T_2' (T_1' + T_2')(T_2' + T_0')(T_0' + T_1' + T_2').$$

The logarithm associated with this equation (and the choice of the cocycle $\{W/T_0' T_1' T_2'\}$) is

$$\ell(\tau) = \sum_{m \geq 0} \left((-1)^m \sum_k \binom{m}{k}^2 \binom{m+k}{k} \right) \frac{\tau^{2m+1}}{2m+1}.$$

The above examples can also be found in [9]. Also the other examples of op. cit., but not its main theorem, are covered by this section.

Appendix. In this appendix (formula A3) we re-interpret the power series $\ell(\tau)$ of Theorem 2. The situation is as in Theorem 2, with the additional hypotheses that one has fixed an embedding $K \hookrightarrow \mathbf{C}$ and that $F(1, 0, \ldots, 0) \neq 0$. We work in the complex analytic setting.

Take affine coordinates $t_m = T_m/T_0$ and set $P(t_1, \ldots, t_N) = F(1, t_1, \ldots, t_N)$. Choose a positive real number ϵ such that the function P vanishes nowhere on the ball $\{(t_1, \ldots, t_N) \in \mathbf{C}^N \mid |t_1|^2 + \cdots + |t_N|^2 \leq N\epsilon^2\}$. Let $I = [0, 1]$ be the unit interval in $\mathbf{R}$. For $k = (k_0, \ldots, k_N) \in \mathbf{Z}^{N+1}$ put $t^k = t_1^{k_1} \cdot \ldots \cdot t_N^{k_N}$. Let $e = (1, \ldots, 1) \in \mathbf{Z}^{N+1}$.

Fix $i, j \in J$. From the definition of $\beta_{m,i,j}$ in Theorem 2 and Cauchy's residue theorem we get, for sufficiently small complex values of τ_j

$$(*) \qquad \sum_{m\geq 1} m^{-1}\beta_{m,i,j}\tau_j^m = (2\pi\sqrt{-1})^{-N} \int \frac{\tau_j t^{i+j-e} dt_1 \wedge \cdots \wedge dt_N \wedge du}{t^{2j} - \tau_j^2 u^2 P}$$

where the domain of integration is

$$\{(t_1, \ldots, t_N, u) \in \mathbf{C}^N \times I \mid |t_1| = \cdots = |t_N| = \epsilon u\}$$

with the appropriate orientation. Integrating first with respect to t_1 one can apply the residue theorem again. If τ_j is small enough one encounters only simple poles. Hence the integral in (*) is equal to the integral

$$(2\pi\sqrt{-1})^{1-N} \int [2j_1 t^{2j} t_1^{-1} - \tau_j^2 u^2 (\partial P/\partial t_1)]^{-1} \tau_j t^{i+j-e} dt_2 \wedge \cdots \wedge dt_N \wedge du$$

with domain of integration

$$\{(t_1, \ldots, t_N, u) \in \mathbf{C}^N \times I \mid |t_1| < |t_2| = \cdots = |t_N| = \epsilon u,$$
$$(t^j \tau_j^{-1} u^{-1})^2 = P\}.$$

The substitution $w = t^j \tau_j^{-1} u^{-1}$ transforms this integral to

$$(2\pi\sqrt{-1})^{1-N} \int_{\Gamma_j(\tau_j)} \omega_i$$

with

$$\text{(A1)} \qquad \omega_i = \frac{t^{i-e} dt_1 \wedge \cdots \wedge dt_N}{2w}$$

and

$$\text{(A2)}\quad \Gamma_j(\tau_j) = \{(w, t_1, \ldots, t_N) \in \mathbf{C}^{N+1} \mid w^2 = P,\ \tau_j^{-1}w^{-1}t^j \in I,$$
$$|t_1| < |t_2| = \cdots = |t_N| = \epsilon\tau_j^{-1}w^{-1}t^j\}.$$

Thus we see that, for sufficiently small complex values of the parameters $\tau_j (j \in J)$, the map $\ell(\tau)$ of Theorem 2 is also given by the system of integrals

$$\text{(A3)}\qquad \ell(\tau) = \left[(2\pi\sqrt{-1})^{1-N} \sum_{j\in J} \int_{\Gamma_j(\tau_j)} \omega_i\right]_{i\in J}.$$

The orientation of the chains $\Gamma_j(\tau_j)$ must agree with $\ell(\tau) \equiv \tau \bmod \deg \geq 2$.

(A4). *Comments.* Assume for simplicity that X is smooth. Then

(a) The forms $\omega_i (i \in J)$ constitute a basis of $H^0(X, \Omega^N_{X/K})$.

(b) The coordinates of the formal group law should apparently be viewed as functions on X: τ_j corresponds with the function T^jW^{-1}.

(c) The domain of integration $\Gamma_j(\tau_j)$ is a homology N-chain in the analytic space associated with X and the embedding $K \subset \mathbf{C}$. In case $N > 1$, its boundary,

$$\gamma_j(\tau_j) = \{(w, t_1, \ldots, t_N) \in \mathbf{C}^{N+1} \mid w^2 = P,\ t^jw^{-1} = \tau_j,$$
$$|t_1| < |t_2| = \cdots = |t_N| = \epsilon\},$$

is contained in the fiber of the function T^jW^{-1} over τ_j. The domain of integration, $\Gamma_j(\tau_j)$, is a sort of cone over $\gamma_j(\tau_j)$. For $N = 1$ one has a similar result, and the domain of integration is a union of paths.

(d) The integrals in (A3) change only by periods, if $\Gamma_j(\tau_j)$ is replaced by another N-chain $\Gamma_j'(\tau_j)$ with boundary $\gamma_j'(\tau_j)$ contained in $X_{\tau_j} = \{(W, T_0, \ldots, T_N) \in X \mid T^jW^{-1} = \tau_j\}$ and with $\gamma_j'(\tau_j)$ homologous to $\gamma_j(\tau_j)$ on X_{τ_j}. Unfortunately, for $N > 1$ the periods only very rarely form a discrete subgroup of $H^{0,N}(X)$.

So, (A3) bears an interesting analogy with the classical case of (hyper-) elliptic integrals. These integrals deserve further study.

Remark. Integrals of the kind appearing in (A3), appear also in R. Friedman's work on the mixed Hodge structure of an open variety [3].

Example. If $N = 1$, $d = 2$ and $F = T_0T_1^3 + aT_0^3T_1 + bT_0^4$, one is actually looking at the Weierstrass model of an elliptic curve, usually written as $y^2 = x^3 + ax + b$. The differential which appears in the above considerations is the standard form $\omega = dx/2y$. The domain of integration $\Gamma(\tau)$ has two connected components, $\Gamma^+(\tau)$ and $\Gamma^-(\tau)$. Take the projective completion of the affine curve $y^2 = x^3 + ax + b$ in $\mathbf{P}^2$. Let $\Gamma^\infty(\tau)$ be a path "close to ∞" which connects the point ∞ with the point whose (x, y)-coordinates satisfy $y^2 = x^3 + ax + b, x/y = \tau$ and $|x| > \epsilon$. By Abel one knows

$$\int_{\Gamma(\tau)} \omega = \int_{\Gamma^+(\tau)} \omega + \int_{\Gamma^-(\tau)} \omega = -\int_{\Gamma^\infty(\tau)} \omega.$$

Thus

$$\ell(\tau) = -\int_{\Gamma^\infty(\tau)} \omega.$$

So, for the Weierstrass model of an elliptic curve we find the logarithm of the standard (formal) group structure on the elliptic curve (cf. [11]), at least if in the model $b \neq 0$. (The latter condition can in fact easily be removed.)

Remark. As a by-product of the previous example we find, that in terms of the coordinate $u = -x/y$ near ∞ the power series expansion of ω is

$$\omega = \sum_{n \geq 0} \beta_{2n+1} u^{2n} du$$

with

$$\beta_{2n+1} = \text{the coefficient of } x^{2n} \quad \text{in} \quad (x^3 + ax + b)^n.$$

In the literature one finds such a formula for the expansion coefficients of ω only for the p-th coefficient mod p (p a rational prime number). That is Deuring's formula for the Hasse invariant of the elliptic curve (cf. [11]). The "global" result giving all coefficients of the expansion of ω apparently remained unnoticed, or unpublished. The first proof I saw was given by Beukers [2].

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