The Ω-Spectrum for Brown-Peterson Cohomology, Part III.

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ABSTRACT In this paper we give a description of the cohomology operations on the module of indecomposibles for the cohomology of each space in the Ω-spectrum for Brown-Peterson cohomology.

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INTRODUCTION

Let $BP = \{BP_k\}$ be the Brown-Peterson spectrum given as an Ω-spectrum. In [9] the cohomologies of all the spaces BP_k were computed. In [6] the homology of these spaces is computed giving much more information, in particular, a technique for computing the comodule structure of H_*BP_k over the Steenrod algebra is given. (All coefficients in this paper are Z_p, where p is the prime associated with the Brown-Peterson spectrum). In this paper an entirely different approach is taken to cohomology operations. We give a description of the structure of the indecomposibles QH^*BP_k as a module over the Steenrod algebra. This may occasionally be easier to compute with than the results of [6].

Let R be a sequence of non-negative integers $(r_1,r_2,\ldots)$. Define $d(R) = \Sigma\, 2r_i(p^i-1)$, the degree of R. Define $\ell(R) = \Sigma\, r_i$. Let V_j be the graded group free over $Z_{(p)}$ with a generator for each sequence R with $\ell(R) = j$ and grade d(R). We can now form the generalized Eilenberg-MacLane spectrum $K(V_j)$ with Ω-spectrum $\{K(V_j,k)\}$. We have fundamental generators $i_R \in H^*K(V_j,k)$ for each R with $\ell(R) = j$ and i_R has degree $d(R) + k$. We have a map of modules over the Steenrod algebra

$$s^*(d_j) : QH^*K(V_j,k+1) \longrightarrow QH^*K(V_j,k)$$

defined by $s^*(d_j)\ (i_R) = \Sigma\ Q_i\, i_{R-\Delta_i}$ where Δ_i is the sequence with one in the $i^{\underline{th}}$ place and zeros everywhere else and Q_i is the Milnor primitive ([4]). Any element a of $QH^*K(V_j,k)$ can be written as $\Sigma\ a_R\, i_R (\ell(R) = j)$ where $a_R \in A_p$ the Steenrod algebra. If a can be written such that each a_R

is in the algebra of reduced $p^{\underline{th}}$ powers (see section 2 for $p = 2$) we say a is without Bocksteins.

<u>Theorem A</u> The kernel of the map $s^*(d_j)$ restricted to all $a \in QH^*K(V_j,2k)$ with no Bocksteins is a module over the Steenrod algebra which is isomorphic to M_j/M_{j-1} in a filtration of QH^*BP_{2k}

$$QH^*BP_{2k} = M_\infty \supset \dots M_j \supset M_{j-1} \supset \dots M_o \supset 0$$

by modules over the Steenrod algebra.

<u>Remark 1</u>. H^*BP_{2k} is a polynomial algebra and H^*BP_{2k-1} is an exterior algebra. (See [9].) Likewise for Y_{2k} and Y_{2k-1} below.

<u>Remark 2</u>. The filtration is finite on each degree.

<u>Remark 3</u>. From [9] we know that the cohomology suspension gives an isomorphism $s^* : QH^*BP_{2k} \longrightarrow QH^*BP_{2k-1}$ of modules over the Steenrod algebra so the theorem determines the structure for BP_{2k-1} as well.

Because the space BP_k can be broken up into the product of much smaller spaces ([10]) this computation can be made even more accessible.

In [10], irreducible spaces Y_k for $k > 0$ were constructed and the following homotopy equivalence was proved:

$$BP_k \simeq Y_k \times \prod_{j>n} Y_{k+2(p^j-1)}$$

when $2(p^{n-1}+\dots+p+1) < k \leq 2(p^n+\dots+p+1)$. For k in the above range we can now give a description of QH^*Y_k as a module over the Steenrod algebra. Let $V(n)_j$ be the graded subgroup of V_j where we only use sequences R with $r_i = 0$ for $i > n$. We have the same map

$$s^*(d_j) : QH^*K(V(n)_j,k+1) \longrightarrow QH^*K(V(n)_{j-1},k)$$

defined by $s^*(d_j)\,(i_R) = \Sigma\, Q_i\, i_{R-\Delta_i}$.

Theorem B. For $2(p^{n-1}+\ldots+p+1) < 2k \leq 2(p^n+\ldots+p+1)$, the kernel of the map $s^*(d_j)$ restricted to all $a \in QH^*K(V(n)_j, 2k)$ with no Bocksteins is a module over the Steenrod algebra which is isomorphic to M_j/M_{j-1} in a filtration of QH^*Y_{2k}

$$QH^*Y_{2k} = M_\infty \supset \ldots M_j \supset M_{j-1} \supset \ldots M_o \supset 0$$

by modules over the Steenrod algebra.

Remark 4. The cohomology suspension homorphism gives an isomorphism of modules over A_p $s^* : QH^*Y_{2k} \longrightarrow QH^*Y_{2k-1}$.

Using this result, one can quite easily compute by hand the explicit Steenrod algebra structure of QH^*BP_k as far as one wants. (Try using [2].)

My apologies to anyone that is actually interested in seeing the proofs of the above theorems. You will learn quickly enough that complete intimacy with the notation and techniques of [9] is necessary for reading past the introduction. From your point of view, [9] and this paper should be one, however, the thoughts in this paper came somewhat later, and, it would be morally inexcusable to afflict the rather nice result of [9] with the presence of detailed proofs of the above theorems.

These results were motivated by three things; first, several people kept asking me what I knew about the cohomology operations on the spaces BP_k and Y_k. This paper is intended to supply an answer to that question. Second, when the results were originally proven, serious applications existed. A much better approach to the applications has since been found. The short-lived applications will be discussed briefly in section 3. Third, I am convinced (through computations) that analogous theorems exist for the homology and homology operations. I had always hoped to intertwine these hypothetical theorems and the present ones in a much more informative proof of the main theorem of [9] ($H^*(BP_k, Z_{(p)})$ has no torsion.) which would eliminate the messy details in [9] and here. However, I have made no progress here and the new proof given in [6] satisfies my personal quest. I do suggest that the analogous result dealing with homology operations could be a very interesting problem to anyone who solves it.

Technical propositions and a proof of theorem A appear in section 1. Theorem B is proven in section 2.

Remark 5. It is interesting to note that the results of this paper imply that every generator for H^*Y_k can be defined by some higher order cohomology operation on the k-dimensional class which arises because β is always zero as there is no torsion.

Section 1. Theorem A

Let us consider BP_{2k}. Here we have:

1.1
$$\begin{array}{ccccc} K(V_j, 2k) & \xrightarrow{i_j} & X^j & & \\ & & \downarrow & & \\ & & X^{j-1} & \xrightarrow{k_{j-1}} & K(V_j, 2k+1) \end{array}$$

where $BP_{2k} = \text{inv. lim. } X^j$. There is a filtration on H^*X^j such that

1.2 $$E_0 H^* X^j \simeq H^* X^{j-1} // k^*_{j-1} \otimes E[\ldots] \otimes TP[\ldots]$$ where E denotes an exterior algebra on odd degrees and TP is a truncated polynomial algebra of height p on even dimensional generators. The generators of E and TP are both determined by $QH^*K(V_j,2k+1)\backslash\backslash(k_{j-1})^*$.

There is a natural map PH $\longrightarrow$ QH, where P and Q denote the primitives and indecomposibles of a Hopf algebra H. When this is onto, H is called primitive.

Lemma 1.3. (1.1 in [9]) $H' \subset H$ a sub-Hopf algebra over Z_p, H primitive, then H' is primitive.

If V is a graded module, let $s^{-1}V$ be the graded module with degrees lowered by one. TP is generated by s^{-1} of the odd degree part of $Q(H^*K(V_j,2k+1)\backslash\backslash(k_{j-1})^*)$, and E is generated by a quotient of the even degree part, $H^*K(V_j,2k+1)$ is primitive so by 1.3, $H^*K(V_j,2k+1)\backslash\backslash(k_{j-1})^*$ is too. So, for $x \in Q(H^*K(V_j,2k+1)\backslash\backslash(k_{j-1})^*)$ we have $x' \longrightarrow x$, $x' \in P(H^*K(V_j,2k+1)\backslash\backslash(k_{j-1})$ Let $x' \longrightarrow x'' \in PH^*K(V_j,2k+1)$. $s^{-1}(x)$ is in the module of indecomposibles for E or TP. It is possibly zero.

Lemma 1.4. (1.3 of [9])

$$(i_j)^*(s^{-1}(x)) = s^*(x'')$$

s^* the cohomology suspension homomorphism.

We need just a couple more lemmas before we get started.

Lemma 1.5. (1.4 of [9]) Let $a \in PH^m K(V_j,2k+1)$. If $s^*(a) = 0$, then $a = P^t x_{2t}$ or $a = \beta P^t x_{2t+1}$ where $P^i \in A$ is the $i^{\underline{th}}$ reduced $p^{\underline{th}}$ power, A is the Steenrod algebra and x_i has degree i.

Lemma 1.6. (2.2 of [9])

If $a \in H^{2m}K(V_j,2k+1)$ and $(i_{j-1})^* \circ (k_{j-1})^*(a) = 0$, then there exists $b \in H^{2m-1}K(V_{j+1},2k+1)$ such that $s^*(a) = (i_j)^* \circ (k_j)^*(b)$.

Now we can get to work and prove our main technical proposition.

Proposition 1.7 (j)

a(j) $H^*X^j \backslash\backslash (i_j)^* = H^*X^{j-1}//(k_{j-1})^*$

b(j) $H^*X^{j-1}//(k_{j-1})^* \subset H^*X^j$ injects into H^*X^{j+1}

c(j) $H^*X^j//(k_j)^*$ is a polynomial algebra on even dimensional generators.

d(j) The maps in b(j) are split as algebras.

Proof of a(j). Assume for j-1. The induction is easy to start. We have 1.2. $H^*X^{j-1}//(k_{j-1})^*$ is clearly in $H^*X^j \backslash\backslash (i_j)^*$, so all we need to show for a(j) is that the generators of E and TP inject into $H^*K(V_j,2k)$. By lemmas 1.4 and 1.5 applied to our description of generators for TP we see that they inject. By the same argument, the only generators of E that can possibly go to zero must be of the form $\beta P^t x_{2t+1}$ or $P^t x_{2t}$ in $Q(H^*K(V_j,2k+1) \backslash\backslash (k_{j-1})^*)$.

Case 1. $\beta P^t x_{2t+1}$

If $(k_{j-1})^*(\beta P^t x_{2t+1}) = 0$, then $(k_{j-1})^*(x_{2t+1}) = 0$. Proof: If $(k_{j-1})^*(x_{2t+1}) \neq 0$, then by b(j-1), it is not in $H^*X^{j-2}//(k_{j-2})^* \subset H^*X^{j-1}$ so since it is an odd dimensional primitive it must land in E (for X^{j-1}). By a(j-1) it goes on through to $H^*K(V_{j-1},2k)$ and there βP^t is injective on

2t+1 dimensional classes ([7]) so $(i_{j-1})^* \circ (k_{j-1})^* (\beta P^t x_{2t+1}) \neq 0$. This contradicts our assumption that $(k_{j-1})^* (\beta P^t x_{2t+1}) = 0$, so $(k_{j-1})^* (x_{2t+1}) = 0$. By [8] there is a differential in the Eilenberg-Moore spectral sequence that gives 1.2 which goes from $\gamma_p(s^{-1}x_{2t+1})$ to $s^{-1}(\beta P^t x_{2t+1})$, i.e., $\beta P^t x_{2t+1}$ is zero in the quotient of $s^{-1}Q(H^*K(V_{j,2k+1}) \backslash\backslash (k_{j-1})^*)$ which gives the generators of E.

<u>Case 2</u> $P^t x_2$.

The same type of argument as above shows that $(k_{j-1})^* (P^t x_{2t}) = 0$ implies $(k_{j-1})^* (x_{2t}) = 0$. In that case, $P^t x_{2t}$ is a p^{th} power in $H^*K(V_{j}, 2k+1) \backslash\backslash (k_{j-1})^*$ and therefore zero in the indecomposibles.

<u>Proof of b(j)</u> If b(j) is not true, then there must be some $a \in H^{2m} K(V_{j+1}, 2k+1)$ with $(k_j)^*(a) \in H^*X^{j-1}//(k_{j-1})^*$. a must be of even degree by c(j-1). Now by a(j), $(i_j)^* \circ (k_j)^*(a) = 0$. If $s^*(a) \neq 0$, by lemma 1.6 there must be a b with $(i_{j+1})^* \circ (k_{j+1})^*(b) = s^*(a)$. But if this is so, then $s^{-1}(a)$ must be in $E_o\, H^*X^{j+1}$ for $(k_{j+1})^*$ (b) to hit, thus $(k_j)^*(a) = 0$. This is a contradiction, so $s^*(a) = 0$. By lemma 1.5 $a = P^t x_{2t}$ or $\beta P^t x_{2t+1}$.

<u>Case 1</u> $a = \beta P^t x_{2t+1}$

If $(k_j)^*(x_{2t+1}) \neq 0$, then by c(j-1) it must lie in E and by a(j), $(i_j)^* \circ (k_j)^* (x_{2t+1}) \neq 0$. By the injectivity of βP^t in $H^*K(V_j, 2k)$ we have that $(i_j)^* \circ (k_j)^* (\beta P^t x_{2t+1} = a) \neq 0$. This contradicts our assumption so $(k_j)^* (x_{2t+1}) = 0$ and this implies $(k_j)^* (\beta P^t x_{2t+1} = a) = 0$.

Case 2 $a = P^t x_{2t}$

Assume $P^t x_{2t} = a$ is the element of minimal degree with $(k_j)^*(a) \in H^*(X^{j-1})//(k_{j-1})^*$. Clearly $(k_j)^*(x_{2t}) \neq 0$ and so lies in TP by our minimality condition. By a(j), x_{2t} goes on to $H^* K(V_j, 2k)$ non-zero but $P^t x_{2t}$ cannot. This contradicts the fact that the p^{th} power of the image of x_{2t} in $H^* K(V_j, 2k)$ will be non-zero.

Proof of c(j)

First we must prove that everything in E for $E_0 H^* X^j$ gets killed by $(k_j)^*$. Use a(j) and lemma 1.6 with techniques similar to those already repeated several times.

For TP, I claim that in $H^* X^j$ (as opposed to $E_0 H^* X^j$) this is a polynomial algebra. A generator for TP comes from $x_{2t+1} \in Q(H^* K(V_j, 2k+1) \backslash\backslash (k_{j-1})^*)$ and by a(j), $(i_j)^*(s^{-1})(x_{2t+1})) \neq 0$. $P^t((i_j)^*(s^{-1}(x_{2t+1}))) \neq 0$ because it is a p^{th} power in a product of Eilenberg-MacLane spaces. By a(j), $P^t(s^{-1}(x_{2t+1})) = (s^{-1}(x_{2t+1}))^p$ must be $s^{-1}(P^t x_{2t+1})$.

The problem is to show that if $(k_j)^*$ gets a p^{th} power, say $s^{-1}(P^t x_{2t+1}) = (k_j)^*(a)$, then it also hits $s^{-1}(x_{2t+1})$. Using the techniques of [9] one can show that x_{2t+1} must be "with Bocksteins". This can be used to show the necessary result.

The proof of d(j) is nothing exciting and is left to the diligent reader of this paper and [9].

We have the following corollaries to the proposition.

Corollary 1.8 The diagram 1.1 gives rise to an exact sequence in the category of Hopf algebras.

Corollary 1.9

$$H^*X^j//(k_j)^* \longrightarrow H^*BP_{2k}$$

is injective and splits as a map of algebras.

Corollary 1.10

$$Q(H^*X^j//(k_j)^*) \longrightarrow Q\, H^*BP_{2k}$$

is injective.

We can now define the filtration:

1.11 $QH^*BP_{2k} = M_\infty \supset \ldots \supset M_j \supset M_{j-1} \supset \ldots M_o \supset 0$ by setting $M_j = \text{image } Q(H^*X^j//(k_j)^*)$. By corollary 1.10 and the construction 1.1 of BP_{2k} this is well-defined. We have also already computed M_j/M_{j-1} although not yet in quite the nice form stated in the theorem in the introduction.

If $a \in PH^*K(V_j,k)$, we say a is with Bocksteins if $a = \Sigma\, a_R i_R$ ($\ell(R) = j$) where each $a_R \in (Q_o)$ the left-right ideal generated by the Bockstein $Q_o = \beta$. We say a is with no Bocksteins if each a_R is in the algebra of reduced $p^{\underline{th}}$ powers. (For p = 2, if each a_R can be written in terms of the Milnor basis $Sq^{(I)}$ with all sequences made of even integers.) We have the map

$$s^*(d_j) : H^*K(V_j,k+1) \longrightarrow H^*K(V_{j-1},k)$$

defined by $s^*(d_j)\,(i_R) = \Sigma\, Q_i i_{R-\Delta_i}$.

Each $a \in PH^*K(V_j,k)$ can be written as $a = b + c$ where b is with Bocksteins and c is with no Bocksteins. In [9,p.50] we showed:

Lemma 1.12 $s^*(d_j)\ (a) = 0$ if and only if both $s^*(d_j)\ (b) = 0$ and $s^*(d_j)\ (c) = 0$.

We also proved a much stronger lemma than we quoted in 1.6.

Lemma 1.13 (2.2 of [9])

If $a \in PH^*K(V_j,k)$ with Bocksteins and $s^*(d_j)\ (a) = 0$, then there exists $b \in PH^*K(V_{j+1},k)$ such that $s^*(d_{j+1})\ (b) = s^*(a)$.

Note 1.14 $s^*(d_j)$ and $(i_{j-1})^* \circ (k_{j-1})^*$ are the same.

Note 1.15 We really needed 1.13 in the proof of c(j) when we referred the reader to techniques of [9].

Now, by the definition of the filtration M_* (1.11), and the proposition 1.7, we can compute M_j/M_{j-1}. We know that $(k_j)^*$ kills the odd dimensional generators in 1.2 and $Q(H^*X^j//(k_j)^*)$ is even dimensional. Even dimensional generators of H^*X^j which do not come from $H^*X^{j-1}//(k_{j-1})^*$ come from TP in 1.2. Everything in Q(TP) comes from an $a \in PH^*K(V_j,2k+1)$ (for BP_{2k}) with $s^*(d_j)\ (a) = 0$. From 1.12, $a = b + c$ and $s^*(d_j)\ (b) = 0 = s^*(d_j)\ (c)$. By 1.5 $s^*(b) \neq 0$ (if $b \neq 0$) and by 1.13, $s^{-1}(b)$ is killed by $(k_j)^*$ and so we need only concern ourselves with $a = c$, i.e., a with no Bocksteins. Furthermore, since anything in the image of $s^*(d_{j+1})$ is with Bocksteins, we have essentially computed M_j/M_{j-1}. Many of the generators of TP are not generators of $H^*X^j//(k_j)^*$ because they are p^{th} powers, but that can be detected in $H^*K(V_j,2k)$. We have shown that M_j/M_{j-1} is the set of all $s^*(a) \in QH^*K(V_j,2k)$ where $a \in PH^*K(V_j,2k+1)$ is with no Bocksteins and $s^*(d_j)\ (a) = 0$. This is not in the form we give in the introduction yet either. We need the following diagram and lemma.

1.16

$$\begin{array}{ccc}
QH^*K(V_{j-1},2k-1) & \xleftarrow{s^*(d_j)} & QH^*K(V_j,2k) \\
\uparrow & & \uparrow \\
PH^*K(V_{j-1},2k-1) & \xleftarrow{s^*(d_j)} & PH^*K(V_j,2k) \\
\uparrow s^* & & \uparrow s^* \\
QH^*K(V_{j-1},2k) & \xleftarrow{s^*(d_j)} & QH^*K(V_j,2k+1) \\
\uparrow & & \uparrow \\
PH^*K(V_{j-1},2k) & \xleftarrow{s^*(d_j)} & PH^*K(V_j,2k+1)
\end{array}$$

In the above diagram the map $P \longrightarrow Q$ is the standard map.

<u>Lemma 1.17</u>. The set of all $a \in QH^*K(V_j,2k)$ with no Bocksteins such that $s^*(d_j)$ (a) is the same as the set of all $s^*(a') \in QH^*K(V_j,2k)$, $a' \in PH^*K(V_j,2k+1)$ with no Bocksteins and $s^*(d_j)\ (a') = 0$.

<u>Sketch proof</u>. We have the commutative diagram 1.16 to aid us in the visualization of the proof. By commutativity, every such $s^*(a')$ is contained in the first set and our only problem is to go the other way. All the vertical maps are onto, in fact, for $a \in QH^*K(V_j,2k)$ with no Bocksteins there exists $a' \in PH^*K(V_j,2k+1)$ with no Bocksteins that goes down to it. All we need to show is that $s^*(d_j)\ (a') = 0$. a' is an odd dimensional element and $P \longrightarrow Q$ is an isomorphism on odd dimensional elements. Also, s^* is injective on odd dimensional elements by 1.5 so we really have $s^*(d_j)\ (a') = 0$ iff the lifting of a to $a'' \in PH^*K(V_j,2k)$ has $s^*(d_j)\ (a'') = 0$. By commutativity, $s^*(d_j(\ (a'') = (x_{2t})^p = P^t x_{2t}$ as this is the only type of element in the kernel of the map $P \longrightarrow Q$

and we must have zero in the upper left hand corner of 1.16. Now, using the techniques of [9] it is easy to show that x_{2t} is with Bocksteins and $s^*(d_{j-1})(x_{2t}) = 0$. Using lemma 2.2(j) of [9] we can find a $y_{2t} \in PH^*K(V_j,2k)$ with no Bocksteins such that $s^*(d_j)(y_{2t}) = x_{2t}$. If we alter our choice for a" by adding $-P^t y_{2t}$ we obtain the desired result.

This concludes the proof of theorem A stated in the introduction.

<u>Section 2</u> The Y_k and Theorem B

As the BP_k can be built up out of smaller spaces (see the theorem of [10] mentioned in the introduction), it is to our advantage to be able to compute with the Y_k instead of the much larger spaces BP_k.

Y_k for $2(p^{n-1}+\ldots+p+1) < k \leq 2(p^n+\ldots+p+1)$ is the $k^{\underline{th}}$ space in the Ω-spectrum for $BP < n >$, a BP module spectrum with $\pi_* BP < n >$ $= \pi_* BP/(v_{n+1}, v_{n+2}, \ldots)$ where $\pi_* BP = Z_{(p)}[v_1, v_2, \ldots]$ with degree $v_i = 2(p^i - 1)$. $H^* BP = A/A(Q_o, Q_1, Q_2, \ldots)$ and $H^* BP < n > = A/A(Q_o, \ldots, Q_n)$.

The theories $BP < n >$ have their applications in [1] and [3] as well as in [10].

For k in the above range, $BP_k = Y_k \times X = BP < n >_k \times X$. The map $BP_k \longrightarrow Y_k$ comes from the map of BP to $BP < n >$. Since we have this splitting, the "Adams-Postnikov" construction for BP_k (1.1)

$$\begin{array}{ccc} K(V_j, k) & \longrightarrow & X_j \\ & & \downarrow \\ & & X_{j-1} \longrightarrow K(V_j, k+1) \end{array}$$

must also split at each stage. So, all we really need to do in order to determine the split off system for Y_k is to identify the homotopy groups of Y_k in terms of those for BP_k. This follows easily from the fact that $v^I = v_1^{i_1} v_2^{i_2} \ldots$ in $\pi_* BP$ corresponds to the I generator in $\pi_* K(V_{\ell(I)}, k)$ modulo higher order products. A little thought (or a little work with the Adams spectral sequence for BP and $BP < n >$) will verify this and convince the reader that the splitting is as in the introduction, i.e., Y_k is built from a system

$$\begin{array}{ccc} K(V(n)_j,k) & \xrightarrow{i(n)_j} & X(n)_j \\ & & \Big| \\ & & X(n)_{j-1} \xrightarrow{k(n)_{j-1}} K(V(n)_j,k+1) \end{array}$$

where $V(n)_j$ is the graded group, free over $Z_{(p)}$ with a generator for each $R = (r_1,r_2,\ldots)$ with $\ell(R) = j$ and $r_i = 0$ for $i > n$. We have the same maps

$$(i(n)_{j-1})^* \circ (k(n)_{j-1})^* \, (i_R) = s^*(d(n)_j) \, (i_R) = \Sigma Q_i i_{R-\Delta_i}$$

Now, by the fact that each stage splits off from that for BP_k we have injections on all of our cohomology and we can just read off theorem B.

As a novel application we have $BU_{(p)} \simeq \prod_{i=1}^{p} Y_{2i}$ and we have therefore described the Steenrod algebra structure of QH^*BU.

Section 3. Past applications.

In [6] we consider the monster space

$$\mathbb{BP} = \text{dir. lim.}(n \to -\infty) \prod_{k \geq n} BP_{2k}$$

Not only is this an H-space because it is a loop space ($\Omega^2 \mathbb{BP} = \mathbb{BP}$), but there is another product which comes from the fact that BP is a ring spectrum

$$BP_{2k} \wedge BP_{2n} \longrightarrow BP_{2(k+n)} .$$

$\Pi_0 \mathbb{BP} = \Pi_* BP = Z_{(p)}[v_1,v_2,\ldots]$. This gives rise to elements

$$[v_n] \in H_0 BP_{-2(p^n-1)} \subset H_0 \mathbb{BP}.$$

These elements generate $H_0 \mathbb{BP}$. We have another select set of elements.

$BP^* \, CP^\infty = BP^*[[T]]$ where $T \in BP^2 \; CP^\infty$. $BP^k(X) = [X, BP_k]$, so let $CP^\infty \longrightarrow BP_2$ represent T. Now, we have generators $\beta_i \in H_{2i} \; CP^\infty$ which we can push into $H_{2i} \; BP_2$. In particular we let

$$b_{(i)} \in H_{2p^i} \, BP_2 \subset H_{2p^i} \, \mathbb{B}P$$

be the image of β_{p^i}. The $b_{(i)}$ generate the stable homology of BP using the second product mentioned above. In [6] we construct a "Hopf-ring" completely algebraically which is isomorphic to $H_* \, \mathbb{B}P$. The proof that they are isomorphic depends on the fact that the $[v_i]$ and $b_{(i)}$ generates $H_* \, \mathbb{B}P$ if both products are allowed. In [6] we solve that problem by computing $H_* \, \mathbb{B}P$ directly in terms of the $[v_i]$ and $b_{(i)}$. However, originally we proved it using the technical proposition 1.7. What follows is a very brief sketch of how we approached the problem then.

If we consider $H_* \, \mathbb{B}P$ as an algebra using the loop space product, it is enough to show that the elements of $QH_* \, \mathbb{B}P$ are linear combinations of elements

$$v^I b^J = \ldots [v_2]^{\circ i_2} \circ [v_1]^{\circ i_1} \circ b_{(o)}^{\circ j_o} \circ b_{(1)}^{\circ j_1} \ldots$$

where the circle denotes the second product we discussed. The element $v_i \in \pi_* BP$ gives rise to a map

$$v_i : S^{2(p^i-1)} BP \longrightarrow BP$$

which is just multiplication by v_i. This also carries over to $\mathbb{B}P \longrightarrow \mathbb{B}P$ and gives a map

$$(v_i)_* : H_* \, \mathbb{B}P \longrightarrow H_* \, \mathbb{B}P$$

In fact, $v^I b^J$ is just $(v^I)_*(b^J)$. By a simple induction we need only prove that the b^J generate QH_*BP_{2k} modulo the image of all the $(v^I)_*$ from the $QH_*BP_{2k+d(I)}$. By duallity, it will be enough to show that either $x \in PH^*BP_{2k}$ is in the image of the cohomology suspension from H^*BP or $(v^I)^*(x) \neq 0$ for some I. We know from [9] and [5] that if x is a primitive it is either a primitive generator or a $p^{\underline{th}}$ power of one. By the fact that H^*BP_{2k} is a polynomial algebra, the $p^{\underline{th}}$ powers will follow if we can prove it for primitive generators. So, $x \in QH^*BP_{2k}$. Find j such that $x \in M_j$ but $x \notin M_{j-1}$. By the definition of the M_j and proposition 1.7 we have $(i_j)^*(x) \neq 0$ and is in fact an element with no Bocksteins in $QH^*K(V_j,2k)$. Find an $i_{R'} \in H^*K(V_j,2k)$ such that the coefficient in $(i_j)^*(x) = \Sigma\, a_R i_R$ has $a_{R'}\ i_{R'} \neq 0$. A little hand-waving should convince the reader that the map

$$v^{R'} : BP_{2k+d(R')} \longrightarrow BP_{2k}$$

actually factors through $K(V_j,2k)$ and $i_{R'}$ pulls back to the generator y of $H^{2k+d(R')}BP_{2k+d(R')}$. This is just the fact mentioned in section 2 that the homotopy of BP_{2k} given by $K(V_j,2k)$ is just the v^I with $\ell(I) = j$ modulo higher order products. In particular, $(v^{R'})^*(x) = a_{R'}y$ which is non-zero and in the image of H^*BP.

Remark: The above notation allows us to define $M_j \subset QH^*BP_{2k}$ directly.

$$M_j = \bigcap_{\ell(I)>j} \ker(v^I)^*$$

REFERENCES

[1] D. C. Johnson and W. S. Wilson : Projective dimension and Brown-Peterson Homology, Topology 12(1973), pp. 327-353.

[2] D. Kraines: On excess in the Milnor basis, Bull. London Math. Soc. 3(1971), pp. 363-365.

[3] P. S. Landweber: Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$, Am. J. Math. (to appear).

[4] J. W. Milnor: The Steenrod algebra and its dual. Ann. Math. 67(1958), pp. 150-171.

[5] D. C. Ravenel and W. S. Wilson: Bipolynomial Hopf algebras, J. Pure and Applied Algebra, (to appear).

[6] ________________ ________: The Hopf ring of complex cobordism, Bull. Am. Math. Soc. (to appear).

[7] L. Smith: Cohomology of two stage Postnikov systems, Ill. J. Math. 11(1967), pp. 310-329.

[8] __________ Lectures on the Eilenberg-Moore spectral sequence, Lecture Notes in Mathematics, 134, Springer-Verlag.

[9] W. S. Wilson: The Ω-spectrum for Brown-Peterson cohomology, Part I, Comm. Math. Helv. 48(1973), pp. 45-55.

[10] _____________: The Ω-spectrum for Brown-Peterson cohomology, Part II, Am. J. Math. (to appear).