

NOTES ON EQUIVARIANT BUNDLES

FOLING ZOU

ABSTRACT. We compare two notions of G -fiber bundles and G -principal bundles in the literature, with an aim to clarify early results in equivariant bundle theory that are needed in current work of equivariant algebraic topology. We also give proofs of some equivariant generalizations of well-known non-equivariant results involving the classifying space.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 2. Equivariant bundles | 5 |
| 2.1. G -CW complexes | 5 |
| 2.2. Non-equivariant bundles | 5 |
| 2.3. Definitions of equivariant bundles | 7 |
| 2.4. Comparisons of definitions | 13 |
| 2.5. Examples: the V -framing bundle | 15 |
| 2.6. Fixed point theorems | 17 |
| 3. Classifying spaces | 20 |
| 3.1. V -trivial bundles | 20 |
| 3.2. Universal equivariant bundles | 21 |
| 3.3. The fiber of the universal bundle and the loop space of the classifying space over a point | 23 |
| 3.4. The gauge group of an equivariant principal bundle | 25 |
| 3.5. Free loop spaces and adjoint bundles | 27 |
| References | 32 |

1. INTRODUCTION

Non-equivariantly, fiber bundles and principal bundles are closely related. Namely, fixing a compact Lie group Π and a space F with an effective Π -action, one can make sense of a fiber bundle with fiber F to have structure group Π , and there is a structure theorem providing an equivalence of categories between such fiber bundles and principal Π -bundles. One key idea involved is the data of admissible maps of a fiber bundle $p : E \rightarrow B$ with fiber F , which are specified homeomorphisms $\psi : F \cong p^{-1}(b)$ for $b \in B$ that come from the local trivialisations.

Equivariantly, let G and Π be compact Lie groups, with G being the ambient action group and Π being the structure group. It is common to assume G to be a compact Lie group in G -equivariant homotopy theory for several reasons. Firstly, the orbit category of a compact Lie group G is more controlled as it has a discrete skeleton.

More precisely, the orbit category has objects G/H for closed subgroups $H \subset G$, and two closed subgroups of G being sufficiently close will imply that one is conjugate to a subgroup of the other by [MZ42]. If one works with a general topological group G , the isotropy subgroups of a G -space may vary continuously. Secondly, nice G -spaces such as smooth G -manifolds are known to allow G -equivariant triangulation by [Ill83]. One can work with more general topological groups Π with careful point set considerations, but we refrain from doing so in this paper.

To obtain a structure theorem relating fiber bundles and principal bundles equivariantly, we need to answer the following two questions: What does it mean for a G -fiber bundle with fiber F to have structure group Π ? What is an equivariant principal Π -bundle?

Section 2.3 is devoted to answering these two questions and establishing the structure theorems. The forthright guess already works for some examples including equivariant vector bundles. We assume that the Π -action on F is effective throughout.

Definition 1.1. (Definition 2.9) Let F be a space with Π -action. A G -fiber bundle with fiber F and structure group Π is a map $p : E \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant fiber bundle with fiber F and structure group Π ;
- (2) Both E and B are G -spaces and p is G -equivariant;
- (3) The G -action is given by morphisms of bundles with structure group Π .

Tom Dieck [TD69] generalized this definition to a twisted version and Lewis–May–Steinberger [LMSM86, IV1] introduced the following further generalization. We fix an extension of compact Lie groups $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ as data.

Definition 1.2. (Definition 2.22) Let F be a space with Γ -action. A G -fiber bundle with fiber F , structure group Π and total group Γ is a map $p : E \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant fiber bundle with fiber F and structure group Π ;
- (2) Both E and B are G -spaces and p is a G -map;
- (3) For any $g \in G$ and admissible maps $\psi : F \rightarrow F_b$ and $\zeta : F \rightarrow F_{gb}$, the composite

$$F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb} \xrightarrow{\zeta^{-1}} F$$

is a lift $y \in \Gamma$ of $g \in G$.

In terms of admissible maps, Definition 1.1 requires that for any admissible map $\psi : F \rightarrow E$ and $g \in G$, the composite $g\psi$ is also admissible; while Definition 1.2 requires that there is a lift $y \in \Gamma$ of g such that $g\psi y^{-1}$ is admissible.

The following gives an interesting example of the generalization .

Example 1.3. (Examples 2.15, 2.27 and 2.38) Atiyah [Ati66] introduced the notion of Real vector bundles: it is a complex vector bundle with C_2 -action such that the non-trivial element of C_2 acts anti-complex-linearly. It is a C_2 -fiber bundle with fiber \mathbb{C}^n , structure group $U(n)$ and total group $\Gamma = U(n) \rtimes_{\alpha} C_2$ in the sense of Definition 1.2, where $\alpha : C_2 \rightarrow \text{Aut}(U(n))$ sends the non-trivial element of C_2 to the entry-wise complex-conjugation of $U(n)$. But it is *not* a C_2 -fiber bundle with fiber \mathbb{C}^n , structure group $U(n)$ in the sense of Definition 1.1.

Note that in [Definition 1.1](#), the fiber F only has a Π -action, but in [Definition 1.2](#), the fiber F has an action of the total group Γ . [Definition 1.1](#) becomes a special case of [Definition 1.2](#) by setting $\Gamma = \Pi \times G$ and imposing the trivial G -action on F (see [Proposition 2.34](#)). Tom Dieck's definition, which is not given above, corresponds to the case of split extensions $\Gamma = \Pi \rtimes_{\alpha} G$ in [Definition 1.2](#). It allows the G -action on the bundle to have a preassigned twisting in the structure group, such as in the Real vector bundles. Conceptually, the most general [Definition 1.2](#) sees a more general twisting specified by the group extension $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$.

The following are companion definitions of equivariant principal bundles.

Definition 1.4. ([Definition 2.12](#)) A principal G - Π -bundle is a map $p : P \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant principal Π -bundle;
- (2) Both P and B are G -spaces and p is G -equivariant;
- (3) The actions of G and Π commute on P .

Definition 1.5. ([Definition 2.17](#)) Let $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ be an extension of compact Lie groups. A principal $(\Pi; \Gamma)$ -bundle is a map $p : P \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant principal Π -bundle;
- (2) The space P is a Γ -space; B is a G -space. Viewing B as a Γ -space by pulling back the action, the map p is Γ -equivariant.

The first definition is a special case of the second definition by setting $\Gamma = \Pi \times G$.

There are structure theorems relating equivariant fiber bundles and equivariant principal bundles ([Theorems 2.14](#) and [2.29](#)). For any G -space B , there is an equivalence of categories between

$$\begin{array}{l} G\text{-fiber bundles over } B \text{ with fiber } F \\ \text{and structure group } \Pi \end{array} \leftrightarrow \text{principal } G\text{-}\Pi\text{-bundles over } B.$$

$$\begin{array}{l} G\text{-fiber bundles over } B \text{ with fiber } F \\ \text{structure group } \Pi \text{ and total group } \Gamma \end{array} \leftrightarrow \text{principal } (\Pi; \Gamma)\text{-bundles over } B.$$

In the first equivalence, F is any Π -effective space; in the second equivalence, F is any Π -effective Γ -space.

As an example, we study the V -framing bundle $\text{Fr}_V(E)$ of a G - n -vector bundle E , where V is a G -representation given by $\beta : G \rightarrow O(n)$. It turns out $\text{Fr}_{\mathbb{R}^n}(E) \cong \text{Fr}_V(E)$ as $(O(n); O(n) \times G) \cong (O(n); O(V) \rtimes G)$ -principal bundles in the sense of [Definition 1.2](#), hinting that the V -framing bundle may not be an interesting notion. However, they can be given different canonical G -actions and the G -action on $\text{Fr}_V(E)$ can be identified with the Λ_{β} -action on $\text{Fr}_{\mathbb{R}^n}(E)$ for some subgroup $\Lambda_{\beta} \subset O(n) \times G$. This is in [Section 2.5](#).

There exists a universal principal $(\Pi; \Gamma)$ -bundle $E(\Pi; \Gamma) \rightarrow B(\Pi; \Gamma)$. It is universal in the sense that there is a bijection of sets between $\{\text{equivalence classes of principal } (\Pi; \Gamma)\text{-bundles over } B\}$ and $\{G\text{-homotopy classes of } G\text{-maps } B \rightarrow B(\Pi; \Gamma)\}$ for any paracompact G -space B . Thus, $B(\Pi; \Gamma)$ is called the classifying space of principal $(\Pi; \Gamma)$ -bundles. In the case $\Gamma = \Pi \times G$, we also denote the universal principal G - Π -bundle by $E_G \Pi \rightarrow B_G \Pi$.

The universal principal bundle can be constructed using homotopy theory techniques. Non-equivariantly, one can construct $E\Pi$, a contractible space with free Π -action, and

the universal principal Π -bundle is modeled by $E\Pi \rightarrow E\Pi/\Pi$. Equivariantly, a family \mathcal{F} of subgroups of Γ is a collection of subgroups that is non-empty and closed under subgroups and conjugations. For each family, there exists a Γ -space $E\mathcal{F}$ with the property that $(E\mathcal{F})^\Lambda \simeq \begin{cases} \emptyset & \Lambda \subset \Gamma \text{ and } \Lambda \notin \mathcal{F} \\ * & \Lambda \subset \Gamma \text{ and } \Lambda \in \mathcal{F} \end{cases}$. We take the family to be $\mathcal{F} = \mathcal{F}(\Pi) := \{\Lambda \subset \Gamma \mid \Lambda \cap \Pi = e\}$. Then the universal principal $(\Pi; \Gamma)$ -bundle is modeled by $E\mathcal{F}(\Pi) \rightarrow E\mathcal{F}(\Pi)/\Pi$ (Theorem 3.7). Note that to recover the non-equivariant case, we can take $G = e$, so that $\Gamma = \Pi$ and $\mathcal{F}(\Pi) = e$, and the fixed-point properties of $E\mathcal{F}(\Pi)$ coincide with the defining properties of $E\Pi$. More details are in Section 3.2.

Remark 1.6. Lück–Uribe worked with general topological groups G and Π and those principal G - Π -bundles such that the isotropy subgroups of the total space are in \mathcal{R} for a prescribed family of subgroups of $\Gamma = \Pi \times G$. This family needs to satisfy conditions in [LU14, Definition 6.1]. The universal bundle of such principal bundles can be modeled by $E\mathcal{R} \rightarrow E\mathcal{R}/\Pi$ ([LU14, Theorem 11.5]). In our case, $\mathcal{R} = \mathcal{F}(\Pi)$, and we will not make use of a general \mathcal{R} . To translate notations, their Γ is our G and their G is our Π .

In Section 2.6, we study the fixed points of a principal G - Π -bundle $p : P \rightarrow B$. Let $H \subset G$ be a subgroup and use $\text{Rep}(H, \Pi)$ to denote group homomorphisms $\rho : H \rightarrow \Pi$ up to Π -conjugation. Each component B_0 of B^H has an associated homomorphism $[\rho] \in \text{Rep}(H, \Pi)$. The $[\rho]$ is determined by the fixed-point behavior of the total space: Let $\Lambda_\rho \subset \Pi \times G$ denote the graph of ρ , then $\{\rho : H \rightarrow \Pi \mid (p^{-1}(B_0))^{\Lambda_\rho} \neq \emptyset\}$ holds for exactly one Π -conjugate class of homomorphisms. Furthermore, the non-equivariant principal Π -bundle $p^{-1}(B_0) \rightarrow B_0$ has a reduction of the structure group from Π to a subgroup $Z_\Pi(\rho) \subset \Pi$ (Theorem 2.46). We apply this theorem to obtain a comparison of principal G - Π -bundles. In a map between principal G - Π -bundles, if \bar{f} , the map of the total spaces, is an $\mathcal{F}(\Pi)$ -equivalence, then f , the map of the base spaces, is a G -equivalence (Theorem 2.48).

In Section 3.3, we study the loop space $\Omega_b B_G \Pi$ of $B_G \Pi$ based at a G -fixed point b . As $(B_G \Pi)^G$ is not connected in general, the G -homotopy type of $\Omega_b B_G \Pi$ depends on the choice of b . Our greatest interest is in the case $\Pi = O(n)$, and it works the same for general Π as discussed in Remark 3.14. Note that $B_G O(n)$ classifies G - n -vector bundles and a homomorphism $\rho : G \rightarrow O(n)$ gives an n -dimensional G -representation V . Suppose $b \in B_G O(n)$ is in the component indexed by $[V]$. In Theorem 3.12, we show that there is a G -homotopy equivalence $\Omega_b B_G O(n) \simeq O(V)$, where $O(V)$ is the isometric self maps of V with G -action by conjugation. Later in Corollary 3.31, we upgrade the G -equivalence to one compatible with the monoid structure: $\Omega_b B_G O(n)$ has the structure of a G - A_∞ -monoid via concatenation of loops. There is a zig-zag of equivalence of G - A_∞ -monoids $\Omega_b B_G O(n) \simeq O(V)$.

In Section 3.4, we study the space of bundle maps to the equivariant universal bundle. It is an equivariant principal bundle with a non-trivial extension in the total group. Let $p : P \rightarrow B$ be a principal G - $O(n)$ -bundle, $\Pi = \text{Aut}_B(P)$ be the topological group of automorphisms of P over B and $\text{Hom}(P, E_G O(n))$ be the space of non-equivariant principal $O(n)$ -bundle maps. We have that G acts on $\text{Hom}(P, E_G O(n))$ and Π by conjugation, and that Π acts on $\text{Hom}(P, E_G O(n))$ by precomposition. This gives a

$(\Gamma = \Pi \rtimes G)$ -action on $\text{Hom}(P, E_G O(n))$. In [Theorem 3.17](#), we show that

$$\pi : \text{Hom}(P, E_G O(n)) \rightarrow \text{Map}_p(B, B_G O(n))$$

is a principal $(\text{Aut}_B P; \text{Aut}_B P \rtimes G)$ -bundle and $\text{Hom}(P, E_G O(n)) \simeq E\mathcal{F}$ for the family $\mathcal{F} = \{\Lambda \subset \Gamma \text{ such that } \Lambda \cap \text{Aut}_B P = e\}$. Here, π sends a bundle map to its map of base spaces and $\text{Map}_p(B, B_G O(n))$ is the image of π in $\text{Map}(B, B_G O(n))$. We conjecture that π is the universal principal $(\text{Aut}_B P; \text{Aut}_B P \rtimes G)$ -bundle. The issue is that $\text{Aut}_B P$ is not necessarily a Lie group, nor does it satisfy the conditions in [\[LU14\]](#), and we have not developed the classification theory in such full generality.

In [Section 3.5](#), we show that there is a weak G -equivalence between the free loop space $LB_G \Pi$ and the adjoint bundle $Ad(E_G \Pi) := E_G \Pi \times_{\Pi} \Pi_{\text{ad}}$ as G -fibrations over $B_G \Pi$ ([Theorem 3.30](#)).

Organization of the paper. We give preliminaries of G -CW complexes in [Section 2.1](#). We review non-equivariant bundles in [Section 2.2](#). We give definitions of equivariant bundles and compare the definitions in [Section 2.3](#) - [Section 2.5](#). We study the fixed points of equivariant bundles in [Section 2.6](#). We prove the aforementioned theorems about classifying spaces in [Section 3](#).

Acknowledgements. This is part of the author's PhD thesis. The author is indebted to her advisor, Peter May, for explaining his past work and for his enormous help with the writing. The author thanks the referee for the useful comments and writing suggestions.

2. EQUIVARIANT BUNDLES

2.1. G -CW complexes. In this subsection, we give some preliminaries. For a compact Lie group G , a G -CW complex X is a union of G -spaces X^n constructed as follows. We start with a disjoint union of orbits $X^0 = \sqcup_K G/K$. Each X^n is inductively obtained by gluing n -cells to X^{n-1} . An n -cell is of the form $G/K \times D^n$ where $K \subset G$ is a closed subgroup and $D^n = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\}$ is a disk. There are G -maps from $\sqcup_K G/K \times S^{n-1}$, the boundary of the n -cells, to X^{n-1} and $X^n = X^{n-1} \cup_{(\sqcup_K G/K \times S^{n-1})} (\sqcup_K G/K \times D^n)$.

Definition 2.1. A map $f : X \rightarrow Y$ between spaces is said to be a weak equivalence if it induces a bijection $\pi_0(X) \rightarrow \pi_0(Y)$ and an isomorphism $\pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ for any $x \in X$ and $n \geq 1$. A G -map $f : X \rightarrow Y$ between G -spaces is said to be a weak G -equivalence if $X^H \rightarrow Y^H$ is a weak equivalence for any subgroup $H \subset G$.

A G -space X is said to have the homotopy type of a G -CW complex if there is a G -CW complex X' and a G -homotopy equivalence $X' \simeq X$.

Theorem 2.2 (Equivariant Whitehead theorem). *A weak G -equivalence between G -spaces having homotopy types of G -CW complexes is a G -homotopy equivalence.*

This theorem allows us to use the induced maps on homotopy groups to detect homotopy equivalences when working with G -CW complexes.

2.2. Non-equivariant bundles. We start with a review of non-equivariant bundles.

A *fiber bundle* with fiber F is a map $p : E \rightarrow B$ with an open cover $\{U_i\}$ of B and homeomorphisms $\phi_i : p^{-1}(U_i) \cong U_i \times F$. The U_i are called *coordinate neighborhoods* and the ϕ_i are called *local trivializations*.

The structure group of a fiber bundle gives information about the change of local trivializations under changes of coordinate neighborhoods. Let Π be a topological group with an effective action on F . Here, *effective* means $\Pi \rightarrow \text{Aut}(F)$ is an injection. A bundle with fiber F is said to have *structure group* Π , if for any two coordinate neighborhoods with $U_i \cap U_j \neq \emptyset$, the composite $\phi_i \phi_j^{-1} : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$ is given by $(b, f) \mapsto (b, g_{ij}(b)(f))$ for some continuous function $g_{ij} : U_i \cap U_j \rightarrow \Pi$, called a *coordinate transformation*. We always topologize $\text{Aut}(F)$ with the compact-open topology of mapping spaces. If F is a compact Hausdorff space, $\text{Aut}(F)$ is a topological group; if F is only locally compact, there are more technical assumptions for the inverse map to be continuous due to Arens (See [Ste51, I.5.4]). Morally, a fiber bundle with fiber F is automatically a fiber bundle with the implicit structure group $\text{Aut}(F)$. Having an explicit structure group Π is extra data to reduce the structure group to a smaller one.

One can associate a principal Π -bundle to a fiber bundle with structure group Π . An *admissible map* of the bundle is a homeomorphism $\psi : F \rightarrow p^{-1}(b)$ for some $b \in U_i$, satisfying $\phi_i \psi \in \Pi$. The set of admissible maps has a natural topology, and the *associated principal Π -bundle* of p is the space of admissible maps.

The following immediate observation about admissible maps hides the local trivializations in the background.

Lemma 2.3. *A map $\psi : F \rightarrow F_b$ is admissible if and only if for any admissible map $\zeta : F \rightarrow F_b$, the composite $\zeta^{-1}\psi$ is in Π . \square*

Let $p_1 : E_1 \rightarrow B_1$ and $p_2 : E_2 \rightarrow B_2$ be two fiber bundles with fiber F and structure group Π . A *morphism* between them is a bundle map $\chi : E_1 \rightarrow E_2$ such that for any local trivializations $\phi_U : p_1^{-1}(U) \cong U \times F$ and $\phi_V : p_2^{-1}(V) \cong V \times F$, the composite

$$(2.4) \quad \phi_V \chi \phi_U^{-1} : (U \cap \chi^{-1}(V)) \times F \rightarrow (\chi(U) \cap V) \times F$$

is given by $(b, f) \mapsto (\chi(b), g_{VU}(b)(f))$, where $g_{VU} : U \cap \chi^{-1}(V) \rightarrow \Pi$ is some continuous function. Such a morphism induces a morphism between the two associated principal Π -bundles.

Assumption 2.5. We always assume that Π has the subspace topology of $\text{Aut}(F)$.

We pause to explain the role of [Assumption 2.5](#). Suppose χ is a bundle map and we would like to check if χ is a morphism, that is, whether it respects the structure group. It seems as if one only needs to check that χ sends all admissible maps to admissible maps. This is not true without [Assumption 2.5](#), since the set of admissible maps does not see the topology. Steenrod [Ste51, I.5] studied this difference carefully and concluded that [Assumption 2.5](#) will resolve the discrepancy.

We include some explanation here for completeness: What the set of admissible maps sees is an Ehresmann-Feldbau bundle with structure group Π , which has now become an obsolete notion. An Ehresmann-Feldbau bundle is a bundle $p : E \rightarrow B$ with fiber F and a set of homeomorphism $\psi : F \cong p^{-1}(b)$ for all $b \in B$, called admissible maps.

It is required that for any $b \in U_i$, the composite $F = \{b\} \times F \rightarrow U_i \times F \xrightarrow{\phi_i^{-1}} p^{-1}(U_i)$ is admissible, and that for any $b \in B$ and any admissible map $\psi : F \rightarrow p^{-1}(b)$, all the admissible maps $F \rightarrow p^{-1}(b)$ are exactly $\psi \circ \nu$ for some $\nu \in \Pi$. While this aligns with [Lemma 2.3](#) when the bundle has a structure group Π , there is a difference between the two notions, which lies exactly in that an Ehresmann-Feldbau bundle does not

require Π to have a topology. In other words, the coordinate transformations g_{ij} are not asked to be continuous, which is equivalent to putting the trivial topology on Π . If Π does start life with a different topology, the coordinate transformations g_{ij} obtained from an Ehresmann-Feldbau bundle may not be continuous. It is shown in [Ste51, I.5.4] that with [Assumption 2.5](#), the g_{ij} 's are automatically continuous, so that a fiber bundle with local trivializations has structure group Π if and only if the admissible maps satisfy [Lemma 2.3](#). We have the following criteria:

Proposition 2.6. *A bundle map $\chi : E_1 \rightarrow E_2$ is a morphism of fiber bundles with structure group Π if and only if either of the two equivalent conditions is true:*

- (1) *If F_1 is a fiber in E_1 and F_2 is a fiber in E_2 such that χ maps F_1 to F_2 , then the composite $\zeta^{-1}\chi\psi$ is in Π for any admissible maps $\psi : F \rightarrow F_1$ and $\zeta : F \rightarrow F_2$.*
- (2) *For any admissible map $\psi : F \rightarrow F_1$ to a fiber in E_1 , the composite $\chi\psi$ is an admissible map to a fiber in E_2 .*

Proof. We need to check that for any ϕ_U, ϕ_V as in [\(2.4\)](#), the desired g_{VU} exists. With [Assumption 2.5](#), it suffices to check that for any $b \in U \cap \chi^{-1}(V)$, there exists a desired $g_{VU}(b) \in \Pi$. This is part [\(1\)](#). Part [\(2\)](#) follows from [Lemma 2.3](#). \square

Example 2.7. The most familiar case is when F is a vector space (\mathbb{R}^n or \mathbb{C}^n) and $\Pi = GL_n$ is the corresponding general linear group. By definition of the general linear group, χ being a bundle map is equivalent to it being fiberwise linear and non-degenerate.

The following well-known structure theorem turns the problem of classifying fiber bundles into classifying principal bundles.

Theorem 2.8. *Let Π be a compact Lie group. Let B, F be spaces. Assume that Π acts effectively on F . Then there is an equivalence of categories between $\{\text{fiber bundles over } B \text{ with fiber } F \text{ and structure group } \Pi\}$ and $\{\text{principal } \Pi\text{-bundles over } B\}$.*

Proof. We have already shown how to construct a principal Π bundle from a fiber bundle with fiber F and structural group Π at the beginning of this subsection. In the other direction, given a principal Π -bundle $P \rightarrow B$, the map $P \times_{\Pi} F \rightarrow B$ is a fiber bundle with fiber F and structure group Π . These two constructions are functorial and inverse of each other. Indeed, [Ste51, I] described both types of bundles using local transformations, called coordinate bundles, where the equivalence becomes transparent. \square

2.3. Definitions of equivariant bundles. When it comes to the equivariant story, there are definitions of different generality, both on the fiber bundle side and on the principal bundle side. The reason is that the ambient group G could interact non-trivially with the structure group Π . We start with the simplest definition where “ G and Π commute” [Las82].

Definition 2.9. A G -fiber bundle with fiber F and structure group Π is a map $p : E \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant fiber bundle with fiber F and structure group Π ;
- (2) Both E and B are G -spaces and p is G -equivariant;
- (3) The G -action is given by morphisms of bundles with structure group Π .

Proposition 2.10. *The requirement in (3) above is equivalent to the following: for any $g \in G$ and admissible map $\psi : F \rightarrow F_b$, the composite $F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb}$ is also admissible.*

Proof. By [Proposition 2.6](#). □

Remark 2.11. Let G be a finite group. We take $F = \mathbb{R}^n$ and $\Pi = \mathrm{GL}_n(\mathbb{R})$ in [Definition 2.9](#). Although $\mathrm{GL}_n(\mathbb{R})$ is not compact, the definition still works, and we obtain a G - n -vector bundle.

Definition 2.12. A principal G - Π -bundle is a map $p : P \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant principal Π -bundle;
- (2) Both P and B are G -spaces and p is G -equivariant;
- (3) The actions of G and Π commute on P .

Remark 2.13. This is called a principal (G, Π) -bundle in [\[LMSM86, IV1\]](#).

As in the non-equivariant case, we write the Π -action on the right of a principal G - Π -bundle P ; for convenience of diagonal action, we consider P to have a left Π -action, that is, $\nu \in \Pi$ acts on $z \in P$ by $\nu z = z\nu^{-1}$.

The structure theorem formally passes to this equivariant context.

Theorem 2.14. *Let G, Π be compact Lie groups and F, B be spaces. Assume that Π acts effectively on F . Then there is an equivalence of categories between $\{G$ -fiber bundles over B with fiber F and structure group $\Pi\}$ and $\{\text{principal } G$ - Π -bundles over $B\}$.*

Proof. The two types of G -bundles in [Definitions 2.9](#) and [2.12](#) are indeed objects with a G -action in the corresponding non-equivariant category. So the equivalence in the non-equivariant structure theorem restricts to give an equivalence on the G -objects. □

However, [Definitions 2.9](#) and [2.12](#) are not ideal for studying some interesting cases. In the most general scenario, we want to study a map $p : E \rightarrow B$ that happens to be both a fiber bundle with structure group Π and a G -map between G -spaces. It is true that p is a G -fiber bundle with structure group $\mathrm{Aut}(F)$, but p is usually not a G -fiber bundle with structure group Π . In other words, we can not reduce the structure group even though we know non-equivariantly it reduces to Π . Below, we give two concrete examples of this sort.

The first example is Atiyah's Real vector bundles [\[Ati66\]](#).

Example 2.15. Let $G = C_2$. A Real vector bundle is a map $p : E \rightarrow B$ such that

- The map p is a complex vector bundle of dimension n ;
- The non-trivial element of C_2 acts anti-complex-linearly.

In this case, p is a C_2 -bundle with structure group $O(2n)$, but not $U(n)$.

The second simple, but illuminating, example is from [\[LMSM86\]](#).

Example 2.16. For G -spaces B and F , the projection $p : B \times F \rightarrow B$ is not a G -bundle with structure group e unless G acts trivially on F .

Proof. The admissible maps for p are only the inclusions of fibers

$$\psi_b : \{b\} \times F \rightarrow B \times F.$$

An element $g \in G$ acts by a bundle map if and only if for all b , the composite

$$\{b\} \times F \xrightarrow{\psi_b} p^{-1}(b) \xrightarrow{g} p^{-1}(gb) \xrightarrow{\psi_{gb}^{-1}} \{gb\} \times F$$

is in the structure group. But this map is merely the g action on F . \square

Consequently, we would like a more general version than Definitions 2.9 and 2.12. To work with Real vector bundles, Tom Dieck [TD69] introduced a complex conjugation action of C_2 on $U(n)$. Lashof–May [LM86] had the idea to further introduce a total group that is the extension of the structure group Π by G . Tom Dieck’s work became a special case of a split extension, or equivalently a semidirect product. One good, but rather brief and sketchy, early reference for both is [LMSM86, IV1]; we shall flesh out that source and come back to the two examples afterwards.

We start with the well-studied principal bundle story.

Definition 2.17. ([LM86]) Let $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ be an extension of compact Lie groups. A principal $(\Pi; \Gamma)$ -bundle is a map $p : P \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant principal Π -bundle;
- (2) The space P is a Γ -space; B is a G -space. Viewing B as a Γ -space by pulling back the action, the map p is Γ -equivariant.

Remark 2.18. The total space P does not have a G -action in general. It only does so when we specify a splitting $G \rightarrow \Gamma$. An example of this sort will be discussed in Section 2.5.

Definition 2.19. A morphism between two principal $(\Pi; \Gamma)$ -bundles $p_1 : P_1 \rightarrow B_1$ and $p_2 : P_2 \rightarrow B_2$ is a pair of maps (\bar{f}, f) fitting in the commutative diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{f}} & P_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

such that f is G -equivariant and \bar{f} is Γ -equivariant.

Example 2.20. Let $y \in \Gamma$ be with image $g \in G$. The action map (y, g) is an automorphism.

Taking $\Gamma = \Pi \times G$, we recover the principal G - Π -bundles of Definition 2.12. In this case we have two names for the same thing. This could be confusing, but since a “principal G - Π -bundle” looks more natural than a “principal $(\Pi; \Pi \times G)$ -bundle” for this thing, we will keep both names.

Taking Γ to be a split extension, or equivalently, $\Gamma = \Pi \rtimes_{\alpha} G$ for some group homomorphism $\alpha : G \rightarrow \text{Aut}(\Pi)$, we recover Tom Dieck’s principal (G, α, Π) -bundles.

Remark 2.21. To be useful later, we write the elements of $\Gamma = \Pi \rtimes_{\alpha} G$ as (ν, g) for $\nu \in \Pi, g \in G$ and write $\alpha(g) \in \text{Aut}(\Pi)$ as α_g . We have the following facts:

- The product in Γ is given by $(\nu, g)(\mu, h) = (\nu\alpha_g(\mu), gh)$ (That is, g acts on μ when they interchange);
- The identity element is (e, e) ;
- The inverse is $(\nu, g)^{-1} = (\alpha_{g^{-1}}(\nu^{-1}), g^{-1})$;
- The elements (e, g) form a subgroup of Γ that is canonically isomorphic to G ;
- A space with Γ -action is a space with both Π and G actions such that

$$g(\nu(-)) = \alpha_g(\nu)(g(-)), \text{ which is indeed } (\alpha_g(\nu), g)(-).$$

The fiber bundle story is not as clear. It turns out that the appropriate fiber of an equivariant fiber bundle is not just the preimage of any point, but rather with a preassigned action of Γ . This is unnatural at first glance, for example in a G -vector bundle we will not expect there to be an $(O(n) \times G)$ -action on the fiber \mathbb{R}^n . We will explain why this is necessary at the end of this subsection. How G -vector bundles fit in this context will be stated in [Example 2.26](#). Let us start with the definition:

Definition 2.22. ([[LMSM86](#), IV1]) Let $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ be an extension of compact Lie groups and F be a space with Γ -action. A G -fiber bundle with fiber F , structure group Π and total group Γ is a map $p : E \rightarrow B$ such that the following statements hold:

- (1) The map p is a non-equivariant fiber bundle with fiber F and structure group Π ;
- (2) Both E, B are G -spaces and p is a G -map;
- (3) For any $g \in G$ and admissible maps $\psi : F \rightarrow F_b$ and $\zeta : F \rightarrow F_{gb}$, the composite

$$F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb} \xrightarrow{\zeta^{-1}} F$$

is a lift $y \in \Gamma$ of $g \in G$. In other words, the y in the following diagram is asked to be a lift of $g \in G$ in Γ :

$$\begin{array}{ccc} F & \xrightarrow{y} & F \\ \psi \downarrow \cong & & \cong \downarrow \zeta \\ F_b & \xrightarrow{g} & F_{gb} \end{array}$$

Proposition 2.23. *The requirement (3) above is equivalent to the following: For each $y \in \Gamma$ with image $g \in G$ and admissible map $\psi : F \rightarrow F_b$, the composite*

$$F \xrightarrow{y^{-1}} F \xrightarrow{\psi} F_b \xrightarrow{g} F_{gb}$$

is also admissible.

Proof. For any two lifts y and y' of g , $y'y^{-1}$ is a lift of $e \in G$, so it is in Π . The claim then follows from [Lemma 2.3](#). \square

Taking $g = e$ in [Proposition 2.23](#), the possible lifts y are exactly the elements of Π , so we just see the non-equivariant structure group (compare with [Lemma 2.3](#)); taking general g , the assignment $\psi \mapsto g\psi y^{-1}$ is mimicking the action by an element of Π on the admissible map ψ , but it changes the fiber from over b to over gb . In this sense, the extension of the structure group Π to the total group Γ is used to regulate admissible maps to fibers over the orbit of b .

Definition 2.24. Let $p_1 : E_1 \rightarrow B_1$ and $p_2 : E_2 \rightarrow B_2$ be two G -fiber bundles with fiber F , structure group Π and total group Γ . A morphism between them is a pair of maps (\bar{f}, f) fitting in the commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

such that the following statements hold:

- (1) The pair (\bar{f}, f) is a non-equivariant morphism between bundles with fiber F and structure group Π .
- (2) Both \bar{f} and f are G -equivariant.

Remark 2.25. By [Proposition 2.6](#), the condition (1) of [Definition 2.24](#) is explicitly the following: For any admissible map $\psi : F \rightarrow F_1$ to a fiber in E_1 , the composite $\bar{f}\psi$ is an admissible map to a fiber in E_2 .

We do not have a requirement on a morphism regarding the condition (3) of [Definition 2.22](#) because it is automatic: if ψ is admissible, we have that $g\psi y^{-1}$ is admissible and so is $\bar{f}(g\psi y^{-1})$. But $\bar{f}g = g\bar{f}$, so $g(\bar{f}\psi)y^{-1}$ is also admissible.

As opposed to [Definition 2.9](#), in [Definition 2.22](#) the Γ -action on the total space E can restrict to a G -action only when there is a splitting of the extension given by $G \rightarrow \Gamma$. The following example illustrates that varying the splitting map can give different G -fiber bundle descriptions of the same bundle. It will be discussed in [Section 2.5](#).

Example 2.26. A G - n -vector bundle is both a G -fiber bundle with fiber \mathbb{R}^n , structure group $O(n)$ and total group $O(n) \times G$ and a G -fiber bundle with fiber V , structure group $O(V)$ and total group $O(V) \rtimes G$. (Here, we take $\Gamma = O(n) \times G \cong O(V) \rtimes G$.)

Example 2.27. A Real vector bundle is a C_2 -fiber bundle with fiber \mathbb{C}^n , structure group $U(n)$ and total group $\Gamma = U(n) \rtimes_{\alpha} C_2$, where $\alpha : C_2 \rightarrow \text{Aut}(U(n))$ sends the non-trivial element of C_2 to the entry-wise complex-conjugation of $U(n)$.

Proof. Let the non-trivial element a of C_2 act by complex conjugation on \mathbb{C}^n . This extends the $U(n)$ -action to a Γ -action by [Remark 2.21](#). We only need to check that [Definition 2.22 \(3\)](#) holds for $g = a$. An automorphism X of \mathbb{C}^n is anti-complex-linear if and only if $A = X \circ a$, the pre-composition of X with conjugation, is complex-linear. So A is an element of $U(n)$, and $X = (A, a)$ is the lift of a in $U(n) \rtimes_{\alpha} C_2$. \square

Example 2.28. For G -spaces B and F , the projection $B \times F \rightarrow B$ is a G -fiber bundle with fiber F , structure group e and total group $\Gamma = G$.

Proof. The proof in [Example 2.16](#) verifies [Definition 2.22 \(3\)](#). \square

When $\Gamma = \Pi \times G$, [Definition 2.9](#) can be considered as a special case of [Definition 2.22](#): The canonical way to extend the Π -action on F to a Γ -action is by taking the trivial G -action. With this convention, a G -fiber bundle in the first definition satisfies the second definition, as we will show shortly in [Proposition 2.34](#). In fact, there could be multiple ways to extend the action, such as in [Example 2.26](#). On the other hand, [Example 2.16](#)

shows that even for a trivial bundle $B \times F \rightarrow F$, the more general [Definition 2.22](#) is needed if G acts non-trivially on F .

We have the following structure theorem in the context of [Definitions 2.17](#) and [2.22](#):

Theorem 2.29. ([\[LMSM86, IV1\]](#)) *For any Π -effective Γ -space F and G -space B , there is an equivalence of categories between $\{G$ -fiber bundles with structure group Π , total group Γ and fiber F over $B\}$ and $\{\text{principal } (\Pi; \Gamma)\text{-bundles over } B\}$.*

Proof. This is an expansion of the sketchy proof in the reference. For brevity, we refer to the two categories as equivariant fiber bundles and equivariant principal bundles.

Given an equivariant fiber bundle $E \rightarrow B$, we take the non-equivariant associated principal bundle $\text{Fr}_F(E) \rightarrow B$. It suffices to give a Γ -action on $\text{Fr}_F(E)$ such that $\text{Fr}_F(E) \rightarrow B$ is a G -map. For $y \in \Gamma$ with image $g \in G$ and an admissible map $\psi : F \rightarrow F_b$, let $y(\psi) = g\psi y^{-1}$. By [Proposition 2.23](#), $g\psi y^{-1}$ is an admissible map to the fiber over gb . This shows that $\text{Fr}_F(E) \rightarrow B$ is an equivariant principal bundle.

Given an equivariant principal bundle $P \rightarrow B$, let $E = (P \times F)/\Pi \rightarrow B$ be the fiber bundle with admissible maps $\psi_p : F \rightarrow E$ of the form $\psi_p(f) = [p, f]$ for some $p \in P$. We verify the three conditions for $E \rightarrow B$ to be an equivariant fiber bundle. Firstly, $E \rightarrow B$ is a non-equivariant fiber bundle with structure group Π . Secondly, we describe the G -action on E . Take the diagonal Γ -action on $P \times F$. For any space with Γ -action X , we can define a $\Gamma/\Pi \cong G$ -action on X/Π by lifting $g \in G$ to $y \in \Gamma$ and let $g[x] = [yx]$ for $x \in X$. Since Π is a normal subgroup of Γ , this is a well-defined action, independent of choice of y or representative x . For $X = P \times F$, this gives $(P \times F)/\Pi$ a G -action. Since $P \rightarrow B$ is Γ -equivariant, it can be checked that $E \rightarrow B$ is G -equivariant. Thirdly, we show that the condition in [Proposition 2.23](#) is satisfied. In fact, for $y \in \Gamma$ lifting $g \in G$ and $p \in P$, we have $g\psi_p y^{-1} = \psi_{yp}$. To see this, evaluating on $f \in F$, we have

$$\begin{aligned} g\psi_p y^{-1}(f) &= g[p, y^{-1}f] && \text{definition of } \psi; \\ &= [yp, yy^{-1}f] && \text{definition of } G\text{-action;} \\ &= [yp, f] = \psi_{yp}(f) && \text{definition of } \psi. \end{aligned}$$

These two constructions give inverse functors. Given an equivariant fiber bundle $E \rightarrow B$, we have a map

$$\xi : (\text{Fr}_F(E) \times F)/\Pi \rightarrow E, \quad \xi([\psi, f]) = \psi(f).$$

Non-equivariantly we already know that (ξ, id_B) is a morphism of fiber bundles with structure group Π and that ξ is a homeomorphism. To check that ξ is G -equivariant, suppose $g \in G$ lifts to $y \in \Gamma$. Then

$$g([\psi, f]) = [y(\psi), yf] = [g\psi y^{-1}, yf]$$

and $\xi([g\psi y^{-1}, yf]) = (g\psi y^{-1})(yf) = g(\psi(f))$. So (ξ, id_B) is a morphism of equivariant fiber bundles by [Definition 2.24](#). It is an isomorphism because the non-equivariant inverse is also an equivariant inverse as it is a homeomorphism. Given an equivariant principal bundle $P \rightarrow B$, we have a map which we abusively denote by

$$\psi : P \rightarrow \text{Fr}_F((P \times F)/\Pi), \quad p \mapsto \psi_p.$$

Here, ψ_p is the previously defined admissible map of $(P \times F)/\Pi$, thus an element of its associated principal bundle. Again, non-equivariantly we know that the map ψ is a homeomorphism (the Π -effectiveness is needed to assure that if $p \neq q$ in P ,

then $\psi_p \neq \psi_q$). To check that ψ is Γ -equivariant, the definition of the Γ -action on admissible maps gives $y\psi_p = g\psi_p y^{-1}$ and we have verified $g\psi_p y^{-1} = \psi_{yp}$, so we have $y\psi_p = \psi_{yp}$. Thus, (ψ, id_B) is a morphism of equivariant principal bundles. It is also an isomorphism. \square

Remark 2.30. The isomorphisms ξ and ψ in the proof are natural and provide the unit and counit maps of the adjunction

$$\text{Hom}((P \times F)/\Pi, E) \cong \text{Hom}(P, \text{Fr}_F(E))$$

$$(- \times F)/\Pi : \left\{ \begin{array}{l} \text{principal } (\Pi; \Gamma)\text{-} \\ \text{bundles over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} G\text{-fiber bundles over } B \\ \text{with structure group } \Pi, \\ \text{total group } \Gamma \text{ and fiber } F \end{array} \right\} : \text{Fr}_F(-)$$

We can see in the proof of [Theorem 2.29](#) that it is essential for F to have a Γ -action. If P were a principal $(\Pi; \Gamma)$ -bundle and the fiber F only had a Π -action, the associated fiber bundle $(P \times F)/\Pi$ would not have a G -action. If we insist on our notion of a G -fiber bundle to be a G -map between G -spaces, this is the price to pay.

2.4. Comparisons of definitions. We have two concepts of G -fiber bundles. One is the G -fiber bundle with fiber F and structure group Π as in [Definition 2.9](#); the other is the G -fiber bundle with fiber F , structure group Π and total group Γ for a specific extension of compact Lie groups $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$, as in [Definition 2.22](#). The differences between the concepts are two-fold: in the first one, G acts by bundle maps, but in the second one, the G -action is regulated by Γ ; in the first one, F has only a Π -action, but in the second one, F has a Γ -action. We compare these two concepts and show that the first concept is a special case of the second when $\Gamma \cong \Pi \times G$ and Γ acts on F via the projection $\Pi \times G \rightarrow \Pi$ ([Proposition 2.34](#)).

We start with some simple group theory observations that will come into play.

Definition 2.31. A retraction $\Gamma \rightarrow \Pi$ is a group homomorphism that restricts to identity on the subgroup Π .

It turns out that Γ admits a retraction to Π if and only if it is isomorphic to $\Pi \times G$. We prove this explicitly in the case of a semidirect product first, then for general Γ .

Proposition 2.32. *Let $\Gamma = \Pi \rtimes_{\alpha} G$ be a split extension. Then*

- (1) *The retractions $\tilde{\beta} : \Gamma \rightarrow \Pi$ are in bijection to homomorphisms $\beta : G \rightarrow \Pi$ satisfying $\alpha_g(\nu) = \beta(g)\nu\beta(g)^{-1}$ for all $g \in G$ and $\nu \in \Pi$. (Note that for a given $\alpha : G \rightarrow \text{Aut}(\Pi)$, the homomorphism β may not exist.)*
- (2) *Each β in (1) specifies an isomorphism $\Pi \rtimes_{\alpha} G \cong \Pi \times G$.*

Proof. To see (1), we use the explicit expression for semidirect product group Γ as in [Remark 2.21](#). Suppose we have a retraction $\tilde{\beta} : \Gamma \rightarrow \Pi$. Let $\beta(g)$ be the image $\tilde{\beta}(e, g)$. Then β is a group homomorphism. We have $\tilde{\beta}(\nu, e) = \nu$ and

$$\tilde{\beta}(\nu, g) = \tilde{\beta}((\nu, e)(e, g)) = \nu\beta(g).$$

In order for $\tilde{\beta}$ to be a homomorphism, it is required that the following two elements are equal for all $g, h \in G$ and $\nu, \mu \in \Pi$:

$$\begin{aligned} \tilde{\beta}(\nu\alpha_g(\mu), gh) &= \nu\alpha_g(\mu)\beta(gh); \\ \tilde{\beta}(\nu, g)\tilde{\beta}(\mu, h) &= \nu\beta(g)\mu\beta(h). \end{aligned}$$

Comparing the two lines gives $\alpha_g(\mu) = \beta(g)\mu\beta(g)^{-1}$. On the other hand, if we have β as required, the formula $\tilde{\beta}(\nu, g) = \nu\beta(g)$ defines a retraction $\tilde{\beta}$.

Given such a β , the group isomorphism in (2) is given by

$$\Pi \rtimes_{\alpha} G \cong \Pi \times G, \quad (\nu, g) \mapsto (\nu\beta(g), g). \quad \square$$

Proposition 2.33. *There is a bijection of sets between $\{\text{retractions } \tilde{\beta} : \Gamma \rightarrow \Pi\}$ and $\{\text{isomorphisms of extensions } \Gamma \cong \Pi \times G\}$.*

Proof. Consider Π as a subgroup of Γ and denote by q the surjection $\Gamma \rightarrow G$. Given a retraction $\tilde{\beta} : \Gamma \rightarrow \Pi$, the map $(\tilde{\beta}, q) : \Gamma \rightarrow \Pi \times G$ is a group isomorphism, and vice versa.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Pi & \xleftarrow{\tilde{\beta}} & \Gamma & \xrightarrow{q} & G \longrightarrow 1 \\ & & \parallel & & \downarrow (\tilde{\beta}, q) & & \parallel \\ 1 & \longrightarrow & \Pi & \longrightarrow & \Pi \times G & \longrightarrow & G \longrightarrow 1 \end{array} \quad \square$$

We now compare Definitions 2.9 and 2.22 in the following propositions. Note that we can think about a retraction $\Gamma \rightarrow \Pi$ as a chosen isomorphism $\Gamma \cong \Pi \times G$ of extensions by Proposition 2.33.

Proposition 2.34. *Let F be a space with an effective Π -action and $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ be an extension of compact Lie groups. Then one can extend the Π -action on F to a Γ -action such that a G -fiber bundle of Definition 2.9 is always a G -fiber bundle of Definition 2.22 if and only if there is a retraction $\Gamma \rightarrow \Pi$ and the extended Γ -action on F is via the retraction.*

Proof. Suppose we have $p : E \rightarrow B$ as in Definition 2.9 and F has an extended Γ -action. Then the only thing to check for p to be a G -fiber bundle of Definition 2.22 is whether it satisfies the condition in Proposition 2.23. That is, it suffices to show for each $y \in \Gamma$ with image $g \in G$ and admissible homeomorphism $\psi : F \rightarrow F_b$, the composite $g\psi y^{-1}$ is also admissible. By Proposition 2.10, $g\psi$ is admissible. So by Lemma 2.3, for $y \in \Gamma$, $g\psi y^{-1}$ is admissible if and only if y acts on F as an element in Π . In other words, the group homomorphism $\Gamma \rightarrow \text{Aut}(F)$ factors through $\Pi \rightarrow \text{Aut}(F)$. \square

The converse is also true.

Proposition 2.35. *Let $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ be an extension of compact Lie groups and F be a Π -effective Γ -space. Then a G -fiber bundle of Definition 2.22 is always a G -fiber bundle of Definition 2.9 if and only if Γ acts on F via a retraction $\Gamma \rightarrow \Pi$.*

Proof. We can reverse the argument in Proposition 2.34. Suppose we have $p : E \rightarrow B$ as in Definition 2.22; to check whether p is a G -fiber bundle of Definition 2.9, we only need to check whether the condition in Proposition 2.10 holds. Take any admissible homeomorphism $\psi : F \rightarrow F_b$. By Proposition 2.23, for any $y \in \Gamma$ with image $g \in G$, $g\psi y^{-1}$ is admissible. By Lemma 2.3, $g\psi$ is admissible if and only if y acts on F as an element in Π . \square

Using Propositions 2.34 and 2.35, we can identify Definition 2.9 as a special case of Definition 2.22.

Example 2.36. Let $\Gamma = \Pi \times G$ and F be a space with an effective Π -action. We give F the trivial G -action. Equivalently, this is viewing F as a space with Γ -action via the projection $\Gamma \rightarrow \Pi$. In this perspective, the structure theorem [Theorem 2.14](#) is a special case of [Theorem 2.29](#).

Example 2.37. In particular, let $\Gamma = O(n) \times G$ and give \mathbb{R}^n the usual $O(n)$ -action and the trivial G -action. We have an equivalence of the two concepts:

- G -vector bundles with fiber \mathbb{R}^n (the classical G -equivariant vector bundles);
- G -fiber bundles with fiber \mathbb{R}^n , structure group $O(n)$ and total group $O(n) \times G$.

Example 2.38 (non-example). For a Real vector bundle as in [Example 2.27](#), Γ does not act on \mathbb{C}^n via $U(n)$ for any n . So a Real vector bundle is not a C_2 -fiber bundle with fiber \mathbb{C}^n and structure group $U(n)$.

Proof. There is no retraction $\Gamma \rightarrow U(n)$, because otherwise by [Proposition 2.32](#), we would need an element $\beta(a)$ of $U(n)$ such that $\beta(a)A = \bar{A}\beta(a)$ for all $A \in U(n)$, where \bar{A} is the complex conjugation of A . But this does not exist for any n . \square

2.5. Examples: the V -framing bundle. In the extension $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$, the group G is redundant because it is just Γ/Π . However, due to the special role of the group G in equivariant homotopy theory, we would like to understand the G -action wherever applicable. Since the total space of a principal $(\Pi; \Gamma)$ -bundle has only a Γ -action, we now focus on the case of split extensions, when we have a specified group homomorphism $G \rightarrow \Gamma$. This becomes relevant when we define and study the V -framing bundle of a G -vector bundle for representations V . It turns out that $\text{Fr}_V(E)$ and $\text{Fr}_{\mathbb{R}^n}(E)$ are the same even as principal $(\Pi; \Gamma)$ -bundles, but they have different G -actions.

Let F be a space with an effective Π -action and one can do some yoga with the fiber. We fix a group homomorphism $\beta : G \rightarrow \Pi$. Then by [Proposition 2.32](#), β determines an isomorphism

$$(2.39) \quad \Pi \rtimes_{\alpha} G \cong \Pi \times G.$$

where $\alpha : G \rightarrow \text{Aut}(\Pi)$ is the group homomorphism given by

$$(2.40) \quad \alpha_g(\nu) = \beta(g)\nu\beta(g)^{-1}.$$

We can let the groups in [\(2.39\)](#) act on F via their retraction to Π . Note that this is the same abstract action but is different for elements of the form (ν, g) on the two sides. For clarity, we denote this space by F' . Explicitly, $(\Pi \times G)$ acts on F' by $(\nu, g)(x) = \nu(x)$ for $x \in F'$; $(\Pi \rtimes_{\alpha} G)$ acts on F' by $(\nu, g)(x) = \nu(\beta(g)(x))$. Inclusion to the second coordinate gives a canonical inclusion of G into both $\Pi \times G$ and $\Pi \rtimes_{\alpha} G$, but this is not compatible with the isomorphism [\(2.39\)](#). The second image is the graph subgroup $\Lambda_{\beta} = \{(\beta(g), g) | g \in G\} \subset \Pi \times G$. Consequently, the two G -actions on F' are different.

In summary, we have an isomorphism of extensions in the situation, but it is not an isomorphism of split extensions, as (e, g) of $\Pi \rtimes_{\alpha} G$ is sent to $(\beta(g), g)$ in $\Pi \times G$.

$$\begin{array}{ccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Pi \rtimes_{\alpha} G & \xrightarrow[(e,g)\leftarrow g]{\cong} & G \longrightarrow 1 \\
& & \parallel & & \downarrow \cong & & \parallel \\
1 & \longrightarrow & \Pi & \longrightarrow & \Pi \times G & \xrightarrow[(e,g)\leftarrow g]{\cong} & G \longrightarrow 1
\end{array}$$

As a consequence, we get the following trivial corollary of Propositions 2.34 and 2.35:

Corollary 2.41. *In the context above, for a group homomorphism $\alpha : G \rightarrow \text{Aut}(\Pi)$ given by (2.40) with associated isomorphism (2.39), the following categories are equivalent:*

- A G -fiber bundle with fiber F and structure group Π ;
- A G -fiber bundle with fiber F' , structure group Π and total group $\Pi \times G$;
- A G -fiber bundle with fiber F' , structure group Π and total group $\Pi \rtimes_{\alpha} G$. \square

Similarly, a principal $(\Pi; \Pi \times G)$ -bundle is literally the same thing as a principal $(\Pi; \Pi \rtimes_{\alpha} G)$ -bundle, but they have different canonical G -actions.

Notation 2.42. For a principal G - Π -bundle, we call it a principal $(\Pi; \Pi \times G)$ -bundle if we let G act on the total space by $G \subset \Pi \times G$; we call it a principal $(\Pi; \Pi \rtimes G)$ -bundle if we let G act on the total space by $\Lambda_{\beta} \subset \Pi \times G$. And similarly for a G -fiber bundle with fiber F and structure group Π .

This trivial observation allows us to define and study the V -framing bundle of an equivariant vector bundle. Let V be an orthogonal G -representation given by $\rho : G \rightarrow O(n)$. In the remainder of this subsection, we write $O(V)$ for the group $O(n)$ with the data $G \rightarrow \text{Aut}(O(n))$ given by $g(\nu) = \rho(g)\nu\rho(g)^{-1}$ for $g \in G$ and $\nu \in O(n)$, so it is clear what $O(V) \rtimes G$ means. This convention coincides with the conjugation G -action on $O(V)$ thought of as a mapping space in Top_G . In this case, taking $F = \mathbb{R}^n$ and pointing aloud the G -action on F' , Corollary 2.41 reads: A G - n -vector bundle is a G -fiber bundle with fiber \mathbb{R}^n , structure group $O(n)$ and total group $O(n) \times G$, as well as a G -fiber bundle with fiber V , structure group $O(n)$ and total group $O(V) \rtimes G$.

Definition 2.43. Let $p : E \rightarrow B$ be a G - n -vector bundle. Let $\text{Fr}_V(E)$ be the space of the admissible maps with the G -action $g(\psi) = g\psi\rho(g)^{-1}$.

In other words, $\text{Fr}_V(E)$ has the same underlying space as $\text{Fr}_{\mathbb{R}^n}(E)$, but we think of admissible maps as mapping out of V instead of \mathbb{R}^n .

Proposition 2.44. *$\text{Fr}_V(E)$ is a principal $(O(n); O(V) \rtimes G)$ -bundle and we have isomorphisms of G -vector bundles:*

$$E \cong (\text{Fr}_V(E) \times V)/O(n).$$

Proof. This is a corollary of the structure theorem Theorem 2.29. Namely, Corollary 2.41 and the explanation afterwards have turned the vector bundle $p : E \rightarrow B$ into a G -fiber bundle with fiber V , structure group $O(n)$ and total group $O(V) \rtimes G$. By examination, $\text{Fr}_V(E)$ in Definition 2.43 agrees with the construction $\text{Fr}_V(E)$ in Theorem 2.29. \square

2.6. Fixed point theorems. Non-equivariantly, the long exact sequence of the homotopy groups of a fiber sequence is a useful tool to study the homotopy group of one term, knowing the other two. To do this equivariantly, we need to know what taking-fixed-points does to equivariant bundles. We focus on $\Gamma = \Pi \times G$ in this subsection.

Let $\text{Rep}(G, \Pi)$ be the set:

$$\text{Rep}(G, \Pi) = \{\text{group homomorphism } \rho : G \rightarrow \Pi\} / \Pi\text{-conjugation.}$$

Any subgroup $H \subset G$ with a group homomorphism $\rho : H \rightarrow \Pi$ gives a subgroup Λ_ρ of $(\Pi \times G)$ via its graph. That is,

$$\Lambda_\rho = \{(\rho(h), h) | h \in H\}.$$

For each $\rho : H \rightarrow \Pi$, denote the centralizer of the image of ρ in Π by

$$Z_\Pi(\rho) = \{\nu \in \Pi | \nu\rho(h) = \rho(h)\nu \text{ for all } h \in H\}.$$

Proposition 2.45. *Let Π be a compact Lie group and H be a subgroup. Then $Z_\Pi(H)$ is a closed subgroup of Π , thus also a compact Lie group.*

Proof. Fix an element $h \in H$. Then the map $c_h : \Pi \rightarrow \Pi$, $\nu \mapsto \nu h \nu^{-1}$ is continuous. Since the singleton $\{h\} \in \Pi$ is closed, the set $c_h^{-1}(\{h\}) = \{\nu \in \Pi | \nu h = h \nu\}$ is also closed. So $Z_\Pi(H) = \bigcap_{h \in H} c_h^{-1}(\{h\})$ is closed. \square

The following theorem is proven in [LM86, Theorem 12] for general Γ . We spell it out for $\Gamma = \Pi \times G$.

Theorem 2.46. *Let G and Π be compact Lie groups. Let $p : E \rightarrow B$ be a principal G - Π -bundle and $H \subset G$ be a subgroup. Assume that E is completely regular.*

(1) *On the base,*

$$B^H = \coprod_{[\rho] \in \text{Rep}(H, \Pi)} p(E^{\Lambda_\rho}).$$

(2) *As sets, the preimages over each component of B^H are*

$$p^{-1}(p(E^{\Lambda_\rho})) = \coprod_{\{\rho' : \Pi\text{-conjugate to } \rho\}} E^{\Lambda_{\rho'}}.$$

As spaces,

$$p^{-1}(p(E^{\Lambda_\rho})) \cong \Pi \times_{Z_\Pi(\rho)} E^{\Lambda_\rho}.$$

(3) *For a fixed representative ρ of $[\rho]$, we have a principal $Z_\Pi(\rho)$ -bundle:*

$$Z_\Pi(\rho) \rightarrow E^{\Lambda_\rho} \xrightarrow{p} p(E^{\Lambda_\rho}).$$

(4) *In particular, the following is a principal Π -bundle:*

$$\Pi \rightarrow E^H \xrightarrow{p} p(E^H).$$

Explanation. In words, part (1) says that the H -fixed points of B are the images of the Λ -fixed points of E for all subgroups $\Lambda \subset \Pi \times G$ that are graphs of a homomorphism $H \rightarrow \Pi$. Furthermore, E^Λ and $E^{\Lambda'}$ share the same projection image when Λ and Λ' are Π -conjugate, or equivalently the corresponding representations $H \rightarrow \Pi$ are Π -conjugate. The assumption that E is completely regular implies that if Λ and Λ' are not Π -conjugate, the images of E^Λ and $E^{\Lambda'}$ are disjoint.

Parts (2) and (3) imply that E restricted on each component of B^H has a reduction of the structure group from Π to $Z_\Pi(\rho)$. In the proof of Theorem 3.12(1), we will

describe in an example how to find the representations ρ when $H = G$. The idea is that the fiber over an H -fixed base has an H -action, and ρ tells what this action is in terms of the native Π -action as a principal bundle. Note that the representation ρ is dependent on the choice of a base point z in the fiber; a different choice gives a conjugate representation. From the description of the action, a point in the same fiber, written uniquely as $z\nu$ for some $\nu \in \Pi$, is Λ_ρ -fixed if and only if $\rho(h)\nu\rho(h)^{-1} = \nu$ for all $h \in H$. This justifies the first statement of part (2) as well as part (3).

For the second statement of part (2), which is not in the reference, we use the map:

$$\Pi \times_{Z_\Pi(\rho)} E^{\Lambda_\rho} \rightarrow E, (\nu, x) \mapsto x\nu^{-1}.$$

Here, $Z_\Pi(\rho)$ is a subgroup of Π and acts on the right of Π by multiplication; the left Π -action on E restricts to a left $Z_\Pi(\rho)$ -action on E^{Λ_ρ} . It is a homeomorphism to its image, which is exactly $p^{-1}(p(E^{\Lambda_\rho}))$:

We have $\Lambda_e = H$ for the trivial representation $e : H \rightarrow \Pi$. Part (4) follows from taking $\rho = e$ in part (3). \square

Remark 2.47. From [Theorem 2.46](#), for a principal G - Π -bundle $p : E \rightarrow B$ and a subgroup $H \subset G$, each component B_0 of B^H has an associated representation class $[\rho] \in \text{Rep}(H, \Pi)$. It is characterized by the fact that for any representation $\rho' : H \rightarrow \Pi$,

$$(p^{-1}(B_0))^{\Lambda_{\rho'}} \neq \emptyset \text{ if and only if } [\rho'] = [\rho].$$

The restricted principal Π -bundle $p^{-1}(B_0) \rightarrow B_0$ has a reduction of the structure group from Π to $Z_\Pi(\rho)$.

Non-equivariantly, a map between two principal G -bundles that is an underlying equivalence on the total spaces will give an equivalence on the base spaces, as can be shown by the long exact sequence of homotopy groups. Equivariantly, we also want this tool of knowing when a map of two principal G - Π -bundles gives a G -equivalence on the base spaces.

Theorem 2.48. *Let G, Π be compact Lie groups and $i : \Pi \rightarrow \Pi'$ be an inclusion of topological groups. Let $p : E \rightarrow B$ be a principal G - Π -bundle, $p' : E' \rightarrow B'$ be a principal G - Π' -bundle and assume that B, B' have the homotopy type of G -CW complexes. Then E' has a $(\Pi \times G)$ -action by i .*

Suppose that there is a $(\Pi \times G)$ -map $\bar{f} : E \rightarrow E'$ over a G -map $f : B \rightarrow B'$, as in the following commutative diagram:

$$\begin{array}{ccc} \Pi & \xrightarrow{i} & \Pi' \\ \downarrow & & \downarrow \\ E & \xrightarrow{\bar{f}} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

Assume

- (1) *The map i includes Π as a deformation retract of Π' in groups, that is, there exists a group homomorphism $j : \Pi' \rightarrow \Pi$ such that $j \circ i = \text{id}$ and $i \circ j \simeq \text{id}$ rel $i(\Pi)$ in topological groups;*

(2) On the total spaces, the map \bar{f} is an $\mathcal{F}(\Pi)$ -equivalence, that is, a Λ -equivalence for any subgroup $\Lambda \subset \Pi \times G$ such that $\Lambda \cap \Pi = e$.

Then, on the base spaces, $f : B \rightarrow B'$ is a G -equivalence.

Proof. To simplify notation in this proof, we use the same letters to denote the restrictions of the corresponding maps to a subspace. By the equivariant Whitehead theorem, it suffices to show that:

For any subgroup $H \subset G$, the map $f : B^H \rightarrow (B')^H$ is an equivalence.

We make the following two claims comparing Π and Π' :

- (a) For any group H , the induced map $i_* : \text{Rep}(H, \Pi) \rightarrow \text{Rep}(H, \Pi')$ is a bijection.
- (b) For any subgroup K of Π , the inclusion $i : Z_\Pi K \rightarrow Z_{\Pi'} i(K)$ is a homotopy equivalence;

These two claims follow from the assumption (1). For (a), we take the functor $F = \text{Rep}(H, -)$ from the category of groups to sets. It has equivalent images on Π and Π' , and we skip the details. For (b), we take the functor $F = Z_{(-)}K$ from the category of groups containing K as a subgroup. It also has equivalent images on Π and Π' , and the details come later in Lemma 2.52.

By Theorem 2.46 (1) and (a), it suffices to show that:

For any H and $\rho \in \text{Rep}(H, \Pi)$, the map $f : p(E^{\Lambda_\rho}) \rightarrow p'((E')^{\Lambda_\rho})$ is an equivalence.

By Theorem 2.46 (3), taking the Λ_ρ -fixed points of E and E' yields a map between principal bundles:

$$\begin{array}{ccc} Z_\Pi(\rho) & \xrightarrow{i} & Z_{\Pi'}(\rho) \\ \downarrow & & \downarrow \\ E^{\Lambda_\rho} & \xrightarrow{\bar{f}} & (E')^{\Lambda_\rho} \\ \downarrow p & & \downarrow p' \\ p(E^{\Lambda_\rho}) & \xrightarrow{f} & p'((E')^{\Lambda_\rho}) \end{array}$$

By the claim (b) and the assumption (2), both i and \bar{f} are equivalences. The long exact sequence of homotopy groups shows that f is an equivalence. \square

Remark 2.49. In Theorem 2.48, the assumption (1) is satisfied in our applications with $\Pi' = \Pi$ or $\Pi' = \Pi^I$. The assumption (2) is satisfied when \bar{f} is a $(\Pi \times G)$ -equivalence, but is weaker. The weaker version is needed in our applications.

From the proof, we also have a version of Theorem 2.48 relaxing the assumption (2).

Corollary 2.50. *Suppose we have (i, \bar{f}, f) in the context of Theorem 2.48, except that instead of the assumption (2), $\bar{f} : E \rightarrow E'$ is only known to be a Λ_ρ -equivalence for a fixed representation $\rho : H \rightarrow \Pi$. Then on the base spaces, $f : p(E^{\Lambda_\rho}) \rightarrow p'((E')^{\Lambda_\rho})$ is an equivalence.*

Note that $p(E^{\Lambda_\rho})$ is the space of components of B^H that are associated to ρ as described in Remark 2.47. In particular, if $(B')^H$ is connected for all subgroups $H \subset G$, then $(B')^H$ has only one associated representation ρ_H . Moreover, ρ_H has to be the restriction of ρ_G . We have:

Corollary 2.51. *Let B' be a G -connected space as explained above and ρ_G be the associated representation. Suppose we have (i, \bar{f}, f) in the context of [Corollary 2.50](#), such that \bar{f} is a Λ_{ρ_G} -equivalence. Then on the base spaces, $f : B \rightarrow B'$ is a G -equivalence.*

Proof. Since the map $f : B^H \rightarrow (B')^H$ preserves the associated representation, we know that B^H only has one associated representation ρ_H as well. The claim then follows by applying [Corollary 2.50](#) to $\rho = \rho_H$ for all H . \square

The following is a lemma for [Theorem 2.48](#):

Lemma 2.52. *Assume $i : \Pi \rightarrow \Pi'$ is an inclusion of topological groups with a deformation retract $j : \Pi' \rightarrow \Pi$, that is, they satisfy condition (1) in [Theorem 2.48](#). Then for any subgroup K of Π , the inclusion $i : Z_{\Pi}K \rightarrow Z_{\Pi'}i(K)$ is a homotopy equivalence.*

Proof. We first check that in general, given any group homomorphism $f : G \rightarrow G'$ and subgroup $K \subset G$, the map f restricts to a map $f_0 : Z_GK \rightarrow Z_{G'}(f(K))$ on subspaces. This is because $xk = kx$ for all $k \in K$ implies $f(x)f(k) = f(k)f(x)$ for all $f(k) \in f(K)$. So, we have

$$i_0 : Z_{\Pi}K \rightarrow Z_{\Pi'}(i(K)) \text{ and } j_0 : Z_{\Pi'}(i(K)) \rightarrow Z_{\Pi}(ji(K)) = Z_{\Pi}K.$$

The map j_0 gives deformation retract data of the inclusion i_0 . It is obvious that $j_0i_0 = \text{id}$. It remains to show $i_0j_0 \simeq \text{id}$. The image of i_0 is the subspace $Z_{i(\Pi)}(i(K)) \subset Z_{\Pi'}(i(K))$. The homotopy $ij \simeq \text{id}$ rel $i(\Pi)$ restricts to a homotopy $i_0j_0 \simeq \text{id}$ rel $Z_{i(\Pi)}(i(K))$. \square

3. CLASSIFYING SPACES

3.1. V -trivial bundles. An equivariant bundle $E \rightarrow B$ is V -trivial for some n -dimensional G -representation V if there is a G -vector bundle isomorphism $E \cong B \times V$. Such an isomorphism is a V -framing of the bundle. This is analogous to the case of non-equivariant vector bundles, except that equivariance adds in the complexity of a representation V that's part of the data.

However, the representation V in the equivariant trivialization of a fixed vector bundle may not be unique. We give a lemma to recognize when two trivial bundles are isomorphic, then a counterexample.

Let $\text{Iso}(V, W)$ be the space of linear isomorphisms $V \rightarrow W$ with the conjugation G -action for G -representations V and W .

Lemma 3.1. *For a G -space B , there exists a G -vector bundle isomorphism $B \times V \cong B \times W$ if and only if there exists a G -map $f : B \rightarrow \text{Iso}(V, W)$.*

Proof. Let $F : B \times V \rightarrow B \times W$ be a vector bundle map. For $b \in B$, let $F_b : V \rightarrow W$ be such that $F_b(v) = F(b, v)$. Then F is a G -vector bundle isomorphism if and only if

- (1) F is fiberwise isomorphism: $F_b \in \text{Iso}(V, W)$;
- (2) F is a G -map: $gF(b, v) = F(gb, gv)$, or equivalently, $F_{gb} = gF_b g^{-1}$, for all $g \in G$.

Taking $f(b) = F_b$, it follows that F is an isomorphism if and only if f is a G -map. \square

Corollary 3.2. *If B has a G -fixed point, then $B \times V \cong B \times W$ only when $V \cong W$.*

Proof. The equivariant map $f : B \rightarrow \text{Iso}(V, W)$ induces $f^G : B^G \rightarrow \text{Iso}_G(V, W)$. The source being nonempty implies that the target is nonempty. \square

Remark 3.3. More generally, for any two n -dimensional G -vector bundles E, E' over B , one can form the non-equivariant bundle $\mathcal{H}om_B(E, E')$ which consists of all bundle maps $E \rightarrow E'$ over B (not necessarily fiberwise isomorphisms). It has a G -action by conjugation and is indeed an n^2 -dimensional G -vector bundle over B . Let $\mathcal{I}so_B(E, E')$ be the subspace consisting of only fiberwise isomorphisms. It is a GL_n -bundle over B . Then tautologically $E \cong E'$ if there is a G -invariant section of $\mathcal{I}so_B(E, E')$.

Example 3.4 (Counterexample). Let $G = C_2$, σ be the sign representation. The unit sphere, $S(2\sigma)$, is S^1 with the 180 degree rotation action. As C_2 -vector bundles,

$$S(2\sigma) \times \mathbb{R}^2 \cong S(2\sigma) \times 2\sigma.$$

Proof. By [Lemma 3.1](#), it suffices to construct a C_2 -map $S(2\sigma) \rightarrow \text{Iso}(\mathbb{R}^2, 2\sigma) \cong GL_2$, where the nontrivial element of C_2 acts on GL_2 by multiplying by $-\text{Id}$. We give $S(2\sigma)$ a G -CW decomposition of a 0-cell C_2/e and a 1-cell $C_2/e \times D^1$ and construct the map by skeleton. It is obvious that any equivariant map on the 0-skeleton extends to the 1-skeleton if and only if the two images lie in the same path component of GL_2 , which is true in this case as $-\text{Id}$ and Id lie in the same path component. \square

The following counterexample is suggested by Gus Longerman.

Example 3.5. (Counterexample) Take G to be any compact Lie group and V and W to be any two representation of G that are of the same dimension. Then $G \times V \cong G \times W$, because $\text{Map}_G(G, \text{Iso}(V, W)) \cong \text{Map}(\text{pt}, \text{Iso}(V, W)) \neq \emptyset$. Indeed, the isomorphism can be constructed explicitly by $F(g, x) = (g, \rho_W(g)\rho_V(g)^{-1}x)$, where $\rho_V, \rho_W : G \rightarrow O(n)$ are matrix representations of V, W .

3.2. Universal equivariant bundles.

Definition 3.6. A principal $(\Pi; \Gamma)$ -bundle $E \rightarrow B$ is called universal if for any paracompact G -space X , there is a bijection of sets between $\{\text{equivalence classes of principal } (\Pi; \Gamma)\text{-bundles over } X\}$ and $\{G\text{-homotopy classes of } G\text{-maps } X \rightarrow B\}$.

The correspondence in one direction is by pulling back the universal bundle along G -maps $X \rightarrow B$. The universal principal $(\Pi; \Gamma)$ -bundle exists and is unique up to homotopy. Moreover, it can be recognized by the following property:

Theorem 3.7. ([\[LM86, Theorem 9\]](#)) *A principal $(\Pi; \Gamma)$ -bundle $p : E \rightarrow B$ is universal if and only if*

$$E^\Lambda \simeq *, \text{ for all subgroups } \Lambda \subset \Gamma \text{ such that } \Lambda \cap \Pi = e.$$

The universal principal $(\Pi; \Gamma)$ -bundle can be constructed in various ways. We denote it as $E(\Pi; \Gamma) \rightarrow B(\Pi; \Gamma)$. One construction generalizes Milnor's infinite join construction. See [\[TD69, Section 3\]](#) for the case $\Gamma = \Pi \rtimes G$ or [\[Las82, Section 2\]](#) for the case $\Gamma = \Pi \times G$. We describe another abstract construction using [Theorem 3.7](#) as outlined in the introduction. Suppose we have a universal Γ -space $E\mathcal{F}$ for the family $\mathcal{F} = \{\Lambda \subset \Gamma \mid \Lambda \cap \Pi = e\}$. Then the restricted Π -action on $E\mathcal{F}$ is free, as any non-trivial element $A \in \Pi$ will generate a subgroup $\langle A \rangle \subset \Gamma$ not in \mathcal{F} , so $\{x \in E\mathcal{F} \mid Ax = x\} \cong E\mathcal{F}^{\langle A \rangle} = \emptyset$. So the quotient map $E\mathcal{F} \rightarrow E\mathcal{F}/\Pi$ is a principal $(\Pi; \Gamma)$ -bundle, and it is equivalent to the universal one by [Theorem 3.7](#). Thus it suffices

to construct $E\mathcal{F}$. This can be done via Elmendorf's construction [Elm83]: The $E\mathcal{F}$ is constructed as a Γ -CW complex (See Section 2.1 for G -CW complexes). First, we take

$$E_0 = \sqcup_{\Lambda \in \mathcal{F}} \Gamma/\Lambda.$$

It has the property that $E_0^\Lambda \neq \emptyset$ exactly when $\Lambda \in \mathcal{F}$. We make the desired fixed-point-spaces contractible by adding in higher cells. To start, we add 1-cells

$$\sqcup_{\Lambda_0, \Lambda_1 \in \mathcal{F}} \text{Map}_\Gamma(\Gamma/\Lambda_1, \Gamma/\Lambda_0) \times \Gamma/\Lambda_1 \times D^1$$

to obtain E_1 . Here, $\text{Map}_\Gamma(\Gamma/\Lambda_1, \Gamma/\Lambda_0)$ is the space of Γ -equivariant maps topologized as homeomorphic to the space $(\Gamma/\Lambda_0)^{\Lambda_1}$. The gluing map on

$$(f, x, t) \in \text{Map}_\Gamma(\Gamma/\Lambda_1, \Gamma/\Lambda_0) \times \Gamma/\Lambda_1 \times S^0$$

sends $(f, x, -1)$ to $f(x) \in \Gamma/\Lambda_0 \subset E_0$ and sends $(f, x, 1)$ to $x \in \Gamma/\Lambda_1 \subset E_0$. Now we have E_1^Λ is connected for $\Lambda \in \mathcal{F}$. The higher cells are selected and glued following the same idea and we skip the details. Technically speaking, when Γ is not a discrete group, $\text{Map}_\Gamma(\Gamma/\Lambda_1, \Gamma/\Lambda_0) \times \Gamma/\Lambda_1 \times D^1$ may not be a 1-cell as defined in Section 2. However, since $\text{Map}_\Gamma(\Gamma/\Lambda_1, \Gamma/\Lambda_0)$ has the homotopy type of a CW-complex, the constructed $E\mathcal{F}$ always has the homotopy type of a Γ -CW complex.

Remark 3.8. When $\Gamma = \Pi \times G$, such a subgroup Λ comes in the form of

$$\{(\rho(h), h) | h \in H\}, \text{ for } H \subset G \text{ and } \rho : H \rightarrow \Pi \text{ is a group homomorphism.}$$

This group was denoted by Λ_ρ in Theorem 2.46.

When $\Gamma = \Pi \rtimes_\alpha G$, such a subgroup Λ comes in the form of

$$\{(\rho(h), h) | h \in H\}, \text{ for } H \subset G \text{ and } \rho : H \rightarrow \Pi \text{ such that } \rho(h_1 h_2) = \rho(h_1) \cdot \alpha_{h_1}(\rho(h_2)).$$

We mostly specialize to the case $\Gamma = O(n) \times G$, when a principal $(O(n); \Gamma)$ is also a principal G - $O(n)$ -bundle. We also denote the universal principal G - $O(n)$ -bundle by $E_G O(n) \rightarrow B_G O(n)$ and denote the universal G - n -vector bundle by $\zeta_n \rightarrow B_G O(n)$ where

$$\zeta_n = E_G O(n) \times_{O(n)} \mathbb{R}^n.$$

As an immediate corollary of Theorems 2.46 and 3.7, one gets the G -homotopy type of the universal base. Recall that

$$\begin{aligned} \text{Rep}(G, O(n)) &= \{\rho : G \rightarrow O(n) \text{ group homomorphism}\} / O(n)\text{-conjugation}; \\ &\cong \{V : n\text{-dimensional orthogonal representation of } G\} / \text{isomorphism} \end{aligned}$$

and $Z_{O(n)}(\rho) = \{a \in O(n) | a\rho(g) = \rho(g)a, \text{ for all } g \in G\}$ is the centralizer of the image of ρ in $O(n)$.

Theorem 3.9. ([Las82, Theorem 2.17])

$$\begin{aligned} (B_G O(n))^G &\simeq \coprod_{[\rho] \in \text{Rep}(G, O(n))} BZ_{O(n)}(\rho); \\ &\simeq \coprod_{[V] \in \text{Rep}(G, O(n))} B(O(V))^G. \end{aligned}$$

Example 3.10. Take $H = G = C_2$ and $\Pi = O(2)$. Then

$$\text{Rep}(C_2, O(2)) = \{\text{id}, \text{rotation}, \text{reflection}\},$$

where rotation is the rotation by 180 degrees and reflection is the equivalence class of all reflections. For $\rho = \text{id}$ or $\rho = \text{rotation}$, $Z_\Pi(\rho) = O(2)$. For $\rho = \text{reflection}$, $Z_\Pi(\rho) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. So

$$(B_{C_2}O(n))^{C_2} \simeq BO(2) \sqcup BO(2) \sqcup B(\mathbb{Z}/2 \times \mathbb{Z}/2).$$

3.3. The fiber of the universal bundle and the loop space of the classifying space over a point. From [Theorem 3.9](#), one can make explicit the classifying maps of V -trivial bundles as follows.

A G -map $\theta : \text{pt} \rightarrow B_GO(n)$ lands in one of the G -fixed components of $B_GO(n)$. Suppose that this component is indexed by $[V]$.

Proposition 3.11. *The pullback of the universal bundle is $\theta^*\zeta_n \cong V$ over pt .*

Proof. It follows from part (1) of the following [Theorem 3.12](#) that

$$\theta^*\zeta_n \cong O(\mathbb{R}^n, V) \times_{O(n)} \mathbb{R}^n \cong V.$$

In fact, the n -planes in a complete G -universe with the tautology n -plane bundle is a model for $B_GO(n)$ and ζ_n ; $\theta(\text{pt})$ is just a G -representation isomorphic to V . \square

Theorem 3.12. *Take a G -fixed base point $b \in B_GO(n)$ in the component indexed by $[V]$. Let $p : E_GO(n) \rightarrow B_GO(n)$ be the universal principal G - $O(n)$ -bundle. Then*

- (1) *The fiber over b , $p^{-1}(b)$, is homeomorphic to $O(\mathbb{R}^n, V)$ as an $(O(n) \times G)$ -space. Here, $(O(n) \times G)$ acts on $O(\mathbb{R}^n, V)$ by G acting on V and $O(n)$ acting on \mathbb{R}^n .*
- (2) *The loop space of $B_GO(n)$ at the base point b , $\Omega_b B_GO(n)$, is G -homotopy equivalent to $O(V)$, the isometric self maps of V with G acting by conjugation.*

Proof. (1) This is due to Lashof and we explain how to find the representation V here. Choose and fix a base point $z \in p^{-1}(b)$. We construct a group homomorphism $\rho_z : G \rightarrow O(n)$ as follows. For any $g \in G$, there exists a unique element, $\rho_z(g) \in O(n)$ such that $gz = z\rho_z(g)$. It is easy to check that $g \mapsto \rho_z(g)$ gives a group homomorphism. Suppose z' is another base point in $p^{-1}(b)$, and $z' = z\nu$ for some unique $\nu \in O(n)$. Then

$$gz' = gz\nu = z\rho_z(g)\nu = z'(\nu^{-1}\rho_z(g)\nu).$$

So $\rho_{z'} = \nu^{-1}\rho_z\nu$ is $O(n)$ -conjugate to ρ_z . The different ρ_z 's are the matrix representations of some vector space representation V . From the proof of [Theorem 2.17](#) of [[Las82](#)], this is exactly the index V . Without loss of generality, we take V to be given by ρ_z as matrix representation.

The following map gives a non-equivariant homeomorphism:

$$\begin{array}{ccc} O(\mathbb{R}^n, V) & \cong & O(n) \xrightarrow{\cong} p^{-1}(b), \\ & & \nu \mapsto z\nu. \end{array}$$

It suffices to check it is an equivariant homeomorphism with the described action. Let $(\mu, g) \in O(n) \times G$. Then

$$z((\mu, g) \circ \nu) = z(\rho_z(g)\nu\mu^{-1}) = (z\rho_z(g))(\nu\mu^{-1}) = (gz)(\nu\mu^{-1}) = (\mu, g) \circ z\nu.$$

(2) The idea is to compare the path space fibration with the universal bundle. Equivariantly, the base point should be G -fixed. Since the space involved is not G -connected,

base points from different components might give inequivalent loop spaces. We use subscripts in path spaces and loop spaces to indicate the base point. For example,

$$P_b B_G O(n) = \{\alpha \in \text{Map}([0, 1], B_G O(n)) \mid \alpha(0) = b\}.$$

Fix $z \in p^{-1}(b)$ and $\rho = \rho_z : G \rightarrow O(n)$ as above. Take z to be the base point of $E_G O(n)$. It is a Λ -fixed point, where

$$\Lambda = \{(\rho(g), g) \mid g \in G\} \subset O(n) \times G.$$

We prove that $E_G O(n)$ is Λ -contractible. In fact, let Λ' be any subgroup of Λ . Then $\Lambda' \cap O(n) = e$, so by [Theorem 3.7](#), $(E_G O(n))^{\Lambda'}$ is contractible.

So, the contraction map gives a based Λ -equivariant homotopy:

$$E_G O(n) \wedge I \rightarrow E_G O(n).$$

Here, $I = [0, 1]$ is based at 0 and has the trivial Λ -action. (The map sends $x \wedge 0$ and $z \wedge t$ to z for all $x \in E_G O(n)$ and $t \in I$.) We take the adjoint of this homotopy to get $E_G O(n) \rightarrow P_z E_G O(n)$, and then compose with $P_z E_G O(n) \rightarrow P_b B_G O(n)$ induced by $p : E_G O(n) \rightarrow B_G O(n)$. The composite is

$$f : E_G O(n) \rightarrow P_z E_G O(n) \rightarrow P_b B_G O(n).$$

It sends a point $x \in E_G O(n)$ to a path in $B_G O(n)$ that starts at b and ends at $p(x)$. This yields a commutative diagram:

$$(3.13) \quad \begin{array}{ccc} E_G O(n) & \xrightarrow{f} & P_b B_G O(n) \\ p \downarrow & & \downarrow p_1 \\ B_G O(n) & \xlongequal{\quad} & B_G O(n) \end{array}$$

Moreover, this diagram is G -equivariant, where the G -action on $P_b B_G O(n)$ is by point-wise action on the path. It is worth noting that the G -action we take on $E_G O(n)$ is not the original one, but via the identification $q : \Lambda \cong G$. In other words, $g \in G$ acts by what $(\rho(g), g)$ acts. The two vertical maps are non-equivariant fibrations and f maps the fiber of p over $b \in B_G O(n)$, denoted F_1 , to the fiber of p_1 over b , denoted F_2 .

We first identify the fibers F_1 and F_2 . From part [\(1\)](#), $F_1 \cong O(\mathbb{R}^n, V)$ as $(O(n) \times G)$ -spaces. So $F_1 \cong O(V)$ as G -spaces. It is clear that $F_2 \cong \Omega_b B_G O(n)$ as G -spaces.

We claim that f restricts to a G -equivalence $F_1 \rightarrow F_2$. The strategy is to show that it induces an isomorphism on homotopy groups of H -fixed points for all $H \subset G$, using the long exact sequences of homotopy groups of fiber sequences. Without dealing with general G -fibrations, it suffices to work out the following:

- Denote by $\Lambda' = q^{-1}(H)$, the subgroup of Λ that is isomorphic to H . The commutative diagram [\(3.13\)](#) restricts to the following commutative diagram:

$$\begin{array}{ccc} (F_1)^H & \xrightarrow{\quad} & (F_2)^H \\ \downarrow & & \downarrow \\ (E_G O(n))^{\Lambda'} & \xrightarrow{f^H} & (P_b B_G O(n))^H \\ p \downarrow & & \downarrow p_1 \\ p((E_G O(n))^{\Lambda'}) & \xrightarrow{\quad} & p_1((P_b B_G O(n))^H) \end{array}$$

- On the total space level, f^H induces isomorphism on homotopy groups. This is true because $E_G O(n)$ is Λ -contractible and $P_b B_G O(n)$ is G -contractible.
- The base spaces are equal. In fact, it is easy to see that they are both the component of $(B_G O(n))^H$ indexed by $[V]$ from Theorems 2.46 and 3.9.
- The two vertical lines are fiber sequences. For the first, we use Theorem 2.46 (3) with $(F_1)^H = (O(V))^H = Z_\Pi(\rho|_H)$; for the second, it is merely the path space fibration $\Omega_b X \rightarrow P_b X \rightarrow X$, where X denotes the component of $(B_G O(n))^H$ containing b . \square

Remark 3.14. The proof of Theorem 3.12 works for general Π placing $O(n)$. Take a G -fixed base point $b \in B_G \Pi$ in the component indexed by $[\rho : G \rightarrow \Pi]$. Let Π_{ad} be the space Π with the adjoint Π -action and consider it as a G -space via ρ . Then there is a G -homotopy equivalence $\Omega_b B_G \Pi \simeq \Pi_{\text{ad}}$.

3.4. The gauge group of an equivariant principal bundle. Let $EO(n) \rightarrow BO(n)$ be the universal principal $O(n)$ -bundle and $p : P \rightarrow B$ be any principal $O(n)$ -bundle. The gauge group of P , $\text{Aut}_B(P)$, is the space of bundle automorphisms of P that are identity on the base space B ([Hus94, Chap 7, Definition 1.1]). It turns out that the space of principal bundle maps, $\text{Hom}(P, EO(n))$, is also universal: The map

$$(3.15) \quad \text{Hom}(P, EO(n)) \rightarrow \text{Map}_p(B, BO(n))$$

that restricts a bundle map to its base spaces is known to be the universal principal $\text{Aut}_B(P)$ -bundle. Here, $\text{Map}_p(B, BO(n))$ denotes the component of the classifying map of p in $\text{Map}(B, BO(n))$. A proof of this result can be found in [Hus94, Chap 7, Corollary 3.5]. In this subsection, we show the equivariant generalization of this result (Theorem 3.17).

Let $E_G O(n) \rightarrow B_G O(n)$ be the universal principal G - $O(n)$ -bundle and $p : P \rightarrow B$ be any principal G - $O(n)$ -bundle. The restricting-to-the-base map

$$(3.16) \quad \pi : \text{Hom}(P, E_G O(n)) \rightarrow \text{Map}_p(B, B_G O(n))$$

is a G -map lifting (3.15). Here, $\text{Map}_p(B, B_G O(n))$ is the (non-equivariant) component of the classifying map of p in $\text{Map}(B, B_G O(n))$; G acts by conjugation on both sides of (3.16). Let $\Gamma = \text{Aut}_B P \rtimes G$, where G acts on $\text{Aut}_B P$ by conjugation. Then the map π in (3.16) is a universal principal $(\text{Aut}_B(P); \Gamma)$ -bundle. Note that this is an equivariant principal bundle not in the sense of Definition 2.12, but of Definition 2.17 - the total group is a non-trivial extension of $\text{Aut}_B(P)$ by G .

Theorem 3.17. *In the context above, the map*

$$\pi : \text{Hom}(P, E_G O(n)) \rightarrow \text{Map}_p(B, B_G O(n))$$

is a principal $(\text{Aut}_B P; \Gamma)$ -bundle, where $\Gamma = \text{Aut}_B(P) \rtimes G$. Moreover, we have

$$\text{Hom}(P, E_G O(n)) \simeq E\mathcal{F}$$

for the family $\mathcal{F} = \{\Lambda \subset \Gamma \text{ such that } \Lambda \cap \text{Aut}_B P = e\}$.

Proof. As stated above, it is known non-equivariantly that π is a universal principal $\text{Aut}_B P$ -bundle. One can use the conjugation G -action to get a principal $(\text{Aut}_B P; \Gamma)$ -bundle structure on π . However, later in this proof we want a Γ -action on the bundle P , so at the risk of elaborating the obvious, we describe the Γ -action on $\text{Hom}(P, E_G O(n))$ by putting a Γ -action on both P and $E_G O(n)$. The group $\text{Aut}_B P$ naturally has a left

action on P ; take its trivial action on $E_G O(n)$. The group G acts on P and $E_G O(n)$ because they are G -vector bundles. One can check by [Remark 2.21](#) that this gives a Γ -action on P and $E_G O(n)$, thus by conjugation on $\text{Hom}(P, E_G O(n))$. Explicitly, the action is

$$\begin{aligned} (\text{Aut}_B P \rtimes G) \times \text{Hom}(P, E_G O(n)) &\rightarrow \text{Hom}(P, E_G O(n)) \\ ((s, g), f) &\mapsto g f s^{-1} g^{-1}. \end{aligned}$$

Since $s \in \text{Aut}_B P$ restricts to identity on B , we have

$$\pi(g f s^{-1} g^{-1}) = g \pi(f) g^{-1}.$$

By [Definition 2.17](#), the map π is a principal $(\text{Aut}_B P; \Gamma)$ -bundle.

It remains to show that

$$\text{Hom}(P, E_G O(n))^\Lambda \simeq * \text{ for any } \Lambda \subset \Gamma \text{ such that } \Lambda \cap \text{Aut}_B P = e.$$

Such subgroups Λ are isomorphic to subgroups H of G . The claim follows from various applications of the postponed [Lemma 3.18](#), and it is essentially a consequence of the universality of $E_G O(n)$.

To see it, we first consider the case $\Lambda = H$, that is, the case where $\rho(h) = e$ for all $h \in H$ in [Remark 3.8](#). By restricting the G -action to an H -action, $E_G O(n)$ is also the universal principal H - $O(n)$ -bundle. Then $\text{Hom}(P, E_G O(n))^H \simeq *$ by taking $\Pi = O(n)$, $G = H$ and $\Gamma = O(n) \times H$ in [Lemma 3.18](#).

In the general case, Λ is isomorphic to a subgroup $H \subset G$ by the projection map $\Gamma \rightarrow G$, with a possibly non-trivial map ρ in [Remark 3.8](#). Here is the crux: the elements in $\text{Aut}_B P$ are $O(n)$ -equivariant maps, so the $(\Gamma = \text{Aut}_B P \rtimes G)$ -action on P defined at the beginning of this proof commutes with the $O(n)$ -action; and we have $\Lambda \subset \Gamma$. In other words, P is also a principal Λ - $O(n)$ -bundle. Since Λ acts by H on $E_G O(n)$, the space $E_G O(n)$ is also the universal principal Λ - $O(n)$ -bundle. Now we are basically in the first case again: $\text{Hom}(P, E_G O(n))^\Lambda \simeq *$ by taking $\Pi = O(n)$, $G = \Lambda$ and $\Gamma = O(n) \times \Lambda$ in [Lemma 3.18](#). \square

The following lemma is a consequence of the universality:

Lemma 3.18. *Let $1 \rightarrow \Pi \rightarrow \Gamma \rightarrow G \rightarrow 1$ be an extension of groups. Let*

$$p_{\Pi; \Gamma} : E(\Pi; \Gamma) \rightarrow B(\Pi; \Gamma)$$

be the universal principal $(\Pi; \Gamma)$ -bundle and let $p : P \rightarrow B$ be any principal $(\Pi; \Gamma)$ -bundle. Then $(\text{Hom}(P, E(\Pi; \Gamma)))^G$ is contractible.

Proof. To clarify the notations, $\text{Hom}(P, E(\Pi; \Gamma))$ is the space of maps of (nonequivariant) principal Π -bundles. By definition,

$$\text{Hom}(P, E(\Pi; \Gamma)) \cong \text{Map}_\Pi(P, E(\Pi; \Gamma)).$$

The space $\text{Hom}(P, E(\Pi; \Gamma))$ has a Γ -action by conjugation. Since $\Pi \subset \Gamma$ acts trivially, it descends to a G -action, and

$$(\text{Hom}(P, E(\Pi; \Gamma)))^G \cong \text{Map}_\Gamma(P, E(\Pi; \Gamma)).$$

By definition, the space $\text{Map}_\Gamma(P, E(\Pi; \Gamma))$ is in fact the space of morphisms between principal $(\Pi; \Gamma)$ -bundles. It is non-empty because it consists of the classifying map of p . It is further path-connected because any two Γ -maps $P \rightarrow E(\Pi; \Gamma)$ will both restrict

to a classifying map $B \rightarrow B(\Pi; \Gamma)$ of p , so they are G -homotopic. The pull back of $p_{\Pi; \Gamma}$ along this homotopy gives a homotopy, or path, between the two maps.

Using the arbitrariness of P in the above argument, one can further show that the space $\text{Map}_{\Gamma}(P, E(\Pi; \Gamma))$ is contractible as follows. Let Y be a random G -space. We denote by $Y \times P$ the principal $(\Pi; \Gamma)$ -bundle $Y \times P \rightarrow Y \times B$. Here, Γ acts on Y by pulling back the G -action and acts $Y \times P$ diagonally. Then we have an adjunction:

$$(3.19) \quad \text{Map}_G(Y, \text{Hom}(P, E(\Pi; \Gamma))) \cong \text{Map}_{\Gamma}(Y \times P, E(\Pi; \Gamma)).$$

By what has been shown, the right hand side, thus the left hand side of (3.19) is always non-empty and path-connected for any Y . Taking $Y = \text{Hom}(P, E(\Pi; \Gamma))$, we obtain that $\text{Map}_G(Y, Y)$ is path-connected. In particular, the identity map and the constant map to a point in Y^G are connected by a path. This implies the contractibility of $Y^G = (\text{Hom}(P, E(\Pi; \Gamma)))^G$. \square

Remark 3.20. Alternatively, one can show $\text{Map}_{\Gamma}(P, E(\Pi; \Gamma)) \simeq *$ using the fact that $E(\Pi; \Gamma)$ is a universal space for a family of subgroups of Γ specified by [Theorem 3.7](#), which contains all the isotropy groups of P .

3.5. Free loop spaces and adjoint bundles. We end this section by showing an equivariant equivalence of the free loop space $LB_G \Pi$ and the adjoint bundle $\text{Ad}(E_G \Pi)$ in [Theorem 3.30](#). This gives [Corollary 3.31](#), which upgrades the G -equivalence $\Omega_b B_G O(n) \simeq O(V)$ to a multiplicative one. Our proof follows the non-equivariant treatment in the appendix of Gruher's thesis [[Gru07](#)] and the key equivariant tool is [Theorem 2.48](#).

We start with G -fibrations.

Definition 3.21. A G -map $p : E \rightarrow B$ between G -spaces is a G -fibration if for all subgroups $H \subset G$, the map $p^H : E^H \rightarrow B^H$ is a Hurewicz fibration.

The first examples of G -fibrations are G -fiber bundles.

Example 3.22. Let $p : E \rightarrow B$ be a principal G - Π -bundle as in [Definition 2.12](#). Then p is also a G -fibration by [Theorem 2.46 \(4\)](#). However, $p : E^H \rightarrow B^H$ is not necessarily surjective. In contrast to the other parts of [Theorem 2.46](#), we do not have control over the components of B^H that are not hit by $p(E^H)$, at least not obviously. In this sense, the notion of a G -fibration is not as rich as a principal G - Π -bundle.

Example 3.23. Let F be an effective Π -space and $q : E' \rightarrow B'$ be a G -fiber bundle with fiber F , structure group Π as in [Definition 2.9](#). Then q is also a G -fibration.

Lemma 3.24. *We have the following results on the fiber of a G -fibration:*

- (1) *Let $p : E \rightarrow B$ be a G -fibration and $b \in B^H$ be an H -fixed point, then the maps $(p^{-1}(b))^H \rightarrow E^H \rightarrow B^H$ form a fiber sequence.*
- (2) *Let $p : D \rightarrow B$ and $q : E \rightarrow B$ be two G -fibrations and $f : D \rightarrow E$ be a G -map over B . Take an H -fixed point $b \in B^H$. If f is a G -equivalence, then $p^{-1}(b) \rightarrow q^{-1}(b)$ is an H -equivalence.*

Proof. Non-equivariantly ($G = \{e\}$), this is the fact that a map over B and homotopy equivalence is a homotopy equivalence of fibrations over B (See [[May99](#), 7.5-7.6]). Equivariantly, the first claim is immediate from the definition; the second claim reduces to the non-equivariant case for each subgroup $H' \subset H$. \square

We adopt the language of fiberwise monoids in [[Gru07](#), Definition 4.2.1].

Definition 3.25. A G -fibration $p : E \rightarrow B$ is a G -fiberwise monoid if there is a unit section map $\eta : B \rightarrow E$ and a multiplication map $m : E \times_B E \rightarrow E$ over B , both G -equivariant, that satisfy the unital and associativity conditions. In other words, E is a monoid in the category of G -fibrations over B .

We can relax the strict associativity condition and define G -fiberwise A_∞ -monoids as well. Let \mathcal{A} be a reduced A_∞ -operad in Top ($\mathcal{A}(0) = *$).

Definition 3.26. A G -fibration $p : E \rightarrow B$ is a G -fiberwise A_∞ -monoid if it is an algebra over \mathcal{A} in the category of G -fibrations over B . In concrete words, there are G -equivariant structure maps over B for each $k \geq 0$

$$\gamma_k : \mathcal{A}(k) \times_{\Sigma_k} \underbrace{(E \times_B E \times_B \cdots \times_B E)}_{k \text{ times}} \rightarrow E$$

that satisfy the unital, associativity and Σ -equivariance conditions of an algebra over an operad. Here, $\mathcal{A}(k)$ is thought to have the trivial G -action; for $k = 0$, $\gamma_0 : B \rightarrow E$ is just a section of p .

Definition 3.27. A morphism of G -fiberwise A_∞ -monoids over B is a morphism of A_∞ -monoids in the category of G -fibrations over B . An equivalence is a morphism and G -equivalence on the total space.

By a G -monoid, we mean a monoid in G -spaces, and similarly for a G - A_∞ -monoid. Notice that the fiber of a G -fiberwise (A_∞)-monoid over a point $b \in B$ is not a G -(A_∞)-monoid. Instead, it is a G_b -(A_∞)-monoid, where $G_b = \{g \in G \mid gb = b\}$ is the isotropy subgroup of b . A morphism of fiberwise G -(A_∞)-monoids induces a morphism of G_b -(A_∞)-monoids on the fibers over b ; An equivalence induces a G_b -equivalence on the fibers by [Lemma 3.24](#).

To clarify this notion, we make the following remarks:

- (1) A G -fiberwise monoid is a G -fiberwise A_∞ -monoid. In this case, the unit section map η is γ_0 and the multiplication map m is γ_2 .
- (2) The relevant examples of G -fiberwise A_∞ -monoids here are mostly G -fibrations over B whose fibers are some sort of loops. The structure maps come from fiberwise- A_∞ structure of loop spaces. We will abuse terms to refer to the structure maps as the unit map and the multiplication map.
- (3) A G -fiberwise monoid or a G -monoid here is not a “genuinely equivariant algebra” as it does not have G -set indexed multiplications.

Construction 3.28. For a G -space X , the free loop space $LX = X^{S^1}$ is a G -fibration over X by evaluating at a base point of S^1 . It is also a G -fiberwise A_∞ -monoid with the unit map given by the constant loop and the multiplication map given by the concatenation of loops.

Construction 3.29. For a principal G - Π -bundle $E \rightarrow B$, the adjoint bundle of E is $Ad(E) = E \times_\Pi \Pi_{\text{ad}}$. Here, Π_{ad} is the space Π with adjoint action: for elements $\mu \in \Pi$ and $\nu \in \Pi_{\text{ad}}$, μ acts on ν by $\mu(\nu) = \mu\nu\mu^{-1}$. As defined, $Ad(E)$ is a G -fiber bundle over B with fiber Π , but no longer a principal G - Π -bundle. We claim that $Ad(E)$ has the structure of a G -fiberwise monoid over B . First, $Ad(E)$ is the fiberwise automorphism bundle $\mathcal{I}so_B(E, E)$, so naturally a fiberwise monoid over B . This is the

bundle version of the observation that for a right Π -space S homeomorphic to Π , there is a homeomorphism

$$\begin{aligned} \text{Aut}_\Pi(S) &\cong S \times_\Pi \Pi_{\text{ad}} \\ f(s) = s\nu &\leftrightarrow [(s, \nu)]. \end{aligned}$$

Moreover, $Ad(E) \cong \mathcal{I}so_B(E, E)$ as G -spaces, where G acts on $Ad(E)$ by acting on E and on $\mathcal{I}so_B(E, E)$ by conjugation. This breaks down to commuting the action of G and Π on E . Just to clarify the notations,

$$\text{Aut}_B(E) = \text{Iso}_B(E, E) \cong \text{Section}(\mathcal{I}so_B(E, E)).$$

Theorem 3.30. *Let G, Π be compact Lie groups. Then there is a G -fiberwise A_∞ -monoid $(\tilde{P}E_G\Pi)/\Pi$ over $B_G\Pi$ and equivalences as G -fiberwise A_∞ -monoids over $B_G\Pi$:*

$$LB_G\Pi \xleftarrow[\cong]{\xi} (\tilde{P}E_G\Pi)/\Pi \xrightarrow[\cong]{\psi} Ad(E_G\Pi)$$

Proof. We first construct the space and the map

$$\tilde{p}: (\tilde{P}E_G\Pi)/\Pi \rightarrow B_G\Pi.$$

Recall that $p: E_G\Pi \rightarrow B_G\Pi$ is the universal principal G - Π bundle. Denote the space of paths in $E_G\Pi$ that start and end in the same fiber over $B_G\Pi$ to be

$$\tilde{P}E_G\Pi = \{\alpha: I \rightarrow E_G\Pi \mid p(\alpha(0)) = p(\alpha(1))\}.$$

Then $\tilde{P}E_G\Pi$ inherits an $(\Pi \times G)$ -action from $E_G\Pi$. The quotient $(\tilde{P}E_G\Pi)/\Pi$ is a G -space over $B_G\Pi$ by $\tilde{p}(\alpha) = p(\alpha(0))$.

The map \tilde{p} has the structure of a G -fiberwise A_∞ -monoid. The unit map η is given by the constant path in the fiber of p . There is only one constant path in each fiber since we have taken quotient of the Π -action. The multiplication map m is given as follows: for two classes of paths $[\alpha], [\beta] \in (\tilde{P}E_G\Pi)/\Pi$, we may choose representatives such that $\alpha(1) = \beta(0)$. Let $m([\alpha], [\beta]) = [\alpha.\beta]$ be the concatenation of the paths:

$$\begin{array}{ccc} & \bullet & \beta(1) \\ & \uparrow \beta & \\ \alpha.\beta & \bullet & \alpha(1) = \beta(0) \\ & \uparrow \alpha & \\ & \bullet & \alpha(0) \end{array}$$

The class $[\alpha.\beta]$ does not depend on the choice of α, β . Both η and m are G -equivariant.

Next, we compare both $LB_G\Pi$ and $Ad(E_G\Pi)$ with $(\tilde{P}E_G\Pi)/\Pi$.

On one hand, we have $LB_G\Pi = (\tilde{P}E_G\Pi)/\Pi^I$. Here, Π^I is the group $\text{Map}([0, 1], \Pi)$ and acts on $\tilde{P}E_G\Pi \subset (E_G\Pi)^I$ pointwise in I . The projection $\tilde{P}E_G\Pi \rightarrow LB_G\Pi$ is a principal G - Π^I -bundle, as the Π^I action commutes with the G -action on $\tilde{P}E_G\Pi$.

The projection $\xi: (\tilde{P}E_G\Pi)/\Pi \rightarrow (\tilde{P}E_G\Pi)/\Pi^I$ commutes with the unit map and multiplication map, so it is a map of G -fiberwise A_∞ -monoids. Moreover, we have the

following commutative diagram:

$$\begin{array}{ccc}
\Pi & \longrightarrow & \Pi^I \\
\downarrow & & \downarrow \\
\tilde{P}E_G\Pi & \xlongequal{\quad} & \tilde{P}E_G\Pi \\
\downarrow & & \downarrow \\
(\tilde{P}E_G\Pi)/\Pi & \xrightarrow{\xi} & (\tilde{P}E_G\Pi)/\Pi^I = LB_G\Pi
\end{array}$$

By [Theorem 2.48](#), ξ is a G -equivalence. (The idea is that Π and Π^I are not so different.)

On the other hand, we may define a $(\Pi \times G)$ -equivariant map

$$\begin{array}{ccc}
\bar{\psi} : \tilde{P}E_G\Pi & \rightarrow & E_G\Pi \times \Pi_{\text{ad}} \\
\alpha & \mapsto & (\alpha(1), \nu)
\end{array}$$

where $\nu \in \Pi$ is the unique element such that $\alpha(1) = \alpha(0)\nu^{-1}$. We give $E_G\Pi \times \Pi_{\text{ad}}$ the G -action on $E_G\Pi$ and the diagonal Π -action. To check the equivariance of $\bar{\psi}$, take any $(\mu, g) \in \Pi \times G$, then $(\mu, g) \circ \alpha(t) = g\alpha(t)\mu^{-1}$ for $t \in [0, 1]$. So,

$$\bar{\psi}((\mu, g) \circ \alpha) = (g\alpha(1)\mu^{-1}, \mu\nu\mu^{-1}) = (\mu, g) \circ \bar{\psi}(\alpha).$$

Since $Ad(E_G\Pi) = (E_G\Pi \times \Pi_{\text{ad}})/\Pi$, we get a map $\psi : (\tilde{P}E_G\Pi)/\Pi \rightarrow Ad(E_G\Pi)$. It is easy to check that ψ commutes with the unit and multiplication maps, and is thus a map of G -fiberwise A_∞ -monoids.

To show that ψ is a G -equivalence, we consider the following morphism of principal G - Π -bundles:

$$\begin{array}{ccc}
\Pi & \xlongequal{\quad} & \Pi \\
\downarrow & & \downarrow \\
\tilde{P}E_G\Pi & \xrightarrow{\bar{\psi}} & E_G\Pi \times \Pi_{\text{ad}} \\
\downarrow & & \downarrow \\
(\tilde{P}E_G\Pi)/\Pi & \xrightarrow{\psi} & Ad(E_G\Pi)
\end{array}$$

By [Theorem 2.48](#), it suffices to show that $\bar{\psi}$ is a Λ -equivalence for any $\Lambda \subset \Pi \times G$ with $\Lambda \cap \Pi = e$.

We can construct a Λ -homotopy inverse for $\bar{\psi} : \tilde{P}E_G\Pi \rightarrow E_G\Pi \times \Pi_{\text{ad}}$, called $\bar{\phi}$. The idea is already in Gruher's proof [[Gru07](#)]. But in the equivariant case, $\bar{\phi}$ is dependent on the subgroup Λ . (In particular, it is not meant to be a $(\Pi \times G)$ -homotopy inverse.) Recall that $\bar{\psi}$ records the two endpoints of a path. So an inverse $\bar{\phi}$ is going to choose a canonical path between any two points in a continuous way. This choice of canonical path exists because of the Λ -contractibility of $E_G\Pi$; it is not meant to be a canonical choice.

The construction of $\bar{\phi}$ is as follows: Since $E_G\Pi$ is Λ -contractible, $(E_G\Pi)^\Lambda$ is non-empty. We fix a Λ -fixed base point $z_0 \in E_G\Pi$. Let $E_G\Pi \times I \rightarrow E_G\Pi$ be a Λ -equivariant contraction of $E_G\Pi$ to z_0 ; the adjoint of it gives a Λ -map $\gamma : E_G\Pi \rightarrow P_{z_0}E_G\Pi$. For $z \in E_G\Pi$, we write $\gamma(z)$ as γ_z . It is a path connecting z to z_0 . Now, recall that for an element $(z, \nu) \in E_G\Pi \times \Pi_{\text{ad}}$, the image $\bar{\phi}(z, \nu) \in \tilde{P}E_G\Pi$ wants to be a path from $z\nu$ to

z in $E_G\Pi$. We define it to be

$$\bar{\phi}(z, \nu) = \text{concatenation of } \gamma_{z\nu} \text{ and the reverse of } \gamma_z,$$

as illustrated in the picture on the left:



It remains to verify that $\bar{\phi}$ is Λ -homotopy inverse of $\bar{\psi}$. It is clear that $\bar{\psi}\bar{\phi} = \text{id}$. The illustration above on the right shows how a Λ -equivariant homotopy $\bar{\phi}\bar{\psi} \simeq \text{id}$ is defined: For a path $\alpha \in \tilde{P}E_G\Pi$ going from a point a to a point b , the path $\bar{\phi}\bar{\psi}(\alpha)$ is the concatenation of γ_a and the reverse of γ_b . A homotopy of paths $\bar{\phi}\bar{\psi}(\alpha) \simeq \alpha$ is a map H out of the square, such that the value of H has been given on the border as indicated. To fill the interior, we connect every point x on the border to the point labeled by z_0 with line segments and use the map $\gamma_{H(x)}$ on each segment. This homotopy H is “functorial” for elements $\alpha \in \tilde{P}E_G\Pi$, so it extends to a homotopy $\bar{\phi}\bar{\psi} \simeq \text{id}$; it is Λ -equivariant because the map γ is. \square

We review the Moore loop space construction. For any space X and base point b , the Moore loop space of X at the base point b , $\mathbf{\Lambda}_b X$, is defined to be

$$\mathbf{\Lambda}_b X = \{(l, \alpha) \in \mathbb{R}_{\geq 0} \times X^{\mathbb{R}_{\geq 0}} \mid \alpha(0) = b, \alpha(t) = b \text{ for } t \geq l\}.$$

It has the same homotopy type as $\Omega_b X$ and it is a (strictly associative) monoid, with $\eta : * \rightarrow \mathbf{\Lambda}_b X$ given by $\eta(*) = (0, b)$ and $m : \mathbf{\Lambda}_b X \times \mathbf{\Lambda}_b X \rightarrow \mathbf{\Lambda}_b X$ given by

$$m((l, \alpha), (s, \beta)) = (l + s, \alpha.\beta) \text{ for } (\alpha.\beta)(t) = \begin{cases} \alpha(t) & t < l \\ \beta(t - l) & l \leq t < l + s \\ b & t \geq l + s \end{cases}.$$

As a corollary of [Theorem 3.30](#), we can upgrade [Theorem 3.12 \(2\)](#) into an equivalence of G - A_∞ -monoids $\Omega_b B_G O(n) \simeq O(V)$. Strictifying $\Omega_b B_G O(n)$ to the Moore loop space $\mathbf{\Lambda}_b B_G O(n)$, there is an equivalence of G -monoids $\mathbf{\Lambda}_b B_G O(n) \simeq O(V)$:

Corollary 3.31. *Take a G -fixed base point $b \in B_G O(n)$ in the component indexed by V . Then $\mathbf{\Lambda}_b B_G O(n)$ is equivalent to $O(V)$ as a G -monoid. Here, G acts on $\mathbf{\Lambda}_b B_G O(n)$ by acting on $B_G O(n)$ and acts on $O(V)$ by conjugation.*

Proof. We explain how the G - A_∞ -monoid statement is a corollary. Take the fiber over b in [Theorem 3.30](#) for $\Pi = O(n)$. Then there are equivalences of the fibers as G - A_∞ -monoids by [Lemma 3.24](#). The fiber of $LB_G O(n)$ is $\Omega_b B_G O(n)$. By [Theorem 3.12 \(1\)](#), the fiber of $Ad(E_G O(n))$ is $O(\mathbb{R}^n, V) \times_{O(n)} O(n)_{\text{ad}} \cong O(V)$ as G -monoid. So there is a zig-zag of equivalences of G - A_∞ -monoids between $\Omega_b B_G O(n)$ and $O(V)$. For the G -monoid statement, just replace the free loop space and path space in [Theorem 3.30](#) by the Moore version, and the proof stays intact.

Explicitly, the zigzag of G -monoids is given by

$$(3.32) \quad \mathbf{\Lambda}_b B_G O(n) \xleftarrow{\xi} (\tilde{\mathbf{\Lambda}}_b E_G O(n))/\Pi \xrightarrow{\psi} O(V).$$

We use p to denote the universal principal G - $O(n)$ -bundle $E_G O(n) \rightarrow B_G O(n)$. We define

$$\tilde{\Lambda}_b E_G O(n) = \{(l, \alpha) \mid l \in \mathbb{R}_{\geq 0}, \alpha : \mathbb{R}_{\geq 0} \rightarrow E_G O(n), p(\alpha(0)) = p(\alpha(t)) = b \text{ for } t \geq l\},$$

so that $(\tilde{\Lambda}_b E_G O(n))/\Pi = [l, \alpha]$ where the equivalence relation is

$$(l, \alpha) \sim (l, \beta) \text{ if there is } \nu \in O(n) \text{ such that } \alpha(t) = \beta(t)\nu \text{ for all } t \geq 0.$$

While $\tilde{\Lambda}_b E_G O(n)$ does not have the structure of a G -monoid, $(\tilde{\Lambda}_b E_G O(n))/\Pi$ does.

Fix a base point $z \in p^{-1}(b) \subset E_G O(n)$. The maps are given by

$$\begin{aligned} \xi([l, \alpha]) &= (l, p(\alpha)) \in \Lambda_b B_G O(n); \\ \psi([l, \alpha]) &\in O(V) \text{ is determined by } \alpha(0)\psi([l, \alpha]) = \alpha(l). \end{aligned} \quad \square$$

REFERENCES

- [Ati66] M. F. Atiyah. *K*-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966.
- [Elm83] A. D. Elmendorf. Systems of fixed point sets. *Trans. Amer. Math. Soc.*, 277(1):275–284, 1983.
- [Gru07] Kate Gruher. *String topology of classifying spaces*. PhD thesis, Stanford University, 2007.
- [Hus94] Dale Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1994.
- [Ill83] Sören Illman. The equivariant triangulation theorem for actions of compact Lie groups. *Math. Ann.*, 262(4):487–501, 1983.
- [Las82] Richard K Lashof. Equivariant bundles. *Illinois Journal of Mathematics*, 26(2):257–271, 1982.
- [LM86] Richard K Lashof and J Peter May. Generalized equivariant bundles. *Bull. Soc. Math. Belg. Sér. A*, 38:265–271, 1986.
- [LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.
- [LU14] Wolfgang Lück and Bernardo Uribe. Equivariant principal bundles and their classifying spaces. *Algebr. Geom. Topol.*, 14(4):1925–1995, 2014.
- [May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.
- [MZ42] Deane Montgomery and Leo Zippin. A theorem on Lie groups. *Bull. Amer. Math. Soc.*, 48:448–452, 1942.
- [Ste51] Norman Steenrod. The topology of fiber bundles. *Princeton Mathematical Series*, page 16, 1951.
- [TD69] Tammo Tom Dieck. Faserbündel mit gruppenoperation. *Archiv der Mathematik*, 20(2):136–143, 1969.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109 UNITED STATES

Email address: folingz@umich.edu