HIGHER DIOPHANTINE APPROXIMATION EXPONENTS
AND CONTINUED FRACTION SYMMETRIES
FOR FUNCTION FIELDS II

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ABSTRACT. We construct families of non-quadratic algebraic laurent series
(over finite fields of any characteristic) which have only bad rational approx-
imations so that their rational approximation exponent is as near to 2 as we
wish and, at the same time, have very good quadratic approximations so that
the quadratic exponent is close to the Liouville bound and thus can be ar-
bitrarily large. In contrast, in the number field case, the Schmidt exponent
(analog of the Roth exponent of 2 for rational approximation) for approx-
imations by quadratics is 3. We do this by exploiting the symmetries of the
relevant continued fractions. We then generalize some of the aspects from the
degree 2 (= $p^0 + 1$)-approximation to degree $p^n + 1$-approximation. We also
calculate the rational approximation exponent of an analog of $\pi$.

1. BACKGROUND

We recall [S80, Chapter 8] some basic definitions, facts and conjectures about
diophantine approximation of real numbers by rationals or (real) algebraic numbers.
(See also [B04, BG06] and [W] for a nice survey of recent developments.)

Definition 1 (Height and higher diophantine approximation exponents). For a
non-zero algebraic number $\beta$, define $H(\beta)$ to be the maximum of the absolute values
of the coefficients of a non-trivial irreducible polynomial with co-prime integral
coefficients that it satisfies.

For $\alpha$ an irrational real number not algebraic of degree $\leq d$, define $E_d(\alpha)$ ($E_{\leq d}(\alpha)$
respectively) as $\lim\sup(-\log |\alpha - \beta|/\log H(\beta))$, where $\beta$ varies through all algebraic
real numbers of degree $d$ ($\leq d$ respectively).

Note that $E_1(\alpha)$ is the usual exponent $E(\alpha) := \lim\sup(-\log |\alpha - P/Q|/\log |Q|)$.

Then for irrational $\alpha$, we have $E(\alpha) \geq 2$ by Dirichlet’s theorem, whereas for
irrational algebraic $\alpha$ of degree $d$, we have $E(\alpha) \leq d$ by Liouville’s theorem and
$E(\alpha) = 2$ by Roth’s theorem, improving Liouville, Thue, Siegel, and Dyson bounds.

For real $\alpha$ not algebraic of degree $\leq d$, Wirsing (generalizing Dirichlet’s result)
conjectured $E_{\leq d}(\alpha) \geq d + 1$ and proved a complicated lower bound (for this ex-
ponent) which is slightly better than $(d + 3)/2$, whereas Davenport and Schmidt
proved his conjecture for $d = 2$. On the other hand, for $\alpha$ of degree $> d$, we have
the Liouville bound \( E_{\leq d}(\alpha) \leq \deg \alpha \). Schmidt (generalizing Roth’s result) proved that for real algebraic \( \alpha \) of degree greater than \( d \), \( E_{\leq d}(\alpha) \leq d + 1 \).

From now on, unless stated otherwise, we only focus on the function field analogs (see, e.g., [T04] for general background and [T04] Cha. 9, [T09] for diophantine approximation, continued fractions background and references), where the role of \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) is played by \( A = \mathbb{F}[t], K = \mathbb{F}(t), K_\infty = \mathbb{F}((1/t)) \) respectively, where \( \mathbb{F} \) is a finite field of characteristic \( p \). With the usual absolute value coming from the degree in \( t \) of polynomials or rational functions, we have exactly the same definitions of heights and exponents. Now by rationals, reals, and algebraic, we mean elements of \( K, K_\infty \), and algebraic over \( K \) respectively. (There should be no confusion between degrees in \( t \) (as above) of rationals and algebraic degree over \( K \) of irrational algebraics.)

Then analogs of Dirichlet and Liouville theorems hold, but a naive analog of Roth’s theorem fails, as shown by Mahler [M49]. For other results, see [dM70] [S00] [T04] and the references therein, and, for example, results [KT00] in the Wirsing direction. We also recall that for almost all real \( \alpha \) (and for almost all \( \alpha \in K_\infty \)), \( E_d(\alpha) = d + 1 \). (Here ‘almost all’ means that the set of exceptions is of measure zero.)

We now deal with the similar phenomena for quadratic approximations. In [T11], for \( p = 2 \) and any integer \( m > 1 \), we constructed algebraic elements \( \alpha \) of degree at most \( 2^m \) having continued fractions with folding pattern symmetries and a bounded sequence of partial quotients, so that \( E(\alpha) = 2 \), but with \( E_2(\alpha) \geq 2^m > 3 \). In this paper, with a different construction, based on some ideas of [T99] [T11], we prove:

**Theorem 1.** Let \( p \) be a prime, \( q \) be a power of \( p \) and \( \epsilon > 0 \) be given. Then we can construct infinitely many algebraic \( \alpha \), with explicit equations and continued fractions, such that

\[
q \leq \deg(\alpha) \leq q + 1, \quad E(\alpha) < 2 + \epsilon, \quad E_2(\alpha) > q - \epsilon,
\]

with an explicit sequence of quadratic approximations realizing the last bound.

**Theorem 2.** Let \( p \) be a prime, \( q \) be a power of \( p \) and \( m, n > 1 \) be given. Then we can construct infinitely many algebraic \( \alpha_{m,n} \), with explicit equations and continued fractions, such that

\[
\deg(\alpha_{m,n}) \leq q^m + 1, \quad \lim_{n \to \infty} E_{\alpha_{m,n}}(\alpha_{m,n}) = 2, \quad \lim_{n \to \infty} E_{q+1}(\alpha_{m,n}) \geq q^{m-1} + \frac{q - 1}{(q + 1)q},
\]

with an explicit sequence of degree \( q + 1 \)-approximations realizing the last bound.

A natural question raised by these considerations is whether there are algebraic \( \alpha \)’s of each degree \( d \), with \( E(\alpha) = 2 \) (or even with bounded partial quotients) and for which the Liouville bound for the lower degree approximations is attained, or whether some of these requirements need to be relaxed.

2. **Continued fractions**

Continued fractions are natural tools of the theory of diophantine approximation. See [dM70] [BS76] [S00] [T04] for the basics in the function field case.

Let us review some standard notation. We write \( \alpha = a_0 + 1/(a_1 + 1/(a_2 + \cdots)) \) in the short-form \([a_0, a_1, \cdots] \). We write \( \alpha_n = [a_n, a_{n+1}, \cdots] \) so that \( \alpha = \alpha_0 \). Let us...
define $p_n$ and $q_n$ as usual in terms of the partial quotients $a_i$ so that $p_n/q_n$ is the $n$-th convergent $[a_0, \ldots, a_n]$ to $\alpha$. Hence $\deg q_n = \sum_{i=1}^n \deg a_i$.

Following the basic analogies mentioned above, we use the absolute value coming from the degree in $t$ to generate the continued fraction in the function field case, and we use the ‘polynomial part’ in place of the ‘integral part’ of the ‘real’ number $\alpha \in K_\infty$. In the function field case, for $i > 0$, $a_i$ can be any non-constant polynomial, and so the degree of $q_i$ increases with $i$, but $a_i$ or $q_i$ need not be monic. As usual, we have

\[ p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, \quad \alpha = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}, \]

implying the usual basic approximation formula

\[ \alpha - p_n/q_n = (-1)^n / ((\alpha_{n+1} + q_{n-1}/q_n)q_n^2), \]

which because of the non-archimedean nature of the absolute value, now implies

\[ |\alpha - p_n/q_n| = 1 / (|a_{n+1}|q_n^2). \]

If we know the continued fraction for $\alpha$, the equation allows us to calculate the exponent, using $\deg q_n = \sum_{i=1}^n \deg a_i$, as

\[ E(\alpha) = 2 + \limsup \frac{\deg a_{n+1}}{\sum_{i=1}^n \deg a_i}. \]

3. Proof of Theorem 1

First, we note that if $\gamma = [a_0, \ldots, a_m, b, \ldots]$ (with arbitrary entries after $b$), then, by (1),

\[ \alpha - \gamma = \frac{\alpha_{m+1} p_m + p_{m-1}}{(\alpha_{m+1} q_m + q_{m-1})} - \frac{\gamma_{m+1} p_m + p_{m-1}}{(\gamma_{m+1} q_m + q_{m-1})} = \pm \frac{\alpha_{m+1} - \gamma_{m+1}}{(\cdots)(\cdots)}. \]

Hence the non-archimedean nature of the absolute values implies that

\[ |\alpha - \gamma| = \frac{1}{|b q_m^2|} \quad \text{if} \quad \deg(a_{m+1} - b) = \deg a_{m+1}. \]

Next, we normalize the absolute value and the logarithm so that $\log |a|$ is a degree of $a$ in $t$ and denote by $h := \log H$ the resulting logarithmic height and give a simple bound on the height of a quadratic irrational $\theta$ in terms of the degrees of the partial quotients of its continued fraction, which is eventually periodic by analogy (see, e.g., [9]) of Lagrange’s theorem. Thus consider $\theta = [b_0, \ldots, b_j, \mu]$, where $\mu = [X]$ is purely periodic continued fraction obtained by repeating period (tuple) $X = (a_0, \ldots, a_n)$. Thus $\mu = (\mu p_n + p_{n-1})/(\mu q_n + q_{n-1})$ implies $h(\mu) \leq \deg(p_n) = \sum \deg a_i$, where the sum runs over $i$ from 0 to $n$. For $a \in A$, comparing the polynomials satisfied by quadratic $\gamma$ and $a + 1/\gamma$, we see that $h(a + 1/\gamma) \leq h(\gamma) + 2 \deg(a)$, whose repeated application gives

\[ h(\theta) \leq 2 \sum_{i=0}^j \deg b_i + \sum_{i=0}^n \deg a_i. \]

Next, we proceed to the construction of the $\beta$’s and their quadratic approximations $\beta_s$. By capital letters $X, Y, \ldots$ etc., we will denote tuples of partial quotients, and we will denote by $X^m$ the tuple resulting from $X$ by raising each of its entries
to the $m$-th power. Let $n + 1 = r\ell$, let $a_0, \ldots, a_{\ell - 1} \in A$ be non-constant polynomials, and let $Y = (a_0, \ldots, a_{\ell - 1})$ and $X = (Y, \ldots, Y) = (a_0, \ldots, a_n)$, obtained by repeating $Y$ $r$ times. Then

$$\alpha := [X, X^q, X^{q^2}, \cdots] = [X, \alpha^q] = \frac{\alpha^q p_n + p_{n-1}}{\alpha^q q_n + q_{n-1}}$$

so that $\alpha$ is algebraic of degree at most $q + 1$. For any $s > 1$, let

$$\beta_s := [X, X^q, \cdots, X^{q^{s-1}}, Y^{q^s}].$$

Let $L = \sum_{i=0}^{\ell} \deg a_i$ so that $\sum_{i=0}^{n} \deg a_i = Lr$. We see using (6) that $h(\beta_s) \leq 2Lr(q^s - 1)/(q - 1) + q^s L$, while (5) implies (since $b = a_0^q$) that $-\log |\alpha - \beta_s| = q^s \deg a_0 + 2(Lr(q^{s+1} - 1)/(q - 1) - \deg a_0)$. Hence, letting $s$ tend to infinity, we see that $E_2(\alpha) \leq (2Lrq + (q - 1) \deg a_0)/(2Lr + L(q - 1))$ (when is this an equality?), which is at most $q$ and tends to $q$ if $r$ tends to infinity. On the other hand, by (4), for some $i$, $E(\alpha) - 2 \leq q^{m+1} \deg(a_i)/(rq^m \deg a_i)$, which tends to 0, as $r$ tends to infinity. Hence, given $\epsilon > 0$, choosing $r$ appropriately large, we satisfy the claims of the theorem, with the Liouville bound implying that $\deg(\alpha) \geq q$, since without loss of generality we can assume that $\epsilon < 1$. This completes the proof.

Finally, we remark that if our equation for $\alpha$ is reducible, then we reach the Liouville bound (of $q$ in that case), at least as $\epsilon$ tends to zero. It might also be possible to tighten the inequalities to get a better lower bound for $E_2$.

4. Proof of Theorem 2

We follow the strategy of the previous section, but now we define

$$\alpha := \alpha_{m,n} := [X, X^{q^m}, X^{q^{2m}}, \cdots, X^{q^{(s-1)m}}, X^{q^{sm}}, \cdots]$$

with $X = (B, B^q, \cdots, B^{q^{n-1}}, C)$, where $B, C \in \mathbb{F}[t]$, with $b := \deg B > 0$, $\deg C = q^n b$, and $\deg(B^{q^n} - C) = q^n b$, which is clearly possible if $\mathbb{F} \neq \mathbb{F}_2$. (We will deal with the case $\mathbb{F} = \mathbb{F}_2$ at the end.) Next we let

$$\beta_s := [X, X^{q^m}, X^{q^{2m}}, \cdots, X^{q^{(s-1)m}}, B^{q^{sm}}, B^{q^{sm+1}}, B^{q^{sm+2}}, \cdots].$$

The mobius transformation expression for $\alpha$ and $\beta_s$, as in the proof of the last theorem, shows that $\deg \alpha \leq q^{m+1}$ and that $\deg(\beta_s) \leq q + 1$. The last inequality is in fact equality by the Liouville theorem by calculating its exponent by (4).

Write

$$D := b \frac{q^{n+1} - 1}{q - 1} q^{sm} - 1, \quad F := q^{ms} b \frac{q^{n+1} - 1}{q - 1},$$

which are the sums of the degrees of entries in $(X, X^{q^m}, \cdots, X^{q^{(s-1)m}})$ and $X^{q^{sm}}$, respectively. Then a straight calculation using (5) shows that

$$-\log |\alpha - \beta_s| = 2(D - b + F) - q^{ms + nb}.$$

Now the height of $\theta := |B^{q^{sm}}, B^{q^{sm+1}}, \cdots| = B^{q^{sm}} + 1/\theta q$ is clearly $q^{sm} b$. For $a \in A$, by comparing heights of $\gamma$ and $\alpha + 1/\gamma$ for a $\gamma$ satisfying an equation of the form $\gamma = (P\gamma^q + Q)/(R\gamma^q + S)$, we see that $h(a + 1/\gamma) \leq h(\gamma) + (q + 1) \deg(a)$, whose repeated application gives

$$h(\beta_s) \leq (q + 1)D + q^{ms} b.$$
Taking the limit of the ratio as $s$ tends to infinity, we get
\[ E_{q+1}(\alpha) \geq 2 \left( \frac{q^{n+1} - 1}{q - 1} \right) \left( \frac{q^m}{q^m - 1} - q^n \right)/((q + 1)\left( \frac{q^{n+1} - 1}{q - 1} \right)\left( \frac{1}{q^m - 1} \right) + 1), \]
and taking the limit of the right side, as $n$ tends to infinity, we get the lower bound claimed in the theorem.

Finally, using (4), we see that
\[ E(\alpha) = 2 + \lim_{s \to \infty} \frac{q^{mb}}{D - b} = 2 + \frac{(q - 1)(q^m - 1)}{q^{n+1} - 1}, \]
implying the claim on the limit as $n$ tends to infinity.

Finally, we consider the case $F = F_2$. We can choose $B, C \in F[t]$, with deg $B = b$, deg($C$) = $q^n b$ and deg($B^{t^n} - C$) = $q^n b - 1$. The whole asymptotic analysis is the same, as $b$ tends to infinity, leading to the same bounds. This finishes the proof.

5. Exponent for an analog of $\pi$

In [T11] Sec. 7, we calculated the exponent of an analog of $\pi$ for $F_q[t]$. (For this section, we take $F = F_q$.) But as discussed in [T04] pp. 47-48, there are a couple of good candidates for analogs of $\pi$ (up to rational multiples, which do not change exponents). We now consider $\pi_1 := \prod (1 - [j]/[j + 1]) \in K_\infty$, where $[j] := t^{q^j} - t$ and the product is over $j$ from 1 to $\infty$. As explained in the reference above, the Carlitz period (good analog of $2\pi i$) is then $(-[1])^{1/(q-1)}\pi_1$.

**Theorem 3.** For $\pi_1$ as above, $E(\pi_1) \geq (q - 1)^2/q$, with equality when $q > 5$.

**Proof.** Note that $[j + 1] - [j] = [1]q^j$ and $[1]$ divides $[n]$ with the quotient coprime to $[1]$. Thus the truncation of the product at $j = N - 1$, which equals $[1]q^j + \cdots + q^{N-1}/([2][3] \cdots [N])$, has a denominator of degree $q^2 + q^3 + \cdots + q^N - (N - 1)q$, whereas it approximates $\pi_1$ with error of degree $q^N - q^{N+1}$ (resulting from $1 - (1 - N/[N + 1])$), showing that the inequality claimed, as the limit of the ratio of these two quantities as $N$ tends to $\infty$, tends to $(q - 1)^2/q$. When $q > 5$, we have $(q - 1)^2/q > \sqrt{q} + 1$, and this implies by a proposition of Voloch (see [V88] Prop. 5 or [T04] Lemma 9.3.3) the equality claimed.

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**References**


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