POLYGONAL BICYCLE PATHS AND THE DARBOUX TRANSFORMATION

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ABSTRACT. A bicycle $(n,k)$-gon is an equilateral $n$-gon whose $k$ diagonals are of equal length. In this paper we introduce periodic bicycle $(n,k)$-paths, which are a natural variation in which the polygon is replaced with a periodic polygonal path, and study their rigidity and integrals of motion.

1. Background

Our motivation comes from three seemingly unrelated problems. The first is the problem of *Floating Bodies of Equilibrium in Two Dimensions*. From 1935 to 1941, mathematicians at the University of Lviv, among them Stefan Banach and Mark Kac, collected mathematical problems in a book, which became known as the Scottish Book, since they often met in the Scottish Coffee House. Stanislaw Ulam posed problem 19 of this book: "Is a sphere the only solid of uniform density which will float in water in any position?" The answer in the two-dimensional case, as it turns out, depends on the density of the solid.

The second problem, known as the *Tire Track Problem*, originated in the story, "The Adventure of the Priory School" by Arthur Conan Doyle, where Sherlock Holmes and Dr. Watson discuss in view of the two tire tracks of a bicycle which way the bicycle went. The problem is: "Is it possible that tire tracks other than circles or straight lines are created by bicyclists going in both directions?" As shown in Figure 1, the answer to this subtle question is affirmative.

The third problem is the problem of describing the trajectories of *Electrons in a Parabolic Magnetic Field*. All three problems turn out to be equivalent [13].

![Figure 1](image)

**Figure 1.** Ambiguous bicycle tracks. The rear-wheel track is the inner curve and the front-wheel track is the outer curve. One cannot tell which way the bicycle went because a bicycle could have followed either one of two trajectories [13].

Often in mathematics it is fruitful to discretize a problem. As such, S. Tabachnikov proposed a “discrete bicycle curve” (also known as a “bicycle polygon”) [9], which is a polygon satisfying discrete analogs of the properties of a bicycle track. The main requirement turns out to be that, in the language of discrete differential geometry, the polygon is “self-Darboux”. That is, the discrete differential geometric notion of a Discrete Darboux Transformation [3, 12], which relates one polygon to another, relates a discrete bicycle curve to itself.

The topic of bicycle curves and polygons belongs to a number of active areas of research. On the one hand, it is part of rigidity theory. As an illustration, R. Connelly and B. Csikós consider the problem of...
classifying first-order flexible regular bicycle polygons [5]. Other work on the rigidity theory of bicycle curves and polygons can be found in [6, 7, 9].

The topic is also part of the subject of Discrete Integrable Systems. This point of view is taken in [10], where the authors find integrals of motion (i.e., quantities which are conserved) of bicycle curves and polygons under the Darboux Transformation and Recutting of polygons [1, 2].

In this paper, in analogy with bicycle polygons, we introduce a new concept called a periodic discrete bicycle path and study both its rigidity and integrals.

2. BICYCLE \((n, k)\)-PATHS

A bicycle \((n, k)\)-gon is an equilateral \(n\)-gon whose \(k\) diagonals are of equal length [9]. We consider the following analog.

**Definition 1.** Define \(P = \{ V_i \in \mathbb{R}^2 : i \in \mathbb{Z} \} \) (for brevity, \(V_0 V_1 \cdots V_{n-1}\)) to be a discrete periodic bicycle \((n, k)\)-path (or discrete \((n, k)\)-path) if the following conditions hold:

(i). \(V_{n+i} = V_i + c_1 \forall i\), where \(c_1 = (1, 0)\) and \(V_0 = (0, 0)\). (Periodicity Condition)

(ii). \(|V_i V_{i+1}| = |V_j V_{j+1}| \forall i, j\). (Equilateralness)

(iii). \(|V_i V_{i+k}| = |V_j V_{j+k}| \forall i, j\). (Equality of \(k\)-diagonals)

![Figure 2. An example of a discrete (6,5)-path.](image)

Definition 1 is meant to model the motion of a bicycle whose trajectory is spatially periodic. The condition that \(|V_j V_{j+1}|\) is independent of \(j\) prescribes a constant speed for the motion of the bike. The condition that \(|V_i V_{i+k}|\) is independent of \(j\) represents the ambiguity of the direction in which the bicycle went (see [9] for details).

Some natural questions regarding periodic \((n, k)\)-paths are for which pairs \((n, k)\) they exist, how many there are and whether they are rigid or flexible. We consider these questions in section 3. A simple example of a bicycle \((n, k)\)-path, analogous to the regular \((n, k)\)-polygon, is \(V_i = (\frac{i}{n}, 0)\), i.e. when all vertices lie at equal intervals on the line. We call this the regular path. Since bicycle \((n, k)\)-paths are discretized bicycle paths, it is also interesting to see if there are any integrals of motion. We show that this is indeed the case in section 4, by showing that area is an integral of motion.

3. RIGIDITY

The following two Lemmas will be helpful when analyzing the rigidity of discrete bicycle paths.

**Lemma 1.** Let \(n \in \mathbb{N}\), \(\chi_i \in \{-1, 1\}\) for every \(i \in \mathbb{Z}/n\mathbb{Z}\) and let

\[ S = \{(x_0, x_1, ..., x_{n-1}) \in \mathbb{R}^n : (x_{i+1} - x_i)^2 = (x_{j+1} - x_j)^2 \ \forall i, j \in \mathbb{Z}/n\mathbb{Z}\}. \]

Then

\[ S = \{(x_0, x_1, ..., x_{n-1}) : x_{i+1} = x_i + \chi_i r \ \text{for} \ i \in \mathbb{Z}/n\mathbb{Z}, \sum_{i=0}^{n-1} r\chi_i = 0 \ \text{and} \ r \geq 0\}. \]

In particular, if \(n\) is odd, then \(S = \{(x_0, x_1, ..., x_{n-1}) : x_i = x_j \ \forall i, j \in \mathbb{Z}/n\mathbb{Z}\}.

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Proof. First note that the candidate set is well-defined since \( x_{j+n} = x_j + \sum_{i=j}^{j+n-1} r \chi_i = x_j + \sum_{i=0}^{n-1} r \chi_i = x_j \).

Let \((x_0, x_1, \ldots, x_{n-1}) \in S\). Recall that

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0
\end{cases}
\]

and that \(\text{sgn}(x) | x = x\). Set \(r := |x_{i+1} - x_i|\) and \(\chi_i = \text{sgn}(x_{i+1} - x_i) + (1 - \text{sgn}(r))\). Then

\[
x_{i+1} = x_i + \chi_i r
\]

and

\[
\sum_{i=0}^{n-1} r \text{sgn}(x_{i+1} - x_i) = 0.
\]

It follows that any \(n\)-tuple in \(S\) satisfies the conditions \(x_{i+1} = x_i + \chi_i r\), \(\sum_{i=0}^{n-1} r \chi_i = 0\) and \(r \geq 0\). The opposite inclusion is clear. \(\square\)

**Lemma 2.** Let \(x_i \in \mathbb{R}\) for every \(i \in \mathbb{Z}\) with \(x_0 = 0\) and let \(k\) and \(n\) be coprime integers. Assume that \(x_{i+k} - x_i = x_i - x_{i-k}\) for each \(i\) and that \(x_{i+n} = 1 + x_i\). Then \(x_i = \frac{i}{n}\) for each \(i\).

Proof. Define \(z_i\) via \(x_i = z_i + \frac{1}{n}\). The hypothesis \(x_{i+n} = 1 + x_i\) implies that

\[
z_{i+n} = z_i.
\]

The difference

\[
\Delta z := z_{i+k} - z_i
\]

is independent of \(i\) due to the assumption that \(x_{i+k} - x_i = x_i - x_{i-k}\) and because \(k\) and \(n\) are coprime. This implies that

\[
0 = z_{i+n} - z_i = z_{i+nk} - z_i = n\Delta z.
\]

It follows that \(z_i = 0\) for every \(i\). \(\square\)

The following Theorem gives a classification of a family of periodic \((n, k)\)-paths.

**Theorem 1.** The discrete \((n, dn - 1)\)-paths \(V_i = (x_i, y_i), i \in \mathbb{N}\) with \(d \neq 0\) are exactly those paths which satisfy

\[
x_j = \frac{j}{n}
\]

and

\[
y_{j+1} = y_j + \chi_j r
\]

for \(j \in \mathbb{Z}/n\mathbb{Z}\) with \(\sum_{i=0}^{n-1} r \chi_i = 0\) and \(r \geq 0\) for each \(j\). In particular, if \(n\) is odd then a discrete \((n, dn - 1)\)-path must be regular.

Proof. For every \(i\),

\[
|V_i V_{i+1}| = |V_{i+dn-1} V_{i+dn}|
\]

and

\[
|V_i V_{i+dn-1}| = |V_{i+1} V_{i+dn}|.
\]

Therefore

\[
(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2
\]

and

\[
(d + x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2 = (d + x_i - x_{i+1})^2 + (y_i - y_{i+1})^2.
\]

It follows that

\[
d(x_{i+1} - x_i) = d(x_i - x_{i-1}).
\]

Since \(d \neq 0\),

\[
x_{i+1} - x_i = x_i - x_{i-1}.
\]

By Lemma 2, \(x_j = \frac{j}{n}\) for each \(j\). Now equation \(|V_i V_{i+1}| = |V_{j} V_{j+1}|\) for all \(i, j\) implies that

\[
(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = (x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2 \forall i, j
\]
so that
\[(y_{i+1} - y_i)^2 = (y_{j+1} - y_j)^2.\]

By Lemma 1, we are done. □

**Theorem 2.** The discrete \((n, dn + 1)\)-paths \(V_i = (x_i, y_i), i \in \mathbb{N}\) with \(d \neq 0\) are exactly those paths which satisfy
\[x_j = \frac{j}{n}\]

and
\[y_{j+1} = y_j + \chi_j r \text{ for } j \in \mathbb{Z}/n\mathbb{Z} \text{ with } \sum_{i=0}^{n-1} \chi_i = 0 \text{ and } r \geq 0\]

for each \(j\). In particular, if \(n\) is odd then a discrete \((n, dn + 1)\)-path must be regular.

**Proof.** Set \(C_1 = |V_i V_{i+dn+1}|^2\) and \(C_2 = |V_i V_{i+1}|^2\). Then
\[(d + x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = C_1\]

and
\[(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = C_2.\]

Substituting, we get
\[d^2 + 2d(x_{i+1} - x_i) + C_2 = C_1,\]

so that \(x_{i+1} - x_i\) is constant. By Lemma 2, \(x_i = \frac{i}{n}\). It follows that \((y_{i+1} - y_i)^2\) is constant, so by Lemma 2 we are done. □

**Corollary 1.** Any \((n, dn + 1)\)-path is an \((n, dn - 1)\)-path and vice versa.

For an example, see Figure 2.

4. **Darboux Transformation and Integrals**

It is important to make a distinction between infinitesimal “trapezoidal” movement and infinitesimal “parallelogram” movement of the bicycle. Consider a pair of conjoined bikes, sharing a back wheel and facing in opposite directions. Since this bicycle moves in such a way that the distance between the turnable wheels is constant, at each moment of time the turnable wheels must enclose equal angles with the line of the frame. When the two turnable wheels are parallel, the trike is gliding, but then the common back wheel of the bikes is slipping, which is not allowed. That is why we exclude parallelogram movements from consideration for the remainder of this paper.

**Definition 2. (Trapezoidal Condition)** We will say that a discrete \((n, k)\)-path satisfies the trapezoidal condition if \(V_i V_{i+k+1} \parallel V_{i+1} V_{i+k}\) for each \(i \in \mathbb{Z}\).

As an illustration of these concepts, consider Figure 2: \(V_0 V_1 V_5 V_6\) is a trapezoidal motion, while \(V_1 V_2 V_6 V_7\) is a parallelogram motion. Consequently, the bicycle path in the figure does not satisfy the Trapezoidal Condition.

Assuming the Trapezoidal Condition, we may view bicycle paths in terms of an important construction in Discrete Differential Geometry called the Darboux Transformation [3, 12].

**Definition 3. (Darboux Transform)** We say that two polygons \(P = P_1 P_2 \ldots\) and \(Q = Q_1 Q_2 \ldots\) are in Darboux Correspondence if for each \(i = 1, 2, \ldots, Q_{i+1}\) is the reflection of \(P_i\) in the perpendicular bisector of the segment \(P_{i+1}Q_i\).

If segment \(P_i Q_i\) is of length \(\ell\) then for each \(i\), \(P_i Q_i\) is of length \(\ell\). We then say that \(P\) and \(Q\) are in Darboux Correspondence with parameter \(\ell\). We also note that each quadrilateral \(P_i Q_i P_{i+1} Q_{i+1}\) is an isosceles trapezoid.
We denote the map taking vertex $P_i$ to $Q_i$ by $D$. We will also refer to the map of polygons $D(P) = Q$ by the same letter, since no confusion ought to occur.

Consider a polygonal line $P$ with vertices $V_0, V_1, \ldots, V_{n-1}$. Let $v_0$ be a vector with its origin at $V_0$. Having a vector $v_i$ at vertex $V_i$, we obtain a vertex $v_{i+1}$ of the same length at $V_{i+1}$ via the trapezoidal condition. For example, in Figure 3, $v_1 = P_1Q_1$ and $v_2 = P_2Q_2$. For a fixed length of $v_0$, we may view the map taking $v_0$ to $v_j$ as a self-map of the circle of radius $|v_0| = |v_j|$ by identifying the circle at $V_0$ with circle at $V_j$ via parallel translation.

**Definition 4.** *(Monodromy map of the Darboux Transformation)* The monodromy map is the map acting on the identified circles at $V_0$ and $V_n$ which takes $v_0$ to $v_n$.

It is known that the monodromy map is a cross-ratio preserving transformation (in terms of affine coordinates, a fractional linear transformation) on a circle of fixed radius after we identify the circle with the real projective line via stereographic projection [10]. We will assume throughout, unless otherwise stated, that the monodromy map is acting on a fixed point; in other words, we will assume that the Darboux transform has been chosen so that the initial vector $v_0$ is equal to the vector $v_n$, where $n$ is the period. This is analogous to applying the Darboux transform to a closed polygon and requiring that its image is closed also.

We mention in passing that in the case of closed polygons, Darboux Correspondence implies that the monodromies of the two polygons are conjugated to each other. The invariants of the conjugacy class of the monodromy, viewed as functions of the length parameter are consequently integrals of the Darboux Correspondence [10].

**Connection between Darboux Transformation and Discrete $(n,k)$-paths.** A discrete $(n,k)$-path satisfying the trapezoidal condition may be interpreted in terms of the Darboux Transform. Indeed, given such a path, we consider the periodic equilateral linkages $L_i = \ldots V_{0+i}V_{k+i}V_{2k+i} \ldots$ for $i = 0, 1, \ldots, k-1$. The trapezoidal condition implies that there is a Darboux Correspondence $D(L_i) = L_{i+1}$ of the same parameter (since the $(n,k)$-path is equilateral) for consecutive linkages (see Figure 4).
Also, Similarly, Using equation (4.1),

From the isosceles trapezoids, it follows that

Theorem 3. The Darboux transformation is area preserving on periodic polygonal paths.

Proof. Let $P$ and $P'$ be two periodic polygonal paths in Darboux correspondence. We show that the difference of the areas of $P$ and $P'$ is zero. We denote the vertex of $P'$ which corresponds via the Darboux Transformation to the vertex $V_i$ in $P$ by $V_i'$ for each $i$. We have

\[ |P| = \sum_{i=0}^{n-1} |\bar{V}_iV_iV_{i+1}\bar{V}_{i+1}|, \]

and similarly for $P'$. Therefore

\[ |P| - |P'| = \sum_{i=0}^{n-1} |\bar{V}_iV_iV_{i+1}\bar{V}_{i+1}| - |\bar{V}_i'V_i'V_{i+1}\bar{V}_{i+1}'|. \]

From the isosceles trapezoids,

\[ |V_iV_{i+1}V_{i+1}'| = |V_i'V_{i+1}'V_i|. \]

Also,

\[ |\bar{V}_iV_iV_{i+1}\bar{V}_{i+1}| = |\bar{V}_iV_iV_{i+1}\bar{V}_{i+1}| + |\bar{V}_{i+1}V_{i+1}\bar{V}_{i+1}V_i| + |V_iV_{i+1}\bar{V}_{i+1}|. \]

Similarly,

\[ |\bar{V}_i'V_i'V_{i+1}\bar{V}_{i+1}'| = |\bar{V}_i'V_i'V_{i+1}\bar{V}_{i+1}'| + |\bar{V}_{i+1}V_{i+1}\bar{V}_{i+1}'| + |V_i'V_{i+1}\bar{V}_{i+1}'|. \]

Using equation (4.1),

\[ |\bar{V}_iV_iV_{i+1}\bar{V}_{i+1}| - |\bar{V}_i'V_i'V_{i+1}\bar{V}_{i+1}'| = |\bar{V}_{i+1}V_{i+1}\bar{V}_{i+1}V_i| - |\bar{V}_{i+1}V_{i+1}\bar{V}_{i+1}V_i|. \]

It follows that

\[ |P| - |P'| = \sum_{i=0}^{n-1} |\bar{V}_{i+1}V_{i+1}'V_iV_{i+1}| - |\bar{V}_{i+1}V_{i+1}'V_iV_{i+1}|, \]

which telescopes to

\[ |P| - |P'| = |\bar{V}_nV_n\bar{V}_nV_{n}\bar{V}_{n}| - |\bar{V}_0V_0\bar{V}_0V_{0}|. \]
Figure 5. Two periodic paths $P$ and $P'$ in Darboux Correspondence. By Theorem 3, the two paths have equal areas under the curve.

Since $V'_n = V'_0 + e_1$ and $V_n = V_0 + e_1$, it follows that $\overrightarrow{V_nV'_n} = \overrightarrow{V_0V'_0}$ and $|\overrightarrow{V'_nV'_nV_n}| = |\overrightarrow{V'_0V'_0V_0}|$, so that $|P| = |P'|$. □

5. Questions

We end our discussion with some research topics and questions of interest concerning bicycle $(n,k)$-paths.

1. Construct interesting families of bicycle $(n,k)$-paths. For example, ones for which the condition $x_j = \frac{j}{n}$ does not hold.

2. What is the $m$th order ($m \in \mathbb{N}$) infinitesimal rigidity theory of bicycle $(n,k)$-paths like?

3. For closed bicycle polygons, there are many integrals of motion [10]. For example, a geometric center called the Circumcenter of Mass [11] is invariant under Darboux Transformation for closed polygons. Are there other integrals of motion for bicycle $(n,k)$-paths?

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