The Limit Shape of the Leaky Abelian Sandpile Model

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The **Leaky Abelian Sandpile Model (Leaky-ASM)** is a cellular automaton defined on a graph $G = (V, E)$.

- An initial sandpile distribution $s_0 : V \rightarrow \mathbb{R}_{\geq 0}$
- Dissipation $d > 1$

The sandpile evolves through toppling unstable vertices. If $s_0(v) > d \cdot \deg(v)$ then the vertex $v$ is unstable and topples resulting in the new sandpile $s_1$ where

$$s_1(v) = s_0(v) - d \cdot \deg(v)$$

and for $u \neq v$

$$s_1(u) = \begin{cases} 
    s_0(u) + 1 & \text{if} \quad u \sim v \\
    s_0(u) & \text{if} \quad u \not\sim v.
\end{cases}$$
Let $G = \mathbb{Z}^2$ with $s_0 = n\delta_{(0,0)}$ and topple until stable. The final sandpile has a limit shape (Pegden-Smart 2013). The limit shape is bounded between circles of radii $c_1\sqrt{n}$ and $c_2\sqrt{n}$ with $c_2/c_1 = \sqrt{3}/\sqrt{2}$ (Levine-Peres 2008).

Figure: ASM with $n = 10^7$ and $d = 1$

Is the limit shape a circle? Is it a polygon?
Main Result

- $D_{n,d}$ is the set of sites which accumulate sand

**Theorem (A.- Mkrtchyan (2020))**

Let $d > 1$ and $r = \log n - \frac{1}{2} \log \log n$. The boundary of $r^{-1} D_{n,d}$ converges to the dual of the boundary of the gaseous phase in the amoeba of

$$P(z, w) = \frac{4d - z - z^{-1} - w - w^{-1}}{4(d - 1)}.$$
Figure: Simulations of the Leaky-ASM

(a) $d = 1.05$   (b) $d = 2$   (c) $d = 1000$

Figure: Limit shapes
The amoeba of a polynomial $P(z, w)$ is the image of
$\{(z, w) \in \mathbb{C}^2 : P(z, w) = 0\}$ under the map

$$(z, w) \mapsto (\log |z|, \log |w|).$$

Figure: The boundary of the amoeba of $P(z, w) = \frac{4d - z - z^{-1} - w - w^{-1}}{4(d-1)}$ and its dual curve. The limit shape is given by the curve in red.
Theorem (A.- Mkrtchyan (2020))

As \( d \to 1 \) the Leaky-ASM converges to the ASM with background height \(-1\).

(a) \( d - 1 = 2.5 \cdot 10^{-6} \)

(b) \( d - 1 = 2.5 \cdot 10^{-7} \)

(c) ASM with height \(-1\)