Problem 37:

a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that $f$ is not 1-1.

Suppose $f$ is 1-1. Then $f$ is also surjective onto its image, $f(\mathbb{R}^2)$ in $\mathbb{R}$. In particular, $f: D \to A \subset \mathbb{R}$ where $D$ is the closed unit disk is a homeomorphism. However, notice that $A - \{x\}$ for $x \in f(f(A))$ has two connected components while $f^{-1}(A - \{x\})$ is still connected, a contradiction.

b) Generalize this result to the case of a continuously differentiable function $f: \mathbb{R}^m \to \mathbb{R}^n$, $m \leq n$.

Okay, we won't try to use topology this time. We may apply the implicit function theorem. The Jacobian of $f$ is an $m \times n$ matrix with rank at most $m$. Also, we may assume that $f' \neq 0$ for at least one point. (Otherwise, $f$ is constant and definitely not 1-1).

In fact, points where $f' = 0$ must be isolated (otherwise $f$ is constant on a set). Thus we may choose a subset $A \subset \mathbb{R}^n$ where $f' \neq 0 \forall x \in A$. Next, define $F: A \to \mathbb{R}^m$ by $F(x_1, ..., x_n) = (f(x_1), ..., f(x_n))$.

We have that

$$F'(x) = \begin{pmatrix} Df(x_1) & Df(x_2) & \cdots & Df(x_m) \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

So we may apply problem 36 to say that $F$ has a differentiable inverse on its image, $F(A)$, i.e. $x \in \mathbb{R}^n$ and set $x = F^{-1}(y)$, $y \in \mathbb{R}^m$. Choose $x$ so that $F(x) = (f(x), x_1, ..., x_n) = (y, x_1, ..., x_n) \in A$. Then $F^{-1}(y, x_1, ..., x_n) = x$. However, there are points $(y, x)$ with $1 \neq x$ in $A$ and $F^{-1}(y, x)$ for such points in $A$ is not defined. => contradiction.

Problem 38:

a) If $f: \mathbb{R} \to \mathbb{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, show that $f$ is 1-1 (on all of $\mathbb{R}$).

Suppose $f$ is continuous and $f'(a) \neq 0 \forall a \in \mathbb{R}$. Suppose also $\exists x, y \in \mathbb{R}$ such that $f(x) = f(y)$.

Then by the mean value theorem, $\exists c \in [x, y]$ such that $f'(c) = \frac{f(y) - f(x)}{y - x} = 0$ which is a contradiction.

b) Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $\det f'(x, y) \neq 0 \forall (x, y)$ but $f$ is not 1-1.

$$\det f'(x) = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x} (\cos^2 y + \sin^2 y) = e^{2x} \text{ which is clearly always positive and never zero. However, } e^x \cos y = e^x \sin \theta \text{ whenever } \cos \theta = \sin \theta, \text{ which happens, for example, when } x = y = \frac{\pi}{2} \text{ so } f \text{ is not 1-1.}$$

* This means that if $\det f' \neq 0$ at $a$, you can only conclude that $f$ is a local homeomorphism (i.e. continuous bijection) but you need stronger conditions to get a global homeomorphism. x
Problem 40: Use the Implicit Function Theorem to re-do Problem 2-15(c) which says:

If \((a(t), b(t)) \neq 0\) for all \(t\) and \(b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R}\) are differentiable, let \(s_1, \ldots, s_n : \mathbb{R} \to \mathbb{R}\) be the functions such that \(s_i(t), \ldots, s_n(t)\) are the solutions of the equations 
\[
\sum_{j=1}^{n} a_{ij}(t) s_j(t) = b_i(t)\quad (i = 1, \ldots, n).
\]

Show that \(s_i\) is differentiable and find \(s_i'(t)\).

* Define \(F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\) by 
\[
F(x_1, \ldots, x_n) = \left( \sum_{j=1}^{n} a_{ij}(t) x_j - b_i(t), \ldots, \sum_{j=1}^{n} a_{nj}(t) x_j - b_n(t) \right).
\]
So we have 
\[
D_i F(x_1, \ldots, x_n) = \left[ a_{ij} \right]
\]
for \(i \neq j\).

Clearly, \([a_{ij}]\) has det \(\neq 0\) \(\Rightarrow\) \(D_i F(x_1, \ldots, x_n)\) is invertible.

To find \(s_i'(t)\), note that 
\[
F'(x, s) = \left( \sum_{j=1}^{n} a_{ij}(t) s_j(t) - b_i(t), \ldots, \sum_{j=1}^{n} a_{nj}(t) s_j(t) - b_n(t) \right) = 0.
\]
Then just take the derivative of both sides. (A similar thing is done after the proof of the implicit function theorem. Read that for details).

Problem 41: Let \(f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) be differentiable. For each \(x \in \mathbb{R}\) define \(g_x : \mathbb{R} \to \mathbb{R}\) by 
\[
g_x(q) = f(x, q).
\]
Suppose that for each \(x\) there is a unique \(q_x\) with \(g_x'(q_x) = 0\). Let \(c(x)\) be this \(q_x\).

a) If \(D_{x_1} f(x_1, q_1) \neq 0\) for all \((x, q)\), show that \(c\) is differentiable and 
\[
c'(x) = -\frac{D_{x_2} f(x, c(x))}{D_{x_1} f(x, c(x))},
\]
First, assume \(f\) is at least twice differentiable.

b) Show that if \(c'(x) = 0\), then for some \(y\) we have 
\[
D_{x_1} f(x, y) = 0, \quad D_{x_2} f(x, y) = 0.
\]

Then \(c'(x) = 0 \Rightarrow D_{x_1} f(x, c(x)) = 0\) by part a). Also \((D_x f)(x, c(x)) = 0\) also follows from a).

c) Let \(f(x, y) = x \log_2 y - y \log x\). Find \(\max_{1/2 \leq x \leq 2} (\min_{1/2 \leq y \leq 2} f(x, y))\).

We must assume that \(x, q_x \geq 0\).

\[
D_2 f(x, q_x) = x \log_2 y - \log x
\]

\[
\Rightarrow D_2 f(x, q_x) = 0 \text{ when } y = x^{1/2} \Rightarrow \text{we may apply a) with } c(x) = x^{1/2}
\]

\[
D_2 f(x, q_x) = \frac{y}{x} > 0 \Rightarrow \min_{q_x} f(x, q_x) = c(x)
\]

but we want to minimize over \(q \in [1/2, 1]\).

\[
c(x) = x^{1/2} - 1 \log_2 x > 0 \text{ for } x < e. \quad \text{Also } c(x) = 1 \Rightarrow x = 1 + 3^{1/b} \text{ with } e^{c(b)} = 1/3.
\]

For \(x_0, \min_{q \in [1/2, 1]} f(x, y)\) happens when \(y = b\) if \(1/2 \leq x \leq 1\) or \(y = c(b) = 1/3\) if \(b \leq x \leq 1\) and at \(y = 1\) if \(1 \leq x \leq 2\). So you have to check several cases.\footnote{In the end, \(\max\) happens at \((1/2, 1/3)\) and \(f(1/2, 1/3) = \frac{1}{6} \log_2 (4/3)\).}

See the page below for a more detailed computation.
\[ f(x,y) = x(y \log y - y) - y \log x \]

\[
\begin{align*}
\frac{\partial}{\partial x} f(x,y) &= \frac{y^2}{y} - x \log y - x - \log x \\
x \log y - \log x &= 0 \Rightarrow x \log y = \log x \\
\log y &= \log(x^{1/2}) \Rightarrow y = x^{1/2} \\
\end{align*}
\]

Furthermore, \( \frac{\partial}{\partial y} f(x,y) = \frac{x}{y} \) which is always positive for \( x,y > 0 \).

\[
\Rightarrow \text{For any fixed } x \in [1/2, 2], \text{ the minimum happens when } y = x^{1/2}. \text{ However, we have to be careful.} \\
y = x^{1/2} \Rightarrow 1 \text{ when } x = 1, \text{ so this minimum will happen outside our green box when } x > 1. \\
\text{Also } (\frac{1}{2})^{1/2} = \frac{1}{\sqrt{2}} \text{ and } x^{1/2} = 1 \Rightarrow \text{The intermediate value theorem says that the} \]
\[
\text{curve } y(x) = x^{1/2} \text{ takes on the value } \frac{1}{2} \text{ inside } (1/2, 1), \text{ say when } x = a. \\
\Rightarrow \frac{\partial}{\partial y} f(x,y) = \frac{x}{y} \text{ crosses the line } y = \frac{1}{2} \text{ somewhere.} \\
\text{When } y > x^{1/2} \text{ (i.e. "above" the blue curve) } \frac{\partial}{\partial y} f(x,y) \text{ for fixed } x \text{ is increasing.} \\
\text{When } y < x^{1/2} \text{ the opposite is true, and the minimum of } \frac{\partial}{\partial y} f(x,y) \text{ happens when } y = x^{1/2} \\
\Rightarrow \text{We want to maximize } f \text{ along the curve } y(x): \]

\[
\gamma(x) = \begin{cases} 
\gamma_1 = \frac{1}{2}, & x \in [1/2, a) \\
\gamma_2 = x^{1/2}, & x \in [a, 1) \\
\gamma_3 = 1, & x \in [1, 2] 
\end{cases}
\]

\[
\frac{\partial}{\partial x} f(x) = \frac{y^2}{y} - x \log y - \frac{1}{2} \log x - \frac{1}{2} \log x = \log \left( \frac{y}{x} \right) - 1
\]

\[
\frac{d}{dx} f_1(x) = -x^{1/2} \log(x^{1/2}) + \frac{x}{x^{1/2}} \log(x) = x^{1/2} (-x^{1/2} \log x - x^{1/2} \log x) = -x^{1/2} \log x
\]

\[
\text{On the other hand, } f_2(x) = f |_{y=1} = x \cdot x^{1/2} \log(x^{1/2}) - x \cdot x^{1/2} - x^{1/2} \log(x) = x^{1/2} \log(x) - x^{1/2} \log(x) - x^{1/2} = -x^{1/2} \log x
\]

\[
\text{which is always decreasing and also negative.} \\
\Rightarrow \max_{x \in [0, 1]} f_2(x) = f_2(0) = 0.
\]

\[
\frac{\partial}{\partial x} f_3(x) = f |_{y=x} = -x - \log(x) \text{ is also decreasing} \Rightarrow \max_{x \in [1, 2]} f_3(x) = f_3(1) = -1.
\]

\[
\text{Note that } f(1/2, 1/2) = \frac{1}{2} \log(1/2) - \frac{1}{2} - \frac{1}{2} \log(1/2) = -\frac{1}{4} + \frac{1}{2} \log(2) - \frac{1}{2} \log(2) = \frac{1}{4} (\log(2) - 1) \\
\Rightarrow 1 > -1 \Rightarrow \frac{1}{2} (\log(1/2) - 1) \text{ is the max over the whole interval.}
\]
Problem 1: Let $f: [0,1] \times [0,1] \rightarrow \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2, \\ 1 & \text{if } 1/2 \leq x \leq 1 \end{cases}$. Show that $f$ is integrable and $\int_{[0,1] \times [0,1]} f = 1/2$.

- We define the obvious partition for $[0,1] \times [0,1]$. $P_1 = \{0, 1/2, 1\}$ and $P_2 = \{0, 1/2, 1\}$. Then clearly $m_S(f) = \inf \{f(x) : x \in S\} = 0$ and $M_S(f) = \sup \{f(x) : x \in S\} = 1$ where $S$ is a subrectangle of $[0,1] \times [0,1]$, and $m_S(f) = M_S(f) = 0$.

- So clearly, $U(f, P) = U(f, P_1) = \text{Volume of a rectangular prism with base } 1/2, \text{height}=1, \text{and width } = 1/2$.

Problem 2: Let $f: A \rightarrow \mathbb{R}$ be integrable and let $g = f$ except at finitely many points. Show $g$ is integrable and $\int_A f = \int_A g$.

- This is clear if you define your partitions so that the look like $(a_i, b_i)$ where all the points at which $g$ differs from $f$ appear as an $a_i$ or $b_i$ and are never contained in the interior of an interval in your partition. (You guys should work out all the details.)

Problem 3: Let $f: A \rightarrow \mathbb{R}$ and let $P$ be a partition of $A$. Show that $f$ is integrable if and only if for each subrectangle $S$, the function $f|_S$ is integrable and that in this case $\int_A f = \sum S f|_S$.

- First suppose that $f$ is integrable on $A$ but there is some subrectangle $S$ over which $f$ is not integrable. Then $\inf U(f|_S, S) - \sup L(f|_S, S) > 0 \implies$ for any partition $U(f, P_1) - L(f, P_1) > 0$ which is a contradiction since $f$ is integrable.

- On the other hand let $\Delta M = \max S \{U(f|_S, S) - L(f|_S, S)\}$ and suppose there are $n$ rectangles in our partition. Then $U(f, P_1) - L(f, P_1) \leq n \Delta M$ but we can make $\Delta M \rightarrow 0$ by refining $P$, since $f|_S$ is integrable. So $U(f, P_1) - L(f, P_1) \rightarrow 0 \implies f$ is also integrable.

Problem 4: Let $f: [0,1] \times [0,1] \rightarrow \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 0 & \text{if } x \text{ irrational, } y \text{ rational, } x/y \text{ irrational} \\ 0 & \text{if } x/y \text{ rational, } y = p/q \text{ in lowest terms} \\ 1 & \text{otherwise} \end{cases}$

- Show that $f$ is integrable and $\int_{[0,1] \times [0,1]} f = 0$.

- Recall that $m_S = \inf \{f(x) : x \in S\}$. So clearly, $m_S = 0$ for any partition of $A = [0,1] \times [0,1]$. Therefore it suffices to show that $M_S \rightarrow 0$ as $S \rightarrow 0$. But that's easy because we can just let $P = \{(0,1/2, \ldots, 0\cdot (1/2^k), 1/2) : k \geq 1\}$.

- So if $x \in [p/q, (p+1)/q]$ with $p < q$, $x = a/b$, $b \geq q$, then $M_S(f|_S) = 1/q^2$ and since $\varepsilon$ can be chosen to be arbitrary large, $\int_A f = \inf U(f, P) = \inf_S \varepsilon \cdot 1/q^2 = 0$. 


Problem 11: Let $A \subseteq [0,1]$ be the union of open intervals $(a_i, b_i)$ such that each rational number in $(0,1)$ is contained in some $(a_i, b_i)$. Then $A = [0,1] - A$. If $\sum_i (b_i - a_i) < 1$, show that $A$ does not have measure zero.

- Recall that a subset $A$ of $\mathbb{R}^n$ has measure 0 if for every $\varepsilon > 0$ there exists a cover $\{U_i\}$ of $A$ by closed rectangles such that $\sum_i v(U_i) < \varepsilon$. In our case, since $A \subseteq [0,1]$, $v(U_i) = (b_i - a_i)$ for every $U_i$.

- Suppose that there was a covering of $[0,1] - A$ by closed intervals $\{[a_i, b_i]\}$ such that $\sum_i (b_i - a_i) < \varepsilon$ for any $\varepsilon$. By problem 18, $[0,1] - A \implies A = [0,1] - 3A \implies \sum_i (b_i - a_i) = 1 - \varepsilon$.

- This is a contradiction because each $[a_i, b_i]$ is “very small” compared to $[0,1]$.

*Here I have left out an important detail. This proof only works when your covering is countable. Make sure you see why that's important.*

Problem 12: Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Show that $\{(x, f(x)) \mid x \in [a, b]\}$ has measure zero.

- Hint: Use problem 1-30 to show that $\sum_{i=1}^n q_i < 1$ is finite for each integer $n$.

- Recall that problem 1-30 asked us to show that if $x_1, \ldots, x_n \in [a, b]$ are distinct, then $\sum_{i=1}^n q_i < 1$.

- I will prove this in a slightly different way but it should be equivalent to using 1-30.

- Step 1: The set of discontinuities $A$ is at most countable. For any $x \in A$, $f(x^-) < f(x) < f(x^+)$ where $f(x^-)$ and $f(x^+)$ are the left and right hand limits near $x$, and $\mathbb{Q}$ is some rational number between them. For $x, x' \in A$, $f(x) < f(x')$, so $f(x^-) < q(x) = q(x')$.

- Thus the map $\mathbb{Q} \to \mathbb{R}$ given by $x \to q(x)$ is injective and surjective onto its image.

- Step 2: Any countable subset of $\mathbb{R}$ has measure zero. Let $\mathcal{A}$ be a countable subset of $\mathbb{R}$, a.e. $A$. Clearly $\mathbb{Q} \cap [a - \delta_i, a + \delta_i] \cap V > \varepsilon$. Thus we have $\sum_i V(U_i)$ covering $A$ such that $\sum_i v(U_i) < \varepsilon \implies$ a countable sum $\sum_{i=1}^\infty v(U_i) \to 0$ as $\varepsilon \to 0 \implies$ any countable subset of $\mathbb{R}$, in particular the set of discontinuities of $f$ has measure zero.