Problem 4.1: Let $e_1, \ldots, e_n$ be the usual basis of $\mathbb{R}^n$ and let $q_1, \ldots, q_n$ be the dual basis.

(a) Show that $q_i, \ldots, q_n$ are linearly independent. What would the right side be if we forgot the coefficient $\frac{(k+1)!}{k!}$?

(b) By Theorem 4.1 we have $q_i, \ldots, q_n$ are linearly independent. What would the right side be if we forgot the coefficient $\frac{(k+1)!}{k!}$?

If $\frac{(k+1)!}{k!}$ did not appear in the definition of $\Lambda$, then we would get $k!$.

(b) Show that $q_i, \ldots, q_n$ is the determinant of the $k \times k$ minor of $\left(\begin{array}{c} v_1 \\ \vdots \\ v_k \end{array}\right)$ obtained by selecting columns $i, \ldots, n$.

(c) Since $\{e_1, \ldots, e_k\}$ is a basis, we can write $v_i = \sum_{j=1}^k a_{ij} e_j$. We have

$\det(a_{ij}) q_i, \ldots, q_n (e_1, \ldots, e_k) = \det(a_{ij}) q_i, \ldots, q_n (e_1, \ldots, e_k)$

Problem 2: If $f: V \to V$ is a linear transformation and $\dim V = n$, then $f^*: \Lambda^n(V) \to \Lambda^n(V)$ must be multiplication by some constant $c$. Show that $c = \det f$.

Recall that $\dim \Lambda^n(V) = 1 + n!$ if $V$ is a vector space. This is why $f^*$ is multiplication by a constant $c$. Let $e = \{e_1, \ldots, e_n\}$ be a basis for $V$ and $e^* = \{e_1^*, \ldots, e_n^*\}$ the dual basis. We just calculate:

$\det f(q_1, \ldots, q_n)(e_1, \ldots, e_n) = q_1, \ldots, q_n(f(e_1), \ldots, f(e_n)) = \det f(q_1, \ldots, q_n)(e_1, \ldots, e_n)$

The fact that forms are multilinear $\Rightarrow$ this is true for any $(x_1, \ldots, x_n) \in V^n$.

Problem 3: If $\omega \in \Lambda^n(V)$ is the volume element determined by the inner product $T$ on the measure $\mu$, and $\omega, \ldots, \omega, \omega, \ldots, \omega$ is a vector, show that $\int \omega \ldots \omega = \det(\omega_{ij})$, where $\omega_{ij} = T(\omega_i, \omega_j)$. Hint: If $\omega_1, \ldots, \omega_n$ is an orthonormal basis and $\omega_i = \sum_{j=1}^n a_{ij} \omega_j$, then:

Following the hint, let $v = \{v_1, \ldots, v_n\}$ be an orthonormal basis and $a_{ij} = \sum_{j=1}^n a_{ij} \omega_j$

Theorem 4.6 $\Rightarrow$ $\omega(\omega_1, \ldots, \omega_n) = \det(\omega_{ij})$, $\omega(\omega_1, \ldots, \omega_n) = \det(\omega_{ij})$

Let $A$ be the matrix $a_{ij} = T(\omega_i, \omega_j)$ and $A$ the matrix $a_{ij}$

Then $A A^T = \mu \Rightarrow \det(A) = \det(A^T) \Rightarrow \text{done}$.

Problem 5: If $c: [0,1] \to (\mathbb{R}^n)^n$ is continuous and each $(c^*(c), \ldots, c^*(c))$ is a basis for $\mathbb{R}^n$, show that $[c^*(c), \ldots, c^*(c)] = [c^*(c), \ldots, c^*(c)]$.

Let $c^*(c) = \sum_{j=1}^n a_{ij} \omega_j$.

Let $A(c) = a_{ij}(c)$.

$\Rightarrow$ take determinants of both sides. Note that $\det(A(c))$ is continuous and doesn't change signs $\Rightarrow$ done.
Problem 10: If $w_1, \ldots, w_n \in \mathbb{R}^n$ show that $|u, x \times x w_n| = \det(g_{ij})$ where $g_{ij} = \langle w_i, w_j \rangle$

* $\langle w, w \rangle = g(e) = \det(v_1, \ldots, v_n)^T$
* Let $q = 0, x \times x w_n \times 2 = \frac{2}{3}g$ Then define $g \in \mathbb{R}^{n+1}(v)$ by $q(x, \ldots, x w_n) = \det(x, \ldots, x w_n, q)$ so that $q(0, \ldots, x w_n) = \frac{2}{3}g$

* Let $V = \text{span}(w_1, \ldots, w_n)$. Let $(v_1, \ldots, v_n)$ be an orthonormal basis for $V$ so that $(v_1, \ldots, v_n, \frac{2}{3}g)$ is an orthonormal basis for $\mathbb{R}^n$ => $q(v_1, \ldots, v_n) = \pm 1$ => if $\det(v_1, \ldots, v_n) = \mu$ $\implies (\langle v, v \rangle) = q$ is a volume element so $\det$

Theorem 4-6: we are done.

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Problem 13:

a) If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^m \to \mathbb{R}^n$, show that $(g \circ f)^* = g^* \circ f^*$ and $(g \circ f)^* = f^* \circ g^*$

$$(g \circ f)^*(v_p) = \langle g(f(p)), v_p \rangle = \langle f(p), g^*(v_p) \rangle = \langle f(p), \text{tr}(Dg(p)) = f^* \circ g^*(v_p) \rangle$$

b) If $g: \mathbb{R}^n \to \mathbb{R}$ show that $d(g \circ f) = f^* \circ g \circ d f$.

This is a direct consequence of Leibnitz rule.

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Problem 21: Prove that, on the set where $\theta$ is defined, we have $d\theta = \frac{x}{x^2+y^2} dx + \frac{y}{x^2+y^2} dy$.

* $\theta$ is defined in problem 3-41 as

\[
\theta(x, y) = \begin{cases} 
\arctan(y/x) & x > 0 \ y > 0 \\
\pi + \arctan(y/x) & x < 0 \ y > 0 \\
\pi - \arctan(y/x) & x > 0 \ y < 0 \\
3\pi/2 & x = 0 \ y > 0 \\
3\pi/2 & x = 0 \ y < 0 
\end{cases}
\]

This is a straightforward application of the definitions and only involves computing $\frac{\partial}{\partial x} \arctan(y/x)$ and $\frac{\partial}{\partial y} \arctan(y/x)$, which is easy.

Problem 23: For $R > 0$ and $n$ an integer, define the singular 1-cube $C_{R,n}: [0,1] \to \mathbb{R}^2$ by $C_{R,n}(t) = (R \cos(2n \pi t), R \sin(2n \pi t))$. Show that there is a singular 2-cube $c: [0,1]^2 \to \mathbb{R}^2$ such that $C_{R,n} - C_{R,n} = \partial c$.

* Define $c: [0,1]^2 \to \mathbb{R}^2$ by $c(x,y) = x C_{R,n} - (1-x) C_{R,n}$

Then $\partial c = 4 \pi^2 (c(0,y) + c(1-y) + c(1-x) + c(x)) = C_{R,n} - C_{R,n}$.