Lemma: $p'$ refines $p$. Then $\mathcal{L}(f, p) \leq \mathcal{L}(f, p')$ and $u(f, p) \geq u(f, p')$

Proof: Let $S \in p$ be a rectangle: $S = S_1 \cup S_2 \cup \ldots \cup S_n$

so $v(S) = v(S_1) + \ldots + v(S_n)$

By definition, $ms(S) \leq ms_i(S)$ since $S_i \subseteq S$, so the values of $f(x)$ could be smaller in $S$.

Thus $ms(S) v(S) = \sum_{i=1}^{d} ms(S) v(S_i) \leq \sum_{i=1}^{d} ms_i(S) v(S_i)$

$\implies \mathcal{L}(f, p) \leq \mathcal{L}(f, p')$.

Same argument works for upper sums.

Idea to remember (1d): refine

$\rightarrow$ $S_1 \quad S_2$

$ms(S) \leq ms_1(S)$

$\leq ms_2(S)$

$\implies Ms(S) \geq Ms_1(S)$

$\geq Ms_2(S)$

Another key idea: If $p, p'$ are partitions and you are proving something about them, it is often useful to consider a partition $p''$ that refines both of them.
Corollary: If \( P, P' \) are partitions, then \( L(f, P) \leq U(f, P') \).

Proof: Let \( P'' \) be a refinement of both \( P \) and \( P' \). (Why does it exist? Then \( L(f, P'') \leq L(f, P) \leq U(f, P'') \leq U(f, P) \).

By above.

Definition: \( f \) is integrable on \( A \) (rectangle) if \( f : A \to \mathbb{R} \)

\[ \text{if } f \text{ is bounded and } \sup_{\overline{E}} f(x) = \inf_{\underline{E}} f(x) \]

Example: \( f(x) = \begin{cases} 0, & x \in \mathbb{Q} - \text{rationals} \\ 1, & x \notin \mathbb{Q} \end{cases} \)

\( A = [0,1] \)

Take any partition of \([0,1]\). Then any \( S \cap P \) contains rationals, which implies that \( m_S(f) = 0 \). Similarly \( M_S(f) = 1 \), so \( f \) is not integrable.

Simple criterion: A bounded function \( f: A \to \mathbb{R} \) is integrable if and only for every \( \varepsilon > 0 \) there exists a partition \( P \) of \( A \)

\[ U(f, P) - L(f, P) < \varepsilon. \]
Proof: If the condition holds, then \( \sup \| L_c'(P_p') \| = \in\{ \sup L_c'(P_p') \} \) by definition.

Conversely, suppose that \( \sup \| L_c'(P_p') \| = \in\{ \sup L_c'(P_p') \} \). Then for any \( \epsilon > 0 \), \( \exists \ p, p' \in U(S, P) - L(S, P') < \epsilon \). Let \( p'' \) be a refinement of both \( p, p' \). Then

\[
U(S, P) - L(S, P) \leq U(S, P) - L(S, P') < \epsilon
\]

and we are done.

Measure 0: \( A \subseteq \mathbb{R}^n \) has measure 0 if there is a cover \( \{ U_i, U_1, \ldots, U_n \} \) of \( A \) by closed rectangles \( \gamma \sum v(U_i) < \epsilon \), like the rationals \( \frac{\epsilon}{2^n} \).

Example: A countable, i.e. \( A = \{ q_i \} \).

Take a rectangle of side-length \( \frac{\epsilon}{2^n} \) centered at each \( q_i \). Then \( \sum v(U_i) \leq \sum \frac{\epsilon}{2^n} = \epsilon \).

But things get more insane. Consider

\[
0 \begin{bmatrix} 3 \ 3 \ 4 \ end{bmatrix} 1 \begin{bmatrix} E_i \ what \ is \ left \ get \ rid \ of \ the \ middle \ part \end{bmatrix}
\]
Let $C = \bigcap_{i=1}^{\infty} E_i$ be closed and bounded, hence compact.

Clearly, $C \subseteq E_n$ for each $n$. Observe that $v(E_n) = 2^{-n} \cdot 3^{-n} \to 0$ as $n \to \infty$.

It follows that $C$ has measure 0!

And yet, $C$ is uncountable. Why? Because $C$ contains all numbers in $[0,1]$ with 0's and 2's in their decimal binary expansion and you get all of $[0,1]$.

Definition: $A \subseteq \mathbb{R}^n$ has $n$-dimensional content 0 if for every $\varepsilon > 0$ a finite cover $\{U_1, \ldots, U_n\}$ of closed rectangles

\[ \sum_{i=1}^{n} v(U_i) < \varepsilon \]
If \( a < b \), then \([a,b]\) does not have content 0.

If \( \{\mathcal{U}_i\}_{i=1}^n \) is a finite cover, then
\[
\sum_{i=1}^n v(\mathcal{U}_i) \geq b-a
\]
by closed intervals.

**Proof:** Let \( a = t_0 < t_1 < \ldots < t_k = b \) endpoints of \( \mathcal{U}_i \)’s. Then
\[
v(\mathcal{U}_i) = \sup \sum_{j=0}^{k-1} (t_j - t_{j+1}) \leq \sum_{j=0}^{k-1} (t_j - t_{j+1}) = b-a
\]
So
\[
\sum_{i=1}^n v(\mathcal{U}_i) \geq \sum_{j=1}^{k-1} (t_j - t_{j-1}) = b-a
\]

What about measure 0?

**Theorem:** If \( A \) is compact and has measure 0, then
\( A \) has content 0.

**Proof:** Let \( \varepsilon > 0 \). Since \( A \) has measure 0, there is a cover \( \mathcal{U}_1, \mathcal{U}_2, \ldots \) of \( A \) by open rectangles such that
\[
\sum_{i=1}^{n} v(\mathcal{U}_i) < \varepsilon.
\]
Since \( A \) is compact, a finite subcollection covers, so
\[
\sum_{i=1}^{n} v(\mathcal{U}_i) < \varepsilon.
\]

**Caution:** \( A = \mathbb{Q} \) has measure 0, especially in \([0,1]\), but
\[
\text{content}(A) \neq 0.
\]