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\[(X, \mathcal{F}, \mu) \quad \text{measure space}\]

\[f \in L^p(X) \quad \text{if} \quad \int_X |f(x)|^p \, dx \leq \infty \]

\[\mu \text{ is presumed} \quad \mathcal{X}\]

\[\|f\|_p = \left( \int_X |f(x)|^p \, dx \right)^{\frac{1}{p}}\]

\[\text{norm}\]

Note: \(L^p\) is a complete normed space by Math 471. Please read up on this if you need to.

\[\text{If} \quad \mu = \text{Lebesgue measure on } \mathbb{R}^d, \text{ we get}\]

\[\|f\|_p = \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}}\]

\[\text{If} \quad X = \mathbb{N}, \mu = \text{counting measure},\]

\[\|f\|_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}, \text{ which yields}\]

\[\ell^p(\mathbb{N})\]
Basic inequalities:

\[ \text{Hölder: } 1 < p < \infty \quad 1 < q < \infty \quad \frac{1}{p} + \frac{1}{q} = 1 \]

Let \( f \) and \( g \) be integrable functions, then \( fg \) is integrable and

\[ \|fg\|_1 \leq \|f\|_p \|g\|_q \]

Proof: \( A, B \geq 0 \quad 0 \leq \theta \leq 1 \)

\[ A \theta B^{1-\theta} \leq \theta A + (1-\theta)B \quad (\star) \]

To see this, note that we may assume that \( B \neq 0 \). Replace \( A \) by \( AB \theta \) and reduce \((\star)\) to

\[ (AB)^{\theta} B^{1-\theta} \leq \theta AB + (1-\theta)B \]

\[ AB \leq \theta AB + (1-\theta)B \]

\[ A \leq \theta A + (1-\theta) \]

Let \( f(x) = x^\theta - \Theta x - (1-\Theta) \)

\[ f'(x) = \Theta x^{\theta-1} - \Theta \text{ so } f' > 0 \text{ for } 0 \leq x < 1 \text{ and } f' < 0 \text{ for } x > 1. \]
We conclude that $f$ attains its maximum at $x=1$, where $f(1) = 0$. We conclude that $g(A) = 0$.

We are now ready to prove Hölder's. Divide $f$ by $\|f\|_p$ and $g$ by $\|g\|_q$, thus reducing to the case $\|f\|_p = \|g\|_q = 1$.

Let $A = |f(x)|$, $B = |g(x)|$, $\theta = \frac{1}{p}$, so

\[
|f(x)|^{\theta} = |g(x)|^{\frac{1}{q}} \leq \theta \cdot |f(x)|^{\theta} + \frac{1}{\theta} \cdot |g(x)|^{\theta} \text{ if the result follows by integration.}
\]

Minkowski: $1 \leq p \leq \infty$, if $g \in \mathbb{L}^p$, then $f + g \in \mathbb{L}^p$, and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof: Before establishing a stronger inequality, note that

\[
|s(x) + g(x)|^p = (s(x) + g(x))^p \chi_{s(x)} \leq |g(x)|^p \quad (s(x) + g(x))^p \chi_{s(x)} > |s(x)|^p \quad 2^p (|s(x)|^p + |g(x)|^p)
\]
The constant is bad \((2^p)\), but this shows that \(f + g \in L^p\). Also, the fact that \(p=1\) has not come up!

To get constant 1 for \(p \geq 1\), note that

\[
|g(x) + g(x)|^p \leq (g(x)|g(x)| + g(x))^p - 1
\]

\[
+ |g(x)| |g(x)|^{p-1}
\]

It follows that

\[
\|g + g\|_p^p \leq \|g\|_p^p \|g + g\|_p^p + \|g\|_p^{p-1} |g|_p^p
\]

\[
= (p-1) \delta = (p-1) \frac{1}{1 - \frac{1}{p}} = \frac{(p-1)p}{p}, \quad \text{so}
\]

\[
\|g + g\|_p \leq \|g\|_p^{p-\frac{p}{p}} \|g + g\|_p^\frac{p}{p} + \|g\|_p^{p-\frac{p}{p}} \|g\|_p^{\frac{p}{p}}
\]

\[
\Rightarrow \|g + g\|_p^{p-\frac{p}{p}} \leq \|g\|_p + \|g\|_p
\]

\[
\|g + g\|_p \leq \|g\|_p + \|g\|_p; \quad \text{as claimed}
\]