Random Fourier Series:
\[ \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \]
complex

Re-index: \[ g_n(t) = \begin{cases} g_{2n+1}(t), & n \geq 0 \\ g_{2n-1}(t), & n < 0 \end{cases} \]

Theorem: i) If \( \sum_{-\infty}^{\infty} |c_n|^2 < \infty \), then for a.e. \( \theta \in [0,1] \), the function
\[ f(\theta) = \sum_{-\infty}^{\infty} g_n(t)e^{in\theta} \]
\( \in L_p([0,2\pi]) \) for every \( p < \infty \).

ii) If \( \sum_{-\infty}^{\infty} |c_n|^2 = \infty \), then for a.e. \( \theta \in [0,1] \), the series above is not a series of an integrable function.

In the proof, we shall exploit the independence of Rademacher functions in yet another way.
Lemma: For each $p < \infty$ there is a bound $A_p$ so that

$$\|F\|_p \leq A_p \|F\|_2,$$

for all $F \in L^p([0,1])$ of the form

$$F(t) = \sum_{n=\infty}^{\infty} a_n s_n(t) \quad \text{(viewed as the $L^2$ limit of the partial sums)}$$

Proof: We may assume that $a_n$'s are real and

$$\|F\|_2^2 = \sum_{n=\infty}^{\infty} a_n^2 = 1.$$

Observation: If $\{s_n^2\}$ is a sequence of mutually independent real-valued functions, then

$$\{\Phi_n^2(s_n)\}$$

is also an independent sequence, where $\Phi_n: \mathbb{R} \to \mathbb{R}$ is continuous.

It follows that $\{e^{a_n s_n(t)}\}$ are mutually independent.

We conclude that if $F_N(t) = \sum_{n=1}^{N} a_n s_n(t)$, then

$$\int_0^1 F_N(t) \, dt = \int_0^N \left( \sum_{n=1}^{N} e^{a_n s_n(t)} \right) \, dt = \int_0^N \left( \sum_{n=1}^{N} \int_0^1 e^{a_n s_n(t)} \, dt \right).$$

mutual independence
Observe that \( \int_0^1 e^{an} \, dt = \cosh(an) \) (why?) because \( \cos \) takes values \(+1\) and \(-1\) on sets of measure \( \frac{1}{2} \).

We also need the inequality \( \cosh(x) \leq e^{x^2} \), \( x \in \mathbb{R} \) just compare the power series.

It follows that \( \int_0^1 e^{F_n(t)} \, dt \leq \frac{N}{11} \sum a_n^2 \leq \sum a_n^2 \leq e \).

Replacing \( an \) by \(-an\) yields the same inequality, so

\( \int_0^1 |F_n(t)| \, dt \leq 2e. \)

Let \( N \to \infty \) and conclude that \( \int_0^1 |F(t)| \, dt \leq 2e \).

In particular, integrable.

We finish off the proof with a simple trick. For each \( p \in [0, \infty) \) let \( u \mapsto e^{pu} \) if \( u \geq 0 \).

This implies that \( \|F\|_p \leq 2e c_p \) \( \Rightarrow \) lemma is proved w/ \( A_p = (2e c_p)^\frac{1}{p} \).
We are now ready to attack the theorem.

Assume that \( \sum_{-\infty}^{\infty} |c_n|^2 = 1 \) and define \( F(t) = \xi(t) \), 
\[ a_n = c_n e^{i n \theta}, \quad \theta \text{ fixed}. \]

Now, 
\[
\int_0^{2\pi} |F(t)|^p \, dt = \int_0^{2\pi} |\xi(t)|^p \, dt \leq A_p^p \]
by Parseval.

Recall that \( \theta \) is fixed, but the upper bound does not depend on \( \theta \). Integrating in \( \theta \) yields
\[
\int_0^{2\pi} \int_0^{2\pi} |\xi(t)|^p \, dt \, d\theta \leq 2\pi A_p^p
\]
by Fubini.

\[
\Rightarrow \quad \int_0^{2\pi} |\xi(t)|^p \, d\theta < \infty, \text{ for a.e. } t \in [0, 2\pi].
\]

The converse relies on elementary facts about Fourier series. Suppose that for a set \( E_1 \subset [0, 2\pi], \, m(E_1) > 0, \)
\[
\int_{E_1} (|0, 2\pi|, d\theta) \quad \text{for } t \in E_1.
\]

Every function in \( L([0, 2\pi]) \) has a Fourier series that is Cesàro summable almost everywhere.
It follows that \( \tilde{E} \subset [0,1] \times [0,2\pi] \) of positive \( L^2 \) measure and \( M < \infty \)

\[
\sup_N 16_N c_N(\theta) \leq M \quad \text{for each } (\varepsilon, \theta) \in \tilde{E}
\]

\[
\delta_N (\varepsilon, \theta) = \sum_{\ln 1 \leq N} s_n(\varepsilon) e^{i n \theta} \left( 1 - \frac{\ln 1}{N} \right)
\]

Fubini \( \Rightarrow \# ) \) holds for at least one \( \varepsilon_0 \)
and all \( \theta \in E \), \( m(E) > 0 \).

Let \( e^{i m \theta} = \alpha_n + i \beta_n \), \( \alpha_n, \beta_n \) real.

Lemma 1.6 implies that \( \exists M \) and \( N_0 \) s.t.

\[
\sup_N \sum_{\ln 1 \leq N} \alpha_n^2 \leq M \quad \Rightarrow \quad \sum_{\ln 1 \leq N} \alpha_n^2 \text{ converges}
\]

Similarly, \( \sum_{-\infty}^{\infty} \beta_n^2 \) converges and the resulting contradiction completes the proof.
Back to Bernoulli trials: Much of the theory we developed still goes through if we replace probability \( \frac{1}{2} \) in the Bernoulli trials by \( P \), \( P + Q = 1 \), \( 0 < P < 1 \).

We start by replacing the probability measure on \( \mathbb{Z}^\infty \) by measure \( m_2 \), where \( E03 \) is assigned measure \( P \) and \( E01 \) is assigned measure \( Q \).

In this setting, \( \frac{S_N}{N} \to P - Q \) and

\[
\Pr(\exists x: a < \frac{S_N(x) - N(P - Q)}{N^2} < b^2) \to \infty
\]

\[
\frac{1}{6^{1/2} \pi} \int_a^b e^{-t^2/2b^2} dt,
\]

where \( b^2 = 1 - (P - Q)^2 \).

We shall soon derive a general form of the Central Limit Theorem that will imply the one above.
Sums of independent random variables:

A sequence $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ is said to be identically distributed if

$$m(\exists x: \xi_n(x) \in B)$$

is the same for all $n$ given any Borel set $B$.

Theorem: Suppose $\xi_n$ is a sequence of functions that are mutually independent, are identically distributed and have mean $\mu_0$. Then

$$\frac{1}{N} \sum_{n=0}^{N-1} \xi_n(x) \rightarrow \mu_0 \quad \text{for a.e. } x \in X \quad \text{as } N \rightarrow \infty.$$

We are going to reduce this result to an "ergodic" theorem. We shall now develop the necessary definitions.

Define the joint distribution measure:

$$M_{\xi_0, \ldots, \xi_N}(B) = m(\exists x: (\xi_0(x), \ldots, \xi_N(x)) \in B)$$

Borel set

Suppose that $\xi_n$ is a sequence on a possibly different probability space $(X, m^* \pi Y)$ measure on $Y$ space

measure space
We say that \( \mathcal{E}_n \) and \( \mathcal{E}_m \) have the same joint distribution if

\[
\mu_{y_1, \ldots, y_n}(B) = \mu_{y_1, \ldots, y_m}(B) \quad \text{for all Borel sets in } \mathbb{R}^N.
\]

The relevant space \( Y \) will be constructed as follows:

Let \( Y = \prod_{j=0}^{\infty} R_j \), where each \( R_j \) is \( \mathbb{R} \).

On each \( R_j \) consider the measure \( m_j \), the common distribution measure of the \( \mathcal{E}_n \)'s. Define \( m^* \) to be the corresponding product measure on \( Y \).

Define a shift \( \tau: Y \to Y \) \( \tau(y) = (y_{n+1})_{n=0}^{\infty} \) if \( y = (y_n)_{n=0}^{\infty} \).

Define coordinate functions \( \langle \mathcal{E}_n \rangle \):

\[
g_n(y) = y_n \quad \text{if} \quad y = (y_n)_{n=0}^{\infty}.
\]

We shall establish the following:

1. \( g_n(\tau(y)) = g_n(y) \quad \forall n \geq 0 \implies g_n(y) = g_n(\tau^n(y)) \)

2. \( \tau \) is measure preserving and ergodic (to be defined in the next lecture)